Noname manuscript No.
(will be inserted by the editor)

A Survey of the Hysteretic Duhem Model

Fayçal Ikhouane

Received: date / Accepted: date

Abstract The Duhem model is a *simulacrum* of a complex and hazy reality: hysteresis. Introduced by Pierre Duhem to provide a mathematical representation of thermodynamical irreversibility, it is used to describe hysteresis in other areas of science and engineering. Our aim is to survey the relationship between the Duhem model as a mathematical representation, and hysteresis as the object of that representation.

Keywords Duhem model · Differential equations · Hysteresis

1 Prolegomenon

Citing a reference allows the author of a scientific article to attribute work and ideas to the correct source. Nonetheless, the process of describing that work and these ideas assumes some interpretation, at least of their relative importance. In order to ensure that the interpretation is reliable, we use, whenever adequate, a quotation from the reference so that the reader has a direct access to the cited source. This direct access is even more important when the source is not easily available like Ref. [67] or is not written in English like Refs. [16]–[22] among others, in which case we provide a translation. This is our approach to the literature review of Section 2.

Supported by grant DPI2016-77407-P (AEI/FEDER, UE) of the Spanish Ministry of Economy, Industry and Competitiveness.

F. Ikhouane
Universitat Politècnica de Catalunya, Department of Mathematics. Barcelona East School of Engineering, carrer Eduard Maristany, 16, 08019, Barcelona, Spain.
E-mail: faycal.ikhouane@upc.edu

In Sections 4–9 we proceed differently since our aim in these sections is to provide an accurate description of the results presented in the references under study. Because of the diversity of notations and nomenclature in these references, quotations may not be the best way to transmit that accurate description. Instead, we summarize the references using a unifying framework provided in Ref. [35]. The references we have chosen in Sections 4–9 are, in our opinion, those that are relevant to the subject of this study.

Our aim in this work is also to shed light on the relationships between the concepts introduced in this paper. To this end, we use a special form of the Duhem model, the scalar semilinear one, as a case study.

2 Introduction and literature review

`A brief history of the Duhem model.` The term *hysteresis* was coined by J. A. Ewing in 1881 to describe a specific relationship between the torsion of a magnetized wire and its polarization (although the phenomenon of hysteresis has been known and described by several authors before that date as shown in the literature review provided in Ref. [65]).

Quoting from Ewing's paper[28]: "*These curves exhibit, in a striking manner, a persistence of previous state, such as might be caused by molecular friction. The curves for the back and forth twists are irreversible, and include a wide area between them. The change of polarization lags behind the change of torsion. To this action ... the author now gives the name Hysterēsis (... to be behind)*".

In 1887 Lord Rayleigh models the relationship between a magnetizing force $F \in [-F_{\max}, F_{\max}]$ and the corresponding magnetization M using two polynomials

[59, p. 240]:

$$M = \alpha F + \beta F_{\max}^2 \left(1 - \frac{1}{2}\left(1 - \frac{F}{F_{\max}}\right)^2\right)$$
when F is decreasing,

$$M = \alpha F + \beta F_{\max}^2 \left(-1 + \frac{1}{2}\left(1 + \frac{F}{F_{\max}}\right)^2\right)$$
when F is increasing,

where α, β and $F_{\max}$ are constants.

However, the first in-depth study of hysteresis is due to Pierre Duhem[1] in the period 1896–1902. A detailed review of Duhem's work on hysteresis may be found in [67, Chapter IV] so that we provide here only those elements of that extensive work that are directly related to the present paper.

To understand the motivation for Duhem's work we quote from [67, p. 306]: *"take a metallic wire under strain by means of a load. We can take the length of the wire and its temperature as variables that define its state. The gravity weight P will represent the external action. At temperature T and under the load P the wire may be at equilibrium with length l. Give P infinitely small variations, the length l and temperature T will also experience infinitely small variations, and a new equilibrium may be achieved. In this last state, give the gravity weight and temperature variations equal in absolute value, but of opposite signs to the previous ones. The length l should experience a variation equal to the previous one with opposite sign. However, experimentation shows that this is not the case. In general, to the expansion of the wire corresponds a smaller contraction, and the difference lasts with time."*

This permanent deformation is the subject of a seven-memoirs research by Duhem, see Refs. [16]–[22]. In his first memoir submitted to the section of sciences of the Académie de Belgique on October 13, 1894, and reviewed by the mathematician Charles Lagrange in Ref. [44],[2] Duhem writes: *"The attempts to make the different kinds of permanent deformations compatible with the principles of thermodynamics have been few up till now. Only one of these attempts, due to M. Marcel Brillouin, appears to us worthy of interest."* [16, p. 3]. Duhem analyzes the work of Brillouin and concludes that it is not compatible with the principles of thermodynamics [16, p. 6] (see also [19, pp. 5–7]).

[1] For a detailed study of the life and work of Pierre-Maurice-Marie Duhem (9 June 1861 – 14 September 1916) see Refs. [39] or [67].

[2] We are indebted to Jean François Stoffel for this information.

As an alternative, Duhem starts a theory of permanent deformations by considering the simplest case: that of a system defined by one normal variable x and its absolute temperature T. Denoting $\mathcal{F}(x,T)$ the internal thermodynamic potential of the system, Duhem writes [16, p. 8]: *"Let X be the external action to which this system is subject. The condition of equilibrium of the system will be*

$$X = \frac{\partial \mathcal{F}(x,T)}{\partial x}. \tag{1}$$

Let (x,T,X) and $(x+dx, T+dT, X+dX)$ be two equilibria of the system, infinitely close to each other; owing to equality [(1)] *we get*

$$dX = \frac{\partial^2 \mathcal{F}(x,T)}{\partial x^2} dx + \frac{\partial^2 \mathcal{F}(x,T)}{\partial x \partial T} dT. \text{ "} \tag{2}$$

Equation (2) does not take into account the fact that the modifications of equilibria are not reversible. So Duhem introduces a term $f(x,T,X)|dx|$ to be added to the right-hand side of Equation (2), where f is a continuous function of the three variables x, T, and X. For an isothermal modification (that is when T is maintained constant) we get [16, pp. 9–10]:

$$\frac{dX}{dx} = \begin{cases} f_1(x,T,X) & \text{for an increasing } x, \\ f_2(x,T,X) & \text{for a decreasing } x, \end{cases} \tag{3}$$

where

$$\begin{aligned} f_1(x,T,X) &= \frac{\partial^2 \mathcal{F}(x,T)}{\partial x^2} + f(x,T,X), \\ f_2(x,T,X) &= \frac{\partial^2 \mathcal{F}(x,T)}{\partial x^2} - f(x,T,X). \end{aligned} \tag{4}$$

Observe that, when the input is piecewise monotone, the model (3) is equivalent to the model (5) proposed in Refs. [3] and [43, p. 282]:

$$\dot{x}(t) = \begin{cases} \phi_\ell\big(x(t),u(t)\big)\dot{u}(t) & \text{for } \dot{u}(t) \le 0, \\ \phi_r\big(x(t),u(t)\big)\dot{u}(t) & \text{for } \dot{u}(t) \ge 0, \end{cases} \tag{5}$$

where ϕ_ℓ and ϕ_r are functions that satisfy some conditions, the function u is the input (which is x using Duhem's notation), the function x the state (which is X using Duhem's notation), and t is time.

To the best of our knowledge, the first reference that called the form (5) "Duhem model" is Ref. [48] in 1993.[3] Indeed, the authors of Ref. [43] attributed erroneously Duhem's model to Madelung [63, p. 797].[4]

[3] Ref. [48] cites a translation into German of the original memoir Ref. [16] which is written in French.

[4] Quoting from Ref. [48, p. 96]: *"the Madelung paper does not use a differential equation or integral operator. In fact, Madelung allows nonuniqueness of trajectories through a point … which would make a differential equation model difficult."*

Between 1916 when P. Duhem dies and 1993 when his model of hysteresis is finally attributed to him, Duhem's work on hysteresis does not have a relevant impact. Major references on hysteresis like Refs. [8], [12] or [56] do not cite his memoirs. Several authors propose different forms of the Duhem model without a direct reference to Duhem's memoirs. This is the case of the Coleman and Hodgdon model of magnetic hysteresis [12], the Dahl model of friction [13], the model (5) in Ref. [3], and a generalized form of the model (5) in Ref. [43, p. 95]. In 1952, Everett cites briefly Duhem's work as follows [24, p. 751]: *"From a thermodynamic standpoint the introduction of an additional variable whose value depends on the history of the system is sufficient for a formal discussion (cf. Duhem*[ref]*). To advance our understanding of the phenomenon* [of hysteresis]*, however, a molecular interpretation is desirable."*

A general theory of physics based on a molecular interpretation was precisely what Duhem rejected. In a review of his work presented in 1913 for his application to the Académie des Sciences, Duhem writes that his *"doctrine should note imitate the numerous mechanical theories proposed by physicists hitherto; to the observable properties that apparatus measure, it will not substitute hidden movements of hypothetical bodies"* [67, p. 74].

In recent times, Duhem's phenomenological approach is becoming more accepted [5,9,46,52,57]. Indeed, *"hysteretic phenomena arising in structural and mechanical systems are so complicated that there has been no well-accepted mathematical model which can describe all observed hysteretic characteristics."* [52, p. 1408]. Moreover, the Preisach model which was believed to describe the constitutive behavior of magnetic hysteresis, has shown to be a phenomenological model [49, p. 2].

Several reasons are invoked for the use of Duhem's model to describe hysteresis. On the one hand, *"differential equation-based models lead to a particularly simple phenomenological description"* [46, p. C8-545]. On the other hand, the *"Duhem models* [sic] *... have the advantage that* [they] *require a small amount of memory so they are suitable in practical and low cost applications."* [9, p. 628]. Finally, many phenomenological models of friction or hysteresis can be seen as particular cases of a more general form of the Duhem model: this is the case for example of the Dahl [13], the LuGre [2,11], or the Maxwell-slip models [30]. Thus *"recast*[ing] *each model in the form of a generalized ... Duhem model ... provide*[s] *a unified framework for comparing the hysteretic nature of these models."* [57, p. 91].

There are several generalizations of the original Duhem model (5). The following generalization is proposed in [43, p. 95]: $\dot{x}(t) = f\big(t, x(t), u(t), \dot{u}(t)\big)$. In [64, p. 141] the terms $\phi_\ell\big(x(t), u(t)\big)$ and $\phi_r\big(x(t), u(t)\big)$ in (5) are replaced by $\big[\phi_\ell(x, u)\big](t)$ and $\big[\phi_r(x, u)\big](t)$ respectively, where ϕ_ℓ and ϕ_r are causal operators. In Ref. [54] Duhem's model is generalized as $\dot{x}(t) = f\big(x(t), u(t)\big)g\big(\dot{u}(t)\big)$ whilst [64, p. 145] proposes the following form for vector hysteresis: $\dot{x}(t) = f\big(x(t), u(t), \pi(\dot{u})\big)|\dot{u}(t)|$ where $\pi(\dot{u} \neq 0) = \dot{u}/|\dot{u}|$.

`Why are there different generalized forms of the Duhem model ?` To answer this question we have to recall the concept of rate independence.[5]

To the best of our knowledge, the earliest author to state clearly rate independence is R. Bouc in Ref. [8], although that property was known before Bouc's work. Due to the importance of rate independence in the study of hysteresis, and the fact that Ref. [8] is not available in English, we quote from [8, p. 17]: *"Consider the graph with hysteresis of Fig. 1 where $\mathcal{F}$ is not a function of x. To the value $x = x_0$ correspond four values of $\mathcal{F}$.*

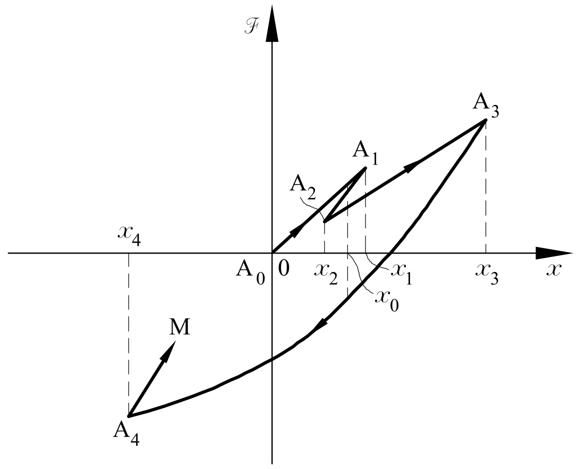

Fig. 1: *Graph "Force–Displacement" with hysteresis.*

... If we consider now x as a function of time, the value of the force at instant t will depend not only on the value $x(t)$, but also on all past values of function x since the origin instant where it is defined. If β is that instant ($x(\beta) = \mathcal{F}(\beta) = 0, \beta \geq -\infty$), then we denote $\mathcal{F}(t) = \mathcal{A}\big(x(\cdot), t\big)$ the value of the force at instant "t", where $x(\cdot)$ "represents" the whole function on the interval $[\beta, t]$[footnote]*. Our aim is to explicit functional $\mathcal{A}\big(x(\cdot), t\big)$.*

To this end, we make the following assumption: the graph of Fig. 1 remains the same for all increasing func-

[5] The term "rate independence" is attributed to Truesdell and Noll (Section 99, Encyclopedia of Phyics, volume III/3, 1965) by Visintin [64, p. 13]. We read Section 99 of the 2004 edition [62] of the original treatise by Truesdell and Noll but found no clear evidence of the correctness of the attribution.

tion $x(\cdot)$ between 0 and x_1, decreasing between the values x_1 and x_2, etc. The functional will no longer depend explicitly on time and we will write $\mathcal{F}(t) = \mathcal{A}\big(x(\cdot)\big)(t)$. We can say: If $x(t_j)$ and $x(t_{j+1})$ are two extremal values, consecutive in time, we have for all $t \in [t_j, t_{j+1}]$

$$\mathcal{A}\big(x(\cdot)\big)(t) = f_j\big(x(t)\big),$$

where f_j is a function of only the variable $x(t)$.

We can also say: If $\phi : \mathbb{R} \to \mathbb{R}$ is a class C^1 function whose derivative is strictly positive for $t \geq \beta$ with $\phi(\beta) = \beta$, and if we consider the function $y(t) = x\big(\phi(t)\big)$ which is a "compression" or an "expansion" of x by intervals, then the graphs $\big(\mathcal{A}(y(\cdot)), y\big)$ and $\big(\mathcal{A}(x(\cdot)), x\big)$ are identical and we have

$$\mathcal{A}\big(x(\cdot)\big)(t) = \mathcal{A}\big(y(\cdot)\big)\big(\phi^{-1}(t)\big). \qquad "$$

The exact definition of rate independence varies from author to author. For example, Visintin requires the time-scale-change ϕ to be a strictly increasing time homeomorphism [64, p. 13] whilst Oh and Bernstein consider that ϕ is continuous, piecewise C^1, nondecreasing, $\phi(0) = 0$, and $\lim_{t\to\infty} \phi(t) = \infty$ [54]. Loosely speaking, rate independence means that the graph of hysteresis (output versus input) is invariant with respect to any change in time scale.

Rate independence is used by Visintin to define hysteresis : *"Definition. Hysteresis = Rate Independent Memory Effect."* [64, p. 13]. However, *"this definition excludes any viscous-type memory"* [64, p. 13] because it leads to rate-dependent effects that increase with velocity. A definition based on rate independence assumes that *"the presence of hysteresis loops is not ... an essential feature of hysteresis."* [64, p. 14].

This point of view is challenged by Oh and Bernstein who consider hysteresis as a *"nontrivial quasi-dc input-output closed curve"* [54, p. 631] and propose a modified version of the Duhem model which can represent rate-dependent or rate-independent effects. A characterization of hysteresis systems using hysteresis loops is also addressed by Ikhouane in Ref. [35] through the concepts of consistency and strong consistency.

In light of what has been said it becomes clear that, in Ref. [64], the generalizations of Duhem's model are done in such a way that rate independence is preserved, whilst a definition of hysteresis based on hysteresis loops in Ref. [54] is compatible with a generalized form of the Duhem model that may be rate dependent or rate independent.

`Why are there different models of hysteresis?` In Ref. [16] Duhem proposes his model to account for the irreversibility in the modifications of equilibria observed experimentally in magnetic hysteresis [16, Chapter IV], sulfur [17], red phosphorus [19, Chapter III], and in different processes of metallurgy [19].

Preisach [56] uses *"plausible hypotheses concerning the physical mechanisms of magnetization"* [49, p. 1] to elaborate a model of magnetic hysteresis. This model is also proposed and studied by Everett and co-workers [24]–[27] who postulate *"that hysteresis is to be attributed in general to the existence in a system of a very large number of independent domains, at least some of which can exhibit metastability."* [24, p. 753].

Krasnosel'skiĭ and Pokrovskiĭ point out to the issue of admissible inputs, as *"it is by no means clear a priori for any concrete transducer with hysteresis, how to choose the relevant classes of admissible inputs"* [43, p. 5]. This is why they introduce the concept of vibro-correctness which allows the determination of the output of a hysteresis transducer that corresponds to any continuous input, once we know the outputs that correspond to piecewise monotone continuous inputs [43, p. 6]. The models that Krasnosel'skiĭ and Pokrovskiĭ propose (ordinary play, generalized play, hysteron) are vibro-correct, although the authors acknowledge the existence of hysteresis models that may not be vibro-correct like the Duhem model.[6]

Hysteresis models based on a feedback interconnection between a linear system and a static nonlinearity are proposed in Ref. [55]. The authors study *"hysteresis arising from a continuum of equilibria ... and hysteresis arising from isolated equilibria"* [55, p 101].

A review of hysteresis models is provided in Ref. [48] and a detailed study of these (and other) models may be found in Refs. [7], [10], [14], [37], [49], [64].

In light of what has been said, the diversity of hysteresis models is due to the wide range of areas to which hysteresis is concomitant, and the diversity of methods and assumptions underlying the elaboration of these models.

Note that all mathematical models of hysteresis share a common property: they model hysteresis. This fact leads us to our next question.

`What is hysteresis?` A description found in many papers is that hysteresis *"refers to the systems that have memory, where the effects of input to the system are experienced with a certain delay in time."* [33, p. 210]. This description is misleading as it applies also to dynamic linear systems. Indeed, when the output y is related to the input u by $\dot{x} = Ax + Bu$ and $y = Cx$ which is a possible description of a linear system, the output is given by $y(t) = C\big[\exp(tA)x_0 + \int_0^t \exp\big((t-\tau)A\big)Bu(\tau)\mathrm{d}\tau\big]$ where x_0 is the initial state and $t \geq 0$ is time. We can see that $y(t)$ depends on the integral of a function that incorporates $u(\tau)$ for all $\tau \in [0, t]$, which means that the linear system does have memory. How-

[6] called the Madelung model in Ref. [43].

ever, *"hysteresis is a genuinely nonlinear phenomenon"* [10, p. vii].

Mayergoyz considers hysteresis as a rate-independent phenomenon which is *"consistent with existing experimental facts."* [49, p. xvi]. However, *"for very fast input variations, time effects become important and the given definition of rate-independent hysteresis fails."* [49, p. xvi]. Also, *"in the existing literature, hysteresis phenomenon is by and large linked with the formation of hysteresis loops (looping). This may be misleading and create the impression that looping is the essence of hysteresis. In this respect, the given definition of hysteresis emphasizes the fact that history dependent branching constitutes the essence of hysteresis, while looping is a particular case of branching."* [49, pp. xvi–xvii].

Following Mayergoyz, *"All rate-independent hysteresis nonlinearities fall into two general classifications: (a) hysteresis nonlinearities with local memories, and (b) hysteresis nonlinearities with nonlocal memories."* [49, pp. xvii]. In a hysteresis with a local memory, the state or output at time $t \geq t_0$ is completely defined by the state or output at instant t_0, and the input on $[t_0, t]$. This is the case for example of a hysteresis given by a differential equation. Hysteresis with a nonlocal memory is a hysteresis which is not with local memory. This is the case for example of the Preisach model. *"However, the notion of hysteresis nonlinearities with local memories is not consistent with experimental facts."* [49, pp. xix–xx]. Hodgdon, on the other hand, writes in relation with the use of a special case of the Duhem model to represent ferromagnetic hysteresis: *"These results are in good agreement with the manufacturer's dc hysteresis data and with experiments"* [34, p. 220].

In Ref. [54], Oh and Bernstein consider the generalized Duhem model $\dot{x} = f(x, u)g(\dot{u})$ and $y = h(x, u)$ with u the input, y the output and x the state. The authors assume the existence of a unique solution of the differential equation on the time interval $[0, \infty[$. They also assume the existence of a T–periodic solution x_T for any T–periodic input u_T with one increasing part and one decreasing part, which means that the graph $\{(u_T, x_T)\}$ is a closed curve. Finally they assume that when $T \to \infty$ the graph $\{(u_T, x_T)\}$ converges with respect to the Hausdorff metric to a closed curve $\mathcal{C}$. If we can find $(a, b_1) \in \mathcal{C}$ and $(a, b_2) \in \mathcal{C}$ with $b_1 \neq b_2$, the curve $\mathcal{C}$ is not trivial and the generalized Duhem model is a hysteresis.

In a PhD thesis advised by Bernstein [15], Drinčić considers systems of the form $\dot{x} = f(x, u)$ and $y = h(x, u)$ for which hysteresis is defined as in Ref. [54]. The system is supposed to be step convergent, that is $\lim_{t\to\infty} x(t)$ exists for all initial conditions and for all constant inputs. It is noted that there exists *"a close relationship"* [15, p. 6] between the curve $\mathcal{C}$ and the input-output equilibria map, that is the set $\mathcal{E} = \big\{\big(u, h(\lim_{t\to\infty} x(t), u)\big)\big\}$ where u is constant and $f\big(\lim_{t\to\infty} x(t), u\big) = 0$. In particular, the *"system ... is hysteretic if the multivalued map $\mathcal{E}$ has either a continuum of equilibria or a bifurcation"* [15, p. 7].

In Ref. [6] Bernstein states that *"a hysteretic system must be multistable; conversely, a multistable system is hysteretic if increasing and decreasing input signals cause the state to be attracted to different equilibria that give rise to different outputs."* Multistability means that *"the system must have multiple attracting equilibria for a constant input value"* [6].

In Ref. [50], Morris presents six examples of hysteresis systems taken from the areas of electronics, biology, mechanics, and magnetics; hysteresis being understood as a *"characteristic looping behavior of the input-output graph"* [50, p. 1]. The author explains the qualitative behavior of these systems from the point of view of multistability. For *"the differential equations used to model the Schmitt trigger, cellular signaling and a beam in a magnetic field"* it is observed that *"these systems, all possess, for a range of constant inputs, several stable equilibrium points."* [50, p. 13]. The author observes that the systems are rate dependent for high input rates.

For the play operator, the Preisach model and the Bouc-Wen model which are rate independent, *"these models present a continuum of equilibrium points."* [50, p. 13]. These observations lead the author to conclude that *"hysteresis is a phenomenon displayed by forced dynamical systems that have several equilibrium points; along with a time scale for the dynamics that is considerably faster than the time scale on which inputs vary."* [50, p. 13]. Morris proposes the following definition.

"A hysteretic system is one which has (1) multiple stable equilibrium points and (2) dynamics that are considerably faster than the time scale at which inputs are varied." [50, p. 13].

In Ref. [35], Ikhouane considers a hysteresis operator *"$\mathcal{H}$ that associates to an input u and initial condition ξ^0 an output $y = \mathcal{H}(u, \xi^0)$, all belonging to some appropriate spaces."* [35, p. 293]. It is assumed that the operator $\mathcal{H}$ is causal and satisfies the property that constant inputs lead to constant outputs. Examples include all rate-independent models [47, Proposition 2.1], some rate-dependent models, models with local memory like the various generalizations of the Duhem model, and models with nonlocal memory like the Preisach model.

The author introduces two changes in time scale: (1) a linear one which is applied to a given input, and (2) a -possibly- nonlinear one which is the total variation of the original input. When the input is composed with the linear time-scale change, both the input and the output are re-scaled with respect to the total variation of the input, which provides a normalized input independent of the linear time-scale change, and a normalized output. Consistency is defined as being the convergence of the normalized outputs in the space L^∞ endowed with the uniform convergence norm. It is shown that consistency implies the convergence to some set of the graphs output versus input of the hysteresis operator when the linear time scale varies [35, Lemma 9].

Strong consistency is defined as the property that the limit of the normalized outputs, seen a parametrized curve, converges to a periodic orbit which characterizes the hysteresis loop.

The author does not propose a definition of hysteresis, but considers that consistency and strong consistency are properties of a class of hysteresis systems.

`Aim of the paper.` The aim of the paper is to survey the research carried out on the Duhem model from the perspective of its hysteretic properties.

`Organization of the paper.` Section 4 presents some results obtained in Ref. [43], namely the concept of vibro-correctness, sufficient conditions to ensure global solutions of the scalar rate independent Duhem model, and a study of the continuity of the model seen as an operator. Section 5 presents a definition of hysteresis proposed in Ref. [54] that uses a generalized form of Duhem's model as a tool to get that formal definition. Section 6 presents the concepts of consistency and strong consistency introduced in Ref. [35]. The tools and notations of Ref. [35] are also used as a unifying framework to present the results of the present paper. Section 7 presents a characterization of the generalized Duhem model obtained in Ref. [51]. Section 8 summarizes the results obtained in Ref. [40] in relation with the study of the dissipativity of the Duhem model. Section 9 summarizes some results obtained in Ref. [64] in relation with the existence of a Duhem operator, its smoothness, and some generalizations of the model. Section 10 is a note that explores the minor loops of hysteresis systems with particular emphasis on the Duhem model. For ease of reference, some results on the existence and uniqueness of the solutions of differential equations are presented in Appendix A.

To illustrate the results obtained in Sections 4–10, and to analyze the relationships between these results, we use the scalar semilinear Duhem model as a case study. The corresponding mathematical analysis is stated in various lemmas and theorems provided in Section 11, whose proofs are given in B–F. The relationships between the results obtained in Sections 4–9 are commented upon in Section 12. These comments lead to the formulation of several open problems in Section 13 and a conjecture in Section 11.9.

3 Terminology and notations

A real number x is said to be strictly positive when $x > 0$, strictly negative when $x < 0$, nonpositive when $x \leq 0$, and nonnegative when $x \geq 0$. A function $h : \mathbb{R} \to \mathbb{R}$ is said to be strictly increasing when $t_1 < t_2 \Rightarrow h(t_1) < h(t_2)$, strictly decreasing when $t_1 < t_2 \Rightarrow h(t_1) > h(t_2)$, nonincreasing when $t_1 < t_2 \Rightarrow h(t_1) \geq h(t_2)$, and nondecreasing when $t_1 < t_2 \Rightarrow h(t_1) \leq h(t_2)$.[7]

An ordered pair a, b is denoted (a, b) whilst the open interval $\{t \in \mathbb{R} \mid a < t < b\}$ is denoted $]a, b[$. The set of nonnegative integers is denoted $\mathbb{N} = \{0, 1, \ldots\}$ and the set of nonnegative real numbers is denoted $\mathbb{R}_+ = [0, \infty[$.

The Lebesgue measure on $\mathbb{R}$ is denoted μ. We say that a subset of $\mathbb{R}$ is measurable when it is Lebesgue measurable. Let $I \subset \mathbb{R}_+$ be an interval, and consider a function $\phi : I \to \mathbb{R}^l$ where $l > 0$ is an integer. We say that ϕ is measurable when ϕ is (M_μ, B)–measurable where B is the class of Borel sets of $\mathbb{R}^l$ and M_μ is the class of measurable sets of $\mathbb{R}_+$ [66]. For a measurable function $\phi : I \to \mathbb{R}^l$, $\|\phi\|_I$ denotes the essential supremum of the function $|\phi|$ on I where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^l$. When $I = \mathbb{R}_+$, this essential supremum is denoted $\|\phi\|$.

$W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^l)$ denotes the Sobolev space of absolutely continuous functions $\phi : \mathbb{R}_+ \to \mathbb{R}^l$. For this class of functions, we have $\|\phi\| < \infty$; the derivative of ϕ is denoted $\dot{\phi}$; this derivative is defined almost everywhere and satisfies $\|\dot{\phi}\| < \infty$. Endowed with the norm $\|\phi\|_{W^{1,\infty}(\mathbb{R}_+,\mathbb{R}^l)} = \max\left(\|\phi\|, \|\dot{\phi}\|\right)$, the vector space $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^l)$ is a Banach space [45, pp. 280–281].

$L^\infty(\mathbb{R}_+, \mathbb{R}^l)$ denotes the Banach space of measurable and essentially bounded functions $\phi : \mathbb{R}_+ \to \mathbb{R}^l$ endowed with the norm $\|\cdot\|$.

$C^0(\mathbb{R}_+, \mathbb{R}^l)$ denotes the Banach space of continuous functions $\phi : \mathbb{R}_+ \to \mathbb{R}^l$ endowed with the norm $\|\cdot\|$.

$\forall \gamma \in]0, \infty[$, the linear time-scale-change $s_\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by the relation $s_\gamma(t) = t/\gamma, \forall t \in \mathbb{R}_+$.

$\lim\limits_{x \uparrow a}$ sets for $\lim\limits_{\substack{x \to a \\ x < a}}$ whilst $\lim\limits_{x \downarrow a}$ sets for $\lim\limits_{\substack{x \to a \\ x > a}}$.

Let U be a set and let $T \in]0, \infty[$. The function $\phi : \mathbb{R}_+ \to U$ is said to be T–periodic if $\phi(t) = \phi(t + T), \forall t \in \mathbb{R}_+$.

[7] In this paper we avoid the words "positive", "negative", "increasing", "decreasing" as they mean different things in different books.

4 A summary of the results obtained in Ref. [43]

This section presents those results obtained in Ref. [43] that are relevant to the present paper. This is in particular the case of the concept of vibro-correctness which allows to extend the set of admissible inputs from continuously differentiable to continuous.

4.1 The concept of vibro-correctness

Consider the differential equation [43, p. 95]

$$\dot{x}(t) = \zeta_1\big(t, x(t), u(t), \dot{u}(t)\big), \tag{6}$$
$$x(t_0) = x_0. \tag{7}$$

In Equations (6)–(7) the initial time $t_0 \in \mathbb{R}$ and the initial state $x_0 \in \mathbb{R}^n$ where $n > 0$ is an integer. Furthermore, the function $\zeta_1 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ is continuous and the input $u \in C^1\big([t_0, \infty[, \mathbb{R}\big)$. Theorem 10 ensures the existence of at least a solution of (6)–(7) on some time interval $[t_0, t_0'[$ where $t_0' > t_0$ may be finite or infinite. Is it possible to extend the set of inputs from continuously differentiable to solely continuous? The answer to this question leads to the concept of vibro-correctness.

Let $t_1 \in]t_0, \infty[$ and $v \in C^0\big([t_0, t_1], \mathbb{R}\big)$. For any $\delta \in]0, \infty[$ define the set

$$E(\delta, v) = \big\{u \in C^1\big([t_0, t_1], \mathbb{R}\big) \mid \|u - v\|_{[t_0,t_1]} \le \delta\big\}. \tag{8}$$

Definition 1 [43, pp. 95–96] The differential equation (6)–(7) is *vibro-correct* if for each $x_0 \in \mathbb{R}^n$ and each input $u_* \in C^0\big([t_0, \infty[, \mathbb{R}\big)$ there exist $t_1 \in]t_0, \infty[$ and $\delta_0 \in]0, \infty[$ such that Propreties (i)–(ii) hold.

(i) $\forall u \in E(\delta_0, u_*)$ the solution $x = \mathcal{W}(u, x_0)$ of Equations (6)–(7) exists and is unique on the time interval $[t_0, t_1]$.

(ii) $\lim\limits_{\delta \to \infty} \sup\limits_{u,v \in E(\delta,u_*)} \|\mathcal{W}(u, x_0) - \mathcal{W}(v, x_0)\|_{[t_0,t_1]} = 0.$

In the following we analyze the consequences of vibro-correctness. Consider a sequence of inputs $u_k \in E(\delta_0, u_*)$ such that $\lim_{k\to\infty} \|u_k - u_*\|_{[t_0,t_1]} = 0$. Then, owing to Proprety (ii) of Definition 1, it follows that $\{\mathcal{W}(u_k, x_0)\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $C^0\big([t_0, t_1], \mathbb{R}\big)$. Thus it converges with respect to the norm $\|\cdot\|$ to a function $x_* \in C^0\big([t_0, t_1], \mathbb{R}\big)$. Note that the function x_* is independent of the particular choice of the sequence u_k owing to Proprety (ii) of Definition 1. Defining $\mathcal{W}(u_*, x_0)$ as being x_* means that the operator $\mathcal{W}$ has been extended to the set of continuous inputs.

Thus, the concept of vibro-correctness allows to extend the definition of the operator $\mathcal{W}$ from the set of continuously differentiable inputs to that of continuous inputs.

Another consequence of Property (ii) is the uniqueness of the solutions of (6)–(7). This means that it is not necessary to state explicitly in Property (i) that the differential equation (6)–(7) has a unique solution (this is what is done in Ref. [43]; see also [43, p. 104]).

Definition 2 [43, p. 98] If we consider only constant inputs u_* in Definition 1 then the differential equation (6)–(7) is said to be *vibro-correct on constant inputs*.

Theorem 1 *[43, p. 98] If the differential equation (6)–(7) is vibro-correct on constant inputs then we can find functions $\zeta_2, \zeta_3 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ such that for all $(t, x, u, v) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ we have $\zeta_1(t, x, u, v) = \zeta_2(t, x, u)v + \zeta_3(t, x, u)$.*

Theorem 1 means that the only differential equations (6)–(7) that may be vibro-correct are the ones that have the following form:

$$\dot{x}(t) = \zeta_2\big(t, x(t), u(t)\big)\dot{u}(t) + \zeta_3\big(t, x(t), u(t)\big), \tag{9}$$
$$x(t_0) = x_0. \tag{10}$$

4.2 Global solutions of the scalar rate-independent Duhem model

Consider the space $S(t_0, t_2)$ of absolutely continuous functions $u : [t_0, t_2] \to \mathbb{R}$ such that

$$\|u\|_S = |u(t_0)| + \int_{t_0}^{t_2} |\dot{u}(t)| \,\mathrm{d}t < \infty, \tag{11}$$

where $t_2 \in]t_0, \infty[$ is fixed. Consider following differential equation [43, p. 286]:

$$\dot{x}(t) = h_\ell\big(x(t), u(t)\big)\dot{u}(t) \text{ for almost all } t \in [t_0, t_2] \text{ such that } \dot{u}(t) \le 0, \tag{12}$$
$$\dot{x}(t) = h_r\big(x(t), u(t)\big)\dot{u}(t) \text{ for almost all } t \in [t_0, t_2] \text{ such that } \dot{u}(t) \ge 0, \tag{13}$$
$$x(t_0) = x_0, \tag{14}$$

where $u \in S(t_0, t_2)$, and $x(t) \in \mathbb{R}$. The functions $h_\ell, h_r : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Borel, locally bounded,[8] and satisfy the following unilateral Lipschitz conditions with respect to the first variable [43, p. 278]:

$$(x_1 - x_2)\big(h_\ell(x_1, v) - h_\ell(x_2, v)\big) \ge -\lambda(v)(x_1 - x_2)^2, \quad \forall x_1, x_2 \in \mathbb{R}, \forall v \in [a_u, b_u], \tag{15}$$
$$(x_1 - x_2)\big(h_r(x_1, v) - h_r(x_2, v)\big) \le \lambda(v)(x_1 - x_2)^2, \quad \forall x_1, x_2 \in \mathbb{R}, \forall v \in [a_u, b_u], \tag{16}$$

[8] If the functions h_ℓ and h_r are continuous then they are Borel and locally bounded. Continuity is the condition that appears in Ref. [48].

where $\lambda : \mathbb{R} \to \mathbb{R}_+$ is continuous, $a_u = \min_{t\in[t_0,t_2]} u(t)$, and $b_u = \max_{t\in[t_0,t_2]} u(t)$. Observe that (15)–(16) are the transcription of (133) for the differential equation (12)–(13). Given that the function λ is continuous, it is bounded on the interval $[a_u, b_u]$ so that the term $\lambda(v)$ in Inequalities (15)–(16) can be replaced by a constant. Thus there exists a unique solution to (12)–(14) whose maximal interval of existence is $[t_0, t_2]$ owing to Lemma 12.

4.3 Continuity of the rate-independent Duhem model seen as an operator

For any given initial condition $x_0 \in \mathbb{R}$ define the operator $\mathcal{Z}_{x_0} : S(t_0, t_2) \to S(t_0, t_2)$ that associates to each input $u \in S(t_0, t_2)$ the solution x of the differential equation (12)–(14). Then,

Theorem 2 *[43, Theorem 29.1] The operator $\mathcal{Z}_{x_0}$ is continuous. Furthermore, let $a \in]0, \infty[$, then* $\sup_{\{u\in S(t_0,t_2)\mid \|u\|_S\le a\}} \|\mathcal{Z}_{x_0}(u)\|_S < \infty$.

5 A summary of the results obtained in Ref. [54]

This section presents those results obtained in Ref. [54] that are relevant to the present paper. In particular, the authors of Ref. [54] propose a definition that decides whether a given generalized Duhem model is a hysteresis or not.

5.1 The generalized Duhem model

The *generalized* Duhem model with input u, state x and output y consists of a differential equation that describes the state x as [54]

$$\dot{x}(t) = f\big(x(t), u(t)\big) g\big(\dot{u}(t)\big), \text{ for almost all } t \in \mathbb{R}_+, \tag{17}$$

$$x(0) = x_0, \tag{18}$$

and an algebraic equation that describes the output y as

$$y(t) = h\big(x(t), u(t)\big), \forall t \in \mathbb{R}_+. \tag{19}$$

In Equations (17)–(19) the input $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$;[9] the function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n\times n'}$ is continuous; n and n' are strictly positive integers; the function $g : \mathbb{R} \to \mathbb{R}^{n'}$ is continuous and satisfies $g(0) = 0$; the function $h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is continuous; and the initial state $x_0 \in \mathbb{R}^n$. The following is assumed in [54, Section II, p. 633].

Assumption 1 For every $(u, x_0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})\times\mathbb{R}^n$ there exists a unique solution $x \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ that satisfies Equations (17)–(18).

From Assumption 1 we get $y \in C^0(\mathbb{R}_+, \mathbb{R})\cap L^\infty(\mathbb{R}_+, \mathbb{R})$.

Define the operator $\mathcal{H}_o : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R}^n \to C^0(\mathbb{R}_+, \mathbb{R}) \cap L^\infty(\mathbb{R}_+, \mathbb{R})$ by the relation $\mathcal{H}_o(u, x_0) = y$; and the operator $\mathcal{H}_s : W^{1,\infty}(\mathbb{R}_+, \mathbb{R})\times\mathbb{R}^n \to W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ by the relation $\mathcal{H}_s(u, x_0) = x$.

5.2 Definition of hysteresis according to Ref. [54]

We stress that Ref. [54] does not propose a definition of hysteresis in general. Instead, the authors of Ref. [54] propose a definition that decides whether *a given generalized Duhem model* is a hysteresis or not (this is Definition 4). We now present the different steps that are followed in Ref. [54] to come to Definition 4.

Definition 3 The nonempty set $\mathcal{C} \subset \mathbb{R}^2$ is a *closed curve* if there exists $T \in]0, \infty[$, a continuous, piecewise C^1, and T–periodic function $\eta : [0, T] \to \mathbb{R}^2$ such that $\eta([0, T]) = \mathcal{C}$ and $\eta(0) = \eta(T)$.

Note that Definition 3 is equivalent to [54, Definition 2.1]. Let $u_{\min}, u_{\max} \in \mathbb{R}$ with $u_{\min} < u_{\max}$ and let $\alpha_1, T \in \mathbb{R}$ with $0 < \alpha_1 < T$. Consider a T–periodic input $u : \mathbb{R}_+ \to [u_{\min}, u_{\max}]$ such that

(i) the function u is continuous on $\mathbb{R}_+$,
(ii) the function u is continuously differentiable on $]0, \alpha_1[$ and on $]\alpha_1, T[$ with $\|\dot{u}\| < \infty$,
(iii) the function u is strictly increasing on $]0, \alpha_1[$ and is strictly decreasing on $]\alpha_1, T[$,
(iv) we have $u(0) = u(T) = u_{\min}$ and $u(\alpha_1) = u_{\max}$.

Let $\Lambda_{u_{\min},u_{\max},\alpha_1,T}$ be the set of all such inputs u, and define the set

$$\Lambda = \bigcup_{\substack{u_{\min}<u_{\max}\\ 0<\alpha_1<T}} \Lambda_{u_{\min},u_{\max},\alpha_1,T}. \tag{20}$$

Let $\gamma \in]0, \infty[$; observe that the input $u \circ s_\gamma$ is $T\gamma$–periodic where s_γ is a linear time-scale change. The following is assumed in [54, Definition 2.2].

Assumption 2 Under Assumption 1, for every $u \in \Lambda$ there exists a unique[10] initial condition $x_{0,u} \in \mathbb{R}^n$ such that $\mathcal{H}_s(u, x_{0,u})$ is also T–periodic.

[9] Ref. [54] considers that u is continuous and piecewise C^1. However, the results that we present here are also valid for inputs belonging to $W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$.

[10] The uniqueness of $x_{0,u}$ is not asked in [54, Definition 2.2]. However without uniqueness the equality in Condition (i) of Definition 4 would have no meaning since $\mathcal{C}_{u,\gamma}$ would not correspond to a single mathematical object.

In the following, to simplify the notations, the initial condition $x_{0,u\circ s_\gamma}$ for $\gamma \in]0,\infty[$ is denoted simply $x_{0,\gamma}$. Note that, owing to the continuity and periodicity of $\mathcal{H}_s(u\circ s_\gamma, x_{0,\gamma})$, we have $[\mathcal{H}_s(u\circ s_\gamma, x_{0,\gamma})](0) = [\mathcal{H}_s(u\circ s_\gamma, x_{0,\gamma})](T\gamma)$. This fact, combined with Equation (19) implies that the output $\mathcal{H}_o(u\circ s_\gamma, x_{0,\gamma})$ is also $T\gamma$–periodic and that $[\mathcal{H}_o(u\circ s_\gamma, x_{0,\gamma})](0) = [\mathcal{H}_o(u\circ s_\gamma, x_{0,\gamma})](T\gamma)$. Define the closed curve

$$\mathcal{C}_{u,\gamma} = \left\{ \left(u\circ s_\gamma(t), [\mathcal{H}_o(u\circ s_\gamma, x_{0,\gamma})](t)\right), t\in[0,T\gamma] \right\}. \tag{21}$$

Now, we introduce the so-called Hausdorff distance. Let $k \geq 2$ be an integer. For any two nonempty compact sets S_1 and S_2 in $\mathbb{R}^k$, define the Hausdorff distance d_k by the relation

$$d_k(S_1,S_2) = \max\left\{ \sup_{\eta_1\in S_1}\left(\inf_{\eta_2\in S_2}|\eta_1-\eta_2|\right), \sup_{\eta_2\in S_2}\left(\inf_{\eta_1\in S_1}|\eta_1-\eta_2|\right)\right\}. \tag{22}$$

Then, the collection of all nonempty compact subsets of $\mathbb{R}^k$ is a complete metric space with respect to the Hausdorff distance d_k [23, p. 67].

Definition 4 [54, Definition 2.2] Under Assumptions 1 and 2, the operator $\mathcal{H}_o$ is a *hysteresis* if Conditions (i) and (ii) hold for all $(u,x_0)\in \Lambda\times\mathbb{R}^n$.

(i) There exists a closed curve $\mathcal{C}_u \subset \mathbb{R}^2$ such that $\lim_{\gamma\to\infty} d_2(\mathcal{C}_u,\mathcal{C}_{u,\gamma}) = 0$.
(ii) There exist $a, b_1, b_2 \in \mathbb{R}$ with $b_1 \neq b_2$ such that $(a,b_1)\in\mathcal{C}_u$ and $(a,b_2)\in\mathcal{C}_u$.

Remark 1 Condition (i) in Definition 4 states that $\lim_{\gamma\to\infty} d_2(\mathcal{C}_u,\mathcal{C}_{u,\gamma}) = 0$. For this reason, it is not necessary that $\gamma\in]0,\infty[$ in Assumption 2, it suffices that $\exists\gamma_0\in]0,\infty[$ such that the condition in Assumption 2 holds for all $\gamma\in]\gamma_0,\infty[$.

Remark 2 Owing to Theorem 1, the generalized Duhem model (17)–(19) is not vibro-correct when the function g is not linear. This implies that it cannot be extended to continuous inputs by the use of the concept of vibro-correctness [43, p. 279]. If g is linear it is shown in [54, Proposition 3.2] that, for $u\in\Lambda$, the state x can be written as a function of the input u which means that Condition (ii) of Definition 4 cannot be met.

5.3 Case study

The semilinear Duhem model is used to illustrate Definition 4 and to analyze the relationship between Definition 4 and the concept of strong consistency presented in Section 6. To this end Section 11.5 provides an analytical study of the conditions under which the scalar semilinear Duhem model is a hysteresis according to Definition 4. This study is illustrated by numerical simulations in Section 11.6. Finally the relationship between Definition 4 and strong consistency is analyzed in Section 12.1.

6 A summary of the results obtained in Ref. [35]

This section presents those results obtained in Ref. [35] that are relevant to the present paper. This is in particuler the case for the concepts of consistency and strong consistency.

6.1 The normalized input

Let $p > 0$ be an integer. For $u\in W^{1,\infty}(\mathbb{R}_+,\mathbb{R}^p)$, let $\rho_u:\mathbb{R}_+\to\mathbb{R}_+$ be the total variation of u on $[0,t]$, that is $\rho_u(t) = \int_0^t |\dot{u}(\tau)|\,\mathrm{d}\tau \in\mathbb{R}_+$, $\forall t\in\mathbb{R}_+$. The function ρ_u is well defined, nondecreasing and absolutely continuous. Observe that ρ_u may not be invertible (this happens when the input u is constant on some interval or intervals). Denote $\rho_{u,\max} = \lim_{t\to\infty}\rho_u(t)$ and let

- $I_u = [0,\rho_{u,\max}]$ if $\rho_{u,\max} = \rho_u(t)$ for some $t\in\mathbb{R}_+$ (in this case the interval I_u is finite),
- $I_u = [0,\rho_{u,\max}[$ if $\rho_{u,\max} > \rho_u(t)$ for all $t\in\mathbb{R}_+$ (in this case the interval I_u may be finite or infinite).

Lemma 1 *[35] Let $u\in W^{1,\infty}(\mathbb{R}_+,\mathbb{R}^p)$ be non constant* [11] *so that the interval I_u is not reduced to a single point. Then there exists a unique function $\psi_u\in W^{1,\infty}(I_u,\mathbb{R}^p)$ that satisfies $\psi_u\circ\rho_u = u$. Moreover, the function ψ_u satisfies $\|\dot{\psi}_u\|_{I_u} = 1$ and*
$\mu\left(\left\{\varrho\in I_u \mid \dot{\psi}_u(\varrho) \text{ is not defined or } |\dot{\psi}_u(\varrho)|\neq 1\right\}\right) = 0.$

The function ψ_u is constructed as follows. Let $\varrho\in I_u$, then there exists $t_\varrho\in\mathbb{R}_+$ such that $\rho_u(t_\varrho) = \varrho$ (note that t_ϱ is not necessarily unique as ρ_u is not necessarily invertible). Then, $u(t_\varrho)$ is independent of the particular choice of t_ϱ, and $\psi_u(\varrho)$ is defined by the relation $\psi_u(\varrho) = u(t_\varrho)$ [35].

Lemma 1 shows that the input u has been "normalized" so that the resulting function ψ_u is such that $\dot{\psi}_u$ has norm 1 with respect to the new time variable ϱ. For this reason, we call function ψ_u the *normalized input*.

For every $\gamma\in]0,\infty[$ recall the linear time-scale-change s_γ.

Lemma 2 *[35] $\forall\gamma\in]0,\infty[$, $I_{u\circ s_\gamma} = I_u$ and $\psi_{u\circ s_\gamma} = \psi_u$.*

[11] u is non constant if $\exists t_1,t_2\in\mathbb{R}_+$ such that $u(t_1)\neq u(t_2)$.

6.2 Class of operators

Let Ξ, U, Y be arbitrary sets. Let $\mathcal{U}$ be the set of functions $u : \mathbb{R}_+ \to U$, and $\mathcal{Y}$ the set of functions $y : \mathbb{R}_+ \to Y$. Consider a function (called operator in this work) $\mathcal{H} : \mathcal{U} \times \Xi \to \mathcal{Y}$. The operator $\mathcal{H}$ is said to be *causal* if the following holds [64, p. 60]: $\forall u_1, u_2 \in \mathcal{U}, \forall x_0 \in \Xi, \forall \tau \in]0, \infty[$, if $\forall t \in [0, \tau], u_1(t) = u_2(t)$, then $\forall t \in [0, \tau], [\mathcal{H}(u_1, x_0)](t) = [\mathcal{H}(u_2, x_0)](t)$.

Assumption 3 [35] Let Ξ be a set of initial conditions. Consider a causal operator $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ where $m \in \mathbb{N}\backslash\{0\}$. For every $(u, x_0, \theta) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p) \times \Xi \times \mathbb{R}_+$, if u is constant on the interval $[\theta, \infty[$, then $\mathcal{H}(u, x_0)$ is constant on the same interval $[\theta, \infty[$.

6.3 The normalized output

Lemma 3 *[35] Let Ξ be a set of initial conditions. Assume that the operator $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ is causal and satisfies Assumption 3. Let $(u, x_0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p) \times \Xi$. Then, there exists a unique function $\varphi_u \in L^\infty(I_u, \mathbb{R}^m)$ that satisfies $\varphi_u \circ \rho_u = \mathcal{H}(u, x_0)$. Moreover, we have $\|\varphi_u\|_{I_u} \leq \|\mathcal{H}(u, x_0)\|$. If $\mathcal{H}(u, x_0)$ is continuous on $\mathbb{R}_+$, then φ_u is continuous on I_u and we have $\|\varphi_u\|_{I_u} = \|\mathcal{H}(u, x_0)\|$.*

The function φ_u, called *normalized output*, is constructed as follows. Let $\varrho \in I_u$, then there exists a not necessarily unique $t_\varrho \in \mathbb{R}_+$ such that $\rho_u(t_\varrho) = \varrho$. Then, $[\mathcal{H}(u, x_0)](t_\varrho)$ is independent of the particular choice of t_ϱ, and $\varphi_u(\varrho)$ is defined by the relation $\varphi_u(\varrho) = [\mathcal{H}(u, x_0)](t_\varrho)$ [35].

Note that the correct notation for function φ_u is $\varphi_{u,x_0,\mathcal{H}}$ to stress that this function depends also on the initial condition x_0 and on the operator $\mathcal{H}$. However, in the definition of consistency (Definition 5), neither the initial condition x_0 nor the operator $\mathcal{H}$ vary, which justifies the simplified notation.

6.4 Definition of consistency

The concept of consistency is introduced in Ref. [35] as follows.[12] Consider that the input u is composed with the time-scale-change s_γ where $\gamma \in]0, \infty[$. Then, consider the set

$$\mathcal{S}_{u,\gamma} = \left\{ \left(u \circ s_\gamma(t), [\mathcal{H}(u \circ s_\gamma, x_0)](t) \right), t \in \mathbb{R}_+ \right\} \tag{23}$$

which is the output $\mathcal{H}(u \circ s_\gamma, x_0)$ versus the input $u \circ s_\gamma$ (observe that the initial condition x_0 does not vary with γ). Using the notations of Sections 6.1 and 6.3 we get $\psi_{u \circ s_\gamma} \circ \rho_{u \circ s_\gamma} = u \circ s_\gamma$ and $\varphi_{u \circ s_\gamma} \circ \rho_{u \circ s_\gamma} = \mathcal{H}(u \circ s_\gamma, x_0)$ for all $\gamma \in]0, \infty[$. Thus, the set $\mathcal{S}_{u,\gamma}$ can be rewritten as

$$\mathcal{S}_{u,\gamma} = \left\{ \left(\psi_{u \circ s_\gamma} \circ \rho_{u \circ s_\gamma}(t), \varphi_{u \circ s_\gamma} \circ \rho_{u \circ s_\gamma}(t) \right), t \in \mathbb{R}_+ \right\}, \tag{24}$$

which leads to

$$\mathcal{S}_{u,\gamma} = \left\{ \left(\psi_{u \circ s_\gamma}(\varrho), \varphi_{u \circ s_\gamma}(\varrho) \right), \varrho \in I_{u \circ s_\gamma} \right\}. \tag{25}$$

Using Lemma 2 it follows from Equation (25) that

$$\mathcal{S}_{u,\gamma} = \left\{ \left(\psi_u(\varrho), \varphi_{u \circ s_\gamma}(\varrho) \right), \varrho \in I_u \right\}. \tag{26}$$

Observe that, in the expression (26) of the set $\mathcal{S}_{u,\gamma}$, the only term that depends on γ is the function $\varphi_{u \circ s_\gamma} \in L^\infty(I_u, \mathbb{R}^m)$.

Definition 5 [35] Let Ξ be a set of initial conditions. Consider a causal operator $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ that satisfies Assumption 3. Let $(u, x_0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p) \times \Xi$. The operator $\mathcal{H}$ is said to be *consistent with respect to* (u, x_0) if there exists a function $\varphi_u^\star \in L^\infty(I_u, \mathbb{R}^m)$ such that $\lim_{\gamma \to \infty} \|\varphi_{u \circ s_\gamma} - \varphi_u^\star\|_{I_u} = 0$.

Define the set $\mathcal{S}_u^\star$ by the relation

$$\mathcal{S}_u^\star = \left\{ \left(\psi_u(\varrho), \varphi_u^\star(\varrho) \right), \varrho \in I_u \right\}. \tag{27}$$

Recall that the Hausdorff distance d_{p+m} is defined by Equation (22).

Lemma 4 *[35] Let Ξ be a set of initial conditions. Assume that the operator $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ is causal and satisfies Assumption 3. If $\mathcal{H}$ is consistent with respect to (u, x_0) then $\lim_{\gamma \to \infty} d_{p+m}\left(\bar{\mathcal{S}}_{u,\gamma}, \bar{\mathcal{S}}_u^\star\right) = 0$, where $\bar{X}$ is the closure of the set X.*

The converse of Lemma 4 is not true in general [35, Example 2].

Definition 6 [13] Let Ξ be a set of initial conditions. Consider a causal operator $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ that satisfies Assumption 3. We say that $\mathcal{H}$ is *rate independent with respect to linear time-scale changes* if $\forall (u, x_0, \gamma) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p) \times \Xi \times]0, \infty[$ we have $\mathcal{H}(u \circ s_\gamma, x_0) = \mathcal{H}(u, x_0) \circ s_\gamma$ almost everywhere.

[12] In the proof of [54, Proposition 5.1] Oh and Bernstein use as input $u \circ s_\gamma$ where $u \in \Lambda$, and obtain by a limiting process a rate-independent semilinear Duhem model. In Ref. [35], Ikhouane extends this idea to causal operators $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ that satisfy Assumption 3, and to inputs that belong to $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p)$.

[13] Definition 6, Assumption 4, and Proposition 1 do not appear in Ref. [35].

Assumption 4 Let Ξ be a set of initial conditions. Consider a causal operator $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m) \cap C^0(\mathbb{R}_+, \mathbb{R}^m)$ that satisfies Assumption 3. Assume that $\mathcal{H}$ is consistent with respect to all $(u, x_0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p) \times \Xi$.

The new element that Assumption 4 introduces is that the output $\mathcal{H}(u, x_0)$ is assumed to be continuous.

Proposition 1 *Under Assumption 4, let the operator* $\mathcal{H}^\star : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m) \cap C^0(\mathbb{R}_+, \mathbb{R}^m)$ *be defined by the relation* $\mathcal{H}^\star(u, x_0) = \varphi_u^\star \circ \rho_u$. *Then* $\mathcal{H}^\star$ *is causal, satisfies Assumption 3, and is rate independent with respect to linear time-scale changes.*

Proof Straightforward.

Under Assumption 4 write the operator $\mathcal{H}$ as

$$\mathcal{H} = \mathcal{H}^\star + \mathcal{H}^\dagger, \tag{28}$$

$$\mathcal{H}^\dagger = \mathcal{H} - \mathcal{H}^\star. \tag{29}$$

For any $(u, x_0, \gamma) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p) \times \Xi \times]0, \infty[$ we have $\mathcal{H}^\dagger(u \circ s_\gamma, x_0) = \mathcal{H}(u \circ s_\gamma, x_0) - \mathcal{H}^\star(u, x_0) \circ s_\gamma$. On the other hand, $\mathcal{H}(u \circ s_\gamma, x_0) = \varphi_{u\circ s_\gamma} \circ \rho_{u\circ s_\gamma}$ and $\mathcal{H}^\star(u, x_0) \circ s_\gamma = \varphi_u^\star \circ \rho_{u\circ s_\gamma}$. By Lemma 3 it follows that $\big\|\mathcal{H}^\dagger(u \circ s_\gamma, x_0)\big\| = \|\varphi_{u\circ s_\gamma} - \varphi_u^\star\|_{I_u}$. Since the operator $\mathcal{H}$ is consistent by Assumption 4 it follows that $\lim_{\gamma\to\infty} \|\varphi_{u\circ s_\gamma} - \varphi_u^\star\|_{I_u} = 0$. We thus conclude that

$$\lim_{\gamma\to\infty} \big\|\mathcal{H}^\dagger(u \circ s_\gamma, x_0)\big\| = 0. \tag{30}$$

The interpretation of Equations (28)–(30) is postponed to Section 12.1.3.

6.5 Definition of strong consistency

Observe that, in Definition 5 of consistency, the input u may be periodic or not. However, to characterize the hysteresis loop of the operator $\mathcal{H}$, the input u needs to be periodic. For this reason, Ref. [35] introduces the concept of strong consistency (this is Definition 7) in relation with periodic inputs.[14]

Lemma 5 *[35] Let* $T \in]0, \infty[$. *If* $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p)$ *is non constant and* T*–periodic, then* $I_u = \mathbb{R}_+$ *and* $\psi_u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p)$ *is* $\rho_u(T)$*–periodic.*

[14] To the best of our knowledge, proposing a formal definition of hysteresis based on the existence of a hysteresis loop was first done by Oh and Bernstein in Ref. [54] for the generalized Duhem model, and for inputs belonging to Λ. Ikhouane used a different perspective to generalize this idea to causal operators $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ that satisfy Assumption 3, and to periodic inputs that belong to $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p)$ [35].

Definition 7 [35] Let Ξ be a set of initial conditions and let $x_0 \in \Xi$. Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p)$ be such that the input u is non constant and T–periodic where $T \in]0, \infty[$. Consider an operator $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ that is causal and that satisfies Assumption 3. Assume furthermore that the operator $\mathcal{H}$ is consistent with respect to (u, x_0). For any nonnegative integer k, define the function $\varphi_{u,k}^\star \in L^\infty\big([0, \rho_u(T)], \mathbb{R}^m\big)$ by $\varphi_{u,k}^\star(\varrho) = \varphi_u^\star\big(\rho_u(T)\,k + \varrho\big), \forall \varrho \in [0, \rho_u(T)]$. The operator $\mathcal{H}$ is said to be *strongly consistent with respect to* (u, x_0) if there exists $\varphi_u^\circ \in L^\infty\big([0, \rho_u(T)], \mathbb{R}^m\big)$ such that $\lim_{k\to\infty} \|\varphi_{u,k}^\star - \varphi_u^\circ\|_{[0,\rho_u(T)]} = 0$.

Definition 8 [35] Let Ξ be a set of initial conditions and $x_0 \in \Xi$. Let $T \in]0, \infty[$. Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p)$ be non constant and T–periodic. Consider an operator $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ that is causal and that satisfies Assumption 3. Assume furthermore that the operator $\mathcal{H}$ is strongly consistent with respect to (u, x_0). We call *hysteresis loop of the operator* $\mathcal{H}$ *with respect to* (u, x_0) the set

$$\mathcal{G}_u = \big\{\big(\psi_u(\varrho), \varphi_u^\circ(\varrho)\big), \varrho \in [0, \rho_u(T)]\big\}. \tag{31}$$

Note that the hysteresis loop $\mathcal{G}_u$ may be independent of the initial condition x_0 (see for example Section 11.3).

Observe that some operators may be strongly consistent but do not describe a hysteresis, like any static nonlinearity $y = f(u)$ where f is a function. This is why the following definition is useful for the characterization of hysteresis systems.

Definition 9 [15] Let Ξ be a set of initial conditions. Consider a causal operator $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ that satisfies Assumption 3. Let $T \in]0, \infty[$ and $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p)$ be a non constant and T–periodic input. Let $x_0 \in \Xi$. We say that the operator $\mathcal{H}$ has a *nontrivial hysteresis loop with respect to* (u, x_0) if Conditions (i) and (ii) hold.

(i) The operator $\mathcal{H}$ is strongly consistent with respect to (u, x_0).
(ii) $\mu\Big(\big\{\varrho_1 \in I_u \mid \exists \varrho_2 \in I_u \text{ such that } \psi_u(\varrho_1) = \psi_u(\varrho_2) \text{ and } \varphi_u^\circ(\varrho_1) \neq \varphi_u^\circ(\varrho_2)\big\}\Big) \neq 0.$

The operator $\mathcal{H}$ has a *trivial hysteresis loop* with respect to (u, x_0) if Condition (i) holds and Condition (ii) does not hold.

[15] Definition 9 does not appear in Ref. [35]. Compare with Condition (ii) of Definition 4.

6.6 Case study

The semilinear Duhem model is used to illustrate the concepts of consistency and strong consistency (Sections 11.1, 11.2, 11.3, 11.4), and to analyze the relationship between these concepts and Definition 4 (Section 12.1).

7 A summary of the results obtained in Ref. [51]

This section presents those results obtained in Ref. [51] that are relevant to the present paper. In particular, Ref. [51] characterizes the function g that appears in Equation (17).

Consider the generalized Duhem model (17)–(18) under Assumption 1. Let $\lambda \in \,]0,\infty[$.

Assumption 5 The limits $\lim_{w\downarrow 0}\frac{g(w)}{w^\lambda}$ and $\lim_{w\uparrow 0}\frac{g(w)}{|w|^\lambda}$ exist, are finite, and at least one of them is nonzero.

Assumption 5 implies that λ is unique, and the function g is said to be of class λ.

Assumption 6 There exists a continuous function $Q : \mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ such that $\|x\| \le Q(|x_0|, \|u\|, \|\dot{u}\|)$ for each initial state x_0 and each input $u \in W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$.

Under Assumptions 1, 5, and 6 we have the following.

Lemma 6 *Suppose that the operator $\mathcal{H}_s$ (see Section 5.1) is consistent with respect to (u, x_0) for each initial state x_0 and each input $u \in W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$, and suppose that function g is of class $\lambda \in \,]0,\infty[$. Then the following holds.*

(i) *If $\lambda \in \,]0,1[$ then $f(\cdot,\cdot)g(\cdot)$ is identically zero.*
(ii) *If $\lambda \in \,]1,\infty[$ then $\varphi_u^\star$ (see Section 6.4) is identically x_0.*
(iii) *If $\lambda = 1$, let $q_u = \varphi_u^\star \circ \rho_u$ (see Section 6.1) then*

$$q_u(t) = x_0 + \int_0^t f\big(q_u(\tau), u(\tau)\big)\hat{g}\big(\dot{u}(\tau)\big)\mathrm{d}\tau, \forall t \in [0,\infty[\tag{32}$$

$$\hat{g}(\vartheta) = \begin{cases} \vartheta \lim\limits_{w\downarrow 0}\dfrac{g(w)}{w} & \vartheta \ge 0, \\ |\vartheta| \lim\limits_{w\uparrow 0}\dfrac{g(w)}{|w|} & \vartheta < 0. \end{cases} \tag{33}$$

Proof (i) follows from Lemma 12 and Remark 14 in Ref. [51], whereas (ii) and (iii) are given in [51, Lemma 12].

Lemma 6 says that if $\lambda \ne 1$ then the corresponding generalized Duhem model does not represent a hysteresis behavior.[16] Thus, the existence of $\lim_{w\downarrow 0}\frac{g(w)}{w}$ and $\lim_{w\uparrow 0}\frac{g(w)}{|w|}$ is a necessary condition for the generalized Duhem model to represent a hysteresis. This necessary condition has been derived from the concept of consistency presented in Section 6.4. Note that this condition has been assumed for the semilinear Duhem model proposed in Ref. [54] (see Equation (68) along with Equations (66)–(67)).

8 A summary of the results obtained in Ref. [40]

This section presents those results obtained in Ref. [40] that are relevant to the present paper. This is the case for the dissipativity of a special form of the Duhem model. The concept of dissipativity/passivity is treated in [42, chapter 6] as an abstracted form of energy dissipation which makes this concept relevant to the study of hysteresis.

8.1 The scalar rate-independent Duhem model

The following scalar rate-independent Duhem model is considered in Ref. [40]:

$$\dot{x}(t) = f_1\big(x(t), u(t)\big)\dot{u}(t) \text{ for almost all } t \in [0,\infty[\text{ such that } \dot{u}(t) \ge 0, \tag{34}$$

$$\dot{x}(t) = f_2\big(x(t), u(t)\big)\dot{u}(t) \text{ for almost all } t \in [0,\infty[\text{ such that } \dot{u}(t) \le 0, \tag{35}$$

$$x(0) = x_0, \tag{36}$$

where $x_0 \in \mathbb{R}$ is the initial condition, functions $f_1, f_2 \in C^1(\mathbb{R}^2,\mathbb{R})$, and the input $u \in AC(\mathbb{R}_+,\mathbb{R})$: the set of absolutely continuous functions defined from $\mathbb{R}_+$ to $\mathbb{R}$. To ensure the existence and uniqueness of the solutions of the differential equation on the time interval $[0,\infty[$, the following unilateral Lipschitz condition is assumed:

$$(x_1 - x_2)\big(f_1(x_1, v) - f_1(x_2, v)\big) \le \lambda_1(v)(x_1 - x_2)^2, \quad \forall x_1, x_2, v \in \mathbb{R}, \tag{37}$$

$$(x_1 - x_2)\big(f_2(x_1, v) - f_2(x_2, v)\big) \ge -\lambda_2(v)(x_1 - x_2)^2, \quad \forall x_1, x_2, v \in \mathbb{R}, \tag{38}$$

[16] Indeed, if $\lambda \in \,]0,1[$, Equations (17)–(18) lead to $x(t) = x_0, \forall t \in \mathbb{R}_+$. If $\lambda \in \,]1,\infty[$, $\varphi_u^\star$ is identically x_0 which implies that φ_u° is identically x_0. In both cases the operator $\mathcal{H}_s$ has a trivial hysteresis loop with respect to all inputs and initial states (see Definition 9).

where $\lambda_1, \lambda_2 : \mathbb{R} \to \mathbb{R}_+$ are bounded on any bounded interval.[17] Using Lemma 12, Inequalities (37)–(38) ensure that $x \in AC(\mathbb{R}_+, \mathbb{R})$.

8.2 Definition of dissipativity

Define the operator $\Phi : AC(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R} \to AC(\mathbb{R}_+, \mathbb{R})$ by the relation $\Phi(u, x_0) = x$ where x is the solution of the differential equation (34)–(36).

Definition 10 [40] The operator Φ is said to be *dissipative with respect to the supply rate* $\dot{x}u$ if there exists a nonnegative function $\varsigma : \mathbb{R}^2 \to \mathbb{R}_+$ such that $\forall (u, x_0) \in AC(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R}$ we have

$$\frac{\mathrm{d}\varsigma\big(x(t), u(t)\big)}{\mathrm{d}t} \leq \dot{x}(t)u(t), \text{ for almost all } t \in \mathbb{R}_+, \quad (39)$$

where $x = \Phi(u, x_0)$.

8.3 Sufficient conditions for the dissipativity of the scalar rate-independent Duhem model

Define the functions $F_1, F_2 : \mathbb{R}^2 \to \mathbb{R}$ by the relations

$$F_1 = \frac{f_1 - f_2}{2}; \qquad F_2 = \frac{f_1 + f_2}{2}. \quad (40)$$

Assumption 7 [40] The implicit function $v \mapsto \{x_1 \in \mathbb{R} \mid F_1(x_1, v) = 0\}$ admits a unique solution $x_1 = f_{\mathrm{an}}(v)$ where $f_{\mathrm{an}} \in C^1(\mathbb{R}, \mathbb{R})$.

Such a function f_{an} is called an *anhysteresis function* and the corresponding graph $\big\{\big(v, f_{\mathrm{an}}(v)\big) \mid v \in \mathbb{R}\big\}$ is called an *anhysteresis curve*.

For every $(x_0, u_0) \in \mathbb{R}^2$, let $\omega_{\Phi,1}(\cdot, x_0, u_0) : [u_0, \infty[\to \mathbb{R}$ be the solution z of $z(v) - x_0 = \int_{u_0}^{v} f_1\big(z(\sigma), \sigma\big)\,\mathrm{d}\sigma$, for all $v \in [u_0, \infty[$. Similarly, let $\omega_{\Phi,2}(\cdot, x_0, u_0) :]-\infty, u_0] \to \mathbb{R}$ be the solution z of the integral equation $z(v) - x_0 = \int_{u_0}^{v} f_2\big(z(\sigma), \sigma\big)\,\mathrm{d}\sigma$, for all $v \in]-\infty, u_0]$.

Define the function $\omega_\Phi(\cdot, x_0, u_0) : \mathbb{R} \to \mathbb{R}$ as follows:

$$\omega_\Phi(v, x_0, u_0) = \begin{cases} \omega_{\Phi,2}(v, x_0, u_0) & \forall v \in]-\infty, u_0[, \\ \omega_{\Phi,1}(v, x_0, u_0) & \forall v \in [u_0, \infty[. \end{cases} \quad (41)$$

Define the function Ω that characterizes the intersection between $\omega_\Phi(\cdot, x_0, u_0)$ and $f_{\mathrm{an}}(\cdot)$ as follows. The function $\Omega : \mathbb{R}^2 \to \mathbb{R}$ is an *intersecting function* that corresponds to ω_Φ and f_{an} if Properties (i)–(iv) hold.

(i) $\omega_\Phi\big(\Omega(x_0, u_0), x_0, u_0\big) = f_{\mathrm{an}}\big(\Omega(x_0, u_0)\big), \forall (x_0, u_0) \in \mathbb{R}^2$,
(ii) $\Omega(x_0, u_0) \geq u_0$ whenever $x_0 \geq f_{\mathrm{an}}(u_0)$,
(iii) $\Omega(x_0, u_0) < u_0$ whenever $x_0 < f_{\mathrm{an}}(u_0)$,
(iv) $\frac{\mathrm{d}\Omega\big(x(t), u(t)\big)}{\mathrm{d}t}$ exists for almost all $t \in \mathbb{R}_+$, and for all $u \in AC(\mathbb{R}_+, \mathbb{R})$ where $x = \Phi(u, x_0)$.

Define the function $\varsigma : \mathbb{R}^2 \to \mathbb{R}$ by

$$\begin{aligned} \varsigma(x_1, v) = x_1 v + \int_v^{\Omega(x_1, v)} \omega_\Phi\big(\sigma, x_1, v\big)\,\mathrm{d}\sigma \\ - \int_0^{\Omega(x_1, v)} f_{\mathrm{an}}(\sigma)\,\mathrm{d}\sigma, \forall (x_1, v) \in \mathbb{R}^2. \end{aligned} \quad (42)$$

Theorem 3 *[40] Suppose that*

(i) *There exists an intersecting function Ω that corresponds to ω_Φ and f_{an},*
(ii) *$F_1(x_1, v) \geq 0$ whenever $x_1 \leq f_{\mathrm{an}}(v)$, and $F_1(x_1, v) < 0$ otherwise.*

Then $\forall (u, x_0) \in AC(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R}$, the function $t \mapsto \varsigma\big(x(t), u(t)\big)$ is right differentiable and satisfies Inequality (39) where $x = \Phi(u, x_0)$. Moreover, if $f_1 \geq 0$ and $f_2 \geq 0$ then $\varsigma \geq 0$ and Φ is dissipative with respect to the supply rate $\dot{x}u$.

A sufficient condition for the existence of an intersecting function is provided in the following lemma.

Lemma 7 *[41] Assume that f_{an} is strictly increasing and that there exists $\epsilon \in]0, \infty[$ such that $\forall (x_1, v) \in \mathbb{R}^2$ we have*

(i) *$f_1(x_1, v) < \frac{\mathrm{d}f_{\mathrm{an}}(v)}{\mathrm{d}v} - \epsilon$ whenever $x_1 > f_{\mathrm{an}}(v)$, and*
(ii) *$f_2(x_1, v) < \frac{\mathrm{d}f_{\mathrm{an}}(v)}{\mathrm{d}v} - \epsilon$ whenever $x_1 < f_{\mathrm{an}}(v)$.*

Then there exists an intersecting function $\Omega \in C^1(\mathbb{R}^2, \mathbb{R})$ corresponding to ω_Φ and f_{an} such that for all $(u, x_0) \in AC(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R}$ the derivative $\frac{\mathrm{d}\Omega\big(x(t), u(t)\big)}{\mathrm{d}t}$ exists for almost all $t \in \mathbb{R}_+$.

8.4 Extension of the results obtained in Ref. [40]

Similar results are given in Ref. [40] when the equation $F_1(x_1, v) = 0$ has a unique solution in the form $v = g_{\mathrm{an}}(x_1)$. The dissipativity property (39) of the scalar rate-independent Duhem model means that it has a counterclockwise input-output dynamics [1]. A dual result for clockwise input-output dynamics is provided in Ref. [53].

[17] The condition that functions λ_1, λ_2 are bounded on any bounded interval does not appear in Ref. [40]. However, without this condition there is no guarantee that the maximal interval of existence of the solutions of (34)–(36) is $[0, \infty[$, see Section 4.2. In [43, p. 278] it is considered that $\lambda_1 = \lambda_2$ is continuous so that the local boundedness condition holds.

8.5 Case study

The scalar rate-independent semilinear Duhem model is used in Section 11.7 to illustrate the concept of dissipativity. To this end, the results of Ref. [40] are used to derive explicit conditions on the model parameters to ensure dissipativity. These conditions are illustrated by numerical simulations in Section 11.8. The relationship between dissipativity and orientation of the hysteresis loop is commented upon in Section 12.3.

9 A summary of the results obtained in Ref. [64]

This section presents those results obtained in Ref. [64] that are relevant to the present paper. In particular, a local Lipschitz property of the Duhem model is provided.

The following scalar rate-independent Duhem model is considered in [64, Chapter V].

$$\dot{x}(t) = f_1\big(x(t), u(t)\big)\dot{u}(t) \text{ for almost all } t \in \mathbb{R}_+ \text{ such that } \dot{u}(t) \geq 0, \tag{43}$$

$$\dot{x}(t) = f_2\big(x(t), u(t)\big)\dot{u}(t) \text{ for almost all } t \in \mathbb{R}_+ \text{ such that } \dot{u}(t) \leq 0, \tag{44}$$

$$x(0) = x_0, \tag{45}$$

where $x_0 \in \mathbb{R}$ is the initial condition, and the functions $f_1, f_2 \in C^0(\mathbb{R}^2, \mathbb{R})$. Let $\mathcal{T} \in [0, \infty[$.[18]

Theorem 4 *[64, Theorem 1.1] Assume that f_1, f_2 fulfil the following one-sided Lipschitz conditions*

$$(x_1 - x_2)\big(f_1(x_1, v) - f_1(x_2, v)\big) \leq \lambda_0(v)(x_1 - x_2)^2, \quad \forall x_1, x_2, v \in \mathbb{R}, \tag{46}$$

$$(x_1 - x_2)\big(f_2(x_1, v) - f_2(x_2, v)\big) \geq -\lambda_0(v)(x_1 - x_2)^2, \quad \forall x_1, x_2, v \in \mathbb{R}, \tag{47}$$

where $\lambda_0 : \mathbb{R} \to \mathbb{R}_+$ is continuous. Then,

(i) For any $u \in W^{1,1}\big([0, \mathcal{T}], \mathbb{R}\big)$ and any $x_0 \in \mathbb{R}$ there exists a unique $x \in W^{1,1}\big([0, \mathcal{T}], \mathbb{R}\big)$ such that Equations (43)–(45) hold. That is, we can define an operator $\mathcal{M} : W^{1,1}\big([0, \mathcal{T}], \mathbb{R}\big) \times \mathbb{R} \to W^{1,1}\big([0, \mathcal{T}], \mathbb{R}\big)$ by the relation $\mathcal{M}(u, x_0) = x$.

(ii) For any $u \in C^1\big([0, \mathcal{T}], \mathbb{R}\big)$ we have $x \in C^1\big([0, \mathcal{T}], \mathbb{R}\big)$. Moreover, for any $x_0 \in \mathbb{R}$, the mapping $\mathcal{M}(\cdot, x_0)$ is continuous in $W^{1,1}\big([0, \mathcal{T}], \mathbb{R}\big)$ with respect to either the strong and the weak topology.

[18] Since all the results of this section are proved for a *finite* time interval, Ref. [64] considers that the differential equation (43)–(44) holds almost everywhere on that finite time interval. We consider that the differential equation (43)–(44) holds almost everywhere on $\mathbb{R}_+$ to simplify the discussion of Section 12.2 without loss of generality.

Proposition 2 *[64, Proposition 1.3] Assume that $\forall R > 0, \exists L(R) > 0 \mid \forall (x_i, v_i) \in \mathbb{R}^2 (i = 1, 2)$ we have the following. If $|v_i| \leq R$, then $\big|f_j(x_1, v_1) - f_j(x_2, v_2)\big| \leq L(R)\big(|v_1 - v_2| + |x_1 - x_2|\big), (j = 1, 2)$.*

Then, $\forall x_0 \in \mathbb{R}$, in any ball $B_R(0)$ of $W^{1,\infty}\big([0, \mathcal{T}], \mathbb{R}\big)$, the operator $\mathcal{M}(\cdot, x_0)$ is Lipschitz continuous with respect to the metric of $W^{1,\infty}\big([0, \mathcal{T}], \mathbb{R}\big)$. That is $\forall R > 0, \exists l(R, \mathcal{T}) > 0 \mid \forall u_1, u_2 \in W^{1,\infty}\big([0, \mathcal{T}], \mathbb{R}\big)$ such that $\|u_i\|_{W^{1,\infty}([0,\mathcal{T}],\mathbb{R})} \leq R, i = 1, 2$, we have $\big\|\mathcal{M}(u_1, x_0) - \mathcal{M}(u_2, x_0)\big\|_{W^{1,\infty}([0,\mathcal{T}],\mathbb{R})} \leq l(R, \mathcal{T})\|u_1 - u_2\|_{W^{1,\infty}([0,\mathcal{T}],\mathbb{R})}$.

It is shown in [64, Theorem 1.5] that the operator $\mathcal{M}$ can be extended to an operator $\bar{\mathcal{M}} : C^0\big([0, \mathcal{T}], \mathbb{R}\big) \cap BV\big([0, \mathcal{T}], \mathbb{R}\big) \times \mathbb{R} \to C^0\big([0, \mathcal{T}], \mathbb{R}\big) \cap BV\big([0, \mathcal{T}], \mathbb{R}\big)$ where BV is the space of functions that have bounded total variation.

Duhem's model (43)–(45) is generalized as follows [64, Section V.2].

$$\dot{x}(t) = \big[\mathcal{F}_1(x, u)\big](t)\dot{u}(t) \text{ for almost all } t \in]0, \mathcal{T}[\text{ such that } \dot{u}(t) \geq 0, \tag{48}$$

$$\dot{x}(t) = \big[\mathcal{F}_2(x, u)\big](t)\dot{u}(t) \text{ for almost all } t \in]0, \mathcal{T}[\text{ such that } \dot{u}(t) \leq 0, \tag{49}$$

$$x(0) = x_0, \tag{50}$$

where $\mathcal{F}_i : C^0\big([0, \mathcal{T}], \mathbb{R}\big)^2 \to C^0\big([0, \mathcal{T}], \mathbb{R}\big)$, $i = 1, 2$ are causal operators. Sufficient conditions are considered for the existence of the operator $\mathcal{M}$. The smoothness properties of $\mathcal{M}$ are studied along with the extension of $\mathcal{M}$ to an operator $\tilde{\mathcal{M}} : C^0\big([0, \mathcal{T}], \mathbb{R}\big) \cap BV\big([0, \mathcal{T}], \mathbb{R}\big) \times \mathbb{R} \to C^0\big([0, \mathcal{T}], \mathbb{R}\big) \cap BV\big([0, \mathcal{T}], \mathbb{R}\big)$.

Also, Duhem's model (43)–(45) is generalized to include vector inputs in [64, Section V.3]. Let $N \in \mathbb{N}\backslash\{0\}$ and set

$$S_{N-1} = \{v \in \mathbb{R}^n \mid |v| = 1\}, \quad \pi(v) = \begin{cases} v/|v| & \text{if } v \neq 0, \\ 0 & \text{if } v = 0. \end{cases}$$

Let $f : \big(\mathbb{R}^N\big)^2 \times S_{N-1} \to \mathbb{R}^N$ be continuous, and let $(u, x_0) \in C^1\big([0, \mathcal{T}], \mathbb{R}^N\big) \times \mathbb{R}^N$. Consider the model

$$\dot{x}(t) = f\big[x(t), u(t), \pi\big(\dot{u}(t)\big)\big]|\dot{u}(t)|, \forall t \in]0, \mathcal{T}[, \tag{51}$$

$$x(0) = x_0. \tag{52}$$

Sufficient conditions are provided for the existence of an operator $\mathcal{M}$ that is causal, rate independent, fulfils a semigroup property, and is piecewise monotone in some sense. An extension of model (51)–(52) following the lines of model (48)–(50) is also proposed.

Section 12.2 provides comments on the relationship between Proposition 2 and the effect of noise on the hysteresis loop of the Duhem model.

10 A note on minor loops

The minor loops of the Duhem model have not been studied formally in the available literature. However, their behavior is important as evidenced by the large number of published works dedicated to their study both from an experimental point of view, and from a mathematical point a view for the Preisach model (see for example Ref. [49] and the references therein).

For this reason, we provide in this section the formal definition of a minor loop and analyze the behavior of the minor loops of the scalar semilinear Duhem model in Section 11.9. The material provided in this section may be used as a platform to attract mathematicians to the formal analysis of the minor loops of the Duhem model.

In magnetic hysteresis, when magnetization M is plotted against magnetic field H the following is observed. The curve $\big(H(t), M(t)\big)$ follows the path $P_1 \to P_2$ when H increases with time t (see Figure 2). Then the path $P_2 \to P_3$ is followed when H decreases. What is important to note is that, when H increases again from the point P_3, the path followed by $\big(H(t), M(t)\big)$ ends *precisely at the point* P_2 (see for example Ref. [31]).

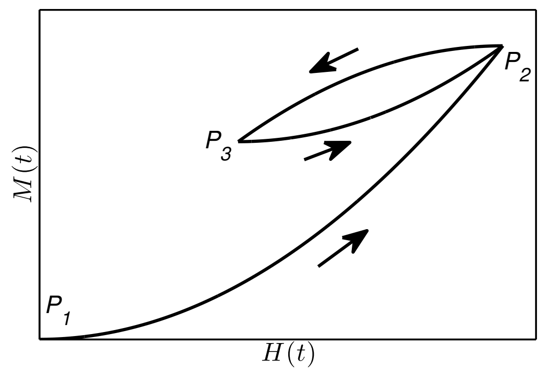

Fig. 2: The path $P_1 \to P_2$ is part of the major loop. The path $P_2 \to P_3 \to P_2$ is a minor loop.

The loop formed by the path $P_2 \to P_3 \to P_2$ is called a minor loop. It occurs in electromagnetic devices when the input is periodic but not exactly sinusoidal. The distortion of the input generates minor loops when hysteresis is involved which causes energy losses. This fact explains the interest of analyzing the behavior of minor loops.

In what follows we formalize mathematically the behavior observed in Figure 2.

Let $u_{\min,1}, u_{\min,2}, u_{\max,1}, u_{\max,2} \in \mathbb{R}$ such that $u_{\min,1} \leq u_{\min,2} < u_{\max,1} \leq u_{\max,2}$ and at least one of the following holds: $u_{\min,1} \neq u_{\min,2}$ or $u_{\max,1} \neq u_{\max,2}$. Let $\alpha_1, \alpha_2, \alpha_3, T \in \mathbb{R}$ with $0 < \alpha_1 < \alpha_2 < \alpha_3 < T$. Consider a T–periodic input $u : \mathbb{R}_+ \to [u_{\min,1}, u_{\max,2}]$ such that

(i) the function u is continuous on $\mathbb{R}_+$,
(ii) the function u is continuously differentiable on $]0, \alpha_1[$, $]\alpha_1, \alpha_2[$, $]\alpha_2, \alpha_3[$, and $]\alpha_3, T[$ with $\|\dot{u}\| < \infty$,
(iii) the function u is strictly increasing on $]0, \alpha_1[$, strictly decreasing on $]\alpha_1, \alpha_2[$, strictly increasing on $]\alpha_2, \alpha_3[$, and strictly decreasing on $]\alpha_3, T[$,
(iv) we have $u(0) = u(T) = u_{\min,1}$, $u(\alpha_1) = u_{\max,1}$, $u(\alpha_2) = u_{\min,2}$, $u(\alpha_3) = u_{\max,2}$.

Let $\mathbb{M}_{u_{\min,1}, u_{\min,2}, u_{\max,1}, u_{\max,2}, \alpha_1, \alpha_2, \alpha_3, T}$ be the set of all such inputs u, and let Ξ be a set of initial conditions. In this section, we consider an operator $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m) \cap C^0(\mathbb{R}_+, \mathbb{R}^m)$ that is causal and that satisfies Assumption 3. We assume that $\mathcal{H}$ is consistent with respect to all $(u, x_0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}) \times \Xi$ and is strongly consistent with respect to all periodic inputs $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ and all initial states $x_0 \in \Xi$.

For $u \in \mathbb{M}_{u_{\min,1}, u_{\min,2}, u_{\max,1}, u_{\max,2}, \alpha_1, \alpha_2, \alpha_3, T}$ define $\varrho_i = \rho_u(\alpha_i)$, $i = 1, 2, 3$. Then we have

$$\varrho_1 = u_{\max,1} - u_{\min,1}, \tag{53}$$
$$\varrho_2 = \varrho_1 + u_{\max,1} - u_{\min,2}, \tag{54}$$
$$\varrho_3 = \varrho_2 + u_{\max,2} - u_{\min,2}, \tag{55}$$
$$\rho_u(T) = \varrho_4 = \varrho_3 + u_{\max,2} - u_{\min,1}. \tag{56}$$

The function $\psi_u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ in ϱ_4–periodic by Lemma 5, and can be determined using Lemma 1 as

$$\psi_u(\varrho) = \varrho + u_{\min,1}, \forall \varrho \in [0, \varrho_1], \tag{57}$$
$$\psi_u(\varrho) = -\varrho + \varrho_1 + u_{\max,1}, \forall \varrho \in [\varrho_1, \varrho_2], \tag{58}$$
$$\psi_u(\varrho) = \varrho - \varrho_2 + u_{\min,2}, \forall \varrho \in [\varrho_2, \varrho_3], \tag{59}$$
$$\psi_u(\varrho) = -\varrho + \varrho_3 + u_{\max,2}, \forall \varrho \in [\varrho_3, \varrho_4]. \tag{60}$$

Define

$$\varrho_5 = u_{\max,1} - u_{\min,2} + \varrho_2 \in]\varrho_2, \varrho_3],$$
$$\varrho_6 = u_{\min,2} - u_{\min,1} \in]0, \varrho_1[,$$
$$\varrho_7 = \varrho_3 + u_{\max,2} - u_{\min,2} \in]\varrho_3, \varrho_4[.$$

Then $\psi_u(\varrho_1) = \psi_u(\varrho_5) = u_{\max,1}$ and $\psi_u(\varrho_2) = \psi_u(\varrho_6) = \psi(\varrho_7) = u_{\min,2}$. Figure 3 illustrates what has been exposed up till now.

Assumption 8
$\forall (u, x_0) \in \mathbb{M}_{u_{\min,1}, u_{\min,2}, u_{\max,1}, u_{\max,2}, \alpha_1, \alpha_2, \alpha_3, T} \times \Xi$ we have $\varphi_u^\circ(\varrho_1) = \varphi_u^\circ(\varrho_5)$.

Definition 11 Under Assumption 8 define the sets

$$\mathcal{V}_u = \big\{\big(\psi_u(\varrho), \varphi_u^\circ(\varrho)\big), \varrho \in [0, \varrho_1] \cup [\varrho_5, \varrho_4]\big\}, \tag{61}$$
$$\mathcal{N}_u = \big\{\big(\psi_u(\varrho), \varphi_u^\circ(\varrho)\big), \varrho \in [\varrho_1, \varrho_5]\big\}. \tag{62}$$

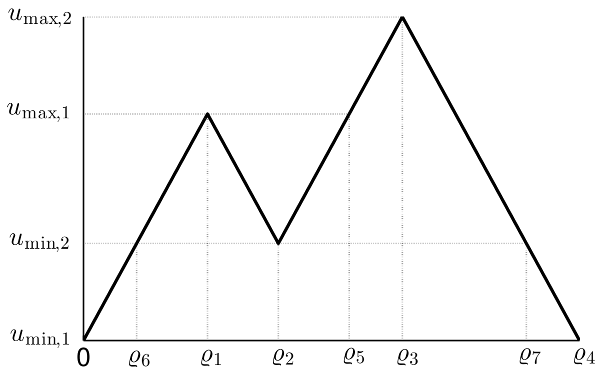

Fig. 3: $\psi_u(\varrho)$ versus ϱ.

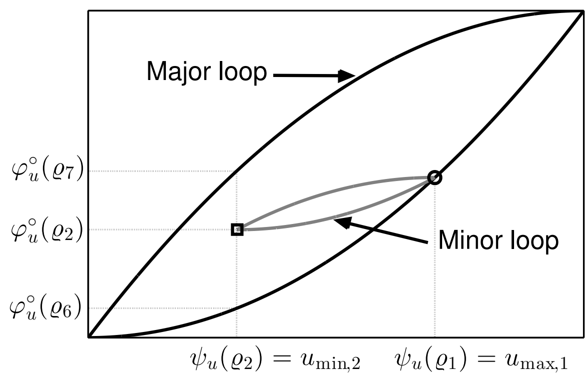

Fig. 4: Hysteresis loop $\varphi_u^\circ(\varrho)$ versus $\psi_u(\varrho)$ for $\varrho \in [0, \varrho_4]$. Black: major loop $\mathcal{V}_u$. Grey: minor loop $\mathcal{N}_u$. The marker $\circ$ corresponds to the point $\big(\psi_u(\varrho_1), \varphi_u^\circ(\varrho_1)\big) = \big(\psi_u(\varrho_5), \varphi_u^\circ(\varrho_5)\big)$. The marker $\square$ corresponds to the point $\big(\psi_u(\varrho_2), \varphi_u^\circ(\varrho_2)\big)$.

The set $\mathcal{V}_u$ is called the *major loop* and the set $\mathcal{N}_u$ a *minor loop* (see Figure 4).

Depending on the particular field in which hysteresis is observed, minor loops may have some additional properties that may be formalized mathematically. As an example, for magnetic hysteresis Assumption 8 holds [31], and we observe that if $u_{\max,1} < u_{\max,2}$ then for all $(u, x_0) \in \mathbb{M}_{u_{\min,1},u_{\min,2},u_{\max,1},u_{\max,2},\alpha_1,\alpha_2,\alpha_3,T} \times \varXi$, Properties (i)–(ii) hold.

(i) $\mathcal{V}_u \cap \mathcal{N}_u = \big\{\big(\psi_u(\varrho_1), \varphi_u^\circ(\varrho_1)\big) = \big(\psi_u(\varrho_5), \varphi_u^\circ(\varrho_5)\big)\big\}$.
(ii) $\varphi_u^\circ(\varrho_6) < \varphi_u^\circ(\varrho_2) < \varphi_u^\circ(\varrho_7)$ or $\varphi_u^\circ(\varrho_7) < \varphi_u^\circ(\varrho_2) < \varphi_u^\circ(\varrho_6)$.

Property (i) says that the major loop and the minor loop intersect at only one point when $u_{\max,1} < u_{\max,2}$. Property (ii) says that the minor loop is located inside the major loop. Both conditions are the transcription of experimental observations in magnetic hysteresis (see for example [4, Figure 7]).

Note that the hysteresis loop $\mathcal{G}_u$ of Equation (31) is such that $\mathcal{G}_u = \mathcal{V}_u \cup \mathcal{N}_u$. Figure 4 provides an example of a minor loop and a major loop that correspond to the normalized input of Figure 3.

The concepts introduced in this section are applied to the scalar semilinear Duhem model in Section 11.9.

11 Case study: the semilinear Duhem model

In this section we use the semilinear Duhem model to illustrate the concepts presented in this paper, and to analyze the relationships between these concepts. Section 11.1 presents the model. In Section 11.2 we provide sufficient conditions for the consistency of the model. Section 11.3 focuses on the study of the strong consistency of the semilinear Duhem model. The results of Sections 11.2 and 11.3 are illustrated by numerical simulations in Section 11.4. In Section 11.5 we specialize into the scalar version of the semilinear Duhem model. Section 11.5 provides the conditions under which the scalar semilinear Duhem model is a hysteresis according to Definition 4. The results of Section 11.5 are illustrated by numerical simuations in Section 11.6. The relationship between Definition 4 and strong consistency is commented upon in Section 12.1. Section 11.7 analyzes the dissipativity of the scalar rate-independent semilinear Duhem model. The results of Section 11.7 are illustrated by numerical simulations in Section 11.8. The relationship between dissipativity and orientation of the hysteresis loop is commented upon in Section 12.3. The minor loops of the scalar semilinear Duhem model are studied and commented upon in Section 11.9.

11.1 The semilinear Duhem model: definition and global existence of solutions

The *semilinear* Duhem model is a special case of the generalized Duhem model (17)–(19). It is called so because, although the model may be nonlinear with respect to the input, the state appears linearly both in the state equation (63) and in the output equation (65). The semilinear Duhem model has been proposed in Ref. [54] as:

$$\begin{aligned} \dot{x}(t) &= g_1\big(\dot{u}(t)\big)\big(A_1 x(t) + B_1 u(t) + E_1\big) \\ &\quad + g_2\big(\dot{u}(t)\big)\big(A_2 x(t) + B_2 u(t) + E_2\big) \\ &\quad \text{for almost all } t \in \mathbb{R}_+, \end{aligned} \tag{63}$$

$$x(0) = x_0, \tag{64}$$

$$y(t) = Cx(t) + Du(t), \forall t \in \mathbb{R}_+. \tag{65}$$

In Equations (63)–(65) the matrix $A_1 \in \mathbb{R}^{n\times n}$ where n is a strictly positive integer, $A_2 \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times 1}$,

$B_2 \in \mathbb{R}^{n\times 1}$, $E_1 \in \mathbb{R}^{n\times 1}$, $E_2 \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$, and $D \in \mathbb{R}$. We consider that $C \neq (0,\ldots,0)$ to avoid having a linear process $y = Du$ that does not describe hysteresis. We consider that $u \in W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$ whereas the properties of $y : \mathbb{R}_+ \to \mathbb{R}$ and $x : \mathbb{R}_+ \to \mathbb{R}^n$ will be analyzed in Theorem 5. The functions $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$ are continuous and satisfy $g_1(w) = 0$ for $w \leq 0$, $g_2(w) = 0$ for $w \geq 0$. Define

$$\bar{g}_1(w) = \frac{g_1(w)}{|w|}, \forall w \neq 0, \tag{66}$$

$$\bar{g}_2(w) = \frac{g_2(w)}{|w|}, \forall w \neq 0. \tag{67}$$

As in Ref. [54] we assume that[19]

$$\lim_{w\downarrow 0} \bar{g}_1(w) = 1 \text{ and } \lim_{w\uparrow 0} \bar{g}_2(w) = -1. \tag{68}$$

In Equation (63), the functions $g_1(\dot{u})$ and $g_2(\dot{u})$ are measurable [60, Theorem 1.12(d)]. Thus, the differential equation (63) can be seen as a linear time-varying system that satisfies all the assumptions of [29, Theorem 3]. This implies that a unique absolutely continuous solution of (63) exists on $\mathbb{R}_+$.

As noted in Ref. [54], the semilinear Duhem model is rate independent when $g_1(w) = \max(0, w)$ and $g_2(w) = \min(0, w), \forall w \in \mathbb{R}$.

11.2 Consistency of the semilinear Duhem model

This section presents the results obtained in Ref. [35] in relation with the consistency of the semilinear Duhem model.

Theorem 5 *[35] Consider the semilinear Duhem model (63)–(65). Assume that both matrices A_1 and $-A_2$ are stable[20] and have a common Lyapunov matrix $P = P^T > 0$ (that is $A_1^T P + PA_1 < 0$ and $-A_2^T P - PA_2 < 0$). Then, $x \in W^{1,\infty}(\mathbb{R}_+,\mathbb{R}^n)$ and $y \in W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$.*

In Equations (63)–(65) consider the operators $\mathcal{H}'_s : L^\infty(\mathbb{R}_+,\mathbb{R}) \times W^{1,\infty}(\mathbb{R}_+,\mathbb{R}) \times \mathbb{R}^n \to W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$ and $\mathcal{H}'_o : L^\infty(\mathbb{R}_+,\mathbb{R}) \times W^{1,\infty}(\mathbb{R}_+,\mathbb{R}) \times \mathbb{R}^n \to W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$ such that $\mathcal{H}'_s(\dot{u}, u, x_0) = x$, and $\mathcal{H}'_o(\dot{u}, u, x_0) = y$.

Observe that the operators $\mathcal{H}'_s$ and $\mathcal{H}'_o$ are causal owing to the uniqueness of the solutions of (63)–(64).

Consider the left-derivative operator $\varDelta_-$ defined on $W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$ by $[\varDelta_-(u)](t) = \lim_{\tau\uparrow t} \frac{u(\tau)-u(t)}{\tau - t}$. The operator $\varDelta_-$ is causal as $[\varDelta_-(u)](t)$ depends only on the values of $u(\tau)$ for $\tau \leq t$. We also have $\varDelta_-(u) = \dot{u}$ almost everywhere since $u \in W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$ so that $\varDelta_-(u) \in L^\infty(\mathbb{R}_+,\mathbb{R})$, that is $\varDelta_- : W^{1,\infty}(\mathbb{R}_+,\mathbb{R}) \to L^\infty(\mathbb{R}_+,\mathbb{R})$.

Consider the operators $\mathcal{H}_s, \mathcal{H}_o : W^{1,\infty}(\mathbb{R}_+,\mathbb{R}) \times \mathbb{R}^n \to W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$ defined by the relations

$$\mathcal{H}_s(u, x_0) = \mathcal{H}'_s(\varDelta_-(u), u, x_0) = x,$$
$$\mathcal{H}_o(u, x_0) = \mathcal{H}'_o(\varDelta_-(u), u, x_0) = y.$$

Then $\mathcal{H}_s$ and $\mathcal{H}_o$ are causal. Observe also that $\mathcal{H}_s$ and $\mathcal{H}_o$ satisfy Assumption 3. These facts mean the the operators $\mathcal{H}_s$ and $\mathcal{H}_o$ belong to the class of operators of Section 6.2 so that the definitions and results of Sections 6.3–6.5 apply.

To study the consistency of the operators $\mathcal{H}_s$ and $\mathcal{H}_o$ we follow the steps given in Section 6.4. If instead of u the input is $u \circ s_\gamma$ where $\gamma \in]0,\infty[$ then Equation (63) becomes

$$\begin{aligned}\dot{x}_\gamma(t) &= g_1\big(\dot{u}_\gamma(t)\big)\big(A_1 x_\gamma(t) + B_1 u_\gamma(t) + E_1\big)\\ &\quad + g_2\big(\dot{u}_\gamma(t)\big)\big(A_2 x_\gamma(t) + B_2 u_\gamma(t) + E_2\big)\\ &\quad \text{for almost all } t \geq 0\end{aligned} \tag{69}$$

where $u_\gamma = u \circ s_\gamma$. The initial state remains the same for all γ as explained in Section 6.4 so that Equation (64) becomes

$$x_\gamma(0) = x_0. \tag{70}$$

Given $\varrho \in I_u$ there exists a not necessarily unique $t_{\varrho,\gamma} \in \mathbb{R}_+$ such that $\rho_{u\circ s_\gamma}(t_{\varrho,\gamma}) = \varrho$. Since the operator $\mathcal{H}_s$ belongs to the class of operators of Section 6.2 it follows that $x_\gamma(t_{\varrho,\gamma})$ is independent of the particular choice of $t_{\varrho,\gamma}$ [35]. Thus, a function $x_{u\circ s_\gamma} : I_u \to \mathbb{R}^n$ can be defined by the relation $x_{u\circ s_\gamma}(\varrho) = x_\gamma(t_{\varrho,\gamma})$ so that $x_{u\circ s_\gamma} \circ \rho_{u\circ s_\gamma} = x_\gamma$ (recall that by Lemma 2 we have $I_{u\circ s_\gamma} = I_u$). We call the function $x_{u\circ s_\gamma}$ the *normalized state*.

Also, if instead of u the input is $u \circ s_\gamma$ then Equation (65) becomes

$$y_\gamma(t) = Cx_\gamma(t) + Du \circ s_\gamma(t), \forall t \in \mathbb{R}_+. \tag{71}$$

Given $\varrho \in I_u$ there exists a not necessarily unique $t_{\varrho,\gamma} \in \mathbb{R}_+$ such that $\rho_{u\circ s_\gamma}(t_{\varrho,\gamma}) = \varrho$. Since the operator $\mathcal{H}_o$ belongs to the class of operators of Section 6.2 it follows that $y_\gamma(t_{\varrho,\gamma})$ is independent of the particular choice of $t_{\varrho,\gamma}$. Thus, the normalized output $\varphi_{u\circ s_\gamma} : I_u \to \mathbb{R}^n$ is defined by the relation $\varphi_{u\circ s_\gamma}(\varrho) = y_\gamma(t_{\varrho,\gamma})$ so that $\varphi_{u\circ s_\gamma} \circ \rho_{u\circ s_\gamma} = y_\gamma$. Taking into account that $\psi_{u\circ s_\gamma} = \psi_u$ by Lemma 2 we get

$$\varphi_{u\circ\gamma}(\varrho) = Cx_{u\circ\gamma}(\varrho) + D\psi_u(\varrho), \forall \varrho \in I_u. \tag{72}$$

Finally, given $\varrho \in I_u$ there exists a not necessarily unique $t_\varrho \in \mathbb{R}_+$ such that $\rho_u(t_\varrho) = \varrho$. Since the operator

[19] If $\lim_{w\downarrow 0} \bar{g}_1(w) = a_1 \neq 0$ and $\lim_{w\uparrow 0} \bar{g}_2(w) = -a_2 \neq 0$, the constants a_1 and a_2 are incorporated into the matrices A_1 and A_2 respectively.

[20] A matrix is stable if all its eigenvalues have strictly negative real parts.

Δ_- belongs to the class of operators of Section 6.2 it follows that $\dot{u}(t_\varrho)$ is independent of the particular choice of t_ϱ. This implies that a function $v_u : I_u \to \mathbb{R}$ can be defined almost everywhere by the relation $v_u(\varrho) = \dot{u}(t_\varrho)$. The function $v_u \in L^\infty(I_u, \mathbb{R})$ by Lemma 3 and we have $v_u \circ \rho_u = \dot{u}$. We call function v_u the *normalized input-derivative*. More about v_u in B.

Theorem 6 *[35] Consider the semilinear Duhem model (63)–(65). Assume that both matrices A_1 and $-A_2$ are stable and have a common Lyapunov matrix $P = P^T > 0$. Then, for all $\gamma \in]0, \infty[$, $x_{u\circ s_\gamma} \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ and $\varphi_{u\circ s_\gamma} \in W^{1,\infty}(I_u, \mathbb{R})$. Moreover*

$$\begin{aligned} x_{u\circ s_\gamma}(\sigma) = x_0 + \int_0^\sigma &\bar{g}_1\left(\frac{v_u(\varrho)}{\gamma}\right)\big[A_1 x_{u\circ s_\gamma}(\varrho) \\ &+ B_1\psi_u(\varrho) + E_1\big] \\ &+ \bar{g}_2\left(\frac{v_u(\varrho)}{\gamma}\right)\big[A_2 x_{u\circ s_\gamma}(\varrho) \\ &+ B_2\psi_u(\varrho) + E_2\big]\,\mathrm{d}\varrho, \forall \sigma \in I_u. \end{aligned} \tag{73}$$

Also $\exists!\, x_u^\star \in W^{1,\infty}(I_u, \mathbb{R}^n)$ such that $\lim_{\gamma\to\infty} \|x_u^\star - x_{u\circ s_\gamma}\|_{I_u} = 0$ which means that the operator $\mathcal{H}_s$ is consistent with respect to all $(u, x_0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \mathbb{R}^n$; and $\exists!\, \varphi_u^\star \in W^{1,\infty}(I_u, \mathbb{R})$ such that $\lim_{\gamma\to\infty} \|\varphi_u^\star - \varphi_{u\circ s_\gamma}\|_{I_u} = 0$ which means that the operator $\mathcal{H}_o$ is consistent with respect to all $(u, x_0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)\times\mathbb{R}^n$. We have:

$$\begin{aligned} \frac{\mathrm{d}x_u^\star}{\mathrm{d}\varrho}(\varrho) = &\frac{\dot{\psi}_u(\varrho)+1}{2}\big(A_1 x_u^\star(\varrho) + B_1\psi_u(\varrho) + E_1\big) \\ &+ \frac{\dot{\psi}_u(\varrho)-1}{2}\big(A_2 x_u^\star(\varrho) + B_2\psi_u(\varrho) + E_2\big) \\ &\text{for almost all } \varrho \in I_u, \end{aligned} \tag{74}$$

$$x_u^\star(0) = x_0, \tag{75}$$

$$\varphi_u^\star(\varrho) = C x_u^\star(\varrho) + D\psi_u(\varrho), \forall \varrho \in I_u. \tag{76}$$

11.3 Strong consistency of the semilinear Duhem model

This section presents the results obtained in Ref. [35] in relation with the strong consistency of the semilinear Duhem model.

To study the strong consistency of the operators $\mathcal{H}_s$ and $\mathcal{H}_o$ we follow the steps given in Section 6.5. Consider an input u that is non constant and T–periodic where $T \in]0, \infty[$. For any nonnegative integer k, define $x_{u,k}^\star \in W^{1,\infty}\big([0, \rho_u(T)], \mathbb{R}^m\big)$ by

$$x_{u,k}^\star(\varrho) = x_u^\star\big(\rho_u(T)k + \varrho\big), \forall \varrho \in [0, \rho_u(T)], \tag{77}$$

and define $\varphi_{u,k}^\star \in W^{1,\infty}\big([0, \rho_u(T)], \mathbb{R}^m\big)$ by

$$\varphi_{u,k}^\star(\varrho) = \varphi_u^\star\big(\rho_u(T)k + \varrho\big), \forall \varrho \in [0, \rho_u(T)]. \tag{78}$$

Theorem 7 *[35] Consider the semilinear Duhem model (63)–(65). Assume that the matrices A_1 and $-A_2$ are both stable and have a common Lyapunov matrix $P = P^T > 0$. Let $(u, x_0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \mathbb{R}^n$ be such that u is non constant and T–periodic. Then there exists a unique function $x_u^\circ \in W^{1,\infty}\big([0, \rho_u(T)], \mathbb{R}^n\big)$ such that $\lim_{k\to\infty} \|x_{u,k}^\star - x_u^\circ\|_{[0,\rho_u(T)]} = 0$ which means that the operator $\mathcal{H}_s$ is strongly consistent with respect to (u, x_0). Also $\exists\ !\ \varphi_u^\circ \in W^{1,\infty}([0, \rho_u(T)], \mathbb{R})$ such that $\lim_{k\to\infty} \|\varphi_{u,k}^\circ - \varphi_u^\circ\|_{[0,\rho_u(T)]} = 0$ which means that the operator $\mathcal{H}_o$ is strongly consistent with respect to (u, x_0). We have $x_u^\circ(0) = x_u^\circ\big(\rho_u(T)\big)$, $\varphi_u^\circ(0) = \varphi_u^\circ\big(\rho_u(T)\big)$, and*

$$\begin{aligned} \frac{\mathrm{d}x_u^\circ}{\mathrm{d}\varrho}(\varrho) = &\frac{\dot{\psi}_u(\varrho)+1}{2}\big(A_1 x_u^\circ(\varrho) + B_1\psi_u(\varrho) + E_1\big) \\ &+ \frac{\dot{\psi}_u(\varrho)-1}{2}\big(A_2 x_u^\circ(\varrho) + B_2\psi_u(\varrho) + E_2\big) \\ &\text{for almost all } \varrho \in [0, \rho_u(T)], \end{aligned} \tag{79}$$

$$\varphi_u^\circ(\varrho) = C x_u^\circ(\varrho) + D\psi_u(\varrho), \forall \varrho \in [0, \rho_u(T)]. \tag{80}$$

Note that the initial condition $x_u^\circ(0)$ may be different from x_0.

Special cases[21]

Special case 1. We consider that $u \in \Lambda_{u_{\min}, u_{\max}, \alpha_1, T}$ (see Equation (20)). In this case it is possible to find the explicit expression for the initial condition $x_u^\circ(0)$. Indeed, from Equation (79) it comes that

$$\frac{\mathrm{d}x_u^\circ}{\mathrm{d}\varrho}(\varrho) = A_1 x_u^\circ(\varrho) + B_1\psi_u(\varrho) + E_1, \forall \varrho \in \big]0, \rho_u(\alpha_1)\big[. \tag{81}$$

The differential equation (81) gives

$$\begin{aligned} x_u^\circ\big(\rho_u(\alpha_1)\big) = &e^{\rho_u(\alpha_1)A_1} x_u^\circ(0) \\ &+ e^{\rho_u(\alpha_1)A_1}\int_0^{\rho_u(\alpha_1)} e^{-\varrho A_1}\big(B_1\psi_u(\varrho) + E_1\big)\,\mathrm{d}\varrho. \end{aligned} \tag{82}$$

On the other hand, using Lemma 1 and the fact that $u \in \Lambda_{u_{\min}, u_{\max}, \alpha_1, T}$ it comes that

$$\psi_u(\varrho) = \varrho + u_{\min}, \forall \varrho \in [0, \rho_u(\alpha_1)], \tag{83}$$

$$\psi_u(\varrho) = -\varrho + 2u_{\max} - u_{\min}, \forall \varrho \in [\rho_u(\alpha_1), \rho_u(T)], \tag{84}$$

$$\rho_u(\alpha_1) = u_{\max} - u_{\min}, \tag{85}$$

$$\rho_u(T) = 2(u_{\max} - u_{\min}). \tag{86}$$

[21] These special cases of are not studied in Ref. [35].

Combining Equations (83), (82) and (85) we get

$$\begin{aligned} x_u^\circ(u_{\max}-u_{\min}) &= e^{(u_{\max}-u_{\min})A_1}x_u^\circ(0) \\ &+\Big(-A_1^{-1}(u_{\max}-u_{\min})-A_1^{-2} \\ &+A_1^{-2}e^{(u_{\max}-u_{\min})A_1}\Big)B_1 \\ &+\Big(-A_1^{-1}+A_1^{-1}e^{(u_{\max}-u_{\min})A_1}\Big) \\ &\times\big(B_1u_{\min}+E_1\big). \end{aligned} \tag{87}$$

Note that the matrix A_1 is invertible as it is stable. Also, the differential equation (79) gives

$$\begin{aligned} x_u^\circ\big(\rho_u(T)\big) &= e^{-\big(\rho_u(T)-\rho_u(\alpha_1)\big)A_2}x_u^\circ\big(\rho_u(\alpha_1)\big) \\ &- e^{-\rho_u(T)A_2}\int_{\rho_u(\alpha_1)}^{\rho_u(T)} e^{\varrho A_2}\big(B_2\psi_u(\varrho)+E_2\big)\,\mathrm{d}\varrho. \end{aligned} \tag{88}$$

Combining Equations (84)–(88) it comes that

$$\begin{aligned} x_u^\circ\big(2(u_{\max}-u_{\min})\big) &= e^{(u_{\min}-u_{\max})A_2}x_u^\circ(u_{\max}-u_{\min}) \\ &+B_2\Big[-A_2^{-2}+2(u_{\max}-u_{\min})A_2^{-1} \\ &+A_2^{-2}e^{(u_{\min}-u_{\max})A_2} \qquad (89)\\ &-A_2^{-1}e^{(u_{\min}-u_{\max})A_2}(u_{\max}-u_{\min})\Big] \\ &+\Big(-A_2^{-1}+A_2^{-1}e^{(u_{\min}-u_{\max})A_2}\Big) \\ &\times\big(B_2(2u_{\max}-u_{\min})+E_2\big). \end{aligned}$$

Note that the matrix A_2 is invertible as $-A_2$ is stable. From Theorem 7 it follows that that $x_u^\circ\big(2(u_{\max}-u_{\min})\big)=x_u^\circ(0)$ owing to Equation (86). This equality combined with Equations (89) and (87) gives

$$\begin{aligned} x_u^\circ(0) &= \theta = D_0^{-1}N_0, \qquad (90)\\ N_0 &= e^{(u_{\min}-u_{\max})A_2}\Big[\Big(-A_1^{-1}(u_{\max}-u_{\min})-A_1^{-2} \\ &+A_1^{-2}e^{(u_{\max}-u_{\min})A_1}\Big)B_1 \\ &+\Big(-A_1^{-1}+A_1^{-1}e^{(u_{\max}-u_{\min})A_1}\Big)\big(B_1u_{\min}+E_1\big)\Big] \\ &+\Big[-A_2^{-2}+2(u_{\max}-u_{\min})A_2^{-1} \\ &+A_2^{-2}e^{(u_{\min}-u_{\max})A_2} \\ &-A_2^{-1}e^{(u_{\min}-u_{\max})A_2}(u_{\max}-u_{\min})\Big]B_2 \\ &+\Big(-A_2^{-1}+A_2^{-1}e^{(u_{\min}-u_{\max})A_2}\Big) \\ &\times\big(B_2(2u_{\max}-u_{\min})+E_2\big), \\ D_0 &= I_n - e^{(u_{\min}-u_{\max})A_2}\cdot e^{(u_{\max}-u_{\min})A_1}, \end{aligned}$$

where I_n is the $n\times n$ identity matrix.

Special case 2. We consider that $u\in\Lambda_{u_{\min},u_{\max},\alpha_1,T}$ and $n=1$. Our aim is to study the conditions for which the hysteresis loop of the scalar semilinear Duhem model is not trivial (see Definition 9).
To this end, combining Equations (81), (83) and (85) we get

$$\dot\xi_1(\nu)=A_1\xi_1(\nu)+B_1\nu+E_1,\ \forall\nu\in\,]u_{\min},u_{\max}[, \tag{91}$$

where $\xi_1:[u_{\min},u_{\max}]\to\mathbb{R}$ is defined by the relation $\xi_1(\nu)=x_u^\circ(\varrho)$ with $\nu=\varrho+u_{\min}$ and $\varrho\in[0,\rho_u(\alpha_1)]$. Similarly, for $\varrho\in[\rho_u(\alpha_1),\rho_u(T)]$ we get

$$\dot\xi_2(\nu)=A_2\xi_2(\nu)+B_2\nu+E_2,\ \forall\nu\in\,]u_{\min},u_{\max}[, \tag{92}$$

where $\xi_2:[u_{\min},u_{\max}]\to\mathbb{R}$ is defined by the relation $\xi_2(\nu)=x_u^\circ(\varrho)$ with $\nu=-\varrho+2u_{\max}-u_{\min}$.

Solving the differential equations (91) and (92) we get for all $\nu\in[u_{\min},u_{\max}]$

$$\begin{aligned} \xi_1(\nu) &= -\frac{B_1}{A_1}\nu-\frac{E_1}{A_1}-\frac{B_1}{A_1^2} \qquad (93)\\ &+\left(\frac{B_1}{A_1}u_{\min}+\frac{E_1}{A_1}+\frac{B_1}{A_1^2}+\theta\right)e^{A_1(\nu-u_{\min})}, \\ \xi_2(\nu) &= -\frac{B_2}{A_2}\nu-\frac{E_2}{A_2}-\frac{B_2}{A_2^2} \qquad (94)\\ &+\left(\frac{B_2}{A_2}u_{\min}+\frac{E_2}{A_2}+\frac{B_2}{A_2^2}+\theta\right)e^{A_2(\nu-u_{\min})}. \end{aligned}$$

The hysteresis loop $\mathcal{G}_u$ of the operator $\mathcal{H}_o$ with respect to (u,x_0) is independent of the initial state x_0 and is given by (see Definition 8):

$$\begin{aligned} \mathcal{G}_u = &\ \big\{\big(\nu,C\xi_1(\nu)+D\nu\big),\nu\in[u_{\min},u_{\max}]\big\} \\ &\cup\big\{\big(\nu,C\xi_2(\nu)+D\nu\big),\nu\in[u_{\min},u_{\max}]\big\}. \end{aligned} \tag{95}$$

Lemma 8 *Consider the semilinear Duhem model (63)–(65) with $n=1$, $A_1<0$ and $A_2>0$. Then, Propositions* (i) *and* (ii) *are equivalent.*

(i) *For all $(u,x_0)\in\Lambda_{u_{\min},u_{\max},\alpha_1,T}\times\mathbb{R}$, the operator $\mathcal{H}_o$ has a trivial hysteresis loop with respect to (u,x_0).*
(ii) *Equalities (96) and (97) hold.*

$$A_2^{-1}B_2=A_1^{-1}B_1, \tag{96}$$
$$B_1A_1^{-1}\left(A_2^{-1}-A_1^{-1}\right)-E_1A_1^{-1}+E_2A_2^{-1}=0. \tag{97}$$

Proof See Appendix E.

11.4 Illustration of the consistency and strong consistency of the scalar semilinear Duhem model

We consider the semilinear Duhem model with the following parameters: $n=1$, $A_1=-1$, $A_2=1$, $B_1=1$, $B_2=-1$, $E_1=0$, $E_2=0$, $C=1$, $D=0$. The function $g_1:\mathbb{R}\to\mathbb{R}$ is defined by the relations $\forall x\in\mathbb{R}, g_1(x)=0$ if $x\le 0$, and $g_1(x)=x+x^2$ if $x\ge 0$. The function

$g_2 : \mathbb{R} \to \mathbb{R}$ is defined by the relations $\forall x \in \mathbb{R}, g_2(x) = 0$ if $x \geq 0$, and $g_2(x) = x$ if $x \leq 0$.

Consider the 2–periodic input u defined as follows: $u(t) = t, \forall t \in [0, 1]$, and $u(t) = 2 - t, \forall t \in [1, 2]$ (see Figure 5). Observe that, since ρ_u is the identity function,

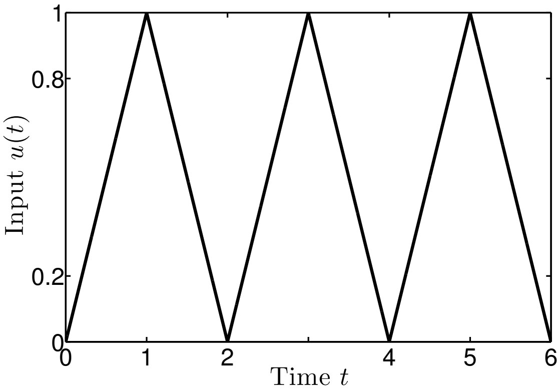

Fig. 5: $u(t)$ versus t.

we have $v_u = \dot{u}$ almost everywhere so that in the differential equation (73) we have $v_u(\varrho) = 1, \forall \varrho \in]0, 1[$ and $v_u(\varrho) = -1, \forall \varrho \in]1, 2[$. The following values of γ are considered: $\gamma = 1$, $\gamma = 10$ and $\gamma = 100$. The differential equation (73) is solved using Matlab solver ode23s for the three values of γ and with the initial condition $x_0 = 0$. For each value of γ we obtain the corresponding $x_{u\circ\gamma}$ which, in this case, is equal to $\varphi_{u\circ\gamma}$ as $C = 1$ and $D = 0$ (see Equation (72)). Figure 6 provides the plot of function $\varphi_{u\circ\gamma}(\varrho)$ versus time ϱ for $\gamma = 1$, $\gamma = 10$ and $\gamma = 100$ (dotted). The same figure provides the plot of function $\varphi_u^\star(\varrho)$ versus time ϱ (solid). The plot of $\varphi_u^\star$ has been obtained by solving the differential equation (74) using Matlab solver ode23s, and taking into account that $\psi_u = u$ and that the initial condition $\varphi_u^\star(0)$ is also $x_0 = 0$ (see Equation (75)). Since $C = 1$ and $D = 0$ we have $\varphi_u^\star = x_u^\star$ (see Equation (76)). We can see that the plots $\varphi_{u\circ\gamma}(\varrho)$ versus ϱ converge to the plot $\varphi_u^\star(\varrho)$ versus ϱ as γ increases which is predicted by Theorem 6.

Now that $\varphi_u^\star$ has been computed, the functions $\varphi_{u,k}^\star$ where $k \in \mathbb{N}$ are determined using Equation (78). Figure 7 provides the plots of function $\varphi_{u,k}^\star(\varrho)$ versus ϱ for $k = 0$, $k = 1$ and $k = 2$ (dotted). The same figure provides the plot of the function $\varphi_u^\circ(\varrho)$ versus time ϱ (solid). The plot of φ_u° is obtained by solving the differential equation (79) using Matlab solver ode23s, and taking into account that $\psi_u = u$. The initial condition $x_u^\circ(0)$ is obtained from Equation (90). Note that we have $\varphi_u^\circ = x_u^\circ$ as $C = 1$ and $D = 0$ (see Equation (80)). As predicted by Theorem 7 it can be seen that the plots

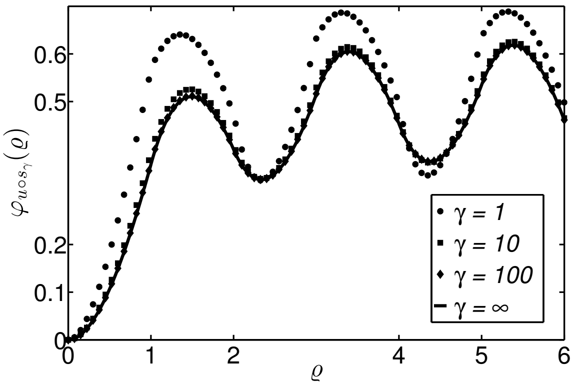

Fig. 6: Dotted: $\varphi_{u\circ s_\gamma}(\varrho)$ versus ϱ for $\gamma = 1$, $\gamma = 10$ and $\gamma = 100$. Solid: $\varphi_u^\star(\varrho)$ versus ϱ (labeled as $\gamma = \infty$). Note that the plot that corresponds to $\gamma = 100$ is practically equal to the one that corresponds to $\gamma = \infty$.

$\varphi_{u,k}^\star(\varrho)$ versus ϱ converge to the plot $\varphi_u^\circ(\varrho)$ versus ϱ as k increases.

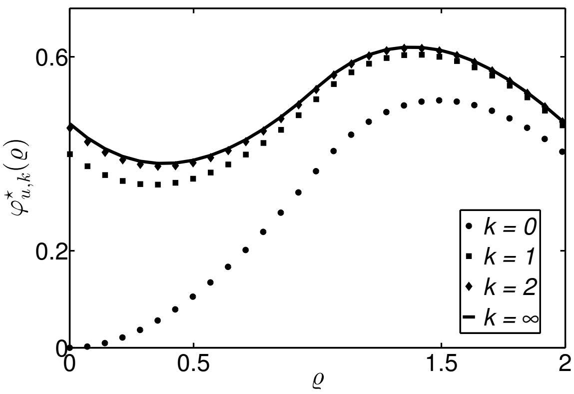

Fig. 7: Dotted: $\varphi_{u,k}^\star(\varrho)$ versus ϱ for $k = 0$, $k = 1$ and $k = 2$. Solid: $\varphi_u^\circ(\varrho)$ versus ϱ (labeled as $k = \infty$). Note that the plot that corresponds to $k = 2$ is practically equal to the one that corresponds to $k = \infty$.

The hysteresis loop of the operator $\mathcal{H}_o$ with respect to (u, x_0), that is the set $\left\{\left(\psi_u(\varrho), \varphi_u^\circ(\varrho)\right), \varrho \in [0, 2]\right\}$ (see Equation (31)), is plotted in Figure 8. It can be seen that the hysteresis loop is not trivial as predicted by Lemma 8 since Equality (97) does not hold.

We now use the value $E_2 = 2$ instead of $E_2 = 0$ so that both Equalities (96) and (97) hold. Lemma 8 predicts that the hysteresis loop is trivial as can be observed in Figure 9.

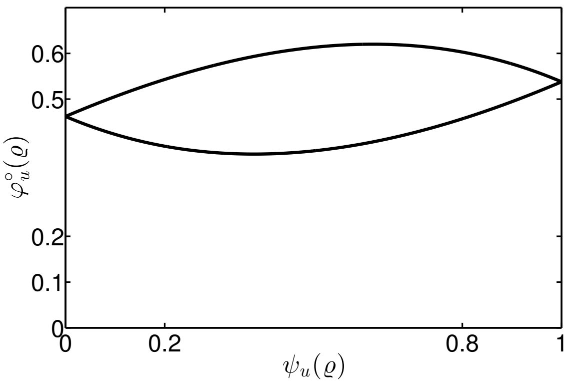

Fig. 8: $\varphi_u^\circ(\varrho)$ versus $\psi_u(\varrho)$ for $\varrho \in [0, 2]$.

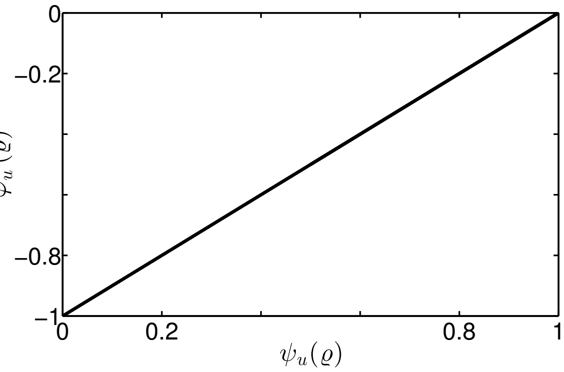

Fig. 9: $\varphi_u^\circ(\varrho)$ versus $\psi_u(\varrho)$ for $\varrho \in [0, 2]$.

11.5 Hysteresis property -according to Definition 4- of the scalar semilinear Duhem model

In this section we focus on the scalar version of the semilinear Duhem model (63)–(65), that is we consider that $n = 1$. We also consider that $A_1 < 0$ and $A_2 > 0$ so that Theorem 5 applies.

Our aim is to check whether the scalar semilinear Duhem model is a hysteresis according to Definition 4. To this end, we need to check whether Assumptions 1 and 2 hold as a prerequisite for Definition 4. Owing to Theorem 5 we can see that $x \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ so that Assumption 1 is satisfied.

Now, we have to check whether Assumption 2 is satisfied. To this end, let $\gamma \in]0, \infty[$ and $u \in \Lambda_{u_{\min}, u_{\max}, \alpha_1, T}$; recall that the input $u \circ s_\gamma$ is $T\gamma$–periodic where s_γ is a linear time-scale change. Assumption 2 will be satisfied if we can find a unique initial condition $x_{0,\gamma} \in \mathbb{R}$ such that $\mathcal{H}_s(u \circ s_\gamma, x_{0,\gamma})$ is also $T\gamma$–periodic.

When the semilinear Duhem model is rate independent, $x_{0,\gamma}$ is independent of γ. In this case Ref. [54] provides the expression of $x_{0,\gamma}$ (see [54, Equations (4.9)–(4.14)]) which means that Assumption 2 is satisfied.

However, Ref. [54] provides no proof that Assumption 2 is satisfied for the rate-dependent semilinear Duhem model. Instead, another argument is used in the proof of [54, Proposition 5.1] to check whether the rate-dependent semilinear Duhem model is a hysteresis according to Definition 4 (or equivalently [54, Definition 2.2]). As shown in Section 12.1.3, that argument does not imply necessarily that Assumption 2 is satisfied.

In what follows we prove that Assumption 2 is satisfied for both the rate-independent and the rate-dependent scalar semilinear Duhem model.

Theorem 8 *Consider the semilinear Duhem model (63)–(65) with $n = 1$, $A_1 < 0$, $A_2 > 0$. Let $u \in \Lambda_{u_{\min}, u_{\max}, \alpha_1, T}$. Then, $\exists \gamma_0 > 0$ such that $\forall \gamma \in]\gamma_0, \infty[$ there exists a unique $x_{0,\gamma} \in \mathbb{R}$ such that $\mathcal{H}_s(u \circ s_\gamma, x_{0,\gamma})$ is also $T\gamma$–periodic.*

Proof See Appendix C.

Theorem 8 shows that Assumption 2 is satisfied (see Remark 1). Our objective now is to prove that Conditions (i) and (ii) of Definition 4 are met. We start with Condition (i).

The authors of Ref. [54] provide no proof that Condition (i) of Definition 4 is satisfied for the rate-dependent semilinear Duhem model (for the rate-independent model, the proof is trivial). To prove that Condition (i) is met we start by finding the explicit expression of the set $\mathcal{C}_{u,\gamma}$ of Equation (21). Let $\gamma \in]\gamma_0, \infty[$ where γ_0 is given by Equation (139). From Equations (21) and (169) it follows that

$$\mathcal{C}_{u,\gamma} = \left\{ \big(u(\sigma), C\bar{z}_\gamma(\sigma) + Du(\sigma)\big), \sigma \in [0, T] \right\}. \tag{98}$$

where $\bar{z}_\gamma$ is defined in Appendix C, Equation (166).

Define the function $h_1 : [0, \alpha_1] \to \mathbb{R}$ by

$$h_1(\sigma) = \left(\bar{z}_\gamma(0) + \int_0^\sigma \gamma \frac{g_1\left(\frac{\dot{u}(\tau)}{\gamma}\right)\left(B_1 u(\tau) + E_1\right)}{\exp\left(\gamma A_1 \int_0^\tau g_1\left(\frac{\dot{u}(t)}{\gamma}\right) \mathrm{d}t\right)} \mathrm{d}\tau \right) \times \exp\left(\gamma A_1 \int_0^\sigma g_1\left(\frac{\dot{u}(\tau)}{\gamma}\right) \mathrm{d}\tau\right), \forall \sigma \in [0, \alpha_1]. \tag{99}$$

It can be checked that h_1 satisfies the following differential equation

$$\begin{aligned} \dot{h}_1(\sigma) &= \gamma g_1\left(\frac{\dot{u}(\tau)}{\gamma}\right)\left(A_1 h_1(\sigma) + B_1 u(\sigma) + E_1\right), \forall \sigma \in]0, \alpha_1[, \\ h_1(0) &= \bar{z}_\gamma(0). \end{aligned} \tag{100}$$

Owing to the uniqueness of the solutions of (167) it comes that

$$\bar{z}_\gamma(\sigma) = h_1(\sigma), \forall \sigma \in [0, \alpha_1]. \tag{101}$$

A similar argument on the interval $[\alpha_1, T]$ shows that

$$\bar{z}_\gamma(\sigma) = \left(\bar{z}_\gamma(\alpha_1) + \int_{\alpha_1}^{\sigma} \gamma \frac{g_2\left(\frac{\dot{u}(\tau)}{\gamma}\right)\left(B_2 u(\tau) + E_2\right)}{\exp\left(\gamma A_2 \int_{\alpha_1}^{\tau} g_2\left(\frac{\dot{u}(t)}{\gamma}\right) \mathrm{d}t\right)} \mathrm{d}\tau\right) \times \exp\left(\gamma A_2 \int_{\alpha_1}^{\sigma} g_2\left(\frac{\dot{u}(\tau)}{\gamma}\right) \mathrm{d}\tau\right), \forall \sigma \in [\alpha_1, T]. \tag{102}$$

Owing to the T–periodicity of $\bar{z}_\gamma$ we have $\bar{z}_\gamma(T) = \bar{z}_\gamma(0)$. This fact along with Equations (99), (101), and (102) gives

$$\bar{z}_\gamma(0) = x_{0,\gamma} = \frac{N}{D}, \tag{103}$$

where

$$\begin{aligned} N =& \exp\left[\int_0^{\alpha_1} \gamma A_1 g_1\left(\frac{\dot{u}(\tau)}{\gamma}\right) \mathrm{d}\tau + \int_{\alpha_1}^{T} \gamma A_2 g_2\left(\frac{\dot{u}(\tau)}{\gamma}\right) \mathrm{d}\tau\right] \\ &\times \int_0^{\alpha_1} \gamma \frac{g_1\left(\frac{\dot{u}(\tau)}{\gamma}\right)\left(B_1 u(\tau) + E_1\right)}{\exp\left(\gamma A_1 \int_0^{\tau} g_1\left(\frac{\dot{u}(t)}{\gamma}\right) \mathrm{d}t\right)} \mathrm{d}\tau \\ &+ \exp\left[\gamma A_2 \int_{\alpha_1}^{T} g_2\left(\frac{\dot{u}(\tau)}{\gamma}\right) \mathrm{d}\tau\right] \\ &\times \int_{\alpha_1}^{T} \gamma \frac{g_2\left(\frac{\dot{u}(\tau)}{\gamma}\right)\left(B_2 u(\tau) + E_2\right)}{\exp\left(\gamma A_2 \int_{\alpha_1}^{\tau} g_2\left(\frac{\dot{u}(t)}{\gamma}\right) \mathrm{d}t\right)} \mathrm{d}\tau, \\ D =& 1 - \exp\left[\gamma A_1 \int_0^{\alpha_1} g_1\left(\frac{\dot{u}(\tau)}{\gamma}\right) \mathrm{d}\tau \right. \\ &\left. + \gamma A_2 \int_{\alpha_1}^{T} g_2\left(\frac{\dot{u}(\tau)}{\gamma}\right) \mathrm{d}\tau\right]. \end{aligned}$$

Define the function $\bar{z} : [0, T] \to \mathbb{R}$ by

$$\bar{z}(\sigma) = \xi_1\big(u(\sigma)\big), \forall \sigma \in [0, \alpha_1], \tag{104}$$

$$\bar{z}(\sigma) = \xi_2\big(u(\sigma)\big), \forall \sigma \in [\alpha_1, T], \tag{105}$$

where the functions ξ_1 and ξ_2 are given by Equations (93) and (94) respectively. It can checked that $\bar{z}(T) = \bar{z}(0) = \theta$ where θ is given by Equation (90). Define the closed curve

$$\mathcal{C}_u = \left\{\big(u(\sigma), C\bar{z}(\sigma) + Du(\sigma)\big), \sigma \in [0, T]\right\}. \tag{106}$$

Theorem 9 $\lim_{\gamma \to \infty} d_2(\mathcal{C}_{u,\gamma}, \mathcal{C}_u) = 0.$

Proof See Appendix D.

Recall that the operator $\mathcal{H}_o$ that characterizes the scalar semilinear Duhem model associates to each input $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ and each initial condition $x_0 \in \mathbb{R}$ the output $y \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ given by Equation (65). Theorem 9 shows that Condition (i) of Definition 4 holds for the operator $\mathcal{H}_o$. Now it remains to check whether Condition (ii) of Definition 4 also holds.

Lemma 9 *Consider the semilinear Duhem model (63)–(65) with $n = 1$, $A_1 < 0$ and $A_2 > 0$. Then Condition* (ii) *of Definition 4 holds for the operator $\mathcal{H}_o$ if and only if at least one of the equalities (107)–(108) does not hold.*

$$A_2^{-1} B_2 = A_1^{-1} B_1, \tag{107}$$

$$B_1 A_1^{-1}\left(A_2^{-1} - A_1^{-1}\right) - E_1 A_1^{-1} + E_2 A_2^{-1} = 0. \tag{108}$$

Proof The proof is similar to that of Lemma 8 *mutatis mutandis* (See Appendix E).

Lemma 9 has not been derived in Ref. [54].

As a conclusion for the present section, when $n = 1$, $A_1 < 0$, and $A_2 > 0$, the operator $\mathcal{H}_o$ is a hysteresis according to Definition 4 if and only if at least one of the equalities (107)–(108) does not hold.

11.6 Illustration of the hysteresis property -according to Definition 4- of the semilinear Duhem model

We consider the same scalar semilinear Duhem model as in Section 11.4, that is we consider that

$$\begin{aligned} &\dot{x}(t) = g_1\big(\dot{u}(t)\big)\big(-x(t) + u(t)\big) + g_2\big(\dot{u}(t)\big)\big(x(t) - u(t)\big) \\ &\qquad\qquad \text{for almost all } t \in \mathbb{R}_+, \\ &x(0) = x_0, \\ &y(t) = x(t), \forall t \in \mathbb{R}_+. \end{aligned}$$

We take as initial condition $x_0 = 0$, and as input the 2–periodic function u defined as follows: $u(t) = t, \forall t \in [0, 1]$, and $u(t) = 2 - t, \forall t \in [1, 2]$ (see Figure 5). Let $\gamma \in]0, \infty[$ and consider the output $\mathcal{H}_o(u \circ s_\gamma, x_0) = x_\gamma$ which is the solution of the differential equation (140). We take $\gamma = 1$ and solve (140) using Matlab solver ode23s. The resulting solution is plotted against the input $u \circ s_\gamma$ in Figure 10 (dotted).

The value $x_{0,\gamma}$ is computed using Equation (103); we get $x_{0,\gamma} \simeq 0.4979$. The fact that $x_{0,\gamma} \neq x_0$ explains why the set $\left\{\big(u \circ s_\gamma(t), [\mathcal{H}_o(u \circ s_\gamma, x_0)](t)\big), t \in \mathbb{R}_+\right\}$ is not a closed curve. We now solve the differential equation (140) taking as initial condition $x(0) = x_{0,\gamma}$. The obtained solution is plotted against the input $u \circ s_\gamma$ in Figure 10 (solid). We can see that the set $\mathcal{C}_{u,\gamma} = \left\{\big(u \circ s_\gamma(t), [\mathcal{H}_o(u \circ s_\gamma, x_{0,\gamma})](t)\big), t \in \mathbb{R}_+\right\}$ is a curve which is closed as predicted by Theorem 8.

In Figure 10 observe that the point $\big(u \circ s_\gamma(t), [\mathcal{H}_o(u \circ s_\gamma, x_0)](t)\big)$ gets closer to the closed curve $\mathcal{C}_{u,\gamma}$ as $t \to \infty$. This is a consequence of the uniform convergence of z_m to $\bar{z}_\gamma$ on the interval $[0, T]$ (see the proof of Theorem 8).

Now we plot the closed curve $\mathcal{C}_{u,\gamma}$ for $\gamma = 1$, $\gamma = 10$ and $\gamma = 100$ (see Figure 11). The closed curve $\mathcal{C}_u$ is

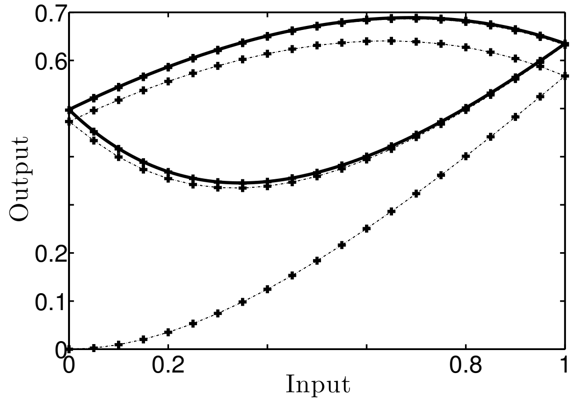

Fig. 10: Dotted: $[\mathcal{H}_o(u \circ s_\gamma, x_0)](t)$ versus $u \circ s_\gamma(t)$ for $\gamma = 1$, $t \in [0, 6]$. Solid: $\mathcal{C}_{u,\gamma}$, that is $[\mathcal{H}_o(u \circ s_\gamma, x_{0,\gamma})](t)$ versus $u \circ s_\gamma(t)$, for $\gamma = 1$ and $t \in [0, 2]$.

plotted using Equation (106) and the explicit expressions of the functions ξ_1 and ξ_2 provided in Equations (93)–(94). We observe that $\mathcal{C}_{u,\gamma}$ gets closer to the closed curve $\mathcal{C}_u$ as γ increases as predicted by Theorem 9 which shows that Condition (i) of Definition 4 is fulfiled.

Regarding Condition (ii) of Definition 4, observe that Equation (108) does not hold in our case. Thus, using Lemma 9, it follows that Condition (ii) of Definition 4 holds. This fact can be observed in Figure 11 since to any input value $\nu \in]u_{\min}, u_{\max}[=]0, 1[$ correspond two different values $\xi_1(\nu)$ ($\bullet$ marker) and $\xi_2(\nu)$ ($\star$ marker).

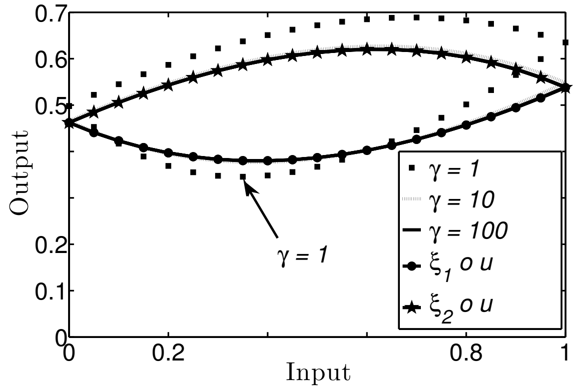

Fig. 11: $\mathcal{C}_{u,\gamma}$ for $\gamma = 1$, $\gamma = 10$, and $\gamma = 100$. Solid with markers: $\mathcal{C}_u$. Note that $\mathcal{C}_{u,100}$ is practically $\mathcal{C}_u$. The markers $\bullet$ on $\mathcal{C}_u$ correspond to $\xi_1 \circ u$ versus u. The markers $\star$ on $\mathcal{C}_u$ correspond to $\xi_2 \circ u$ versus u.

11.7 Dissipativity of the scalar rate-independent semilinear Duhem model

The aim of this section is to apply the results of Ref. [40] provided in Section 8 to study the dissipativity of the scalar semilinear Duhem model. To this end, we follow Section 8 by considering the model

$$\dot{x}(t) = \big(A_1 x(t) + B_1 u(t) + E_1\big)\dot{u}(t) \quad \text{for almost all } t \in [0, \infty[\text{ such that } \dot{u}(t) \geq 0, \tag{109}$$

$$\dot{x}(t) = \big(A_2 x(t) + B_2 u(t) + E_2\big)\dot{u}(t) \quad \text{for almost all } t \in [0, \infty[\text{ such that } \dot{u}(t) \leq 0, \tag{110}$$

$$x(0) = x_0, \tag{111}$$

$$y(t) = Cx(t) + Du(t), \forall t \in \mathbb{R}_+, \tag{112}$$

where $A_1, A_2, B_1, B_2, E_1, E_2, C, D \in \mathbb{R}$ are the model parameters, $x_0 \in \mathbb{R}$ is the initial condition, the function $u \in AC(\mathbb{R}_+, \mathbb{R})$ is the input, the function $x : \mathbb{R}_+ \to \mathbb{R}$ is the state, and the function $y : \mathbb{R}_+ \to \mathbb{R}$ is the output. Note that Inequalities (37)–(38) hold for any values of A_1 and A_2. This fact ensures the existence and uniqueness of solutions of the differential equation (109)–(111) on $\mathbb{R}_+$ so that $x, y \in AC(\mathbb{R}_+, \mathbb{R})$.

Observe that the functions $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ in (63) are defined by $g_1(v) = \max(0, v)$ and $g_2(v) = \min(0, v)$ for all $v \in \mathbb{R}$. Thus, it follows from Ref. [54] that the semilinear Duhem model (109)–(112) is rate independent.

Define the operators $\Phi, \Phi_1 : AC(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R} \to AC(\mathbb{R}_+, \mathbb{R})$ by $\Phi(u, x_0) = x$ and $\Phi_1(u, x_0) = y$. Note that, if Φ is dissipative with respect to the supply rate $\dot{x}u$, then there exists a nonnegative function $\varsigma : \mathbb{R}^2 \to \mathbb{R}_+$ such that $\forall (u, x_0) \in AC(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R}$, Inequality (39) holds. If $C > 0$ and $D \geq 0$ define the function $\varsigma_1 : \mathbb{R}^2 \to \mathbb{R}_+$ by

$$\varsigma_1(Cx_1 + Dv, v) = C\varsigma(x_1, v) + \frac{1}{2}Dv^2, \forall (x_1, v) \in \mathbb{R}^2. \tag{113}$$

Then, it can be checked that Inequality (39) holds for ς_1 and Φ_1, that is Φ_1 is dissipative with respect to the supply rate $\dot{y}u$.

Lemma 10 *Consider the model (109)–(112). Suppose that*

$$A_1 < 0, A_2 > 0, B_1 > 0, C > 0, D \geq 0, \tag{114}$$

$$A_2^{-1} B_2 = A_1^{-1} B_1, \tag{115}$$

$$B_1 A_1^{-1}\left(A_2^{-1} - A_1^{-1}\right) - E_1 A_1^{-1} + E_2 A_2^{-1} < 0. \tag{116}$$

Then, the intersecting function Ω is obtained explicitly by Equation (203). The function ς is obtained explicitly by Equations (204)–(205), and is such that Inequality

(39) holds for any $(u, x_0) \in AC(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R}$. However, ς is not nonnegative. If $\forall t \in \mathbb{R}_+, u(t) \in \left[\frac{1}{A_1}, \frac{1}{A_2}\right]$ then $\forall t \in \mathbb{R}_+, \varsigma\big(x(t), u(t)\big) \geq 0$.

Proof See Appendix F.

From Inequality (39) it follows that $\varsigma\big(x(t), u(t)\big) - \varsigma\big(x(0), u(0)\big) \leq \int_0^t \dot{x}(\tau)u(\tau)\mathrm{d}\tau$ for all $t \in \mathbb{R}_+$. If $\forall t \in \mathbb{R}_+, \varsigma\big(x(t), u(t)\big) \geq 0$ then, for all $t \in \mathbb{R}_+$ we have $-\varsigma\big(x(0), u(0)\big) \leq \int_0^t \dot{x}(\tau)u(\tau)\mathrm{d}\tau$ which means that the curve $t \mapsto \big(u(t), x(t)\big)$ is counterclockwise [1].

Theorem 3 provides sufficient conditions for the function ς to be nonnegative: $f_1 \geq 0$ and $f_2 \geq 0$. For the model (109)–(112) these sufficient conditions do not hold. Lemma 10 says that the curve $t \mapsto \big(u(t), x(t)\big)$ is counterclockwise when the input u is small enough.

Remark 3 Note that the condition $f_1 \geq 0$ and $f_2 \geq 0$ for the curve $t \mapsto \big(u(t), x(t)\big)$ to be counterclockwise has also been proposed by Duhem in 1896. Indeed, in [16, p. 11] Duhem assumes that *"if (x, X) and $(x + dx, X + dx)$ are two infinitely close equilibria relatively to the same temperature T of the system, dx and dX have always the same sign:*

$$dX\,dx > 0. \tag{117}$$

... inequality (117) translates geometrically as follows:
All upward lines go up from left to right;
All downward lines go down from right to left."
In Duhem's notations, x is the input and X the output so that Condition (117), which is the same as $\frac{dX}{dx} > 0$, is equivalent to $f_1 > 0$ and $f_2 > 0$ using the notations of Ref. [40].

Remark 4 In Ref. [58] sufficient conditions are provided for the rate-independent semilinear Duhem model to have counterclockwise dynamics. However, unlike Ref. [40], these conditions depend on the explicit solution of the model, which may not be easy to translate into conditions on the model's parameters.

11.8 Illustration of the dissipativity of the scalar rate-independent semilinear Duhem model

Consider the model (109)–(112) with parameters $A_1 = -1$, $A_2 = 1$, $B_1 = 1$, $B_2 = -1$, $E_1 = E_2 = 0$, $C = 1$, $D = 0$. With these values the relations (114)–(116) hold. The anhysteresis function is given by $f_{\mathrm{an}}(v) = v$, and it is possible to find the intersecting function Ω explicitly. We get

$$\Omega(x_0, u_0) = \begin{cases} u_0 + \log(x_0 - u_0 + 1) & \text{if} \quad x_0 \geq u_0, \\ u_0 - \log(-x_0 + u_0 + 1) & \text{if} \quad x_0 \leq u_0, \end{cases} \tag{118}$$

where log sets for the natural logarithm. The function ω_Φ in (41) is given by

$$\omega_\Phi(\sigma, x_1, v) = \begin{cases} \sigma - 1 + (x_1 - v + 1)e^{v-\sigma} & \text{if} \quad \sigma \geq v, \\ \sigma + 1 + (x_1 - v - 1)e^{\sigma - v} & \text{if} \quad \sigma \leq v, \end{cases} \tag{119}$$

and the function ς in (42) is given by

$$\varsigma(x_1, v) = \begin{cases} x_1 v - v - \log(x_1 - v + 1) - \frac{v^2}{2} + x_1 \\ \text{if} \quad x_1 \geq v, \\ x_1 v + v - \log(-x_1 + v + 1) - \frac{v^2}{2} - x_1 \\ \text{if} \quad x_1 \leq v. \end{cases} \tag{120}$$

We take as initial condition $x_0 = 0$. Now, consider the 2–periodic input u defined as follows: $u(t) = t, \forall t \in [0, 1]$, and $u(t) = 2 - t, \forall t \in [1, 2]$ (see Figure 5). Note that $\forall t \in \mathbb{R}_+, u(t) \in \left[\frac{1}{A_1}, \frac{1}{A_2}\right] = [-1, 1]$. The curve $x(t)$ $(= y(t))$ versus $u(t)$ is plotted in Figure 12. As predicted by Lemma 10 it can be seen that $t \mapsto \big(u(t), x(t)\big)$ is counterclockwise.

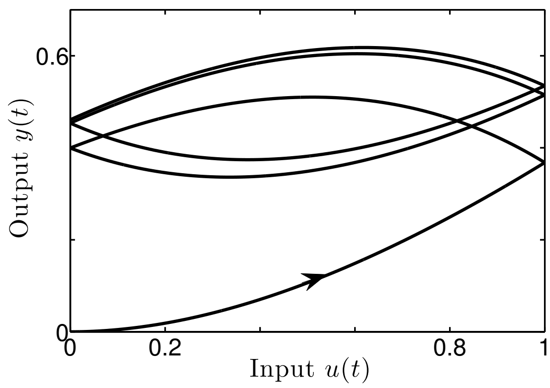

Fig. 12: $y(t)$ $(= x(t))$ versus $u(t)$

Now take as new input the 2–periodic function u defined as follows: $u(t) = t - 3, \forall t \in [0, 1]$, and $u(t) = -1 - t, \forall t \in [1, 2]$ (see Figure 13). Observe that the input is not in the interval $[-1, 1]$. The curve $t \mapsto \big(u(t), y(t)\big)$ is provided in Figure 14. It can be seen that $t \mapsto \big(u(t), y(t)\big)$ is not counterclockwise.

11.9 Minor loops of the scalar semilinear Duhem model

In this section we apply the concepts introduced in Section 10 to the scalar semilinear Duhem model.

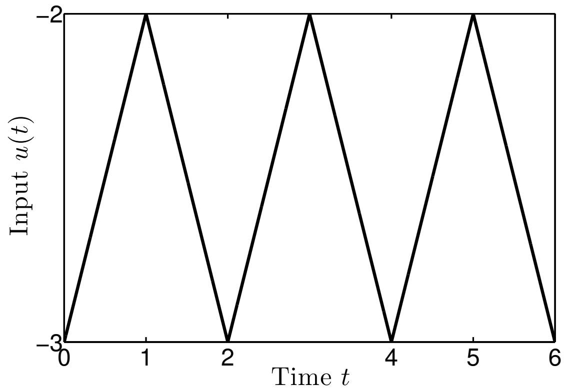

Fig. 13: Input $u(t)$ versus time t.

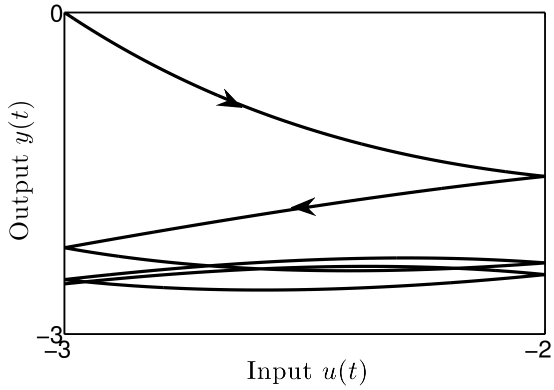

Fig. 14: $y(t)$ versus $u(t)$

Lemma 11 *Consider the semilinear Duhem model (63)–(65) with $n = 1$, $A_1 < 0$, $A_2 > 0$. If Assumption 8 holds, then Equalities (96)–(97) hold, and $\forall(u, x_0) \in \Lambda \times \mathbb{R}$ the operator $\mathcal{H}_o$ has a trivial hysteresis loop with respect to (u, x_0) (see Definition 9).*

Proof See Appendix G.

To illustrate Lemma 11 consider the semilinear Duhem model of Section 11.4 with $E_2 = 0$, and the input $u = \psi_u$ given by Equations (209)–(212) for $\alpha = 0.5$ (see Figure 15).

The corresponding hysteresis loop is the set $\{(\psi_u(\varrho), \varphi_u^\circ(\varrho))), \varrho \in [0, \varrho_4 = 3]\}$ where φ_u° obeys Equations (79)–(80), and the initial condition is given by Equation (234). The hysteresis loop is provided in Figure 16. Observe that $\psi_u(\varrho_1) = \psi_u(\varrho_3 = \varrho_5)$ and that $\varphi_u^\circ(\varrho_1) \neq \varphi_u^\circ(\varrho_3)$. This is due to the fact that Equality (97) does not hold so that Assumption 8 is not valid by Lemma 11.

We now use the value $E_2 = 2$ instead of $E_2 = 0$ so that both equalities (96) and (97) hold, which is a necessary condition for Assumption 8 to hold. We consider the input $u \in \Lambda$ of Figure 5. The corresponding

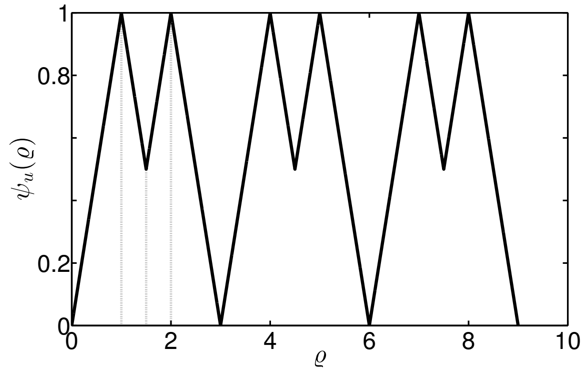

Fig. 15: $\psi_u(\varrho)$ versus ϱ for $\varrho \in [0, 3]$. We have $\varrho_1 = 1$, $\varrho_2 = 1.5$, $\varrho_3 = \varrho_5 = 2$, $\varrho_4 = 3$.

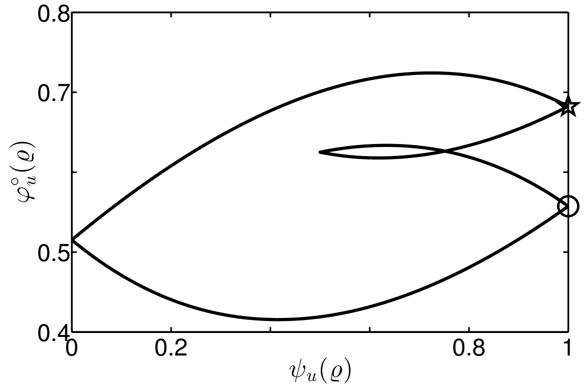

Fig. 16: $\varphi_u^\circ(\varrho)$ versus $\psi_u(\varrho)$ for $\varrho \in [0, \varrho_4]$. The marker $\circ$ corresponds to the point $(\psi_u(\varrho_1), \varphi_u^\circ(\varrho_1))$ whilst the marker $\star$ corresponds to the point $(\psi_u(\varrho_3 = \varrho_5), \varphi_u^\circ(\varrho_3 = \varrho_5))$.

hysteresis loop is reported in Figure 9: it is a line. This means that the operator $\mathcal{H}_o$ has a trivial hysteresis loop with respect to (u, x_0) as predicted by Lemma 11.

Lemma 11 says that the scalar semilinear Duhem model cannot represent the hysteresis behavior observed in magnetic hysteresis. Indeed, to produce minor loops that satisfy Assumption 8, the hysteresis loop of the model should be trivial.

This observation leads to the following conjecture.

Conjecture 1 Consider the generalized Duhem model (17)–(19). Assume that the corresponding operators $\mathcal{H}_o$ and $\mathcal{H}_s$ are consistent with respect to all $(u, x_0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R}^n$ and are strongly consistent with respect to all periodic inputs $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ and all initial states $x_0 \in \mathbb{R}^n$. If Assumption 8 holds, then $\forall(u, x_0) \in \Lambda \times \mathbb{R}^n$, the operators $\mathcal{H}_o$ and $\mathcal{H}_s$ have a trivial hysteresis loop with respect to (u, x_0) (see Definition 9).

If true, the conjecture would mean that the Duhem model -in its generalized form- is not able to describe the minor loops in magnetic hysteresis.

However, in several engineering problems, the Duhem model is not used to reproduce the behavior of minor loops in magnetic hysteresis. For example, in control problems, it is not necessary to have an accurate model that describes the controlled process with precision. Instead, an *approximate* model may be appropriate if it captures some essential features of the controlled plant, and at the same time, is simple enough to allow the design of a relatively simple controller (see for example Ref. [36]).

12 Relationships between concepts

In this section we explore the connections that exist between the concepts presented in this paper. We use the case study of the semilinear Duhem model to illustrate these connections and motivate the open problems proposed in Section 13.

12.1 Relationship between Definition 4 and strong consistency

In this section we compare the definitions of hysteresis loop implied by Definition 4 and the concept of strong consistency.

12.1.1 Comments on Definition 4

We have seen in Section 5.2 that Ref. [54] proposes a definition that aims to decide whether a given generalized Duhem model is a hysteresis or not. According to Definition 4 we have to proceed as follows.

(i) Check whether Assumption 1 holds.
(ii) Check whether Assumption 2 holds.
(iii) Check whether Condition (i) of Definition 4 holds.
(iv) Check whether Condition (ii) of Definition 4 holds.

In the process of checking Assumption 2 we do not need to find the explicit expression of the initial condition $x_{0,\gamma}$. Indeed, the concept of Cauchy sequence can be used to prove the existence of $x_{0,\gamma}$ without actually having to find the explicit expression of $x_{0,\gamma}$. This is what has been done in the proof of Theorem 8.

Similarly it is not necessary to get the explicit expression of the closed curve $\mathcal{C}_u$ to check Condition (i) of Definition 4. Again, the concept of Cauchy sequence may be used to prove the convergence of the sets $\mathcal{C}_{u,\gamma}$, although this is not how we proceed in the proof of Theorem 9. However, if we do not have the explicit expression of $\mathcal{C}_{u,\gamma}$ then it may be difficult to prove this convergence.

Knowing the explicit expression of $\mathcal{C}_{u,\gamma}$ is equivalent to knowing the explicit expression of the initial condition $x_{0,\gamma}$. Indeed, for the generalized Duhem model (17) the closed curve $\mathcal{C}_{u,\gamma}$ is characterized by the same differential equation (17) where the input u is replaced by $u \circ s_\gamma$, and the initial condition x_0 is replaced by $x_{0,\gamma}$.

Let us illustrate that statement. To prove that Condition (i) of Definition 4 holds for the scalar semilinear Duhem model we have demonstrated Equality (174). This equality is obtained thanks to the explicit expression (103) of the initial condition $x_{0,\gamma}$. That explicit expression is derived from the explicit solution (99) and (102) of the differential equation (167). We get an explicit solution because the differential equation (167) is linear with respect to the state.

To sum up, the linearity with respect to the state in the differential equation that describes the scalar semilinear Duhem model, is crucial to prove that Condition (i) of Definition 4 holds. For a generalized Duhem model (17) that does not enjoy this linearity property it may not be easy to check analytically whether Condition (i) of Definition 4 holds.

12.1.2 Comments on strong consistency

To check whether a given generalized Duhem model is strongly consistent we have first to check whether it is consistent. The analysis of the consistency of the semilinear Duhem model is provided in Section 11.2, and it uses both the linearity with respect to the state, and the fact that the initial condition in Equation (70) does not change with γ. For the generalized Duhem model (17) that may not be linear with respect to the state, Lemma 6 provides sufficient conditions that provide the expression of the corresponding rate independent Duhem model. However, ensuring these sufficient conditions may not be easy if the model is nonlinear with respect to the state.

Also checking the strong consistency of the semilinear Duhem model in Section 11.3 is made possible because it is not necessary to find the explicit expression of the initial state $x_u^\circ(0)$. Instead, the concept of Cauchy sequence is used in Ref. [35] to prove the desired convergence property. Again, the linearity of the model is used to derive a Lyapunov function which allows mathematical analysis. For the generalized Duhem model, finding a Lyapunov function may not be easy if the model is nonlinear with respect to the state.

12.1.3 Relationship between the hysteresis loop derived from Definition 4 and the one derived from strong consistency

The hysteresis loop derived from Definition 4 is the set $\mathcal{C}_u$ defined as the limit of the sets $\mathcal{C}_{u,\gamma}$ with respect to Hausdorff distance d_2 as $\gamma \to \infty$. The hysteresis loop derived from strong consistency is the set $\mathcal{G}_u$ of Equation (31).

Do we have $\mathcal{C}_u = \mathcal{G}_u$?
For the scalar semilinear Duhem model the answer is positive. Indeed, the set $\mathcal{C}_u$ is given by Equation (106) and the set $\mathcal{G}_u$ is given by Equation (95). It can be checked that, for the scalar semilinear Duhem model, we have $\mathcal{C}_u = \mathcal{G}_u$.

However, for the generalized Duhem model, at the time of the submission of the present paper we *do not know* whether the sets $\mathcal{C}_u$ and $\mathcal{G}_u$ are equal or not. This statement leads to formulating Open problem 1 in Section 13.1.

Note that the authors of Ref. [54] assume tacitly that, for the semilinear Duhem model, we have $\mathcal{C}_u = \mathcal{G}_u$ (see the proof of [54, Proposition 5.1]).

For the Preisach model, defining the concept of a hysteresis loop is simple because the model does not have a transient response under the usual conditions. This means that the hysteresis loop is simply the graph output versus input. For the -possibly- rate-dependent generalized Duhem model, the output contains typically a transient term and a steady-state term. This is why there are two possibilities for defining a hysteresis loop: as the set $\mathcal{C}_u$ or as the set $\mathcal{G}_u$. From the discussion of Sections 12.1.1 and 12.1.2, it is not clear which of these two definitions is easier to check from the point of view of the mathematical analysis.

The following comment sheds more light on the question.

Consider an operator $\mathcal{H}$ that satisfies Assumption 4. From Equation (30) it comes that the operator $\mathcal{H}^\dagger$ is such that $\left(\mathcal{H}^\dagger\right)^\star = 0$. This implies that the hysteresis loop of $\mathcal{H}^\dagger$ with respect to all $(u, x_0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p) \times \mathit{\Xi}$ is trivial (see Definition 9).

From Equations (28)–(29) it follows that the operator $\mathcal{H}$ has been decomposed into the sum of two operators:

(i) An operator $\mathcal{H}^\star$ that is rate independent with respect to linear time-scale changes,
(ii) and an operator $\mathcal{H}^\dagger$ such that the output $\mathcal{H}^\dagger(u \circ s_\gamma, x_0)$ vanishes when $\gamma \to \infty$ (loosely speaking, the output vanishes when the frequency of the input goes to zero).

The decomposition (28)–(29) is compatible with experimental observations of hysteresis processes. Indeed, quoting from [64, p. 14]: *"in several cases the rate independent component prevails, provided that evolution is not too fast."* Additionally, the hysteresis loop of the operator $\mathcal{H}^\dagger$ is trivial (loosely speaking, $\mathcal{H}^\dagger$ does not represent a hysteresis behavior).

For all these reasons, we call Equations (28)–(29) the *canonical decomposition* of the operator $\mathcal{H}$, the operator $\mathcal{H}^\star$ the *rate-independent component* of $\mathcal{H}$, and the operator $\mathcal{H}^\dagger$ the *nonhysteretic component* of $\mathcal{H}$.

This canonical decomposition was possible owing to the use of the concept of consistency.

12.2 Relationship between the Lipschitz property and the effect of perturbations

In this section we analyze the effect of a perturbation of the input and the initial condition on the hysteresis loop.

Consider a causal operator $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p) \times \mathit{\Xi} \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ where $\mathit{\Xi}$ is a Banach space. Suppose that $\mathcal{H}$ satisfies Assumption 3, is consistent with respect to all $(u, x_0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p) \times \mathit{\Xi}$, and is strongly consistent with respect to all periodic inputs $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p)$ and all initial states $x_0 \in \mathit{\Xi}$.

Let the T–periodic input $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p)$ and the initial state $x_0 \in \mathit{\Xi}$ be given. The hysteresis loop of the operator $\mathcal{H}$ with respect to (u, x_0) is the set $\mathcal{G}_u$ defined by Equation (31).

Let $\varepsilon \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p)$ be a function that represents a perturbation of the input, and $\epsilon \in \mathit{\Xi}$ a vector that represents a perturbation of the initial condition. The perturbed input $v = u + \varepsilon \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p)$ may not be periodic which means that $\mathcal{H}$ may not have a hysteresis loop when v is the input. The perturbed initial state is $x_0' = x_0 + \epsilon$. The perturbed output that corresponds to (v, x_0') is $\mathcal{H}(v, x_0')$. To evaluate the effect of (ε, ϵ) on $\mathcal{G}_u$ we need the following assumptions.

Assumption 9 $I_v = \mathbb{R}_+$.

Assumption 10 For any $(w, y_0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p) \times \mathit{\Xi}$ the function $\mathcal{H}(w, y_0)$ is continuous on $\mathbb{R}_+$. That is $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^p) \times \mathit{\Xi} \to L^\infty(\mathbb{R}_+, \mathbb{R}^m) \cap C^0(\mathbb{R}_+, \mathbb{R}^m)$.

Since the operator $\mathcal{H}$ is consistent with respect to (v, x_0') there exists a function $\varphi_v^\star$ as in Definition 5. Combining Assumptions 9, 10 and Lemma 3 it comes that $\varphi_v^\star \in L^\infty(\mathbb{R}_+, \mathbb{R}^m) \cap C^0(\mathbb{R}_+, \mathbb{R}^m)$. For all $k \in \mathbb{N}$ define the function $\varphi_{v,k}^\star \in C^0\big([0, \rho_v(T)], \mathbb{R}^m\big)$ by $\varphi_{v,k}^\star(\varrho) = \varphi_v^\star\big(\rho_v(T)\,k + \varrho\big), \forall \varrho \in [0, \rho_v(T)]$. Define the set

$$P_{v,k} = \left\{ \big(\psi_v(\varrho), \varphi_{v,k}^\star(\varrho)\big), \varrho \in [0, \rho_v(T)] \right\}. \tag{121}$$

Note that $P_{v,k}$ and $\mathcal{G}_u$ are compact owing to Assumption 10. Thus we can define

$$q(u, x_0, \varepsilon, \epsilon) = \limsup_{k\to\infty} d_{p+m}(P_{v,k}, \mathcal{G}_u) \tag{122}$$

where d_{p+m} is the Hausdorff distance defined by Equation (22). The quantity $q(u, x_0, \varepsilon, \epsilon)$ measures the effect of the perturbation (ε, ϵ) on the hysteresis loop $\mathcal{G}_u$.

Our aim now is to apply these concepts to the scalar rate-independent Duhem model (43)–(45) where the output is the state x. To do so we need to change the time variable from t to ϱ. Following the same steps as in Section 11.2 and using the same set of notations, Equation (43) becomes

$$\begin{aligned} v_u(\varrho)\dot{x}_{u\circ s_\gamma}(\varrho) = v_u(\varrho) f_1\big(x_{u\circ s_\gamma}(\varrho), \psi_u(\varrho)\big),\\ \text{for almost all } \varrho \in \mathbb{R}_+. \end{aligned} \tag{123}$$

We can eliminate $v_u(\varrho)$ since, by Lemma 13, the function v_u is nonzero almost everywhere on $\mathbb{R}_+$. Note that Equation (123) is independent of γ so that we use the simplified notation x_u instead of $x_{u\circ s_\gamma}$. Thus, for the input u and the initial state x_0 the scalar rate-independent Duhem model (43)–(45) in terms of t–variable can be written in terms of ϱ–variable as

$$\dot{x}_u(\varrho) = f_1\big(x_u(\varrho), \psi_u(\varrho)\big), \tag{124}$$
$$\text{for almost all } \varrho \in \mathbb{R}_+ \text{ such that } \dot{\psi}_u(\varrho) = 1,$$
$$\dot{x}_u(\varrho) = f_2\big(x_u(\varrho), \psi_u(\varrho)\big), \tag{125}$$
$$\text{for almost all } \varrho \in \mathbb{R}_+ \text{ such that } \dot{\psi}_u(\varrho) = -1,$$
$$x_u(0) = x_0. \tag{126}$$

Observe that $\varphi_v^\star = \varphi_v = x_v$ so that $d_{p+m}(P_{v,k}, \mathcal{G}_u)$ includes terms of the form $|\varphi_{v,k}(\varrho_1) - \varphi_u^\circ(\varrho_2)|$ for $(\varrho_1, \varrho_2) \in [0, \rho_v(T)] \times [0, \rho_u(T)]$ by Equation (22). Note that $\varphi_{v,k}$ obeys Equations (124)–(125) with u substituted by v and with the initial condition $x_v\big(k\rho_v(T)\big)$. Also φ_u° obeys Equations (124)–(125) with the initial condition $x_u^\circ(0)$. It is to be noted that we cannot use Proposition 2 to get a bound on $|\varphi_{v,k}(\varrho_1) - \varphi_u^\circ(\varrho_2)|$ because the initial conditions $x_v\big(k\rho_v(T)\big)$ and $x_u^\circ(0)$ may be different. This means that, in order to evaluate the effect of perturbations on the hysteresis loop of the model (43)–(45), Proposition 2 needs to be enhanced to take into account different initial conditions.

This observation leads to formulating Open Problem 2 in Section 13.2.

We now consider the effect of perturbations on the hysteresis loop of the generalized Duhem model (17). Observe that, from Equation (122) it comes that the quantity $q(u, x_0, \varepsilon, \epsilon)$ depends on $\varphi_v^\star$ and φ_u° which obey Equations (124)–(125) by Lemma 6. This means that there is no need to look for an extension of Proposition 2 to the generalized Duhem model.

12.3 Relationship between dissipativity and orientation of the hysteresis loop

For the scalar rate-independent Duhem model (34)–(36), dissipativity is the property of Definition 10. Dissipativity is studied in Ref. [40] mainly because of its interest in control. In this section, we focus on the relationship between dissipativity and the orientation of the hysteresis loop, as this orientation is easy to obtain experimentally.

At the time of the submission of this paper, we do not know whether a dissipative model (34)–(36) is strongly consistent. This observation leads to the formulation of Open Problem 3 in Section 13.3.

If the model (34)–(36) is dissipative and strongly consistent, then the hysteresis loop is oriented counterclockwise [1].

Theorem 3 provides sufficient conditions to ensure dissipativity. One of these conditions is $f_1 \geq 0$ and $f_2 \geq 0$. For the scalar semilinear rate-independent Duhem model, the conditions $f_1 \geq 0$ and $f_2 \geq 0$ do not hold so that Theorem 3 could not be used directly to study the dissipativity of the model. Instead, an *ad-hoc* analysis combined with Theorem 3 showed that, when the input is small in some sense, the hysteresis loop is counterclockwise (see Lemma 10).

The question of how to generalize Lemma 10 to encompass the model (34)–(36) leads to formulating Open Problem 4 in Section 13.4.

Note that there is no need to generalize Lemma 10 to encompass the generalized Duhem model (17) since the hysteresis loop is characterized by the rate-independent Duhem model (124)–(125).

13 Open problems

13.1 Open Problem 1

The motivation for Open Problem 1 is provided in Section 12.1.3.

Consider that the generalized Duhem model (17)–(19) satisfies Assumption 1 so that we can define the operators $\mathcal{H}_o$ and $\mathcal{H}_s$ of Section 5.1. Suppose that Assumption 2 holds and that Conditions (i) and (ii) of Definition 4 hold for all $(u, x_0) \in \Lambda \times \mathbb{R}^n$.

Furthermore, suppose that the operators $\mathcal{H}_o$ and $\mathcal{H}_s$ are strongly consistent with respect to all $(u, x_0) \in \Lambda \times \mathbb{R}^n$.

(i) Find sufficient conditions that ensure $\mathcal{C}_u = \mathcal{G}_u$ for all $(u, x_0) \in \Lambda \times \mathbb{R}^n$.
(ii) Find a generalized Duhem model such that there exist an input $u \in \Lambda$ and an initial condition $x_0 \in \mathbb{R}^n$ that satisfy $\mathcal{C}_u \neq \mathcal{G}_u$.

13.2 Open Problem 2

The motivation for Open Problem 2 is provided in Section 12.2.

Consider the scalar rate-independent Duhem model (43)–(45) where the output is the state x. Suppose that Assumption 1 holds so that we can define the operator $\mathcal{H}_s$ of Section 5.1. Let $u, v \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ and $x_0, x_0' \in \mathbb{R}$.

(i) Find sufficient conditions that provide an upper bound on $\|\mathcal{H}_s(u, x_0) - \mathcal{H}_s(v, x_0')\|_{W^{1,\infty}([0,\mathcal{T}],\mathbb{R})}$ for some finite real number $\mathcal{T} > 0$. Can we obtain an upper bound that is a continuous function of $\big(\|u - v\|_{W^{1,\infty}([0,\mathcal{T}],\mathbb{R})}, |x_0 - x_0'|\big)$ and that becomes the bound obtained in Proposition 2 when $x_0 = x_0'$?
(ii) Let $T \in\,]0, \infty[$ and assume that u is T–periodic. Find an upper bound on $q(u, x_0, \varepsilon, \epsilon)$ as tight as possible.
(iii) Find sufficient conditions so that if $|x_0 - x_0'| + \|u - v\|_{W^{1,\infty}([0,\mathcal{T}],\mathbb{R})}$ is small then $q(u, x_0, \varepsilon, \epsilon)$ is small.
(iv) Generalize the obtained results to the *vector* rate-independent Duhem model (32)–(33).

13.3 Open Problem 3

The motivation for Open Problem 3 is provided in Section 12.3.

Consider the scalar rate-independent Duhem model (34)–(36) where the output is the state x. Suppose that Assumption 1 holds so that we can define the operator $\mathcal{H}_s$ of Section 5.1. Suppose that we can find a nonnegative function $\varsigma : \mathbb{R}^2 \to \mathbb{R}$ such that $\forall (u, x_0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R}$ Inequality (39) holds.

(i) Can we conclude that $\mathcal{H}_s$ is strongly consistent with respect to all periodic inputs $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ and all initial states $x_0 \in \mathbb{R}$?

13.4 Open Problem 4

The motivation for Open Problem 4 is provided in Section 12.3.

Consider the scalar rate-independent Duhem model (34)–(36) where the output is the state x. Suppose that Assumption 1 holds so that we can define the operator $\mathcal{H}_s$ of Section 5.1. Suppose that all conditions of Theorem 3 hold except $f_1 \geq 0$ and $f_2 \geq 0$.

(i) Find a set S as large as possible of pairs $(u, x_0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R}$ for which (i)–1 and (i)–2 hold.
(i)–1. The operator $\mathcal{H}_s$ is strongly consistent with respect to all $(u, x_0) \in S$.
(i)–2. The curve $\varrho \mapsto \big(\psi_u(\varrho), \varphi_u^\circ(\varrho)\big)$ is counterclockwise for all $(u, x_0) \in S$.
(ii) Generalize the obtained results to the *vector* rate-independent Duhem model (32)–(33).

14 Epilogue

More research is needed to better understand Duhem's model seen as a class of differential equations, and also as a representation of hysteresis. In particular, it is important to get answers to the open problems -and to the conjecture- proposed in this paper.

A On the existence and uniqueness of solutions of differential equations

In this section we present some existence and uniqueness theorems for the solutions of ordinary differential equations. To this end, let $\mathcal{D}$ be a *domain*, that is an open connected subset of $\mathbb{R} \times \mathbb{R}^n$ where $n > 0$ is an integer. Let $(t_0, x_0) \in \mathcal{D}$ and let $a, b \in\,]0, \infty[$. Define the parallelepiped $Q_{a,b}$ by

$$Q_{a,b} = \{(t, w) \in \mathbb{R} \times \mathbb{R}^n \mid |t - t_0| \leq a, |w - x_0| \leq b\}. \quad (127)$$

We say that the map $F : \mathcal{D} \to \mathbb{R}^n$ satisfies the Carathéodory conditions on the domain $\mathcal{D}$ if Conditions (i)–(iii) hold on any parallelepiped $Q_{a,b} \subset \mathcal{D}$ [61, p. 68].
(i) The function F is defined and continuous in w for almost all t;
(ii) the function F is measurable in t for each fixed w;
(iii) for each $Q_{a,b} \subset \mathcal{D}$ there exists a measurable function $m_{Q_{a,b}} \in L^1\big([t_0 - a, t_0 + a], \mathbb{R}\big)$ such that

$$\begin{aligned} &|F(t, w)| \leq m_{Q_{a,b}}(t),\ \forall w \in \mathbb{R}^n \text{ and for almost all} \\ &t \in [t_0 - a, t_0 + a] \text{ satisfying } (t, w) \in Q_{a,b}. \end{aligned} \quad (128)$$

Now, consider the differential equation

$$\dot{x}(t) = F\big(t, x(t)\big), \quad (129)$$
$$x(t_0) = x_0, \quad (130)$$

where $F : \mathcal{D} \to \mathbb{R}^n$ satisfies the Carathéodory conditions on the domain $\mathcal{D} \subset \mathbb{R} \times \mathbb{R}^n$ and $(t_0, x_0) \in \mathcal{D}$.

Theorem 10 *[61, p. 68] The differential equation (129)–(130) has a solution on some nonempty open interval $I \ni t_0$, in the sense that there exists an absolutely continuous function $x : I \to \mathbb{R}^n$ such that the following properties* (i)–(iii) *are satisfied.*
(i) *The initial condition (130) holds;*
(ii) *$\forall t \in I$ we have $\big(t, x(t)\big) \in \mathcal{D}$;*
(iii) *and the differential equation (129) is satisfied almost everywhere in I.*

A lower bound on the size of the interval I is obtained by solving the inequality

$$\int_{t_0 - c}^{t_0 + c} m_{Q_{a,b}}(t)\, \mathrm{d}t \leq b, \quad (131)$$

where $a, b \in\,]0, \infty[$ are chosen so that $(t_0, x_0) \in Q_{a,b} \subset \mathcal{D}$. Observe that the function $c \to \int_{t_0-c}^{t_0+c} m_{Q_{a,b}}(t)\, dt$ is continuous and is zero at $c = 0$. This implies that there exists at least a $0 < c \leq a$ such that (131) holds. Then we have $]t_0 - c, t_0 + c[\, \subset I$ [61, p. 69].

Theorem 11 *[61, p. 70 and p. 80] Assume that $F : \mathcal{D} \to \mathbb{R}^n$ satisfies the Carathéodory conditions on the domain $\mathcal{D}$. Let x be a solution of the differential equation (129)–(130) defined on some interval I. Then x may be extended as a solution of (129)–(130) to a maximal interval of existence $]\omega_-, \omega_+[$ and $\big(t, x(t)\big) \to \partial\mathcal{D}$ as $t \to \omega_\pm$, where $\partial\mathcal{D}$ is the boundary of $\mathcal{D}$.*

Theorem 12 *[29, p. 5] Assume that $F : \mathcal{D} \to \mathbb{R}^n$ satisfies the Carathéodory conditions on the domain $\mathcal{D}$. Assume that there exists a function $l \in L^1\big(J, \mathbb{R}_+\big)$ for every finite interval $J \subset \mathbb{R}$ which satisfies the following. For almost all $t \in \mathbb{R}$ and $\forall w_1, w_2 \in \mathbb{R}^n$ such that $(t, w_1), (t, w_2) \in \mathcal{D}$ we have*

$$\big|F(t, w_1) - F(t, w_2)\big| \leq l(t)|w_1 - w_2|. \tag{132}$$

Then in the domain $\mathcal{D}$ there exists at most one solution to the differential equation (129)–(130).

The local Lipschitz condition (132) can be relaxed as follows [29, p. 5].

$$\big(F(t, w_1) - F(t, w_2)\big) \cdot (w_1 - w_2) \leq l_1(t)|w_1 - w_2|^2, \quad \text{for almost all } t \geq t_0, \tag{133}$$

$$\big(F(t, w_1) - F(t, w_2)\big) \cdot (w_1 - w_2) \geq -l_2(t)|w_1 - w_2|^2, \quad \text{for almost all } t \leq t_0, \tag{134}$$

where the product is understood as the scalar product if $F(t, w_1), F(t, w_2), w_1, w_2$ are vectors; the functions $l_1, l_2 \in L^1\big(J, \mathbb{R}_+\big)$ for every finite interval $J \subset \mathbb{R}$, and $w_1, w_2 \in \mathbb{R}^n$ are such that $(t, w_1), (t, w_2) \in \mathcal{D}$.

Finally we provide a result we could not find in the literature, and which is useful to the present paper.

Lemma 12 *Suppose that the application $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions on the domain $\mathbb{R}^2$. Assume that there exists $k \in [0, \infty[$ such that*

$$\big(F(t, w_1) - F(t, w_2)\big) \cdot (w_1 - w_2) \leq k|w_1 - w_2|^2, \quad \textit{for almost all } t \geq t_0, \forall w_1, w_2 \in \mathbb{R}. \tag{135}$$

Then the differential equation (129)–(130) has exactly one solution defined on $[t_0, \infty[$.

Proof From Theorems 10, 11, and 12 it follows that there exists a unique solution x to the differential equation (129)–(130) defined on a maximal interval of existence $[t_0, \omega_+[$ where $\omega_+ \in]t_0, \infty]$. Assume that $\omega_+ < \infty$, and let $w \in \mathbb{R}$ be fixed. It comes from Theorem 11 that $\exists\, t_w \in]t_0, \omega_+[$ such that $\forall t \in [t_w, \omega_+[$ we have $|x(t)| > |w|$. Consider the case $\forall t \in [t_w, \omega_+[$, $x(t) > |w| \geq w$ (a similar proof holds for the case $\forall t \in [t_w, \omega_+[$, $x(t) < -|w|$). Then Inequality (135) leads to

$$F(t, x(t)) \leq F(t, w) + k\big(x(t) - w\big), \text{ for almost all } t \in [t_w, \omega_+[. \tag{136}$$

Integrating both sides of (136) on the time interval $[t_w, t]$ it follows that

$$\begin{aligned} |x(t)| = x(t) &= x(t_w) + \int_{t_w}^{t} F(s, x(s))\, \mathrm{d}s \\ &\leq C + k \int_{t_w}^{t} |x(s)|\, \mathrm{d}s, \forall t \in [t_w, \omega_+[, \end{aligned} \tag{137}$$

$$C = x(t_w) + \int_{t_w}^{\omega_+} |F(s, w)|\, \mathrm{d}s + k|w|(\omega_+ - t_w) < \infty.$$

Using Gronwall's lemma [32, p. 24] it comes from Inequality (137) that

$$|x(t)| \leq Ce^{t - t_w} \leq Ce^{\omega_+ - t_w}, \forall t \in [t_w, \omega_+[. \tag{138}$$

Inequality (138) contradicts the fact that $|x(t)| \to \infty$ as $t \to \omega_+$.

B Proof of Lemma 13

Lemma 13 *Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ be non constant. There exists a unique function $v_u \in L^\infty(I_u, \mathbb{R})$ that is defined by $v_u \circ \rho_u = \dot{u}$. Moreover, $\|v_u\|_{I_u} \leq \|\dot{u}\|$ and v_u is nonzero almost everywhere on I_u.*

Proof The operator Δ_- defined in Section 11.2 is causal and satisfies Assumption 3. Using Lemma 3 it follows that $v_u \in L^\infty(I_u, \mathbb{R})$ and $\|v_u\|_{I_u} \leq \|\dot{u}\|$. Now, define the following sets:

$$\begin{aligned} A &= \{\varrho \in I_u \mid v_u(\varrho) = 0\}, \\ B &= \{t \in \mathbb{R}_+ \mid \dot{u}(t) = 0\}, \\ B_1 &= \{t \in \mathbb{R}_+ \mid \dot{\rho}_u(t) \text{ is not defined at } t\}, \\ B_2 &= \{t \in \mathbb{R}_+ \mid \dot{u}(t) \text{ is defined, } \dot{\rho}_u(t) \text{ is defined, and} \\ &\qquad |\dot{u}(t)| \neq \dot{\rho}_u(t)\}, \\ C &= \{t \in \mathbb{R}_+ \mid \dot{\rho}_u(t) = 0\}. \end{aligned}$$

Since ρ_u is absolutely continuous on $\mathbb{R}_+$, we get from [45, Corollary 3.41] that $\mu(B_1) = 0$. Since $\dot{u} \in L^\infty(\mathbb{R}_+, \mathbb{R})$ we get from [45, Lemma 3.31] that $\dot{\rho}_u = |\dot{u}|$ almost everywhere on $\mathbb{R}_+$, which implies that $\mu(B_2) = 0$. Also, from [45, Corollary 3.14] it follows that $\mu\big(\rho_u(C)\big) = 0$. Since ρ_u is absolutely continuous on $\mathbb{R}_+$, and since $\mu(B_1) = \mu(B_2) = 0$ it follows from [45, Corollary 3.41] that $\mu\big(\rho_u(B_1)\big) = \mu\big(\rho_u(B_2)\big) = 0$. Now, observe that $B \subset C \cup B_1 \cup B_2$, thus $\rho_u(B) \subset \rho_u(C) \cup \rho_u(B_1) \cup \rho_u(B_2)$ which implies that $\mu\big(\rho_u(B)\big) = 0$. Since $A = \rho_u(B)$ it follows that $\mu(A) = 0$.

C Proof of Theorem 8

We get from Equation (68) that $\exists \delta_1 > 0$ such that $\forall w \in (0, \delta_1)$ we have $|\bar{g}_1(w) - 1| < \frac{1}{2}$, and $\exists \delta_2 > 0$ such that $\forall w \in (-\delta_2, 0)$ we have $|\bar{g}_2(w) + 1| < \frac{1}{2}$. Define

$$\gamma_0 = \frac{\|\dot{u}\|}{\min(\delta_1, \delta_2)}. \tag{139}$$

Observe that $0 < \gamma_0 < \infty$ since $u \in \Lambda_{u_{\min}, u_{\max}, \alpha_1, T}$. Let $\gamma \in]\gamma_0, \infty[$ be fixed, and define $x_\gamma = \mathcal{H}_s(u \circ s_\gamma, x_0)$. From Equations (63) and (64) we get

$$\begin{aligned} x_\gamma(t) = x_0 &+ \int_0^t g_1\big(\dot{u}_\gamma(\tau)\big)\big(A_1 x_\gamma(\tau) + B_1 u_\gamma(\tau) + E_1\big) \\ &+ g_2\big(\dot{u}_\gamma(\tau)\big)\big(A_2 x_\gamma(\tau) + B_2 u_\gamma(\tau) + E_2\big)\, \mathrm{d}\tau, \forall t \in \mathbb{R}_+ \end{aligned} \tag{140}$$

where $u_\gamma = u \circ s_\gamma$. Consider the change of variable $\tau' = \frac{\tau}{\gamma}$, then

$$\begin{aligned} x_\gamma(t) = x_0 &+ \gamma \int_0^{\frac{t}{\gamma}} g_1\left(\frac{\dot{u}(\tau')}{\gamma}\right)\big[A_1 x_\gamma(\gamma\tau') + B_1 u(\tau') + E_1\big] \\ &+ g_2\left(\frac{\dot{u}(\tau')}{\gamma}\right) \times \big[A_2 x_\gamma(\gamma\tau') + B_2 u(\tau') + E_2\big]\, \mathrm{d}\tau', \\ &\forall t \in \mathbb{R}_+. \end{aligned}$$

(141)

Define $\sigma = \frac{t}{\gamma}$ and $z : \mathbb{R}_+ \to \mathbb{R}$ by $z(\sigma) = x_\gamma(\gamma\sigma), \forall \sigma \in \mathbb{R}_+$; then

$$\begin{aligned} z(\sigma) = x_0 + \gamma \int_0^\sigma g_1\left(\frac{\dot{u}(\tau')}{\gamma}\right)\left(A_1 z(\tau') + B_1 u(\tau') + E_1\right) \\ + g_2\left(\frac{\dot{u}(\tau')}{\gamma}\right)\left(A_2 z(\tau') + B_2 u(\tau') + E_2\right) \mathrm{d}\tau', \\ \forall \sigma \in \mathbb{R}_+. \end{aligned} \tag{142}$$

For any $m \in \mathbb{N}$ define $z_m : [0,T] \to \mathbb{R}$ by

$$z_m(\sigma) = z(\sigma + mT), \forall \sigma \in [0,T]. \tag{143}$$

The objective of the following analysis is to show that the sequence $\{z_m\}_{m\in\mathbb{N}}$ converges in the Banach space $C^0([0,T],\mathbb{R})$ endowed with the norm $\|\cdot\|_{[0,T]}$. To this end, we prove that $\{z_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence. For any $m_1, m_2 \in \mathbb{N}$ define

$$z_{m_1,m_2} = z_{m_1} - z_{m_2}. \tag{144}$$

Owing to the T–periodicity of both u and $\dot{u}$ it follows from Equations (142)–(144) that

$$\begin{aligned} \dot{z}_{m_1,m_2}(\sigma) = \gamma\left(A_1 g_1\left(\frac{\dot{u}(\sigma)}{\gamma}\right) + A_2 g_2\left(\frac{\dot{u}(\sigma)}{\gamma}\right)\right) z_{m_1,m_2}(\sigma), \\ \forall \sigma \in]0,\alpha_1[\, \cup\,]\alpha_1, T[. \end{aligned} \tag{145}$$

Let $\sigma \in (0,\alpha_1)$ then $\dot{u}(\sigma) \geq 0$ since $u \in \Lambda_{u_{\min},u_{\max},\alpha_1,T}$. We study two cases: $\dot{u}(\sigma) > 0$ and $\dot{u}(\sigma) = 0$.

Case $\dot{u}(\sigma) > 0$**.** Since $0 < \frac{\dot{u}(\sigma)}{\gamma} < \frac{\|\dot{u}\|}{\gamma_0} \leq \delta_1$ it follows that $\left|\bar{g}_1\left(\frac{\dot{u}(\sigma)}{\gamma}\right) - 1\right| < \frac{1}{2}$ which, using Equation (66), leads to

$$\frac{3A_1}{2}\dot{u}(\sigma) \leq \gamma A_1 g_1\left(\frac{\dot{u}(\sigma)}{\gamma}\right) \leq \frac{A_1}{2}\dot{u}(\sigma). \tag{146}$$

Case $\dot{u}(\sigma) = 0$**.** In this case, Inequality (146) holds by definition of the function g_1. That is we have

$$\frac{3A_1}{2}\dot{u}(\sigma) \leq \gamma A_1 g_1\left(\frac{\dot{u}(\sigma)}{\gamma}\right) \leq \frac{A_1}{2}\dot{u}(\sigma), \forall \sigma \in]0,\alpha_1[. \tag{147}$$

Similarly, it can be shown that

$$\frac{3A_2}{2}\dot{u}(\sigma) \leq \gamma A_2 g_2\left(\frac{\dot{u}(\sigma)}{\gamma}\right) \leq \frac{A_2}{2}\dot{u}(\sigma), \forall \sigma \in]\alpha_1,T[. \tag{148}$$

Now, define the function $V : [0,T] \to \mathbb{R}$ by

$$V(\sigma) = \frac{1}{2} z_{m_1,m_2}^2(\sigma), \forall \sigma \in [0,T]. \tag{149}$$

Then, V is continuous on $[0,T]$ and is C^1 on $]0,\alpha_1[\,\cup\,]\alpha_1,T[$. From Equation (145) we obtain

$$\begin{aligned} \dot{V}(\sigma) = 2\gamma\left(A_1 g_1\left(\frac{\dot{u}(\sigma)}{\gamma}\right) + A_2 g_2\left(\frac{\dot{u}(\sigma)}{\gamma}\right)\right) V(\sigma), \\ \forall \sigma \in]0,\alpha_1[\,\cup\,]\alpha_1,T[. \end{aligned} \tag{150}$$

Combining Equations (150), (147) and (148) it follows that

$$\dot{V}(\sigma) \leq A_1 \dot{u}(\sigma) V(\sigma), \forall \sigma \in]0,\alpha_1[, \tag{151}$$

$$\dot{V}(\sigma) \leq A_2 \dot{u}(\sigma) V(\sigma), \forall \sigma \in]\alpha_1,T[. \tag{152}$$

Define the continuous function $W : [0,\alpha_1] \to \mathbb{R}$ as being the solution of the following differential equation

$$\dot{W}(\sigma) = A_1 \dot{u}(\sigma) W(\sigma), \forall \sigma \in]0,\alpha_1[, \tag{153}$$

$$W(0) = V(0). \tag{154}$$

Integrating (153)–(154) gives

$$W(\sigma) = V(0)\exp\left(\frac{A_1}{\gamma}\left(u(\sigma) - u_{\min}\right)\right), \forall \sigma \in [0,\alpha_1]. \tag{155}$$

Using the Comparison Lemma [42, p. 102] it comes from Equations (151), (153), (154), and (155) that

$$V(\alpha_1) \leq W(\alpha_1) = V(0)\exp\left(A_1\left(u_{\max} - u_{\min}\right)\right). \tag{156}$$

Using a similar argument on the interval $[\alpha_1, T]$ it follows that

$$V(T) \leq W(\alpha_1)\exp\left(A_2\left(u_{\min} - u_{\max}\right)\right). \tag{157}$$

As a conclusion, we have proved that

$$V(T) \leq rV(0), \tag{158}$$

$$0 < r = \exp\left((A_1 - A_2)(u_{\max} - u_{\min})\right) < 1, \tag{159}$$

$$\|V\|_{[0,T]} \leq V(0). \tag{160}$$

Note that (160) is due to the inequality $\dot{V}(\sigma) \leq 0$, $\forall \sigma \in]0,\alpha_1[\,\cup\,]\alpha_1,T[$ because of Inequalities (151)–(152).

Combining Equations (158), (149), (144), and (143) we get

$$\begin{aligned} \Big[z\big((m_1+1)T\big) - z\big((m_2+1)T\big)\Big]^2 \leq r\Big[z\big(m_1 T\big) - z\big(m_2 T\big)\Big]^2, \\ \forall m_1, m_2 \in \mathbb{N}. \end{aligned} \tag{161}$$

An argument by induction shows that from (161) we get

$$\begin{aligned} V(0) &= \frac{1}{2}\Big[z\big(m_1 T\big) - z\big(m_2 T\big)\Big]^2 \\ &\leq \frac{1}{2} r^{\min(m_1,m_2)}\Big[z(0) - z\big(|m_2 - m_1|T\big)\Big]^2 \\ &\leq 2r^{\min(m_1,m_2)}\|z\|^2, \forall m_1, m_2 \in \mathbb{N}. \end{aligned} \tag{162}$$

Observe that, owing to Theorem 5, we have $\|z\| < \infty$. Hence, from Equations (162), (160), (149), (144), and (159) it comes that $\{z_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence. Therefore there exists $z_\infty \in C^0([0,T],\mathbb{R})$ such that $\lim_{m\to\infty}\|z_m - z_\infty\|_{[0,T]} = 0$. Thus we get $\lim_{m\to\infty}|z_m(0) - z_\infty(0)| = 0$ and $\lim_{m\to\infty}|z_m(T) - z_\infty(T)| = 0$. Note that $z_m(0) = z(mT)$ and $z_m(T) = z\big((m+1)T\big)$ by (143). Take $m_1 = m$ and $m_2 = m+1$ in Inequality (162). Then we get $\lim_{m\to\infty}\left|z(mT) - z\big((m+1)T\big)\right| = 0$. All these facts show that we have

$$z_\infty(0) = z_\infty(T). \tag{163}$$

Combining Equations (142) and (143) it comes that

$$\begin{aligned} z_m(\sigma) &= z_m(0) + \gamma\int_0^\sigma g_1\left(\frac{\dot{u}(\tau)}{\gamma}\right)\left[A_1 z_m(\tau) + B_1 u(\tau) + E_1\right] \\ &\quad + g_2\left(\frac{\dot{u}(\tau)}{\gamma}\right)\left(A_2 z_m(\tau) + B_2 u(\tau) + E_2\right)\mathrm{d}\tau, \\ &\quad \forall \sigma \in [0,T], \forall m \in \mathbb{N}. \end{aligned} \tag{164}$$

Note that $\|z_m\| \leq \|z\| < \infty$. Also, $\left|\frac{\dot{u}(\tau)}{\gamma}\right| \leq \frac{\|\dot{u}\|}{\gamma_0}$ so that, by the continuity of the functions g_1 and g_2 we have $\left|g_1\left(\frac{\dot{u}(\tau)}{\gamma}\right)\right| \leq$

k_1 and $\left|g_2\left(\frac{\dot u(\tau)}{\gamma}\right)\right| \le k_2$, where $k_1, k_2 \in \mathbb{R}_+$ are independent of τ and m. This means that the term under the integral in Equation (164) is bounded by a constant independent of τ and m. Using the Lebesgue Dominated Convergence Theorem it follows from (164) that

$$\begin{aligned} z_\infty(\sigma) = z_\infty(0) + \gamma\int_0^\sigma g_1\left(\frac{\dot u(\tau)}{\gamma}\right)\left[A_1 z_\infty(\tau) + B_1 u(\tau) + E_1\right] \\ + g_2\left(\frac{\dot u(\tau)}{\gamma}\right)\left[A_2 z_\infty(\tau) + B_2 u(\tau) + E_2\right]\mathrm{d}\tau, \\ \forall\sigma \in [0, T]. \end{aligned} \tag{165}$$

Define $\bar z_\gamma : \mathbb{R}_+ \to \mathbb{R}$ by

$$\bar z_\gamma(\sigma + mT) = z_\infty(\sigma), \forall\sigma \in [0, T], \forall m \in \mathbb{N}. \tag{166}$$

Then it comes from Equations (166), (165) and (163) that $\bar z_\gamma$ is T–periodic and

$$\begin{aligned} \bar z_\gamma(\sigma) = \bar z_\gamma(0) + \gamma\int_0^\sigma g_1\left(\frac{\dot u(\tau)}{\gamma}\right)\left[A_1 \bar z_\gamma(\tau) + B_1 u(\tau) + E_1\right] \\ + g_2\left(\frac{\dot u(\tau)}{\gamma}\right)\left[A_2 \bar z_\gamma(\tau) + B_2 u(\tau) + E_2\right]\mathrm{d}\tau, \\ \forall\sigma \in \mathbb{R}_+. \end{aligned} \tag{167}$$

As a conclusion, we have proved that there exists

$$x_{0,\gamma} = \bar z_\gamma(0) \tag{168}$$

such that

$$\mathcal{H}_s(u \circ s_\gamma, x_{0,\gamma}) = \bar z_\gamma \circ s_\gamma \tag{169}$$

is $T\gamma$–periodic.

To prove the uniqueness of $x_{0,\gamma}$ we use an argument similar to the one used for the proof of the existence. Take $\gamma > \gamma_0$ and suppose that there exists $x'_{0,\gamma}$ such that $\mathcal{H}_s(u \circ s_\gamma, x'_{0,\gamma})$ is $T\gamma$–periodic. Define $\bar z'_\gamma : \mathbb{R}_+ \to \mathbb{R}$ by $\bar z'_\gamma = \mathcal{H}_s(u \circ s_\gamma, x'_{0,\gamma}) \circ s_{\frac{1}{\gamma}}$. Then, $\bar z'_\gamma(0) = x'_{0,\gamma}$ and $\bar z'_\gamma$ satisfies Equation (167) with $\bar z_\gamma$ replaced by $\bar z'_\gamma$. Considering the difference $\varepsilon = \bar z_\gamma - \bar z'_\gamma$ it follows that ε satisfies Equation (145) with z_{m_1,m_2} replaced by ε. A function V can be defined as in Equation (149) with z_{m_1,m_2} replaced by ε which leads to Inequality (158). Since $V(0) = V(T)$ owing to the T–periodicity of V, it follows that $V(0) = 0$ as V is nonnegative. Thus $x'_{0,\gamma} = x_{0,\gamma}$.

D Proof of Theorem 9

Let $\gamma \in]\gamma_0, \infty[$ where γ_0 is given by Equation (139). From Equation (147) it follows that

$$\frac{3A_1}{2}(u_{\max} - u_{\min}) \le \int_0^\tau \frac{3A_1}{2}\dot u(t)\mathrm{d}t \le \int_0^\tau \gamma A_1 g_1\left(\frac{\dot u(t)}{\gamma}\right)\mathrm{d}t, \tag{170}$$

and

$$\left|\gamma A_1 g_1\left(\frac{\dot u(\tau)}{\gamma}\right)\right| \le \frac{3|A_1|}{2}\|\dot u\|, \forall\tau \in]0, \alpha_1[. \tag{171}$$

Also, From Equation (148) it follows that

$$\frac{3A_2}{2}(u_{\min} - u_{\max}) \le \int_{\alpha_1}^\tau \frac{3A_2}{2}\dot u(t)\mathrm{d}t \le \int_{\alpha_1}^\tau \gamma A_2 g_2\left(\frac{\dot u(t)}{\gamma}\right)\mathrm{d}t, \tag{172}$$

and

$$\left|\gamma A_2 g_2\left(\frac{\dot u(\tau)}{\gamma}\right)\right| \le \frac{3A_2}{2}\|\dot u\|, \forall\tau \in]\alpha_1, T[. \tag{173}$$

Equations (170)–(173) show that we can apply the Lebesgue Dominated Convergence Theorem in (103) so that we get

$$\lim_{\gamma\to\infty} \bar z_\gamma(0) = \bar z(0) = \theta. \tag{174}$$

Observe that using the same theorem we can show that $\forall\sigma \in [0, T]$ we have $\lim_{\gamma\to\infty} |\bar z_\gamma(\sigma) - \bar z(\sigma)| = 0$. However, this simple convergence does not imply Theorem 9; we need to prove the *uniform* convergence of $\bar z_\gamma$ to $\bar z$ on the interval $[0, T]$. This is the aim of the following analysis.

Inequalities (170)–(173) along with Equations (99), (101) and (102) lead to

$$\|\bar z_\gamma\|_{[0,T]} \le c_1, \forall\gamma \in]\gamma_0, \infty[\tag{175}$$

where $c_1 \in \mathbb{R}_+$ is independent of γ.

On the other hand, it can be checked that Equations (93), (94), (90), (104), (105) lead to

$$\dot{\bar z}(\sigma) = \dot u(\sigma)\left(A_1 \bar z(\sigma) + B_1 u(\sigma) + E_1\right), \ \forall\sigma \in]0, \alpha_1[, \tag{176}$$

$$\dot{\bar z}(\sigma) = \dot u(\sigma)\left(A_2 \bar z(\sigma) + B_2 u(\sigma) + E_2\right), \ \forall\sigma \in]\alpha_1, T[. \tag{177}$$

Define the function $V : [0, T] \to \mathbb{R}$ by the relation

$$V_\gamma(\sigma) = \frac{1}{2}\left(\bar z(\sigma) - \bar z_\gamma(\sigma)\right)^2, \forall\sigma \in [0, T]. \tag{178}$$

Take $\sigma \in]0, \alpha_1[$, then it comes from Equations (167) and (176) that

$$\begin{aligned} \dot V_\gamma(\sigma) &= \left(\bar z(\sigma) - \bar z_\gamma(\sigma)\right)\Big[\dot u(\sigma)\left(A_1 \bar z(\sigma) + B_1 u(\sigma) + E_1\right) \\ &\quad - \gamma g_1\left(\frac{\dot u(\sigma)}{\gamma}\right)\left(A_1 \bar z_\gamma(\sigma) + B_1 u(\sigma) + E_1\right)\Big] \\ &= \left(\bar z(\sigma) - \bar z_\gamma(\sigma)\right)\left(B_1 u(\sigma) + E_1\right)\left(\dot u(\sigma) - \gamma g_1\left(\frac{\dot u(\sigma)}{\gamma}\right)\right) \\ &\quad + A_1\left(\bar z(\sigma) - \bar z_\gamma(\sigma)\right)\left(\dot u(\sigma) - \gamma g_1\left(\frac{\dot u(\sigma)}{\gamma}\right)\right)\bar z(\sigma) \\ &\quad + 2A_1\gamma g_1\left(\frac{\dot u(\sigma)}{\gamma}\right)V_\gamma(\sigma), \forall\sigma \in]0, \alpha_1[, \forall\gamma > \gamma_0. \end{aligned} \tag{179}$$

Let $\varepsilon > 0$. From Equations (66) and (68) it follows that $\exists\delta_\varepsilon > 0$ such that $\forall w \in]0, \delta_\varepsilon[$ we have $|\bar g_1(w) - 1| < \frac{\varepsilon}{\|\dot u\|}$. Thus, $\exists\gamma_\varepsilon = \min\left(\gamma_0, \frac{\|\dot u\|}{\delta_\varepsilon}\right)$ such that $\forall\gamma > \gamma_\varepsilon$ we have

$$\left|\gamma g_1\left(\frac{\dot u(\sigma)}{\gamma}\right) - \dot u(\sigma)\right| \le \varepsilon, \forall\sigma \in]0, \alpha_1[. \tag{180}$$

Combining Equations (178)–(180) along with Inequalities (175) and (147) it comes that

$$\dot V_\gamma(\sigma) \le A_1 \dot u(\sigma) V_\gamma(\sigma) + c_2\varepsilon\sqrt{V_\gamma(\sigma)}, \forall\sigma \in]0, \alpha_1[, \forall\gamma > \gamma_\varepsilon. \tag{181}$$

where $c_2 \in \mathbb{R}_+$ is independent of γ. Define the continuous function $W : [0, \alpha_1] \to \mathbb{R}_+$ as the solution of the following differential equation

$$\dot W(\sigma) = A_1 \dot u(\sigma) W(\sigma) + c_2\varepsilon\sqrt{W(\sigma)}, \forall\sigma \in]0, \alpha_1[, \tag{182}$$

$$W(0) = V(0). \tag{183}$$

Integrating (182)–(183) gives

$$W(\sigma) = e^{A_1 u(\sigma)}\Big(\sqrt{V(0)}e^{-\frac{A_1}{2}u_{\min}} + \frac{c_2}{2}\varepsilon\int_0^\sigma e^{-\frac{A_1}{2}u(\tau)}\mathrm{d}\tau\Big)^2,$$
$$\forall\sigma\in[0,\alpha_1],$$
$$\le\ e^{A_1 u_{\min}}\Big(\sqrt{V(0)}e^{-\frac{A_1}{2}u_{\min}} + \frac{c_2}{2}\varepsilon\int_0^{\alpha_1} e^{-\frac{A_1}{2}u(\tau)}\mathrm{d}\tau\Big)^2,$$
$$\forall\sigma\in[0,\alpha_1]. \tag{184}$$

Using the Comparison Lemma [42, p. 102] it follows from (181)–(184) that

$$V_\gamma(\sigma) \le e^{A_1 u_{\min}}\Big(\sqrt{V(0)}e^{-\frac{A_1}{2}u_{\min}} + \frac{c_2}{2}\varepsilon\int_0^{\alpha_1} e^{-\frac{A_1}{2}u(\tau)}\mathrm{d}\tau\Big)^2,$$
$$\forall\sigma\in[0,\alpha_1], \forall\gamma>\gamma_\varepsilon. \tag{185}$$

Equations (185), (174) and (178) show that $\lim_{\gamma\to\infty}\|V_\gamma\|_{[0,\alpha_1]} = 0$. A similar argument on the interval $[\alpha_1, T]$ shows that $\lim_{\gamma\to\infty}\|V_\gamma\|_{[0,T]} = 0$. The uniform convergence of $\bar{z}_\gamma$ (restricted to the interval $[0,T]$) to $\bar{z}$ has thus been demonstrated, which completes the proof.

E Proof of Lemma 8

(i) $\Rightarrow$ (ii). From Equation (80) and $C \neq 0$ it comes that $\forall\varrho_1,\varrho_2\in[0,\rho_u(T)]$ we have $\varphi_u^\circ(\varrho_1) = \varphi_u^\circ(\varrho_2) \Leftrightarrow x_u^\circ(\varrho_1) = x_u^\circ(\varrho_2)$. Condition (i) implies that $\forall\nu\in[u_{\min},u_{\max}]$ we have $\xi_1(\nu) = \xi_2(\nu)$. Therefore $\forall\nu\in\,]u_{\min},u_{\max}[$ we have $\dot{\xi}_1(\nu) = \dot{\xi}_2(\nu)$. Thus we get from (91)–(92) that

$$\xi_1(\nu) = \xi_2(\nu) = \frac{B_1-B_2}{A_2-A_1}\nu + \frac{E_1-E_2}{A_2-A_1}, \forall\nu\in\,]u_{\min},u_{\max}[. \tag{186}$$

Consider the functions $f_1, f_2, f_3, \mathbf{0} :]u_{\min},u_{\max}[\ \to\mathbb{R}$ defined by $\forall\nu\in\,]u_{\min},u_{\max}[, f_1(\nu) = 1$, $f_2(\nu) = \nu$, $f_3(\nu) = e^{A_1(\nu-u_{\min})}$, and $\mathbf{0}(\nu) = 0$. Then Equation (186) along with (93)–(94) lead to

$$\begin{aligned}&\left(\frac{E_1-E_2}{A_2-A_1} + \frac{E_1}{A_1} + \frac{B_1}{A_1^2}\right)f_1 + \left(\frac{B_1-B_2}{A_2-A_1} + \frac{B_1}{A_1}\right)f_2\\ &-\left(\frac{B_1}{A_1}u_{\min} + \frac{E_1}{A_1} + \frac{B_1}{A_1^2} + \theta\right)f_3 = \mathbf{0},\end{aligned} \tag{187}$$

$$\begin{aligned}&\left(\frac{E_1-E_2}{A_2-A_1} + \frac{E_2}{A_2} + \frac{B_2}{A_2^2}\right)f_1 + \left(\frac{B_1-B_2}{A_2-A_1} + \frac{B_2}{A_2}\right)f_2\\ &-\left(\frac{B_2}{A_2}u_{\min} + \frac{E_2}{A_2} + \frac{B_2}{A_2^2} + \theta\right)f_3 = \mathbf{0}.\end{aligned} \tag{188}$$

Consider the vector space of functions $\{p :]u_{\min},u_{\max}[\ \to\mathbb{R}\}$ with its usual binary operations of vector addition and scalar multiplication. Then the functions f_1, f_2, f_3 are linearly independent vectors so that, owing to Equations (187)–(188), we must have

$$\frac{E_1-E_2}{A_2-A_1} + \frac{E_1}{A_1} + \frac{B_1}{A_1^2} = 0, \tag{189}$$
$$\frac{B_1-B_2}{A_2-A_1} + \frac{B_1}{A_1} = 0, \tag{190}$$
$$\frac{B_1}{A_1}u_{\min} + \frac{E_1}{A_1} + \frac{B_1}{A_1^2} + \theta = 0, \tag{191}$$
$$\frac{E_1-E_2}{A_2-A_1} + \frac{E_2}{A_2} + \frac{B_2}{A_2^2} = 0, \tag{192}$$
$$\frac{B_1-B_2}{A_2-A_1} + \frac{B_2}{A_2} = 0, \tag{193}$$
$$\frac{B_2}{A_2}u_{\min} + \frac{E_2}{A_2} + \frac{B_2}{A_2^2} + \theta = 0. \tag{194}$$

Simple calculations show that Equations (189)–(194) lead to (96)–(97).

(ii) $\Rightarrow$ (i). It can be checked that Equations (96)–(97) lead to (189)–(194) so that the opertor $\mathcal{H}_o$ has a trivial hysteresis loop with respect to all $(u, x_0) \in \Lambda_{u_{\min},u_{\max},\alpha_1,T}\times\mathbb{R}$.

F Proof of Lemma 10

Using Equation (40) the functions $F_1, F_2 : \mathbb{R}^2\to\mathbb{R}$ are given by

$$F_1(x_1, v) = \frac{A_1-A_2}{2}x_1 + \frac{B_1-B_2}{2}v + \frac{E_1-E_2}{2}, \tag{195}$$
$$F_2(x_1, v) = \frac{A_1+A_2}{2}x_1 + \frac{B_1+B_2}{2}v + \frac{E_1+E_2}{2}. \tag{196}$$

Then Assumption 7 holds since $A_1 \neq A_2$. The anhysteresis function is

$$f_{\mathrm{an}}(v) = -\frac{B_1}{A_1}v + \frac{E_2-E_1}{A_1-A_2}, \forall v\in\mathbb{R} \tag{197}$$

where (114) has been used. For every pair $(x_0, u_0)\in\mathbb{R}^2$, let $\omega_{\Phi,1}(\cdot, x_0, u_0) : [u_0,\infty)\to\mathbb{R}$ be the solution z of $z(\sigma) - x_0 = \int_{u_0}^\sigma A_1 z(\tau) + B_1\tau + E_1\,\mathrm{d}\tau$, for all $\sigma\in[u_0,\infty[$ and let $\omega_{\Phi,2}(\cdot, x_0, u_0) :]-\infty, u_0]\to\mathbb{R}$ be the solution z of $z(\sigma)-x_0 = \int_{u_0}^\sigma A_2 z(\tau) + B_2\tau + E_2\,\mathrm{d}\tau$, for all $\sigma\in\,]-\infty, u_0]$. Then

$$\begin{aligned}\omega_{\Phi,1}(\sigma, x_0, u_0) = &\frac{A_1B_1u_0 + A_1E_1 + B_1}{A_1^2}e^{(\sigma-u_0)A_1}\\ &-\frac{A_1B_1\sigma + A_1E_1 + B_1}{A_1^2}\\ &+ e^{(\sigma-u_0)A_1}x_0, \forall\sigma\in[u_0,\infty[,\end{aligned} \tag{198}$$

$$\begin{aligned}\omega_{\Phi,2}(\sigma, x_0, u_0) = &\frac{A_2B_2u_0 + A_2E_2 + B_2}{A_2^2}e^{(\sigma-u_0)A_2}\\ &-\frac{A_2B_2\sigma + A_2E_2 + B_2}{A_2^2}\\ &+ e^{(\sigma-u_0)A_2}x_0, \forall\sigma\in\,]-\infty, u_0].\end{aligned} \tag{199}$$

Equations (198)–(199) are valid since $A_1 \neq 0$ and $A_2 \neq 0$. Define the function $\omega_\Phi(\cdot, x_0, u_0)$ by Equation (41). Then, the intersecting function Ω should satisfy

$$\omega_\Phi\big(\Omega(x_1, v), x_1, v\big) = f_{\mathrm{an}}\big(\Omega(x_1, v)\big), \forall(x_1, v)\in\mathbb{R}^2. \tag{200}$$

Define

$$M_1 = \left(B_1 A_1^{-1}\left(A_2^{-1} - A_1^{-1}\right) - E_1 A_1^{-1} + E_2 A_2^{-1}\right)\frac{A_2}{A_1 - A_2}, \tag{201}$$

$$M_2 = \left(B_1 A_1^{-1}\left(A_2^{-1} - A_1^{-1}\right) - E_1 A_1^{-1} + E_2 A_2^{-1}\right)\frac{A_1}{A_1 - A_2}. \tag{202}$$

Note that $M_1 > 0$ and $M_2 < 0$ owing to (114)–(116). Combining (197)–(200) and (114)–(116) it follows from the definition of function Ω (in Section 8.3) that

$$\Omega(x_1, v) = \begin{cases} v - \frac{1}{A_1}\log\left(1 + \frac{x_1 - f_{\mathrm{an}}(v)}{M_1}\right) & \text{if} \quad x_1 \geq f_{\mathrm{an}}(v), \\ v - \frac{1}{A_2}\log\left(1 - \frac{f_{\mathrm{an}}(v) - x_1}{M_2}\right) & \text{if} \quad x_1 \leq f_{\mathrm{an}}(v), \end{cases} \tag{203}$$

where log sets for the natural logarithm. The function ς in Equation (42) can be determined explicitly as

$$\varsigma(x_1, v) = x_1 v - \frac{1}{A_1}x_1 + \frac{E_1}{A_1}v + \frac{B_1}{2A_1}v^2 - M_1\Omega(x_1, v) + \frac{E_2 - E_1}{A_1(A_1 - A_2)} \quad \text{if} \quad x_1 \geq f_{\mathrm{an}}(v), \tag{204}$$

$$\varsigma(x_1, v) = x_1 v - \frac{1}{A_2}x_1 + \frac{E_2}{A_2}v + \frac{B_1}{2A_1}v^2 - M_2\Omega(x_1, v) + \frac{E_2 - E_1}{A_2(A_1 - A_2)} \quad \text{if} \quad x_1 \leq f_{\mathrm{an}}(v). \tag{205}$$

It can be checked that

$$\varsigma(x_1, v) = -\frac{B_1}{2A_1}v^2 \quad \text{if} \quad x_1 = f_{\mathrm{an}}(v). \tag{206}$$

The fact that Inequality (39) holds for any input $u \in AC(\mathbb{R}_+, \mathbb{R})$ and any initial condition $x_0 \in \mathbb{R}$ follows from Theorem 3. However, ς is not nonnegative: it can be checked that for any fixed x_1 we have $\lim_{v \to \pm\infty} \varsigma(x_1, v) = -\infty$.

The aim of the following analysis is to show that $\forall (x_1, v) \in \mathbb{R} \times \left[\frac{1}{A_1}, \frac{1}{A_2}\right]$ we have $\varsigma(x_1, v) \geq 0$. To this end, observe that, from (114) and (206), we have

$$\varsigma(x_1, v) \geq 0 \quad \text{whenever} \quad x_1 = f_{\mathrm{an}}(v). \tag{207}$$

Now, fix $v \in \left[\frac{1}{A_1}, \frac{1}{A_2}\right]$. From (203)–(204) and (114)–(116) it follows that

$$\lim_{x_1 \to \infty} \varsigma(x_1, v) = \infty. \tag{208}$$

Suppose that there exists $x_2 \in]f_{\mathrm{an}}(v), \infty[$ such that $\varsigma(x_2, v) < 0$. Then, from (207)–(208) it follows that $\varsigma(\cdot, v)$ should have a minimum at $x_3 \in]f_{\mathrm{an}}(v), \infty[$ such that $\varsigma(x_3, v) < 0$. A necessary condition for this to happen is $\frac{\partial \varsigma}{\partial x_1}(x_3, v) = 0$. It can be checked from Equation (204) that this last equality cannot hold. A similar argument can be used for Equation (205).

G Proof of Lemma 11

Observe that, for Theorem 5 to hold, it is needed that A_1 and $-A_2$ are both stable. Since $n = 1$, this condition translates into $A_1 < 0$ and $A_2 > 0$ so that the results of Theorems 5, 6, and 7 apply.

The proof is done in two steps. In Step 1 we consider a specific T–periodic input $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ and an arbitrary initial condition x_0. Using Theorem 7 it follows that the function φ_u° that characterizes the hysteresis loop satisfies the differential state equation (79) and the output equation (80). The aim of Step 1 is to find the initial state $x_u^\circ(0)$ since the latter may be different from x_0. In Step 2 we use the knowledge of $x_u^\circ(0)$ to prove that, if Assumption 8 holds, then the relations (251)–(252) hold.

STEP 1. Let $\alpha \in]0, 1[$; define $\varrho_1 = 1$, $\varrho_2 = 2 - \alpha$, $\varrho_3 = 3 - 2\alpha$, $\varrho_4 = 4 - 2\alpha$. Note that $0 < \varrho_1 < \varrho_2 < \varrho_3 < \varrho_4$. We consider the ϱ_4-periodic input $u : \mathbb{R}_+ \to \mathbb{R}$ defined on the interval $[0, \varrho_4]$ by

$$u(\varrho) = \varrho, \forall \varrho \in [0, \varrho_1], \tag{209}$$

$$u(\varrho) = 2 - \varrho, \forall \varrho \in [\varrho_1, \varrho_2], \tag{210}$$

$$u(\varrho) = 2\alpha - 2 + \varrho, \forall \varrho \in [\varrho_2, \varrho_3], \tag{211}$$

$$u(\varrho) = 4 - 2\alpha - \varrho, \forall \varrho \in [\varrho_3, \varrho_4]. \tag{212}$$

Observe that $u(0) = 0$, $u(\varrho_1) = 1$, $u(\varrho_2) = \alpha$, $u(\varrho_3) = 1$, $u(\varrho_4) = 0$, and that $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$. Observe also that $|\dot{u}(\varrho)| = 1$ for almost all $\varrho \in \mathbb{R}_+$ so that ρ_u is the identity function which gives $\psi_u = u$. Let $x_0 \in \mathbb{R}$ and consider the scalar semilinear Duhem model with input u and initial condition x_0 (Equations (63)–(65)). Since all conditions of Theorem 7 hold, we get from Equality (80) that

$$\varphi_u^\circ(\varrho) = C x_u^\circ(\varrho) + D u(\varrho), \forall \varrho \in [0, \varrho_4], \tag{213}$$

where x_u° satisfies the differential equation (79). To find the initial condition $x_u^\circ(0)$ we compute $x_u^\circ(\varrho_k), k = 1, \ldots, 4$ as a function of $x_u^\circ(0)$ and we use the fact that, by Theorem 7, we have $x_u^\circ(0) = x_u^\circ(\varrho_4)$. We start by computing $x_u^\circ(\varrho_1)$ as a function of $x_u^\circ(0)$. In the interval $[0, \varrho_1]$, the differential equation (79) becomes

$$\frac{\mathrm{d}x_u^\circ}{\mathrm{d}\varrho}(\varrho) = A_1 x_u^\circ(\varrho) + B_1 u(\varrho) + E_1, \forall \varrho \in]0, \varrho_1[. \tag{214}$$

Equation (214) can be solved explicitly and it gives

$$x_u^\circ(\varrho_1) = e^{\varrho_1 A_1} x_u^\circ(0) + e^{\varrho_1 A_1} \int_0^{\varrho_1} e^{-\tau A_1}\left(B_1 u(\tau) + E_1\right) \mathrm{d}\tau. \tag{215}$$

Taking into account Equation (209) it follows that

$$x_u^\circ(1) = e^{A_1} x_u^\circ(0) + \beta_{11}, \tag{216}$$

$$\beta_{11} = A_1^{-1}\left[\left(-1 - A_1^{-1} + A_1^{-1} e^{A_1}\right) B_1 + \left(-1 + e^{A_1}\right) E_1\right]. \tag{217}$$

In the interval $[\varrho_1, \varrho_2]$, the differential equation (79) becomes

$$\frac{\mathrm{d}x_u^\circ}{\mathrm{d}\varrho}(\varrho) = -A_2 x_u^\circ(\varrho) - u(\varrho) B_2 - E_2, \forall \varrho \in]\varrho_1, \varrho_2[. \tag{218}$$

Equation (218) can be solved explicitly and it gives

$$x_u^\circ(\varrho_2) = e^{-(\varrho_2 - \varrho_1)A_2} x_u^\circ(\varrho_1) - e^{-\varrho_2 A_2} \int_{\varrho_1}^{\varrho_2} e^{\tau A_2}\left(u(\tau) B_2 + E_2\right) \mathrm{d}\tau. \tag{219}$$

Taking into account Equation (210) it follows that

$$x_u^\circ(\varrho_2) = e^{(\alpha - 1)A_2} x_u^\circ(1) + \beta_{21} e^{A_2 \alpha} + \beta_{22}\alpha + \beta_{23}, \tag{220}$$

$$\beta_{21} = A_2^{-1} e^{-A_2}\left(B_2(1 + A_2^{-1}) + E_2\right), \tag{221}$$

$$\beta_{22} = -A_2^{-1} B_2, \tag{222}$$

$$\beta_{23} = -A_2^{-1}\left(A_2^{-1} B_2 + E_2\right). \tag{223}$$

In the interval $[\varrho_2, \varrho_3]$, the differential equation (79) becomes

$$\frac{\mathrm{d}x_u^\circ}{\mathrm{d}\varrho}(\varrho) = A_1 x_u^\circ(\varrho) + B_1 u(\varrho) + E_1, \forall \varrho \in]\varrho_2, \varrho_3[. \tag{224}$$

Equation (224) can be solved explicitly and it gives

$$\begin{aligned} x_u^\circ(\varrho_3) &= e^{(\varrho_3-\varrho_2)A_1} x_u^\circ(\varrho_2) \\ &\quad + e^{\varrho_3 A_1} \int_{\varrho_2}^{\varrho_3} e^{-\tau A_1} \left(B_1 u(\tau) + E_1\right) \mathrm{d}\tau. \end{aligned} \tag{225}$$

Taking into account Equation (211) it follows that

$$x_u^\circ(\varrho_3) = e^{(1-\alpha)A_1} x_u^\circ(\varrho_2) + \beta_{31}\alpha e^{-A_1\alpha} + \beta_{32} e^{-A_1\alpha} + \beta_{33}, \tag{226}$$

$$\beta_{31} = A_1^{-1} B_1 e^{A_1}, \tag{227}$$

$$\beta_{32} = A_1^{-1} e^{A_1} \left(A_1^{-1} B_1 + E_1\right), \tag{228}$$

$$\beta_{33} = A_1^{-1} \left(-(1 + A_1^{-1}) B_1 - E_1\right). \tag{229}$$

In the interval $[\varrho_3, \varrho_4]$, the differential equation (79) becomes

$$\frac{\mathrm{d}x_u^\circ}{\mathrm{d}\varrho}(\varrho) = -A_2 x_u^\circ(\varrho) - u(\varrho) B_2 - E_2, \forall \varrho \in]\varrho_3, \varrho_4[. \tag{230}$$

Equation (230) can be solved explicitly and it gives

$$\begin{aligned} x_u^\circ(\varrho_4) &= e^{-(\varrho_4-\varrho_3)A_2} x_u^\circ(\varrho_3) \\ &\quad - e^{-\varrho_4 A_2} \int_{\varrho_3}^{\varrho_4} e^{\tau A_2} \left(u(\tau) B_2 + E_2\right) \mathrm{d}\tau. \end{aligned} \tag{231}$$

Taking into account Equation (212) it follows that

$$x_u^\circ(\varrho_4) = e^{-A_2} x_u^\circ(\varrho_3) + \beta_{41}, \tag{232}$$

$$\begin{aligned} \beta_{41} &= -A_2^{-1} \Big[B_2 \left(-e^{-A_2} - A_2^{-1} e^{-A_2} + A_2^{-1}\right) \\ &\quad + E_2 \left(1 - e^{-A_2}\right) \Big]. \end{aligned} \tag{233}$$

Now we use the relation $x_u^\circ(0) = x_u^\circ(\varrho_4)$ to find $x_u^\circ(0)$ using Equations (216)–(217), (220)–(223), (226)–(229) and (232)–(233). We get

$$x_u^\circ(0) = \frac{\beta_{51} + \beta_{52} e^{(A_2-A_1)\alpha} + \beta_{53}\alpha e^{-A_1\alpha} + \beta_{54} e^{-A_1\alpha}}{1 + \beta_{55} e^{(A_2-A_1)\alpha}}, \tag{234}$$

$$\beta_{51} = \beta_{33} e^{-A_2} + \beta_{41}, \tag{235}$$

$$\beta_{52} = \beta_{21} e^{A_1-A_2} + \beta_{11} e^{A_1-2A_2}, \tag{236}$$

$$\beta_{53} = \beta_{22} e^{A_1-A_2} + e^{-A_2} \beta_{31}, \tag{237}$$

$$\beta_{54} = \beta_{23} e^{A_1-A_2} + e^{-A_2} \beta_{32}, \tag{238}$$

$$\beta_{55} = -e^{2(A_1-A_2)}. \tag{239}$$

Note that, since $0 < \alpha < 1$, $A_1 < 0$ and $A_2 > 0$ it follows that $0 < e^{(2-\alpha)(-A_2+A_1)} < 1$ so that the denominator in Equation (234) is nonzero.

STEP 2. By Assumption 8 it follows that $\varphi_u^\circ(\varrho_1) = \varphi_u^\circ(\varrho_3)$. This means that $x_u^\circ(1) = x_u^\circ(\varrho_3)$ because $C \neq 0$. Since $x_u^\circ(0)$ has been computed explicitly, $x_u^\circ(1)$ and $x_u^\circ(\varrho_3)$ are available explicitly using Equations (216)–(217) and (226)–(229) respectively. We get

$$\begin{aligned} x_u^\circ(1) &= \frac{e^{A_1}}{1 + \beta_{55} e^{(A_2-A_1)\alpha}} \cdot \Big(\beta_{51} + \beta_{52} e^{(A_2-A_1)\alpha} \\ &\quad + \beta_{53}\alpha e^{-A_1\alpha} + \beta_{54} e^{-A_1\alpha}\Big) + \beta_{11}, \end{aligned} \tag{240}$$

$$\begin{aligned} x_u^\circ(\varrho_3) = {} & \frac{e^{2A_1-A_2}}{1 + \beta_{55} e^{(A_2-A_1)\alpha}} \cdot \Big(\beta_{51} e^{(A_2-A_1)\alpha} \\ & + \beta_{52} e^{2(A_2-A_1)\alpha} + \beta_{53}\alpha e^{(A_2-2A_1)\alpha} \\ & + \beta_{54} e^{(A_2-2A_1)\alpha}\Big) + \beta_{61} e^{(A_2-A_1)\alpha} \\ & + \beta_{62}\alpha e^{-A_1\alpha} + \beta_{63} e^{-A_1\alpha} + \beta_{33}, \end{aligned} \tag{241}$$

where

$$\beta_{61} = \left(\beta_{21} + \beta_{11} e^{-A_2}\right) e^{A_1}, \tag{242}$$

$$\beta_{62} = \beta_{22} e^{A_1} + \beta_{31}, \tag{243}$$

$$\beta_{63} = \beta_{23} e^{A_1} + \beta_{32}. \tag{244}$$

Our aim in the following analysis is to find the conditions under which we have $x_u^\circ(1) = x_u^\circ(\varrho_3)$ for all inputs u that satisfy the relations (209)–(212). This means finding the conditions under which we have $x_u^\circ(1) = x_u^\circ(\varrho_3)$ for all $\alpha \in]0, 1[$. In the equality $x_u^\circ(1) = x_u^\circ(\varrho_3)$ we multiply both terms with $1 + \beta_{55} e^{(A_2-A_1)\alpha}$ so that we get from Equalities (240)–(244) that

$$\begin{aligned} \beta_{71} + \beta_{72} e^{(A_2-A_1)\alpha} + \beta_{73}\alpha e^{-A_1\alpha} + \beta_{74} e^{-A_1\alpha} = 0, \\ \forall \alpha \in]0, 1[, \end{aligned} \tag{245}$$

where

$$\beta_{71} = e^{A_1}\beta_{51} + \beta_{11} - \beta_{33}, \tag{246}$$

$$\beta_{72} = e^{A_1}\beta_{52} + \beta_{11}\beta_{55} - e^{2A_1-A_2}\beta_{51} - \beta_{61} - \beta_{55}\beta_{33}, \tag{247}$$

$$\beta_{73} = e^{A_1}\beta_{53} - \beta_{62}, \tag{248}$$

$$\beta_{74} = e^{A_1}\beta_{54} - \beta_{63}. \tag{249}$$

Consider the functions $f_1, f_2, f_3, f_4, \mathbf{0} :]0, 1[\to \mathbb{R}$ defined by $\forall \alpha \in]0, 1[$, $f_1(\alpha) = 1$, $f_2(\alpha) = e^{(A_2-A_1)\alpha}$, $f_3(\alpha) = \alpha e^{-A_1\alpha}$, $f_4(\alpha) = e^{-A_1\alpha}$, and $\mathbf{0}(\alpha) = 0$. Then Equation (245) can be written as

$$\beta_{71} \cdot f_1 + \beta_{72} \cdot f_2 + \beta_{73} \cdot f_3 + \beta_{74} \cdot f_4 = \mathbf{0}. \tag{250}$$

Consider the vector space of functions $\{p :]0, 1[\to \mathbb{R}\}$ with its usual binary operations of vector addition and scalar multiplication. Then the functions f_1, f_2, f_3, f_4 are linearly independent vectors so that, owing to Equation (250), we have $\beta_{71} = \beta_{72} = \beta_{73} = \beta_{74} = 0$ since β_{ij} is independent of α for all possible i and j.

We start by solving Equation $\beta_{73} = 0$. Combining Equations (248), (237), (243), (222), and (227) it comes that

$$A_2^{-1} B_2 = A_1^{-1} B_1. \tag{251}$$

Now we solve Equation $\beta_{71} = 0$. Combining Equations (251), (246), (235), (229), (233), and (217) it follows that

$$B_1 A_1^{-1} \left(A_2^{-1} - A_1^{-1}\right) - E_1 A_1^{-1} + E_2 A_2^{-1} = 0. \tag{252}$$

It can be checked that Equalities (251)–(252) imply that $\beta_{71} = \beta_{72} = \beta_{73} = \beta_{74} = 0$.

Lemma 11 follows from Lemma 8.

Compliance with Ethical Standards

Funding: This study was funded by the Spanish Ministry of Economy, Industry and Competitiveness (grant number DPI2016-77407-P (AEI/FEDER, UE)).
Conflict of Interest: The author declares that he has no conflict of interest.

References

1. D. Angeli. Systems with counterclockwise input-ouput dynamics. *IEEE Trans. Aut. Contr.*, **51** (2006) 1130–1143.
2. K. J. Åström and C. Canudas de Wit. Revisiting the LuGre friction model. *IEEE Contr. Syst. Mag.* **28** (2008) 101–114.
3. I. Babuška. Die nichtlineare theorie der inneren reibung. *Aplikace matematiky*, **4** (1952) 303–321.
4. J. A. Barker, D. E. Schreiber, B. G. Huth, D. H. Everett. Magnetic hysteresis and minor loops: models and experiments. *Proc. Royal Soc. London A*, **386** (1983) 251–261.
5. D. Bernardini, and F. Vestroni. Hysteretic modeling of shape memory alloy vibration reduction devices. *J. Mater. Proc. & Manufact. Sci. 9* (2000) 101–112.
6. D. S. Bernstein. Ivory ghost [ask the experts]. *IEEE Contr. Syst. Mag.* **27** (2007) 16–17.
7. G. Bertotti and I. Mayergoyz, editors. *The Science of Hysteresis*, 3-volume set (Elsevier, 2006).
8. R. Bouc. Modèle mathématique d'hystérésis. *Acustica* **21** (1971) 16–25.
9. M. Branciforte, A. Meli, G. Muscato, and D. Porto. ANN and non–integer order modeling of ABS solenoid valves. *IEEE Trans. Contr. Syst. Tech.* **19** (2011) 628–635.
10. M. Brokate and J. Sprekels. *Hysteresis and Phase Transitions*, (Springer–Verlag, 1996).
11. C. Canudas de Wit, H. Olsson, K. J. Åström, and P. Lischinsky. A new model for control of systems with friction. *IEEE Trans. Aut. Contr.* **40** (1995) 419–425.
12. B. D. Coleman and M. L. Hodgdon. A constituve relation for rate-independent hysteresis in ferromagnetically soft materials. *Intern. J. Eng. Sci.* **24** (1986) 897–919.
13. P. Dahl. Solid friction damping of mechanical vibration. *The American Institute of Aeronautics and Astronautics (AIAA)* **14** (1976) 1675–1682.
14. E. Della Torre. *Magnetic Hysteresis*, (IEEE Press, 1999).
15. B. Drinčić. *Mechanical Models of Friction That Exhibit Hysteresis, Stick–Slip, and the Stribeck Effect*, PhD thesis, University of Michigan, 2012.
16. P. Duhem. Sur les déformations permanentes et l'hystérésis. Premier Mémoire, tome LIV, *Mémoires couronnés et Mémoires des savants étrangers*, l'Académie royale des sciences, des lettres et des beaux–arts de Belgique, 1896 [The Cornell University library digital collections].
17. P. Duhem. Les modifications permanentes du soufre. Deuxième Mémoire, tome LIV, *Mémoires couronnés et Mémoires des savants étrangers*, l'Académie royale des sciences, des lettres et des beaux–arts de Belgique, 1896 [The Cornell University library digital collections].
18. P. Duhem. Théorie générale des déformations permanentes. Troisième Mémoire, tome LIV, *Mémoires couronnés et Mémoires des savants étrangers*, l'Académie royale des sciences, des lettres et des beaux–arts de Belgique, 1896 [The Cornell University library digital collections].
19. P. Duhem. Étude de divers systèmes dépendant d'une seule variable. Quatrième Mémoire, tome LVI, *Mémoires couronnés et Mémoires des savants étrangers*, l'Académie royale des sciences, des lettres et des beaux–arts de Belgique, 1898 [The Cornell University library digital collections].
20. P. Duhem. Étude de divers systèmes dépendant de deux variables. Cinquième Mémoire, tome LVI, *Mémoires couronnés et Mémoires des savants étrangers*, l'Académie royale des sciences, des lettres et des beaux–arts de Belgique, 1898 [The Cornell University library digital collections].
21. P. Duhem. L'inégalité de Clausius et l'hystérésis. Sixième Mémoire, tome LXII, *Mémoires couronnés et autres Mémoires*, l'Académie royale de Belgique, 1902 [The Cornell University library digital collections].
22. P. Duhem. Hystérésis et viscosité. Septième Mémoire, tome LXII, *Mémoires couronnés et autres Mémoires*, l'Académie royale de Belgique, 1902 [The Cornell University library digital collections].
23. G. A. Edgar. *Measure, Topology and Fractal Geometry*, (Springer-Verlag, 1990).
24. D. H. Everett and W. I. Whitton. A general approach to hysteresis. *Trans. Faraday Soc.* **48** (1952) 749–757.
25. D. H. Everett and F. W. Smith. A general approach to hysteresis. Part 2: Development of the domain theory. *Trans. Faraday Soc.* **50** (1954) 187–197.
26. D. H. Everett. A general approach to hysteresis. Part 3: A formal treatment of the independent domain model of hysteresis. *Trans. Faraday Soc.* **volume 50** (1954) 1077–1096.
27. D. H. Everett. A general approach to hysteresis. Part 4: An alternative formulation of the domain model. *Trans. Faraday Soc.* **51** (1955) 1551–1557.
28. J. A. Ewing. On the production of transient electric currents in iron and steel conductors by twisting them when magnetised or by magnetising them when twisted. *Proc. Royal Soc. London* **33** (1881) 21–23.
29. A. F. Filippov. *Differential Equations with Discontinuous Righthand Sides*, (Kluwer Academic Publishers, 1988).
30. M. Goldfarb and N. Celanovic. A lumped parameter electromechanical model for describing the nonlinear behavior of piezoelectric actuators. *Trans. ASME J. Dyn. Syst., Measur., Contr.* **119** (1997) 478–485.
31. M. Hamimid, S. M. Mimoune, M. Feliachi, K. Atallah. Non centered minor hysteresis loops evaluation based on exponential parameters transforms of the modified inverse Jiles–Atherton model. *Physica B*, **451** (2014) 16–19.
32. P. Hartman. *Ordinary Differential Equations*. SIAM Classics In Applied Mathematics (SIAM, 2002).
33. V. Hassani, T. Tjahjowidodo, and T. N. Do. A survey on hysteresis modeling, identification and control. *Mech. Syst. Sign. Proces.* **49** (2014) 209–233.
34. M. L. Hodgdon. Applications of a theory of ferromagnetic hysteresis. *IEEE Trans. Magnet.* **24** (1988) 218–221.
35. F. Ikhouane. Characterization of hysteresis processes. *Math. Contr., Sign., Syst.* **25** (2013) 294–310.
36. F. Ikhouane and J. Rodellar. A linear controller for hysteretic systems. *IEEE Trans. Aut. Contr.*, **51**:2 (2006) 340–344.
37. F. Ikhouane and J. Rodellar. *Systems with Hysteresis: Analysis, Identification and Control using the Bouc–Wen model*, (John Wiley & Sons, 2007).
38. M. Ismail, F. Ikhouane, and J. Rodellar. The hysteresis Bouc-Wen model, a survey. *Arch. Comput. Meth. Eng.* **16** (2009) 161–188.
39. S. L. Jaki. *Uneasy genius: The Life and Work of Pierre Duhem*, (Martinus Nijhoff Publishers, 1987).
40. B. Jayawardhana, R. Ouyang, and V. Andrieu. Stability of systems with the Duhem hysteresis operator: The dissipativity approach. *Automatica*, **48** (2012) 2657–2662.
41. B. Jayawardhana, R. Ouyang, and V. Andrieu. Dissipativity of general Duhem hysteresis models. *50th IEEE Conf. Decis. Contr., Europ. Contr. Conf.*, Orlando, Florida, USA, December 12–15, (2011) 3234–3239.

42. H. Khalil. *Nonlinear Systems*. Third Edition (Prentice Hall, 2002).
43. M. A. Krasnosel'skiĭ and A. V. Pokrovskiĭ. *Systems with Hysteresis*, (Springer–Verlag, 1989)
44. C. Lagrange. Rapport de M. Ch. Lagrange, premier commisaire. *Bulletins de l'Académie royale des sciences, des lettres et des beaux-arts de Belgique*, 65e année, 3e série, t. XXIX, n°6, (1895) 805–820.
45. G. Leoni. *A First Course in Sobolev Spaces*, (The American Mathematical Society, 2009). USA, 2009.
46. R. Loloee, T. J. Pence, and D. S. Grummon. Phase–fraction evolution during incomplete cyclic transformation in TiNi: correlation and analytical models with magnetic susceptibility measurements. *Journal de Physique IV*, Colloque C8, supplément au Journal de Physique III, **5** (1995) C8-545–C8-550.
47. H. Logemann, E. P. Ryan, and I. Shvartsmann. A class of differential-delay systems with hysteresis: Asymptotic behaviour of solutions. *Nonlin. Anal.*, **69** (2008) 363–391.
48. J. W. Macki, P. Nistri, and P. Zecca. Mathematical models for hysteresis, *SIAM Review*, **35** (1993) 94–123.
49. I. Mayergoyz. *Mathematical Models of Hysteresis*, Elsevier Series in Electromagnetism (Elsevier, 2003).
50. K. A. Morris. What is Hysteresis? *Appl. Mech. Rev.* **64** (2011) Article Number: 050801, 14 pages.
51. M. F. M. Naser and F. Ikhouane. Consistency of the Duhem model with hysteresis. *Math. Prob. Eng.* **2013** (2013) Article ID 586130.
52. Y. Q. Ni, Z. G. Ying, J. M. Ko, and W. Q. Zhu. Random response of integrable Duhem hysteretic systems under non–white excitation. *Non-Lin. Mech.* **37** (2002) 1407–1419.
53. R. Ouyang, V. Andrieu, and B. Jayawardhana. On the characterization of the Duhem hysteresis operator with clockwise input-output dynamics. *Syst. Contr. Lett.* **62** (2013) 286–293, .
54. J. Oh and D. S. Bernstein. Semilinear Duhem model for rate-independent and rate-dependent hysteresis. *IEEE Trans. Aut. Contr.* **50** (2005) 631–645.
55. J. Oh, B. Drinčić and D. S. Bernstein. Nonlinear feedback models of hysteresis. *IEEE Contr. Syst. Mag.* **29** (2009) 100–119.
56. F. Preisach. Über die magnetische Nachwirkung. *Zeitschrift für Physik*, **94** (1935) 277–302.
57. A. K. Padthe, B. Drincic, J. Oh, D. D. Rizos, S. D. Fassois, and D. S. Bernstein. Duhem modeling of friction–induced hysteresis. *IEEE Contr. Syst. Mag.* **28** (2008) 90–107.
58. A. K. Padthe, J. Oh, and D. S. Bernstein. Counterclockwise dynamics of a rate-independent semilinear Duhem model. *44th IEEE Conf. Decis. Contr., Europ. Contr. Conf.*, Seville, Spain, December 12–15, (2005) 8000–8005.
59. Lord Rayleigh. On the behavior of iron and steel under the operations of feeble magnetic forces. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, **23** (1887) 225–245.
60. W. Rudin. *Real and Complex Analysis*. McGraw-Hill Series in Higher Mathematics, Third Edition (McGraw-Hill, 1987).
61. K. Schmitt and R. C. Thompson. *Nonlinear Analysis and Differential Equations. An Introduction.* Free E–Book, 2000.
62. C. Truesdell and W. Noll. *The Non-Linear Field Theories of Mechanics.* Edited by S. S. Antman, Third Edition (Springer–Verlag, 2004).
63. A. Visintin. P.D.E.s with hysteresis 30 years later. *Discr. Cont. Dyn. Syst. Series S*, **8** (2015) 793–816, .
64. A. Visintin. *Differential Models of Hysteresis*, (Springer–Verlag, 1994) Berlin, Heidelberg, 1994.
65. G. Wiedemann. Magnetic researches. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, **22** (1886) 50–70.
66. E. H. Yen and H. R. Van Der Vaart. On measurable functions, continuous functions and some related concepts. *Amer. Math. Monthly*, **73** (1966) 991–993.
67. *Pierre Duhem. Sa vie – Ses œuvres.*[22] Gauthier–Villars, Mémoires de la Société des sciences physiques et naturelles de Bordeaux, 7ème série, Paris, France, 1917.

[22] There are no specified authors. The one-page preface written by M. V. cites the following people as co–authors or co–authors to be: M. Edouard Jordan, M. J. Hadamard, M. L. Marchis, M. H. Pélabon, M. Ed. Le Roy, and M. Darbon. However, the chapters bear the following names. The biography of P. Duhem is written by E. Jordan. The "Notice sur les titres et travaux scientifiques de Pierre Duhem" is the note Duhem wrote himself when he applied to the Académie des Sciences. The following chapter "La physique de P. Duhem" is written by Octave Manville. The chapter "L'œuvre de Pierre Duhem dans son aspect mathématique" is authored by J. Hadamard. Finally "L'histoire des sciences dans l'œuvre de P. Duhem" is written by A. Darbon.