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A SYSTEM OF GAPS IN THE EXPONENT SET OF PRIMITIVE MATRICES

BY

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1. Introduction

A matrix is *nonnegative* (*positive*) if all its entries are nonnegative (positive). A nonnegative square matrix A is *primitive* if $A^k > 0$ for some positive integer k. The smallest such k for the given matrix A is $\gamma(A)$, the *exponent* (of primitivity) of A. Since 1950 when Wielandt [10] first proclaimed the exact general upper bound for γ, there has been a considerable number of papers establishing bounds for special families of primitive matrices. The interested reader is referred to [1], [2], [3], [4], [6], all of which use graph theory as a major tool in the search for γ.

In [2] Dulmage and Mendelsohn reveal what they refer to as *gaps* in the exponent set of primitive matrices. Each gap is a set S of consecutive integers below Wielandt's general bound $W_n = n^2 - 2n + 2$, such that no n-square primitive matrix has an exponent in S. The gaps displayed are

$$n^2 - 3n + 4 < \gamma < (n-1)^2 \quad \text{and} \quad n^2 - 4n + 6 < \gamma < n^2 - 3n + 2.$$

For even n a gap contains the union of the two gaps just mentioned: $n^2 - 4n + 6 < \gamma < (n-1)^2$.

It is the purpose of this paper to disclose a system of such gaps containing the two general gaps just mentioned as special cases. We show that for any integral n and t there is no primitive matrix A of order n for which

$$n^2 - tn + \tfrac{1}{4}(t+1)^2 < \gamma(A) < n^2 - (t-1)n + t - 2.$$

For $t = 3, 4$ these are the gaps shown in [2]. For even n an additional gap is supplied indicating how further gaps may be obtained.

2. Definitions and notations

Let $G(A)$ be the directed graph defined by the nonnegative matrix A. A graph is *primitive* with exponent γ, if it is a graph of a primitive matrix with exponent γ. Let $L(G)$ denote the set of lengths of the simple circuits of G and let $\lambda(G)$ denote the number of the distinct lengths.

It is well known that G is primitive if and only if it is strongly connected and

$$\text{g.c.d. } \{c \mid c \in L(G)\} = 1.$$

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Let u and v be two vertices in G. A path from u to v of length s (number of edges) will be referred to as a *$(u, v \mid s)$-path*. If s is irrelevant, we shall write *(u, v)-path*; if u and v are irrelevant we shall simply write *s-path*. Let the double comma ", ," stand for ", ...,". An s-path will be denoted by $[x_0, x_1, , x_s]$, or shortly by $[x_0, , x_s]$. The length of an (x, y)-path is denoted by $l[x, y]$, the length of a shortest (x, y)-path by $d[x, y]$. $d[x, x]$ is the length of the minimal circuit to which x belongs. Let $\{c_1, , c_k\} \subset L(G)$. Denote by $d(c_1, , c_k)[x, y]$ the shortest (x, y)-path which meets at least one circuit of each length c_j. Define the circum-diameter $d(c_1, , c_k)$ as the minimum of $d(c_1, , c_k)[x, y]$ over all pairs x, y of vertices. V, E will stand for the set of vertices, edges respectively, of the graph in question.

A matrix $A = (a_{ij})$ is said to be of *symmetric pattern*, (sometimes referred to as positively symmetric) if $a_{ji} = 0$ if and only if $a_{ij} = 0$.

Let C_s denote a simple circuit of length s. Let H_n denote the Hamiltonian circuit C_n of the form $[1, 2, , n, 1]$ where $\{1, 2, , n\} = V(G)$. Let $H'_n = [1, 2, , n]$ be the Hamiltonian $(1, n)$-path.

Let $\{c_1, , c_k\}$ be a set of distinct positive integers. Define $(c_1, , c_k) =$ g.c.d. $(c_1, , c_k)$. Let $R(c_1, , c_k)$ denote the set of all integers which are representable as a nonnegative linear combination of elements of $\{c_1, , c_k\}$. Let $(c_1, , c_k) = 1$. Then there exists a largest integer not in R. Denote it by $F(c_1, , c_k)$. Let $F(c_1, , c_k) + 1 = \phi(c_1, , c_k)$. It is well known that for $k = 2$ we have $\phi(c_1, c_2) = (c_1 - 1)(c_2 - 1)$.

3. The dependence of γ on λ

We shall state a preliminary result giving a bound for γ in case $\lambda > 2$.

THEOREM 3.1. *Let G be a primitive graph with $\lambda(G) \geq 3$. Then*

$$\gamma(G) \leq [\tfrac{1}{2}(n^2 - 2n + 4] = [\tfrac{1}{2}W_n] + 1.$$

This bound is attained for every $n \geq 3$.

This theorem is an improvement upon a result of Heap and Lynn [3] in which they prove

$$\gamma \leq n^2 - 2n + 2 - \tfrac{1}{2}(\lambda - 2)(2n - \lambda - 3).$$

We first prove a lemma.

LEMMA 3.1. *Let G be an n-graph and let P, Q, R be simple circuits of lengths p, q, r respectively with $p < q < r$. Let us further assume that any two such circuits intersect. Then $d(p, q, r) \leq 2n - p - 2$.*

Proof. Let $x, y \in V(G)$, and such that $d(p, q, r)[x, y] = d(p, q, r)$. We may assume that x does not belong to circuits of all three lengths. If $x \in R$ let $z = x$; if $x \notin R$ let $z \in V(R)$ satisfy $d[x, z] = \min(u \in R) d[x, u]$.

Case 1. $d[z, y] \geq n - p$. Then every (z, y)-path meets P and Q, so that

$$d(p, q, r) \leq d[x, z] + d[z, y] \leq n - r + n - 1 \leq 2n - p - 2.$$

Case 2. $d[z, y] \leq n - p - 2$. Then $d(p, q, r) \leq (n - r) + r + d[z, y] \leq 2n - p - 2$.

Case 3. $d[z, y] = n - p - 1$. Then on its course from z to y the path meets Q. Adding Q at the appropriate place we get $d(p, q, r) \leq n - r + q + n - p - 1 \leq 2n - p - 2$. This proves the lemma.

We now prove Theorem 3.1.

Case 1. Let $\{p, q, r\} \subset L(G)$ with $p < q < r$ and $(p, q, r) = 1$.

Subcase 1.1. $p \leq \frac{1}{2}(n - 2)$. Then, by a result of Sedlaćek [8] (see also [2, Theorem 1]) we get

$$\gamma(G) \leq n + p(n - 2) \leq \tfrac{1}{2}(n^2 - 2n + 4).$$

Subcase 1.2. $p = \frac{1}{2}(n - 1)$, $q = \frac{1}{2}(n + 1)$. Then $(p, q) = 1$. Clearly every C_p meets every C_r; so does every C_q. If every C_p meets every C_q, then we may use Lemma 3.1 to obtain $d(p, q) \leq d(p, q, r) < 2n - p - 1$. If some C_p does not meet some C_q, then every vertex belongs either to a C_p or to a C_q. Arguing as in the proof of Lemma 3.2 (i) we get $d(p, q) \leq 2n - p - 1$.

$$\begin{aligned}\gamma &\leq d(p, q) + (p - 1)(q - 1)\\ &= 2n - \tfrac{1}{2}(n + 1) + \tfrac{1}{4}(n - 3)(n - 1) \leq \tfrac{1}{2}(n^2 - 2n + 4)\end{aligned}$$

for $n \geq 5$. The only other possibility is $p = 1, q = 2, r = n = 3$ and this is easily checked.

Subcase 1.3. $p = [n/2]$, $\frac{1}{2}(n + 1) < q \leq n - 2$. By [7] we have $\phi(p, q, r) \leq \frac{1}{2}(q - 1)(r - 2)$. The conditions of Lemma 3.1 are satisfied, so that $d(p, q, r) \leq 2n - p - 2$ and hence

$$\begin{aligned}\gamma(G) &\leq 2n - p - 2 + \tfrac{1}{2}(q - 1)(r - 2)\\ &\leq 3n/2 - \tfrac{3}{2} + \tfrac{1}{2}(n - 3)(n - 2) \leq \tfrac{1}{2}(W_n + 1).\end{aligned}$$

Subcase 1.4. $p = [n/2]$, $q = n - 1$. Then $r = n$, so that $\phi(p, q, r) = [\frac{1}{2}(n - 2)^2]$. Let x, y be such that there is no $(x, y|\gamma - 1)$-path in G. We have

Subcase 1.4.1. $d[x, y] \geq n - p$. Then every (x, y)-path meets circuits of all three lengths, so that

$$\gamma(G) \leq n + [\tfrac{1}{2}(n - 2)^2] = [\tfrac{1}{2}W_n] + 1.$$

Subcase 1.4.2. $d[x, y] < n - p$. If $x = y$, then $d[x, y] = p$, so that $n = 2p + 1$. We have $x \in C_p \cap C_n$, so that there is an $(x, x|N)$-path for every $N \geq [\frac{1}{2}(n - 2)^2]$.

If $x \neq y$, then $\{x, y\} \cap C_q \neq \emptyset$. It is therefore possible to add the necessary circuits en passant from x to y, so that

$$\gamma(G) \leq n - p - 1 + [\tfrac{1}{2}(n-2)^2] < [\tfrac{1}{2}W_n].$$

Subcase 1.5. $p > n/2$. Then by [9, Theorem 2] we have $\phi(p, q, r) \leq \frac{1}{2}p(r-2)$, so that by Lemma 3.1 we get

$$\begin{aligned}\gamma \leq 2n - p - 2 + \tfrac{1}{2}p(r-2) &= 2n - 2 + \tfrac{1}{2}p(r-4)\\ &\leq 2n - 2 + \tfrac{1}{2}(n-2)(n-4)\\ &= \tfrac{1}{2}W_n + 1.\end{aligned}$$

Case 2. Let $L(G) = \{a_0, a_1, , a_s\}$. Assume that $(a_i, a_j, a_k) > 1$ for every three members of $L(G)$. Then every pair a_i, a_j must necessarily have at least two prime common divisors, hence $|a_i - a_j| \geq 2 \cdot 3 = 6$. Therefore

$$(*) \qquad n \geq 6(s+1), \quad a_0 \leq n - 6s.$$

Now, applying Theorem 4 in [9] we get

$$\begin{aligned}\gamma(G) \leq d(a_0, , a_s) + \phi(a_0, a_s) &\leq \sum_{t=0}^{s}(n - a_t) + (n-1) + [\tfrac{1}{2}a_0](a_s - 2)\\ &< (s+2)n - (s+1)a_0 + \tfrac{1}{2}a_0(n-2).\end{aligned}$$

Using (*) we have

$$\begin{aligned}\gamma &< \tfrac{1}{2}a_0(n - 2s - 4) + (s+2)n \leq \tfrac{1}{2}(n-6s)(n-2s-4) + (s+2)n\\ &= \tfrac{1}{2}(n^2 - 2n + 4) - ((3s-1)n - 6s(s+2) + 2)\\ &< \tfrac{1}{2}(n^2 - 2n + 4) - ((3s-1)6(s+1) - 6s(s+2) + 2)\\ &= \tfrac{1}{2}(n^2 - 2n + 4) - 4(3s^2 - 1) < \tfrac{1}{2}(n^2 - 2n + 4).\end{aligned}$$

To show that the theorem is best possible, consider $G = (V, E)$ with

$$E = H_n \cup \{(n-1, 1), ([n/2], 1)\}.$$

There are three simple circuits of lengths $p = [n/2]$, $q = n - 1$, $r = n$, so that

$$\phi(p, q, r) = [\tfrac{1}{2}(n-2)^2].$$

The shortest non-empty (n, n)-path is the $(n, n|n)$-path and hence there is no $(n, n|n + \phi(p, q, r) - 1)$-path with

$$n + \phi(p, q, r) - 1 = [\tfrac{1}{2}(n^2 - 2n + 4)] - 1.$$

This completes the proof of the theorem. From Theorem 3.1 it follows that if G is a primitive graph and $\gamma(G) \geq [\frac{1}{2}W_n] + 2$, then necessarily $\lambda = 2$. We shall supply a test enabling us to decide for a given integer m, such that $[\frac{1}{2}W_n] + 2 \leq m < W_n$ whether $m = \gamma(G)$ for some G or not. If not, m belongs to a gap. If there is G for which $m = \gamma(G)$, we supply such a G, and hence such a matrix, which need not be unique.

LEMMA 3.2. *Let G be a primitive n-graph with $L(G) = \{p, q\}$, $p < q$. We then have:*

(i) $d(p, q) \leq 2n - p - 1$.
(ii) *If $p + q > n$, then $d(p, q) \leq n - 1 + q - p$.*

Proof of (i). Every vertex of G lies either on a p-circuit or on a q-circuit. If $x \in C_p$, then it needs at most $n - q$ edges to reach a C_q. Otherwise it needs at most $n - p$ edges to reach a C_p. Then $d(p, q) \leq n - p + n - 1 = 2n - p - 1$.

Proof of (ii). Let $D(p, q)$ be an $(x, y\,|\,d(p, q))$-path in G. Let $\{p, q\} = \{f, g\}$. Let $x \in C_f$. (If $y = x$ we have $d[x, y] \geq p$.)

Case 1. $d[x, y] \geq n - g$. Then $D(p, q)$ meets every g-circuit and hence $d(p, q) = d[x, y] \leq n - 1$.

Case 2. $d[x, y] < n - g$. By completing an f-circuit we meet a g-circuit, since $f + g > n$. Then

$$d(p, q) \leq f + d[x, y] \leq f + n - g - 1 \leq n + q - p - 1.$$

This proves the lemma. It might be of interest to note that the strict inequality sign in (ii) is absolutely necessary, for assuming equality, we can produce for every n and p, $0 < p < n/2$, an n-graph for which $d(p, q) = 2n - p - 1$. Let n and p be given. Put $q = n - p$. Consider the n-graph $G = (V, E)$ where

$$V = \{1, , n\}, \quad E = H'_n \cup \{(p, 1), (q, 1), (n, p + 1)\}.$$

The $(q + 1, n)$-circumpath is then realised by $P_0 = [q + 1, , n, p + 1, , q, 1, , n]$ of length $2n - p - 1$. (Remark: If $p = 1$, then $(p, 1)$ is a loop, and if $n = 2p + 1$, then $p + 1 = q$ and the subpath $(p + 1, , q)$ is void.)

THEOREM 3.2. *Let A be a primitive matrix not of symmetric pattern and let $L(G(A)) = \{p, q\}$, $p < q$. Then there is an edge (x, y) in $G(A)$ such that $d[y, x] = q - 1$. In fact there is such an edge on every q-circuit which is not doubly directed.*

Proof. For $p = 1, 2$ the theorem is clear so we assume $p \geq 3$. Let $E = \{1, 2, , n\}$, let $Q = [1, 2, q, , \bar{1}]$ be a q-circuit and let P_0 be a simple $(1, t\,|\,s)$-path with vertices disjoint from Q except for the endvertices $1, t$. We also require $2 < t < q$ and $1 \leq s < t - 1$. We show that in case such a path exists, there is i_0, $2 \leq i_0 \leq t$, such that $d[i_0, i_0 - 1] = q - 1$. We have clearly $d[i, i - 1] \leq q - 1$ for all i, $i \leq q$ and if $d[i, i - 1] < q - 1$ for some i, then necessarily $d[i, i - 1] = p - 1$. Now, the path $P_0 + [t, , q, 1]$ is a circuit of length less than q so that it is of length p. It follows that $s + q - t + 1 = p$ and hence $t = q - p + s + 1$. Suppose that for all i, $2 \leq i < t$ we have $d[i, i - 1] = p - 1$. Let P_i be an $(i, i - 1)$-path. Consider the path $\sum_{i=t}^{2} P_i + P_0$. This is a closed path and hence a nonnegative linear combination of simple circuits of lengths p and q. The length of this closed path is

$$r = (t-1)(p-1) + s = (q-p+s)(p-1) + s = pq - p - q - p(p-s-2).$$

Since $t < q$, we have $p \geq s+2$ and hence $r \notin R(p, q)$, a contradiction. We assume from now on that there is no $(1, t \mid s)$-path with $s < t-1$. Suppose there is a $(1, t \mid s)$-path with $s \geq t-1$. The inequality $s > t-1$ implies the existence of a simple circuit of length greater than q, a contradiction. The only remaining possibility is $s = t-1$ which means that the $(1, t \mid s)$-path is equal in its length to the $(1, t)$-path along Q. Such paths are the only paths we shall assume to exist (if at all) between nonneighbouring vertices of Q. Suppose there is no (x, y) on Q such that $d[y, x] = q-1$. Let S be a $(2, 1 \mid p-1)$-path. We show $Q \cap S = \{1, 2\}$. Suppose otherwise. Let

$$S = (2, , v_1) + (v_1, , v_2) + \cdots + (v_{k-1}, , v_k) + (v_k, , 1),$$

where $\{v_1, , v_k\} \subset Q$. Put $2 = v_0$, $1 = v_{k+1}$. Consider some subpath $(v_r, , v_{r+1})$ of S. Its length is necessarily equal to the length of the corresponding subpath along Q. We may therefore replace all the subpaths by their corresponding counterparts along Q. We thus obtain a $(2, 1 \mid p-1)$-path along Q, an obvious contradiction, so that $Q \cap S = \{1, 2\}$. Consider a $(1, q \mid p-1)$-path S_1 and a $(q, q-1 \mid p-1)$-path S_2.

Let S_1 and S_2 have vertex $v \neq q$ in common. Let

$$S_1 = (1, , v) + (v, , q), \quad S_2 = (q, , v) + (v, , q-1)$$

with $l(1, , v) = a$, $l(v, , q) = b$, $l(q, , v) = c$, $l(v, , q-1) = d$. We then have $a + b = p-1$, $c + d = p-1$, $b + c \geq p$, since $(v, , q) + (q, , v)$ is a circuit and hence $a + d \leq p-2 < q-2$. It follows that $(1, , v) + (v, , q-1)$ is a $(1, t \mid s)$-path with $t = q-1$, $0 < s < t-1$, a contradiction, so that S_1 and S_2 have only vertex q in common. Now let S_3 be a $(j, j-1 \mid p-1)$-path, $2 < j < q$. Let S_1 and S_3 have vertex w in common. Again let $l(1, , w) = a$, $l(w, , q) = b$, $l(j, , w) = c$, $l(w, , j-1) = d$. We have $a + b = c + d = p-1$, $b + c = q - j$, $a + d = j-2$ so that $2p - 2 = a + b + c + d = q-2$ and hence $2p = q$, so that $(p, q) > 1$, a contradiction. It follows that all $(j, j-1 \mid p-1)$-paths for distinct j are disjoint except for end vertices of neighboring paths. By adding all q such paths we obtain a simple circuit of length $q(p-1)$. Since $p > 2$, this is a contradiction and the theorem is proved.

COROLLARY 3.1. *Let A be a primitive matrix not of symmetric pattern and such that $L(G(A)) = \{p, q\}$. Then $\gamma(A) \geq p(q-1)$.*

Proof. Consider $G(A)$. Let (x, y) be an edge such that $d[y, x] = q-1$. Then there is no $(y, x \mid l)$-path in $G(A)$ with $l = q - 1 + pq - p - q = p(q-1) - 1$ and the corollary follows.

By reasoning similar to those used in this section the following theorem may be derived.

THEOREM 3.3. *Let A be a primitive n-square matrix of symmetric pattern and let r be the length of the maximal odd circuit in $G(A)$. Then $\gamma(A) \leq 2n - r - 1$. Since $r \geq 1$ we have the result of* [5, *Corollary* 2].

COROLLARY 3.2. *If A is a primitive n-square matrix of symmetric pattern, then $\gamma(A) \leq 2(n-1)$.* (See also [6, Corollary 2].)

4. The gaps

We first state and prove some preparatory theorems.

THEOREM 4.1. *Let A be a primitive n-square matrix not of symmetric pattern and let $L(G(A)) = \{p, q\}$, with $p < q$ and $p + q > n$. Then*

$$p(q-1) \leq \gamma(A) \leq n + p(q-2).$$

Moreover, given positive integers p, q, n, m such that $p < q \leq n$, $p + q > n$, $(p, q) = 1$ and satisfying the double inequality $p(q-1) \leq m \leq n + p(q-2)$, there exists a primitive n-square matrix A for which $L(G(A)) = \{p, q\}$ and $\gamma(A) = m$.

Proof. The upper bound follows from Lemma 3.2:

$$\begin{aligned} \gamma(A) &\leq d(p, q) + \phi(p, q) \\ &\leq n + q - p - 1 + (p-1)(q-1) \\ &= n + p(q-2). \end{aligned}$$

The lower bound is Corollary 3.1.

Now let p, q, n, m be positive integers satisfying the given conditions. Let $m = p(q-1) + a$, $0 \leq a \leq n - p$.

Case 1. $p = 1$. Then $q = n$ and $m = q - 1 + a$, $0 \leq a \leq n - 1$. Let $G(A)$ be H_n and the set of loops $\{(i, i) \mid a + 1 \leq i \leq n\}$. There is no $(1, a \mid n + a - 2)$-path so that $\gamma(A) = \gamma(G) = n - 1 + a$.

Case 2. $p = 2$. Then q is odd.

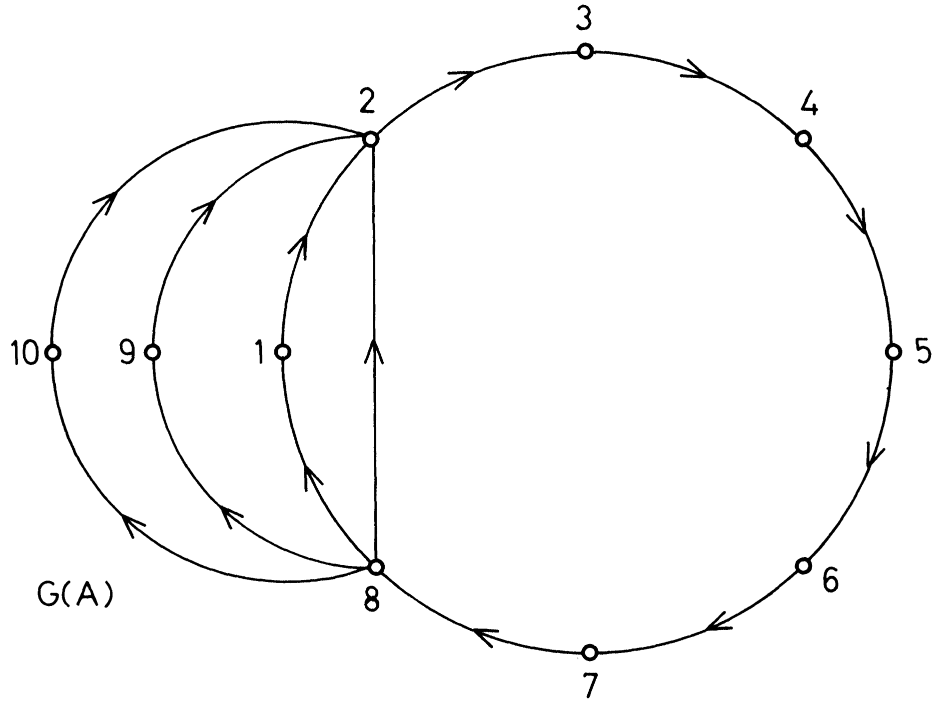
Subcase 2.1. $q = n$. Consider G consisting of H_n and the set of edges

$$\{(i, i-1) \mid a + 2 \leq i \leq q = n\}.$$

Then there is a $(1, a \mid 2(q-1) + a)$-path but no $(1, a \mid 2(q-1) + a - 1)$-path.

Subcase 2.2. $q = n - 1$. Let $V = \{1, , q, 1'\}$. For $a < n - 2$ let the subgraph induced by $V \backslash 1'$ be exactly like the graph in subcase 2.1. Further let $1'$ be a copy of 1 with respect to adjacency and the result follows as in subcase 2.1. For $a = n - 2$ let

$$E = \{(i, i+1) \mid i = 1, 2, , n-1\} \cup \{(1, 1'), (1', 1)\}.$$



There exists a $(2, q\,|\,2(q-1)+n-2)$-path but not a $(2, q\,|\,2(q-1)+n-3)$-path. This completes Case 2.

Case 3. $p \geq 3$.

Subcase 3.1. $q - p \leq a \leq n - p$. Let

$$E = \{[1, 2, , q, 1]\} \cup \{[q, q+1, , a+p, a+1]\} \cup \{[q, j, 2]\,|\,a+p+1 \leq j \leq n\}.$$

(Note that $a+1$ belongs to the q-circuit.) Since $a \leq n-p$ and $q > n-p$, we have $a < q$. There is a $(1, a\,|\,l)$-path in G with $l = a - 1 + q + (p-1)\times(q-1) = p(q-1) + a$, but no $(1, a\,|\,l-1)$-path.

Subcase 3.2. $0 \leq a < q - p$. Consider the circuit $(1, 2, , q, 1)$ with additional edges $(p, 1)$, $(p+1, 2), \ldots, (q-a, q-a-p+1)$. For the vertices $\{j\,|\,q < j \leq n\}$ add further edges $(1, j)$, $(j, 3)$, $(p+1, j)$. There is no $(q-a+1, q\,|\,a-1+p(q-1))$-path in G and hence $\gamma(G) \geq p(q-1) + a$. On the other hand we have $d(p, q) = a - 1 + q$, so that

$$\gamma(G) \leq a - 1 + q + (p-1)(q-1) = p(q-1) + a.$$

It follows that $\gamma(G) = p(q-1) + a$. This proves the theorem.

Choose $q = n$, $p = n - 1$. By Theorem 4.1 we have $(n-1)^2 \leq \gamma \leq W_n$ which may be regarded as yet another proof for Wielandt's bound.

By using the same argument the second part of Theorem 4.1 may be extended in the following manner.

THEOREM 4.1′. *Let p, q, n, m be positive integers such that $p < q \leq n$, $(p, q) = 1$, and satisfying the double inequality*

$$p(q-1) \leq m \leq p(q-2) + h$$

where $h = \min(p+q-1, n)$. Then there exists a primitive n-square matrix A for which $L(G(A)) = \{p, q\}$ and $\gamma(A) = m$.

THEOREM 4.2. *Let G be a primitive n-graph and let $L(G) = \{p, q\}$ with $p + q \leq n$. Then $\gamma(G) \leq [\frac{1}{4}(n+1)^2]$.*

Proof. By Lemma 3.2, we have $d(p, q) \leq 2n - p - 1$. Let $p + q = n - \delta$. Then

$$\begin{aligned}\gamma(G) &\leq 2n - p - 1 + (p-1)(q-1) = n + \delta + p(q-1)\\ &\leq n + \delta + \tfrac{1}{4}(n - \delta - 1)^2 = \psi(\delta).\end{aligned}$$

It may be verified that $\psi(\delta)$ is maximum for $\delta = 0$ and hence $\gamma(G) \leq \frac{1}{4}(n+1)^2$. The extremal graphs for which the bound is attained are the extremal graphs of Lemma 3.2.

THEOREM 4.3. *Let G be a primitive graph and let $L(G) = \{p, q\}$. Then:*

(i) *If $\gamma(G) > \frac{1}{4}(n+1)^2$, then $2(1 + \sqrt{\gamma - n}) \leq p + q \leq n + [\gamma/(n-1)]$.*
(ii) *If $\gamma(G) > (n-1)(n + 1 - 2\sqrt{n-2})$. then $p + q = n + [\gamma/(n-1)]$.*

Proof of (i). Let $p < q$. Theorem 4.2 implies $p + q > n$ and hence by Theorem 4.1 we get $p(q-1) \leq \gamma \leq n + p(q-2)$. We have

$$p(q-1) \geq p(q-1) - (n-q)(n-p-1) = (n-1)(p+q-n).$$

On the other hand $n + p(q-2) \leq n + \frac{1}{4}(p+q-2)^2$, so that

$$(n-1)(p+q-n) \leq \gamma \leq n + \tfrac{1}{4}(p+q-2)^2.$$

The last double inequality is equivalent to (i).

Proof of (ii). Let

$$\gamma > (n-1)(n+1-2\sqrt{n-2}). \tag{1}$$

It may be verified that for $n \geq 2$ we have $(n-1)(n+1-2\sqrt{n-2}) \geq \frac{1}{4}(n+1)^2$ so that (i) implies

$$2(1 + \sqrt{\gamma - n}) \leq p + q \leq [n + \gamma/(n-1)].$$

Put $n + \gamma/(n-1) = a$, $2(1 + \sqrt{\gamma - n}) = b$. It is easy to verify that $b < a$ for every n and γ. The inequality

$$a < b + 1 \tag{2}$$

is equivalent to (1). But (i) implies that

$$b \le p + q \le [a]. \tag{3}$$

The combination of (2) and (3) yields

$$b \le p + q \le [a] \le a < b + 1. \tag{4}$$

The strict inequality $p + q < [a]$ now leads to a contraction in (4), thus proving (ii).

The following example will illustrate the application of Theorem 4.3. Let $n = 10$. Then $W_n = 82$. Is there a 10-square primitive matrix with exponent 50? Since $50 > \frac{1}{2}W_n + 1$, we have $\lambda(G(A)) = 2$. Let $L(G) = \{p, q\}$, $p < q$. Theorem 4.3 implies

$$2(1 + \sqrt{42}) \le p + q \le 10 + [50/9],$$

so that $p + q = 15$. The only possibility such that $(p, q) = 1$ is $p = 7$, $q = 8$. Then

$$p(q-1) = 49 < 50 < 52 = n + p(q-2).$$

By Theorem 4.1 the matrix in question exists. It is exhibited here together with its graph:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Practically the same argument as that for $\gamma = 50$ reveals the gap 46, 47, 48. We now come to our main theorem.

THEOREM 4.4. *Let n and t be arbitrary integers. There does not exist a primitive n-square matrix with exponent γ such that*

$$n^2 - tn + \tfrac{1}{4}(t+1)^2 < \gamma < n^2 - (t-1)n + t - 2. \tag{5}$$

Proof. Suppose there exists a primitive n-square matrix A such that for some positive integer t, inequality (5) is satisfied. Then, as may be verified,

$$3 \le t < 1 + 2\sqrt{n-3},$$

so that $n \ge 3$. By the left inequality of (5) we have

$$\begin{aligned} \gamma &> n^2 - tn + \tfrac{1}{4}(t+1)^2 = n + (n - \tfrac{1}{2}(t+1))^2 \\ &> n + (n - 1 - \sqrt{n-3})^2 > \tfrac{1}{2}W_n + 1 > \tfrac{1}{4}(n+1)^2, \end{aligned}$$

so that by Theorem 3.1 we have $\lambda(G(A)) = 2$ and the conditions of Theorem 4.3, are satisfied. The left and right inequalities of (5) imply

$$2n - t + 2 < 3 + 2\sqrt{\gamma - n} \tag{6}$$

and

$$n + \gamma/(n-1) < 2n - t + 2 \tag{7}$$

respectively. Therefore there exists an integer m such that $b < a < m < b + 1$ and hence by Theorem 4.3 (i) we have $b \leq [a] < m < b + 1$, an obvious contradiction. (The terms a and b are used as in the proof of Theorem 4.3.) Theorem 4.4 is thus proved.

We shall use Theorem 4.3 in order to obtain the gap of Dulmage and Mendelon in [2] for even n which is $n^2 - 4n + 6 < \gamma < (n-1)^2$.

This gap is the union of two systematic gaps corresponding to $t = 3, 4$ (Theorem 4.4) together with the extra interval $I = [n^2 - 3n + 2, n^2 - 3n + 4]$. Now suppose $\gamma(A) \in I$, so $\gamma = (n-1)(n-2) + a, 0 \leq a \leq 2$. For $n > 2$ we have

$$\gamma \geq (n-1)(n-2) > (n-1)(n + 1 - 2\sqrt{n-2}) > \tfrac{1}{2}W_n + 1.$$

Hence Theorems 3.1, 4.3 (ii) may be used. Thus $\lambda(G(A)) = 2$. Let $L(G(A)) = \{p, q\}$, then since $\gamma \in I$, we get $p + q = n + [\gamma/(n-1)] = 2n - 2$, which implies $p = n - 2$, $q = n$. But n is even so $(p, q) = 2$ and there is no such primitive graph.

Similar arguments lead to other special cases, yet it seems that the system in Theorem 4.4 is the maximal system of gaps one could obtain without additional information on n.

It is our belief that there are no gaps below $\frac{1}{2}W_n + 1$; however there seems no easy way to prove our conjecture. On the other hand if our conjecture turns out to be correct, then Theorem 4.3 reveals all possible gaps when n is given explicitly.

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