A survey on enhanced power graphs of finite groups

Xuanlong Ma[a], Andrei Kelarev[b], Yuqing Lin[b], Kaishun Wang[c]

[a]*School of Science, Xi'an Shiyou University, Xi'an 710065, P.R. China*
[b]*Discipline of Computing & Information Technology, School of Information & Phys. Sci., College of Engineering, Science and Environment, The University of Newcastle, Callaghan, 2308, NSW, Australia*
[c]*Lab. of Math. & Complex Systems (MOE), School of Math. Sci., Beijing Normal University, Beijing 100875, P.R. China*

xuanlma@mail.bnu.edu.cn, andrei.kelarev@gmail.com, yuqing.lin@newcastle.edu.au, wangks@bnu.edu.cn

Abstract

We survey known results on enhanced power graphs of finite groups. Open problems, questions and suggestions for future work are also included.

1. Introduction

The study of graphical representations of algebraic structures, especially groups, has been an energizing and fascinating research area originating from the pioneering paper by Arthur Cayley [26] with many recent results (cf. [11, 14, 40, 54, 79]). In particular, graphs associated to groups and other algebraic constructions have valuable applications (cf. [47, 53]) and are related to automata theory (cf. [48]). The majority of these types of graphs have natural associated labelings. Therefore, this direction can also be considered as a part of the broader field of research – the investigation of graph labelings. For more information, refer to examples of publications devoted to graph labelings ([10, 46, 57, 66, 71]).

The power graphs of groups are a fairly recent development in the realm of graphs from groups. The power graph of a group was introduced by Kelarev and Quinn [49]. For a group G, the *directed power graph* of G is denoted by $\overrightarrow{\mathcal{P}}(G)$ and is defined as a digraph with vertex set G and there is a directed edge from x to y, for $x, y \in G$, $x \neq y$, if and only if y is a power of x, i.e., $y = x^k$ for some integer k, which is also equivalent to saying that y belongs to the subgroup $\langle x \rangle$ generated by x in G. The power graphs of semigroups were investigated in [50, 51, 52]. All the above papers [49, 50, 51, 52] use the brief term "power graphs" to refer to the directed power graphs.

In 2009, Chakrabarty et al. [27] explored the undirected power graphs of semigroups. The *undirected power graph* $\mathcal{P}(G)$ of a group G coincides with the underlying undirected graph of the directed power graph defined in [49]. It has the vertex set G where two distinct elements are adjacent if one of them is a power of the other. Cameron et al. [22, 24] examined the undirected power graphs of finite groups and initiated the use of the brief term "power graph" with the second meaning of an undirected power graph.

Notice that the survey [2] introduced the most general definition of a power graph defined for all power-associative *magmas*, i.e., sets with a power-associative operation (Definition 1 on p. 126 in [2]). It is well known that magmas play essential roles in symbolic computation (cf. [76, 77, 78]). Let us refer to two surveys [2, 55] for more information pertaining to the research results and open problems on the power graphs of groups.

Enhanced power graphs are another important class of graphs that have been actively investigated recently. The term *enhanced power graph* of a group was introduced by Aalipour et al. [1] as a graph "lying in between" the power graph and the commuting graph, where the *commuting graph* of a group G, denoted by $\mathcal{C}(G)$, is the graph whose vertex set is G, and two distinct elements x, y are adjacent if $xy = yx$. The *enhanced power graph* of a group G is denoted by $\mathcal{P}_e(G)$ and is defined as a simple graph with vertex set consisting of all elements of G, where two distinct vertices x, y are adjacent if and only if $\langle x, y \rangle$ is a cyclic subgroup of G. The commuting graph was introduced by Brauer and Fowler in their seminal paper [21] establishing that only finitely many groups of even order can have a prescribed centraliser. The commuting graph is the complement of the *non-commuting* graph, first considered by Paul Erdös (cf. [65] and [40, 79]). Likewise, the enhanced power graph is the complement of the noncyclic graph, introduced in [3, 4]. A *noncyclic graph* of a group G is the graph, where two vertices x and y are adjacent if $\langle x, y \rangle$ is noncyclic. For technical reasons, the papers [3, 4] excluded isolated vertices from consideration. Several papers also call the enhanced power graph using the terms *cyclic graph* or the *deleted enhanced power graph*, if the vertex corresponding to the identity of the group is deleted (cf. [31, 32, 61]).

Clearly, for any group G, we have $E(\mathcal{P}(G)) \subseteq E(\mathcal{P}_e(G)) \subseteq E(\mathcal{C}(G))$. As a concrete example, we display the power graph and the enhanced power graph of abelian group $\mathbb{Z}_2 \times \mathbb{Z}_6$ in Fig. 1. Note that $\mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_6)$ is a complete graph.

Many researchers have contributed to the understanding of enhanced power graphs of groups, especially after the publication of [1], and many interesting new theorems have appeared in the literature recently. The present article is a survey of the results on enhanced power graphs of finite groups. Various questions, open problems and suggestions for future work are also included. This is the first survey devoted to the enhanced power graphs of finite groups.
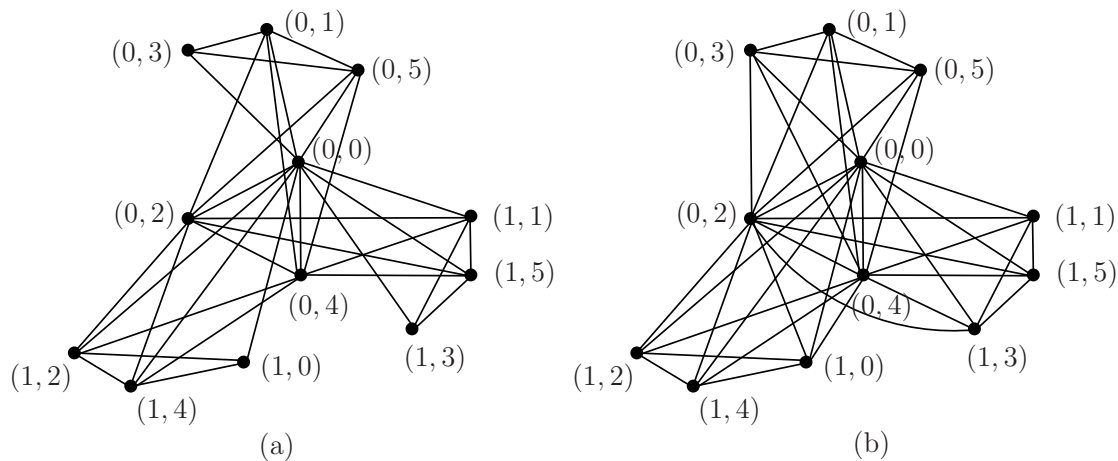
Figure 1. (a) $\mathcal{P}(\mathbb{Z}_2 \times \mathbb{Z}_6)$; (b) $\mathcal{P}_e(\mathbb{Z}_2 \times \mathbb{Z}_6)$.

1.1. Outline of the paper

This survey is divided into the following sections. Section 2 presents the results on the groups with dominatable enhanced power graphs, which derive from an open problem in [15]. Section 3 describes the metric dimension and strong metric dimension of an enhanced power graph. The exact values of these dimensions are computed for the enhanced power graphs of abelian groups, dihedral, semidihedral groups, and generalized quaternion groups. Section 4 deals with the perfect enhanced power graphs of groups. It gives a complete characterization of nilpotent groups that have perfect enhanced power graphs and also gives some groups whose enhanced power graphs are perfect. Section 5 is devoted to the forbidden subgraphs of enhanced power graphs. It presents a classification of all finite groups whose enhanced power graphs are split and threshold, and also classifies all finite nilpotent groups whose enhanced power graphs are chordal graphs and cographs. Several families of non-nilpotent groups whose enhanced power graphs are chordal graphs and cographs are also included. Section 6 contains the results on enhanced power graphs and other related graphs (such as power graphs and commuting graphs). Section 7 presents relevant results related to miscellaneous properties of the enhanced power graphs, including vertex connectivity, diameter, connectedness, clique number, Eulerian graphs, and so on. Section 8 deals with open problems, questions and conjectures on enhanced power graphs. The paper concludes with a brief overview of other related work supplementing the presented results.

1.2. Terminology and notation

A *graph* is a pair (V, E), where V is a set with elements called *vertices* and E is a set of *edges*, i.e., unordered pairs of vertices. As usual, for a graph Γ, we use $V(\Gamma)$ and $E(\Gamma)$ to denote its vertex and edge sets, respectively. The cardinality $|V(\Gamma)|$ is called the *order* of Γ. The *degree* of u in Γ, denoted by $d_\Gamma(u) = d(u)$, is defined as the number of edges of Γ incident to u. A graph is *simple* if it has no loops or parallel edges.

Likewise, a *digraph* can be defined by specifying a set of vertices and a collection of ordered pairs of vertices. Each ordered pair (u, v) in the collection is called an *arc* (or a *directed edge*) from

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u to v. The *underlying graph* of a digraph is the simple undirected graph obtained by replacing each arc by an edge with the same end-vertices.

All groups considered in this survey are finite. Throughout the paper, unless stated otherwise, the word 'graph' means a finite undirected graph without loops and multiple edges. The *distance* between two vertices u and v in a graph Γ, denoted by $d_\Gamma(u,v) = d(u,v)$, is defined as the length of a shortest path between them in Γ. Note that $d_\Gamma(u,v) = 0$ if $u = v$. The *diameter* of Γ, denoted by $\mathrm{diam}(\Gamma)$, is the maximum distance between two vertices of G. The *strong product* $\Gamma \boxtimes \Delta$ of two graphs Γ and Δ is the graph with vertex set $V(\Gamma) \times V(\Delta)$ in which (u,v) and (x,y) are adjacent if and only if one of $u = x$ and $vy \in E(H)$, $v = y$ and $ux \in E(G)$, and $ux \in E(G)$ and $vy \in E(H)$ holds. An *automorphism* of a graph Γ is a permutation of $V(\Gamma)$ which maps adjacent vertices to adjacent vertices and nonadjacent vertices to nonadjacent vertices. The *automorphism group* of Γ, denoted by $\mathrm{Aut}(\Gamma)$, is the group of automorphisms of Γ under the usual composition of permutations. The vertex connectivity of a graph Γ, denoted by $\kappa(\Gamma)$, is the minimum number of vertices which need to be removed from the vertex set of Γ so that the induced subgraph of Γ on the remaining vertices is disconnected.

Let Γ and Δ be graphs. The graph Γ is said to be Δ-*free* if Γ has no induced subgraph isomorphic to Δ. A graph is a *star* if it is a tree with n vertices in which one vertex has degree $n - 1$ and every other vertex has degree 1. Denote by P_n, C_n, and K_n the path with n vertices, the cycle of length n, and the complete graph of order n, respectively. The symbol $2K_2$ denotes the bipartite graph with four vertices and two disjoint edges.

An *edge colouring* or *edge labeling* of Γ is an assignment of some colours or labels to the edges of Γ. An edge colouring of a graph is called a *rainbow colouring* if every pair of distinct vertices of the graph is connected by at least one path with no two edges sharing the same colour. The *rainbow connection number* of Γ, denoted by $\mathrm{rc}(\Gamma)$, is the minimum positive integer k for which there exists a rainbow colouring with k colours in Γ. A vertex of a graph Γ is called a *dominating vertex* if this vertex is adjacent to every other vertex of Γ. The vertex set of Γ is denoted by $V(\Gamma)$, and the edge set is $E(\Gamma)$. The *distance* between two vertices x and y in Γ, denoted $d_\Gamma(x,y)$, is the length of a shortest path from x to y. If the situation is unambiguous, we denote $d_\Gamma(x,y)$ simply by $d(x,y)$. The greatest distance between any two vertices in Γ is called the *diameter* of Γ.

A subset of $V(\Gamma)$ is called a *clique* if any two distinct vertices in this subset are adjacent in Γ. The *clique number* of Γ, denoted by $\omega(\Gamma)$, is the maximum cardinality of a clique in Γ. The *chromatic number* of a graph Γ, denoted by $\chi(\Gamma)$, is the smallest number of colours for $V(\Gamma)$ so that adjacent vertices are coloured differently. A graph is *perfect* if every induced subgraph has clique number equal to chromatic number. A graph Γ is called *weakly perfect* if $\chi(\Gamma) = \omega(\Gamma)$.

We always use G to denote a finite group, and let e denote its identity element. The *order* $o(g)$ of an element g of G is the cardinality of the cyclic subgroup $\langle g \rangle$. The set of orders of elements of G is denoted by $\pi_e(G)$. A *maximal cyclic subgroup* of G is a cyclic subgroup, which is not a proper subgroup of any other cyclic subgroup of G. Denote by $\mathcal{M}_G$ the set of all maximal cyclic subgroups of G. Note that $|\mathcal{M}_G| = 1$ if and only if G is cyclic. The cyclic group of order n is denoted by $\mathbb{Z}_n$. A group G is said to be *nilpotent* if G has an upper central series that terminates with G. Equivalently, its central series is of finite length or its lower central series terminates with $\{e\}$. It is well known that all finite p-groups and abelian groups are nilpotent. It is well known that each finite nilpotent group is isomorphic to the direct product of its Sylow p-subgroups.

The *alternating group* of degree n, denoted by A_n, is the group of even permutations of a set of n elements. For $n \geq 3$, the *dihedral group* of order $2n$ is denoted by D_{2n}. Recall that it is defined by the presentation

$$D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle.$$

Note that $D_{2n} = \langle a \rangle \cup \{b, ab, a^2b, \ldots, a^{n-1}b\}$, $o(a^ib) = 2$ for any $1 \leq i \leq n$, and

$$\mathcal{M}_{D_{2n}} = \{\langle a \rangle, \langle ab \rangle, \langle a^2b \rangle, \ldots, \langle a^nb \rangle\}.$$

For $m \geq 2$, the generalized quaternion group Q_{4m} of order $4m$ is given by

$$Q_{4m} = \langle x, y : x^m = y^2, x^{2m} = e, y^{-1}xy = x^{-1} \rangle.$$

For any $i \leq 1 \leq m$ and any $x, y \in Q_{4m}$, it is well known that $o(x^m) = 2$, $o(x^iy) = 4$, $x^m \in \cap_{M \in \mathcal{M}_{Q_{4m}}} M$, and $\mathcal{M}_{Q_{4m}} = \{\langle x \rangle, \langle xy \rangle, \langle x^2y \rangle, \ldots, \langle x^my \rangle\}$.

2. Dominatable enhanced power graphs

The concept of domination is very important in graph theory. Recently, it has been considered, for example, in [13, 18, 37, 42, 45, 58, 68]. It is obvious that the identity element e is a dominating vertex of every enhanced power graph $\mathcal{P}_e(G)$. Bera and Bhuniya [15] defined a *dominatable enhanced power graph* as an enhanced power graph with a dominating vertex other than e. For example, $\mathcal{P}_e(\mathbb{Z}_2 \times \mathbb{Z}_6)$ is a dominatable, since $(0, 2)$ is a dominating vertex (see Fig. 1).

In [15], the authors characterized all abelian groups and all p-groups G such that $\mathcal{P}_e(G)$ is dominatable.

Theorem 2.1. ([15, Theorem 3.1]) *Let G be a group and $n \in \mathbb{N}$. If $\gcd(|G|, n) = 1$, then the enhanced power graph $\mathcal{P}_e(G \times \mathbb{Z}_n)$ is dominatable.*

Theorem 2.2. ([15, Theorem 3.2]) *Let G be a abelian group. Then $\mathcal{P}_e(G)$ is dominatable if and only if G has a cyclic Sylow subgroup.*

Theorem 2.3. ([15, Theorem 3.3]) *Let G be a non-abelian p-group. Then the enhanced power graph $\mathcal{P}_e(G)$ is dominatable if and only if G is a generalized quaternion group.*

Ma and She [59] gave a characterization of dominatable enhanced power graphs using the following concepts. For $x \in G$, the *cyclicizer* [70] of x in G, denoted by $\mathrm{Cyc}(x)$, is defined by

$$\mathrm{Cyc}(x) = \{y \in G : \langle x, y \rangle \text{ is cyclic}\}.$$

The *cycel* of G, denoted by $K(G)$, is defined by $K(G) = \cap_{x \in G} \mathrm{Cyc}(x)$. By [67, Theorem 2], $K(G) = \cap_{M \in \mathcal{M}_G} M$ is a normal subgroup of G.

Theorem 2.4. ([59, Theorem 4.2]) *Let G be a group. Then $\mathcal{P}_e(G)$ is dominatable if and only if $K(G)$ is nontrivial.*

Since $K(G)$ is a normal subgroup of G, the above theorem yields the following result.

Corollary 2.1. ([15, Theorem 3.4]) *Let G be a non-abelian simple group. Then $\mathcal{P}_e(G)$ is not dominatable.*

In [59], the authors characterized all finite nilpotent groups with dominatable enhanced power graphs. This result extends [15, Theorems 3.2 and 3.3].

Theorem 2.5. ([59, Proposition 4.5]) *Let G be a nilpotent group. Then $\mathcal{P}_e(G)$ is dominatable if and only if*

$$G \cong Q_{4 \cdot 2^n} \times H \text{ or } \mathbb{Z}_{p^n} \times K,$$

where $n \geq 1$, p is a prime, and H and K are nilpotent groups with $2 \nmid |H|$ and $p \nmid |K|$.

Theorem 2.6. ([59, Proposition 4.7]) (i) *For $n \geq 3$, $\mathcal{P}_e(D_{2n})$ is not dominatable.*
(ii) *For $m \geq 2$, $\mathcal{P}_e(Q_{4m})$ is dominatable.*

Recently, Mahmoudifar & Babai [63] gave another characterization for finite groups with dominatable enhanced power graphs.

Theorem 2.7. ([63, Main theorem]) *Let G be a finite group. Then $\mathcal{P}_e(G)$ is dominatable if and only if there exists a prime p such that $p \mid |Z(G)|$ and every Sylow p-subgroup of G is isomorphic to either a cyclic group or a generalized quaternion group, where $Z(G)$ is the center of G.*

The above result implies that if G is a group with $|Z(G)| = 1$, then $\mathcal{P}_e(G)$ is not dominatable.

Theorem 2.8. ([63, Corollary 3.17]) *Suppose that G is a non-abelian group of order pq, where p and q are distinct primes. Then $\mathcal{P}_e(G)$ is not dominatable.*

Theorem 2.9. ([63, Theorem 3.18]) *Suppose that $G = G_1 \times G_2 \times \cdots \times G_r$ with $\gcd(|G_i|, |G_j|) = 1$ for distinct indices i and j. Then $\mathcal{P}_e(G)$ is dominatable if and only if there exists $k \in \{1, 2, \ldots, r\}$ such that $\mathcal{P}_e(G_k)$ is dominatable.*

It was noted in [63] that if G is a finite group such that all its proper subgroups are dominatable, then $\mathcal{P}_e(G)$ is not necessarily dominatable. Also, if there exists a nontrivial normal subgroup N of G such that both N and the quotient group G/N are dominatable, then $\mathcal{P}_e(G)$ is not necessarily dominatable. Moreover, $\mathcal{P}_e(G)$ is dominatable if G is isomorphic to the special linear group $\mathrm{SL}(2, q)$, where $q \geq 3$ is a natural power of an odd prime.

Theorem 2.10. ([63, Corollary 3.16]) *If G is one of the following groups:*
(a) *A non-abelian simple group;*
(b) *The symmetric group S_n, where $n \geq 3$;*
(c) *The projective general linear group $\mathrm{PGL}(n, q)$, where $n \geq 2$;*
(d) *A Frobenius group,*
then $\mathcal{P}_e(G)$ is not dominatable.

3. Metric dimension and strong metric dimension of enhanced power graphs

Metric dimension was introduced by Harary and Melter [44] and, independently, by Slater [74]. It is defined as follows. A vertex z in a graph Γ is said to *resolve* a pair of distinct vertices x and y if $d(x, z) \neq d(y, z)$. A *resolving set* of Γ is a subset S of $V(\Gamma)$ such that every pair of distinct vertices of Γ is resolved by some vertex in S. The *metric dimension* of Γ, denoted by $\dim(\Gamma)$, is the minimum cardinality of a resolving set of Γ. It was considered, for example, in [6].

Strong metric dimension of a graph was introduced by Sebő and Tannier [72], in relation to the applications of strong resolving sets to combinatorial searching. A vertex z of a graph Γ is said to *strongly resolve* vertices x and y in Γ, if there exists a shortest path from z to x containing y, or a shortest path from z to y containing x. A subset S of $V(\Gamma)$ is a *strong resolving set* of Γ if every pair of vertices of Γ is strongly resolved by some vertex in S. The smallest cardinality of a strong resolving set of Γ is denoted by $\mathrm{sdim}(\Gamma)$ and is called the *strong metric dimension* of Γ. The problem of computing strong metric dimension is NP-hard [56].

Ma and She [59] gave an explicit formula for the metric dimension of the enhanced power graph of a group, using the following definition. For $x, y \in G$, we write $x \equiv y$ if $N[x] = N[y]$ or $N(x) = N(y)$ in $\mathcal{G}_e(G)$. Note that $\equiv$ is an equivalence relation. Denote by $\overline{x}$ the $\equiv$-class containing the vertex $x \in G$, and let $\overline{G} = \{\overline{x} : x \in G\}$.

Theorem 3.1. ([59, Theorem 3.1]) *Let G be a group of order n. Then $\dim(\mathcal{P}_e(G)) = n - |\overline{G}|$.*

As a corollary of Theorem 3.1, the metric dimensions of the enhanced power graphs of an elementary abelian p-group, a dihedral group, and a generalized quaternion group have been determined in [59].

Corollary 3.1. ([59, Example 3.11]) (i) *Let $k \geq 2$ and p a prime. Then*

$$\dim(\mathcal{P}_e(\mathbb{Z}_p^k)) = \begin{cases} 2^k - 2, & \text{if } p = 2, \\ p^k - \frac{p^k + p - 2}{p - 1}, & \text{otherwise.} \end{cases}$$

(ii) $\dim(\mathcal{P}_e(D_{2n})) = 2n - 3$.
(iii) $\dim(\mathcal{P}_e(Q_{4m})) = 3m - 2$.

Ma and Zhai [62] characterized the strong metric dimension of the enhanced power graph of a finite group. For $x, y \in G$, denote by $\approx$ the equivalence relation defined by $N[x] = N[y]$ in $\mathcal{P}_e(G)$. It is easy to see that $\approx$ is an equivalence relation over G. The $\approx$-class containing the element $x \in G$ is denoted by $\widehat{x}$. For a subset S of G and an element $g \in G$, let

$$\widehat{S} = \{\widehat{x} : x \in S\}, \quad \mathcal{M}_g = \{M \in \mathcal{M}_G : g \in M\}.$$

Theorem 3.2. ([62, Theorem 4.5]) *Let G be a group of order n. Then*

$$\mathrm{sdim}(\mathcal{P}_e(G)) = n - \max\{|\widehat{M}| : M \in \mathcal{M}_G\}$$
$$= n - \max\{|S| : S \subseteq M \in \mathcal{M}_G \text{ and for any } x, y \in S, \mathcal{M}_x \neq \mathcal{M}_y\}.$$

Panda et al. [69] and Dalal & Kumar [34] computed $\text{sdim}(\mathcal{P}_e(G))$ for some special groups G.

Theorem 3.3. ([62, Corollarys 4.6 and 4.7] , [69, Theorems 3.15, 3.34], [34, Theorem 3.22]) (i)
$\text{sdim}(\mathcal{P}_e(G)) = |G| - 1$ *if and only if G is cyclic;*
(ii) *If G is a noncyclic $\mathcal{P}$-group of order n, then $\text{sdim}(\mathcal{P}_e(G)) = n - 2$;*
(iii) $\text{sdim}(\mathcal{P}_e(D_{2n})) = 2n - 2$;
(iv) $\text{sdim}(\mathcal{P}_e(Q_{4n})) = 4n - 2$;
(v) *Let G be a noncyclic abelian p-group with order n and exponent p^m. Then $\text{sdim}(\mathcal{P}_e(G)) = n - m - 1$.*

For $n \geq 1$, the group U_{6n} of order $6n$ is defined as the group generated by the elements a and b such that $a^{2n} = b^3 = e, ba = ab^{-1}$, that is

$$U_{6n} = \langle a, b : a^{2n} = b^3 = e, ba = ab^{-1} \rangle.$$

Theorem 3.4. ([69, Theorem 3.32]) *Let k and t be integers such that $k \geq 0$, $t > 0$ and $3 \nmid t$. Then*
$\text{sdim}(\mathcal{P}_e(U_{6(3^k t)})) = 6(3^k t) - k - 2$.

For $n \geq 2$, the *semidihedral* group SD_{8n} is a group of order $8n$ defined by the presentation

$$SD_{8n} = \langle a, b : a^{4n} = b^2 = e, ba = a^{2n-1}b \rangle.$$

Theorem 3.5. ([69, Theorem 3.36]) $\text{sdim}(\mathcal{P}_e(SD_{8n})) = 8n - 3$.

For a positive integer n, the group V_{8n} of order $8n$ is defined by

$$V_{8n} = \langle a, b : a^{2n} = b^4 = e, ba = a^{-1}b^{-1}, b^{-1}a = a^{-1}b \rangle.$$

Theorem 3.6. ([34, Theorem 3.17]) *Let k and t be integers such that $k \geq 0$, $t > 0$, and $2 \nmid t$. Then*

$$\text{sdim}(\mathcal{P}_e(V_{8(2^k t)})) = \begin{cases} 8(2^k t) - k - 3, & \text{if } k > 0; \\ 8(2^k t) - 4, & \text{if } k = 0. \end{cases}$$

4. Perfect enhanced power graphs

It was proved in [43] that every power graph of a group is perfect. However, there exists a finite nilpotent group whose enhanced power graph is not perfect. For example, let

$$G = \langle a_1 \rangle \times \langle a_2 \rangle \times \langle b_1 \rangle \times \langle b_2 \rangle \times \langle c_1 \rangle \times \langle c_2 \rangle,$$

where $o(a_1) = o(a_2) = 2$, $o(b_1) = o(b_2) = 3$, and $o(c_1) = o(c_2) = 5$. Then the elements $a_1 b_1, b_1 c_2, a_2, b_2,$ and $a_1 c_1$ form a pentagon as an induced subgraph of $\mathcal{P}_e(G)$. It follows that $\mathcal{P}_e(G)$ is not perfect, since a pentagon is not perfect.

Zahirović et al. [80] gave a characterization of finite nilpotent groups with perfect enhanced power graphs.

Theorem 4.1. ([80, Theorem 6.2]) *A finite nilpotent group has perfect enhanced power graph if and only if this nilpotent group has at most two noncyclic Sylow subgroups.*

On the other hand, the following theorem implies that there are many groups which do not have perfect enhanced power graphs, but whose clique number and chromatic number are equal.

Theorem 4.2. ([80, Theorem 6.3]) *If a finite group G has an element with order equal to the exponent of G, then $\mathcal{P}_e(G)$ is weakly perfect.*

Theorem 4.3. ([80, Corollary 6.1]) *If G is a finite nilpotent group, then $\mathcal{P}_e(G)$ is weakly perfect.*

Theorem 4.4. ([20, Theorem 4.4]) *Let G be a finite group, and let $p_1, p_2, \ldots, p_n$ be all prime divisors of $|G|$ for some $n \geq 2$. If, for each $i > 2$, G has the unique Sylow p_i-subgroup, which is cyclic, then G has perfect enhanced power graph.*

The above result implies the following theorem.

Theorem 4.5. ([20, Corollary 4.5]) *Let G be a finite group of order $p^n q^m$, for some $n, m \geq 0$ and some primes p and q. Then G has perfect enhanced power graph.*

The following example shows that the sufficient condition of Theorem 4.4 is not necessary.

Theorem 4.6. ([20, Example 4.6]) *No Sylow subgroup of the alternating group A_5 is unique, and the enhanced power graph of A_5 is perfect.*

The condition of Theorem 4.4 cannot be weakened by removing the requirement for the Sylow subgroups to be cyclic, nor by removing the demand for the Sylow subgroups to be unique. For example, the group $\mathbb{Z}_{30} \times \mathbb{Z}_{30}$ is abelian, and its Sylow subgroups are unique. However, $\mathcal{P}_e(\mathbb{Z}_{30} \times \mathbb{Z}_{30})$ is not a perfect graph (see [20]).

Theorem 4.7. ([20, Proposition 4.7]) *The symmetric group S_n has a perfect enhanced power graph if and only if $n \leq 7$, and the alternating group A_n has perfect enhanced power graph if and only if $n \leq 8$.*

Theorem 4.8. ([69, Theorems 3.31, 3.33 and 3.35]) *The enhanced power graphs of U_{6n}, D_{2n} and SD_{8n} are perfect.*

Theorem 4.9. ([34, Theorems 3.16 and 3.18]) *The enhanced power graphs of V_{8n} and Q_{4n} are perfect.*

5. Forbidden subgraphs in the enhanced power graph

This section contains the results on the forbidden subgraphs of enhanced power graphs. Chordal graphs, threshold graphs, cographs and split graphs form important classes that can be defined in terms of forbidden induced subgraphs.

A graph is said to be *chordal* if it contains no induced cycles of length greater than 3. In other words, a chordal graph is a graph in which every cycle of length at least 4 has a chord. This means that if a chordal graph has an induced cycle C, then C is isomorphic to C_3. A graph is called a *threshold graph* if it has no induced subgraph isomorphic to the path P_4, the complete graph K_4, or $2K_2$. Chordal graphs and threshold graphs were considered, for example, in [41] and [29]. A graph is called a *cograph* if this graph has no induced subgraph isomorphic to the path P_4. A graph is said to be *split* if its vertex set is the disjoint union of two subsets A and B such that A induces a complete graph and B induces an empty graph.

Theorem 5.1. ([60, Theorems 3.1]) *The following are equivalent for a finite group G:*
(a) $\mathcal{P}_e(G)$ *is split;*
(c) $\mathcal{P}_e(G)$ *is $2K_2$-free;*
(b) G *is a cyclic group, a dihedral group, or an elementary abelian 2-group.*

Theorem 5.2. ([60, Theorems 3.2]) $\mathcal{P}_e(G)$ *is threshold if and only if G is isomorphic to one of the following groups:*

$$\mathbb{Z}_2, \quad \mathbb{Z}_3, \quad D_6, \quad \mathbb{Z}_2^n,$$

where n is a positive integer at least 2.

Theorem 5.3. ([60, Theorems 4.1]) *The following are equivalent for a nilpotent group G:*
(I) $\mathcal{P}_e(G)$ *is a chordal graph;*
(II) $\mathcal{P}_e(G)$ *is a cograph;*
(III) G *has at most one noncyclic Sylow subgroup;*
(IV) $G \cong P \times \mathbb{Z}_n$, *where P is a Sylow p-subgroup of G and $(|P|, n) = 1$.*

A group is called a *P-group* if every nonidentity element of the group has prime order. For example, D_{2q} is a P-group for some odd prime q. A group is called a *CP-group* if every nontrivial element of the group has prime power order. For example, any p-group is a CP-group, and D_{2q^n} is a CP-group for some odd prime q and positive integer n. Clearly, a P-group is also a CP-group. The *prime graph* of a finite group G is a simple graph whose vertex set consists of all prime divisors of $|G|$, and two distinct vertices p and q are adjacent if there exists an element of order pq in G. The prime graph of a group was first introduced by Gruenberg and Kegel in an unpublished manuscript studying integral representations of groups in 1975.

Theorem 5.4. ([60, Propositions 4.6, 4.8 and 4.11]) *Let G be a group. $\mathcal{P}_e(G)$ is a chordal graph as well as a cograph if one of the following holds:*
(a) G *is a p-group;*
(b) *Given a positive integer k, if $|M \cap N| = k$ for any two distinct $M, N \in \mathcal{M}_G$;*
(c) G *is a dihedral group;*
(d) G *is a generalized quaternion group;*
(e) G *is a CP-group;*
(f) G *is a group with null prime graph;*
(g) G *is isomorphic to one of the $PSL(2, q)$ ($q = 4, 7, 8, 9, 17$), $PSL(3, 4)$, $Sz(8)$, and $Sz(32)$.*

Theorem 5.5. ([60, Propositions 4.14, 4.15, 4.17, and 4.18]) *The following hold:*
(a) $\mathcal{P}_e(S_n)$ *is a cograph if and only if $\mathcal{P}_e(S_n)$ is a chordal graph, if and only if $n \leq 5$;*
(b) $\mathcal{P}_e(A_n)$ *is a cograph if and only if $n \leq 6$;*
(c) $\mathcal{P}_e(A_n)$ *is a chordal graph if and only if $n \leq 7$.*

Theorem 5.6. ([60, Corollary 4.20]) *Let G be a group, and let H be a subgroup of G such that $G = H \times \mathbb{Z}_n$, where $(|H|, n) = 1$. Then $\mathcal{P}_e(G)$ is a cograph (resp. a chordal graph) if and only if $\mathcal{P}_e(H)$ is a cograph (resp. a chordal graph).*

Theorem 5.7. ([60, Lemma 4.22]) *Let G be a group and let p, q be two distinct primes. Then $\mathcal{P}_e(G)$ is a chordal graph as well as a cograph if one of the following holds:*
(a) $\pi_e(G) = \{1, p, q, pq\}$ *and G has a unique subgroup of order p or q;*
(b) $\pi_e(G) = \{1, p, q, pq\}$ *and either G has a unique cyclic subgroup of order pq, or the intersection of all cyclic subgroups of order pq has size p or q;*
(c) $\pi_e(G) = \{1, p, q, pq, p^2\}$ *and either G has a unique cyclic subgroup of order pq, or the intersection of all cyclic subgroups of order pq is $\langle a \rangle$ where $a \in K(G)$ and $o(a) \in \{p, q\}$.*

Theorem 5.8. ([60, Theorem 4.25]) *Let G be a group of order at most 24. Then $\mathcal{P}_e(G)$ is a cograph if and only if G is isomorphic to neither $\mathbb{Z}_2 \times Q_{12}$ nor the semidirect product of $\mathbb{Z}_3$ and D_8 with action kernel $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

Theorem 5.9. ([60, Theorem 4.26]) *If G is a minimum order group such that $\mathcal{P}_e(G)$ is not a chordal graph, then $|G| = 36$. In particular, $\mathcal{P}_e(\mathbb{Z}_6 \times \mathbb{Z}_6)$ is not chordal.*

6. Enhanced power graph and other graphs

In this section, we collect some results on the relationships between enhanced power graph and other graphs.

Theorem 6.1. ([1, Theorem 26]) *Let G and H be two groups. If the power graphs of G and H are isomorphic, then their enhanced power graphs are also isomorphic.*

Theorem 6.2. ([1, Theorem 28]) *For a group G, the following conditions are equivalent:*
(a) *the power graph of G is equal to the enhanced power graph of G;*
(b) *every cyclic subgroup of G has prime power order;*
(c) *the prime graph of G is a null graph.*

It was showed in [1] that a group G with these properties is one of the following: a p-group; a Frobenius group whose kernel is a p-group and complement a q-group; a 2-Frobenius group where F_1 and G/F_2 are p-groups and F_2/F_1 is a q-group; or G has a normal 2-subgroup with quotient group H, where $S \leq H \leq \mathrm{Aut}(S)$ and $S \cong A_5$ or A_6. All these types of group exist.

Clearly, if the vertices x and y are joined in the power graph of G, then they are joined in the commuting graph. Therefore, the power graph is a spanning subgraph of the commuting graph.

Theorem 6.3. ([1, Theorem 30]) *For a finite group G, the following are equivalent:*
(a) *the enhanced power graph of G is equal to its commuting graph;*
(b) *G has no subgroup $\mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p;*
(c) *the Sylow subgroups of G are cyclic or (for $p = 2$) generalized quaternion.*

It was showed in [1] that a group satisfying these conditions is either a cyclic p-group for some prime p, or satisfies the following: if $O(G)$ denotes the largest normal subgroup of G of odd order, then $O(G)$ is metacyclic, $H = G/O(G)$ is a group with a unique involution, say z, and $H/\langle z \rangle$ is a cyclic or dihedral 2-group, a subgroup of $\mathrm{P\Gamma L}(2, q)$ containing $\mathrm{PSL}(2, q)$ for q an odd prime power, or A_7. An example for the second case is the direct product of the Frobenius group of order 253 and $\mathrm{SL}(2, 5)$.

Theorem 6.4. ([80, Theorem 3.2]) *Let G and H be two groups. If $\mathcal{P}_e(G) \cong \mathcal{P}_e(H)$, then $\overrightarrow{\mathcal{P}}(G) \cong \overrightarrow{\mathcal{P}}(H)$.*

Theorem 6.5. ([80, Corollary 3.1]) *For all groups G and H, the following are equivalent:*
(a) *Power graphs of G and H are isomorphic;*
(b) *Directed power graphs of G and H are isomorphic;*
(c) *Enhanced power graphs of G and H are isomorphic.*

Theorem 6.6. ([80, Corollary 4.1]) *Let G be a group. Then*

$$\mathrm{Aut}(G) \subseteq \mathrm{Aut}(\overrightarrow{\mathcal{P}}(G)) \subseteq \mathrm{Aut}(\mathcal{P}(G)) \subseteq \mathrm{Aut}(\mathcal{P}_e(G)).$$

Theorem 6.7. ([80, Theorem 4.1]) *Let G and H be two groups. Then any isomorphism from $\mathcal{P}(G)$ to $\mathcal{P}(H)$ is an isomorphism from $\mathcal{P}_e(G)$ to $\mathcal{P}_e(H)$.*

The *deep commuting graph* of a group G is the graph $\mathcal{DC}(G)$ whose vertex set is G, two distinct vertices are adjacent if and only if their pre-images in every central extension of G (that is, every group H with a central subgroup Z such that $H/Z \cong G$) commute. Note that $E(\mathcal{DC}(G))$ is contained in the set of edges of the commuting graph of G and contains the set of edges of the enhanced power graph of G.

Theorem 6.8. ([25, Theorem 4.3]) *Let G be a group. $\mathcal{DC}(G) = \mathcal{P}_e(G)$ if and only if G has the following property: Let H be a Schur cover of G, with $H/Z = G$, Then for any subgroup A of G, with B the corresponding subgroup of H (so $Z \leq B$ and $B/Z = A$), if B is abelian, then A is cyclic.*

Let $\mathcal{P}^*(G)$ be the graph obtained by deleting only the identity element of G in $\mathcal{P}(G)$ and this is called *deleted power graph* of G. Similarly, $\mathcal{P}_e^*(G)$ denotes the graph obtained by deleting only the identity element of G in $\mathcal{P}_e(G)$ and this is called *deleted enhanced power graph* of G.

Theorem 6.9. ([17, Theorem 2.8]) *For any group G, $\mathcal{P}^*(G)$ is connected if and only if the $\mathcal{P}_e^*(G)$ is connected.*

7. Miscellaneous properties of the enhanced power graph

In this section, we present some miscellaneous properties of the enhanced power graph of a group, including vertex connectivity, diameter, automorphism group, independence number, Eulerian, matching number and so on.

Theorem 7.1. ([17, Theorem 1.1]) *Let G be a finite p-group such that G is neither cyclic nor generalized quaternion group. Then $\kappa(\mathcal{P}_e(G)) = 1$.*

Theorem 7.2. ([17, Theorem 1.2]) *Let G be a finite noncyclic abelian group. Then $\kappa(\mathcal{P}_e(G)) = 1$ if and only if G is a p-group.*

Theorem 7.3. ([17, Theorem 1.7]) *Let G be a noncyclic abelian non-p-group such that $G \cong G_1 \times \mathbb{Z}_n$, $\gcd(|G_1|, n) = 1$ and G_1 is a p-group with no cyclic Sylow subgroup. Then $\kappa(\mathcal{P}_e(G)) = n$.*

Theorem 7.4. ([17, Theorems 4.1, 4.2, 4.5, and 4.8]) *For* $n \geq 3$, $\kappa(\mathcal{P}_e(D_{2n})) = 1$, $\kappa(\mathcal{P}_e(Q_{2^n})) = 2$, *and* $\kappa(\mathcal{P}_e(S_n)) = 1$ *if and only if either* n *or* $n - 1$ *is prime. For* $n \geq 7$, $\kappa(\mathcal{P}_e(A_n)) = 1$ *if and only if one of* $n, n-1, n-2, n/2, (n-1)/2, (n-2)/2$ *is prime.*

Recently, Costanzo et al. [31] studied the deleted enhanced power graph of a direct product.

Theorem 7.5. ([31, Theorem A]) *If* G *and* H *are nontrivial groups such that* $\mathcal{P}_e^*(G \times H)$ *is connected, then* $\mathrm{diam}(\mathcal{P}_e^*(G \times H)) \leq 7$.

Theorem 7.6. ([31, Theorem B]) *If* G *and* H *are groups such that* $\mathcal{P}_e^*(G \times H)$ *is connected with* $\mathrm{diam}(\mathcal{P}_e^*(G \times H)) \leq 2$, *then* $(\mathcal{P}_e^*(G))$ *and* $(\mathcal{P}_e^*(H))$ *are connected, and* $\mathrm{diam}(\mathcal{P}_e^*(G)) \leq 2$ *or* $\mathrm{diam}(\mathcal{P}_e^*(H)) \leq 2$.

Recall that a Z-group is a group where every Sylow subgroup is cyclic. Recently, Costanzo et al. [32] investigated $\mathcal{P}_e^*(G)$ for a Z-group G.

Theorem 7.7. ([32, Theorem A]) *Let* G *be a* Z*-group. Then* $\mathcal{P}_e^*(G)$ *is disconnected if and only if* G *is a Frobenius group.*

Theorem 7.8. ([32, Theorem B]) *If* G *is a* Z*-group and* $\mathcal{P}_e^*(G)$ *is connected, then* $\mathrm{diam}(\mathcal{P}_e^*(G)) \leq 4$.

Theorem 7.9. ([32, Theorem C]) *If* G *is a* Z*-group, then* $\mathrm{diam}(\mathcal{P}_e^*(G)) \leq 2$ *if and only if* $Z(G) \neq \{e\}$.

For a positive integer n, let $\pi(n)$ denote the set of prime divisors of n.

Theorem 7.10. ([32, Theorem D]) *Let* G *be a group,* $g \in G$, *and* $\pi = \pi(o(g))$. *Write*

$$g = \prod_{p \in \pi} g_p,$$

where each g_p *is a* p*-element for* $p \in \pi$ *and* $g_p g_q = g_q g_p$ *for all* $p, q \in \pi$. *Then* g *is a dominating vertex for* $\mathcal{P}_e^*(G)$ *if and only if, for each* $p \in \pi$, *a Sylow* p*-subgroup* P *of* G *is cyclic or generalized quaternion and* $\langle g_p \rangle \leq P \cap Z(G)$.

The above result offered a generalization of Theorem 3.2 in [15] (see Theorem 2.2). That is, for a nilpotent group G, the graph $\mathcal{P}_e^*(G)$ is dominatable if and only if G has a cyclic or generalized quaternion Sylow subgroup.

Theorem 7.11. ([1, Corollary 35]) *Let* G *be a finite group. Then*

$$\omega(\mathcal{P}_e(G)) = \max\{o(g) : g \in G\}.$$

Theorem 7.12. ([15, Theorem 2.4]) *The enhanced power graph of a group* G *is complete if and only if* G *is cyclic.*

Theorem 7.13. ([15, Theorem 2.5]) *Let* G *be a group of order* n. *Then* $\mathcal{P}_e(G)$ *is Eulerian if and only if* n *is odd.*

In [80], the authors defined graph $S_p(\overline{a})$, where p is a prime and $\overline{a}$ is a tuple consisting of some positive integers.

Theorem 7.14. ([80, Theorem 5.1]) *A graph Γ is the enhanced power graph of a finite abelian group if and only if*

$$\Gamma \cong \boxtimes_{i=1}^{n} S_{p_i}(\overline{a_i})$$

for some tuples $\overline{a_i}$ of positive integers and for some pairwise different primes $p_1, p_2, \ldots, p_n$.

Theorem 7.15. ([80, Theorem 4.2]) $\mathbb{Z}_2$ *is the unique nontrivial finite group whose enhanced power graph has an abelian automorphism group.*

Theorem 7.16. ([80, Theorem 4.3]) $\mathbb{Z}_2$, $\mathbb{Z}_3$, *and the Klein group are the only nontrivial finite groups whose automorphism groups of their enhanced power graphs have square free orders.*

Theorem 7.17. ([80, Theorem 4.4]) $\mathbb{Z}_2$ *and $\mathbb{Z}_2 \times \mathbb{Z}_4$ are the only nontrivial finite groups whose automorphism groups of their enhanced power graphs have prime power orders*

Theorem 7.18. ([69, Theorem 3.2]) *For any finite group G, the minimum degree of $\mathcal{P}_e(G)$ is $m-1$, where m is the order of the smallest maximal cyclic subgroup of G.*

For a group G, a subset $M(G) := \{x_1, x_2, \ldots, x_m\}$ of G is called an *essential cyclic set* if the following hold:
(i) For any $g \in G$, there exists $x_i \in M(G)$ such that $\langle g \rangle = \langle x_i \rangle$;
(ii) $\langle x_i \rangle \neq \langle x_j \rangle$ for distinct indices i, j;
(iii) For every i, $\langle x_i \rangle$ is a maximal cyclic subgroup of G.
Denote by $i(G)$ the subset

$$\{x \in M(G) : \langle x \rangle \cap \langle y \rangle = \{e\} \text{ for any } y \in M(G) \setminus \{x\}\}$$

of $M(G)$.

Theorem 7.19. ([38, Theorem 3.6]) *Suppose that G is a group with $|i(G)| = 0$. Then the rainbow connection number*

$$\mathrm{rc}(\mathcal{P}_e(G)) = \begin{cases} 1, & \text{if } G \text{ is cyclic;} \\ 2, & \text{if } G \text{ has an awning and is noncyclic;} \\ 3, & \text{if } G \text{ has no awnings.} \end{cases}$$

where the definition of an awning can be found in [38, Definition 2.11].

An *independent set* of Γ is a set of vertices none of two are adjacent in Γ. The *independence number* of Γ, denoted by $\alpha(\Gamma)$, is the largest cardinality of an independent set of Γ.

Theorem 7.20. ([69, Theorem 3.3]) *For any finite group G, the independence number of $\mathcal{P}_e(G)$ is equal to the number of maximal cyclic subgroups of G. Furthermore, if G is nilpotent and the prime factors of $|G|$ are $p_1, p_2, \ldots, p_r$, then*

$$\alpha(\mathcal{P}_e(G)) = m_1 m_2 \cdots m_r,$$

where m_i is the number of maximal cyclic subgroups of a Sylow p_i-subgroup.

A *matching* of Γ is a set of edges such that no two of them are incident to the same vertex. The matching number of Γ, denoted by $\beta(\Gamma)$, is the maximum cardinality of a matching.

Theorem 7.21. ([69, Theorem 3.5]) *For any finite group G. If G has odd order, then $\beta(\mathcal{P}_e(G)) = (|G| - 1)/2$. If G has even order, then*

$$\frac{|G| - t + 1}{2} \leq \beta(\mathcal{P}_e(G)) \leq \frac{|G|}{2},$$

where t is the number of all involutions of G.

Theorem 7.22. ([69, Theorem 3.14]) *For any finite p-group G,*

$$\beta(\mathcal{P}_e(G)) = \begin{cases} \frac{|G|-1}{2}, & p > 2; \\ \frac{|G|-t+1}{2}, & p = 2. \end{cases}$$

where t is the number of all involutions of G.

A group G is a 2-*Frobenius group* if this group has two normal subgroups K and L such that L is a Frobenius group with Frobenius kernel K and G/K is a Frobenius group with Frobenius kernel L/K. The best known example of a 2-Frobenius group is the symmetric group S_4 of order 24.

Theorem 7.23. ([30, Theorem A]) *Let G be a 2-Frobenius group with K as in the definition. If $|K|$ is divisible by at least two distinct primes, then $\mathcal{P}_e^*(G)$ has $|K| + 1$ connected components.*

Let $m_p(G)$ denote the number of subgroups of order p of a group G.

Theorem 7.24. ([30, Theorem B]) *Let G be a 2-Frobenius group, and assume that K and G/L are p-groups for some prime p, where K and L are as in the definition and K has prime power order. Then $\mathcal{P}_e^*(G)$ has $|K| + m_p(G)$ connected components.*

Denote by $m_p^*(G)$ the number of subgroups of order p in G that are not centralized by an element of prime order other than p.

Theorem 7.25. ([30, Theorem C]) *Let G be a 2-Frobenius group, and let p be a prime. Assume that K is a p-group for some prime p and that G/L is not a p-group, where K and L are as in the definition. Then $\mathcal{P}_e^*(G)$ has $|K| + |L/K| + m_p^*(G)$ connected components.*

8. Open questions and other work

This section presents open problems on enhanced power graphs of groups. Note that the enhanced power graph of a group is a union of complete subgraphs on the maximal cyclic subgroups of this group. Cameron [23] put forward the following question.

Problem 1. ([23, Question 2]) *Is there a simple algorithm for constructing the directed power graph or the enhanced power graph from the power graph, or the directed power graph from the enhanced power graph?*

The deep commuting graph of a group was defined by Cameron and Kuzma [25].

Problem 2. ([23, Question 4]) *What can be said about groups G for which $\mathcal{DC}(G) = \mathcal{P}_e(G)$?*

Problem 3. ([15, Open problem]) *Characterize all finite non-abelian groups G such that $\mathcal{P}_e(G)$ is dominatable.*

In fact, Theorems 2.4 and 2.7 are two solutions to Problem 3. Maybe, by other methods, one can characterize all finite groups G such that $\mathcal{P}_e(G)$ is dominatable.

It is clear that $\omega(\mathcal{P}(G)) \leq \omega(\mathcal{P}_e(G))$. It was proved in [23, Theorem 4.2] that there exists a function f on the natural numbers such that, for any finite group G,

$$\omega(\mathcal{P}_e(G)) \leq f(\omega(\mathcal{P}(G))). \tag{1}$$

For example, let $f(\omega(\mathcal{P}(G)))$ be the least common multiple of $\{1, 2, \ldots, \omega(\mathcal{P}(G))\}$.

Problem 4. ([23, Question 8]) *Find the best possible function f in* (1).

An upper bound for the chromatic number of the enhanced power graph is given in [1, Theorem 12] and [23, Theorem 4.3]. For a finite group G, let S be the set of orders of elements of G, then

$$\chi(\mathcal{P}_e(G)) \leq \sum_{n \in S} \varphi(n), \tag{2}$$

where φ is Euler's function. Note that the bound of (2) holds for abelian groups. In fact, if G is an abelian group, then S is the set of divisors of the exponent of G, and so the sum on the right of (2) is the exponent of G, which is the largest element order in G.

Problem 5. ([23, Question 9]) *Find a formula for the chromatic number of the enhanced power graph of a finite group.*

If a finite graph Γ is isomorphic to an induced subgraph of the enhanced power graph of some finite group G, then we say that Γ is *embeddable* in the enhanced power graph of G.

Problem 6. ([23, Question 10]) (a) *What is the smallest group for which a given graph is embeddable in the enhanced power graph of the group?*
(b) *What is the smallest group in which every graph on n vertices can be embedded the enhanced power graph of the group?*

Problem 7. ([23, Question 14]) *For which finite groups is the enhanced power graph a perfect graph, or a cograph, or a chordal graph, or a split graph, or a threshold graph?*

Bošnjak et al. [20] and Zahirović et al. [80] studied perfect enhanced power graphs of groups. Ma, Lv, and She [60] classified finite groups whose enhanced power graphs are split graphs and threshold graphs. They also classified finite nilpotent groups whose enhanced power graphs are chordal graphs and cographs, and gave some families of non-nilpotent groups whose enhanced power graphs are chordal graphs and cographs. These results provide a partial solution to Problem 7.

Problem 8. ([23, Question 20]) *Is it true that the complement of the enhanced power graph has just one connected component, apart from isolated vertices?*

Two vertices u, v of a graph Γ are *open twins* if they have the same open neighbourhoods; they are *closed twins* if they have the same closed neighbourhoods. If x and y are twins (open or closed twins), then we may collapse them to a single vertex; this process is called *twin reduction*. Clear, the automorphism group $\mathrm{Aut}(\Gamma)$ of a graph Γ preserves twin relations on Γ, and that vertices in a twin equivalence class can be permuted arbitrarily. It follows by induction that the group induces an automorphism group on the cokernel Γ^* of Γ, say $\mathrm{Aut}^-(\Gamma^*)$. The twin reduction on Γ is *faithful* if $\mathrm{Aut}^-(\Gamma^*) = \mathrm{Aut}(\Gamma^*)$.

Problem 9. ([23, Question 21]) *Let G be a group.*
(a) *When is twin reduction on $\mathcal{P}_e(G)$ faithful?*
(b) *What is the automorphism group of the cokernel of $\mathcal{P}_e(G)$?*

Problem 10. ([23, Question 22]) *For which non-abelian finite simple group G is it the case that the twin reduction on the enhanced power graph of G is faithful?*

Problem 11. ([1, Question 27]) *For which finite groups is the power graph equal to the enhanced power graph?*

Problem 12. ([1, Question 29]) *For which (finite) groups is the enhanced power graph equal to the commuting graph?*

In [1], the authors solved Problems 11 and 12 for the special case of finite groups (see [1], Theorems 28 and 30, respectively).

Problem 13. ([1, Question 31]) *What can be said about the difference of the enhanced power graph and the power graph, or the difference of the commuting graph and the enhanced power graph? In particular, for which groups is either of these graphs connected?*

The multiset dimension was introduced by Rinovia Simanjuntak et al. [73] as a variation of metric dimension. It was considered in [7, 19]. Here we suggest the following open problem.

Problem 14. *For any finite group G, describe the multiset dimension of the enhanced power graph of G.*

The locating chromatic number of a graph was introduced by Chartrand et al. [28]. It is equal to the minimum number of colours needed in a locating colouring of the graph. Let us refer to [12] for more information on the locating-chromatic number. We propose the following open problem.

Problem 15. *For any finite group G, describe the locating-chromatic number of the enhanced power graph of G.*

The investigation of graph labelings forms a large and important research direction. Many valuable classes of graph labelings have been investigated in the literature, including cordial labelings [10], distance labelings [66], face-antimagic labelings [9], graceful labelings [57], integer additive labelings [71], irregular total labelings [46], supermagic labelings [64] and vertex irregular reflexive labelings [8, 75]. The following general problem is an interesting suggestion for future work.

Problem 16. *Investigate enhanced power graphs with labelings of various essential classes. For any finite group G, describe α-labelings, cordial labelings, distance labelings, face-antimagic labelings, graceful labelings, integer additive set-labelings, irregular total labelings, magic and antimagic graph labelings, super antimagic labelings, supermagic labelings, and vertex irregular reflexive labelings of the enhanced power graph of G.*

In conclusion, we mention a few pertinent references which complement the results presented above and can also be used in the future work. The *proper enhanced power graph* of a group G is the induced subgraph of the enhanced power graph on the set $G \setminus D$, where D is the set of dominating vertices of the enhanced power graph of G. Bera and Dey [16] classified all nilpotent groups such that their proper enhanced power graphs are connected and determined their diameters. They also explicitly found the domination number of the proper enhanced power graph of a finite nilpotent group. The *enhanced power graph* or *cyclic graph* of a semigroup S is a simple graph whose vertex set is S and two vertices $x, y \in S$ are adjacent if and only if $x, y \in \langle z \rangle$ for some $z \in S$, where $\langle z \rangle$ is the subsemigroup of S generated by z. In [33, 35, 36, 5], several graph theoretical properties of the enhanced power graph of semigroups were studied, including the dominating number, independence number, genus, connectedness, minimum degree, chromatic number and so on. Moreover, the authors characterized the semigroups S such that the enhanced power graph of S is complete, bipartite, regular, is a tree, or is a null graph. Moreover, the structure of the enhanced power graph of any semigroup was described. Dupont et al. [39] introduced the enhanced quotient graph of the quotient of a finite group, and gave necessary and sufficient conditions for the graph to be complete and Eulerian.

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