

A $T\bar{T}$ -like deformation of the Skyrme model and the Heisenberg model of nucleon-nucleon scattering

Horatiu Nastase^a and Jacob Sonnenschein^b

^a*Instituto de Física Teórica, UNESP-Universidade Estadual Paulista,
R. Dr. Bento T. Ferraz 271, Bl. II, Sao Paulo 01140-070, SP, Brazil*

^b*School of Physics and Astronomy, The Raymond and Beverly Sackler
Faculty of Exact Sciences, Tel Aviv University,
Ramat Aviv 69978, Israel*

E-mail: horatiu.nastase@unesp.br, cobi@post.tau.ac.il

ABSTRACT: The Skyrme model, though it admits correctly a wide range of static properties of the nucleon, does not seem to reproduce properly the scattering behavior of nucleons at high energies. In this paper we present a $T\bar{T}$ -like deformation of it, inspired by a 1+1 dimensional model, in which boosted nucleons behave like shock waves. The scattering of the latter saturates the Froissart bound. We start by showing that 1+1 dimensional $T\bar{T}$ deformations of the free abelian pion action are in fact generalizations of the old Heisenberg model for nucleon-nucleon scattering, yielding the same saturation of the Froissart bound. We then deform the strong coupling limit of the bosonized action of multi-flavor QCD in two dimensions using the $T\bar{T}$ deformation of the WZW action with a mass term. We derive the classical soliton solution that corresponds to the nucleon, determine its mass and discuss its transformation into a shock-wave upon boosting. We uplift this action into a 3+1 dimensional $T\bar{T}$ -like deformation of the Skyrme action. We compare this deformed action to that of chiral perturbation theory. A possible holographic gravity dual interpretation is explored.

KEYWORDS: Chiral Lagrangians, Effective Field Theories, Nonperturbative Effects, Solitons Monopoles and Instantons

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1 Introduction

An important question that has been intensively investigated for about a half a century is what happens when nucleons (or, more generally, hadrons) collide at very high energy? It is well known that the asymptotic total cross section is bounded by the Froissart bound [1, 2]

$$\sigma_{\text{tot}}(\tilde{s}) \leq C \ln^2 \frac{\tilde{s}}{s_0}, \quad C \leq \frac{\pi}{m^2}, \quad (1.1)$$

where \tilde{s} is the Mandelstam kinematical variable, s_0 a constant and m is the mass of the lightest particle that can be exchanged by the scattering projectiles. But before the Froissart bound, and in fact even QCD were derived, Heisenberg proposed a simple semi-classical model [3] that deals with the regime of high-energy nucleon scattering, leading to the saturation of the bound. The model, which is defined in terms of a single pion field, described by a massive DBI scalar action in 3+1 dimensions takes the following form:

$$S = \int d^{3+1}x l^{-4} \left[1 - \sqrt{1 + l^4 [(\partial_\mu \phi)^2 + m^2 \phi^2]} \right]. \quad (1.2)$$

This action is then reduced to 1+1 dimensions (time and the direction of propagation).

The colliding Lorentz-contracted nuclei and the pion field around them were described by classical shockwave solutions of this action. Invoking some simple assumptions it was shown that the total cross section $\sigma_{\text{tot}}(s)$ saturates the Froissart bound. In the modern context, the Heisenberg model was updated and generalized in [4]. Further work on the Heisenberg model was hindered by the puzzle of how to understand the action (1.2) from the point of view of QCD? Possible effective actions for QCD are described in terms of pions that are SU(2)-valued. Skyrme-type actions admit static solitons that can be identified with the nucleons, but do not have shockwave solutions. On the other hand SU(2)-valued generalizations of (1.2) that have shockwave solutions were found not to have static solitons [5], which presents another puzzle.

Recently [6–8] it was discovered that in 1+1 dimensions there are other interesting generalizations of the DBI scalar action, namely $T\bar{T}$ deformations of a canonical scalar action with a potential V . $T\bar{T}$ deformations have served as laboratories for investigating various aspects of field theory. See for instance [9–16].

A natural question that follows from the generalization of Heisenberg’s action is: can such an action be considered instead of the action (1.2) for modeling nucleon scattering, and can it be related to QCD? After all, in [4] we have already found that various generalizations of (1.2) work as well, and some have an interpretation from the point of view of AdS/CFT, in possible gravity duals of QCD.

With that in mind, we show in this paper that certain $T\bar{T}$ deformed actions, indeed admit shockwave solutions which result in the saturation of the Froissart bound, as does the original model of Heisenberg. Moreover, they also have soliton solutions, which upon boosting allow us to interpret the shockwave as a soliton collision.

Our next step is to elevate the 1+1 dimensional abelian deformed theories into a non-Abelian one. For that purpose we make use of the proposal of [7] for a $T\bar{T}$ deformation of the two dimensional WZW model with a mass term. The deformed WZW action is written

down for $U \in U(N_f)$ with a level N_c where N_f is the number of flavors and N_c the number of colors. It was shown in [17] that this describes the strong coupling effective action of multi-flavor bosonized QCD in two dimensions. It was further found in [17] that this action reduces, for the lowest energy configuration, to a sine-Gordon action. In a similar way the deformed WZW action reduces to a $T\bar{T}$ deformation of the sine-Gordon action. In [18] we found the soliton solutions of that system and we showed that their masses are the same as of the un-deformed ones. We argue that these may be used to describe the physics of two dimensional nucleons.

We then uplift this system into a 3+1 dimensional $T\bar{T}$ -like deformation of the Skyrme action. This action admits both Skyrme-like solitons representing the nucleons, and shockwave solutions corresponding to their collisions. We argue that the latter can be gotten from boosting the former. Upon expanding the deformed action as a power series in the deformation parameter, we get that the leading order terms are those of the Skyrme model followed by terms with higher power of derivative of the group element field U . These higher order terms are then compared with the corresponding terms of the chiral perturbation theory action.

We also briefly discuss the description of the model using the holographic gravity/gauge duality. It turns out that as a result of the deformation the holographic model admits a change of the sign of a brane tension.

The paper is organized as follows. In section 2 we review Heisenberg's model and its generalizations. In section 3 we consider $T\bar{T}$ deformed scalar actions, solitons and shockwave solutions, argue that the latter can be understood as boosting the former, and find that using the mode results in a saturation of the Froissart bound. In section 4 we use the $T\bar{T}$ deformation of the WZW action plus a mass term as a deformation of the strong coupling limit of multi-flavor QCD in 2 dimensions. We determine the baryon as a soliton solution by transforming the action to that of a deformed sine-Gordon mode. We discuss boosting this solution into a shockwave solution. In section 5 we consider how to find both Skyrme-like solitons to non-Abelian 3+1 dimensional generalizations of the $T\bar{T}$ deformed action, and shockwave-like solutions, and how to understand the latter from boosting the former. In section 6 we consider the possible interpretations of the $T\bar{T}$ deformed actions in QCD, and in AdS/CFT, from the point of view of gravity duals, and in section 7 we conclude and present several open questions. In appendix A we show that the $T\bar{T}$ deformation of the *3+1 dimensional* scalar actions can't be used for the Heisenberg model (unlike the oxidation to 3+1 dimensions of the 1+1 dimensional actions), and in appendix B we show that the $T\bar{T}$ deformations of Abelian and non-Abelian gauge theories (Maxwell and Yang-Mills) don't have the needed soliton solutions, so hence cannot be used as a replacement of the Heisenberg model.

2 Review of Heisenberg model for high-energy nucleon-nucleon scattering and its generalizations

In this section we review the model by Heisenberg written in 1952 [3] to describe high-energy nucleon-nucleon scattering, which gives a saturation of the Froissart bound [1, 2], and its modern implementation and generalizations in [4].

Heisenberg starts with the observation that when boosting nucleons to ultra-relativistic speeds, they Lorentz contract, becoming first pancake-shaped, until finally they look like delta-function shockwaves in the direction of propagation. Moreover, the pion field around them also Lorentz contracts, so finally we can consider that it becomes a delta-function sourced pion field shockwave.

The process of high-energy nucleon scattering then is described by the collision of two such shockwaves, and as such can be described by a (classical) shockwave solution of an effective action, which was chosen to be (1.2), for reasons to be explained shortly. In the next section, we will see that in fact the shockwave solution we consider describes the collision of two such nucleon-sourced pion shockwaves, and not just one pion shockwave, a fact that was not clear in previous analyses of the model.

The shockwave solution of (1.2) is described as a solution depending only on $s = t^2 - x^2$, where x is the direction of propagation, $\phi = \phi(s)$, with $\phi(s < 0) = 0$. To constrain the action, Heisenberg imposes that, while $\phi(s)$ must be continuous, so $\phi(0+) = 0$, $(\partial_\mu \phi)^2$ must be finite and nonzero on the nontrivial side ($s > 0$) and zero on the trivial side, the jump in the Lorentz-invariant derivative signalling the existence of the (delta function) nucleon-sourced shock. It is found that a canonical scalar with any polynomial potential does not solve it, but the action (1.2) does, as the solution near $s = 0$ is

$$\phi(s) \simeq l^{-2} \sqrt{s} + \dots, \tag{2.1}$$

so that

$$(\partial_\mu \phi)^2 \simeq -l^{-4} + \dots \tag{2.2}$$

On the other hand, in [4] it was found that the condition of jump in derivative, correlated with the existence of saturation of the Froissart bound, while it is highly nontrivial, it also admits certain generalizations. The simplest one is to an arbitrary potential V inside the square root,

$$\mathcal{L} = l^{-4} \left[1 - \sqrt{1 + l^4 [(\partial_\mu \phi)^2 + 2V(\phi)]} \right], \tag{2.3}$$

another being to several scalars with a sigma model, potential and 2 more functions,

$$\mathcal{L} = l^{-(d+1)} \left[h(\phi^i) - f(\phi) \sqrt{1 + l^{d+1} [g_{ij}(\phi^k) (\partial_\mu \phi^i) (\partial_\mu \phi^j) + 2V(\phi^i)]} \right]. \tag{2.4}$$

We can also consider an action with an AdS/CFT interpretation, as a D-brane action moving in a curved background, with

$$S_{Dd} = l^{-(d+1)} \int d^{d+1} x e^{-\tilde{\phi}} \left[\sqrt{-\det(\eta_{\mu\nu} \tilde{g}(\phi) + l^{d+1} \partial_\mu \phi^i \partial_\nu \phi^j g_{ij}(\phi))} - 1 \right], \tag{2.5}$$

and one can also add some vector mesons (understood as vectors on the D-brane), with Lagrangian

$$\mathcal{L} = l^{-4} \left[1 - \sqrt{\det(\eta_{ab} + l^4 \partial_a \phi \partial_b \phi + l^2 F_{ab}) + m^2 \phi^2 + M_V^2 A_a^2} \right]. \tag{2.6}$$

The Hamiltonian (energy density) coming from (1.2) is

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{l^{-4} + (\nabla\phi)^2 + m^2\phi^2}{\sqrt{1 + l^4[(\partial_\mu\phi)^2 + m^2\phi^2]}} - l^{-4} . \quad (2.7)$$

We interpret the classical shockwave solution $\phi(s)$, which, as we will explain in the next section, describes the *collision* of two pion shockwaves around the high-energy nucleons, as describing in a general direction in (x, t) the pion radiation coming from the collision. Indeed, a classical bosonic field is nothing but the quantum field in the case of many bosons in each occupied state, so the $\phi(s)$ solution can describe radiation from a collision. Then the Hamiltonian above corresponds to the energy radiated in the collision.

Following Heisenberg [3], we can assume that the denominator becomes a non-vanishing constant near $s = 0$, thus avoiding the unphysical divergence in the energy density near the shock. Fourier transforming in x to momentum k the solution near $s = 0$ in (2.1), we obtain

$$\begin{aligned} \phi(k, t) &= l^{-2} \int_0^t dx e^{ikx} \sqrt{t^2 - x^2} \\ &\simeq l^{-2} \frac{\pi}{2} \frac{|t|}{|k|} (J_1(|k||t|) + i\mathbf{H}_1(|\mathbf{k}||\mathbf{t}|)) , \end{aligned} \quad (2.8)$$

where J_1 is a Bessel function and \mathbf{H}_1 is a Struve function. Expanding at large k , we obtain

$$\phi \simeq l^{-2} i \frac{|t|}{|k|} \propto |t|^{1/2} |k|^{-3/2} e^{-i|k||t|} , \quad (2.9)$$

and dropping the non-oscillatory (non-radiative) piece in ϕ , we get $\phi(k) \propto k^{-3/2}$.

However, in reality (and also as we will see in an example in the next section), the shockwave should have a finite thickness of the order of the Lorentz contracted $1/m$, i.e., $\sqrt{s_{0m}} \equiv \sqrt{s_{\min}} = \sqrt{1 - v^2}/m$, so at sufficiently large times t , the momenta k are cut off at $k_{0m} = 1/\sqrt{s_{0m}} = \gamma m$, the relativistic mass of the pion. Then the energy of radiated pions per unit of spatial momentum, derived from the momentum space Hamiltonian in (2.7) with constant denominator, is

$$\frac{dE}{dk} \propto k^2 \phi(k)^2 \sim \frac{\text{const.}}{k} , \quad (2.10)$$

for $k \leq k_{0m}$.

Next we relate to the quantum mechanical description of the classical field, by identifying k with the momentum k_0 of a pion, and identifying through canonical quantization the pion field energy E with the radiated energy \mathcal{E} of the pions, the radiated energy per unit of pion energy is

$$\frac{d\mathcal{E}}{dk_0} = \frac{B}{k_0} , \quad m \leq k \leq k_{0m} , \quad (2.11)$$

integrated to

$$\mathcal{E} = B \ln \frac{k_{0m}}{m} = B \ln \gamma . \quad (2.12)$$

This also leads to a relation for the number of radiated pions per unit of radiated energy, since $dE = k_0 dn$ leads to

$$\frac{dn}{dk_0} = \frac{B}{k_0^2}, \quad m \leq k_0 \leq k_{0m}, \quad (2.13)$$

integrated to

$$n = \frac{B}{m} \left(1 - \frac{m}{k_{0m}}\right). \quad (2.14)$$

Finally, the average emitted energy per pion is

$$\langle k_0 \rangle \equiv \frac{\mathcal{E}}{n} = m \frac{\ln(k_{0m}/m)}{1 - m/k_{0m}} = m \frac{\ln \gamma}{1 - \frac{1}{\gamma}} \simeq m \ln \gamma, \quad (2.15)$$

which is approximately constant (the dependence on the energy is only logarithmic). On the contrary, for a canonical scalar with polynomial potential (and most other canonical scalar actions), through a similar calculation, we find at leading order in γ , $\langle k_0 \rangle \propto \gamma$.

To connect to the saturation of the Froissart bound, we need to consider the fully 4-dimensional form of the action (1.2). For a Mandelstam variable \tilde{s} , the total energy of the colliding nucleons is $\sqrt{\tilde{s}}$, and we assume that the emitted energy is proportional to it, with constant of proportionality given by the wave function overlap. Moreover, since at large transverse distance r ($= \sqrt{y^2 + z^2}$), the wave function is small, so satisfies the free massive KG equation, with solution $\phi \sim e^{-mr}$, the wave function overlap will be e^{-mb} , where b is the impact parameter of the colliding nucleons. Then

$$\mathcal{E} \sim \sqrt{\tilde{s}} e^{-mb}. \quad (2.16)$$

The maximum impact parameter, b_{\max} , arises when we emit a single pion, that is, when the emitted energy \mathcal{E} equals the average per pion emitted energy $\langle k_0 \rangle$, so

$$\begin{aligned} \sqrt{\tilde{s}} e^{-mb_{\max}} = \langle k_0 \rangle &\Rightarrow b_{\max} = \frac{1}{m} \ln \frac{\sqrt{\tilde{s}}}{\langle k_0 \rangle} \Rightarrow \\ \sigma_{\text{tot}} &= \frac{\pi}{m^2} \ln^2 \frac{\sqrt{\tilde{s}}}{\langle k_0 \rangle}. \end{aligned} \quad (2.17)$$

We see that the saturation of the Froissart bound is equivalent with the (approximate) independence of $\langle k_0 \rangle$ of the collision energy $\sqrt{\tilde{s}}$, or in other words, on γ .

3 $T\bar{T}$ -like actions as generalizations of the Heisenberg model

One definition of the $T\bar{T}$ deformation of a Lagrangian, proposed by Zamolodchikov [19, 20] is that, for a full quantum theory in 1+1 dimensions, the variation of the Lagrangian with respect to the deformation parameter equals the determinant of the deformed energy-momentum tensor, i.e, after Wick rotating to Euclidean space,

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -4[T_{zz}^\lambda T_{\bar{z}\bar{z}}^\lambda - (T_{z\bar{z}}^\lambda)^2]. \quad (3.1)$$

Note that here all objects are renormalized and UV finite, and on the right-hand side we have an operator regularized by point-splitting. We can solve the equation by expansion

in a series in λ (by treating the fields as numbers), if we give a starting point (unperturbed Lagrangian \mathcal{L}_0) and a perturbation parameter λ (then by (3.1), the first order in the series, \mathcal{L}_1 is given by the unperturbed $\det T_{\mu\nu}$ above).

Consider as a starting point a real scalar with potential V (still in Euclidean space),

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)^2 + V, \tag{3.2}$$

and a perturbation parameter λ .

Then, at first one obtained a complicated expression, with an infinite series of complicated hypergeometric functions (see eq. 6.34 in [6]), but the series can be summed, to obtain a simple expression [7], giving

$$\begin{aligned} \mathcal{L}(\lambda, X) &= -\frac{1}{2\lambda} \frac{1-2\lambda V}{1-\lambda V} + \frac{1}{2\lambda} \sqrt{\frac{(1-2\lambda V)^2}{(1-\lambda V)^2} + 2\lambda \frac{X+2V}{1-\lambda V}} \\ &= \frac{-(1-2\lambda V) + \sqrt{1+2\bar{\lambda}(\partial_\mu\phi)^2}}{2\bar{\lambda}} \\ &\equiv \tilde{V}(\phi) + \frac{1}{2\bar{\lambda}} \sqrt{1+2\bar{\lambda}(\partial_\mu\phi)^2}, \end{aligned} \tag{3.3}$$

where we have used the notation $X = (\partial_\mu\phi)^2$, as well as defined

$$\bar{\lambda} \equiv \lambda(1-\lambda V). \tag{3.4}$$

Note that when $\lambda V \rightarrow 0$, $\tilde{V} \rightarrow V - 1/(2\lambda)$, which cancels the a constant coming from the square root, and gives the unperturbed potential. This can then also be written as in [8] (since $4\partial\phi\bar{\partial}\phi = (\partial_\mu\phi)^2$),

$$\mathcal{L}_E = \frac{V}{1-\lambda V} + \frac{-1 + \sqrt{1+8\bar{\lambda}\partial\phi\bar{\partial}\phi}}{2\bar{\lambda}}. \tag{3.5}$$

This is the Lagrangian in Euclidean space. Going back to the Minkowski signature, we obtain

$$\mathcal{L}_M = -\frac{V}{1-\lambda V} + \frac{1 - \sqrt{1+2\bar{\lambda}\partial_\mu\phi\partial^\mu\phi}}{2\bar{\lambda}}, \tag{3.6}$$

with $\partial_\mu\phi\partial^\mu\phi = -(\dot{\phi})^2 + (\phi')^2 \equiv X$, as usual.

We now examine whether these Lagrangians (for various possible potentials $V(\phi)$) can be used for generalizations of the Heisenberg model (other than the previously considered ones reviewed in the previous section).

3.1 Review of soliton solutions

In this subsection we review soliton solutions described in a companion paper [18] (after the paper was posted, we became aware of [21, 22], which have some overlap with ours, though here we present the solutions in the form we obtain).

We have considered $T\bar{T}$ and Heisenberg deformations of a canonical scalar with a potential V .

3.1.1 Heisenberg deformations

Starting with the Heisenberg case, the deformed Lagrangian in 1+1 dimensional (Minkowski) space is

$$\mathcal{L}_M = 1 - \sqrt{1 + (\partial_\mu \phi)^2 + 2V}, \quad (3.7)$$

and, by use of a modified virial theorem, we have found that the generic static solution is written implicitly as

$$x - x_0 = \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi}{\sqrt{(1 + 2V) \left[\frac{1+2V}{C^2} - 1 \right]}}, \quad (3.8)$$

where C is an arbitrary constant.

By comparing with the canonical case, we have found that we need to put $C = 1$ to find a modified soliton solution, so that the implicit soliton solution is

$$x - x_0 = \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi}{\sqrt{2V(1 + 2V)}}. \quad (3.9)$$

For the pure DBI scalar case, with $V(\phi) = 0$, the solution is

$$x - x_0 = \int_{\phi(x_0)}^{\phi(x)} \frac{2d\phi}{\sqrt{\frac{1}{C^2} - 1}} = K(\phi(x) - \phi(x_0)), \quad (3.10)$$

and by gluing two such solutions we find the solution to the Poisson equation in one dimension,

$$\phi(x) = \phi(x_0) + K^{-1}|x - x_0|. \quad (3.11)$$

We note that, if we redefine V such that $V(0) = 0$, this solution is valid also for the case of nonzero $V(\phi)$, just that only near $x = x_0$. After an infinite, or rather very large, boost, with $K^{-1}\gamma \equiv \tilde{K}$, we have the propagating shockwave solution

$$\phi(x^-) = \tilde{K}|x^-|. \quad (3.12)$$

But more relevant, perhaps, is the case of the pure DBI action *in 3+1 dimensions*, in which case the conservation equation becomes (using the same formalism as in 1+1 dimensions, but for the one-dimensional Lagrangian in the radial direction $r = x$, $\mathcal{L}(r) = 4\pi r^2[1 - \sqrt{1 + \phi'^2}]$)

$$\frac{d}{dx} \left(x^2 \frac{1 + 2V}{\sqrt{1 + 2V + \phi'^2}} \right) = -2x \sqrt{1 + 2V + \phi'^2}. \quad (3.13)$$

Here it is harder to write the general solution, but one can find the specific solution ("catenoid", see later sections) with a delta function source at $r = 0$, solution of $\vec{\nabla} \cdot \vec{C} = q\delta^3(r)$, with $\vec{C} = \partial\mathcal{L}/\partial(\vec{\nabla}\phi)$, which is

$$\phi'^2 = \frac{q^2}{x^4 - q^2} \Rightarrow \phi(r) = q \int_r^\infty \frac{dx}{\sqrt{x^4 - q^2}}. \quad (3.14)$$

3.1.2 $T\bar{T}$ deformations

In the case of $T\bar{T}$ deformations of a canonical scalar with a potential V , we can find the implicit general static solution for $\phi(x)$ as

$$x - x_0 = \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi \sqrt{2\lambda(1 - \lambda V)}}{\sqrt{\frac{1}{[2\lambda(1 - \lambda V)(C - \tilde{V})]^2} - 1}}, \quad (3.15)$$

where again C is an arbitrary integration constant.

The relevant case for a soliton, that arises as a deformation of the soliton for the canonical scalar with potential V is obtained for $C = 0$, as

$$x - x_0 = \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi |1 - 2\lambda V|}{\sqrt{2V}}. \quad (3.16)$$

In particular, for the sine-Gordon potential,

$$V_{\text{sG}} = -\frac{\mu^2}{\beta} [\cos(\beta\phi) - 1], \quad (3.17)$$

we find the deformed sine-Gordon soliton, defined implicitly as

$$\pm \mu \sqrt{\beta} (x - x_0) = \left[\ln \left(\tan \frac{\beta\phi}{4} \right) + 4 \frac{\lambda \mu^2}{\beta} \cos \frac{\beta\phi}{2} \right] \Big|_{\phi(x_0)}^{\phi(x)}, \quad (3.18)$$

and for the Higgs-type potential,

$$V(\phi) = \alpha(\phi^2 - a^2)^2, \quad (3.19)$$

we find the deformed kink soliton, defined implicitly as

$$x - x_0 = \pm \frac{1}{a\sqrt{2\alpha}} \left[\tanh^{-1} \frac{\phi}{a} - 4\lambda\alpha a^3 \left| \frac{\phi^3}{3a^3} - \frac{\phi}{a} \right| \right]. \quad (3.20)$$

One again finds the same approximate static solution, near a point $x = x_0$, the solution to the Poisson equation in one dimension,

$$\phi(x) = \phi(x_0) + K^{-1}|x - x_0|, \quad (3.21)$$

and as before, after an infinite boost goes over to the same

$$\phi(x^-) = \tilde{K}|x^-|. \quad (3.22)$$

We should also consider the case of soliton solutions of the $3+1$ dimensional extension of the $T\bar{T}$ Lagrangian, for a spherically symmetric solution, when we can write a one-dimensional radial Lagrangian for the radius $r = x$ via $\mathcal{L}(x) \rightarrow 4\pi r^2 \mathcal{L}(r)$. Then, doing exactly the same steps as before, we obtain the conservation equation

$$\frac{d}{dx} \left[\frac{x^2}{2\lambda(1 - \lambda V)\sqrt{1 + 2\lambda(1 - \lambda V)\phi'^2}} + x^2 \tilde{V} \right] = -2x \frac{\sqrt{1 + 2\lambda(1 - \lambda V)\phi'^2}}{2\lambda(1 - \lambda V)}, \quad (3.23)$$

or, formally integrating to an integro-differential equation for ϕ and ϕ' ,

$$\frac{x^2}{2\lambda(1-\lambda V)\sqrt{1+2\lambda(1-\lambda V)\phi'^2}} + x^2\tilde{V} = C - \int dx \, 2x \frac{\sqrt{1+2\lambda(1-\lambda V)\phi'^2}}{2\lambda(1-\lambda V)}. \quad (3.24)$$

Alternatively, we could obtain an integro-differential equation by defining $\vec{E} \equiv \vec{\nabla}\phi$ and

$$\vec{D} \equiv \frac{\partial\mathcal{L}}{\partial\vec{E}}, \quad \rho(\vec{r}) \equiv \frac{\partial\mathcal{L}}{\partial\phi(\vec{r})}, \quad (3.25)$$

so that the equation of motion is $\vec{\nabla} \cdot \vec{D}(\vec{r}) = \rho(\vec{r})$, like for electromagnetism in a medium. Since for $\rho(\vec{r}) = \delta^3(\vec{r} - \vec{r}')$ we have as solution for \vec{D} the Green's function

$$\vec{D}_0(\vec{r} - \vec{r}') = \frac{\widehat{\vec{r} - \vec{r}'}}{4\pi|\vec{r} - \vec{r}'|^2}, \quad (3.26)$$

the solution for \vec{D} is

$$\vec{D}(\vec{r}) = \int d^3\vec{r}' \rho(\vec{r}') D_0(\vec{r} - \vec{r}'), \quad (3.27)$$

so we obtain the integro-differential equation for ϕ and $\vec{\nabla}\phi$ (ϕ' on the radial ansatz)

$$\frac{\partial\mathcal{L}}{\partial\vec{\nabla}\phi(\vec{r})} = \int d^3\vec{r}' \frac{\partial\mathcal{L}}{\partial\phi(\vec{r}')} \frac{\widehat{\vec{r} - \vec{r}'}}{4\pi|\vec{r} - \vec{r}'|^2}. \quad (3.28)$$

However, this is more complicated than the integro-differential equation obtained before, so we don't gain anything (in the pure DBI case, on the right-hand side one had only $qD_0(\vec{r})$, allowing us to completely solve the equation).

3.2 The shockwave solution

In the previous subsection we have considered static solutions $K|x - x_0|$ near $x = x_0$, infinitely boosted to solutions $\tilde{K}|x^-|$, representing shockwaves propagating at the speed of light.

However, there are other shockwave solutions, namely ones described by a function of only $s = t^2 - x^2 = x^+x^-$ ($x^\pm = t \pm x$), and not independently on x^+, x^- . In this case, in [18] we have found that both for the Heisenberg and $T\bar{T}$ deformations, we find the same perturbative shockwave solution as Heisenberg: near $s = 0$, but $s > 0$, the solution is

$$\phi(s) = l^{-2}\sqrt{s} + \mathcal{O}(s^{3/2}), \quad (3.29)$$

while $\phi(s < 0) = 0$.

We have also found that the solution remains true if we change the square root in the action with another rational power less than 1, $p/q < 1$.

In the next subsection, we turn to the interpretation of this solution.

3.3 The shockwave solution as collision of *two* ultrarelativistic waves representing high energy nucleons

In the previous section, as well as in [4], the 1+1 dimensional solution was described as a shockwave, and it is, in the sense of being a function of s that has discontinuous derivatives at $s = 0$. However, from a physical point of view, we will see now that it actually describes *two* shockwaves colliding, which means it really corresponds to the ultra-relativistic limit of the field of colliding moving particles sourcing it, identified with the high energy nucleons.

To see this, we first calculate the energy momentum tensor. Including a nontrivial metric in Heisenberg's Lagrangian for the pion, of the type of a DBI scalar with a mass term in the square root,

$$\mathcal{L} = l^{-4} \sqrt{-g} \left[1 - \sqrt{1 + l^4 (\partial_\mu \phi \partial_\nu \phi g^{\mu\nu} + m^2 \phi^2)} \right], \quad (3.30)$$

and varying it, gives the Belinfante tensor

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \frac{\partial_\mu \phi \partial_\nu \phi}{\sqrt{1 + l^4 (\partial_\mu \phi \partial_\nu \phi g^{\mu\nu} + m^2 \phi^2)}} + g_{\mu\nu} l^4 \left[1 - \sqrt{1 + l^4 (\partial_\mu \phi \partial_\nu \phi g^{\mu\nu} + m^2 \phi^2)} \right]. \quad (3.31)$$

But, as we found in [4], for $s = t^2 - x^2 \rightarrow 0$, on the solution, the square root Lagrangian diverges (and we argued that it must be regularized somehow, since divergencies will lead to quantum fluctuations). But, on the solution, with $x^\pm = x \pm t$, so $s = -x^+ x^-$,

$$\frac{\partial \phi}{\partial x^+} = \frac{d\phi}{ds} \frac{\partial s}{\partial x^+} = -x^- \frac{d\phi}{ds} \quad \frac{\partial \phi}{\partial x^-} = -x^+ \frac{d\phi}{ds}. \quad (3.32)$$

Then, on the solution, near $s = 0$, the term with $g_{\mu\nu} l^{-4}$ is finite, the term with $g_{\mu\nu}$ times the square root goes to zero, and the leading term diverges, as 1 over the square root, so

$$\begin{aligned} T_{++} &\simeq - \frac{\partial_+ \phi \partial_+ \phi}{\sqrt{1 + l^4 (\partial_\mu \phi \partial_\nu \phi g^{\mu\nu} + m^2 \phi^2)}} \propto \left(\frac{d\phi}{ds} \right)^2 (x^-)^2, \\ T_{--} &\simeq - \frac{\partial_- \phi \partial_- \phi}{\sqrt{1 + l^4 (\partial_\mu \phi \partial_\nu \phi g^{\mu\nu} + m^2 \phi^2)}} \propto \left(\frac{d\phi}{ds} \right)^2 (x^+)^2, \\ T_{+-} &\simeq - \frac{\partial_+ \phi \partial_- \phi}{\sqrt{1 + l^4 (\partial_\mu \phi \partial_\nu \phi g^{\mu\nu} + m^2 \phi^2)}} \propto \left(\frac{d\phi}{ds} \right)^2 s. \end{aligned} \quad (3.33)$$

The 1 over the square root diverges, but must be regularized to a finite, yet large value. But then we note that, near $s = 0$ (which means $x^+ = 0$ or $x^- = 0$), we have $T_{+-} \rightarrow 0$, and, for $x^+ = 0$, $T_{--} \rightarrow 0$, thus only T_{++} is nonzero, and is a regularized divergence, and for $x^- = 0$, $T_{++} \rightarrow 0$, thus only T_{--} is nonzero, and is a regularized divergence.

That means that the shockwave solution of Heisenberg represents *two* colliding shockwaves (instead of one), one at $x^+ = 0$ (going in the negative x direction) with only T_{++}

and diverging (like the A-S shockwave in gravity), and one at $x^- = 0$ (going in the positive x direction), with only T_{--} and diverging.

This is a solution in 1+1 dimensions, so there is no dependence on the 2 transverse directions, and the collision of the two shockwaves happens at $x = t = 0$.

If we view it as a limit $r \rightarrow 0$ of a solution $\phi(s, r)$, then implicitly we assume that the collision happens at $x = r = 0$, i.e., head-on collision (no impact parameter b). Yet it also means that the $\phi(s \simeq 0, r \neq 0)$ represents the wave function *overlap* of the two nucleons, as indeed Heisenberg claims in his paper, and this overlap also goes like $e^{-m\pi r}$ at large r , as he says also.

Moreover, as observed both by him implicitly and by us [4] implicitly, this $\phi(s \simeq 0, r \simeq 0) \simeq \phi(s \simeq 0)$ is consistent with large spatial momenta k , if we consider x small and r small, which one did in order to calculate the emitted spectrum (and moreover, we need large t in order for it to be the emission spectrum at large time after the collision, i.e., emitted radiation). All of these were conditions imposed on the radiation derived from the classical solution, so their presence is consistent.

But we can view the shockwave solution also as part of a solution $\phi(x^+, x^-, r)$, at $r \rightarrow 0$, but *only* at $x^- = 0, x^+ < 0$, which means $t < 0$ (before the collision), but $x < 0$, so near one of the shockwaves, before the other one arrives. In that case, we can assume the other one (the one going in the negative x direction) can be situated at some nonzero impact parameter, $r = b \neq 0$. That is why the solution can be also thought of as *representing the collision at some impact parameter b* .

This explains why the solution represents collision of 2 waves, and why we can get the maximum impact parameter b_{\max} by using it.

3.4 On boosting a soliton into a shockwave

Before continuing with the analysis of the cross section for high energy scattering, an issue to consider is: how to understand the relation between static soliton solutions and shockwave solutions for the same action. One would expect that by infinitely boosting the soliton solution, one would get the shockwave, or some limit of it. However, as we just saw, the shockwave solution actually shows the *collision* of two waves (one at $x^+ = 0$, another at $x^- = 0$), so that is not what one wants. In fact, we have seen that by boosting a soliton solution we (rather generically) obtain the solution $\phi(x^-) = \tilde{K}|x^-|$.

Rather, if we are to compare with the shockwave solution, we should try to compare the energy-momentum tensors, more specifically the one coming out of a single wave, say T_{--} . Considering the Heisenberg model with a mass term only (neglecting possible higher orders in ϕ in the potential V), which is the only thing that matters near the shock at $s = 0$, the near shock solution is

$$\phi \simeq \frac{\sqrt{s}}{l^2} \left(1 + a s m^2 + \mathcal{O}(s^2) \right), \tag{3.34}$$

where a is an arbitrary constant. Then the square root defining the Lagrangian, and the Lorentz-invariant denominator of the Hamiltonian and of $T_{\mu\nu}$, is

$$\sqrt{1 + l^4(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2)} \simeq m\sqrt{s}\sqrt{1 - 6a}, \tag{3.35}$$

so it diverges as $s \rightarrow 0$. We also note the condition $a \leq 1/6$ necessary for reality of the Lagrangian. Then more precisely, we have

$$T_{+-} \simeq \frac{1}{4\sqrt{1-6a}} \frac{l^{-4}}{m\sqrt{s}}, \quad T_{--} \simeq \frac{1}{4\sqrt{1-6a}} \frac{l^{-4}}{m\sqrt{s}} \frac{x^+}{x^-}, \quad T_{++} \simeq \frac{1}{4\sqrt{1-6a}} \frac{l^{-4}}{m\sqrt{s}} \frac{x^-}{x^+}. \quad (3.36)$$

Then, if x^+ is finite but $x^- \rightarrow 0$, $T_{++} \rightarrow 0$, $T_{+-} \propto 1/\sqrt{x^-}$, and $T_{--} \propto 1/(x^-)^{3/2}$. But that means that at the shock $T_{+-} \rightarrow \infty$ also, whereas it should really not diverge, at least in the case of a single wave. So, it is natural to assume that the square root should somehow be cut off near $s = 0$ such that it stays finite (in fact, this was implicitly assumed in Heisenberg's case; for the Hamiltonian density H , but it is the same denominator). In that case we would have $T_{++} \rightarrow 0$, T_{+-} finite, $T_{--} \propto 1/x^-$.

But consider the generic soliton solution of $\phi(x) = \tilde{K}|x - x_0|$ near x_0 , which was infinitely boosted to $\phi(x) = \tilde{K}\gamma|x^-|$ (for $\gamma \rightarrow \infty$), near $x^- = 0$. Moreover, consider the Lorentz invariant $l^4 g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ which can be calculated on the static solution as $l^4 \tilde{K}^2$. For the infinite boost to make sense, we need that $\tilde{K} \rightarrow 0$, $\tilde{K}\gamma$ finite, in which case the invariant is $l^4 \tilde{K}^2 \rightarrow 0$. Then the square root on the soliton solution is

$$\sqrt{1 + l^4 m^2 \tilde{K}^2 \gamma^2 (x^-)^2} \simeq (l^2 m \tilde{K} \gamma) |x^-|, \quad (3.37)$$

where we have assumed that the second term is larger than the 1 (we can have $l^2 \tilde{K} \gamma$ large enough for that). Then

$$T_{--} \simeq \frac{\partial_- \phi \partial_- \phi}{\sqrt{1 + l^4 (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2)}} \simeq \frac{\tilde{K} \gamma l^{-2}}{m |x^-|} \quad (3.38)$$

has the same behavior as that expected from the shock near one wave (with the cut-off in the square root).

It would seem like the total energy, $\int dx^- T_{--}$, has a log divergence at $x^- = 0$. However, note that for this correct boosted soliton solution, there is a (very small) x^- at which T_{--} remains finite. More precisely, since

$$T_{--} = \frac{\tilde{K} \gamma l^{-2}}{m \sqrt{(x^-)^2 + \frac{1}{l^4 m^2 \tilde{K}^2 \gamma^2}}}, \quad (3.39)$$

we have that

$$E \sim \int dx^- T_{--} = \frac{\tilde{K} \gamma l^{-2}}{m} \sinh^{-1}(x^- l^2 m \tilde{K} \gamma), \quad (3.40)$$

which doesn't have a divergence at $x^- = 0$, and the (log) divergence at large x^- is spurious also, because the original boosted solution was valid only near $x = x_0$, or near $x^- = 0$, when boosted. We also note the correct relativistic relation $E \propto \gamma$.

3.5 The cross section and saturation of the Froissart bound

Now that we saw that the (1+1-dimensional) $T\bar{T}$ action gives the same shockwave near $s = 0$, and moreover the shockwave solution represents a collision of two ultrarelativistic

waves representing the high energy nucleons, we can trivially generalize the action to 3+1 dimensions, by making $(\partial_\mu\phi)^2$ 3+1-dimensional in the Lagrangian, as well as replacing d^2x with d^4x , and use it as Heisenberg did, for the high-energy nucleon-nucleon collision.

We want to see if the collision cross section can still saturate the Froissart bound, if we use this 3+1 dimensional generalization of the $T\bar{T}$ action instead of the action used by Heisenberg (which, in any case, was just a guess, as the *simplest*, but not necessarily most useful or correct; one that gave him the needed properties of the shockwave for high energy scattering of nucleons).

In the Heisenberg analysis reviewed in section 2, which we made more precise in [4], the next step in getting to the cross section of Froissart, is to calculate the radiated energy per unit of k , so first we need the Hamiltonian. At this point, we are still in 1+1 dimensions, so $(\partial_\mu\phi)^2 = -\dot{\phi}^2 + \phi'^2$. Then

$$\begin{aligned} \mathcal{H} &= \pi\dot{\phi} - \mathcal{L} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}\dot{\phi} - \mathcal{L} \\ &= -\frac{1}{2\bar{\lambda}}\frac{\partial}{\partial\dot{\phi}}\sqrt{1+8\bar{\lambda}(-\dot{\phi}^2+\phi'^2)}\dot{\phi} + \frac{1}{2\bar{\lambda}}\sqrt{1+8\bar{\lambda}(-\dot{\phi}^2+\phi'^2)} + \tilde{V}(\phi) \\ &= \frac{1}{2\bar{\lambda}}\frac{1+8\bar{\lambda}\phi'^2}{\sqrt{1+8\bar{\lambda}(-\dot{\phi}^2+\phi'^2)}} + \tilde{V}(\phi). \end{aligned} \tag{3.41}$$

But the energy is

$$E = \int dx\mathcal{H}(x) = \int dk\mathcal{H}(k), \tag{3.42}$$

where now the Hamiltonian density is

$$\mathcal{H}(k) = \frac{1}{2\bar{\lambda}}\frac{1+8\bar{\lambda}k^2\phi^2(k)}{\sqrt{1+8\bar{\lambda}(-\dot{\phi}^2+k^2\phi^2)}} - \tilde{V}(\phi(k)). \tag{3.43}$$

But, like in the case of [4] (and of Heisenberg), we see that the first term has the square root in the denominator, and on the solution near $s = 0$, the square root vanishes (since V vanishes there, and then the rest is the same square root as for Heisenberg). But, as Heisenberg implicitly noted (and we explained in detail), this is unphysical, and we expect that some quantum fluctuations or corrections make the square root a small, but nonvanishing constant near $s = 0$.

Then, neglecting the subleading term $\tilde{V}(\phi(k))$, we obtain

$$\mathcal{H}(k) \simeq \frac{k^2\phi^2(k)}{\text{const.}}. \tag{3.44}$$

But, since the same $\phi(x,t) \simeq l^{-2}\sqrt{s}$ solution is valid near $s = 0$ here, as in the case of [4], we have the same Fourier transform $\phi(k,t)$ at large k , eqs. (2.14) and (2.15) there. There is a non-oscillatory piece, which can be dropped as there, since we are focusing in on the radiative solution. Then $\phi(k) \sim |k|^{-3/2}$, so we have

$$\frac{dE}{dk} = \mathcal{H}(k) \sim \frac{\text{const.}}{k}, \tag{3.45}$$

as before. This then leads, as we saw in [4] and in the previous section, after several steps, to the average per pion energy being approximately a constant,

$$\langle k_0 \rangle \simeq m \ln \gamma, \tag{3.46}$$

where γ is the Lorentz factor for the collision.

Only now do we need to consider a fully 3+1 dimensional version of the $T\bar{T}$ action. More precisely, we need to use the small field value for it, which is just the canonical free massive scalar action. Then, the solution at small field, in terms of transverse radius $r = \sqrt{y^2 + z^2}$, is

$$\phi \sim e^{-mr}, \tag{3.47}$$

and then the same argument as Heisenberg's follows: the radiated energy \mathcal{E} is proportional to the total collision energy \sqrt{s} , and to the overlap of wave functions, $\sim e^{-mb}$, where b is the impact parameter; and at maximum b , we radiate only $\langle k_0 \rangle$, so $\sqrt{s}e^{-mb_{\max}} = \langle k_0 \rangle$, from which we get the cross section $\sigma_{\text{tot}} = \pi b_{\max}^2 = \frac{\pi}{m^2} \ln^2 \frac{\sqrt{s}}{\langle k_0 \rangle}$.

4 From Abelian to non-Abelian Heisenberg and $T\bar{T}$ deformed models

Next, we would like to consider non-Abelian versions of the Abelian actions considered in the Heisenberg type model. In this section we will consider still 1+1 dimensional models, and in the next one we will consider the full 3+1 dimensional models needed.

For the generalization of the abelian field to non-abelian ones, we can follow two approaches:

- (a) Straightforwardly utilizing a $U \in U(2)$ group element instead of the single real scalar field ϕ .
- (b) Using the proposal of [7] for a $T\bar{T}$ deformation of the two dimensional WZW model, deformed also by the potential (mass) term. In this paper, the deformation of the WZW model was found, and the kinetic term is replaced by a square root term, while the WZW term is unchanged by the $T\bar{T}$ deformation, being topological.

We first consider the case (a), of Heisenberg and $T\bar{T}$ deformations, with a potential of

$$V = -m^2 \left[\cos \left(\sqrt{\frac{4\pi}{N_c}} \phi(t, x) \right) - 1 \right], \tag{4.1}$$

where N_c is the number of colors. To generalize to a non-abelian case we make the following replacements

$$\begin{aligned} \partial_\mu \phi(t, x) \partial^\mu \phi(t, x) &\rightarrow \frac{N_c}{4\pi} \text{Tr} \left[\partial_\mu U(t, x) \partial^\mu U^\dagger(t, x) \right] \\ \cos \left(\sqrt{\frac{4\pi}{N_c}} \phi \right) - 1 &\rightarrow \frac{1}{2} \text{Tr} \left[U + U^\dagger - 2 \right]. \end{aligned}$$

The non-Abelian Lagrangian density corresponding to the Abelian $T\bar{T}$ deformation (3.6) (Minkowskian) then reads

$$\mathcal{L}_M = \frac{\left[1 - \sqrt{1 + \frac{N_c}{2\pi} \lambda \left(1 + \frac{\lambda m^2}{2} \text{Tr}[U + U^\dagger - 2] \right) \text{Tr}[\partial_\mu U(t, x) \partial^\mu U^\dagger(t, x)]} \right]}{2\lambda \left(1 + \frac{\lambda m^2}{2} \text{Tr}[U + U^\dagger - 2] \right)} + \frac{m^2 \text{Tr}[U + U^\dagger - 2]}{2 \left(1 + \frac{\lambda m^2}{2} \text{Tr}[U + U^\dagger - 2] \right)}. \quad (4.2)$$

Note that whereas the kinetic part of the action (4.2) is invariant under $U_L(2) \times U_R(2)$ symmetries associated with the transformations $U \rightarrow A_L U$, $U \rightarrow U A_R$, where A_L and A_R are constant $U(2)$ matrices, the potential term and hence the whole action is only invariant under the diagonal $U(2)$ transformation (with $A_L = A_R^\dagger = A$)

$$U \rightarrow A U A^\dagger. \quad (4.3)$$

In particular the diagonal $U(1)_D$ symmetry should be associated with the baryon number, according to the general theory of QCD. The corresponding Noether currents, for $U(2)$ generators $T^A = (\frac{\sigma^a}{2}, \mathbb{1})$ (where σ^a are the Pauli matrices) are given by

$$j_\mu^A = -i \frac{N_c}{4\pi} \frac{\text{Tr}[\partial_\mu U T^A U^\dagger - U^\dagger T^A \partial_\mu U]}{\sqrt{1 + \frac{N_c}{2\pi} \lambda \left(1 + \frac{\lambda m^2}{2} \text{Tr}[U + U^\dagger - 2] \right) \text{Tr}[\partial_\mu U(t, x) \partial^\mu U^\dagger(t, x)]}}. \quad (4.4)$$

Note that in the Skyrme model, the presence of the extra Skyrme term means that the Noether current has an extra contribution equal to the topological current (for the topological number associated with the hedgehog ansatz). On a static ansatz $\partial_0 U = 0$, the above zero component of the Noether current, as the corresponding term coming from the kinetic term in the Skyrme model, vanishes, meaning that the Noether charge equals to just the topological charge, and is associated with baryon number. Here, however, on the static ansatz we just get a zero Noether charge.

This suggests that perhaps it is better to use the case (b), for the proposal in [7], of deforming the WZW model, which will have a nonzero Noether current, thus a nonzero Noether charge, that will turn out to be equal to the topological charge, and thus can be equated with the baryon number, for the dimensional reduction of an effective model for QCD (QCD_2). The model in [7] will also be deformed by a potential (mass) term.

In conclusion, in this section we will analyze the deformations of (i) the WZW model (ii) the massive WZW model and (iii) baryonic sector of bosonized QCD_2 in the strong coupling limit.

4.1 The deformation of the WZW theory

It is well known that in 1+1 dimensions the theory of a free massless real scalar described by the action

$$S = \frac{1}{8\pi} \int d^2x \partial_\mu \phi \partial^\mu \phi = \frac{1}{4\pi} \int d^2z \partial \phi \bar{\partial} \phi, \quad (4.5)$$

where the second form of the action is taken in complex plane, is invariant under transformations with parameters which are holomorphic and anti-holomorphic with the corresponding $U(1)$ affine Lie algebra conserved currents

$$J = \partial\phi, \quad \bar{J} = \bar{\partial}\phi, \quad \partial\bar{J} = \bar{\partial}J = 0, \quad (4.6)$$

and energy momentum tensor

$$T = -\frac{1}{2}\partial\phi\partial\phi, \quad \bar{T} = -\frac{1}{2}\bar{\partial}\phi\bar{\partial}\phi, \quad \partial\bar{T} = \bar{\partial}T = 0. \quad (4.7)$$

The $U(N) \times U(N)$ non-abelian generalization of this system is the well-known WZW action given by

$$S = S_\sigma + S_{\text{WZW}} = \frac{1}{2} \int d^2x \frac{k}{4\pi} \text{Tr}[\partial_\mu U \partial^\mu U^\dagger] + \frac{k}{12\pi} \int_{B^3} [\text{Tr} U^{-1} dU]^3, \quad (4.8)$$

where k is the level of the Kac Moody algebras associated with the holomorphic (anti-holomorphic) conserved currents \bar{J} (J) given by

$$J = \frac{k}{4\pi} \partial U U^{-1}, \quad \bar{J} = \frac{k}{4\pi} U^{-1} \bar{\partial} U, \quad \partial\bar{J} = \bar{\partial}J = 0. \quad (4.9)$$

The corresponding energy momentum tensor is given by the Sugawara construction

$$T = \frac{1}{k} \text{Tr}[JJ], \quad \bar{T} = \frac{1}{k} \text{Tr}[\bar{J}\bar{J}], \quad \partial\bar{T} = \bar{\partial}T = T_{z\bar{z}} = 0. \quad (4.10)$$

The first term in the action, the sigma term, follows from a map

$$\partial_\mu \phi(t, x) \partial^\mu \phi(t, x) \quad \rightarrow \quad \frac{k}{4\pi} \text{Tr} \left[\partial_\mu U(t, x) \partial^\mu U^\dagger(t, x) \right]. \quad (4.11)$$

The $T\bar{T}$ deformation of the Euclidean WZW action was worked out in [7]. It takes the following form

$$S = \int d^2x \left[-\frac{1}{2\lambda} + \frac{1}{2\lambda} \sqrt{1 + \frac{N_c}{2\pi} \lambda X + \frac{N_c}{2\pi} \frac{N_c}{4\pi} \lambda^2 \tilde{X}^2} \right] + \frac{i}{4\pi} \int_{B^3} \frac{N_c}{4\pi} [\text{Tr} U^{-1} dU]^3, \quad (4.12)$$

where

$$\begin{aligned} X &\equiv \text{Tr}[\partial_\mu U \partial^\mu U^\dagger] = -\text{Tr}[L_\mu L^\mu] \\ L_\mu &\equiv U^{-1} \partial_\mu U \\ \tilde{X}^2 &\equiv \epsilon^{\mu\rho} \epsilon^{\nu\sigma} \text{Tr}[L_\mu L_\nu] \text{Tr}[L_\rho L_\sigma]. \end{aligned} \quad (4.13)$$

Notice that the WZW term is undeformed. We have also identified $k = N_c$, we will see in the following subsections why. When expanding in small λ the action takes the form

$$\begin{aligned} S &= \frac{N_c}{4\pi} \int d^2x \left[\frac{X}{2} + \frac{\lambda N_c}{2\pi} \left(\frac{(\tilde{X})^2}{4} - \frac{X^2}{8} \right) + \frac{1}{16} \left(\frac{\lambda N_c}{2\pi} \right)^2 (X^3 - 2X(\tilde{X})^2) + \mathcal{O}(\lambda^3) \right] \\ &+ S_{\text{WZ}}. \end{aligned} \quad (4.14)$$

It is easy to check that both X and \tilde{X}_{ij}^2 are invariant under the left and right $U(N)$ transformations

$$U \rightarrow A_L U, \quad A_L \in U_L(N), \quad U \rightarrow U A_R, \quad A_R \in U_R(N), \quad (4.15)$$

and hence these are symmetry transformations of the full deformed action. The corresponding currents J_L^μ and J_R^μ are given by

$$J_{\mu(L,R)}^A = i \frac{N_c}{4\pi} \left(\frac{\text{Tr} \left[-\partial_\mu U T^A U^\dagger + U^\dagger T^A \partial_\mu U \right]}{\sqrt{1 + \lambda \frac{k}{2\pi} \text{Tr} [\partial_\mu U(t, x) \partial^\mu U^\dagger(t, x)]}} \pm \epsilon_{\mu\nu} \text{Tr} \left[\partial^\nu U T^A U^\dagger - U^\dagger T^A \partial^\nu U \right] \right), \quad (4.16)$$

where T^A are the generators of the $U(N)$ group. The left and right currents correspond to the different signs \pm , *are not conserved holomorphically and anti-holomorphically*, but rather in the ordinary covariant form

$$\partial_\mu J_L^\mu = 0, \quad \partial_\mu J_R^\mu = 0. \quad (4.17)$$

Similarly for the energy momentum tensor, and furthermore for the deformed theory we have $T_{z\bar{z}} \neq 0$.

4.2 WZW with a “mass term”

So far we discussed the deformation of the WZW theory, the non-abelian generalization of single free massless scalar field. Next we would like to address the same procedure for an interacting scalar field, namely, a scalar field with a potential, since this is closer to the model we seek. More specifically we consider a sine-Gordon potential given by

$$V = -m^2 [\cos(\beta\phi(t, x)) - 1]. \quad (4.18)$$

To map the abelian theory to a non-abelian we generalize the map of (4.11) with the following replacements

$$\begin{aligned} \partial_\mu \phi(t, x) \partial^\mu \phi(t, x) &\rightarrow \frac{k}{4\pi} \text{Tr} [\partial_\mu U(t, x) \partial^\mu U^\dagger(t, x)] \\ \cos(\beta\phi) - 1 &\rightarrow \frac{1}{2} \text{Tr} [U + U^\dagger - 2]. \end{aligned}$$

The *Minkowski space* action that corresponds to the $T\bar{T}$ deformation of this system is given by

$$S_M = \int d^2x \left\{ \frac{\left[1 - \sqrt{1 + \frac{k}{2\pi} \lambda \left(1 + \frac{\lambda m^2}{2} \text{Tr} [U + U^\dagger - 2] \right) \left(\text{Tr} [\partial_\mu U(t, x) \partial^\mu U^\dagger(t, x)] + \frac{k}{4\pi} \lambda \tilde{X}^2 \right)} \right]}{2\lambda \left(1 + \frac{\lambda m^2}{2} \text{Tr} [U + U^\dagger - 2] \right)} + \frac{m^2 \text{Tr} [U + U^\dagger - 2]}{2 \left(1 + \frac{\lambda m^2}{2} \text{Tr} [U + U^\dagger - 2] \right)} \right\} + \frac{i}{4\pi} \int_{B^3} \frac{k}{4\pi} [\text{Tr} U^{-1} dU]^3. \quad (4.19)$$

This is the 1+1 dimensional action we consider for the non-Abelian Heisenberg model. In the next section we will see how to uplift it to 3+1 dimensions. Upon expanding in small λ , and as before identifying $k = N_c$, we find now

$$S_M \simeq \int d^2x \left[\left(-\frac{N_c}{8\pi} X + Z \right) + \frac{1}{16} \lambda \left(\left(\frac{N_c}{2\pi} \right)^2 (X^2 - 2(\tilde{X})^2) - 16Z^2 - 8\frac{N_c}{2\pi} ZX \right) + \mathcal{O}(\lambda^2) \right] + S_{WZ}, \quad (4.20)$$

where we denoted by $Z = \frac{m^2}{2} \text{Tr} [U + U^\dagger - 2]$. As was mentioned in the previous subsection, both X and X_{ij}^2 are invariant under the full flavor chiral symmetry $U(N)_L \times U(N)_R$ transformations. However this is not a symmetry transformation of the term $\text{Tr}[U + U^\dagger]$. This term is only invariant under the diagonal $U(N)$ transformation (with $A_L = A_R^\dagger = A$)

$$U \rightarrow AUA^\dagger. \quad (4.21)$$

The corresponding current is

$$J^{\mu A} = \frac{ik}{4\pi} \left(\frac{\text{Tr} [UT^A \partial_\mu U^\dagger + U^\dagger T^A \partial_\mu U]}{\sqrt{1 + \frac{N_c}{2\pi} \lambda \left(1 + \frac{\lambda m^2}{2} \text{Tr} [U + U^\dagger - 2] \right) \text{Tr} [\partial_\mu U(t, x) \partial^\mu U^\dagger(t, x)]}} + \epsilon_{\mu\nu} \text{Tr} [UT^A \partial_\mu U^\dagger - U^\dagger T^A \partial_\mu U] \right). \quad (4.22)$$

In particular the diagonal $U(1) \subset U(N)$, which will play an important role below, is given by

$$J^\mu = \frac{ik}{4\pi} \left(\frac{\text{Tr} [U \partial_\mu U^\dagger + U^\dagger \partial_\mu U]}{\sqrt{1 + \frac{N_c}{2\pi} \lambda \left(1 + \frac{\lambda m^2}{2} \text{Tr} [U + U^\dagger - 2] \right) \text{Tr} [\partial_\mu U(t, x) \partial^\mu U^\dagger(t, x)]}} + \epsilon_{\mu\nu} \text{Tr} [U \partial_\mu U^\dagger - U^\dagger \partial_\mu U] \right). \quad (4.23)$$

Next we would search for soliton solutions of this deformed system. The first point to notice is that for static configurations X_{ij}^2 vanishes, as does the WZ term, and therefore the Lagrangian density for static configurations takes the form

$$\mathcal{L}_M = \frac{\left[1 - \sqrt{1 + \frac{k}{2\pi} \lambda \left(1 + \frac{\lambda m^2}{2} \text{Tr} [U + U^\dagger - 2] \right) \text{Tr} [\partial_x U(x) \partial^x U^\dagger(x)]} \right]}{2\lambda \left(1 + \frac{\lambda m^2}{2} \text{Tr} [U + U^\dagger - 2] \right)} + \frac{m^2 \text{Tr} [U + U^\dagger - 2]}{2 \left(1 + \frac{\lambda m^2}{2} \text{Tr} [U + U^\dagger - 2] \right)}. \quad (4.24)$$

We now take a particular ansatz for the group element $U(N)$

$$U(x) = \text{diag} \left(1, 1, \dots e^{-i\sqrt{\frac{k}{4\pi}}\phi(x)} \right), \quad (4.25)$$

This ansatz, which takes us back from the non-Abelian setting to an Abelian one, will be relevant in the next subsection. For this ansatz, we get that

$$\begin{aligned} \frac{k}{4\pi} \text{Tr} \left[\partial_\mu U(t, x) \partial^\mu U^\dagger(t, x) \right] &= \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \\ \frac{1}{2} \text{Tr} \left[U + U^\dagger - 2 \right] &= \cos(\beta\phi) - 1. \end{aligned}$$

Substituting this into the action (4.19), we end up with the action of the $T\bar{T}$ deformation of the scalar field with a sine-Gordon potential. In [18] the soliton solution for this action was derived and its properties were analyzed. We will describe some of them in the next subsection.

4.3 Baryonic sector of bosonized QCD_2

In [23] it was shown that the low energy effective action of multiflavor bosonized QCD_2 , namely the theory with $SU(N_c)$ gauge symmetry and $U(N_f)$ flavor symmetry converges in the strong limit to

$$S_{sc} = N_c S_{WZW}(U) + m^2 \int d^2x \text{Tr}[U + U^\dagger], \quad U \in U(N_f), \quad (4.26)$$

where the explicit expression for m^2 is given in [23]. Thus, the action is a $U(N_f)$ WZW of level $k = N_c$, plus a potential term.

Solitons of this action should correspond to baryons of QCD_2 . It was shown in [23] that the lowest energy soliton is the one with the following structure of $U(x)$:

$$U(x) = \text{diag} \left(1, 1, \dots e^{-i\sqrt{\frac{4\pi}{N_c}}\phi(x)} \right). \quad (4.27)$$

If we substitute this ansatz into (4.26) we get an action for static configurations that takes the form

$$S_s = \int d^2x \left[- \left(\frac{d\phi}{dx} \right)^2 - 2m^2 \left(\cos \left(\sqrt{\frac{4\pi}{N_c}} \phi \right) - 1 \right) \right]. \quad (4.28)$$

In the terminology of the sine-Gordon model, this action has $\beta = \sqrt{\frac{4\pi}{N_c}}$.

Alternatively, we could have followed the route taken in the previous subsection, of deforming the non-Abelian action, and substituting into it the form of $U(x)$ given in (4.27).

As was mentioned in the previous subsection, the $T\bar{T}$ deformation of the sine-Gordon action and its corresponding soliton were analyzed in [18].

The soliton solutions for this action were derived in [18]. For our system, the soliton takes the form

$$\pm \sqrt{2}m\beta(x - x_0) = \left[\ln \left(\tan \left(\sqrt{\frac{\pi}{4N_c}}\phi \right) \right) + 8\lambda m^2 \cos \left(\sqrt{\frac{\pi}{N_c}}\phi \right) \right] \Big|_{\phi(x_0)}^{\phi(x)}. \quad (4.29)$$

Using the expression for the baryon number current (4.23) we find that the baryon number of this soliton solution is

$$Q_B = \int dx J_{B0} = \frac{N_c}{4\pi} (\phi(x \rightarrow \infty) - \phi(x \rightarrow -\infty)) = N_c, \quad (4.30)$$

as it should be in the notation where each quark has unit baryon number.

Moreover in [23] it was proven that the classical mass of the soliton of the deformed theory is exactly the same as in the undeformed one which is given by [23]

$$M_B = 4m\sqrt{\frac{4N_c}{\pi}} = 4 \left[N_c c m_q \left(\frac{g_c \sqrt{N_f}}{\sqrt{2\pi}} \right) \right]^{\frac{1}{1+\Delta_c}} \sqrt{\frac{4N_c}{\pi}}, \quad (4.31)$$

where $\Delta_c = \frac{N_c^2 - 1}{N_c(N_c + N_f)}$.

4.4 Boosting the 2d baryon to a shock wave

The strong coupling effective action of bosonized 2d QCD [17] does not admit a shock wave solution. This follows from the fact that the necessary condition to have such a solution [4], namely to have an infinite tower of higher derivative terms in the action, is not fulfilled in that case. However, as was shown in [21] and more explicitly in [18] the deformed sine-Gordon action does admit a shockwave behavior. Hence, following the discussion above also the deformed QCD_2 with the particular ansatz of the group element, also admits a shockwave behavior. Thus, the deformed QCD_2 action is a framework in which one can discuss the static properties of the baryonic soliton but also its scattering in the form of shockwave collisions.

5 Boosted Skyrmion-like solitons as shockwaves

As we saw in the section 3, the shockwave solution can be used to describe the (saturation limit for the cross section for the) collision of high-energy nucleons. The nucleons, together with the pion field they generate, when boosted will become first pancake-like, then delta function sources surrounded by shockwave pion field, for which the Heisenberg shockwave is a model.

On the other hand, there is one way to deal with nucleons *at rest* not as sources for the pion field, but rather as sourceless topological pion field configurations, representing both the nucleon and the surrounding pion field, as solutions for the low energy expansion in QCD (chiral perturbation theory). That is the Skyrme solution for the nonlinear sigma model (for the $SU(2)$ -valued field corresponding to the three physical pions, π^\pm and π^0) with a Skyrme term. It is also known that a topological solution still exists when we replace the Skyrme term with other higher order terms in the action.

It is a reasonable hope then that by boosting a Skyrme-like solution we should get a shockwave of the type found by Heisenberg (at least in the region when there is a single wave), as it was argued also in [5, 24–26]. Here we will show that by proposing an action that is a natural nonabelian generalization of the $T\bar{T}$ action we obtain a Skyrme-like solution that when boosted looks like Heisenberg’s shockwave, near $s = 0$ and for $t < 0$.

5.1 Abelian analogue of solitons and boosting

In this subsection, after reviewing previous results on various solitons, we will argue for the existence of a soliton of the $T\bar{T}$ action that can pass smoothly through the $1 - \lambda V = 0$ singularity, and then argue about what happens when we infinitely boost such a solution.

5.1.1 BIons and catenoids

We start this subsection by reviewing the previous attempt for the Abelian scalar theory, in [5], to realize the above program. As we said, the hope was initially that a single scalar action would generate both the shockwave-like solution and the Skyrme-like solution, so that we can we could infinitely boost the latter into the former.

The first observation is that, by restricting to an abelian analog (a single real scalar), the Skyrme-like solution would be a solution of finite energy, since the nucleon represented as a Skyrme has a finite energy also. There is one such solution, the BIon solution found by Born and Infeld in 1934 in [27], in their attempt to replace the electron solution of classical electrodynamics, which has an infinite field energy (the infinite energy shows the need to go to quantum field theory, i.e., to QED, but Born and Infeld sought a *classical* replacement).

The Born-Infeld Lagrangian (putting the relevant length scale l to 1),

$$\begin{aligned} \mathcal{L} &= -\sqrt{-\det\left(\eta_{\mu\nu} + \frac{F_{\mu\nu}}{\sqrt{2}}\right)} = -\sqrt{1 + \frac{F_{\mu\nu}F^{\mu\nu}}{2} - \left(\frac{F_{\mu\nu} * F^{\mu\nu}}{2}\right)^2} \\ &= -\sqrt{1 - \vec{E}^2 + \vec{B}^2 - (\vec{E} \cdot \vec{B})^2}, \end{aligned} \tag{5.1}$$

where in the last form we wrote the electric and magnetic fields, has the equation of motion in the presence of a static source with charge density ρ (so zero magnetic field) of

$$\vec{\nabla} \cdot \vec{D} = \rho, \tag{5.2}$$

where as usual we defined

$$\vec{D} = \frac{\partial \mathcal{L}}{\partial \vec{E}}. \tag{5.3}$$

The electric field \vec{E} is then finite (≤ 1), since we obtain

$$\vec{E} = -\frac{\vec{D}}{\sqrt{1 + \vec{D}^2}}. \tag{5.4}$$

In terms of $\vec{E} = -\vec{\nabla}\phi$, with ϕ the electric potential, the Lagrangian on the time-independent solution is

$$\mathcal{L} = -\sqrt{1 - (\vec{\nabla}\phi)^2}, \tag{5.5}$$

and the solution for point-like source $\rho = q\delta^3(r)$ is

$$\phi(r) = q \int_r^\infty \frac{1}{\sqrt{q^2 + x^2}}, \tag{5.6}$$

and has *diverging energy density at $r = 0$* (where $|\vec{E}| = |\phi'(r)| = 1$), *but finite energy*,

$$E_\phi = 4\pi \int_0^\infty r^2 dr \left[\frac{1}{\sqrt{1-\phi'(r)^2}} - 1 \right] = 4\pi q^{3/2} \int_0^\infty \frac{dx}{x^2 + \sqrt{x^4+1}} = 4\pi q^{3/2} \frac{\Gamma[1/4]^2}{6\sqrt{\pi}}. \quad (5.7)$$

Note that the on-shell Lagrangian looks like one for a scalar DBI action, but *with the wrong sign inside the square root* (and there is no corresponding time derivative term, since the scalar is not a real scalar, but is part of a vector).

On the other hand, the shockwave-like solution can be understood as a solution with a jump in the derivative for a true scalar theory, as Heisenberg did, which (as Heisenberg also did) restricts us to the DBI action,

$$\mathcal{L} = -\sqrt{-\det(\eta_{\mu\nu} + \partial_\mu X \partial_\nu X)} = -\sqrt{1 + (\partial_\mu X)^2}. \quad (5.8)$$

If we want a static solution for it, define (similarly to the Born and Infeld case) $\vec{F} = \vec{\nabla} X$, so the on-shell Lagrangian on this ansatz is

$$\mathcal{L} = -\sqrt{1 + \vec{F}^2}. \quad (5.9)$$

The equation of motion in the static case with a source ρ is

$$\vec{\nabla} \cdot \vec{C} = \rho, \quad (5.10)$$

where we have defined

$$\vec{C} = \frac{\partial \mathcal{L}}{\partial \vec{F}}. \quad (5.11)$$

Then

$$\vec{F} = \frac{\vec{C}}{\sqrt{1 - \vec{C}^2}}. \quad (5.12)$$

The solution for point-like source $\rho = q\delta^3(r)$ is the “catenoid”,

$$X(r) = q \int_r^\infty \frac{dx}{\sqrt{x^4 - q^2}}, \quad (5.13)$$

which however, as we see, has a “horizon” at $r_0 = \sqrt{q}$, where $X(r)$ is finite but $X'(r)$ diverges. When we think of the DBI action as the action for the position of a D-brane, this is understood as one half of a solution of D-brane-anti-D-brane, connected by a funnel (solution obtained by adding a mirror image of the catenoid). The solution also has a diverging energy density, yet a finite energy, but now at this “horizon”,

$$E_X = 4\pi \int_{r_0}^\infty r^2 dr \left[\sqrt{1 + X'^2} - 1 \right] = 4\pi q^{3/2} \int_1^\infty x^2 dx \left[\frac{x^2}{\sqrt{x^4 - 1}} - 1 \right]. \quad (5.14)$$

Note that both BIon and catenoid solutions become at large r just

$$\phi(r) \simeq \frac{q}{r}. \quad (5.15)$$

Perhaps one can find a boosting limit, where one takes $q \rightarrow 0$, so as to have no observable horizon with diverging energy density, but keeping $q\gamma$ finite in the limit (where γ is the relativistic factor $1/\sqrt{1-v^2}$), just like when we infinitely boost the Schwarzschild metric we obtain the Aichelburg-Sexl shockwave metric [28], by keeping $p = M\gamma$ finite in the limit where $M \rightarrow 0, \gamma \rightarrow \infty$), but we have not been able to show this. In any case, we will see that in the nonabelian case, the generalization with this positive sign inside the square root doesn't have a Skyrmion-like solution, whereas the one with the negative sign does.

Next, it was found that in the D-brane action, both X and ϕ are present, as

$$S_{\text{DBI}} = - \int d^4x \sqrt{[1 - (\vec{\nabla}\phi)^2][1 + (\vec{\nabla}X)^2] + \vec{\nabla}\phi \cdot \vec{\nabla}X}. \quad (5.16)$$

Therefore this action has both BIon (in ϕ) and catenoid (in X) static solutions. One can hope then that perhaps by infinitely boosting a more general, BIon plus catenoid solution, one can obtain the shockwave solution, but again that is not clear (and we still have the problem of the nonabelian generalization).

5.1.2 BIon-like solution from $T\bar{T}$ action

This was the case so far. But with the $T\bar{T}$ action replacing the DBI action (with mass term inside the square root, which is irrelevant for the solution near $s = 0$), one more possibility arises. For the $T\bar{T}$ action, we have a single scalar ϕ . We can have a BIon-like solution, namely, a solution of finite energy defined for all r until 0, and yet be defined in terms of the same scalar field X (now called ϕ , but meaning a true scalar, not the electric potential). And then infinitely boosting it we can obtain the shockwave solution.

The catenoid solution has $\phi'(r) \rightarrow \infty$ at some $r = r_0$, but $\phi(r_0)$ finite, but to get to $r = 0$ with a finite energy like for the BIon, we need the opposite sign inside the square root. However, that can happen at large field ϕ , since the coupling of the $(\partial_\mu\phi)^2$ term is $\bar{\lambda} = \lambda(1 - \lambda V)$. Then we expect to have both catenoid-like and BIon-like solutions. Note that the positive sign of $\bar{\lambda}$ is associated not just to the catenoid solution, but also to the shockwave solution.

Both catenoid-like and Bion-like go to q/r at $r \rightarrow \infty$, since then also $V \simeq 0$, so $\bar{\lambda} \simeq \lambda$. As we go to smaller r , they start to differ. Then ϕ becomes large, and then so $V(\phi)$, at least in the case that we have only a mass term (and maybe also a ϕ^4 term), so the question is whether $\bar{\lambda}$ reaches zero or not. If it stays positive, we have a catenoid-like solution (and also shockwave solution, in the boosted case), which develops a horizon at $r_0 \simeq \sqrt{q_0}$, where $\phi'(r_0) = \infty$, $\phi(r_0)$ finite. You might think, but the catenoid is singular, so how come for him, V can stay small enough? The answer is that the singularity is in the *derivative*, not the field; the field can stay small. If $\bar{\lambda}$ reaches zero, and changes sign, we have a BIon-like solution (which stands for the Skyrmion-like one in the nonabelian case).

The question we need to answer is whether it is possible to have a continuous solution, for the BIon-like case, that (must!) change sign for $\bar{\lambda}$ from $\bar{\lambda} > 0$ at $r \rightarrow \infty$ (ϕ small) to $\bar{\lambda} < 0$ at $r \rightarrow 0$ (ϕ large). If we can, then we should be able to infinitely boost such a BIon-like solution to the shockwave solution, something that seemed impossible before.

To see whether we can have a continuous solution for the BIon-like case, we check whether the ansatz for having a *finite* (of order 1, neither zero nor infinity) $\phi'(\phi_0)$ for ϕ_0 a *finite* value for which $\bar{\lambda} = 0$ is consistent for the solutions of the equations of motion.

Consider the value $\phi = \phi_0$ of the field for which $\lambda V(\phi_0) = 1$ (so that $\bar{\lambda} = 0$). Then

$$1 - \lambda V(\phi) \simeq -\lambda V'(\phi_0)(\phi - \phi_0) = -\lambda V'(\phi_0)\delta\phi, \quad (5.17)$$

where we have defined $\delta\phi \equiv \phi - \phi_0$. Then we find that the Lagrangian on the above ansatz becomes

$$-\mathcal{L} \simeq \frac{1 + \sqrt{1 - 8\lambda^2 V'(\phi_0)\delta\phi(\delta\phi')^2}}{-2\lambda^2 V'(\phi_0)\delta\phi}. \quad (5.18)$$

Its equation of motion, $2\lambda^2 V'(\phi_0)\delta\phi \times \delta S/\delta(\delta\phi)$, is

$$\begin{aligned} -\frac{1}{\delta\phi} \left(1 + \sqrt{1 - 8\lambda^2 V'(\phi_0)\delta\phi(\delta\phi')^2} \right) - \frac{4\lambda^2 V'(\phi_0)\delta\phi'^2}{\sqrt{1 - 8\lambda^2 V'(\phi_0)\delta\phi(\delta\phi')^2}} \\ + 2\lambda^2 V'(\phi_0)\delta\phi \left[\frac{4\delta\phi'}{\sqrt{1 - 8\lambda^2 V'(\phi_0)\delta\phi(\delta\phi')^2}} \right]' = 0. \end{aligned} \quad (5.19)$$

Multiplying with $\delta\phi\sqrt{1 - 8\lambda^2 V'(\phi_0)\delta\phi(\delta\phi')^2}$, and after some (longish) algebra, we get

$$\begin{aligned} -1 + 12\lambda^2 V'(\phi_0)\delta\phi\delta\phi'^2 + 8\lambda^2 V'(\phi_0)\delta\phi^2\delta\phi'' \\ + [1 - 8\lambda^2 V'(\phi_0)\delta\phi\delta\phi'^2]^{3/2} = 0. \end{aligned} \quad (5.20)$$

The ideal solution is when there is nothing special at this point ϕ_0 , reached at position $x = x_0$, meaning that we can have a regular Taylor expansion of the field ϕ ,

$$\delta\phi(x) = A\delta x = \phi'(x_0)(x - x_0), \quad (5.21)$$

with $A = \phi'(x_0)$ a nonzero and finite constant. Substituting this ansatz into the above equation, the leading term near $x = x_0$, is $-1 + 1 = 0$, therefore the equation is consistent! That means that the ansatz above was correct (if it was incorrect, we could have tried instead, for instance, $\delta\phi(x) = A\delta x^\alpha$, with α a power less than 1, and maybe then we obtained consistency, $0=0$, for some α ; as it is, we had the correct ansatz).

Finally, that means that we *can* find a BIon-like solution, for which the derivative of the field at the point that $\bar{\lambda}$ changes sign stays finite.

Note that we have really considered the approximate Lagrangian (5.18) in order to obtain the solution. But this Lagrangian is not necessarily obtained from the $T\bar{T}$ deformation Lagrangian, but can come from any Lagrangian of the type

$$-\mathcal{L} = \frac{h(\phi) + \sqrt{1 + 2\lambda(1 - \lambda\tilde{g}(\phi))(\partial_\mu\phi)^2}}{2\lambda(1 - \lambda\tilde{g}(\phi))}, \quad (5.22)$$

if for $1 - \lambda\tilde{g}(\phi_0) = 0$, then $h(\phi) = 1$. The notion of having a *potential* $V(\phi)$, and of $\tilde{g}(\phi)$ to be related to $h(\phi)$ is not needed.

In this subsection, we considered a single scalar, but by the embedding of the single scalar action into the nonabelian one, we know that the same conclusions can be reached about the latter. The BIon-like solution becomes a Skyrmion-like solution, and the shockwave still a shockwave. Thus, we have shown that there is a potential for the Skyrmion-like solution to be boosted to a shockwave. Exactly how that is done will be examined next, but the existence of a horizon hiding a region with wrong sign inside the square root (which allows for a topological solution, of finite energy), yet which can be scaled down to zero in a certain limit, is essential.

5.1.3 Infinitely boosting and indirect limit

Next, we need to understand what is the condition needed to show that the infinitely boosted BIon-like solution (which we still don't know, we just proved it exists) goes to the shockwave solution. To do that, we consider how we know that an infinitely boosted Schwarzschild black hole is the Aichelburg-Sexl shockwave.

In the paper of Aichelburg and Sexl [28], they took a nontrivial limit on the Schwarzschild solution for a black hole, which has a singularity (for the point mass M) at $r = 0$ and a horizon at $r_h = 2M$. The horizon exists because the solution is written in terms of the harmonic function (in 3 spatial dimensions) $f(r) = 1 - 2M/r$. The Einstein equations reduce to the Poisson equation for $f(r)$. In the nontrivial limit of $M \rightarrow 0$, $\gamma = 1/\sqrt{1-v^2} \rightarrow \infty$, but $p \simeq M\gamma = \text{finite}$ (for $v \rightarrow 1$), we obtain the Aichelburg-Sexl shockwave.

A general pp wave would be

$$ds^2 = 2dx^+dx^- + dx_id x_i + (dx^+)^2 H(dx^+, x_i). \tag{5.23}$$

For a shockwave, we have

$$H(x^+, x^i) = \delta(x^+) \Phi(x_i). \tag{5.24}$$

For the Aichelburg-Sexl shockwave, we have $\Phi(x_i)$ harmonic in the transverse dimensions, with Poisson source p , so in $D = 4$ dimensions,

$$\Phi(x_i) = -4G_{N,4} p \ln r^2, \tag{5.25}$$

where $r^2 = x_i x_i$, and in general

$$\Phi(x_i) = \frac{16\pi G_{N,d}}{\Omega_{d-3}(d-4)} \frac{p}{r^{d-4}}. \tag{5.26}$$

In previous works, in particular in [25], it was argued that for a similar A-S shockwave in the background of the gravity dual to QCD (say, in cut-off AdS_5), $\Phi(x_i)$ should be associated with the pion field profile, at least at large r , which means that what is true for obtaining the A-S shockwave, should also be true for obtaining the pion shockwave, which is why we are reviewing the gravitational shockwave limiting procedure.

The limiting boost procedure in the A-S paper is complicated, and also involves a certain nontrivial change of coordinates, but one can easily get the qualitative picture without calculations: since $M \rightarrow 0$, the horizon goes to zero, $r_h \rightarrow 0$, which is why there is

no horizon in the shockwave. Then anything gravitational (curving space) moving at the speed of light must be a pp wave, and if it is point-like (like the mass in the Schwarzschild metric is), it is a shockwave (delta function in x^+).

But instead of the complicated limiting procedure, one can use a shortcut, and find the same result in an easier way. Instead of taking the limit on the metric, we take it on the equation it satisfies. We can check (and find also without explicit calculations) that the only nonzero component of the Ricci tensor $R_{\mu\nu}$ for the pp wave metric is R_{++} . Then this must be proportional to g_{++} , and (since $R_{\mu\nu}$ has 2 derivatives), we must have $R_{++} \propto \partial_i \partial_i g_{++}$ (because of general coordinate invariance, it cannot have $\partial + \partial_i$, or d_+^2 , the only other possibilities for 2 derivatives acting on $g_{++} = H(x^+, x_i)$). In fact, with the correct coefficient,

$$R_{++} = -\frac{1}{2} \partial_i^2 H(x^+, x_i), \tag{5.27}$$

which in fact equals the linearized result (it is the only case we know of, when the Einstein equation is exactly equal to its linearization). The Einstein equation makes this equal to T_{++} .

Now a point particle moving at the speed of light has only T_{++} , and moreover has momentum p , as we defined, so

$$T_{++} = p \delta^{d-2}(x_i) \delta(x^+), \tag{5.28}$$

leading in turn to the shockwave ansatz $H = d\delta(x^+) \Phi$, which finally leads to

$$\partial_i^2 \Phi(x_i) = -16\pi G_{N,d} p \delta^{d-2}(x_i), \tag{5.29}$$

with solution the A-S shockwave. Thus after the infinite boost, the Einstein equations reduced to the equation for the field $\Phi(x_i)$ in the pp shockwave ansatz. We see that the pp wave ansatz is a very interesting one: the usually very nonlinear Einstein equations linearize on the ansatz (the linearized equation is actually exact).

We want to mimic the procedure in the case of the abelian pion action. In this case, there is no reduction of the degrees of freedom by the ansatz ($g_{\mu\nu}$ in the general Einstein equations, Φ on the ansatz). Instead, the variable is a single real scalar ϕ before and after the infinite boost.

Then all we need to show is that the equations of motion, or equivalently, the Lagrangian, for the BIon-like ansatz, go over to the Lagrangian for the shockwave, under the infinite boost.

First, we need to understand the limiting procedure. To do that, we go back to the shockwave energy-momentum tensor (3.33), to find the total momentum of one of the colliding shockwaves, for instance $x^- = 0$. For the A-S gravitational shockwave on $x^+ = 0$, only T_{++} was nonzero, but it was a constant times a delta function, integrating to 1, so finite total momentum. Yet now, as we said, $x^- = 0$ implies only T_{--} , which is a large, yet finite constant, times $(d\phi/ds)^2 (x^+)^2$. Since also $(x^+)^2$ is a constant, we have (since $\phi \sim \sqrt{s}$)

$$T_{--} \sim \left(\frac{d\phi}{ds}\right)^2 \sim \frac{1}{s} \propto \frac{1}{\sqrt{x^+ x^-}}. \tag{5.30}$$

By integrating over x^- , this means that there is a vanishing momentum located at $x^- = 0$.

On the other hand, using

$$\dot{\phi} \equiv \frac{\partial \phi}{\partial t} = \frac{d\phi}{ds} \frac{t}{s}, \tag{5.31}$$

the energy is (ignoring the finite and vanishing terms in the energy-momentum tensor)

$$\begin{aligned} E &= \int dx T_{00} \simeq \int dx \frac{\dot{\phi}^2}{\sqrt{1 + l^4(\partial_\mu \phi \partial_\nu \phi g^{\mu\nu} + m^2 \phi^2)}} \\ &\propto \int dx \left(\frac{d\phi}{ds}\right)^2 \frac{t^2}{s^2} \propto \int dx \frac{1}{s^3} = \int dx \frac{1}{(t-x)^{3/2}(t+x)^{3/2}}, \end{aligned} \tag{5.32}$$

and integrating over x at fixed t , around $x=t$, we get a divergence. So the total energy of the field is divergent, which is consistent with a object of finite mass M (the Skyrmion) boosted with an infinite γ . But that means that now $M\gamma$ is not kept finite, but is still infinite.

In fact, now we realize M *should* be kept fixed in the limit. Indeed, M can only depend on the parameters of the theory, and those are fixed by experiment, and cannot be varied.

Now the limit only makes use of Lorentz contraction: the horizon in the x direction has zero size now. But the action on the BIon-like ansatz included $\phi'(r)$ and $\phi(r)$, where r was the total radial coordinate, which is now (since $v \simeq 1$)

$$r^2 \simeq [(x-t)\gamma]^2 + \tilde{r}^2 = (x^-)^2 \gamma^2 + \tilde{r}^2, \tag{5.33}$$

where \tilde{r} represents now the transverse coordinate. That means that the dependence on \tilde{r} becomes negligible. Of course, now also $\dot{\phi}$ becomes nonzero in the ansatz. Thus trivially, we have that the equations of motion on the BIon-like ansatz become the ones on the shockwave ansatz under the infinite boost, therefore the BIon-like solution *should* become the shockwave one (in the single wave region, $t < 0$, $x < 0$, for instance), as wanted.

It is just that we can only “see” the outside horizon solution (with $\bar{\lambda} > 0$), the “inside horizon” part has become obscured. We can ask: where is this “inside horizon” region? Its extent in the x direction has become zero by the infinite boost, that is why we don’t see it. Of course, its extent in the transverse directions has remained the same, but it is situated *on the singularity* $s = 0$, so it is impossible to reach.

5.2 A nonabelian version of $T\bar{T}$ -like actions and chiral perturbation theory

In this subsection, after reviewing Skyrmion-like solitons of non-Abelian actions and chiral perturbation theory, we will propose a 3+1 dimensional, non-Abelian version of the $T\bar{T}$ action, and argue that it consistent with chiral perturbation theory confronted by experiment.

5.2.1 Review of Skyrmion-like solutions in non-Abelian generalizations, and chiral perturbation theory

In [5], one of us has considered the possibility of a certain nonabelian generalization of the D-brane action that can have Skyrmion-like solutions. The reason why we considered the D-brane action is that, as we saw, the Skyrmion-like solution is the analog of the BIon-like solution.

Indeed, in a paper by Pavlovsky [29], it was shown that we can have Skyrme-like solutions for a nonabelian DBI-like scalar action, *but with the wrong sign inside the square root*.

For effective actions for QCD in the low energy expansion (note that the various nonlinear sigma models used for QCD are, at least for the relevant first order in chiral perturbation theory, all equivalent through field redefinitions, etc.), one way of defining the theory is by an SU(2) matrix U containing the linearized pions $\pi^i \sigma_i$ in an expansion of U around the identity, from which we form the quantities

$$L_\mu = U^{-1} \partial_\mu U. \tag{5.34}$$

Then the action of [29] is

$$\mathcal{L}_P = \text{Tr} \sqrt{1 + \frac{1}{2\beta^2} L_\mu L^\mu}. \tag{5.35}$$

Considering the spherically symmetric (“hedgehog”) ansatz

$$U = e^{iF(r)\vec{n}\cdot\vec{\sigma}}, \tag{5.36}$$

where $\vec{n} = \vec{r}/r$ is the radial unit vector and σ_i are the Pauli matrices), we find

$$\begin{aligned} L_i &= in_i \vec{n} \cdot \vec{\sigma} \left(F' - \frac{\sin F \cos F}{r} \right) + \frac{i\sigma_i}{r} \sin F \cos F - i \frac{\sin^2 F}{r} \epsilon_{ijk} n_j \sigma_k \Rightarrow \\ L_i^2 &= -F'^2 - 2 \frac{\sin^2 F}{r^2} - 4\vec{n} \cdot \vec{\sigma} \frac{\sin^3 F \cos F}{r^2} \Rightarrow \\ \text{Tr}[L_i^2] &= 2 \left[-F'^2 - 2 \frac{\sin^2 F}{r^2} \right]. \end{aligned} \tag{5.37}$$

Now, both in [29] (implicitly, from the equations deduced from it) and in [5] (explicitly), the term with $\vec{n} \cdot \vec{\sigma}$ was erroneously forgotten, so that it was claimed that on the hedgehog ansatz, the $L_\mu L^\mu$ appearing inside the square root was already proportional to the identity. For all the calculations in these papers to be valid, however, we need then to change the Lagrangian to one with square root inside,

$$\mathcal{L}'_P = \sqrt{1 + \frac{1}{4\beta^2} \text{Tr}[L_\mu L^\mu]}, \tag{5.38}$$

since then the Lagrangian on the hedgehog ansatz is unmodified with respect to the results there, and is

$$\mathcal{L}'_P = \sqrt{1 - \frac{1}{2\beta^2} F'^2 - \frac{\sin^2 F}{\beta^2 r^2}}. \tag{5.39}$$

We note then that indeed, the sign inside the square root is wrong (minus, instead of plus) with respect to the scalar DBI action. But, together with the wrong sign in front of the action, this means that at low fields, the action still becomes the same canonical scalar, and only the higher nonlinear terms have different signs.

This action [29] has a Skyrminion-like solution, which can be also thought, comparing with the D-brane action above, as a BIon-like solution (since $F(r)$ is like $\phi(r)$, it has the

same sign inside the square root). But, since the sign inside the square root is wrong, we can say (and prove rigorously, see [5]) that it *doesn't* have a catenoid-like solution (with a “horizon”), and so also not a shockwave solution. The Skyrmion solution has $F(0) = N\pi$ and $F(\infty) = 0$, because of the topological constraint, which means that the energy and baryon number of the solution is quantized.

The baryon number density on the Skyrmion solution can be defined. On any solution defined in terms of $U = e^{iF(r)\vec{n}\cdot\vec{\sigma}}$, we have the baryon number (winding number, or topological number)

$$B = -\frac{1}{24\pi^2} \int d^3r \epsilon^{ijk} \text{Tr}[U^{-1}\partial_i U U^{-1}\partial_j U U^{-1}\partial_k U], \quad (5.40)$$

so the baryon number density is what is inside $\int d^3r$. Since we get

$$B = -\frac{1}{2\pi^2} \int d^3r F'(r) \frac{\sin^2 F(r)}{r^2} = -\frac{2}{\pi} \int_0^\infty dr F'(r) \sin^2 F(r), \quad (5.41)$$

it means that the baryon number density is

$$\rho_B(r) = -\frac{1}{2\pi^2} F'(r) \frac{\sin^2 F(r)}{r^2}. \quad (5.42)$$

Then, if $F(0) = \pi$, we have a baryon, and if $F(0) = -\pi$, we have an anti-baryon. Moreover, we can see in the density, that for the baryon, $F'(r) < 0$ (F decreases until $F = 0$ at infinity), so $\rho_B(0) > 0$, whereas for the antibaryon $F'(r) > 0$ (F increases until $F = 0$ at infinity), so $\rho_B(0) < 0$.

But boosting nucleons should lead, according to Heisenberg, to pion shockwave solutions. But nucleons are described by Skyrmons, and so perhaps Skyrmons should be boosted to shockwaves. But in the case described in [29], we have one, but not the other solution, while in the DBI D-brane action, we have both solutions, but in terms of different *fields*.

As we saw, the DBI D-brane action is Abelian, whereas the Skyrme-type Pavlovsky one is non-Abelian, which is essential in order to obtain a topological Skyrme-type solution, so we need to write a non-Abelian generalization. In [5] an attempt at generalization was made only for the space dependence (without the time dependence), and only for $|\vec{\nabla}\phi| \ll \bar{M}$ (energies below some cut-off), so the fields are $A_0 = \phi$, or more precisely $\vec{E} = \vec{\nabla}\phi$, and $L_i = U^{-1}\partial_i U$ replacing X . The action was

$$\frac{\mathcal{L}}{\text{const.}} = 1 - \text{Tr} \sqrt{\left(1 - \frac{\vec{E}^2}{M_2^2}\right) \left(1 - \frac{L_i^2}{M_1^2}\right) - \frac{(E_i L_i)^2}{M_1^2 M_2^2} + \frac{L_0^2}{M_1^2} + \frac{m_\pi^2}{M_1^2} (U + U^\dagger - 2) + M_A^2 \phi^2}, \quad (5.43)$$

but it is not very satisfying. It was found that it has a BIon-like solution *in terms of* ϕ , and at $\phi = 0$ it has a catenoid-like solution (with a “horizon”) in terms of F , but no Skyrmion-like solution for $F(r)$, since it has the wrong sign with respect to [29] (the correct sign is for ϕ , but that cannot have the Skyrmion radial ansatz, since it is not in the fundamental representation of $SU(2)$).

But as we said, one should rather have an action that corresponds to a particular form of chiral perturbation theory. To see what that means, we review a few relevant points about the latter.

The Lagrangian for low-energy QCD contains the u and d quarks (and maybe the s quark), and some mass terms for the quarks (u and d certainly, s if we include it), and in the massless quark limit is invariant under global $U(2)_L \times U(2)_R$ (extended to $U(3)_L \times U(3)_R$ if s is included). The masses break the symmetry, so the Goldstone bosons for the symmetry breaking, the pions $\pi^a = (\pi^+, \pi^-, \pi^0)$ become pseudo-Goldstone bosons.

The diagonal part of the $U(1)_L \times U(1)_R$ component is baryon number, as used in section 4, and the other $U(1)$ is a chiral symmetry broken by anomalies. The $SU(2)_L \times SU(2)_R$ is broken spontaneously to the diagonal $SU(2)_V$, and the other $SU(2)$ is a chiral symmetry, that must be broken spontaneously, but its mechanism in QCD is unknown.

To deal with this spontaneous breaking phenomenologically, one constructs a low energy Lagrangian in terms of the low energy physical (gauge invariant) mesonic states π^a and σ , put together into the matrix (τ^a are the Pauli matrices)

$$\Sigma = \sigma \mathbb{1} + i\tau^a \pi^a, \tag{5.44}$$

and constructs the *linear sigma model* Lagrangian for it, similar to the Higgs Lagrangian. This restricts Σ to its vacuum value v , if we ignore the Higgs-like fluctuation $s(x)$,

$$\Sigma = (v + s(x))U(x), \quad U(x) = e^{i\tau^a \pi^a(x)/v}. \tag{5.45}$$

This results in the *NonLinear Sigma Model* for the unitary matrix U , $U^\dagger U = \mathbb{1}$,

$$\mathcal{L}_{\text{NL}} = -\frac{v^2}{4} \text{Tr}[\partial_\mu U \partial^\mu U^\dagger]. \tag{5.46}$$

This is however only the lowest term (second order) in a low energy expansion in derivatives ∂_μ , or momenta p_μ , which is usually taken together with pion masses m_π , because of the on-shell relation $p^2 = m_\pi^2$. The next term would be a p^4 term, so

$$\mathcal{L} = \mathcal{L}_2(p^2) + \mathcal{L}_4(p^4) + \mathcal{L}_6(p^6) + \dots \tag{5.47}$$

The NLSM Lagrangian (5.46) is the most general chiral invariant effective Lagrangian with 2 derivatives written in terms of the unitary U , however, there are many way to re-express it via field redefinitions, for instance as an $SO(4)$ vector model, etc.

Since in this work we have dealt only with unitary matrices U , we consider it to be the basic first term in the expansion. Indeed, it is the first term in the expansion in derivatives of the non-Abelian version of the Heisenberg or $T\bar{T}$ deformation, see for instance (4.19).

At the next order (p^4) however, there are many possible terms. If we also consider the coupling to external fields (that makes the global symmetries local, by considering a covariant derivative), adding the terms

$$\bar{q}\gamma^\mu (V_\mu + \gamma_5 A_\mu) q - \bar{q}(s - i\gamma_5 p)q \tag{5.48}$$

to the QCD low energy action, we find corresponding terms that can be added to the effective action, now written in terms of U , $\chi = s + ip$, and also in terms of a_μ^L and a_μ^R , equal to $V_\mu \pm A_\mu$, with their corresponding field strengths, $f_{\mu\nu}^L$ and $f_{\mu\nu}^R$.

5.2.2 Nonabelian action with Skyrmion-like solutions and comparison with chiral perturbation theory

We now will write a nonabelian version of the $T\bar{T}$ action extended to 3+1 dimensions, from the point of view of chiral perturbation theory, and then we will compare with the general expectations about the latter discussed in the review [30]. That means that we want to write an action in terms of the matrix U above, and the associated $L_\mu = U^{-1}\partial_\mu U$.

To do that, we note the standard replacement, when going from one scalar to the SU(2)-valued scalars U ,

$$(\partial_\mu \phi)^2 \rightarrow -L_\mu L^\mu, \tag{5.49}$$

that leads one from the free massless scalar to the nonlinear sigma model action

$$\mathcal{L}_{\text{NLSM}} = \frac{f_\pi^2}{4} \text{Tr}[L_\mu L^\mu] = -\frac{f_\pi^2}{4} \text{Tr}[\partial_\mu U \partial^\mu U^{-1}]. \tag{5.50}$$

But this is the starting point of the $T\bar{T}$ deformed action in 1+1 dimensions considered in section 4, trivially extended to 3+1 dimensions. Moreover, as we argued there, we need to also add a WZW term, in order for the Noether charge to match the topological charge, and be equated to the baryon charge. That means that we extend to 3+1 dimensions the analysis of section 4.

Then at zero potential ($V = 0$), the Abelian $T\bar{T}$ action (now equal to DBI) extended to 3+1 dimensions and generalized to a nonabelian action relevant for QCD in the low energy expansion is

$$\mathcal{L}_{\text{DBI,NLSM}} = -f_\pi^2 \beta^2 \left[\sqrt{1 - \frac{1}{4\beta^2} \frac{N_c}{2\pi} \left(\text{Tr}[L_\mu L^\mu] - \frac{1}{4\beta^2} \frac{N_c}{4\pi} \tilde{X}^2 \right)} - 1 \right] + \mathcal{L}_{\text{WZW}}, \tag{5.51}$$

which has the opposite sign inside the square root with respect to the Pavlovsky case (that has a Skyrmion solution, as we saw), but otherwise it is the same. We need to define \tilde{X}^2 and \mathcal{L}_{WZW} in 3+1 dimensions. The latter is standard, and the former can be extended most naturally (so as to go back to the 1+1 dimensional result upon dimensional reduction) as

$$\tilde{X}_1^2 = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \epsilon_{\lambda\tau\rho\sigma} \text{Tr}[L_\mu L_\lambda] \text{Tr}[L_\nu L_\tau], \tag{5.52}$$

which then gives a nonzero contribution *on the static ansatz* $L_0 = 0$,

$$\begin{aligned} \tilde{X}_1^2 &= \epsilon_{imp} \epsilon_{jnp} \text{Tr}[L_i L_j] \text{Tr}[L_m L_n] \\ &= 8 \frac{\sin^2 F}{r^2} \left\{ 2F'^2 + \frac{\sin^2 F}{r^2} + \frac{\sin^4 F \cos^2 F}{r^2} \right\}. \end{aligned} \tag{5.53}$$

With this choice, the Lagrangian on the ansatz is therefore modified by \tilde{X}^2 . However, another possibility would be

$$\tilde{X}_2^2 = \epsilon^{\mu\nu\rho\sigma} \epsilon^{\mu'\nu'\rho'\sigma'} \text{Tr}[L_\mu L_{\mu'}] \text{Tr}[L_\nu L_{\nu'}] \text{Tr}[L_\rho L_{\rho'}] \text{Tr}[L_\sigma L_{\sigma'}], \tag{5.54}$$

which would vanish on the static ansatz $L_0 = 0$. With this extension to 3+1 dimensions then, the Lagrangian on the ansatz would be unmodified by this term.

It remains to add the potential (mass) term. In chiral perturbation theory, a mass term for one scalar is generalized in the SU(2) case to

$$m_\pi^2 \phi^2 \rightarrow -f_\pi^2 m_\pi^2 \frac{1}{2} \text{Tr}[U + U^\dagger - 2], \quad (5.55)$$

the same that we used in section 4 (the 1/2 is there to cancel the factor of 2 from the trace).¹

Then the $T\bar{T}$ action in the 3+1 dimensional nonabelian case for $V \neq 0$ will be (for a potential that is just a mass term, and for $\lambda = 1/(f_\pi^2 \beta^2)$), but note that $F = \phi/f_\pi$ for dimensions to work, where ϕ is the usual dimension 1 field)

$$\begin{aligned} \mathcal{L}_{\text{DBI,NLSM}}^\lambda = & + \frac{f_\pi^2 m_\pi^2 \text{Tr}(U + U^\dagger - 2)}{4 \left(1 + \frac{m_\pi^2}{2\beta^2} \text{Tr}(U + U^\dagger - 2)\right)} \\ & - f_\pi^2 \beta^2 \frac{\left[\sqrt{1 + \frac{1}{4\beta^2} \frac{N_c}{2\pi} \left(-\text{Tr}[L_\mu L^\mu] + \frac{N_c}{4\pi} \frac{1}{4\beta^2} \tilde{X}^2\right) \left(1 + \frac{m_\pi^2}{4\beta^2} \text{Tr}(U + U^\dagger - 2)\right)} - 1 \right]}{\left(1 + \frac{m_\pi^2}{4\beta^2} \text{Tr}(U + U^\dagger - 2)\right)} \\ & + \mathcal{L}_{\text{WZW}}, \end{aligned} \quad (5.56)$$

which is basically the non-Abelian action (4.19) rewritten in 3+1 dimensions. We refer to this action as a $T\bar{T}$ -like deformation of the Skyrme action. It does not fit to the proposal of [12] for the deformation in dimensions higher than 1+1.

In analogy to the expansion in terms of small λ in eq. (4.19) we find now an approximate expression for the Lagrangian density

$$\begin{aligned} \mathcal{L} \simeq & \left(\frac{f_\pi^2}{16} \text{Tr}[L_\mu L^\mu] + \frac{f_\pi^2 m_\pi^2}{4} \text{Tr}[U + U^\dagger - 2] \right) \\ & + \frac{1}{f_\pi^2 \beta^2} \left\{ \frac{1}{8} \left(\frac{f_\pi^2}{4} \right)^2 \left(\frac{N_c}{2\pi} \right)^2 \left[(\text{Tr}[L_\mu L^\mu])^2 - 2(\tilde{X})^2 \right] - \left(\frac{f_\pi^2 m_\pi^2}{2} \text{Tr}[U + U^\dagger - 2] \right)^2 \right. \\ & \left. + \frac{1}{4} \left(\frac{f_\pi^2}{4} \right) \left(\frac{N_c}{2\pi} \right) \left(\frac{f_\pi^2 m_\pi^2}{2} \text{Tr}[L_\mu L^\mu] \text{Tr}[U + U^\dagger - 2] \right) \right\} + \mathcal{O} \left((1/(f_\pi^2 \beta^2))^2 \right). \end{aligned} \quad (5.57)$$

With respect to the Abelian pion case, we therefore identify the mass potential as

$$V \leftrightarrow -\frac{m_\pi^2 f_\pi^2}{4} \text{Tr}(U + U^\dagger - 2) = m_\pi^2 f_\pi^2 2 \sin^2(F/2). \quad (5.58)$$

However, note that $\lambda V = (m_\pi^2/\beta^2) 2 \sin^2(F/2)$, and $m_\pi = 134 \text{ MeV}$ (for π^0) or 139 MeV (for π^\pm), $f_\pi = 130 \text{ MeV}$ (or, depending on convention with respect to multiplication or division by $\sqrt{2}$'s, 93 MeV or 184 MeV) while β can only be $\sim \Lambda_{\text{QCD}}$. Now it depends how this quantity is defined, but if it is defined as the string tension, we would expect it

¹Note that we can perhaps be more general, and also replace a generic potential,

$$V(\phi) = m_\pi^2 f(\phi^2) \rightarrow -m_\pi^2 f \left(\frac{1}{2} \text{Tr}[U + U^\dagger - 2] \right).$$

to be around 0.4 GeV, or perhaps even a factor of 2 smaller. In that case, the factor of $(m_\pi^2/\beta^2)2\sin^2 F/2$ can reach 1, and the horizon can be achieved.²

We can make a comparison with the chiral perturbation theory at order p^4 (first nontrivial order).

If we use \tilde{X}_2^2 , this term is of order p^8 , so doesn't contribute here. If we use the more likely \tilde{X}_1^2 , we find

$$\begin{aligned}\tilde{X}_1^2 &= (\text{Tr}[L_\mu L_\mu])^2 - \text{Tr}[L_\mu L_\nu] \text{Tr}[L_\mu L_\nu] \Rightarrow \\ (\text{Tr}[L_\mu L^\mu])^2 - 2\tilde{X}_1^2 &= -(\text{Tr}[L_\mu L_\mu])^2 + 2\text{Tr}[L_\mu L_\nu] \text{Tr}[L^\mu L^\nu] \\ &= -\left(\text{Tr}[\partial_\mu U \partial^\mu U^\dagger]\right)^2 + 2\text{Tr}[\partial_\mu U \partial^\nu U^\dagger] \text{Tr}[\partial^\mu U \partial^\nu U^\dagger].\end{aligned}\quad (5.59)$$

In [30], section 4.7, is written the most general Lagrangian at order p^4 (like the first nontrivial order in the expansion of the square root for us), in the case $m_\pi = 0$ (thus with some difference with respect to our case)

$$\begin{aligned}-\mathcal{L} &= L_1 \left\{ \text{Tr}[D_\mu U (D^\mu U)^\dagger] \right\}^2 + L_2 \text{Tr} \left[D_\mu U (D_\nu U)^\dagger \right] \text{Tr} \left[D^\mu U (D^\nu U)^\dagger \right] \\ &+ L_3 \text{Tr} \left[D_\mu U (D^\mu U)^\dagger D_\nu U (D^\nu U)^\dagger \right] + L_4 \text{Tr} \left[D_\mu U (D^\mu U)^\dagger \right] \text{Tr} \left(\chi U^\dagger + U \chi^\dagger \right) \\ &+ L_7 \left[\text{Tr}(\chi U^\dagger - U \chi^\dagger) \right] + L_8 \text{Tr} \left(U \chi^\dagger U \chi^\dagger + \chi U^\dagger \chi U^\dagger \right) \\ &- iL_9 \text{Tr} \left[f_{\mu\nu}^R D^\mu U (D^\nu U)^\dagger + f_{\mu\nu}^L (D^\mu U)^\dagger D^\nu U \right] + L_{10} \text{Tr} \left(U f_{\mu\nu}^L U^\dagger f_R^{\mu\nu} \right) \\ &+ H_1 \text{Tr} \left(f_{\mu\nu}^R f_R^{\mu\nu} + f_{\mu\nu}^L f_L^{\mu\nu} \right) + H_2 \text{Tr} \left(\chi \chi^\dagger \right).\end{aligned}\quad (5.60)$$

The covariant derivatives $D_\mu U$ are related to turning on external a_μ^L, a_μ^R flavor gauge fields. Needless to say that we can also uplift the ordinary derivatives in (5.57) to covariant ones. Besides the terms with scalar sources χ and external gauge fields a_μ^L, a_μ^R , there are 10 terms, and comparison with experiment is done in their table 4.3. In it, the coefficients

²However, if one would take as $\beta \sim \Lambda_{\text{QCD}}$ understood as the glueball mass, $\sim 1\text{GeV}$, then $1 - \lambda V \sim 1 - (1/100)2\sin^2 F/2$, which clearly cannot be $=0$. That means this choice doesn't lead to a horizon. In fact, even if we choose $V \sim m_\pi^2 f_\pi^2 F^2/2 = m_\pi^2 \phi^2/2$, that still leads to $1 - \lambda V \sim 1 - (1/100)F^2$, and a topological solution without a horizon has $F(0) = N\pi$ (with $N = 1$ for one Skyrmion) and $F(\infty) = 0$, so $\Delta F = \pi < 100$, so no horizon either.

The only choice, given the parameters, is to have also a $\lambda_0 \phi^4 = \lambda_0 f_\pi^4 F^4$ potential, with $\lambda \lambda_0 f_\pi^4 \sim 1/10$ (meaning a perturbative potential), so that $(F(0) - F(\infty))^4 = \pi^4 > 10$, giving a horizon. Then we replace in (5.56),

$$V = -\frac{m_\pi^2 f_\pi^2}{4} \text{Tr}(U + U^\dagger - 2) \rightarrow -\frac{m_\pi^2 f_\pi^2}{4} \text{Tr} \left[\log^2 \frac{U}{U_0} \right] + \frac{\lambda_0}{2} f_\pi^4 \text{Tr} \left[\log^4 \frac{U}{U_0} \right].$$

In terms of the Skyrmion ansatz, we have

$$V \rightarrow \frac{m_\pi^2}{2} (\phi - \phi_0)^2 + \lambda_0 (\phi - \phi_0)^4,$$

so

$$1 - \lambda V = 1 - \frac{m_\pi^2}{2\beta^2} (F - F_0)^2 - \lambda_0 \frac{f_\pi^2}{\beta^4} (F - F_0)^4,$$

allowing for the possibility of a horizon. Thus in this case, we would need to consider a rather nontrivial potential, but it is also possible to obtain one that has a horizon.

that are best understood as nonzero (experimentally, at more than 3σ away from zero) are 2,3 and 9,10. However, the only terms without external fields are 1, 2 and 3, with $L_1 = 0.4 \pm 0.3$, $L_2 = 1.35 \pm 0.3$ and $L_3 = -3.5 \pm 1.1$. We see that expanding the square root in our Lagrangian (5.56) we obtain the terms first two terms above, with coefficients either 1 and 0 (for \tilde{X}_2^2), or -1 and +2 (for \tilde{X}_1^2), the last one consistent with the experimental results above for about 2σ in L_1 and 2σ in L_2 (re-normalizing the coefficient of L_1 to match), however, the third term would also have zero coefficient, which is within slightly more than 3σ . Thus, at this point it is not yet clear if this is consistent with our model. More importantly it is premature to make any claims about the relations between the prediction of our model and the experimental data.

5.3 Boosted Skyrme-like soliton vs. shockwave solution

We now consider the boost of the Skyrme-like solution for the non-Abelian action, and show that we get the shockwave solution (in the region where there is only one wave). We already showed this in the Abelian analogue case, but now we generalize to the real case (non-Abelian).

We ignore the potential, since as we saw, the horizon of the Skyrme-like solution will shrink onto the singularity $s = 0$, so we can safely ignore V , and consider only the non-abelian DBI of the type considered by Pavlovsky. On the Skyrme ansatz, this Lagrangian becomes

$$\sqrt{1 + \left(\frac{1}{2\beta^2} F'^2 + \frac{\sin^2 F}{\beta^2 r^2} \right)} (1 - \lambda V(F - F_0)). \quad (5.61)$$

Under the boost, F'^2 goes back to $F'^2 - \dot{F}^2$ and, since $r^2 \rightarrow \infty$ under $\gamma \rightarrow \infty$ (at finite x^- , so finite s , meaning for our $\phi(s)$ shockwave), the term with $\sin^2 F/(\beta^2 r^2)$ vanishing.

That means that on the ansatz, the Lagrangian goes over the Heisenberg Lagrangian, the Abelian scalar DBI one, with $f_\pi(F - F_0) = f_\pi(F - F(r = 0))$ identified with the scalar ϕ of the abelian action. Then ϕ increases away from $r = 0$, so $x^- = 0$ (with $\tilde{r} = 0$), as does the shockwave solution.

Then by the logic explained in the case of the Abelian analogue, the Skyrme-like solution should go over to the shockwave one under the infinite boost.

We can also ask whether in the shockwave solution we have a baryon-baryon or baryon-antibaryon collision. We saw that the baryon density $\rho_B(r)$ can be calculated, and Then, if $F(0) = \pi$, we have a baryon, and if $F(0) = -\pi$, we have an antibaryon. Moreover, we can see in the density, that for the baryon, $F'(r) < 0$ (F decreases until $F = 0$ at infinity), so $\rho_B(0) > 0$, whereas for the antibaryon $F'(r) > 0$ (F increases until $F = 0$ at infinity), so $\rho_B(0) < 0$.

But we also see that, almost by definition, if we have a single, continuous, $F(r)$, or, in the DBI picture, field $\phi(s)$, we have a baryon-baryon collision, not a baryon-antibaryon collision, for which ϕ (which stands in for $F(r)$, up to an additive constant, as we saw) would have to be discontinuous at the collision point $x = 0, t = 0$. But we could consider it also to be a baryon-antibaryon solution if we could define different $\phi(x = +\infty)$ and $\phi(x = -\infty)$ at fixed t . But we can't, since the solution depends on $s = -x^+x^-$ only, not independently on x and t (or x^+ and x^-). This means that the solution does not probe the region $x = \pm\infty$ at fixed t , in this region the solution would depend on both variables.

6 Interpretation and discussions

Finally, we come to the possible interpretation of our results for QCD.

6.1 Why $T\bar{T}$?

The first thing to ask is: why can we use the $T\bar{T}$ action in the context of Heisenberg’s model? The simplest answer is that Heisenberg himself just *guessed* an action with the right properties namely, a jump in $(\partial_\mu\phi)^2$ for the shockwave solution and saturation of the Froissart bound, so we can choose another one that has the same properties and, like we showed, has the added property that it can have both a shockwave and a soliton solutions, in such a way that the solitons can be boosted into shockwaves.

A second point is that the $T\bar{T}$ deformation action has unique properties, as being the only deformation of simple models that can be completely solved, like in the case of integrable models. We could think perhaps of the $T\bar{T}$ deformation of the 3+1 dimensional canonical scalar action (instead of the deformation in 1+1 dimensions). After all, the system we are describing is 3+1 dimensional. But we show in appendix A that this is not possible. We could also think perhaps of the $T\bar{T}$ deformation of the Maxwell or Yang-Mills action for a vector (be it the fundamental gluon, or a low energy vector particle). But we show in appendix B that is also not possible.

Accepting this apparent “uniqueness property”, we are still left with the question how to relate this construction to the fundamental QCD description of the colliding nucleons. As we mentioned at the beginning, in the defining relation of the $T\bar{T}$ deformation Lagrangian (3.1) all objects are quantum renormalized and UV finite, and on the right-hand side we have an operator regularized by point-splitting. Yet we treated the resulting Lagrangian as a classical one, giving rise to classical fields, which is in accordance with the reasoning of Heisenberg for his model.

The simplest possibility of relating the $T\bar{T}$ deformed Lagrangian to QCD then is to think of it as being a quantum effective action in the Wilsonian sense for the pions (or rather, a single (pseudo)scalar version of it, that we tried to extend to SU(2)-valued fields). The parameter λ of the deformation (corresponding to l^4 for the usual Heisenberg model (1.2)) *must be related to the QCD string tension*.

The quantum theory for this Wilsonian effective theory, now reduced (the Wilsonian theory, not its quantum theory) to 1+1 dimensions (the direction of propagation and time) should give the defining equation of the $T\bar{T}$ deformation (3.1), as a function of λ . Note that this would not mean that the same equation (3.1) is satisfied for the full quantum theory of the 3+1 dimensional Wilsonian effective theory (in fact, we saw in appendix A that the $T\bar{T}$ deformed 3+1 dimensional theory doesn’t have the needed classical solutions).

The defining relation (3.1) can be rewritten, with $\lambda =$ the string tension σ ,

$$\frac{\partial \mathcal{L}^\sigma}{\partial \sigma} = \det T_{\mu\nu}^{(\sigma)}. \tag{6.1}$$

One possible interpretation of this relation is that as we vary the string tension, the variation of this Wilson effective Lagrangian is proportional to the density square of the

pion field, since the pion field breaks up into a quark at one end and an anti-quark at the other, but both are proportional to the density at that point (note that this is how we understand the defining relation anyway: by point splitting). And then the Lagrangian changes by the variation of this pion particle energy density.

The Wilsonian effective theory reduced to 1+1 dimensions admits a classical limit as well, which is when the value of its field is large (many pions in the same state), so we can talk about a classical profile. For 1+1 dimensions, we have concentrated on the shockwave, which is a statement about the field around a nucleus, moving at high speed. There is a legitimate question as to why the Wilsonian theory is still valid, if we are at high energies, but perhaps the answer is simply that we consider the field around a nucleus, and the nucleus has high energy, yet the pions do not, since they are *virtual* (part of the field). Indeed, the DBI action is used for the saturation of the Froissart bound for σ_{tot} , which by the optical theorem is $\sim \text{Im } \mathcal{M}(\tilde{s}, t = 0) < \sim \tilde{s} \ln^2 \tilde{s}$, (see also around eq. 28 in section 4 of [31]) which are both dominated by emission of large number of soft (low energy) pions, which would make a classical (many particles), but low energy field.

6.2 Gravity dual interpretation

Like in [4], we can ask what is the interpretation from a gravity dual point of view? As usual, the ϕ scalar is the position of some D-brane in the fifth dimension. But we have already analyzed this in [4], even for the nonlinear sigma model $g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j$, in eq. (4.15) there. Because of eq. (4.18) there, we have a single $g(\phi)$, and by comparison with our case (3.6), we have

$$g(\phi) = \bar{\lambda}(\phi) = \lambda(1 - \lambda V(\phi)). \tag{6.2}$$

We had analyzed in detail the case $g(\phi) = 1/\phi^4$, which corresponds to motion in AdS space in the gravity dual. But here, we see that the metric in the fifth dimension is dependent of $V(\phi)$, and moreover has a kind of “horizon”, but only in the sense that the metric $g(\phi)$ changes sign at ϕ_0 .

Until now, we have compared with the our holographic model, eq. (4.15) in [4], only in the metric $g(\phi)$. But the $T\bar{T}$ action has more terms. In particular, there is an $e^{-\varphi}$ factor (where φ is the dilaton) multiplying the square root, just like $1/\bar{\lambda}$ in (3.6), so we identify those two. Then there is another term, outside the square root, $-\tilde{V}$, that is just like the WZ term $A_{0123}e^\varphi$, so again we must identify the two, leading to

$$\begin{aligned} g(\phi) &= \bar{\lambda} = \lambda(1 - \lambda V(\phi)) \\ T_p e^{\varphi(\phi)} &= \bar{\lambda} = \lambda(1 - \lambda V(\phi)) \\ \mu_p A_{0123} e^\varphi(\phi) &= -\tilde{V}(\phi) = \frac{1 - 2\lambda V(\phi)}{2\lambda(1 - \lambda V(\phi))}. \end{aligned} \tag{6.3}$$

Note that since the dilaton $\varphi(\phi)$ must be positive, and since at the “horizon” $1 - \lambda V = 0$, $T_p e^{-\varphi(\phi)}$ changes sign, it follows that the tension T_p changes sign, from D-brane type, to orientifold-type. However, a orientifold does not fluctuate like a D-brane, so it is a negative tension brane in the sense of Randall-Sundrum (which however doesn’t have a string theory interpretation). This is then a truly phenomenological interpretation of AdS/CFT: just

use the map, without worrying about string theory rules and derivations. Yet, note that since at this “horizon”, the brane becomes tensionless, so states are massless, our probe interpretation for the D-brane becomes meaningless, hence the singularity is a fake one, that needs to be resolved. What we do have instead is a phase transition in the string-like description (between D-branes and “orientifold-like objects”). In terms of the field theory side, the Wilsonian description in terms of only pions is no longer complete, we have to introduce the nucleons as well. We “cheat it in” by using the Skyrme-like solution for the pion field to describe the nucleons.

7 Conclusions and open questions

In this paper we have analyzed actions of $T\bar{T}$ deformations of the actions of canonical scalars. The purpose was to use them as effective actions, both in the Heisenberg model for high energy nucleon scattering, and in a Skyrme-like action that admits (topological) solitons that can be identified with the nucleons themselves.

We have first found that for the 1+1 dimensional $T\bar{T}$ deformation of a canonical scalar, which we have found in a companion paper [18] (see also [21, 22]) that it has soliton solutions, as well as shockwave solutions, the latter can be used to describe nucleons and their scatterings that saturate of the Froissart bound. Moreover, we can understand the shockwave solution as corresponding to the collision of two boosted solitons.

We have found then that the same action, when extended to 3+1 dimensions, has both shockwave solutions and BIon-like solutions, with finite radial derivative, while the Heisenberg action (1.2) has catenoid solutions, with horizons where the radial derivative blows up. We have argued that the boosted BIon-like solutions should describe the shockwave solutions.

The 3+1 dimensional nonabelian scalar version of the $T\bar{T}$ deformation action, was found to have Skyrme-like solutions and shockwave solutions describing the collision of two boosted Skyrme-like solutions. It was also found to be possibly consistent with the phenomenological fit of chiral perturbation theory to experiment.

We interpreted these results as saying that the $T\bar{T}$ deformation action should be understood as a quantum Wilsonian effective action for QCD, reduced to 1+1 dimensions, though why do we have this equality is unclear. We also attempted an interpretation from the point of view of a gravity dual, but the interpretation could not be completed.

There are plenty of open questions that are awaiting further investigation. Here we list some of them:

- The $T\bar{T}$ deformation at higher than 1+1 dimensions is not unique. In [12] it was proposed to use $|det T|^{\frac{1}{\alpha}}$ with $\alpha = d - 1$ and in [7] it was further generalized to different values of α . An interesting question is what is the deformation condition that yields the $T\bar{T}$ -like deformation that we are using (5.51).
- The relation between $T\bar{T}$ deformed actions and QCD. We view the current paper as just a starting point for this analysis.

- The standard theory of Skyrmions has been plagued with several significant problems. In particular (i) The results show that if the pion mass is set to its experimental value then the nucleon and delta masses can not be reproduced for any values of the Skyrme parameters; the commonly used Skyrme parameters are simply an artifact of the rigid body approximation [32]. (ii) It yields nuclear binding energies that are an order of magnitude larger than experimental nuclear data 15% instead of 1% [33]. (iii) It predicts intrinsic shapes for nuclei that fail to match the clustering structure of light nuclei. It will be very interesting to explore whether these problems can be circumvented in the deformed Skyrme theory.
- Expanding the deformed Skyrme action in terms of the deformation parameter yields the Skyrme action plus additional terms that are higher order powers in the derivative of the fields. Such terms are part of the chiral Lagrangian. Thus, it will be very interesting to perform a comprehensive comparison between the coefficients of these terms that determined by the experimental data and those that follow from the expansion of the deformed action.

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A 3+1 dimensional $T\bar{T}$ deformation and the Heisenberg model

In this appendix we will consider one possible $T\bar{T}$ deformation of the free scalar in 3+1 dimensions, defined in [7], and we will analyze its applicability to the Heisenberg model.

The deformed Lagrangian in the presence of a general potential V was defined implicitly, as a solution of an equation, which is impossible to solve in a (simple) closed form in the general case. This would be needed for $V = m^2\phi^2/2$, for instance to analyze the radiation arising from a possible shockwave solution, as seen in the text.

However, we will content ourselves with the case of $V = 0$ ($m = 0$), since this suffices in order to find the leading order shockwave solution near $s = 0$, where any mass scale is irrelevant. In that case, the equation for the Lagrangian can be solved exactly for any spacetime dimension D , with result

$$-\mathcal{L}_{D,1}(t, X) = \left\{ \frac{1}{2t} \left[\sqrt{1 + 4t \left[\frac{(\partial_\mu \phi)^2}{2} \right]^{D-1}} - 1 \right] \right\}^{\frac{1}{D-1}}. \tag{A.1}$$

Consider, as in the text, a solution $\phi(s)$ depending only on $s = t^2 - x^2$, and independent also of y, z . If the solution is a shockwave, then moreover $\phi(s) = 0$ for $s < 0$. Then

$$X \equiv (\partial_\mu \phi)^2 = -4s \left(\frac{d\phi}{ds} \right)^2, \tag{A.2}$$

so the Lagrangian on the shockwave solution is

$$-\mathcal{L}_{D,1}(t, X) = \left\{ \frac{1}{2t} \left[\sqrt{1 + 4t \left[-\frac{4s}{2} \left(\frac{d\phi}{ds} \right)^2 \right]^{D-1}} - 1 \right] \right\}^{\frac{1}{D-1}}. \quad (\text{A.3})$$

The equations of motion are

$$\frac{d}{ds} \left\{ \left\{ \frac{1}{2t} \left[\sqrt{1 + 4t \left[-\frac{4s}{2} \left(\frac{d\phi}{ds} \right)^2 \right]^{D-1}} - 1 \right] \right\}^{\frac{2-D}{D-1}} \left\{ \frac{1}{2t} \frac{4t(D-1)}{\sqrt{1 + 4t \left[-\frac{4s}{2} \left(\frac{d\phi}{ds} \right)^2 \right]^{D-1}}} \times \right. \right. \right. \\ \left. \left. \left. \times \left[-2s \left(\frac{d\phi}{ds} \right)^2 \right]^{D-2} \left(-4s \frac{d\phi}{ds} \right) \right\} \right\} = 0. \quad (\text{A.4})$$

Define

$$\frac{4t}{2^{D-1}} \equiv l^{4(D-1)}, \quad (\text{A.5})$$

so that the Lagrangian on the shockwave solution reads

$$-\mathcal{L}_{D,1}(t, X) = \left\{ \frac{1}{2t} \left[\sqrt{1 + \left[-l^4 4s \left(\frac{d\phi}{ds} \right)^2 \right]^{D-1}} - 1 \right] \right\}^{\frac{1}{D-1}}, \quad (\text{A.6})$$

and the equations of motion read

$$\frac{d}{ds} \left\{ \left\{ \left[\sqrt{1 + \left[-l^4 4s \left(\frac{d\phi}{ds} \right)^2 \right]^{D-1}} - 1 \right] \right\}^{\frac{2-D}{D-1}} \left\{ \frac{(D-1)}{\sqrt{1 + \left[-l^4 4s \left(\frac{d\phi}{ds} \right)^2 \right]^{D-1}}} \times \right. \right. \right. \\ \left. \left. \left. \times \left[-4s \left(\frac{d\phi}{ds} \right)^2 \right]^{D-2} \left(-8s \frac{d\phi}{ds} \right) \right\} \right\} = 0. \quad (\text{A.7})$$

The we check whether Heisenberg's near $s = 0$ solution

$$\phi(s) = l^{-2} \sqrt{s} \quad (\text{A.8})$$

is still valid for the equations of motion (A.7) for the D dimensional Lagrangian. We first note that, if it is, then the square root in the Lagrangian and equations of motion still vanishes.

One can check explicitly, that in the Heisenberg case, corresponding to $D = 2$, the solution comes from the leading terms, coming from derivatives of the square root, and in the end, the leading terms in the equation of motion on the solution reduce to

$$\frac{d}{ds} \left[s \left(\frac{d\phi}{ds} \right)^2 \right] = 0, \quad (\text{A.9})$$

which is still true in the D dimensional case. Moreover, like in the $D = 2$ case, the leading terms, coming from the derivatives of the square root, will also result in the above equation. It is then clear that (A.8) continues to be a solution in the D dimensional case.

However, in that case, the on-shell Lagrangian for the shockwave solution is

$$\mathcal{L}_{\text{on-shell}} = (-1)^{\frac{1}{D-1}} = e^{\frac{2\pi i}{D}} \in \mathbb{C}. \quad (\text{A.10})$$

This is only real in the $D = 2$ case, but the Lagrangian is supposed to be real on classical solutions. The only possible exceptions are quantum solutions like instantons, describing some transition from the quantum path integral.

Since the Lagrangian is supposed to be an effective one, describing a regime of low energy, it cannot be used to describe quantum transitions, so there is no possibility to use for $D \neq 2$, in particular cannot be used for $D = 4$.

B On the Abelian and non-Abelian $T\bar{T}$ deformations of Maxwell and Yang-Mills theories

In this appendix we consider the Abelian and non-Abelian $T\bar{T}$ deformations of Maxwell and Yang-Mills theories in [15], and find that the BIon solution of Born and Infeld is not a solution, and moreover, there is no analogue of this soliton solution for these theories.

The Lagrangian is given in an implicit form, in terms of

$$\chi \equiv \lambda \text{Tr} F_{\mu\nu} F^{\mu\nu}, \quad \lambda \mathcal{L} \equiv h(\chi) \quad (\text{B.1})$$

by the equation

$$\chi = \frac{1}{256} \left(1 - \sqrt{1 - 8h(\chi)} \right) \left(3 + \sqrt{1 - 8h(\chi)} \right). \quad (\text{B.2})$$

By applying $d/d\chi$ on the above equation, we obtain

$$1 \leq \frac{dh}{d\chi} = \frac{16}{(3 + \sqrt{1 - 8h})^2} \leq \frac{16}{9}. \quad (\text{B.3})$$

On the other hand, it is easy to see that the equation of motion (with respect to the gauge field A_μ) is

$$D_\mu \left(\frac{dh}{d\chi} F_{\mu\nu} \right) = D_\mu \left(\frac{16}{(3 + \sqrt{1 - 8h})^2} F_{\mu\nu} \right) = 0. \quad (\text{B.4})$$

Another way to think about this is to defined, as in the main text (and as in a medium),

$$\frac{\partial h}{\partial E_i} \equiv D_i, \quad (\text{B.5})$$

such that the equation of motion for a static solution is

$$\partial_i D_i = \vec{\nabla} \cdot \vec{D} = 0, \tag{B.6}$$

or rather, $q\delta^3(r)$ on the right-hand side, for a delta function source. Here

$$\vec{D} = \frac{dh}{d\chi} \vec{E}. \tag{B.7}$$

On the other hand, \vec{E} remains finite, just like in the usual Born-Infeld theory. Indeed, in 2 dimensions, [15] show that, defining $F^2 = F_{\mu\nu}F^{\mu\nu} = -2F_{01}^2 = -2E^2$ in Minkowski space, the Born-Infeld Lagrangian is

$$\mathcal{L} = \sqrt{1 + 4\lambda F^2} = \sqrt{1 - 8\lambda F_{01}^2}, \tag{B.8}$$

so that

$$E = F_{01} \leq \frac{1}{\sqrt{8\lambda}}. \tag{B.9}$$

For the $T\bar{T}$ theory, maximizing the right-hand side of (B.2), which happens at $\mathcal{L} = 1/(8\lambda)$, where the right-hand side takes the value $27/256$, so that

$$F^2 \leq \frac{27}{256\lambda} \Rightarrow E = F_{01} \leq \sqrt{\frac{27}{512\lambda}}, \tag{B.10}$$

which is just very slightly smaller than the maximum value for Born-Infeld.

Coming back to our issue, we see that D_i remains finite, since both terms on the right-hand side of (B.7) remain finite. This is unlike the case of the Born-Infeld soliton, where $\vec{E} \leq 1$ (in units of $E_{\max} = 1/\sqrt{8\lambda}$ here), but

$$\vec{D} = -\frac{\vec{E}}{\sqrt{1 - \vec{E}^2}} \rightarrow 0, \tag{B.11}$$

the singularity coming from it being the singular solution of the $\vec{\nabla} \cdot \vec{D} = \delta^2(r)$ equation. This was found first in 3+1 dimensions, but the same continues to be valid in 1+1 dimensions, where for Born-Infeld we have

$$D = \frac{\partial \mathcal{L}}{\partial E} = -\frac{E}{\sqrt{1 - 8\lambda E^2}}, \tag{B.12}$$

and again the singularity comes from it being a solution of the $\partial D = \delta(r)$ equation.

In conclusion, since \vec{D} is non-singular for the case of the $T\bar{T}$ action, we cannot have BI-type soliton solutions for it.

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References

- [1] M. Froissart, *Asymptotic behavior and subtractions in the Mandelstam representation*, *Phys. Rev.* **123** (1961) 1053 [INSPIRE].
- [2] L. Lukaszuk and A. Martin, *Absolute upper bounds for $\pi\pi$ scattering*, *Nuovo Cim. A* **52** (1967) 122 [INSPIRE].
- [3] W. Heisenberg, *Production of mesons as a shockwave problem* (in German), *Z. Phys.* **133** (1952) 65 [INSPIRE].
- [4] H. Nastase and J. Sonnenschein, *More on Heisenberg's model for high energy nucleon-nucleon scattering*, *Phys. Rev. D* **92** (2015) 105028 [arXiv:1504.01328] [INSPIRE].
- [5] H. Nastase, *DBI skyrmion, high energy (large s) scattering and fireball production*, [hep-th/0512171](#) [INSPIRE].
- [6] A. Cavaglià, S. Negro, I.M. Szécsényi and R. Tateo, *$T\bar{T}$ -deformed 2D Quantum Field Theories*, *JHEP* **10** (2016) 112 [arXiv:1608.05534] [INSPIRE].
- [7] G. Bonelli, N. Doroud and M. Zhu, *$T\bar{T}$ -deformations in closed form*, *JHEP* **06** (2018) 149 [arXiv:1804.10967] [INSPIRE].
- [8] V. Rosenhaus and M. Smolkin, *Integrability and renormalization under $T\bar{T}$* , *Phys. Rev. D* **102** (2020) 065009 [arXiv:1909.02640] [INSPIRE].
- [9] M. Guica, *An integrable Lorentz-breaking deformation of two-dimensional CFTs*, *SciPost Phys.* **5** (2018) 048 [arXiv:1710.08415] [INSPIRE].
- [10] C. Chen, P. Conkey, S. Dubovsky and G. Hernández-Chifflet, *Undressing Confining Flux Tubes with $T\bar{T}$* , *Phys. Rev. D* **98** (2018) 114024 [arXiv:1808.01339] [INSPIRE].
- [11] O. Aharony, S. Datta, A. Giveon, Y. Jiang and D. Kutasov, *Modular invariance and uniqueness of $T\bar{T}$ deformed CFT*, *JHEP* **01** (2019) 086 [arXiv:1808.02492] [INSPIRE].
- [12] J. Cardy, *The $T\bar{T}$ deformation of quantum field theory as random geometry*, *JHEP* **10** (2018) 186 [arXiv:1801.06895] [INSPIRE].
- [13] S. Datta and Y. Jiang, *$T\bar{T}$ deformed partition functions*, *JHEP* **08** (2018) 106 [arXiv:1806.07426] [INSPIRE].
- [14] M. Taylor, *TT deformations in general dimensions*, [arXiv:1805.10287](#) [INSPIRE].
- [15] T.D. Brennan, C. Ferko and S. Sethi, *A Non-Abelian Analogue of DBI from $T\bar{T}$* , *SciPost Phys.* **8** (2020) 052 [arXiv:1912.12389] [INSPIRE].
- [16] H. Babaei-Aghbolagh, K.B. Velni, D.M. Yekta and H. Mohammadzadeh, *$T\bar{T}$ -like flows in non-linear electrodynamic theories and S -duality*, *JHEP* **04** (2021) 187 [arXiv:2012.13636] [INSPIRE].
- [17] Y. Frishman and J. Sonnenschein, *Non-perturbative field theory: From two-dimensional conformal field theory to QCD in four dimensions*, Cambridge Monographs on Mathematical Physics, Cambridge University Press (2014) [DOI] [INSPIRE].
- [18] H. Nastase and J. Sonnenschein, *Soliton, breather and shockwave solutions of the Heisenberg and the $T\bar{T}$ deformations of scalar field theories in 1+1 dimensions*, *JHEP* **04** (2021) 106 [arXiv:2010.12413] [INSPIRE].
- [19] A.B. Zamolodchikov, *Expectation value of composite field $T\bar{T}$ in two-dimensional quantum field theory*, [hep-th/0401146](#) [INSPIRE].

- [20] F.A. Smirnov and A.B. Zamolodchikov, *On space of integrable quantum field theories*, *Nucl. Phys. B* **915** (2017) 363 [[arXiv:1608.05499](#)] [[INSPIRE](#)].
- [21] R. Conti, L. Iannella, S. Negro and R. Tateo, *Generalised Born-Infeld models, Lax operators and the $T\bar{T}$ perturbation*, *JHEP* **11** (2018) 007 [[arXiv:1806.11515](#)] [[INSPIRE](#)].
- [22] R. Conti, S. Negro and R. Tateo, *The $T\bar{T}$ perturbation and its geometric interpretation*, *JHEP* **02** (2019) 085 [[arXiv:1809.09593](#)] [[INSPIRE](#)].
- [23] G. Date, Y. Frishman and J. Sonnenschein, *The Spectrum of Multiflavor QCD in Two-dimensions*, *Nucl. Phys. B* **283** (1987) 365 [[INSPIRE](#)].
- [24] K. Kang and H. Nastase, *High energy QCD from Planckian scattering in AdS and the Froissart bound*, *Phys. Rev. D* **72** (2005) 106003 [[hep-th/0410173](#)] [[INSPIRE](#)].
- [25] K. Kang and H. Nastase, *Heisenberg saturation of the Froissart bound from AdS-CFT*, *Phys. Lett. B* **624** (2005) 125 [[hep-th/0501038](#)] [[INSPIRE](#)].
- [26] H. Nastase, *AdS-CFT and the RHIC fireball*, *Prog. Theor. Phys. Suppl.* **174** (2008) 274 [[arXiv:0805.3579](#)] [[INSPIRE](#)].
- [27] M. Born and L. Infeld, *Foundations of the new field theory*, *Proc. Roy. Soc. Lond. A* **144** (1934) 425.
- [28] P.C. Aichelburg and R.U. Sexl, *On the Gravitational field of a massless particle*, *Gen. Rel. Grav.* **2** (1971) 303 [[INSPIRE](#)].
- [29] O.V. Pavlovsky, *Chiral Born-Infeld theory: Topological spherically symmetrical solitons*, *Phys. Lett. B* **538** (2002) 202 [[hep-ph/0204313](#)] [[INSPIRE](#)].
- [30] S. Scherer, *Introduction to chiral perturbation theory*, *Adv. Nucl. Phys.* **27** (2003) 277 [[hep-ph/0210398](#)] [[INSPIRE](#)].
- [31] A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis and R. Rattazzi, *Causality, analyticity and an IR obstruction to UV completion*, *JHEP* **10** (2006) 014 [[hep-th/0602178](#)] [[INSPIRE](#)].
- [32] R.A. Battye, S. Krusch and P.M. Sutcliffe, *Spinning skyrmions and the skyrme parameters*, *Phys. Lett. B* **626** (2005) 120 [[hep-th/0507279](#)] [[INSPIRE](#)].
- [33] C. Naya and P. Sutcliffe, *Skyrmions and clustering in light nuclei*, *Phys. Rev. Lett.* **121** (2018) 232002 [[arXiv:1811.02064](#)] [[INSPIRE](#)].