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SURVEY PAPER

A SURVEY OF USEFUL INEQUALITIES IN FRACTIONAL CALCULUS

Ahmed Alsaedi [1], **Bashir Ahmad** [2], **Mokhtar Kirane** [3]

Abstract

We present a survey on inequalities in fractional calculus that have proven to be very useful in analyzing differential equations. We mention in particular, a "Leibniz inequality" for fractional derivatives of Riesz, Riemann-Liouville or Caputo type and its generalization to the d-dimensional case that become a key tool in differential equations; they have been used to obtain upper bounds on solutions leading to global solvability, to obtain Lyapunov stability results, and to obtain blowing-up solutions via diverging in a finite time lower bounds. We will also mention the weakly singular Gronwall inequality of Henry and its variants, principally by Medved, that plays an important role in differential equations of any kind. We will also recall some "traditional" inequalities involving fractional derivatives or fractional powers of the Laplacian.

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1. Introduction

Inequalities of any kind (pointwise, integral, etc) are the lifeblood of ordinary or partial differential equations, and of integral equations. Without them, the advance of differential and integral equations would not be at its present stage. However, inequalities are scattered in the literature; they are too important to be gathered in one review paper and be available to the very large community of researchers in differential equations.

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So, in this paper, we present some inequalities in fractional calculus that are used in differential or integral equations/systems.

In differential equations or systems, when one want to use Lyapunov functionals or Moser's scheme to obtain a priori estimations, Leibniz' rule of differentiation is needed; as is well known, in fractional calculus, such rule has a not a very tractable form.

Quite recently, the inequality obtained by Cordoba and Cordoba [15] for the one-dimensional fractional Laplacian and its twin inequality for the Riemann-Liouville or Caputo fractional derivative due to Diaz, Pierantozzi and Vazquez [16], Alikhanov [1], and the general inequality of Ahmad, Alsaedi, and Kirane [4], Zacher [47] turned out be useful in many situations.

Many other inequalities that have shown to de useful, especially with relation to fractional order operators and equations, are presented. We want to point out that some inequalities have already appeared in books like [42] (see the luxury 17. Bibliographical Remarks and Additional Information to Chapter 3, [43]; we recall them here in order, for researchers, to have one working document under hand.

Among the recently published articles on topics of inequalities in fractional calculus, we can mention also [21, 22], [29], [8], references therein, etc.

2. Eilertsen equality and its consequences

For a function u in the Schwartz space or in $C_0^\infty(\mathbb{R}^n)$,

$$(-\Delta)^{\frac{s}{2}}u(x) = C_s P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+s}}, \tag{2.1}$$

where C_s is a normalizing constant.

Eilertsen in [18] proved the following interesting result that has important consequences.

THEOREM 2.1. *If $u, v \in C_0^\infty(\mathbb{R}^n)$ and $0 < s < 1$, then*

$$u(-\Delta)^s v + v(-\Delta)^s u - (-\Delta)^s(uv) = A_s \int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\, dy, \tag{2.2}$$

where $A_s > 0$ and $A_s/s(1-s)$ has finite, positive limits as $s \to 0$ and $s \to 1$.

If we take $u = v$ in (2.2), multiply by $\Gamma_s(x) = C_s|x|^{2s-n}$ and integrate, the identity

$$2\int_{\mathbb{R}^n} ((-\Delta)^s u)u\Gamma_s\, dx = u(0)^2 + A_s \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\Gamma_s(x)\, dxdy, \tag{2.3}$$

is valid for $0 < s < 1$. Hence,

$$\int_{\mathbb{R}^n} ((-\Delta)^s u) u \Gamma_s(x)\, dx \geq 0. \tag{2.4}$$

An other consequence is the following inequality obtained by Cordoba and Cordoba in [15].

3. "Cordoba-Cordoba" type inequalities

As a first consequence of the Eilertsen equality, we present the inequality obtained by Cordoba and Cordoba in [15].

Set $\Lambda = (-\Delta)^{\frac{1}{2}}$.

THEOREM 3.1. *Let $0 \leq \alpha \leq 2, x \in \mathbb{R}^n$ or $x \in \mathbb{T}^n$ (the torus) ($n = 1, 2, 3 \ldots$) and $\theta \in C_0^2(\mathbb{R}^n)$ or $\theta \in C^2(\mathbb{T}^n)$. Then the following inequality holds*

$$2\theta \Lambda^\alpha \theta(x) \geq \Lambda^\alpha \theta^2(x). \tag{3.1}$$

Cordoba-Cordoba's inequality follows from Eilertsen's identity (2.3) by setting $u = v = \theta$.

This inequality enabled Cordoba and Cordoba to obtain the L^∞-decay estimate for the viscosity solutions of the quasi-geostrophic equation.

This inequality has been generalized by Ju [25] as follows.

THEOREM 3.2. *Let $0 \leq \alpha \leq 2, x \in \mathbb{R}^n$ or $x \in \mathbb{T}^n$ (the torus) ($n = 1, 2, 3 \ldots$) and $\theta \in C_0^2(\mathbb{R}^n)$ or $\theta \in C^2(\mathbb{T}^n)$. Then the following inequality holds*

$$p\theta^{p-1} \Lambda^\alpha \theta(x) \geq \Lambda^\alpha \theta^p(x). \tag{3.2}$$

Wu [46] proved the following version.

THEOREM 3.3. *Let $0 \leq \alpha \leq 2$. Let $p_1 = k_1/l_1 \geq 0$ and $p_2 = k_2/l_2 \geq 1$ be rational numbers with l_1 and l_2 being odd, and with $k_1 l_2 + k_2 l_1$ being even. Then, for any $x \in \mathbb{R}^n$ and any function $\theta \in C^2(\mathbb{R}^n)$ that decays sufficiently fast at infinity, Then the following inequality holds*

$$(p_1 + p_2)\theta^{p_1}(x) \Lambda^\alpha \theta^{p_2}(x) \geq p_2 \Lambda^\alpha \theta^{p_1+p_2}(x). \tag{3.3}$$

Ju's and Wu's inequalities have also been used for the quasi-geostrophic equation.

A further generalization has been achieved by Constantin [13].

THEOREM 3.4. *Let $\theta \in C_0^2(\mathbb{R}^n)$ or $\theta \in C^2(\mathbb{T}^n)$ and Φ be a convex function of one variable. Then*

$$\Phi'(\theta)\Lambda^\alpha\theta(x) \geq \Lambda^\alpha\Phi(\theta)(x). \tag{3.4}$$

Ju, Caffarelli and Vasseur [12], and Constantin [13] used the "convexity" inequality for the quasi-geostrophic equation too.

Ye and Xu [45] derived an other variant; it reads:

$$2\nabla u(x)\cdot(-\Delta)^{\frac{\alpha}{2}}\nabla u(x) \geq (-\Delta)^{\frac{\alpha}{2}}(|\nabla u(x)|^2 + \frac{|\nabla u(x)|^{2+\frac{\alpha p}{p+2}}}{c\|u\|_{L^p}^{\frac{\alpha p}{p+2}}}. \tag{3.5}$$

They used this inequality for the 2-D Boussinesq equations.

Recently, Alsaedi, Ahmad and Kirane [5] derived the "convexity" inequality in the Heisenberg group thanks to a result of Ferrari and Franchi [20] concerning an integral representation of the fractional powers of the Laplacian.

THEOREM 3.5. *Let $F \in C^2(\mathbb{R})$ be a convex function, $0 \leq \alpha \leq 2$. Assume that $\varphi \in C_0^2(\mathbb{R}^{2N+1})$. Then*

$$F'(\varphi)(-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}\varphi \geq (-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}F(\varphi) \tag{3.6}$$

holds point-wise. In particular, if $F(0) = 0$ and $\varphi \in C_0^\infty(\mathbb{R}^{2N+1})$, then

$$\int_{R^{2N+1}} F'(\varphi)(-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}\varphi\, d\eta \geq 0. \tag{3.7}$$

Here $(-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}$ is the fractional Laplacian on the Heisenberg group $\mathbb{H}$.

In [5], the convexity inequality is used to prove nonexistence results via the nonlinear capacity method [37] for hyperbolic, parabolic, and hyperbolic equations with linear damping.

Constantin and Vlad [14] derived the following inequality.

THEOREM 3.6. *Let $f \in \mathcal{S}(\mathbb{R}^n)$. For a $k \in \{1, ..., n\}$, let $g(x) = \partial_k f(x)$. Assume that $\overline{x} \in \mathbb{R}^n$ is such that $g(\overline{x}) = \max_{x\in\mathbb{R}^n} g(x) > 0$. Then*

$$\Lambda^\alpha g(\overline{x}) \geq \frac{g(\overline{x})^{1+\alpha}}{C\|f\|_\infty^\alpha}, \tag{3.8}$$

where $\|f\|_\infty$ is the norm of $L^\infty(\mathbb{R}^n)$, for $\alpha \in (0, 2)$, and some universal positive constant $cC = C(n, \alpha)$ which may be computed explicitly.

After the appearance of the "Cordoba-Cordoba inequality", Diaz, Pieranttozi and Vazquez [16] proved a similar inequality for the Riemann-Liouville fractional time derivative.

THEOREM 3.7. *Let* $0 < \alpha < 1$ *and* $u \in C([0,T];\mathbb{R}), u' \in L^1(0,T;\mathbb{R})$ *and* u *be* monotone. *Then*

$$2u(t)D^\alpha_{0,t}u(t) \geq D^\alpha_{0,t}u^2(t), \qquad t \in (0,T]. \tag{3.9}$$

They conjectured that the inequality (3.9) still holds true without the monotonicity condition imposed on u.

They used inequality (3.9) to obtain finite time extinction for some nonlinear fractional in time equations.

In the same paper, they provided a more general version of Theorem 3.7.

THEOREM 3.8. *Given the Hilbert space* $\mathcal{H}$ *with inner product* $(\,,\,)_{\mathcal{H}}$, *let* $0 < \alpha < 1$ *and* $u \in L^\infty(0,T;\mathcal{H})$ *such that* $D^\alpha_{0,t}u \in L^1(0,T;\mathcal{H})$. *Assume that* $\|u(.)\|_{\mathcal{H}}$ *is non-increasing (i.e.* $\|u(t_2)\|_{\mathcal{H}} \leq \|u(t_1)\|_{\mathcal{H}}$ *for a.e.* $t_1, t_2 \in (0,T)$ *such that* $t_1 \leq t_2$*). Then there exists* $k(\alpha) > 0$ *such that for almost every* $t \in (0,T)$, *we have*

$$(u(t), D^\alpha_{0,t}u(t)) \geq k(\alpha)\, D^\alpha_{0,t}\|u(t)\|. \tag{3.10}$$

In [47], Zacher derived the following inequality.

THEOREM 3.9. *Let* $\alpha \in (0,\,1), T > 0$ *and* $\mathcal{H}$ *be a Hilbert space with inner product* $(\,,\,)_{\mathcal{H}}$. *Suppose that* $v \in L^2(0,T;\mathcal{H})$ *and there exists* $x \in \mathcal{H}$ *such that* $v - x \in {}_0H^\alpha_2([0,\,T];\,\mathcal{H}) := \left\{g_\alpha * w :\, w \in L^2(0,T;\,\mathcal{H})\right\}$. *Then*

$$2(u(t), \frac{d}{dt}(g_{1-\alpha} * v)(t))_{\mathcal{H}} \geq \frac{d}{dt}(g_{1-\alpha} * |v|^2_{\mathcal{H}} + g_{1-\alpha}(t)|v|^2_{\mathcal{H}}), \tag{3.11}$$

for a.a. $t \in (0,\,T)$, *where* $g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$, $t > 0, \beta > 0$.

Zacher used inequality (3.11) to obtain some decay estimate for a nonlinear homogeneous time fractional evolution equation.

Alsaedi, Ahmad, and Kirane [4] looking for stability estimates for various diffusion equations with time-fractional derivatives, derived the following results.

THEOREM 3.10. *Let one of the following conditions be satisfied:*

- $u \in C([0,T]), \quad v \in C^\beta([0,T]),\ \alpha < \beta \leq 1;$
- $v \in C([0,T]), \quad u \in C^\beta([0,T]),\ \alpha < \beta \leq 1;$
- $u \in C^\beta([0,T]), v \in C^\delta([0,T]),\ \alpha < \beta + \delta,\ 0 < \beta < 1,\ 0 < \delta < 1.$

Then we have:

$$D_{0+}^{\alpha}(uv)(t) = u(t)D_{0+}^{\alpha}v(t) + v(t)D_{0+}^{\alpha}u(t)$$
$$- \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{(u(s)-u(t))(v(s)-v(t))}{(t-s)^{\alpha+1}}\,ds - \frac{u(t)v(t)}{\Gamma(1-\alpha)t^{\alpha}}, \tag{3.12}$$

pointwise.

The immediate consequences are:

1. If u and v have the same sign and are both increasing or both decreasing, then

$$D_{0+}^{\alpha}(uv)(t) \le u(t)D_{0+}^{\alpha}v(t) + v(t)D_{0+}^{\alpha}u(t). \tag{3.13}$$

By setting $u = v$ in inequality (3.13) and taking only $u \in C^{\beta}([0,T])$, $\alpha < 2\beta$, $\beta \le 1$ we obtain the inequality conjectured by J. I. Diaz, T. Pierantozi and L. Vázquez [16]

$$2u(t)D_{0+}^{\alpha}u(t) \ge D_{0+}^{\alpha}u^2(t). \tag{3.14}$$

In the case $\beta < 1$, our requirement on u is weaker than the one of [16] as u is not differentiable. However, in the case $\beta < 1$, by Rademacher's theorem, u is almost everywhere differentiable [19].

2. By induction, one can show that, for any integer $p \ge 2$,

$$pu^{(p-1)}(t)D_{0+}^{\alpha}u(t) \ge D_{0+}^{\alpha}u^p(t), \tag{3.15}$$

for p even, or p odd whenever $u \ge 0$.

REMARK 3.1. For the Caputo derivative, inequality (3.13) reads

$${}^{c}D_{0+}^{\alpha}(uv)(t) \le u(t)\,{}^{c}D_{0+}^{\alpha}v(t) + v(t)\,{}^{c}D_{0+}^{\alpha}u(t)$$
$$+ \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\left(u(t)v(0) + v(t)u(0) - u(0)v(0)\right).$$

Alikhanov [1], looking for some stability estimates in L^2 for diffusion equations with time-fractional derivative, derived the following equality.

THEOREM 3.11. *Let $0 < \alpha < 1$ and u absolutely continuous on $[0,T]$. Then*

$$2u(t)D_{0,t}^{\alpha}u(t) = D_{0,t}^{\alpha}u^2(t) + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \left(\int_0^s \frac{u'(\eta)}{(t-\eta)^{\alpha}}\,d\eta \right)^2 \frac{ds}{(t-s)^{1-\alpha}}, \tag{3.16}$$

holds true. As a consequence, one obtains

$$2u(t)D_{0,t}^{\alpha}u(t) \ge D_{0,t}^{\alpha}u^2(t).$$

The vectorial case:

In [17], the vectorial version of the "Leibniz' inequality" is presented.

Theorem 3.12. *Let $X : [t_0, +\infty) \to \mathbb{R}^{\mathbb{N}}$ be a vectorial differentiable function. Then, for any time instant $t \geq t_0$,*

$$ {}^c D^{\alpha}_{t_0|t}(X^T X)(t) \leq x^T(t) {}^c D^{\alpha}_{t_0|t} X(t),\ \alpha \in (0,\, 1). \tag{3.17}$$

This inequality is used in [17] to prove Lyapunov uniform stability for fractional order systems.

4. Fermat's Fractional inequality

In his valuable book [39], Nakhushev derived an analogue of the Fermat theorem and an extremum principle for the Riemann-Liouville operator of order $0 < \alpha < 1$.

Theorem 4.1. (Analogue of Fermat's theorem [39], p. 56)
Let the function $u(t) \in L^1([A,\, B])$ attain at $x \in (A,\, B)$ its extremum and let there exists $\delta > 0$ such that $u(t)$ in the one-sided neighborhood ω_δ of the point x satisfies Hölder's condition with exponent $h > \alpha$. Then for any $\alpha \in [0,\, 1]$ and $a \in (A,\, B), a \neq x$, we have

$$ (D^{\alpha}_{a,x} u)(t) \geq \frac{u(x)|x-a|^{-\alpha}}{\Gamma(1-\alpha)} \tag{4.1}$$

in case of a maximum value and

$$ (D^{\alpha}_{a,x} u)(t) \leq \frac{u(x)|x-a|^{-\alpha}}{\Gamma(1-\alpha)} \tag{4.2}$$

in case of a minimum value.

Here $\omega_\delta = [x-\delta,\, x]$ when $x \geq a$ and $\omega_\delta = [x,\, x+\delta]$ when $x \leq a, \delta > 0$.

Corollary 4.1. *If x is a point of extremum of the function $u(t)$ defined in some $\varepsilon-$neighborhood $S^x_\varepsilon = [x-\varepsilon,\, x+\varepsilon]$ of x, then either $D^{\alpha}_{a,x} u(t)$, $0 < \alpha < 1$, does not exist, or it satisfies one of the inequalities (4.1), (4.2), where a is any point of S^x_ε. In particular,*

$$ D^{\alpha}_{a,x} u(t) \geq 0, \quad \forall \alpha \in (0,\, 1), \tag{4.3}$$

if x is a point of local positive maximum.

Nakhushev [39] also derived the following result that may be useful for various fractional differential equations.

THEOREM 4.2. *Let:*
1) the function $u(t) \in L^1([A, B])$ and it attains a maximum value at a point $x \in (A, B)$ where it is differentiable;
2) there exists $\delta > 0$ such that $u'(t)$ on the segment ω_δ satisfies the Hölder condition with exponent $h > \alpha - 1$.

Then for each number $\alpha \in (1, 2)$ and any $a \in [A, B], a \neq x$ the following inequality holds true:

$$D^\alpha_{ax} u(t) \leq \frac{u(x)|x-a|^{-\alpha}}{\Gamma(1-\alpha)}. \tag{4.4}$$

If $u'(t) \in Lip^h([A, B])$ (the space of functions satisfying Hölder's condition with exponent $h \in (0, 1]$) and $h > \alpha - 1$, then in a neighborhood S^x_ε of the point x there exists a point a distinct from x such that for all $\alpha \in (1, 2)$ the following equality holds true:

$$D^\alpha_{ax} u(t) = \frac{u(x)|x-a|^{-\alpha}}{\Gamma(1-\alpha)}. \tag{4.5}$$

Similar results also appeared in the papers of Al-Refai and Luchko [2], [3].

5. Hardy-Landau-Littlewood type inequalities

R.J. Hughes in two papers [27] and [26] derived a Hardy-Landau-Littlewood inequality [24] for the Riemann-Liouville fractional integral $I^\alpha f(x) = \int_0^x (x-t)^{\alpha-1} f(t)\, dt$, then a Hardy-Landau-Littlewood inequality for fractional derivatives in weighted L^p spaces.

THEOREM 5.1. *Let $1 < p < \infty$, and let $I^\alpha, \Re\alpha > 0$ ($\Re$ for real part), with domain $D(I^\alpha) = \{f \in L^p(0, \infty) : I^\alpha f \in L^p(0, \infty)\}$. If $f \in D(I^\gamma), \Re\gamma > 0$, then $f \in D(I^\alpha)$ whenever $0 < \Re\alpha < \Re\gamma$ and that, if γ is real and $0 < \alpha < \beta < \gamma < L$, then*

$$\|I^\beta f\| \leq K(p, L) \| I^\alpha f\|^{(\gamma-\beta)/(\gamma-\alpha)} \, \|I^\gamma f\|^{(\beta-\alpha)/(\gamma-\alpha)}, \tag{5.1}$$

where $\| . \|$ is the usual L^p norm.

An inequality similar to (5.1) for the Weyl fractional inetgral was first derived by Hardy, Landau and Littlewood [24].

Based on Theorem 5.1, Hughes deduced the following theorem.

THEOREM 5.2. *Let D^β denote the β-th Riemann-Liouville fractional derivative acting in the weighted space $L^p_\omega(0, \infty)$, $0 < \beta < \alpha$, and let the weight w satisfy the Muckenhoupt (A_p) condition [27]. Then the following Hardy-Landau-Littlewood inequality is valid:*

$$\|D^{\beta}f\| \leq K(\alpha,\beta,p,\omega)\,\|f\|^{1-\beta/\alpha}\,\|D^{\alpha}f\|^{\beta/\alpha}, \tag{5.2}$$

where $\|f\| = \left(\int_{\mathbb{R}_+} |f|^p \omega\, dx\right)^{1/p}$.

These two theorems are useful in intermediary estimates, for example, for equations with forcing terms containing fractional derivative of order less than the leading derivative in the equation.

Geisberg [23] proved the following inequality for the Marchaud fractional derivative

$$(\mathbf{D}_{+}^{\alpha})(x) = \int_0^{\infty} \frac{f(x)-f(x-t)}{t^{1+\alpha}}\, dt.$$

THEOREM 5.3. *Let $0 < \alpha < 1$. The Marchaud fractional derivative $\boldsymbol{D}_{+}^{\alpha}$ enjoys the following inequality*

$$\|\boldsymbol{D}_{+}^{\alpha}f\|_C \leq K\,\|f\|_C^{1-\alpha/r}\,\|f^{(r)}\|_{\infty}^{\alpha/r}, \tag{5.3}$$

with the usual norm in $C(\mathbb{R})$ in the case $0 < \alpha < r < 1$ for functions $f \in C(\mathbb{R})$, which satisfies the Lipschitz condition of order $\gamma = \gamma(x) > r$, $\|f\|_{\infty} = ess\,sup\{|f(x)|,\, x \in \mathbb{R}\}$.

Other inequalities have been proved by Arestov [7], Babenko and col. [9], [10].

6. Opial inequalities for fractional derivatives

Anastassiou, Koliha and Pečarić in [6] proved a series of Opial inequalities for fractional derivatives to solve fractional differential equations with nonlinearities depending of some fractional derivatives of the unknown. We cite here only three of them.

THEOREM 6.1. *Let $1/p + 1/q = 1$ with $p, q > 1$, let $\gamma \geq 0, \nu > \gamma + 1 - 1/p$, and let $f \in L(0,\, x)$ have an integrable fractional derivative $D^{\nu}f \in L^{\infty}(0,\, x)$, and let $D^{\nu-j}f(0) = 0$ for $j = 1, \ldots, [\nu]+1$. Then*

$$\int_0^x |D^{\gamma}f(s)\, D^{\nu}f(s)|\, ds \leq \Omega(x)\, \left(|D^{\nu}f(s)|^q\, ds\right)^{2/q}, \tag{6.1}$$

where

$$\Omega(x) = \frac{x^{(rp+2)/p}}{2^{1/q}\Gamma(r+1)((rp+1)(rp+2))^{1/p}}, \quad r = \nu - \gamma - 1.$$

THEOREM 6.2. *Let $\nu > \gamma \leq 0$, and let $f \in L(0, x)$ have an integrable fractional derivative $D^\nu f \in L^\infty(0, x)$, and let $D^{\nu-j} f(0) = 0$ for $j = 1, \ldots, [\nu] + 1$. Then*

$$\int_0^x |D^\gamma f(s)\, D^\nu f(s)|\, ds \leq \Omega_1(x)\, \text{ess sup}_{s\in[0,x]} |D^\nu f(s)|^2, \tag{6.2}$$

where

$$\Omega_1(x) = \frac{x^{(r+2)/p}}{\Gamma(r+3)}, \quad r = \nu - \gamma - 1.$$

THEOREM 6.3. *Let $1/p + 1/q = 1$ with $p, q > 1$, let $\gamma \geq 0, \nu > \gamma + 1 - 1/p$, and let $f \in L(0, x)$ have an integrable fractional derivative $D^\nu f \in L^\infty(0, x)$, and let $D^{\nu-j} f(0) = 0$ for $j = 1, \ldots, [\nu] + 1$. Then for any $m > 0$,*

$$\int_0^x |D^\gamma f(s)|^m\, ds \leq \Omega_2(x)\, \left(|D^\nu f(s)|^q\, ds\right)^{m/q}, \tag{6.3}$$

where

$$\Omega_2(x) = \frac{x^{(rm+1+m/p)}}{\Gamma(r+1)^m((rm+1+m/p)(rp+1))^{m/p}}, \quad r = \nu - \gamma - 1.$$

7. The moment inequality of Trebels and Westphal

This section concerns the *moment inequality* for operators.

DEFINITION 7.1. Let A be a closed operator densely defined in the complex Banach space $\mathcal{X}$. The operator A is said to be of type (ω, M) if there exist $0 < \omega < \pi$ and $M \geq 1$ such that $\rho(A) \supset \{\lambda : |arg(\lambda) > \omega|\}$ and $\|\lambda(A-\lambda)^{-1}\| \leq M$ for $\lambda < 0$, and if there exists a number M_ε such that $\|\lambda(A-\lambda)^{-1}\| \leq M_\varepsilon$ holds in $|arg(\lambda)| > \omega + \varepsilon$ for all > 0.

THEOREM 7.1. *Let A be of type (ω, M) and suppose that $0 \in \rho(A)$. For $0 \leq \alpha < \beta \leq 1$, there exists a constant $C_{\alpha,\beta}$ depending only on M, α, and β, such that, for all $u \in D(A^\beta)$,*

$$\|A^\alpha u\| \leq C_{\alpha,\beta}\, \|A^\beta u\|^{\alpha/\beta}\, \|u\|^{1-\alpha/\beta}. \tag{7.1}$$

REMARK 7.1. ([43], p. 39) A more general form of the moment inequality can be described as follows. For any $\alpha < \beta < \gamma$ and for any $u \in D(A^\gamma)$,

$$\|A^\beta u\| \leq C_{\alpha,\beta,\gamma}\, \|A^\gamma u\|^{(\beta-\alpha)/(\gamma-\alpha)}\, \|A^\alpha u\|^{(\gamma-\beta)/(\gamma-\alpha)}. \tag{7.2}$$

For more details, we refer to Krein [33], p. 115.

8. Space-fractional inequalities

The following fractional Gagliardo-Nirenberg inequality is derived by Park [41].

THEOREM 8.1. *Let $m, q, \theta \in \mathbb{R}\backslash\{0\}$ with $q \neq m\theta > 0$, $0 < s < n$, $1 < p < n/s$ and $1 < r/(q_m\theta)$. Then the inequality*

$$\int_{\mathbb{R}^n} |u(x)|^q \, dx \leq C_0 \left(\int_{\mathbb{R}^n} |\Lambda^s u(x)|^p \, dx \right)^{\frac{m\theta}{p}} \left(\int_{\mathbb{R}^n} |u(x)|^r \, dx \right)^{\frac{q-m\theta}{r}} \tag{8.1}$$

holds for

$$m\theta \left(\frac{1}{p} - \frac{s}{n} \right) + \frac{q - m\theta}{r} = 1.$$

The sharp constant satisfies

$$C_0^{\frac{1}{m\theta}} \leq \frac{2^{1-s}}{\pi^{\frac{s}{2}}} \frac{\left[\Gamma(\frac{n}{2}+1)\right]^{\frac{s}{n}} \Gamma(\frac{n-s}{2})}{\Gamma(\frac{s}{2})\Gamma(\frac{n}{2})} \frac{1 - \frac{1}{p} + \frac{s}{n}}{sp} \left(\left[\frac{1-\frac{s}{n}}{1-\frac{1}{p}} \right]^{1-\frac{s}{n}} + \left[\frac{1-\frac{s}{n}}{\frac{1}{p}-\frac{1}{p}} \right]^{1-\frac{s}{n}} \right).$$

In particular, when $m = q$, we have a fractional version of Gagliardo-Nirenberg inequality

$$\left(\int_{\mathbb{R}^n} |u(x)|^q \, dx \right)^{\frac{1}{q}} \leq C_0^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} |\Lambda^s u(x)|^p \, dx \right)^{\frac{\theta}{p}} \left(\int_{\mathbb{R}^n} |u(x)|^r \, dx \right)^{1-\frac{\theta}{r}} \tag{8.2}$$

provided

$$\theta \left(\frac{1}{p} - \frac{s}{n} \right) + \frac{1-\theta}{r} = \frac{1}{q}.$$

COROLLARY 8.1. (Fractional Sobolev inequality) *For $0 < s < n$, $1 < p < \frac{n}{s}$ and $q = \frac{np}{n-ps}$, we have*

$$\|u\|_{L^q} \leq C_0^{\frac{1}{q}} \|\Lambda^s u\|_{L^p}. \tag{8.3}$$

The sharp constant for the inequality for $p = \frac{2n}{n+s}$ and $q = \frac{2n}{n-s}$ is

$$\left[\frac{\pi^{\frac{n}{2}-s}\Gamma(\frac{n}{q})}{2^s} \frac{\Gamma(\frac{n}{q})}{\Gamma(\frac{n}{p})} \left\{ \frac{\Gamma(n)}{\Gamma(\frac{n}{2})} \right\}^{\frac{s}{n}} \right]^{\frac{1}{q}}.$$

Mitrovic derived in [38] the following inequality.

THEOREM 8.2. *Let $v \in C_c^\infty(\mathbb{R}^d)$, $u \in L^2 \cap L^1(\mathbb{R}^d)$, $\alpha > 0, \alpha$ not in $\mathbb{N}$, and $k \in \mathbb{N}$ such that $\alpha - k > 0$ and $\alpha - k - 1 < 0$. Assume that $D_{x_i}^{\alpha-m} \in L^2 \cap L^1(\mathbb{R}^d)$ for every $i \in \{1, \ldots, d\}$ and $m = 0, 1, \ldots, k+1$. Then there exists a positive constant C such that for every $M > 0$,*

$$\begin{aligned}\|D_{x_i}^{\alpha}(uv)\|_2^2 \leq C\Big(\| \sum_{m=0}^{k} D_{x_i}^{m} v D_{x_i}^{\alpha-m} u\|_2^2 + \|D_{x_i}^{\alpha-k} u\|_2^2 \|\xi_i^k \hat{v}\|_1^2 \\ + M^{d+\alpha-k-1}\|u\|_1^2 \|D_{x_i}^{k+1} v\|_2^2 + M^{2(d+\alpha-k-1)}\|u\|_2^2 \|\xi_i^{k+1}\hat{v}\|_1^2\Big),\end{aligned} \tag{8.4}$$

where $\| \,.\, \|_p = \| \,.\, \|_{L^p}(\mathbb{R}^d)$, $\hat{v}$ *is the Fourier transform of* v.

9. **Kato and Ponce type inequalities**

In [28], Kato and Ponce obtained the following commutator estimate that proved to be very useful in partial differential equations.

THEOREM 9.1.

$$\|J^s(fg) - f(J^s(g))\|_p \leq C\left[\|\nabla f\|_\infty \|J^{s-1} g\|_p + \|J^s(f)\|_p \|g\|_\infty\right] \tag{9.1}$$

for $1 < p < \infty$ *and* $s > 0$, *where* $J^s := (I - \Delta)^{s/2}$ *is the Bessel potential,* ∇ *is the n-dimensional gradient,* f, g *are Schwartz functions, and* C *is a constant depending on* n, p *and* s, $\|.\|_p$ *is the norm of* $L^p(\mathbf{R}^n)$, $1 \leq p \leq \infty$.

Using the homogeneous symbol $D^s := (-\Delta)^{s/2}$, Kenig, Ponce and Vega [30] obtained the following estimate.

THEOREM 9.2.

$$\|D^s(fg) - fD^s g - gD^s f\|_r \leq C\|D^s f\|_p \|D^s g\|_q, \tag{9.2}$$

where $C = C(s, s_1, s_2, r, p, q), s = s_1 + s_2$ *for* $s, s_1, s_2 \in (0, 1)$, *and* $1 < p, q, r < \infty$ *such that* $1/r = 1/p + 1/q$.

An other variant of the Kato and Ponce inequality known also as *fractional Leibniz rule* is given by the following theorem.

THEOREM 9.3.

$$\|J^s(fg)\|_r \leq C\left[\|f\|_{p_1} \|J^s g\|_{q_1} + \|J^s(f)\|_{p_2} \|g\|_{q_2}\right] \tag{9.3}$$

where $s > 0$ *and* $1/r = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$ *for* $1 < r < \infty, 1 < p_1, p_2, q_1, q_2 \leq \infty$ *and* $C = C(s, n, r, p_1, p_2, q_1, q_2)$.

10. **Fractional integral inequalities**

The fractional Chebyshev type inequalities started with a paper by Belarbi and Dahmani [11]; they derived the following inequality.

Theorem 10.1. *Let f and g be defined on $[0, \infty)$ such that for all $\tau \geq 0, \rho \geq 0$, $(f(\tau) - f(\rho))((g(\tau) - g(\rho)) \geq 0$ (in this case f and g are said synchronous), then*

$$J^{\alpha}(fg)(t) \geq \frac{\Gamma(\alpha+1)}{t^{\alpha}} J^{\alpha}(f)(t) J^{\alpha}(g)(t), \tag{10.1}$$

where $J^{\alpha}(f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau$, $\alpha > 0, , t > 0$.

Remark 10.1. The inequality (10.1) is reversed if the functions f and g are asynchronous.

They also proved the following result.

Theorem 10.2. *Let f and g be defined on $[0, \infty)$ such that f is increasing, g is differentiable with bounded derivative, $m := \min_{t \geq 0} g'(t)$, $M := \max_{t \geq 0} g'(t)$, then*

$$J^{\alpha}(fg)(t) \geq \left(J^{\alpha}(1)\right)^{-1} J^{\alpha}(f)(t) J^{\alpha}(g)(t) - \frac{mt}{\alpha+1} J^{\alpha}(f)(t) + M J^{\alpha}(tf)(t). \tag{10.2}$$

Many variants then appeared. Here after, one concerning the Hadamard fractional integral.

The Hadamard fractional integral of order $\alpha > 0$ of a function $f(t)$, for all $t > 1$, is defined as

$${}_H J^{\alpha}(f)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(ln\left(\frac{t}{\tau}\right)\right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau}.$$

Theorem 10.3. *Let p be a positive function and let f and g be two differentiable functions on $[1, \infty)$. If $f' \in L^r([1, \infty)), g' \in L^s([1, \infty)), r > 1, r + s = rs$, then for all $t > 1$ and $\alpha > 0, \beta > 0$*

$$\begin{aligned} &|{}_H J^{\alpha}(p(t))\, {}_H J^{\beta}(p(t)f(t)g(t)) - {}_H J^{\alpha}(p(t)f(t))\, {}_H J^{\beta}(p(t)g(t)) \\ &+ {}_H J^{\beta}(p(t))\, {}_H J^{\alpha}(p(t)f(t)g(t)) - {}_H J^{\beta}(p(t)f(t))\, {}_H J^{\alpha}(p(t)g(t)) \\ &\leq \|f'\|_r \|g'\|_s t \left({}_H J^{\alpha}(p(t))\right) \left({}_H J^{\beta}(p(t))\right). \end{aligned} \tag{10.3}$$

11. Singular integral inequalities

Medved [36] (see also [32]) obtained the following inequality that served for nonlinear singular integral equalities and for parabolic equations.

Let $u(t)$ satisfy the integral inequality

$$u(t)^r \le a(t) + \int_0^t (t-s)^{\beta-1} F(s)\omega(u(s))\, ds. \tag{11.1}$$

THEOREM 11.1. *Let $a(t) \ge 0$ be a nondecreasing C^1-function on $[0, T]$ ($0 < T < \infty$), let $F(t) \ge 0$ be continuous on $[0, T], 0 < \beta < 1, r \ge 1$, and let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous, nondecreasing, positive function. Assume that $u(t) \ge 0$ is a continuous function on $[0, T]$ satisfying the inequality* (11.1). *Then*

$$\mathcal{G}_{qr}(u(t)^{qr}) \le \mathcal{G}_{qr}(2^{q-1}a^q) + K_q \int_0^t e^{-qs} F(s)^q\, ds, \tag{11.2}$$

or

$$u(t) \le \left\{ \mathcal{G}_{qr}^{-1} \left[\mathcal{G}_{qr}(2^{q-1}a^q) + K_q \int_0^t e^{-qs} F(s)^q\, ds \right] \right\}^{1/qr}, \tag{11.3}$$

for $0 \le t \le T_1 \le T$, where $\beta = 1/(1+z), z > 0, q = (1/\beta) + \varepsilon = 1 + z + \varepsilon$, $p = (1+z+\varepsilon)/(z+\varepsilon),\ \ \varepsilon > 0$,

$$\mathcal{G}_{qr}(v) = \int_{v_0}^{v} \frac{d\sigma}{\omega(\sigma^{1/rq})^q},$$

$2^{q-1}a(0)^q \ge v_0 > 0, \mathcal{G}_{qr}^{-1}$ is the inverse of $\mathcal{G}_{qr}, a = a(t)$

$$K_q = \frac{2^{q-1}e^{pT}}{p^{1-\alpha p}}\Gamma(1-\alpha p),$$

$\alpha = 1 - \beta = z/(1+z), \Gamma$ is the Eurelian gamma function, and $T_1 > 0$ is such that

$$\mathcal{G}_{qr}(2^{q-1}a^q) + K_q \int_0^t e^{-qs} F(s)^q\, ds \in Dom(\mathcal{G}_{qr}^{-1}),\ \ t \in [0, T_1].$$

A modified version has been proved by Ma and Pecaric [35].

THEOREM 11.2. *Let $u(t), a(t), b(t)$ and $f(t)$ be nonnegative continuous functions for $t > 0$. Let p and q be constants with $p \ge q \ge 0$. If $u(t)$ satisfies*

$$u^p(t) \le a(t) + b(t) \int_0^t (t^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1} f(s) u^q(s) ds,\ t > 0. \tag{11.4}$$

Then for any $K > 0$ we have:

(i) *if* $\alpha \in (0,1], \beta \in (1/2,1), \gamma \geq 3/2-\beta$,

$$u(t) \leq \Big\{a(t)+M_1^{\beta} t^{(\alpha+1)(\beta-1)+\gamma} b(t)\Big[A_1^{1-\beta}(t)+K^{\frac{q-p}{p}} M_1^{\beta}[1-(1-V_1(t))^{1-\beta}]^{-1} \times\Big(\int_0^t s^{\frac{(\alpha+1)(\beta-1)+\gamma}{1-\beta}} f^{\frac{1}{1-\beta}}(s) b^{\frac{1}{1-\beta}}(s) A_1(s) V_1(s)\, ds\Big)^{1-\beta}\Big]\Big\}^{\frac{1}{p}}, \tag{11.5}$$

where

$$M_1 = \frac{1}{\alpha} B\left[\frac{\beta+\gamma-1}{\alpha\beta}, \frac{2\beta-1}{\beta}\right],$$

$$A(t) = \frac{q}{p} K^{\frac{q-p}{p}} a(t) + \frac{p-q}{p} K^{\frac{q}{p}},$$

$$A_1(t) = \int_0^t f^{\frac{1}{1-\beta}}(s) A^{\frac{1}{1-\beta}}(s)\, ds,$$

$$V_1(t) = \exp\left(-K^{\frac{p-q}{p(1-\beta)}} M_1^{\frac{\beta}{1-\beta}} \int_0^t s^{\frac{(\alpha+1)(\beta-1)+\gamma}{1-\beta}} f^{\frac{1}{1-\beta}}(s) b^{\frac{1}{1-\beta}}(s) ds\right),$$

and

$$B[\sigma;\, \eta] = \int_0^1 s^{\sigma-1}(1-s)^{\eta-1}\, ds.$$

(ii) *if* $\alpha \in (0,1], \beta \in (0,1/2], \gamma > (1-2\beta^2)/(1-\beta^2)$, *then*

$$u(t) \leq \left\{a(t) + M_2^{\frac{1+3\beta}{1+4\beta}} t^{\frac{[\alpha(\beta-1)+\gamma](1+4\beta)-\beta}{1+4\beta}} b(t)\left[A_2^{\frac{\beta}{1+4\beta}}(t) + K^{\frac{q-p}{p}} M_2^{\frac{1+3\beta}{1+4\beta}} \times [1–(1–V_2)^{\frac{\beta}{1+4\beta}}]^{-1}\left(\int_0^t s^{\frac{[\alpha(\beta-1)+\gamma](1+4\beta)-\beta}{\beta}} f^{\frac{1+4\beta}{\beta}}(s) b^{\frac{1+4\beta}{\beta}}(s) A_2 V_2\, ds\right)^{\frac{\beta}{1+4\beta}}\right]\right\}^{\frac{1}{p}}, \tag{11.6}$$

where

$$M_2 = \frac{1}{\alpha} B\left[\frac{\gamma(1+4\beta)-\beta}{\alpha(1+3\beta)}, \frac{4\beta^2}{1+3\beta}\right],$$

$$A_2(t) = \int_0^t f^{\frac{1+4\beta}{\beta}}(s) A^{\frac{1+4\beta}{\beta}}(s) ds$$

and

$$V_2(t) = \exp\left(-K^{\frac{(q-p)(1+4\beta)}{p\beta}} M_2^{\frac{1+3\beta}{\beta}} \int_0^t s^{\frac{[\alpha(\beta-1)+\gamma](1+4\beta)-\beta}{\beta}} f^{\frac{1+4\beta}{\beta}}(s) b^{\frac{1+4\beta}{\beta}}(s) ds\right).$$

Thiramanus, Tariboon and Ntouyas [44] obtained the following result.

THEOREM 11.3. *Suppose that the following conditions are satisfied:*

(H_1) *The functions p and $r \in C([t_0,T),\mathbb{R}_+)$.*

(H_2) *The function $\phi \in C([\beta t_0, t_0], \mathbb{R}_+)$ with $\max_{s\in[\beta t_0,t_0]} \phi(s) > 0$, where $0<\beta<1$.*

(H_3) *The function $u \in C([\beta t_0, T), \mathbb{R}_+)$ with*

$$u(t) \le r(t) + \int_{t_0}^{t} (t-s)^{\alpha-1} p(s) \max_{\xi\in[\beta s,s]} u(\xi) ds, \quad t\in[t_0,T), \tag{11.7}$$

$$u(t) \le \phi(t), \quad t \in [\beta t_0, t_0], \quad \text{where } \alpha > 0. \tag{11.8}$$

Then the following assertions hold:

(R_1) *Suppose $\alpha > \frac{1}{2}$, then*

$$u(t) \le e^t \left[c_1 r^2(t) + h_1(t) \exp\left(K_1 \int_{t_0}^{t} p^2(s) ds \right) \right]^{\frac{1}{2}}, \tag{11.9}$$

for $t\in[t_0,T)$, where

$$c_1 = \max\left\{2e^{-2t_0}, e^{-2\beta t_0}\right\}, \tag{11.10}$$

and

$$K_1 = \frac{\Gamma(2\alpha-1)}{4^{\alpha-1}}, \tag{11.11}$$

$$h_1(t) = c_1 \max_{s\in[\beta t_0,t_0]} \phi^2(s) + c_1 K_1 \int_{t_0}^{t} p^2(s) \max_{\xi\in[\beta s,s]} m_1^2(\xi) ds, \tag{11.12}$$

for $t\in[t_0,T)$, with

$$m_1(t) = \begin{cases} r(t), & t\in[t_0,T), \\ \phi(t), & t\in[\beta t_0, t_0]. \end{cases} \tag{11.13}$$

Moreover, if $r\in C([t_0,T),(0,\infty))$ is a nondecreasing function, then

$$u(t) \le \sqrt{c_1 N_1}\, r(t) \exp\left(t + \frac{1}{2} K_1 \int_{t_0}^{t} p^2(s) ds \right), \tag{11.14}$$

for $t\in[t_0,T)$, where

$$N_1 = \max\left\{1, \frac{\max_{s\in[\beta t_0,t_0]} \phi^2(s)}{r^2(t_0)}\right\}. \tag{11.15}$$

(R_2) *Suppose $0<\alpha\le\frac{1}{2}$, then*

$$u(t) \le e^t \left[c_2 r^b(t) + h_2(t) \exp\left(2^{b-1} K_2^b \int_{t_0}^{t} p^b(s) ds \right) \right]^{\frac{1}{b}}, \tag{11.16}$$

for $t\in[t_0,T)$, where

$$\begin{aligned} a &= \alpha+1, && (11.17)\\ b &= 1+\frac{1}{\alpha}, && (11.18)\\ c_2 &= \max\left\{2^{b-1}e^{-bt_0}, e^{-b\beta t_0}\right\}, && (11.19)\\ K_2 &= \left(\frac{\Gamma\left(\alpha^2\right)}{a^{\alpha^2}}\right)^{\frac{1}{a}}, && (11.20) \end{aligned}$$

and

$$h_2(t) = c_2 \max_{s\in[\beta t_0,t_0]} \phi^b(s) + 2^{b-1}c_2K_2^b \int_{t_0}^t p^b(s) \max_{\xi\in[\beta s,s]} m_1^b(\xi)\, ds, \quad (11.21)$$

for $t \in [t_0, T)$*. Moreover, if* $r \in C([t_0,T),(0,\infty))$ *is a nondecreasing function, then*

$$u(t) \le (c_2N_2)^{\frac{1}{b}} r(t) \exp\left(t + \frac{2^{b-1}}{b}K_2^b \int_{t_0}^t p^b(s)ds\right), \quad (11.22)$$

for $t \in [t_0, T)$*, where*

$$N_2 = \max\left\{1, \frac{\max_{s\in[\beta t_0,t_0]} \phi^b(s)}{r^b(t_0)}\right\}. \quad (11.23)$$

Lin [34] obtained the following result.

THEOREM 11.4. *Suppose* $u(t)$ *satisfies the inequality*

$$u(t) \le a(t) + \sum_{i=1}^n b_i(t) \int_0^t (t-s)^{\beta_i-1}u(s)\, ds, \quad t\in[0,T), \quad (11.24)$$

where all functions are nonnegative and continuous. The constants $\beta_i > 0$, $b_i(i = 1,2,...,n)$ *are bounded and monotonic increasing functions on* $[0,T)$*, then*

$$u(t) \le a(t)+\sum_{k=1}^{\infty} \left(\sum_{1',2',\dots,k'=1}^{n} \frac{\prod_{i=1}^k [b_{i'}(t)\Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^k \beta_{i'})} \int_0^t (t-s)^{\sum_{i=1}^k \beta_{i'}-1} a(s)\, ds \right). \quad (11.25)$$

THEOREM 11.5. *Suppose that* $u(t)$ *satisfies the inequality*

$$u(t) \le a(t) + \sum_{i=1}^n b_i(t) \int_1^t \left(\ln\frac{t}{s}\right)^{\beta_i-1} \frac{u(s)}{s}\, ds, \quad t\in[1,T), \quad (11.26)$$

where all functions are nonnegative and continuous. The constants $\beta_i > 0$, $b_i(i = 1,2,...,n)$ *are bounded and monotonic increasing functions on* $[1,T)$*, then*

$$u(t) \le a(t) + \sum_{k=1}^{\infty} \Big(\sum_{1',2',\dots,k'=1}^{n} \frac{\prod_{i=1}^{k} [b_{i'}(t)\Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k} \beta_{i'})} \times \int_1^t \Big(\Big(\ln \frac{t}{s} \Big)^{\sum_{i=1}^{k} \beta_{i'}-1} a(s) \Big) \frac{ds}{s} \Big). \tag{11.27}$$

Lin [34] applied his results in the resolution of multi-fractional derivatives problems.

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[1,2] *NAAM Research Group, Department of Mathematics*
Faculty of Science, King Abdulaziz University
P.O. Box 80203, Jeddah 21589, SAUDI ARABIA
e-mails: [1] *aalsaedi@hotmail.com,* [2] *bashirahmad_qau@yahoo.com*

[3] *LaSIE, Pôle Sciences et Technologies, Université de La Rochelle*
Avenue M. Crépeau 17042, La Rochelle Cedex, FRANCE
and
NAAM Research Group, Department of Mathematics
Faculty of Science, King Abdulaziz University
P.O. Box 80203, Jeddah 21589, SAUDI ARABIA
e-mail: mkirane@univ-lr.fr
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