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A Symmetric/Asymmetric Bimodal Extension Based on the Logistic Distribution: Properties, Simulation and Applications

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Abstract: In this paper, we introduce bimodal extensions, one symmetric and one asymmetric, of the logistic distribution. We define this new density and study some basic properties. We draw inferences from the moment estimator and maximum likelihood approaches. We present a simulation study to assess the behaviour of the moment and maximum likelihood estimators. We also study the singularity of the Fisher information matrix for particular cases. We offer applications in real data and compare them with a mixture of logistics distributions.

Keywords: bimodal distribution; logistic distribution; maximum likelihood; symmetry; asymmetric

MSC: 62E15; 62E20



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1. Introduction

Data sets with bimodal behaviour appear in many disciplines, for example mining (see Bolfarine et al. [1]), medicine (see Ely et al. [2]; Elal-Olivero et al. [3]), meteorology (see Zhang et al. [4]), etc. The mixture of normal distributions is usually used to model bimodal data, but these models present some identifiability problems as discussed in McLachan and Peel [5] and Marin et al. [6].

Various recent works show bimodal distributions based on skew-symmetric distributions, for example Arnold and Beaver [7], Azzalini and Capitanio [8], Ma and Genton [9], Arellano-Valle et al. [10], Kim [11], Elal-Olivero et al. [3], Gómez et al. [12], Bolfarine et al. [1] and Venegas et al. [13]. These bimodal distributions offer a good alternative to statistical models based on finite mixtures of distributions.

The generalized bimodal (GB) distribution was introduced and studied by Rao et al. [14] and Sarma et al. [15], with density given by:

$$f_X(x; \gamma) = \left(\frac{\gamma + x^2}{1 + \gamma} \right) \phi(x), \quad x \in \mathbb{R}, \quad \gamma \geq 0,$$

where γ is the shape parameter and ϕ is the density $N(0, 1)$.

A simple reparametrization of the GB distribution leads us to the expression given by:

$$f_X(x; \alpha) = \left(\frac{1 + \alpha x^2}{1 + \alpha} \right) \phi(x), \quad x \in \mathbb{R}, \quad \alpha \geq 0. \quad (1)$$

It is observed that the distribution $N(0, 1)$ is obtained when $\alpha = 0$ in the GB distribution.

This idea can be extended using elliptic distributions; in the univariate case these are distributions of symmetric random variables. A random variable Y is symmetric about zero if its pdf (g_Y) holds with

$$g_Y(-y) = g_Y(y),$$

for all $y \in \mathbb{R}$. For further discussion of the family of elliptic distributions, see Kelker [16] and Cambanis et al. [17], and the books Fang et al. [18] and Gupta and Varga [19], etc. Then using this methodology it can be extended (1), as follows

$$f_X(x; \alpha) = \left(\frac{1 + \alpha x^2}{1 + \alpha \kappa} \right) g_Y(x), \quad x \in \mathbb{R},$$

where $\alpha \geq 0$, Y is a random variable symmetric about zero and $\kappa = E(Y^2)$.

Values of k are reported in Table 1 for some particular cases of the density g_Y :

1. $g_Y(x) = \phi(x)$ (Normal);
2. $g_Y(x) = \frac{1}{2} \exp(-|x|)$ (Laplace);
3. $g_Y(x) = c_2(1 + x^2/\nu)^{-(1+\nu)/2}$ with $c_2 = (\nu\pi)^{-1/2}\Gamma(\mu/2)^{-1}\Gamma((\nu + 1)/2)$ (Student- t);
4. $g_Y(x) = \exp(x)/(1 + \exp(x))^2$ (Logistic).

Table 1. Values of k for some distributions.

Type	k
Normal	1
Laplace	2
Student- t	$\frac{\nu}{\nu-2}, \nu > 2$
Logistic	$\frac{\pi^2}{3}$

One of the principal motivations of the present article is to consider the symmetric bimodal logistic (BL) distribution, since the logistic distribution has greater kurtosis than normal distribution, on account of the simplicity of its probability density function, given in item 4 of Table 1 and denoted here by $Y \sim L(0, 1)$. This distribution is therefore an alternative for modelling bimodal data when the distribution tails are a little heavier.

In other situations data are observed with greater flexibility in their modes; on this basis we introduce an asymmetric bimodal extension of the logistic distribution.

The paper is organised as follows. In Section 2 we show the density and some properties of the BL distribution. In Section 3 we carry out parameter estimation using the moments (M) and maximum likelihood (ML) methods, and a simulation to show the behaviour of the M and ML estimations. In Section 4 we apply the methodology to one real data set. Section 5 presents an asymmetric extension of the proposed model and two illustrations with real data sets. We finish with some conclusions in Section 6.

2. Density Function and Properties

Definition 1. We will say that a random variable Z has a BL distribution if its density function is given by

$$f_Z(z; \alpha) = 3 \left(\frac{1 + \alpha z^2}{3 + \pi^2 \alpha} \right) \frac{\exp(z)}{(1 + \exp(z))^2}, \quad z \in \mathbb{R}, \tag{2}$$

where $\alpha \geq 0$ is the shape parameter. We denote by $Z \sim BL(\alpha)$.

By a straightforward computation we have

$$\int_{-\infty}^{\infty} f_Z(z; \alpha) dz = \frac{3}{3 + \pi^2 \alpha} \left[\int_{-\infty}^{\infty} \frac{\exp(z)}{(1 + \exp(z))^2} dz + \alpha \int_{-\infty}^{\infty} \frac{z^2 \exp(z)}{(1 + \exp(z))^2} dz \right] = 1.$$

We observe that the first term on the right side of the first equality is the integral of the standard Logistic distribution, and the second is the second moment of the same distribution; therefore their values are 1 and $\pi^2/3$ respectively, verifying the second equality.

Figure 1 shows the graph of the BL density for different values of parameter α .

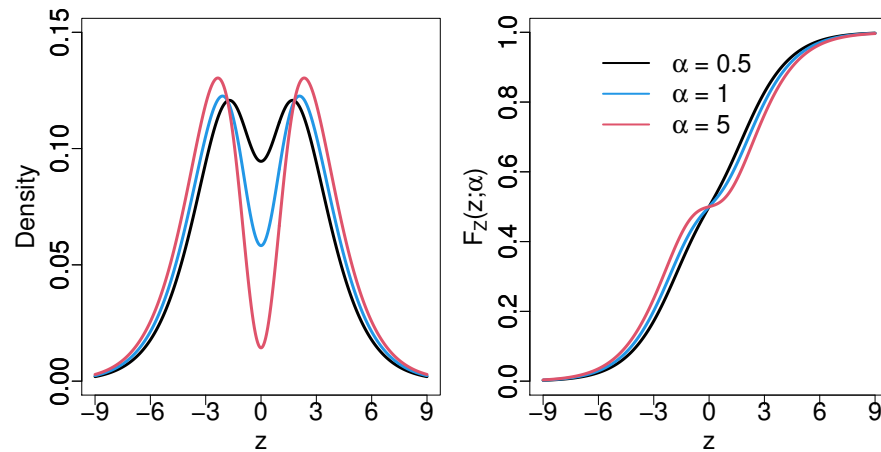


Figure 1. Plots of the BL distribution (left panel) and its corresponding cdf (right panel) for different values of the α parameter: $\alpha = 0.5$ (black line), $\alpha = 1$ (light blue line) and $\alpha = 5$ (red line).

2.1. Some Properties

The following properties arise directly from the above definition. Let $Z \sim BL(\alpha)$ then the following basic properties are obtained:

Property 1. If $\alpha = 0$ the standard logistic distribution, $Logistic(0, 1)$, is obtained.

Property 2. As α tends to infinity the limiting distribution is obtained, defined by

$$f_Z(z) = \frac{3z^2 \exp(z)}{\pi^2(1 + \exp(z))^2}.$$

Property 3. $f_Z(-z; \alpha) = f_Z(z; \alpha)$, for all $z \in \mathbb{R}$.

Property 4. Consider $Y = |Z|$ the Half BL distribution is obtained. In particular, as $\alpha = 0$ the known Half Logistic distribution is obtained.

Property 5. The cumulative distribution function (cdf) of the pdf f_Z defined in (2) is given by

$$F_Z(z, \alpha) = \frac{3}{3 + \alpha\pi^2} \left[\frac{\exp(z)}{1 + \exp(z)} + \alpha \left(\frac{z^2 \exp(z)}{1 + \exp(z)} - 2z \log(1 + \exp(z)) - 2Li_2(-\exp(z)) \right) \right].$$

We note that F_Z is represented based on the poly-logarithm function that is defined as $Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$. More precisely, it refers to the dilogarithm function Li_2 (taking $s = 2$). For more details about this function we refer the reader to Prudnikov et al. [20].

The functions $Li_2(-\exp(z))$ and $z \log(1 + \exp(z))$ tend to zero when z tends to minus infinity; this can be proved using L'Hopital's Rule, concluding that $F_Z(z, \alpha) \rightarrow 0$ as $z \rightarrow -\infty$.

A well-known reflection property of the dilogarithm function is given by $Li_2(t) + Li_2(1/t) = -\pi^2/6 - 1/2 \log^2(-t)$, in which taking $t = -\exp(z)$ we get the following relation

$$Li_2(-\exp(z)) + Li_2(-\exp(-z)) = -\frac{\pi^2}{6} - \frac{1}{2}z^2.$$

Multiplying this equality by -2 and adding $z^2 - 2z \log(1 + \exp(z))$ we have that

$$z^2 - 2z \log(1 + \exp(z)) - 2\text{Li}_2(-\exp(z)) = \frac{\pi^2}{3} + 2\text{Li}_2(-\exp(-z)) + 2z^2 - 2z \log(1 + \exp(z)). \tag{3}$$

We note that

$$\begin{aligned} \lim_{z \rightarrow +\infty} 2z^2 - 2z \log(1 + \exp(z)) &= \lim_{z \rightarrow +\infty} 2z^2 - 2z \log(\exp(z)(1 + \exp(-z))) \\ &= \lim_{z \rightarrow +\infty} -2z \log(1 + \exp(-z)), \end{aligned}$$

then applying L'Hopital's rule it follows that this limit is zero when z tends to infinity. Thus, making $z \rightarrow +\infty$ in Equation (3) we obtain that the expression of the left-hand side tends to $\pi^2/3$.

We rewrite F_Z , given in Property 5, as follows

$$\frac{3}{3 + \alpha\pi^2} \left[\frac{\exp(z)}{1 + \exp(z)} + \alpha \left(\frac{-z^2}{1 + \exp(z)} + z^2 - 2z \log(1 + \exp(z)) - 2\text{Li}_2(-\exp(z)) \right) \right],$$

then this latter, combined with the previous argument, concludes that $F_Z \rightarrow 1$ as $z \rightarrow +\infty$. Figure 1 shows graphs of the cdf for different values of parameter α .

We present the following technical Lemma that will allow us to prove the uni/bimodality of the proposed distribution.

Lemma 1. *Let g be a function defined by:*

$$g(x) = 2\alpha(1 + x) - (\alpha x^2 + 1 - 2\alpha)e^x, \quad \alpha > 0, x \in \mathbb{R}.$$

The following holds:

1. g tends to $-\infty$ as $x \rightarrow \pm\infty$, $g'(x) = 2\alpha - (\alpha x^2 + 2\alpha x + 1 - 2\alpha)e^x$ y $g''(x) = -(\alpha x^2 + 4\alpha x + 1)e^x$.
2. g is negative for all $x \in \mathbb{R}$ whenever $0 < \alpha \leq 1/4$.
3. If $\alpha > 1/4$, then g has two zeros.

Proof. A direct calculation gives the result for part 1. For part 2, we assume that in x_0 we reach a maximum for g , as $g'(x_0) = 0$ we obtain that

$$e^{x_0} = \frac{2\alpha}{\alpha x_0^2 + 2\alpha x_0 + 1 - 2\alpha} \tag{4}$$

with $\alpha < 1/4$ as the hypothesis. If $x_0 \geq 0$, then from relation (4) we obtain that $2\alpha / (\alpha x_0^2 + 2\alpha x_0 + 1 - 2\alpha) > 1$, or $0 < \alpha x_0^2 + 2\alpha x_0 + 1 < 4\alpha$. From the latter inequality we obtain that $1 < 4\alpha$ since $\alpha x_0^2 + 2\alpha x_0 + 1 > 1$; this produces a contradiction with the fact that $0 < \alpha \leq 1/4$. Therefore the maximum is obtained in $x_0 < 0$ and its value is $g(x_0) = 2\alpha x_0(1 + e^{x_0})$, obtained using the definition of g and relation (4); this turns out to be negative, given the desired result.

To prove part 3, we first state that there is a single maximum value of g . In fact, we assume that there are two in order to obtain a contradiction. Let us consider two reached in x_0 and x_1 with $x_0 < x_1$, then at least one $\tilde{x} \in (x_0, x_1)$ where a minimum of g is reached, so we have that

$$g(x_0) - g(\tilde{x}) = 2\alpha x_0(1 + e^{x_0}) - 2\alpha \tilde{x}(1 + e^{\tilde{x}}) = 2\alpha(x_0 - \tilde{x}) + 2\alpha(x_0 e^{x_0} - \tilde{x} e^{\tilde{x}}),$$

the final expression, which is negative, since $x_0 < \tilde{x}$, producing a contradiction with the fact that in \tilde{x} a minimum of g is reached.

Now, let us assume that the maximum value of g is negative and is reached in x_0 . As $\alpha > 1/4$, $g(0) = 4\alpha - 1 > 0$ and $g \rightarrow -\infty$ as $x \rightarrow \pm\infty$; this implies that another maximum

value for g , which is a contradiction as in the previous paragraph. Thus, the maximum for g is positive, which gives the desired result. \square

Proposition 1. Let $X \sim BL(\alpha)$, then the density function is unimodal if $0 < \alpha \leq 1/4$ and bimodal as $\alpha > 1/4$.

Proof. From Equation (2) it follows

$$f'(x) = \frac{\alpha x^2 + 2\alpha x + 1 - (\alpha x^2 - 2\alpha x + 1)e^x}{(1 + \alpha x^2)(1 + e^x)} f(x).$$

Hence, the zeros of f' are the zeros of $h(x) := \alpha x^2 + 2\alpha x + 1 - (\alpha x^2 - 2\alpha x + 1)e^x$. As $h(0) = 0$, we have that $x = 0$ is a critical point of f .

We Note that $h'(x) = g(x)$, where g is the function defined in Lemma 1. Then using part 2 of Lemma 1, we obtain that $h'(x) < 0$ for all $x \in \mathbb{R}$. Therefore, h is strictly decreasing and as $h(0) = 0$ then $x = 0$ is the only zero for f' . By direct calculation together with the fact that $0 < \alpha < 1/4$ we have that $f''(0) = 3(4\alpha - 1)/8(3 + \pi^2\alpha) < 0$, which implies that f is unimodal; its mode is reached in $x = 0$ and its value is $f(0)$.

When $\alpha = 1/4$, by the above argument and Lemma 1, we have that zero is the only critical point for f . Consequently, its mode is reached at zero.

Now, using part 3 of Lemma 1 we have that h' has two zeros when $\alpha > 1/4$ and as $h(0) = 0$ we have that h has three zeros; one of these is $x = 0$ and the others are $\pm x_0$ with $x_0 \neq 0$ (since $h(-x_0) = -h(x_0)e^{-x_0}$). Consequently, f has three zeros and from the previous calculation we have that $f''(0) > 0$, then in $x = 0$ a minimum of f is obtained as well as the other zeros in which f reaches its maximum value, which completes the proof. \square

2.2. Moments

In this subsection we calculate the moments of a random variable with BL distribution and its kurtosis coefficient.

Proposition 2. Let $Z \sim BL(\alpha)$, then for $r = 0, 1, 2, \dots$ the r -th moments of the random variable Z are:

(a) $E[Z^{2r+1}] = 0,$

(b) $E[Z^{2r}] = \frac{6}{3 + \pi^2\alpha} \{ (1 - 2^{1-2r})\Gamma(2r + 1)\zeta(2r) + \alpha(1 - 2^{-2r-1})\Gamma(2r + 3)\zeta(2r + 2) \},$

where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann Zeta Function and satisfies that $\zeta(s) = Li_s(1)$.

Proof. (a) We can express the odd moments of $Z \sim BL(\alpha)$ as

$$E[Z^{2r+1}] = \frac{3}{3 + \pi^2\alpha} \left\{ \int_{-\infty}^{\infty} \frac{z^{2r+1} \exp(z)}{(1 + \exp(z))^2} dz + \alpha \int_{-\infty}^{\infty} \frac{z^{2r+3} \exp(z)}{(1 + \exp(z))^2} dz \right\} = 0. \tag{5}$$

Note that the first and second integral represent the odd moments $2r + 1$ and $2r + 3$ of the standard logistic distribution, for $r = 0, 1, 2, \dots$. It is well known that the odd moments of this distribution are equal to zero (see Johnson et al. [21]).

(b) Let $X \sim \text{Logistic}(0, 1)$ the even moments of X (see Balakrishnan [22]) are given by:

$$E(X^k) = 2(1 - 2^{1-k})\Gamma(k + 1)\zeta(k), \quad \text{if } k \text{ is even.}$$

Then using these moments, we can calculate the even moments of $Z \sim BL(\alpha)$ as

$$\begin{aligned} E[Z^{2r}] &= \frac{3}{3 + \pi^2\alpha} \left\{ \int_{-\infty}^{\infty} \frac{z^{2r} \exp(z)}{(1 + \exp(z))^2} dz + \alpha \int_{-\infty}^{\infty} \frac{z^{2r+2} \exp(z)}{(1 + \exp(z))^2} dz \right\} \\ &= \frac{3}{3 + \pi^2\alpha} \left\{ 2(1 - 2^{1-2r})\Gamma(2r + 1)\zeta(2r) + \alpha 2(1 - 2^{-2r-1})\Gamma(2r + 3)\zeta(2r + 2) \right\} \\ &= \frac{6}{3 + \pi^2\alpha} \left\{ (1 - 2^{1-2r})\Gamma(2r + 1)\zeta(2r) + \alpha(1 - 2^{-2r-1})\Gamma(2r + 3)\zeta(2r + 2) \right\}. \end{aligned}$$

□

Definition 2. Let $Z \sim BL(\alpha)$ and $Y = \mu + \sigma Z$, then the density of Y is given by

$$f_Y(y; \mu, \sigma, \alpha) = \frac{3}{\sigma^3} \left(\frac{\sigma^2 + \alpha(y - \mu)^2}{3 + \pi^2\alpha} \right) \frac{\exp\left(\frac{y - \mu}{\sigma}\right)}{\left(1 + \exp\left(\frac{y - \mu}{\sigma}\right)\right)^2}, \quad y \in \mathbb{R}, \tag{6}$$

where $\mu \in \mathbb{R}$ is the location parameter, $\sigma > 0$ is the scale parameter and $\alpha \geq 0$ is the bimodality parameter. It is denoted as $Y \sim BL(\mu, \sigma, \alpha)$.

Corollary 1. Let $Y \sim BL(\mu, \sigma, \alpha)$. Then for $k = 1, 2, \dots$, the k -th moments of the random variable of Y are:

$$E[Y^k] = \sum_{j=0}^k \binom{k}{j} \sigma^j E(Z^j) \mu^{k-j} = \sum_{j=0}^{2r} \binom{2r}{2j} \sigma^{2j} E(Z^{2j}) \mu^{2(r-j)},$$

where $r = \text{int}(k/2)$, $\text{int}(\cdot)$ denotes the integer part function.

Proof. Using binomial theorem, we have that

$$E[Y^k] = E[(\sigma Z + \mu)^k] = \sum_{j=0}^k \binom{k}{j} \sigma^j E(Z^j) \mu^{k-j}.$$

□

Corollary 2. Let $Y \sim BL(\mu, \sigma, \alpha)$. Then

$$E[Y] = \mu, \quad \text{Var}[Y] = \sigma^2 \left(\frac{7\alpha\pi^4 + 5\pi^2}{5\alpha\pi^2 + 15} \right).$$

Proof. Considering the results of Corollary 1, we have that $E[Y] = \mu$ and $E[Y^2] = \mu^2 + \sigma^2 \left(\frac{7\alpha\pi^4 + 5\pi^2}{5\alpha\pi^2 + 15} \right)$. Thus, to obtain the second central moment the equality $\text{Var}[Y] = E[Y^2] - (E[Y])^2$ must be used. □

Corollary 3. Let $Y \sim BL(\mu, \sigma, \alpha)$. Then the kurtosis coefficient (β_2) of Y is

$$\beta_2(\alpha) = \frac{E[Y - E[Y]]^4}{(\text{Var}[Y])^2} = \frac{5(155\alpha\pi^2 + 49)(\alpha\pi^2 + 3)}{7(7\alpha\pi^2 + 5)^2}. \tag{7}$$

We note that:

1. Considering β_2 as a function of α , we have that it is increasing in the interval $[0, \frac{1}{4\pi^2})$ and is decreasing in $(\frac{1}{4\pi^2}, \infty)$.
2. By straightforward calculation we show that $\lim_{\alpha \rightarrow 0^+} \beta_2 = 4.2$ and $\lim_{\alpha \rightarrow \infty} \beta_2 = \frac{775}{343}$.

Proof. Considering the first derivative with respect to α of the $\beta_2(\alpha)$, $\beta'_2 = -\frac{2560\pi^2(4\alpha\pi^2-1)}{7(7\alpha\pi^2+5)^3}$, we have that the critical value is obtained at $\alpha = \frac{1}{4\pi^2}$. Taking into account the intervals $[0, \frac{1}{4\pi^2})$ and $(\frac{1}{4\pi^2}, \infty)$ it follows that: $\beta'_2(\alpha) > 0$ in $[0, \frac{1}{4\pi^2})$ and it is therefore increasing in this interval; $\beta'_2(\alpha) < 0$ in $(\frac{1}{4\pi^2}, \infty)$ and it is therefore decreasing in this interval. Finally, when we evaluate $\beta''_2(1/4\pi^2) = -\frac{655.360}{137.781}\pi^4 < 0$ and it is therefore a maximum. \square

Remark 1. Figure 2 presents a graph of the kurtosis coefficient of the BL distribution. As the BL distribution is symmetric, the asymmetry coefficient is zero. The values of the kurtosis coefficient of the BL model are smaller than those of the Logistic model; this frequently occurs since the presence of more than one mode means that the distribution tails are lighter.

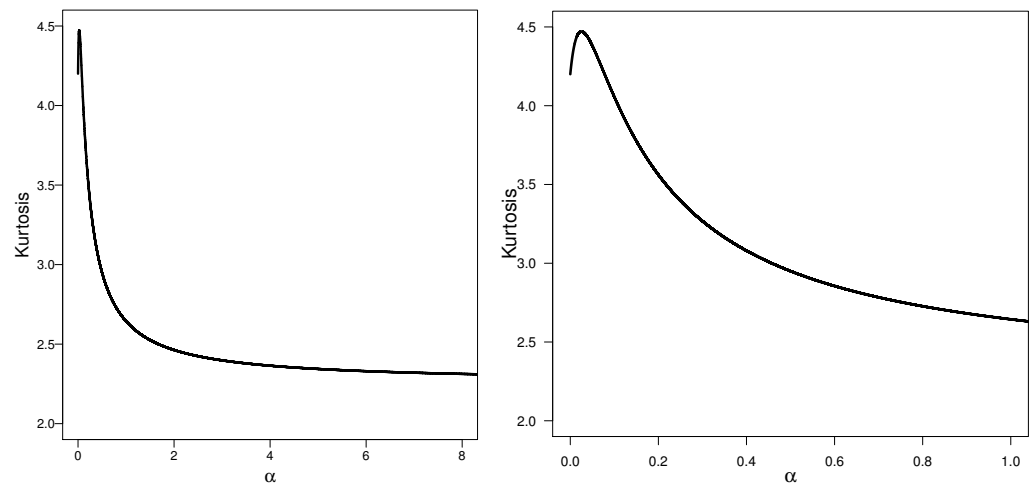


Figure 2. Graph of the kurtosis coefficient in BL(α) distribution.

3. Inference

In this section, we develop inferences from the location-scale version of the BL distribution studied above. The inferences are based mainly on M and ML estimation and a simulation study to see the behaviour of the M estimates (ME) and ML estimates (MLE); we will also derive the Fisher information matrix, studying the singularity of a special case.

3.1. Moment Estimators

Proposition 3. Let Y_1, \dots, Y_n be a random sample of the random variable $Y \sim BL(\theta)$; to find the M estimators for $\theta = (\mu, \sigma, \alpha)$, the following three equations must be solved simultaneously:

$$\hat{\mu}_M = \bar{Y}, \tag{8}$$

$$(343b_2 - 775)\pi^4\hat{\alpha}_M^2 + (490b_2 - 2570)\pi^2\hat{\alpha}_M = 735 - 175b_2, \tag{9}$$

$$\hat{\sigma}_M = \sqrt{S^2 \left(\frac{5\hat{\alpha}_M\pi^2 + 15}{7\hat{\alpha}_M\pi^4 + 5\pi^2} \right)}, \tag{10}$$

where \bar{Y} is the mean of the sample (first sample moment), S^2 is the sample variance and b_2 is the sample kurtosis coefficient. For a solution to Equation (9) to exist it must be the case $\frac{775}{343} < b_2 < \frac{845}{189}$.

Proof. From Corollaries 2 and 3, replace $E[Y]$ with \bar{Y} , $Var[Y]$ with S^2 and β_2 with b_2 , the M estimators $\hat{\theta} = (\hat{\mu}_M, \hat{\sigma}_M, \hat{\alpha}_M)$ are obtained. \square

3.2. ML Estimators

Given a random sample Y_1, \dots, Y_n of the BL(μ, σ, α) distribution, the log-likelihood function can be written as:

$$\ell(\theta) = c(\sigma, \alpha) + \sum_{i=1}^n z_i + \sum_{i=1}^n \log(1 + \alpha z_i^2) - 2 \sum_{i=1}^n \log\{1 + \exp(z_i)\}, \tag{11}$$

where $c(\sigma, \alpha) = n\{\log(3) - \log(\sigma) - \log(3 + \pi^2\alpha)\}$, $\theta = (\mu, \sigma, \alpha)$, $z_i = \frac{y_i - \mu}{\sigma}$ and $i = 1, \dots, n$.

To calculate the estimation for $\theta = (\mu, \sigma, \alpha)$ it is necessary to maximize (11), by solving the following system of equations $\frac{\partial \ell(\theta)}{\partial \mu} = 0$; $\frac{\partial \ell(\theta)}{\partial \sigma} = 0$ and $\frac{\partial \ell(\theta)}{\partial \alpha} = 0$. More precisely,

$$\sum_{i=1}^n \frac{\exp(z_i)}{1 + \exp(z_i)} - \alpha \sum_{i=1}^n \frac{z_i}{1 + \alpha z_i^2} = \frac{n}{2}, \tag{12}$$

$$2 \sum_{i=1}^n \frac{1}{1 + \alpha z_i^2} - \sum_{i=1}^n z_i + 2 \sum_{i=1}^n \frac{z_i \exp(z_i)}{1 + \exp(z_i)} = 3n, \tag{13}$$

$$\sum_{i=1}^n \frac{z_i^2}{1 + \alpha z_i^2} = \frac{\pi^2 n}{3 + \pi^2 \alpha}, \tag{14}$$

must be solved.

The solution for Equations (12)–(14) can be obtained using numerical procedures like Newton-Raphson, which can be implemented in the R software [23].

3.3. Simulation Study

In this section we present the results of a simulation study designed to assess the performance of the BL distribution. The acceptance-rejection method was used to carry out the simulations in this study. Each simulation consisted of 1000 samples of sizes 50, 100 and 200, and the following calculations were obtained for each: mean, root mean squared error (RMSE), mean absolute deviation (MAD) of the ME and MLE. Table 2 shows the results for the samples simulated by acceptance-rejection of sizes 50, 100 and 200 respectively.

Table 2. MLE, RMSE and MAD for the BL model with sample size 50, 100 and 200, respectively.

Parameter	Value	n = 50			n = 100			n = 200		
		MLE	RMSE	MAD	MLE	RMSE	MAD	MLE	RMSE	MAD
μ	68.500	68.491	0.412	0.326	68.491	0.289	0.228	68.503	0.201	0.161
$\log(\sigma)$	0.000	−0.057	0.125	0.089	−0.028	0.091	0.068	−0.010	0.064	0.050
$\log(\alpha)$	−0.693	−0.182	1.751	0.605	−0.499	0.478	0.346	−0.622	0.333	0.262
μ	68.500	68.521	0.389	0.305	68.510	0.272	0.217	68.509	0.193	0.154
$\log(\sigma)$	0.000	−0.041	0.118	0.087	−0.017	0.086	0.067	−0.007	0.065	0.051
$\log(\alpha)$	−0.511	−0.024	2.204	0.692	−0.345	0.536	0.396	−0.446	0.339	0.265
μ	68.500	68.503	0.403	0.319	68.501	0.270	0.211	68.498	0.188	0.150
$\log(\sigma)$	0.000	−0.032	0.112	0.086	−0.011	0.084	0.066	−0.004	0.061	0.048
$\log(\alpha)$	−0.357	0.239	3.060	0.884	−0.239	0.546	0.412	−0.302	0.351	0.277
μ	68.500	68.474	0.628	0.495	68.470	0.417	0.332	68.489	0.295	0.237
$\log(\sigma)$	0.405	0.344	0.125	0.087	0.372	0.085	0.061	0.389	0.060	0.046
$\log(\alpha)$	−0.693	−0.076	2.052	0.613	−0.436	0.496	0.338	−0.561	0.326	0.238
μ	68.500	68.483	0.572	0.451	68.475	0.390	0.31	68.486	0.291	0.232
$\log(\sigma)$	0.405	0.358	0.115	0.084	0.382	0.083	0.064	0.397	0.059	0.047
$\log(\alpha)$	−0.511	0.126	2.758	0.772	−0.283	0.512	0.363	−0.431	0.331	0.26

Table 2. Cont.

Parameter	Value	n = 50			n = 100			n = 200		
		MLE	RMSE	MAD	MLE	RMSE	MAD	MLE	RMSE	MAD
μ	68.500	68.497	0.574	0.452	68.500	0.416	0.331	68.503	0.285	0.228
$\log(\sigma)$	0.405	0.369	0.110	0.082	0.383	0.084	0.065	0.397	0.058	0.046
$\log(\alpha)$	-0.357	0.341	3.316	0.949	-0.158	0.527	0.385	-0.296	0.338	0.269
μ	69.000	69.007	0.425	0.336	68.996	0.282	0.223	68.988	0.202	0.161
$\log(\sigma)$	0.000	-0.053	0.125	0.091	-0.029	0.091	0.068	-0.010	0.062	0.049
$\log(\alpha)$	-0.693	-0.263	1.141	0.516	-0.485	0.514	0.367	-0.623	0.319	0.252
μ	69.000	69.006	0.414	0.324	68.995	0.268	0.219	69.003	0.196	0.156
$\log(\sigma)$	0.000	-0.038	0.117	0.088	-0.020	0.085	0.067	-0.007	0.060	0.048
$\log(\alpha)$	-0.511	-0.046	2.019	0.659	-0.301	1.053	0.406	-0.447	0.354	0.277
μ	69.000	68.992	0.393	0.309	68.991	0.254	0.203	68.995	0.181	0.143
$\log(\sigma)$	0.000	-0.031	0.112	0.085	-0.012	0.081	0.063	-0.008	0.057	0.046
$\log(\alpha)$	-0.357	0.178	2.704	0.810	-0.220	0.533	0.41	-0.296	0.342	0.269
μ	69.000	68.988	0.620	0.49	69.003	0.417	0.335	69.001	0.287	0.228
$\log(\sigma)$	0.405	0.343	0.130	0.092	0.370	0.088	0.064	0.386	0.062	0.047
$\log(\alpha)$	-0.693	-0.098	1.996	0.610	-0.435	0.503	0.336	-0.559	0.321	0.236
μ	69.000	68.962	0.602	0.481	68.995	0.405	0.323	68.988	0.277	0.219
$\log(\sigma)$	0.405	0.354	0.123	0.087	0.382	0.082	0.063	0.395	0.058	0.046
$\log(\alpha)$	-0.511	0.096	2.606	0.745	-0.276	1.041	0.402	-0.427	0.335	0.258
μ	69.000	69.028	0.581	0.449	69.030	0.395	0.312	69.005	0.271	0.216
$\log(\sigma)$	0.405	0.369	0.112	0.083	0.387	0.080	0.062	0.402	0.057	0.045
$\log(\alpha)$	-0.357	0.401	3.393	0.994	-0.173	0.506	0.377	-0.290	0.344	0.270

In Table 2 it is observed that as the sample size increases the MLE of μ , $\log(\sigma)$ and $\log(\alpha)$ approach the true values of the parameters. Furthermore, we note that RMSE and MAD for the parameters μ , $\log(\sigma)$ and $\log(\alpha)$ decrease as the sample size increases, as expected. However we observe that the MLE of the parameters $\log(\sigma)$ and $\log(\alpha)$ underestimate and overestimate the true values in all cases.

In Table 2, RMSE denotes the square root of the empirical mean squared error: for instance, for $\hat{\mu}$, it is calculated as

$$RMSE(\hat{\mu}) = \sqrt{\frac{1}{n} \sum_{i=1}^n (\hat{\mu}_i - \mu)^2},$$

and so on.

The expression of Equation (2) can be rewritten as a mixture of two densities, as follows

$$\begin{aligned} f_Z(z; \alpha) &= \frac{3}{(3 + \pi^2\alpha)} \frac{\exp(z)}{(1 + \exp(z))^2} + \frac{\alpha\pi^2}{(3 + \pi^2\alpha)} \frac{3z^2 \exp(z)}{\pi^2(1 + \exp(z))^2} \\ &= \frac{3}{(3 + \pi^2\alpha)} f_1(z) + \frac{\alpha\pi^2}{(3 + \pi^2\alpha)} f_2(z) \\ &= p_1 f_1(z) + p_2 f_2(z), \end{aligned}$$

where $f_1(z)$ is the pdf of standard logistic distribution, $f_2(z)$ is the pdf present in Property 2, p_1 , and p_2 are the respective weights. Thus, the algorithm for generating random numbers from the $BL(\mu, \sigma, \alpha)$ distribution is shown in Algorithm 1.

Algorithm 1 Generation of random numbers from the distribution $BL(\mu, \sigma, \alpha)$

```

repeat  $i = 1$ 
   $U_{1i} \sim U(0,1)$ 
  if  $U_{1i} \leq 3/(3 + \pi^2\alpha)$  then
     $Z_i \sim \text{Logis}(0,1)$ 
     $X_i = Z_i\sigma + \mu$ 
  else
     $k=0$ 
    while  $k == 0$  do
       $U_{2i} \sim U(0,1)$ 
       $Y_i \sim \text{Cauchy}(0,1)$ 
      if  $U_{2i} \leq [3y_i^2 \exp(y_i)(1 + y_i^2)]/[c\pi(1 + \exp(y_i))^2]$  then
         $Z_i = Y_i$ 
         $X_i = Z_i\sigma + \mu$ 
         $k=1$ 
      end if
    end while
  end if
until  $i = n$ 

```

3.4. Fisher Information Matrix

Below we present the Fisher information matrix for the random variable $Y \sim BL(\mu, \sigma, \alpha)$, which can be written as:

$$\mathcal{I}_f(\boldsymbol{\theta}) = \begin{pmatrix} \mathcal{I}_{\mu\mu} & \mathcal{I}_{\mu\sigma} & \mathcal{I}_{\mu\alpha} \\ & \mathcal{I}_{\sigma\sigma} & \mathcal{I}_{\sigma\alpha} \\ & & \mathcal{I}_{\alpha\alpha} \end{pmatrix},$$

with

$$\begin{aligned} \mathcal{I}_{\mu\mu} &= \frac{3 - 24\alpha + \alpha\pi^2}{3\sigma^2(3 + \pi^2\alpha)} + 4\alpha^2\eta_2, & \mathcal{I}_{\mu\sigma} &= 0, & \mathcal{I}_{\mu\alpha} &= 0, \\ \mathcal{I}_{\sigma\sigma} &= \frac{7\alpha\pi^4 - 15\alpha\pi^2 + 5\pi^2 - 165}{15\sigma^2(3 + \alpha\pi^2)} + 4\sigma^2\eta_0, & \mathcal{I}_{\sigma\alpha} &= 2\sigma\eta_2, & \mathcal{I}_{\alpha\alpha} &= \eta_4 - \frac{\pi^4}{(3 + \pi^2\alpha)^2}, \end{aligned}$$

where $\eta_j = E\left[\frac{(Y-\mu)^j}{(\sigma^2 + \alpha(Y-\mu)^2)^2}\right]$ and the first and second derivatives are given in Appendix A.

3.5. Special Case

Now we will consider the Fisher information matrix for a special case of the BLlocation-scale model. A situation of particular interest is the case when $\alpha = 0$ in Equation (2) resulting from the logistic model. In this case, the elements of the Fisher information matrix are given by

$$\mathcal{I}_f(\mu, \sigma, \alpha = 0) = \begin{pmatrix} \frac{1}{3\sigma^2} & 0 & 0 \\ 0 & \frac{(\pi^2 + 3)}{9\sigma^2} & \frac{2\pi^2}{3\sigma} \\ 0 & \frac{2\pi^2}{3\sigma} & \frac{16\pi^4}{45} \end{pmatrix},$$

its determinant is given by

$$|\mathcal{I}_f| = \frac{16\pi^6 - 132\pi^4}{1215\sigma^4} \neq 0.$$

From this we may conclude that the Fisher information matrix is non-singular when $\alpha = 0$. Then, for large samples, the ML estimators $\hat{\theta}$ of θ have an asymptotically normal distribution, i.e.,

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{\mathcal{D}} N_3(\mathbf{0}, \mathcal{I}_f^{-1}(\theta)),$$

where the asymptotic variance of the ML estimators $\hat{\theta}$ is $\frac{\mathcal{I}_f^{-1}(\theta)}{n}$. As the parameters are unknown, the Fisher information matrix is estimated using the MLE for the unknown parameters.

4. Application

In this section the fit of the BL model is illustrated in a data set that has been analysed previously by Gui et al. [24], Hassan and Hijazi [25], and others. To do this, we used the M and ML methods described in Section 3.

The fit of the BL model with this data set is compared with that of the logistic mixture (LM) model. The respective density is given by:

$$f(x; \mu_1, \sigma_1, \mu_2, \sigma_2, p) = \frac{p \exp\left(\frac{x-\mu_1}{\sigma_1}\right)}{\sigma_1 \left(1 + \exp\left(\frac{x-\mu_1}{\sigma_1}\right)\right)^2} + \frac{(1-p) \exp\left(\frac{x-\mu_2}{\sigma_2}\right)}{\sigma_2 \left(1 + \exp\left(\frac{x-\mu_2}{\sigma_2}\right)\right)^2}.$$

This data set consists of the height in inches of 126 students of the University of Pennsylvania, [26]. Table 3 presents the basic descriptive statistics of this data set, showing that the asymmetry of the sample is very close to zero. Furthermore, the mean and median values are very similar, indicating that the data are symmetrical.

Table 3. Descriptive statistics for the height data set.

<i>n</i>	Median	Mean	Standard Deviation	Range	Skewness
126	68.44	68.54	4.16	23.5	−0.05

The estimations of the parameters of the BL model were obtained by the M method. These results are shown in Table 4.

Table 4. ME for the BL model.

Model	$\hat{\alpha}_M$	$\hat{\mu}_M$	$\hat{\sigma}_M$
BL	0.418	68.546	1.357

Using the estimations in Table 4 as the initial values, the height data were fitted with the BL distribution by ML. The standard errors (SE) were calculated from the Fisher information matrix presented in Section 3.4 by numerical integration with the integral function of the *pracma* package in the R software (see Borchers [27]). The estimations and standard errors were also obtained for the LM model in order to compare the fits. Akaike’s information criterion (AIC) was used to measure the selection in both models, which is defined as $AIC = 2k - 2\log \text{lik}$, where k is the number of parameters in the model and $\log \text{lik}$ is the maximum value for the log-likelihood function (see Akaike [28]). These results are presented in Table 5, where we observe that the BL model presents a better fit as its AIC value is lower than that of the LM model.

Table 5. MLE, with SE in parentheses of the BL and LM models.

Model	$\hat{\alpha}$	\hat{p}	$\hat{\mu}_1$	$\hat{\sigma}_1$	$\hat{\mu}_2$	$\hat{\sigma}_2$	loglik	AIC
BL	0.646 (0.272)	- -	68.627 (0.299)	1.268 (0.097)	- -	- -	-355.118 -	716.236 -
LM	- -	0.534 (0.119)	65.867 (0.661)	1.589 (0.282)	71.628 (0.729)	1.589 (0.306)	-355.75 -	721.5 -

Figure 3 shows the fit of the two densities to the data, and Figure 4 shows the respective QQ-plots for the BL and LM models.

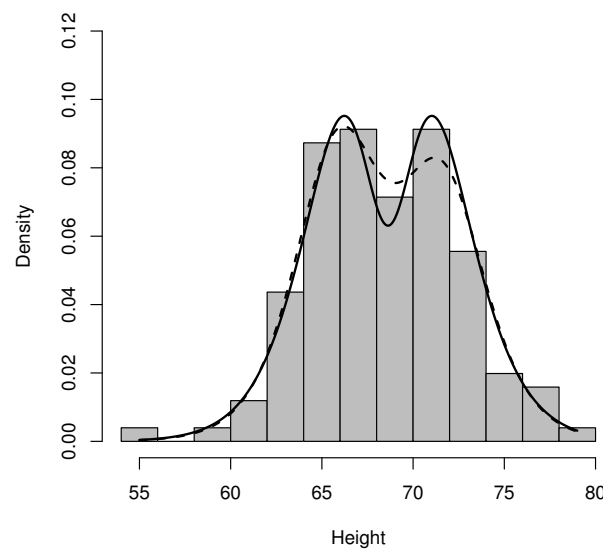


Figure 3. Histogram and models fitted for the height data set. The lines represent the fits using maximum likelihood estimation: LM (dashed line) and BL (solid line).

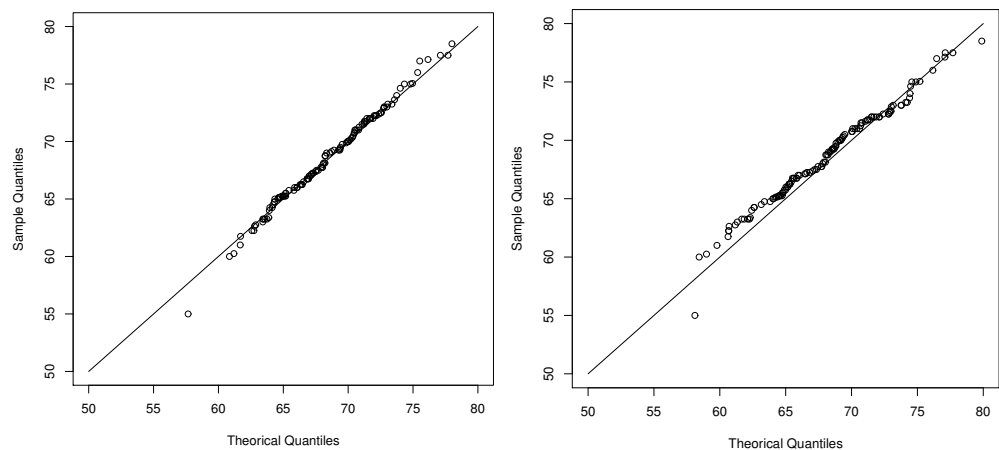


Figure 4. (Left panel): QQ-plot for BL model. (Right panel): QQ-plot for LM model.

5. An Asymmetric Extension

To make the modes in the BL distribution more flexible, we introduce an extension based on the methodology presented by Azzalini [29]. Namely, let f_0 be a pdf symmetric about zero, and G is a cdf such that G' exists and is a pdf symmetric about zero, then

$$f(z; \lambda) = 2f_0(z)G(\lambda z) \quad , z \in \mathbb{R},$$

is a pdf for any $\lambda \in \mathbb{R}$. Using the BL distribution in this result obtains the skew BL (SBL) distribution and its pdf is given by

$$f(z; \mu, \sigma, \alpha, \lambda) = \frac{6}{\sigma^3} \left(\frac{\sigma^2 + \alpha(z-\mu)^2}{3 + \pi^2 \alpha} \right) \frac{\exp((z-\mu)/\sigma)}{(1 + \exp((z-\mu)/\sigma))^2} \frac{\exp(\lambda(z-\mu)/\sigma)}{(1 + \exp(\lambda(z-\mu)/\sigma))}, \tag{15}$$

where $z \in \mathbb{R}, \mu \in \mathbb{R}, \sigma > 0, \alpha \geq 0$ and $\lambda \in \mathbb{R}$. We denote it by $Z \sim \text{SBL}(\mu, \sigma, \alpha, \lambda)$.

Remark 2. Figure 5 shows the flexibility in the modes for different values of λ . When $\lambda = 0$ the BL distribution is obtained. Furthermore, when $\alpha = 0$ the skew-Logist distribution studied in Nadarajah [30] and Gupta and Kundu [31] is obtained. All the properties studied in the BL distribution can be extended to the SBL distribution.

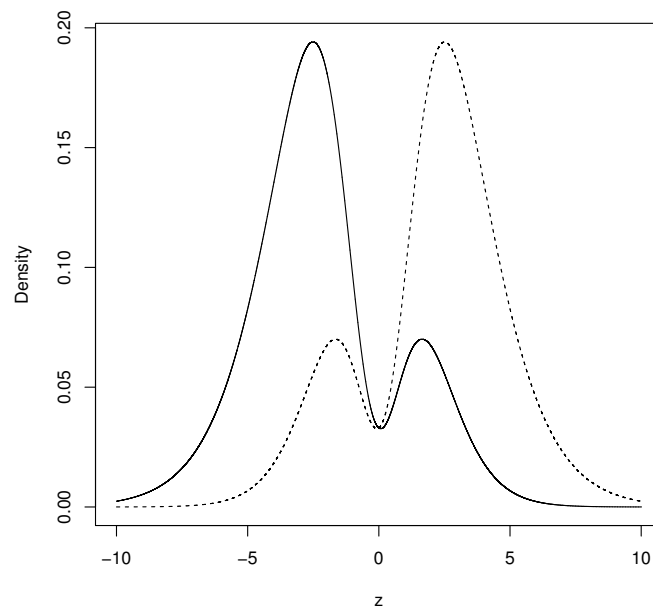


Figure 5. Graph of SBL(0, 1, 2, −0.5) (solid line) and SBL(0, 1, 2, 0.5) (dashed line).

5.1. Application 2

In this subsection we fit the data considered in the previous application. Table 6 shows the ML fits of the BL and SBL distributions, from which, using the AIC criterion, we can observe that the distribution offering the best fit with the data is the BL distribution. Figure 6 shows the graphs of the QQ-plots of the BL and SBL distributions.

Table 6. MLE, with SE in parentheses of the BL and SBL models.

Model	$\hat{\alpha}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\lambda}$	AIC
BL	0.646 (0.272)	68.627 (0.299)	1.268 (0.097)	- -	716.236 -
SBL	0.644 (0.277)	68.767 (0.499)	1.271 (0.100)	−0.033 (0.092)	718.106 -

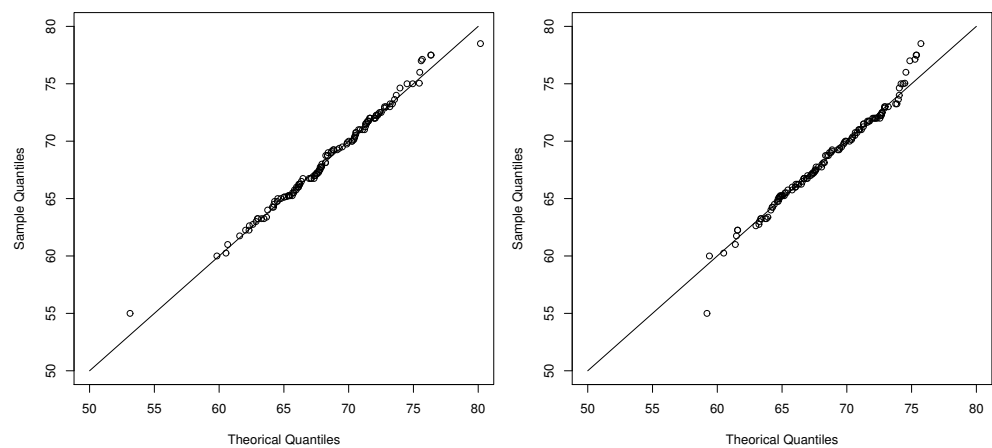


Figure 6. (Left panel): QQ-plot for BL model. (Right panel): QQ-plot for SBL model.

We use the results of the estimations given in Table 6 to test the null hypothesis ($\lambda = 0$) that the data are obtained from the BL distribution against the SBL distribution, and the following statistical test

$$-2 \log \Lambda(\mathbf{z}) = -2 \log \frac{L_{BL}(\hat{\mu}, \hat{\sigma}, \hat{\alpha})}{L_{SBL}(\hat{\mu}, \hat{\sigma}, \hat{\alpha}, \hat{\lambda})}$$

where $\Lambda(\mathbf{z})$ is the likelihood-ratio statistic. The value of the study data set is $-2 \log \Lambda(\mathbf{z}) = 0.13$, which is comparable with the critical value of $\chi_1^2 = 3.84$ at 95%. There is therefore no evidence to reject the null hypothesis, in other words the BL distribution presents a better fit than the SBL distribution. Consequently, it is concluded that the data in Application 1 have a symmetrical bimodal behaviour.

5.2. Application 3

We consider fitting the SBL distribution to a data set available from the web address <http://lib.stat.cmu.edu/datasets/pollen.data> (accessed on 15 March 2022). More specifically, we analyse the 481 observations of the variable “Nub” in the data file POLLEN2.DAT. This variable measures a geometric characteristic of a specific type of pollen. In Table 7, the descriptive analysis is provided.

Table 7. Descriptive statistics for the pollen data set.

<i>n</i>	Median	Mean	Standard Deviation	Range	Skewness
481	−0.298	−0.055	5.071	30.609	−0.149

Table 8 shows the ML fits of the LM and SBL distributions; we observe that by the AIC criterion, the distribution with the best fit to the data is the SBL distribution. Figures 7 and 8 show the fits for the pollen data sets and the graphs of the QQ-plots of the LM and SBL distributions respectively.

Table 8. MLE, with SE in parentheses of the SBL and LM models.

Model	$\hat{\mu}_1$	$\hat{\sigma}_1$	$\hat{\mu}_2$	$\hat{\sigma}_2$	\hat{p}	$\hat{\alpha}$	$\hat{\lambda}$	AIC
LM	−1.826 (0.466)	2.391 (0.172)	5.429 (0.564)	1.508 (0.220)	0.751 (0.064)	-	-	2924.36
SBL	1.716 (0.417)	1.700 (0.089)	-	-	-	0.506 (0.117)	−0.235 (0.071)	2922.72

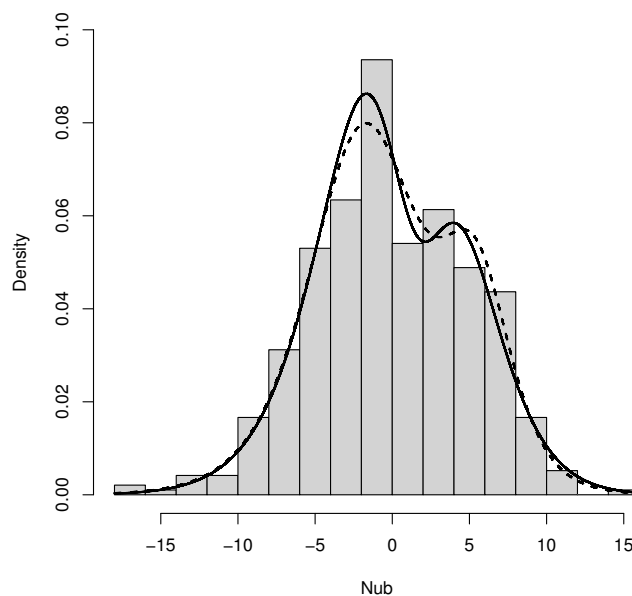


Figure 7. Histogram and models fitted for the pollen data set. The lines represent the fits using maximum likelihood estimation: SBL (solid line) and LM (dashed line).

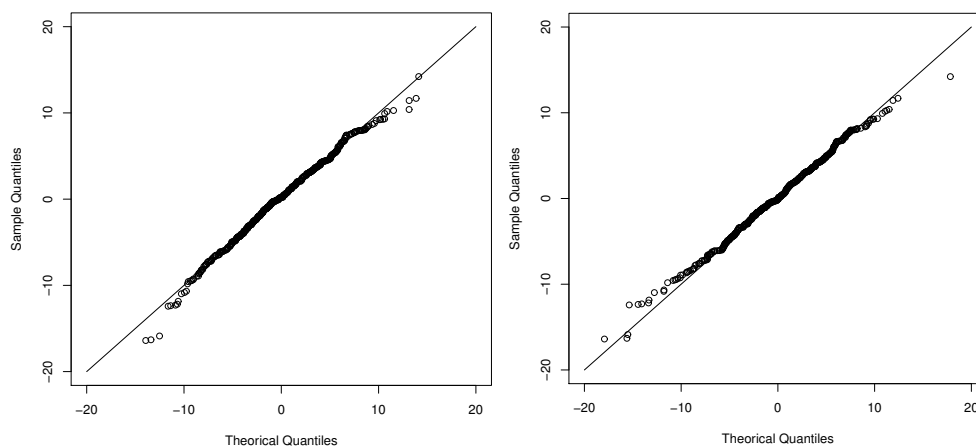


Figure 8. (Left panel): QQ-plot for SBL model. (Right panel): QQ-plot for LM model.

6. Concluding Remarks

We present bimodal extensions of the Logistic distribution, one symmetric and the other asymmetric. These distributions are useful for modelling symmetric/asymmetric bimodal data sets. We present some important properties and discuss two methods of estimation based on the M and ML estimators, obtained using numerical procedures such as the Newton-Raphson procedure. Three applications in the real data set are presented which show the usefulness of the BL and SBL distributions in comparison with two other bimodal distributions. Some other characteristics of the new models are:

- The proposed model has a closed-form expression and contains the logistic model as a particular case.
- The even moments are based on the Riemann Zeta Function.
- The simulation study shows that the behaviour of the M and ML estimators is good.
- The Fisher information matrix is non-singular in the symmetric unimodal special case; this is very important for drawing asymptotic inference from the ML estimators. This result, i.e., a non-singular Fisher information matrix, cannot be obtained with other asymmetric extensions of the logistic model.

- In the application we observe that the BL model shows a better fit with the data than the typical mixture model.
- We also introduce an asymmetric bimodal extension of the Logistic distribution for situations where there are bimodal data with flexible modes. This extension is an alternative to the finite mixture of logistic distribution, as is shown by the first and third application.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Derivatives

The first derivatives of $\ell(\theta)$, taking $z = (y - \mu)/\sigma$, are given by

$$\frac{\partial \ell(\theta)}{\partial \mu} = -\frac{1}{\sigma} + \frac{2 \exp(z)}{\sigma(1 + \exp(z))} - \frac{2\alpha z}{\sigma(1 + \alpha z^2)},$$

$$\frac{\partial \ell(\theta)}{\partial \sigma} = -\frac{3}{\sigma} - \frac{z}{\sigma} + \frac{2}{\sigma(1 + \alpha z^2)} + \frac{2z \exp(z)}{\sigma(1 + \exp(z))},$$

$$\frac{\partial \ell(\theta)}{\partial \alpha} = -\frac{\pi^2}{(3 + \pi^2 \alpha)} + \frac{z^2}{(1 + \alpha z^2)},$$

and the second derivatives of $\ell(\theta)$ are

$$\frac{\partial^2 \ell(\theta)}{\partial \mu^2} = \frac{2 \exp(2z)}{\sigma^2(1 + \exp(z))^2} - \frac{2 \exp(z)}{\sigma^2(1 + \exp(z))} - \frac{4\alpha^2 z^2}{\sigma^2(1 + \alpha z^2)^2} + \frac{2\alpha}{\sigma^2(1 + \alpha z^2)},$$

$$\frac{\partial^2 \ell(\theta)}{\partial \mu \partial \sigma} = \frac{1}{\sigma^2} - \frac{2 \exp(z)}{\sigma^2(1 + \exp(z))} + \frac{2z \exp(2z)}{\sigma^2(1 + \exp(z))^2} - \frac{2z \exp(z)}{\sigma^2(1 + \exp(z))} + \frac{4\alpha z}{\sigma^2(1 + \alpha z^2)^2},$$

$$\frac{\partial^2 \ell(\theta)}{\partial \mu \partial \alpha} = \frac{2\alpha z^3}{\sigma(1 + \alpha z^2)^2} - \frac{2z}{\sigma(1 + \alpha z^2)},$$

$$\begin{aligned} \frac{\partial^2 \ell(\theta)}{\partial \sigma^2} &= \frac{3}{\sigma^2} + \frac{2z}{\sigma^2} - \frac{4z \exp(z)}{\sigma^2(1 + \exp(z))} + \frac{2z^2 \exp(2z)}{\sigma^2(1 + \exp(z))^2} \\ &\quad - \frac{2z^2 \exp(z)}{\sigma^2(1 + \exp(z))} - \frac{4}{\sigma^2(1 + \alpha z^2)^2} + \frac{2}{\sigma^2(1 + \alpha z^2)}, \end{aligned}$$

$$\frac{\partial^2 \ell(\theta)}{\partial \sigma \partial \alpha} = -\frac{2z^2}{\sigma(1 + \alpha z^2)^2}, \quad \frac{\partial^2 \ell(\theta)}{\partial \alpha^2} = \frac{\pi^4}{(3 + \alpha \pi^2)^2} - \frac{z^4}{(1 + \alpha z^2)^2}.$$

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