

# A Table of Integrals of the Error Functions\*

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This is a compendium of indefinite and definite integrals of products of the Error function with elementary or transcendental functions. A substantial portion of the results are new.

Key Words: Astrophysics; atomic physics; Error functions; indefinite integrals; special functions; statistical analysis.

## 1. Introduction

Integrals of the error function occur in a great variety of applications, usually in problems involving multiple integration where the integrand contains exponentials of the squares of the arguments. Examples of applications can be cited from atomic physics [16],<sup>1</sup> astrophysics [13], and statistical analysis [15]. This paper is an attempt to give an up-to-date exhaustive tabulation of such integrals.

All formulas for indefinite integrals in sections 4.1, 4.2, 4.5, and 4.6 below were derived from integration by parts and checked by differentiation of the resulting expressions. Section 4.3 and the second half of 4.5 cover all formulas given in [7], with omission of trivial duplications and with a number of additions; section 4.4 covers essentially formulas given in [4], Vol. I, pp. 233–235. All these formulas have been re-derived and checked, either from the integral representation or from the hypergeometric series of the error function. Sections 4.7, 4.8 and 4.9 originated in a more varied way. Some formulas were derived from multiple integrals involving elementary functions, others from existing formulas for integrals of confluent hypergeometric functions, and still others, a small portion, were compiled directly from existing literature. In connection with the last three sections, the reader should refer to [3] and [4], Vol. II, pp. 402, 409–411.

Throughout this paper, we have adhered to the notations used in the NBS Handbook [9] and we have also assumed the reader's familiarity with the properties of the error functions, for which he is referred to [5]. In addition, the reader should also attend to the following conventions:

(i)  $z = x + iy = r \exp(i\theta)$  is a complex variable,

$$\mathcal{R}(z) = x, \mathcal{I}(z) = y, |z| = r, \arg z = \theta;$$

(ii) the parameters  $a$ ,  $b$ , and  $c$  are real and positive except where otherwise stated;

(iii) unless otherwise specified, the parameters  $n$  and  $k$  represent the integers 0, 1, 2 . . . , whereas the parameters  $p$ ,  $q$ , and  $\nu$  may be nonintegral;

(iv) the integration constants have been omitted for the indefinite integrals;

(v) when  $x$  is used (instead of  $z$ ) as the integration variable, it means that the formula has been established only for real  $x$ , though it may still be valid for certain complex values;

(vi) the integration symbol  $\int$  denotes a Cauchy principal value.

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<sup>1</sup>Figures in brackets indicate the literature references at the end of this paper.

## 2. Glossary of Functions and Notation

$A(x)$	Gaussian Probability Function	$\frac{1}{\sqrt{2\pi}} \int_{-x}^x e^{-t^2/2} dt$
$C(z)$	Fresnel Integral	$\int_0^z \cos\left(\frac{\pi}{2} t^2\right) dt$
$Ci(z)$	Cosine Integral	$-\int_z^\infty \frac{\cos t}{t} dt$
$D_\nu(z)$	Parabolic Cylinder Function	
$e_n(z)$	Truncated Exponential	$\sum_{k=0}^n \frac{z^k}{k!}$
$-Ei(-z) \equiv E_1(z)$	Exponential Integral	$\int_z^\infty \frac{e^{-t}}{t} dt$
$Ei(x)$	Exponential Integral	$-\int_{-x}^\infty \frac{e^{-t}}{t} dt$
${}_1F_1(a; b; z) \equiv M(a, b, z)$	Confluent Hypergeometric Function	$\sum_{n=0}^\infty \frac{(a)_n z^n}{(b)_n n!}$
${}_kF_l(a_1 \dots a_k; b_1 \dots b_l; z)$	Generalized Hypergeometric Function	$\sum_{n=0}^\infty \frac{(a_1)_n \dots (a_k)_n z^n}{(b_1)_n \dots (b_l)_n n!}$
$H_n(x)$	Hermite Polynomial	
$\mathbf{H}_\nu(x)$	Struve Function	
$I_\nu(z)$	Modified Bessel Function	
$J_\nu(z)$	Bessel Function	
$K_\nu(z)$	Modified Bessel Function	
$L_n^\alpha$	Generalized Laguerre Polynomial	
$M_{p,q}(z)$	Whittaker Function	
$Y_\nu(z)$	Neumann Function (Bessel Function of Second Kind)	
$P(x)$	Probability Function	$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$
$(p)_n$	Pochhammer's Symbol	$\Gamma(p+n)/\Gamma(p)$
$P_n(x)$	Legendre Polynomial	
$P_\nu^\mu(z)$	Associated Legendre Function of the First Kind	
$S(z)$	Fresnel Integral	$\int_0^z \sin\left(\frac{\pi}{2} t^2\right) dt$
$si(z)$	Sine Integral	$-\int_z^\infty \sin t \frac{dt}{t}$
$U(a, b, z) \equiv \Psi(a, b, z)$	Confluent Hypergeometric Function	
$W_{p,q}(z)$	Whittaker Function	

$\gamma$	Euler's Constant	0.5772156649 . . .
$\Gamma(p)$	Gamma Function	
$\gamma(p, z)$	Incomplete Gamma Function	$\int_0^z e^{-t} t^{p-1} dt$
$\Gamma(p, z)$	Incomplete Gamma Function	$\int_z^\infty e^{-t} t^{p-1} dt$
$\zeta(z)$	Riemann's Zeta Function	$\sum_{k=1}^{\infty} k^{-z}$
$\psi(z)$	Psi Function	$\frac{d}{dz} [\ln \Gamma(z)]$

### 3. Definition and Integral Representations

#### 3.1. Definitions and Other Notations

$$1. \operatorname{erf}(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt,$$

$$2. \operatorname{erfc}(z) \equiv \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt = 1 - \operatorname{erf}(z),$$

$$3. \operatorname{erfi}(z) \equiv -i \operatorname{erf}(iz) = \frac{2}{\sqrt{\pi}} \int_0^z e^{t^2} dt.$$

Some authors use the above notations without the factor  $\frac{2}{\sqrt{\pi}}$ , and some use  $\Phi(z)$  for  $\operatorname{erf}(z)$ .

$$4. w(z) \equiv e^{-z^2} \operatorname{erfc}(-iz).$$

For real  $x$ , Dawson's integral is defined as

$$5. F(x) \equiv \frac{\sqrt{\pi}}{2} e^{-x^2} \operatorname{erfi}(x) = \frac{\sqrt{\pi}}{2} \mathcal{S}[w(x)].$$

The error function is also closely related to the Gaussian probability functions:

$$6. \operatorname{erf}(x) = 2P(x\sqrt{2}) - 1 = A(x\sqrt{2}).$$

#### 3.2. Integral Representations

$$1. \operatorname{erf}(az) = \frac{2az}{\sqrt{\pi}} \int_0^1 e^{-a^2 z^2 t^2} dt$$

$$2. \operatorname{erfc}(az) = \frac{2}{\sqrt{\pi}} e^{-a^2 z^2} \int_0^\infty e^{-(t^2 + 2azt)} dt$$

$$3. \operatorname{erf}\left(az + \frac{w}{a}\right) = \frac{2a}{\sqrt{\pi}} \exp\left(c - \frac{w^2}{a^2}\right) \int e^{-(a^2 z^2 + 2wz + c)} dz.$$

4.  $\operatorname{erfc}\left(\frac{z}{a}\right) = \frac{2a}{\sqrt{\pi}} \exp\left(c - \frac{z^2}{a^2}\right) \int_0^\infty e^{-(a^2t^2 + 2zt + c)} dt$
5.  $e^{2ab} \operatorname{erf}\left(ax + \frac{b}{x}\right) + e^{-2ab} \operatorname{erf}\left(ax - \frac{b}{x}\right) = \frac{4a}{\sqrt{\pi}} \int e^{-a^2x^2 - b^2/x^2} dx$
6.  $\operatorname{erfc}(az) = \frac{2a}{\sqrt{\pi}} e^{-a^2z^2} \int_0^\infty \frac{e^{-a^2t^2} dt}{(z^2 + t^2)^{1/2}}, \quad \Re(a) > 0, \Re(z) > 0.$
7.  $\operatorname{erfc}(az) = \frac{2z}{\pi} e^{-a^2z^2} \int_0^\infty \frac{e^{-a^2t^2} dt}{(t^2 + z^2)}, \quad \Re(a) > 0, |\arg z| < \pi, z \neq 0.$
8.  $1 - [\operatorname{erf}(x)]^2 = \frac{4}{\pi} e^{-x^2} \int_0^1 \frac{e^{-x^2t^2} dt}{(t^2 + 1)}, \quad x > 0$
9.  $\operatorname{erf}\left(x + \frac{iy}{2}\right) + \operatorname{erf}\left(x - \frac{iy}{2}\right) = \frac{4}{\sqrt{\pi}} e^{y^2/4} \int e^{-x^2} \cos xy dx$
10.  $\operatorname{erf}\left(x + \frac{iy}{2}\right) - \operatorname{erf}\left(x - \frac{iy}{2}\right) = \frac{4}{i\sqrt{\pi}} e^{y^2/4} \int e^{-x^2} \sin xy dx$
11.  $\operatorname{erf}(x) = \frac{x}{\sqrt{\pi}} \int_0^\pi e^{x^2 \cos \theta} \cos(x^2 \sin \theta + \theta/2) d\theta, \quad x \neq 0$
12.  $\operatorname{erf}(x) = \frac{1}{\pi} \int_0^\infty e^{-t} \sin(2x\sqrt{t}) \frac{dt}{t}.$

## 4. Integrals of Product of Error Functions With Other Functions

### 4.1. Combination of Error Function With Powers

1.  $\int \operatorname{erf}(az) dz = z \operatorname{erf}(az) + \frac{1}{a\sqrt{\pi}} \exp(-a^2z^2)$
2.  $\int \operatorname{erfc}(az) dz = z \operatorname{erfc}(az) - \frac{1}{a\sqrt{\pi}} \exp(-a^2z^2)$
3.  $\int_0^\infty \operatorname{erfc}(ax) dx = \frac{1}{a\sqrt{\pi}}, \quad |\arg a| < \frac{\pi}{4}$
4.  $\int \operatorname{erf}(az) z dz = \frac{1}{2} z^2 \operatorname{erf}(az) - \frac{1}{4a^2} \operatorname{erf}(az) + \frac{z}{2a\sqrt{\pi}} \exp(-a^2z^2)$
5.  $\int \operatorname{erfc}(az) z dz = \frac{1}{2} z^2 \operatorname{erfc}(az) + \frac{1}{4a^2} \operatorname{erf}(az) - \frac{z}{2a\sqrt{\pi}} \exp(-a^2z^2)$
6.  $\int_0^\infty \operatorname{erfc}(ax) x dx = \frac{1}{4a^2}, \quad |\arg a| < \frac{\pi}{4}$

7. 
$$\int \operatorname{erf}(az)z^n dz = \frac{z^{n+1}}{n+1} \operatorname{erf}(az) + \frac{e^{-a^2 z^2}}{a\sqrt{\pi}(n+1)} \sum_{k=0}^{l-1} \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}-k+1\right)} \frac{z^{n-2k}}{a^{2k}}$$

$$- \frac{1-j}{n+1} \frac{\Gamma\left(l+\frac{1}{2}\right)}{a^{n+1}\sqrt{\pi}} \operatorname{erf}(az), \quad j=0 \text{ or } 1, 2l-j=n+1$$
8. 
$$\int \operatorname{erf}(az)z^{n+2} dz = \frac{(n+2)(n+1)}{2(n+3)a^2} \int \operatorname{erf}(az)z^n dz + \left(z^2 - \frac{n+2}{2a^2}\right) \frac{z^{n+1}}{(n+3)} \operatorname{erf}(az)$$

$$+ \frac{1}{a\sqrt{\pi}(n+3)} z^{n+2} e^{-a^2 z^2}$$
9. 
$$\int \operatorname{erfc}(az)z^n dz = \frac{z^{n+1}}{(n+1)} \operatorname{erfc}(az) - \frac{e^{-a^2 z^2}}{a\sqrt{\pi}(n+1)} \sum_{k=0}^{l-1} \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}-k+1\right)} \frac{z^{n-2k}}{a^{2k}}$$

$$+ \frac{1-j}{n+1} \frac{\Gamma\left(l+\frac{1}{2}\right)}{a^{n+1}\sqrt{\pi}} \operatorname{erf}(az), \quad j=0 \text{ or } 1, 2l-j=n+1$$
10. 
$$\int \operatorname{erfc}(az)z^{n+2} dz = \frac{(n+2)(n+1)}{2(n+3)a^2} \int \operatorname{erfc}(az)z^n dz + \left(z^2 - \frac{n+2}{2a^2}\right) \frac{z^{n+1}}{(n+3)} \operatorname{erfc}(az)$$

$$- \frac{1}{a\sqrt{\pi}(n+3)} z^{n+2} e^{-a^2 z^2}$$
11. 
$$\int_0^\infty \operatorname{erfc}(ax)x^n dx = \frac{\Gamma\left(\frac{n}{2}+1\right)}{(n+1)\sqrt{\pi}a^{n+1}}, \quad |\arg a| < \frac{\pi}{4}$$
12. 
$$\int \operatorname{erf}(az)z^{-1} dz = \ln z \operatorname{erf}(az) - \frac{2a}{\sqrt{\pi}} \int \ln ze^{-a^2 z^2} dz$$
13. 
$$\int \operatorname{erfc}(az)z^{-1} dz = \ln z \operatorname{erfc}(az) + \frac{2a}{\sqrt{\pi}} \int \ln ze^{-a^2 z^2} dz$$
14. 
$$\int \operatorname{erf}(az)z^{-n} dz = -\frac{\operatorname{erf}(az)}{(n-1)z^{n-1}} + \frac{2a}{(n-1)\sqrt{\pi}} \int \frac{1}{z^{n-1}} e^{-a^2 z^2} dz, \quad n \geq 2$$
15. 
$$\int \operatorname{erfc}(az)z^{-n} dz = -\frac{\operatorname{erfc}(az)}{(n-1)z^{n-1}} - \frac{2a}{(n-1)\sqrt{\pi}} \int \frac{1}{z^{n-1}} e^{-a^2 z^2} dz, \quad n \geq 2$$
16. 
$$\int \operatorname{erf}(az)z^p dz = \frac{z^{p+1}}{p+1} \operatorname{erf}(az) - \frac{1}{(p+1)a^{p+1}\sqrt{\pi}} \gamma\left(\frac{p}{2}+1, a^2 z^2\right), \quad p > -2, p \neq -1$$

<sup>2</sup> See appendix for integrals on the right-hand sides of eqs (12 to 15).

$$17. \int \operatorname{erfc}(az)z^p dz = \frac{z^{p+1}}{p+1} \operatorname{erfc}(az) + \frac{1}{(p+1)a^{p+1}\sqrt{\pi}} \gamma\left(\frac{p}{2}+1, a^2 z^2\right), \quad p > -1$$

$$18. \int_0^\infty \operatorname{erfc}(ax)x^p dx = \frac{1}{(p+1)a^{p+1}\sqrt{\pi}} \Gamma\left(\frac{p}{2}+1\right), \quad |\arg a| < \frac{\pi}{4}, p > -1$$

$$19. \int_0^\infty \operatorname{erf}(ax)x^{p-2} dx = \frac{a^{1-p}}{\sqrt{\pi}(1-p)} \Gamma\left(\frac{p}{2}\right), \quad |\arg a| < \frac{\pi}{4}, 0 < p < 1.$$

#### 4.2. Combination of Error Functions With Exponentials and Powers<sup>3</sup>

$$1. \int \operatorname{erf}(az)e^{bz} dz = \frac{1}{b} e^{bz} \operatorname{erf}(az) - \frac{1}{b} \exp\left(\frac{b^2}{4a^2}\right) \operatorname{erf}\left(az - \frac{b}{2a}\right)$$

$$2. \int \operatorname{erfc}(az)e^{bz} dz = \frac{1}{b} e^{bz} \operatorname{erfc}(az) + \frac{1}{b} \exp\left(\frac{b^2}{4a^2}\right) \operatorname{erf}\left(az - \frac{b}{2a}\right)$$

$$3. \int_0^\infty \operatorname{erf}(ax)e^{-bx} dx = \frac{1}{b} \exp\left(\frac{b^2}{4a^2}\right) \operatorname{erfc}\left(\frac{b}{2a}\right), \quad \Re(b) > 0, |\arg a| < \pi/4$$

$$4. \int_0^\infty \operatorname{erfc}(ax)e^{bx} dx = \frac{1}{b} \exp\left(\frac{b^2}{4a^2}\right) \left[1 + \operatorname{erf}\left(\frac{b}{2a}\right)\right] - \frac{1}{b}, \quad |\arg(b-a)| < \frac{\pi}{4}$$

$$5. \int \operatorname{erf}(az)e^{bz} z dz = \frac{1}{b} \operatorname{erf}(az)e^{bz} \left(z - \frac{1}{b}\right) - \frac{1}{b} \exp\left(\frac{b^2}{4a^2}\right) \left\{ \left(\frac{b}{2a^2} - \frac{1}{b}\right) \operatorname{erf}(t) - \frac{1}{a\sqrt{\pi}} e^{-t^2} \right\}, \quad t = az - \frac{b}{2a}$$

$$6. \int \operatorname{erfc}(az)e^{bz} z dz = \frac{1}{b} \operatorname{erfc}(az)e^{bz} \left(z - \frac{1}{b}\right) + \frac{1}{b} \exp\left(\frac{b^2}{4a^2}\right) \left\{ \left(\frac{b}{2a^2} - \frac{1}{b}\right) \operatorname{erf}(t) - \frac{1}{a\sqrt{\pi}} e^{-t^2} \right\}, \quad t = az - \frac{b}{2a}$$

$$7. \int_0^\infty \operatorname{erf}(ax)e^{-bx} x dx = \frac{1}{b} \exp\left(\frac{b^2}{4a^2}\right) \left[\frac{1}{b} - \frac{b}{2a^2}\right] \operatorname{erfc}\left(\frac{b}{2a}\right) + \frac{1}{ab\sqrt{\pi}}, \quad \Re(b) > 0, |\arg a| < \frac{\pi}{4}$$

$$8. \int_0^\infty \operatorname{erfc}(ax)e^{bx} x dx = \frac{1}{b} \exp\left(\frac{b^2}{4a^2}\right) \left[\frac{b}{2a^2} - \frac{1}{b}\right] \left[1 + \operatorname{erf}\left(\frac{b}{2a}\right)\right] + \frac{1}{b^2} + \frac{1}{ab\sqrt{\pi}}, \quad |\arg(b-a)| < \frac{\pi}{4}$$

$$9.^4 \int \operatorname{erf}(az)e^{bz} z^n dz = (-1)^n \frac{n!}{b^{n+1}} e^{bz} \operatorname{erf}(az) e_n(-bz)$$

$$- \frac{2a}{\sqrt{\pi}} (-1)^n \frac{n!}{b^{n+1}} \sum_{k=0}^n \frac{(-b)^k}{k!} \int z^k \exp(-a^2 z^2 + bz) dz$$

<sup>3</sup>In this section  $a$  and  $b$  can take any value on the complex plane other than the origin, except where otherwise stated.

<sup>4</sup>See appendix for the integrals on the right-hand sides of eqs (9 to 12).

$$\begin{aligned}
10. \quad b \int \operatorname{erf}(az)e^{bz}z^n dz + n \int \operatorname{erf}(az)e^{bz}z^{n-1} dz \\
= e^{bz}z^{n-1} \left[ z \operatorname{erf}(az) + \frac{1}{a\sqrt{\pi}} e^{-a^2z^2} \right] \\
- \frac{1}{a\sqrt{\pi}} \int (bz^n + nz^{n-1}) \exp(-a^2z^2 + bz) dz
\end{aligned}$$

$$\begin{aligned}
11. \quad \int \operatorname{erfc}(az)e^{bz}z^n dz = (-1)^n \frac{n!}{b^{n+1}} e^{bz} \operatorname{erfc}(az) e_n(-bz) \\
+ \frac{2a}{\sqrt{\pi}} \frac{(-1)^n n!}{b^{n+1}} \sum_{k=0}^n \frac{(-b)^k}{k!} \int z^k \exp(-a^2z^2 + bz) dz
\end{aligned}$$

$$\begin{aligned}
12. \quad b \int \operatorname{erfc}(az)e^{bz}z^n dz + n \int \operatorname{erfc}(az)e^{bz}z^{n-1} dz \\
= e^{bz}z^{n-1} \left[ z \operatorname{erfc}(az) - \frac{1}{a\sqrt{\pi}} e^{-a^2z^2} \right] \\
+ \int (bz^n + nz^{n-1}) \exp(-a^2z^2 + bz) dz
\end{aligned}$$

$$\begin{aligned}
13. \quad \int_0^\infty \operatorname{erf}(ax)e^{-bx}x^n dx = \left(\frac{2}{\pi}\right)^{1/2} \sum_{k=0}^n \frac{n!}{b^{k+1}} (2a)^{\frac{k-n}{2}} \exp\left(\frac{b^2}{8a^2}\right) D_{-n+k-1}\left(\frac{b}{a\sqrt{2}}\right) \\
= \sum_{k=0}^n \frac{(-1)^{n-k}}{b^{k+1}} \frac{n!(a)^{k-n}}{(n-k)!} \frac{d^{n-k}}{dq^{n-k}} \left[ \exp\left(\frac{q^2}{4}\right) \operatorname{erfc}\left(\frac{q}{2}\right) \right],
\end{aligned}$$

$$q = \frac{b}{a}, \quad \Re(b) > 0, \quad |\arg a| < \frac{\pi}{4}$$

$$\begin{aligned}
14. \quad \int_0^\infty \operatorname{erfc}(ax)e^{bx}x^n dx = (-1)^{n+1} \frac{n!}{b^{n+1}} \\
+ \left(\frac{2}{\pi}\right)^{1/2} \sum_{k=0}^n (-1)^k \frac{n!}{b^{k+1}} (2a^2)^{\frac{k-n}{2}} \exp\left(\frac{b^2}{8a^2}\right) D_{-n+k-1}\left(-\frac{b}{a\sqrt{2}}\right)
\end{aligned}$$

$$|\arg(b-a)| < \frac{\pi}{4}$$

### 4.3. Combination of Error Function With Exponentials of More Complicated Arguments

$$1. \quad \int_0^a e^{-x^2} \operatorname{erf}(x) dx = \frac{\sqrt{\pi}}{4} (\operatorname{erf} a)^2$$

$$2. \quad \int_0^\infty \operatorname{erf}(ax)e^{-b^2x^2} dx = \frac{\sqrt{\pi}}{2b} - \frac{1}{b\sqrt{\pi}} \tan^{-1} \frac{b}{a}$$

$$3. \quad \int_0^\infty \operatorname{erfc}(ax)e^{b^2x^2} dx = \frac{1}{2\sqrt{\pi}b} \ln \left[ \frac{a+b}{a-b} \right], \quad b \text{ may be complex, } |\arg a| < \frac{\pi}{4}$$

4.  $\int_0^{\infty} \operatorname{erf}(ax)e^{-b^2x^2}dx = \frac{a}{2b^2}(a^2+b^2)^{-1/2}, \quad \Re(b^2) > \Re(a^2), \quad \Re(b^2) > 0$
5.  $\int_0^{\infty} \operatorname{erfc}(ax)e^{b^2x^2}dx = \frac{1}{2b^2} \left[ \frac{a}{(a^2-b^2)^{1/2}} - 1 \right], \quad \Re(a^2) > \Re(b^2)$
6.  $\int_0^{\infty} \operatorname{erf}(ax)e^{-b^2x^2}x^2dx = \frac{\sqrt{\pi}}{4b^3} - \frac{1}{2\sqrt{\pi}} \left[ \frac{1}{b^3} \tan^{-1} \frac{b}{a} - \frac{a}{b^2(a^2+b^2)} \right], \quad |\arg a| < \frac{\pi}{4}$
7.  $\int_0^{\infty} \operatorname{erfc}(ax)e^{-b^2x^2}x^2dx = \frac{1}{2\sqrt{\pi}} \left[ \frac{1}{b^3} \tan^{-1} \frac{b}{a} - \frac{a}{b^2(a^2+b^2)} \right], \quad |\arg a| < \frac{\pi}{4}$
8.  $\int_0^{\infty} \operatorname{erf}(ax)e^{-b^2x^2}x^pdx = \frac{a}{\sqrt{\pi}} b^{-p-2} \Gamma\left(\frac{p}{2}+1\right) {}_2F_1\left(\frac{1}{2}, \frac{p}{2}+1; \frac{3}{2}; -\frac{a^2}{b^2}\right),$   
 $\Re(b^2) > 0, \quad \Re(p) > -2$
9.  $\int_0^{\infty} \operatorname{erfc}(ax)e^{b^2x^2}x^pdx = \frac{\Gamma(\frac{1}{2}p+1)}{\sqrt{\pi}(p+1)a^{p+1}} {}_2F_1\left(\frac{p+1}{2}, \frac{p+2}{2}; \frac{p+3}{2}; \frac{b^2}{a^2}\right),$   
 $\Re(b^2) < \Re(a^2), \quad \Re(p) > -1$
10.  $\int_0^{\infty} \operatorname{erfc}(x)e^{x^2}x^{p-1}dx = \frac{1}{2} \sec\left(\frac{p\pi}{2}\right) \Gamma\left(\frac{p}{2}\right), \quad 0 < p < 1$
11.  $\int_0^{\infty} \operatorname{erf}(ax)e^{-b^2x^2} \frac{dx}{x} = \frac{1}{2} \ln \frac{(a^2+b^2)^{1/2}+a}{(a^2+b^2)^{1/2}-a}$   
 $= \ln \frac{a+(a^2+b^2)^{1/2}}{b}, \quad \Re(b^2) > 0$
12.  $\int_0^{\infty} \operatorname{erf}(iax)e^{-a^2x^2-bx}dx = \frac{1}{2ai\sqrt{\pi}} \exp\left(\frac{b^2}{4a^2}\right) Ei\left(-\frac{b^2}{4a^2}\right), \quad \Re(b) > 0, \quad |\arg a| < \frac{\pi}{4}$
13.  $\int_{-\infty}^{\infty} \operatorname{erf}(x)e^{-(ax+b)^2}dx = -\frac{\sqrt{\pi}}{a} \operatorname{erf}\left(\frac{b}{\sqrt{a^2+1}}\right), \quad \Re(a^2) > 0$
14.  $\int_0^{\infty} \operatorname{erf}(ix)e^{-(x^2+ax)}x dx = \frac{i}{\sqrt{\pi}} \left[ \frac{1}{a} + \frac{a}{4} Ei\left(-\frac{a^2}{4}\right) \exp\left(\frac{a^2}{4}\right) \right], \quad \Re(a) > 0$
15.  $\int_0^{\infty} \operatorname{erf}(ax)e^{-bx^4}x^3dx = \frac{a^2}{8\sqrt{\pi}b^{3/2}} \exp\left(\frac{a^4}{8b}\right) K_{1/4}\left(\frac{a^4}{8b}\right)$
16.  $\int_1^{\infty} \operatorname{erfc}(ax)e^{a^2x}x^{-3}dx = \frac{1}{2}(1-2a^2)e^{a^2} \operatorname{erfc}(a) + \frac{a}{\sqrt{\pi}}$
17.  $\int_1^{\infty} \operatorname{erfc}(ax)e^{a^2x} \frac{(x-1)^{p/2-1}}{x^{p+1}} dx = \frac{1}{\sqrt{\pi}} e^{a^2/2} 2^{5p/4-2} a^{p/2-1} \Gamma\left(\frac{p}{2}\right) D_{-1-p}(a\sqrt{2}), \quad p > 0$
18.  $\int_0^1 \operatorname{erfc}\left(\frac{ax}{\sqrt{2}}\right) e^{\frac{a^2x^2}{2}} x^{p-1} (1-x^2)^{-(p+1)/2} dx = \frac{1}{\sqrt{\pi}} \Gamma(p) \Gamma\left(\frac{1-p}{2}\right) 2^{-p/2} e^{a^2/4} D_{-p}(a), \quad 0 < p < 1$



19.  $\int_0^a \operatorname{erf}(x) e^{x^2} (a^2 - x^2)^{-1/2} x dx = \frac{\sqrt{\pi}}{2} (e^{a^2} - 1), \quad a > 0$
20.  $\int_0^a \operatorname{erfc}(x) e^{x^2} (a^2 - x^2)^{-1/2} x dx = \frac{\sqrt{\pi}}{2} [1 - e^{a^2} \operatorname{erfc}(a)], \quad a > 0$
21.  $\int_0^a \operatorname{erf}(x) e^{x^2} (a^2 - x^2)^{p-1} x dx = \frac{\sqrt{\pi}}{2} e^{a^2} \frac{\Gamma(p)}{\Gamma(p + \frac{1}{2})} \gamma(p + \frac{1}{2}, a^2), \quad p > 0$
22.  $\int_0^\infty \operatorname{erf}\left(\frac{a}{x}\right) e^{-b^2 x^2} x dx = \frac{1}{2b^2} (1 - e^{-2ab}), \quad \Re(a) > 0, \quad \Re(b^2) > 0$
23.  $\int_0^\infty \operatorname{erfc}\left(\frac{a}{x}\right) e^{-b^2 x^2} x dx = \frac{1}{2b^2} e^{-2ab}, \quad \Re(a) > 0, \quad \Re(b^2) > 0$
24.  $\int_0^\infty \operatorname{erfc}\left(\frac{a}{x}\right) e^{-b^2 x^2} \frac{dx}{x} = -Ei(-2ab), \quad \Re(a) > 0, \quad \Re(b^2) > 0$
25.  $\int_0^\infty \operatorname{erfc}(ax) e^{-\frac{b^2}{4x^2}} x dx = \frac{1}{4a^2} e^{-ab} (1 + ab) - \frac{b^2}{4} [-Ei(-ab)]$
26.  $\int_0^\infty \operatorname{erf}(ax) [1 - e^{-\frac{b^2}{4x^2}}] \frac{dx}{x} = \gamma + \ln(ab) + [-Ei(-ab)]$
27.  $\int_0^\infty \operatorname{erfc}(ax) e^{-\frac{b^2}{x^2}} \frac{dx}{x} = -Ei(-2ab), \quad \Re(a) > 0, \quad \Re(b^2) > 0$
28.  $\int_0^\infty \operatorname{erf}(ax) e^{-\frac{b^2}{4x^2}} \frac{dx}{x^3} = \frac{2}{b^2} (1 - e^{-ab})$
29.  $\int_0^\infty \operatorname{erfc}(ax) e^{-\frac{b^2}{4x^2}} \frac{dx}{x^3} = \frac{2}{b^2} e^{-ab}$
30.  $\int_0^\infty \operatorname{erfc}\left(\frac{1}{x}\right) \exp\left(\frac{1}{x^2} - b^2 x^2\right) dx = \frac{1}{b\sqrt{\pi}} [\sin 2bCi(2b) - \cos 2bSi(2b)]$
31.  $\int_0^\infty \operatorname{erfc}\left(\frac{1}{x}\right) \exp\left(\frac{1}{x^2} - b^2 x^2\right) x dx = \frac{\pi}{2b} [\mathbf{H}_1(2b) - Y_1(2b)] - \frac{1}{b}, \quad |\arg b| < \frac{\pi}{4}$
32.  $\int_0^\infty \operatorname{erfc}\left(\frac{1}{x}\right) \exp\left(\frac{1}{x^2} - b^2 x^2\right) \frac{dx}{x} = \frac{\pi}{2} [\mathbf{H}_0(2b) - Y_0(2b)], \quad |\arg b| < \frac{\pi}{4}$
33.  $\int_0^\infty \left[ \operatorname{erfc}\left(\frac{a}{x}\right) (x^2 + 2a^2) - \frac{2}{\sqrt{\pi}} a x e^{-a^2/x^2} \right] e^{-b^2 x^2} x dx = \frac{1}{2b^4} e^{-2ab}, \quad |\arg b| < \frac{\pi}{4}, \quad \Re(a) > 0$
34.  $\int_0^\infty \operatorname{erfc}\left(ax + \frac{b}{x}\right) e^{-c^2 x^2} x dx$   
 $= \frac{1}{2}(a^2 + c^2)^{-1/2} [a + (a^2 + c^2)^{1/2}]^{-1} \exp[-2b(a + \sqrt{a^2 + c^2})], \quad \Re(b) > 0, \quad \Re(a^2 + c^2) > 0$
35.  $\int_0^\infty \left\{ 2 \cosh ab - e^{-ab} \operatorname{erf}\left(\frac{b-2ax^2}{2x}\right) - e^{ab} \operatorname{erf}\left(\frac{b+2ax^2}{2x}\right) \right\} e^{-(c^2 - a^2)x^2} x dx = \frac{1}{c^2 - a^2} e^{-bc},$   
 $a > 0, b > 0, \Re(c^2) > 0$

$$36. \int_0^{\infty} \cosh(2bx) \exp[(a \cosh x)^2] \operatorname{erfc}(a \cosh x) dx$$

$$= \frac{1}{2} \sec(b\pi) e^{a^2/2} K_b(a^2), \quad \Re(a) > 0, -\frac{1}{2} < \Re(b) < \frac{1}{2}$$

$$37. \int_0^{\infty} \{\exp[-(x-a)^2] - \exp[-(x+a)^2]\} \operatorname{erf}(x) dx = \sqrt{\pi} \operatorname{erf}(a/\sqrt{2})$$

$$38. \int_0^{\infty} \{\exp[-(x-a)^2] + \exp[-(x+a)^2]\} \operatorname{erf}(x) dx = \frac{\sqrt{\pi}}{2} \{1 + [\operatorname{erf}(a/\sqrt{2})]^2\}.$$

#### 4.4 Definite Integrals From Laplace Transforms Involving $\operatorname{Erf}(\sqrt{ax})$

$$1. \int_0^{\infty} e^{-bx} [e^{ax} \operatorname{erf}(\sqrt{ax})] dx = (b-a)^{-1} (a/b)^{1/2}, \quad b > a$$

$$2. \int_0^{\infty} e^{-bx} [e^{ax} \operatorname{erfc}(\sqrt{ax})] dx = b^{-1/2} (\sqrt{b} + \sqrt{a})^{-1}$$

$$3. \int_0^{\infty} e^{-bx} [e^{ax} \operatorname{erf}(\sqrt{ax}) - 2(ax/\pi)^{1/2}] dx = (b-a)^{-1} (a/b)^{3/2}, \quad b > a$$

$$4. \int_0^{\infty} e^{-bx} [(\pi x)^{-1/2} - a^{1/2} e^{ax} \operatorname{erfc}(\sqrt{ax})] dx = (\sqrt{b} + \sqrt{a})^{-1}$$

$$5. \int_0^{\infty} e^{-bx} [1 - e^{ax} \operatorname{erfc}(\sqrt{ax})] dx = \sqrt{ab}^{-1} (\sqrt{b} + \sqrt{a})^{-1}$$

$$6. \int_0^{\infty} e^{-bx} [e^{ax} \operatorname{erfc}(\sqrt{ax}) + 2(ax/\pi)^{1/2} - 1] dx = a(\sqrt{b} + \sqrt{a})^{-1} b^{-3/2}$$

$$7. \int_0^{\infty} e^{-bx} [1 - 2(ax/\pi)^{1/2} + (2ax-1)e^{ax} \operatorname{erfc}(\sqrt{ax})] dx = ab^{-1} (\sqrt{b} + \sqrt{a})^{-2}$$

$$8. \int_0^{\infty} e^{-bx} [(x/\pi)^{1/2} - a^{1/2} x e^{ax} \operatorname{erfc}(\sqrt{ax})] dx = (4b)^{-1/2} (\sqrt{b} + \sqrt{a})^{-2}$$

$$9. \int_0^{\infty} e^{-bx} [8axe^{ax} \operatorname{erfc}(\sqrt{ax}) - 8(ax/\pi)^{1/2} + 1] dx = b^{-1} \left( \frac{\sqrt{b} - \sqrt{a}}{\sqrt{b} + \sqrt{a}} \right)^2$$

$$10. \int_0^{\infty} e^{-bx} [e^{ax} \sqrt{ax} (2ax+3) \operatorname{erfc}(\sqrt{ax}) - 2(x/\pi)^{1/2} (ax+1)] dx = -(\sqrt{b} + \sqrt{a})^{-3}$$

$$11. \int_0^{\infty} e^{-bx} [(2a^2x^2 + 5ax + 1)e^{ax} \operatorname{erfc}(\sqrt{ax}) - 2(ax+2)(ax/\pi)^{1/2}] dx = \sqrt{b}(\sqrt{b} + \sqrt{a})^{-3}$$

$$12. \int_0^{\infty} e^{-bx} [2(8a^2x^2 + 8ax + 1)e^{ax} \operatorname{erf}(\sqrt{ax}) - 8(ax/\pi)^{1/2}(2ax+1) - 1] dx$$

$$= b^{-1} (\sqrt{b} - \sqrt{a})^3 (\sqrt{b} + \sqrt{a})^{-3}, \quad b > a$$

$$13. \int_0^{\infty} e^{-bx} [(2a^{1/2}x^{1/2} + 1)xe^{ax} \operatorname{erf}(\sqrt{ax}) - 2(ax^3/\pi)^{1/2}] dx = b^{-1/2} (\sqrt{b} + \sqrt{a})^{-3}, \quad b > a$$

$$14. \int_0^{\infty} e^{-bx} [(4a^2x^2 + 12ax + 3)xe^{ax} \operatorname{erfc}(\sqrt{ax}) - 2(a^3x^5/\pi)^{1/2}(2ax + 5)] dx = 3(\sqrt{b} + \sqrt{a})^{-4}$$

$$15. \int_0^{\infty} e^{-bx} [ae^{ax} \operatorname{erfc}(\sqrt{ax}) + \sqrt{ace^{cx}} \operatorname{erfc}(\sqrt{cx}) - ce^{cx}] dx = (a-c)(b-c)^{-1} \sqrt{b}(\sqrt{b} + \sqrt{a})^{-1}$$

$$16. \int_0^{\infty} e^{-bx} [a^{1/2}e^{cx} \operatorname{erf}(\sqrt{bx}) + c^{1/2}e^{ax} \operatorname{erfc}(\sqrt{ax}) - c^{1/2}e^{cx}] dx \\ = (a-c)\sqrt{c}(b-c)^{-1}b^{-1/2}(\sqrt{b} + \sqrt{a})^{-1}, \quad b > c$$

$$17. \int_0^{\infty} e^{-bx} \left[ 2\left(\frac{x}{\pi}\right)^{1/2} e^{-a^2/(4x)} - a \operatorname{erfc}\left(\frac{a}{2\sqrt{x}}\right) \right] dx = \frac{e^{-a\sqrt{b}}}{b^{3/2}}$$

$$18. \int_0^{\infty} e^{-bx} \left[ a\left(\frac{x}{\pi}\right)^{1/2} e^{-a^2/(4x)} + \left(x + \frac{a^2}{2}\right) \operatorname{erf}\left(\frac{a}{2\sqrt{x}}\right) - \frac{a^2}{2} \right] dx = \frac{1}{b^2} (1 - e^{-a\sqrt{b}})$$

#### 4.5. Combination of Error Function With Trigonometric Functions

$$1. \int \operatorname{erf}(az) \sin bzdz = -\frac{1}{b} \cos bz \operatorname{erf}(az) + \frac{1}{2b} \exp\left(-\frac{b^2}{4a^2}\right) \left\{ \operatorname{erf}\left(az - i\frac{b}{2a}\right) + \operatorname{erf}\left(az + \frac{ib}{2a}\right) \right\}$$

$$2. \int \operatorname{erf}(az) \cos bzdz = \frac{1}{b} \sin bz \operatorname{erf}(az) + \frac{i}{2b} \exp\left(-\frac{b^2}{4a^2}\right) \left\{ \operatorname{erf}\left(az - \frac{ib}{2a}\right) - \operatorname{erf}\left(az + \frac{ib}{2a}\right) \right\}$$

$$3. \int \operatorname{erfc}(az) \sin bzdz = -\frac{1}{b} \cos bz \operatorname{erfc}(az) - \frac{1}{2b} \exp\left(-\frac{b^2}{4a^2}\right) \left\{ \operatorname{erf}\left(az - \frac{ib}{2a}\right) + \operatorname{erf}\left(az + \frac{ib}{2a}\right) \right\}$$

$$4. \int \operatorname{erfc}(az) \cos bzdz = \frac{1}{b} \sin bz \operatorname{erfc}(az) - \frac{i}{2b} \exp\left(-\frac{b^2}{4a^2}\right) \left\{ \operatorname{erf}\left(az - \frac{ib}{2a}\right) - \operatorname{erf}\left(az + \frac{ib}{2a}\right) \right\}$$

$$5. \int_0^{\infty} \operatorname{erfc}(ax) \sin bxdx = \frac{1}{b} \left[ 1 - \exp\left(-\frac{b^2}{4a^2}\right) \right], \quad |\arg a| < \frac{\pi}{4}$$

$$6. \int_0^{\infty} \operatorname{erfc}(ax) \cos bxdx = -\frac{i}{b} \exp\left(-\frac{b^2}{4a^2}\right) \operatorname{erf}\left(\frac{ib}{2a}\right), \quad |\arg a| < \frac{\pi}{4}$$

$$7. \int_0^{\infty} \operatorname{erfc}(ax) \cos(bx)xdx = \frac{1}{2a^2} \exp\left(-\frac{b^2}{4a^2}\right) - \frac{1}{b^2} \left[ 1 - \exp\left(-\frac{b^2}{4a^2}\right) \right]$$

$$8. \int_0^{\infty} \operatorname{erfc}(\sqrt{ax}) \sin bxdx = \frac{1}{b} - \left(\frac{a/2}{a^2 + b^2}\right)^{1/2} [(a^2 + b^2)^{1/2} - a]^{-1/2}, \quad \Re(a) > |\Im(b)|$$

$$9. \int_0^{\infty} \operatorname{erfc}(\sqrt{ax}) \cos bxdx = \left(\frac{a/2}{a^2 + b^2}\right)^{1/2} [(a^2 + b^2)^{1/2} + a]^{-1/2}, \quad \Re(a) > |\Im(b)|$$

$$10. \int_0^{\infty} \operatorname{erf}(ax) \sin b^2x^2dx = \frac{1}{4b\sqrt{2\pi}} \left( \ln \frac{a^2 + b^2 + ab\sqrt{2}}{a^2 + b^2 - ab\sqrt{2}} + 2 \tan^{-1} \frac{ab\sqrt{2}}{b^2 - a^2} \right), \quad a > 0$$

11. 
$$\int_0^\infty \operatorname{erfc}(ax) \sin bxx^p dx = \frac{\Gamma\left(\frac{p+3}{2}\right) b}{a^{p+2}\sqrt{\pi}(p+2)} {}_2F_2\left(\frac{p+2}{2}, \frac{p+3}{2}; \frac{3}{2}, \frac{p+4}{2}; -\frac{b^2}{4a^2}\right),$$

$$\Re(a) > 0, \Re(p) > 0$$
12. 
$$\int_0^\infty \operatorname{erfc}(ax) \cos bxx^p dx = \frac{\Gamma\left(\frac{p+1}{2}\right)}{a^{p+1}\sqrt{\pi}(p+1)} {}_2F_2\left(\frac{p+1}{2}, \frac{p+2}{2}; \frac{1}{2}, \frac{p+3}{2}; -\frac{b^2}{4a^2}\right),$$

$$\Re(a) > 0, \Re(p) > 0$$
13. 
$$\int_0^\infty \operatorname{erf}(ax) \frac{\sin bx}{x^2} dx = \frac{b}{2} \left[ -Ei\left(-\frac{b^2}{4a^2}\right) \right] + \sqrt{\pi} \operatorname{erf}\left(\frac{b^2}{4a^2}\right)$$
14. 
$$\int_0^\infty \operatorname{erf}(ax) \frac{\cos bx}{x} dx = \frac{1}{2} \left[ -Ei\left(-\frac{b^2}{4a^2}\right) \right]$$
15. 
$$\int_0^\infty [\operatorname{erfc}(ax) - \operatorname{erfc}(bx)] \frac{\cos px}{x} dx = \frac{1}{2} \left\{ Ei\left(-\frac{p^2}{4a^2}\right) - Ei\left(-\frac{p^2}{4b^2}\right) \right\}$$
16. 
$$\int_0^\infty \operatorname{erfc}\left(\sqrt{\frac{a}{x}}\right) \sin bxdx = \frac{1}{b} \exp[-(2ab)^{1/2}] \cos[(2ab)^{1/2}], \quad \Re(a) > 0, \Re(b) > 0$$
17. 
$$\int_0^\infty \operatorname{erf}\left(\sqrt{\frac{a}{x}}\right) \cos bxdx = -\frac{1}{b} \exp[-(2ab)^{1/2}] \sin[(2ab)^{1/2}], \quad \Re(a) > 0, \Re(b) > 0$$
18. 
$$\int_0^\infty \operatorname{erfc}(ax) \tan x dx = \sum_{k=1}^\infty \frac{(-1)^k}{k} \exp\left(-\frac{k^2}{a^2}\right) + \ln 2$$
19. 
$$\int_0^\infty \operatorname{erfc}(ax) \sin^2 bx \frac{dx}{x} = \frac{1}{4} \left\{ \gamma + 2 \ln\left(\frac{b}{a}\right) + \left[ -Ei\left(-\frac{b^2}{a^2}\right) \right] \right\}$$
20. 
$$\int_0^\infty \operatorname{erf}(ax) \sin bx \sin cx \frac{dx}{x} = \frac{1}{4} \left\{ \left[ -Ei\left(-\frac{(c-b)^2}{4a^2}\right) \right] - \left[ -Ei\left(-\frac{(c+b)^2}{4a^2}\right) \right] \right\}, \quad c \neq b$$
21. 
$$\int_0^\infty \operatorname{erfc}(ax) \sin bx \sin cx \frac{dx}{x} = \frac{1}{2} \ln\left|\frac{c+b}{c-b}\right| - \int_0^\infty \operatorname{erf}(ax) \sin bx \sin cx \frac{dx}{x}, \quad c \neq b$$
22. 
$$\int_0^\infty \operatorname{erf}(ax) \cos bx \cos cx \frac{dx}{x} = \frac{1}{4} \left\{ \left[ -Ei\left(-\frac{(c-b)^2}{4a^2}\right) \right] + \left[ -Ei\left(-\frac{(c+b)^2}{4a^2}\right) \right] \right\}, \quad c \neq b$$
23. 
$$\int_0^\infty \operatorname{erfc}(ax) \sin bxe^{a^2x^2} dx = \frac{\sqrt{\pi}}{2a} \exp\left(\frac{b^2}{4a^2}\right) \operatorname{erfc}\left(\frac{b}{2a}\right)$$
24. 
$$\int_0^\infty \operatorname{erf}(iax) \sin bxe^{-a^2x^2} dx = \frac{i\sqrt{\pi}}{2a} \exp\left(-\frac{b^2}{4a^2}\right)$$
25. 
$$\int_0^\infty \operatorname{erfc}(ax) \cos bxe^{a^2x^2} dx = \frac{1}{2a\sqrt{\pi}} \exp\left(\frac{b^2}{4a^2}\right) \left[ -Ei\left(-\frac{b^2}{4a^2}\right) \right]$$
26. 
$$\int_0^\infty \operatorname{erfc}(ax) \sin x \cosh x dx = \frac{1}{2} \left[ \sin\left(\frac{1}{2a^2}\right) - \cos\left(\frac{1}{2a^2}\right) + 1 \right]$$

$$27. \int_0^{\infty} \operatorname{erfc}(ax) \cos x \sinh x dx = \frac{1}{2} \left[ \sin\left(\frac{1}{2a^2}\right) + \cos\left(\frac{1}{2a^2}\right) - 1 \right]$$

$$28. \int_0^{\infty} \operatorname{erfc}(ax) x \cos x \cosh x dx = \left[ \frac{1}{2a^2} \cos\left(\frac{1}{2a^2}\right) - \frac{1}{2} \sin\left(\frac{1}{2a^2}\right) \right]$$

$$29. \int_0^{\infty} \operatorname{erfc}(ax) x \sin x \sinh x dx = \left[ \frac{3}{2} \cos\left(\frac{1}{2a^2}\right) + \frac{1}{a^2} \sin\left(\frac{1}{2a^2}\right) - \frac{3}{2} \right]$$

$$30. \int_0^{\infty} \left[ e^{-bx} \operatorname{erfc}\left(ab - \frac{x}{2a}\right) - e^{bx} \operatorname{erfc}\left(ab + \frac{x}{2a}\right) \right] \sin px dx = \frac{2p}{(p^2 + b^2)} \exp[-a^2(b^2 + p^2)].$$

#### 4.6. Combination of Error Function With Logarithms and Powers

$$1. \int \operatorname{erf}(az) \ln z dz = (\ln z - 1) \left[ z \operatorname{erf}(az) + \frac{1}{a\sqrt{\pi}} e^{-a^2 z^2} \right] - \frac{1}{2a\sqrt{\pi}} \operatorname{Ei}(-a^2 z^2)$$

$$2. \int \operatorname{erfc}(az) \ln z dz = (\ln z - 1) \left[ z \operatorname{erfc}(az) - \frac{1}{a\sqrt{\pi}} e^{-a^2 z^2} \right] + \frac{1}{2a\sqrt{\pi}} \operatorname{Ei}(-a^2 z^2)$$

$$3. \int_0^{\infty} \operatorname{erfc}(ax) \ln x dx = -\frac{1}{a\sqrt{\pi}} \left[ 1 + \frac{\gamma}{2} + \ln a \right]$$

$$4.5 \int \operatorname{erf}(az) z \ln z dz = \frac{1}{2} z^2 \operatorname{erf}(az) \ln z + \frac{1}{2a\sqrt{\pi}} z \ln ze^{-a^2 z^2} \\ - \frac{1}{2} \int z \operatorname{erf}(az) dz - \frac{1}{4a^2} \operatorname{erf}(az) - \frac{1}{2a\sqrt{\pi}} \int \ln ze^{-a^2 z^2} dz$$

$$5. \int \operatorname{erfc}(az) z \ln z dz = \frac{1}{2} z^2 \operatorname{erfc}(az) \ln z - \frac{z \ln z}{2a\sqrt{\pi}} e^{-a^2 z^2} \\ - \frac{1}{2} \int z \operatorname{erfc}(az) dz + \frac{1}{4a^2} \operatorname{erf}(az) + \frac{1}{2a\sqrt{\pi}} \int \ln ze^{-a^2 z^2} dz$$

$$6. \int_0^{\infty} \operatorname{erfc}(ax) x \ln x dx = \frac{1}{8a^2} + \frac{1}{2a\sqrt{\pi}} \int_0^{\infty} \ln xe^{-a^2 x^2} dx$$

$$7. (k+1) \int \operatorname{erf}(az) z^k \ln z dz = z^{k+1} \operatorname{erf}(az) \ln z + \frac{1}{a\sqrt{\pi}} z^k \ln ze^{-a^2 z^2} \\ - \int z^k \operatorname{erf}(az) dz - \frac{1}{a\sqrt{\pi}} \int z^{k-1} e^{-a^2 z^2} dz \\ - \frac{k}{a\sqrt{\pi}} \int z^{k-1} \ln ze^{-a^2 z^2} dz$$

$$8. (k+1) \int \operatorname{erfc}(az) z^k \ln z dz = z^{k+1} \operatorname{erfc}(az) \ln z - \frac{1}{a\sqrt{\pi}} z^k \ln ze^{-a^2 z^2} \\ - \int z^k \operatorname{erfc}(az) dz + \frac{1}{a\sqrt{\pi}} \int z^{k-1} e^{-a^2 z^2} dz \\ + \frac{k}{a\sqrt{\pi}} \int z^{k-1} e^{-a^2 z^2} \ln z dz$$

<sup>5</sup> For the elementary integrals in eqs. (4 to 8), see appendix.

$$9. (k+1) \int_0^{\infty} \operatorname{erfc}(ax) x^k \ln x dx = \frac{\Gamma(k/2)}{2\sqrt{\pi} a^{k+1}} \left[ \frac{1}{k+1} + \frac{k}{2} \psi\left(\frac{k}{2}\right) - k \ln a \right].$$

#### 4.7. Combination of Two Error Functions

$$1. \int_0^{\infty} \operatorname{erf}(bx) \operatorname{erfc}(ax) dx = \frac{1}{b\sqrt{\pi}} \left( \frac{\sqrt{a^2+b^2}}{a} - 1 \right)$$

$$2. \int_0^{\infty} \operatorname{erfc}(bx) \operatorname{erfc}(ax) dx = \frac{1}{ab\sqrt{\pi}} (a+b - \sqrt{a^2+b^2})$$

$$3. \int_0^{\infty} \operatorname{erf}\left(\frac{b}{x}\right) \operatorname{erfc}(ax) dx \\ = \int_0^{\infty} \operatorname{erf}(bx) \operatorname{erfc}\left(\frac{a}{x}\right) \frac{dx}{x^2} = \frac{1}{a\sqrt{\pi}} (1 - e^{-2ab}) - \frac{2b}{\sqrt{\pi}} [-Ei(-2ab)]$$

$$4. \int_0^{\infty} \operatorname{erfc}\left(\frac{b}{x}\right) \operatorname{erfc}(ax) dx \\ = \int_0^{\infty} \operatorname{erfc}(bx) \operatorname{erfc}\left(\frac{a}{x}\right) \frac{dx}{x^2} = \frac{1}{a\sqrt{\pi}} e^{-2ab} + \frac{2b}{\sqrt{\pi}} [-Ei(-2ab)]$$

$$5. \int_0^{\infty} \operatorname{erfc}(ax) \operatorname{erf}(bx) e^{b^2 x^2} dx = \frac{-1}{2b\sqrt{\pi}} \ln\left(1 - \frac{b^2}{a^2}\right), \quad a^2 > b^2$$

$$6. \int_0^{\infty} \operatorname{erfc}(ax) \operatorname{erfc}(bx) e^{b^2 x^2} dx = \frac{1}{b\sqrt{\pi}} \ln\left(1 + \frac{b}{a}\right), \quad a+b > 0$$

$$7. \int_0^1 \operatorname{erf}(x) \operatorname{erf}(\sqrt{1-x^2}) x dx = \frac{1}{4} \left( \frac{3}{e} - 1 \right)$$

$$8. \int_0^{\infty} \operatorname{erfc}(bx) \left[ (x^2+a^2) \operatorname{erfc}\left(\frac{a}{x\sqrt{2}}\right) - \frac{2}{\sqrt{\pi}} ax e^{-\frac{a^2}{2x^2}} \right] dx \\ = \frac{1}{3b^3\sqrt{\pi}} \{ e^{-2ab}(2a^2b^2 - ab + 1) - [-Ei(-2ab)] \}$$

$$9. \int_0^{\infty} \operatorname{erfc}(cx) e^{a^2 x^2} \left[ 2 \cosh ab - e^{-ab} \operatorname{erf}\left(\frac{b-2ax^2}{2x}\right) - e^{ab} \operatorname{erf}\left(\frac{b+2ax^2}{2x}\right) \right] dx \\ = \frac{1}{a\sqrt{\pi}} \{ e^{-ab} [-Ei(-bc+ba)] - e^{ab} [-Ei(-bc-ba)] \}.$$

#### 4.8. Combination of Error Function With Bessel Functions

$$1. \int_0^{\infty} \operatorname{erf}(ax) J_0(bx) dx = \frac{1}{b} \operatorname{erfc}\left(\frac{b}{2a}\right)$$

$$2. \int_0^{\infty} \operatorname{erfc}(ax) J_0(bx) dx = \frac{1}{b} \operatorname{erf}\left(\frac{b}{2a}\right)$$

3.  $\int_0^\infty \left[ 2 \operatorname{erfc}(x) - 1 \right] J_0(bx) dx = -\frac{1}{b} \left[ 2 \operatorname{erfc}\left(\frac{b}{2}\right) - 1 \right]$
4.  $\int_0^\infty \operatorname{erf}(x) J_1(bx) dx = \frac{1}{b} e^{-b^2/8} I_0\left(\frac{b^2}{8}\right)$
5.  $\int_0^\infty \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) J_{3/2}(bx) x^{-1/2} dx = b^{-3/2} \operatorname{erf}\left(\frac{b}{\sqrt{2}}\right)$
6.  $\int_0^\infty \operatorname{erf}(ax) J_p(bx) x^p dx = \frac{1}{\sqrt{\pi}} \left(\frac{2}{b}\right)^p \frac{1}{b} \Gamma\left(p + \frac{1}{2}, \frac{b^2}{4a^2}\right), \quad -1 < p < \frac{1}{2}$
7.  $\int_0^\infty \operatorname{erfc}(ax) J_p(bx) x^p dx = \frac{1}{\sqrt{\pi}} \left(\frac{2}{b}\right)^p \frac{1}{b} \gamma\left(p + \frac{1}{2}, \frac{b^2}{4a^2}\right), \quad p > -1$
8.  $\int_0^\infty \operatorname{erfc}(ax) J_p(bx) x^{p+1} dx = \frac{1}{2\sqrt{\pi}} \left(\frac{b}{2}\right)^p \frac{1}{a^{2p+2}} \frac{\Gamma\left(p + \frac{3}{2}\right)}{\Gamma(p+2)} {}_1F_1\left(p + \frac{3}{2}; p+2; -\frac{b^2}{4a^2}\right), \quad p > -1$
9.  $\int_0^\infty \operatorname{erf}(ax) J_p(bx) x^{1-p} dx = \frac{1}{b} \left(\frac{b}{2}\right)^{p-1} \frac{1}{\Gamma(p)} {}_1F_1\left(\frac{1}{2}; p; -\frac{b^2}{4a^2}\right), \quad p > \frac{1}{2}$
10.  $\int_0^\infty \operatorname{erfc}(ax) J_\nu(bx) x^\nu dx = \frac{2^{-(p+2\nu+1)} b^\nu \Gamma(p+\nu+1)}{a^{p+\nu+1} \Gamma(\nu+1) \Gamma\left(\frac{p+\nu+3}{2}\right)} \times {}_2F_2\left(\frac{p+\nu+1}{2}, \frac{p+\nu+2}{2}; \nu+1, \frac{p+\nu+3}{2}; -\frac{b^2}{4a^2}\right), \quad p+\nu > -1$
11.  $\int_0^\infty \operatorname{erfc}(ax) J_0(bx) e^{a^2 x^2} dx = \frac{1}{ab\sqrt{\pi}} \left[ 1 - \sqrt{\pi} \left(\frac{b}{2a}\right) e^{b^2/4a^2} \operatorname{erfc}\left(\frac{b}{2a}\right) \right]$
12.  $\int_0^\infty \operatorname{erfc}(ax) J_p(bx) e^{a^2 x^2} x^{p+1} dx = \frac{1}{2\pi} \left(\frac{b}{2}\right)^p \frac{1}{a^{2p+2}} \Gamma\left(p + \frac{3}{2}\right) e^{b^2/4a^2} \Gamma\left(-p-1, \frac{b^2}{4a^2}\right),$   
 $-1 < p < \frac{1}{2}$
13.  $\int_0^\infty \operatorname{erfc}(ax) J_p(bx) e^{a^2 x^2} x^p dx = \frac{b^{p+1/2} \Gamma\left(p + \frac{1}{2}\right)}{\sqrt{\pi} a^{3p/2+1} 2^{p+1}} U\left(p + \frac{1}{2}, p+1, \frac{b^2}{4a^2}\right)$   
 $= \frac{1}{\sqrt{\pi}} \frac{1}{ba^p} \Gamma\left(p + \frac{1}{2}\right) e^{b^2/8a^2} W_{-p/2, p/2}\left(\frac{b^2}{8a^2}\right), \quad -1 < p < \frac{3}{2}$
14.  $\int_0^\infty \operatorname{erfc}(ax) J_p(bx) e^{a^2 x^2} x^{p+1} dx = \frac{1}{2\pi} \left(\frac{b}{2}\right)^p \frac{1}{a^{2p+2}} \Gamma\left(p + \frac{3}{2}\right) e^{b^2/4a^2} \Gamma\left(-p - \frac{1}{2}, \frac{b^2}{4a^2}\right),$   
 $-1 < p < \frac{1}{2}$
15.  $\int_0^\infty \operatorname{erfc}(x) Y_\nu(bx) e^{x^2} x^{\nu+1} dx = \frac{1}{2\pi} \left(\frac{b}{2}\right)^\nu \Gamma(p+1) e^{b^2/4} \Gamma\left(-p, \frac{b^2}{4}\right), \quad -1 < p < \frac{1}{2}$
16.  $\int_0^\infty \operatorname{erfc}(x) Y_\nu(bx) e^{x^2} x^{\nu+3} dx = -\frac{1}{b\pi} \Gamma(p+2) e^{b^2/8} W_{-(p+3)/2, p/2}\left(\frac{b^2}{4}\right), \quad -2 < p < -\frac{3}{2}$

$$17. \int_0^{\infty} \operatorname{erfc}(x) I_p \left( \frac{1}{2} x^2 \right) e^{x^{2/2} x^{2p+1}} dx = \frac{\Gamma \left( 2p + \frac{3}{2} \right) \Gamma(-p)}{2\pi^{3/2} \left( p + \frac{1}{2} \right)}, \quad -\frac{1}{2} < p < 0$$

$$18. \int_0^{\infty} \operatorname{erfc}(x) I_p \left( \frac{1}{2} x^2 \right) e^{x^{2/2} x^{2p}} dx = \frac{\Gamma \left( 2p + \frac{1}{2} \right)}{2\Gamma(p+1) \cos p\pi}, \quad -\frac{1}{4} < p < \frac{1}{2}$$

$$19. \int_0^{\infty} \operatorname{erfc}(ax) I_p(x^2) e^{-(1-a^2)x^2} x^{2p+1} dx$$

$$= \frac{\Gamma \left( 2p + \frac{3}{2} \right) \Gamma(-p)}{\pi \left( p + \frac{1}{2} \right)} 2^{p-2} a^{-4p-2} {}_2F_1 \left( p + \frac{1}{2}, 2p + \frac{3}{2}; p + \frac{3}{2}; 1 - \frac{2}{a^2} \right),$$

$$p \neq -\frac{1}{2}, \quad \Re(a^2) > 1, \quad -1 < \Re(p) < 0$$

$$20. \int_0^{\infty} \operatorname{erfc} \left( \frac{a}{\sqrt{2x}} \right) K_p(x) e^{a^{2/2} x - x} \frac{dx}{x} = \frac{\pi^{5/2}}{4} \sec p\pi \{ [J_p(a)]^2 + [Y_p(a)]^2 \}, \quad -\frac{1}{2} < p < \frac{1}{2}$$

$$21. \int_0^{\infty} \operatorname{erfc}(x) J_{\lambda+\nu}(ax) J_{\lambda-\nu}(ax) x^p dx$$

$$= \frac{a^{2\lambda} \Gamma \left( \lambda + \frac{1}{2} p + 1 \right) {}_4F_4 \left( 1 + \lambda, \frac{1}{2} + \lambda, 1 + \lambda + \frac{p}{2}, \frac{1}{2} + \lambda + \frac{p}{2}; 1 + \lambda + \nu, 1 + \lambda - \nu, 1 + 2\lambda, \frac{3}{2} + \lambda + \frac{p}{2}; -a^2 \right)}{\sqrt{\pi} 2^{2\lambda+1} \Gamma(\lambda + \nu + 1) \Gamma(\lambda - \nu + 1) \left( \lambda + \frac{1}{2} p + \frac{1}{2} \right)},$$

$$\lambda + \frac{1}{2} p > 0, \quad 1 + 2\lambda \neq -n.$$

#### 4.9. Combination of Error Function With Other Special Functions

$$1. \int_0^{\infty} \operatorname{erfc}(ax) \left[ -Ei \left( -\frac{1}{4} x^2 \right) \right] \frac{dx}{x} = (\gamma + \ln a)^2 + \zeta(2) + 2 \sum_{k=0}^{\infty} \frac{(-a)^{k+1}}{k!(k+1)^3}$$

$$2. \int_0^{\infty} \operatorname{erfc}(ax) \left[ -Ei \left( -\frac{b^2}{x^2} \right) \right] \frac{dx}{x^3} = \frac{1}{2b^2} (1 - 2ab) e^{-2ab} + 2a^2 [-Ei(-2ab)]$$

$$3. \int_0^{\infty} \operatorname{erfc}(x) \operatorname{si}(2px) dx = (e^{-1/4a^2} - 1) - \frac{\sqrt{\pi}}{2a} \operatorname{erfc} \left( \frac{1}{2a} \right)$$

$$4. \int_0^{\infty} \operatorname{erf}(ax) \operatorname{Ci}(x) \frac{dx}{x} = -\frac{1}{8} [\zeta(2) + (\gamma - \ln 4a^2)^2] - \frac{1}{4} \sum_{k=0}^{\infty} \left( -\frac{1}{4a^2} \right)^{k+1} \frac{1}{k!(k+1)^3}$$

$$5. \int_1^{\infty} \operatorname{erfc}(ax) P_{\nu}^{\mu}(x) e^{a^2 x^2} (x^2 - 1)^{-\mu/2} dx$$

$$= \frac{2^{\mu-1}}{\pi} a^{\mu-3/2} e^{a^2/4} \Gamma \left( \frac{\mu + \nu + 1}{2} \right) \Gamma \left( \frac{\mu - \nu}{2} \right) W_{(1-2\mu)/4, (1+2\mu)/4}(a^2),$$

$$\mu < 1, \quad \mu < \nu; \quad \mu + \nu > -1, \quad \mu - \nu \neq -2n$$



6.  $\int_0^\infty \operatorname{erfc}(x) L_\nu^{(p)}(x^2) x^{2\nu+1} dx = \frac{\Gamma\left(p+\frac{3}{2}\right) \Gamma\left(\nu+\frac{1}{2}\right)}{2\pi\nu!(p+\nu+1)}, \quad p > -1$
7.  $\int_0^\infty \operatorname{erfc}(x) L_\nu^{(p)}(ax^2) e^{-(a-1)x^2} x^{2\nu+1} dx$   
 $= \frac{\Gamma\left(p+\frac{3}{2}\right) \Gamma\left(\nu+\frac{1}{2}\right)}{2\pi(p+\nu+1)\Gamma(\nu+1)} {}_2F_1\left(p+\nu+1, p+\frac{3}{2}; p+\nu+2; 1-a\right), \quad |1-a| < 1, \quad p > -1$
8.  $\int_0^\infty \operatorname{erfc}(x) {}_1F_1\left(b-a+\frac{1}{2}; 2b+1; x^2\right) x^{4b} dx = \frac{\Gamma(4b+1) \Gamma\left(a-b+\frac{1}{2}\right)}{2^{4b+1} \Gamma(a+b+1)}$   
 $\mathcal{R}(b) > -\frac{1}{4}, \mathcal{R}(b-a) < \frac{1}{2}.$
9.  $\int_0^\infty \operatorname{erf}(px) {}_1F_1(a; b; -p^2x^2) x^{2(b-1)} dx = \frac{\Gamma(b)}{\sqrt{\pi} p^{2b-1} (2b-1)} {}_2F_1\left(a, b-\frac{1}{2}; b+\frac{1}{2}; -1\right), \quad b > \frac{1}{2}, a < \frac{1}{2}$
10.  $\int_0^\infty \operatorname{erf}(px) {}_1F_1(a; b; -q^2x^2) x^{2(b-1)} dx = \frac{\Gamma(b)}{\sqrt{\pi} p^{2b-1} (2b-1)} {}_2F_1\left(a, b-\frac{1}{2}; b+\frac{1}{2}; -\frac{q^2}{p^2}\right), \quad b > \frac{1}{2}, p^2 > q^2$
11.  $\int_0^\infty \operatorname{erf}(px) {}_1F_1(a; b; -q^2x^2) x^\nu dx$   
 $= \frac{1}{p^{\nu+1} \sqrt{\pi}} \frac{\Gamma\left(\frac{1}{2}\nu+1\right)}{(\nu+1)} {}_3F_2\left(a, \frac{\nu+2}{2}, \frac{\nu+1}{2}; b, \frac{\nu+3}{2}; -\frac{q^2}{p^2}\right), \quad p \neq 0, \nu > -1, \quad p^2 > q^2$
12.  $\int_0^\infty \operatorname{erf}(x) \Psi\left(a-\frac{1}{2}; b-\frac{1}{2}; px^2\right) x^{2b-2} dx$   
 $= \frac{1}{2p^b} \frac{\Gamma(b)}{(a-b)\Gamma(a)} {}_2F_1\left(\frac{1}{2}, b; a; 1-\frac{1}{p}\right), \quad \mathcal{R}(p) \geq \frac{1}{2}, a \neq b$
13.  $\int_0^\infty \operatorname{erf}(x) {}_1F_1\left(a; \frac{3}{2}; qx^2\right) e^{-px^2} x dx$   
 $= \frac{(p+1)^{a-1/2}}{2p(p+1-q)^a} {}_2F_1\left[1, a; \frac{3}{2}; \frac{q}{p(p+1-q)}\right], \quad p \neq 0, p+1 \neq q, \mathcal{R}(p) > \mathcal{R}(q)$
14.  $\int_0^\infty \operatorname{erfc}(x) {}_1F_1(a; b; -px^2) e^{x^2} x^{2b-1} dx = \frac{\Gamma(2b) \Gamma\left(a-b+\frac{1}{2}\right)}{\sqrt{\pi} 2^{2b} \Gamma(a+1)} {}_2F_1\left(a; b+\frac{1}{2}; a+1; 1-p\right),$   
 $\mathcal{R}(b) > 0, \quad \mathcal{R}(b-a) < \frac{1}{2}, \quad |1-p| < 1$
15.  $\int_0^1 \operatorname{erf}(x) {}_1F_1(a; b; 1-x^2) e^{x^2} (1-x^2)^{b-1} x dx = \frac{\Gamma(b)}{2\Gamma\left(b+\frac{3}{2}\right)} {}_1F_1\left(a+1; b+\frac{3}{2}; 1\right),$   
 $\mathcal{R}(b) > 0$

$$16. \int_0^q \operatorname{erf}(ax) {}_1F_1\left[p + \frac{1}{2}; p; b(q^2 - x^2)\right] e^{a^2 x^2} (q^2 - x^2)^{p-1} x dx$$

$$= \frac{a}{2} q^{2p+1} e^{a^2 q^2} \frac{\Gamma(p)}{\Gamma\left(p + \frac{3}{2}\right)} {}_1F_1\left[1; p + \frac{3}{2}; q^2(b - a^2)\right], \quad p \geq 1$$

$$17. \int_0^\infty \operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right) D_\nu(\pm x) e^{x^2/4} dx = \frac{2^{(\nu+1)/2}}{\nu+1} \left[ \frac{1}{\Gamma\left(\frac{1-\nu}{2}\right)} \mp \frac{\sqrt{\pi}}{\Gamma\left(-\frac{\nu}{2}\right)} \right],$$

$\Re(\nu) \neq -1; \quad \Re(\nu) > -1 \quad \text{for lower sign}$

$$18. \int_0^\infty \operatorname{erfc}\left(\frac{a}{\sqrt{2}} x\right) [D_\nu(x) - D_\nu(-x)] e^{(2a-1)x^2/4} x dx$$

$$= \frac{2^{\frac{\nu}{2}+2} a^{-3/2}}{(\nu\pi)} {}_2F_1\left(\frac{1}{2}\nu + 1, 2; \frac{1}{2}\nu + 2; 1 - \frac{1}{a}\right),$$

$\Re(a) > \frac{1}{2}, \quad \Re(\nu) > 0$

$$19. \int_0^\infty \operatorname{erfc}(x) M_{\mu, \nu}(ax^2) e^{(2-a)x^2/2} x^{2\nu-1} dx$$

$$= \frac{\Gamma(4\nu+1)\Gamma(\mu - \nu + \frac{1}{2}) a^{\nu+1/2}}{\Gamma(\mu + \nu + 1)2^{4\nu+1}} {}_2F_1\left(\mu + \nu + \frac{1}{2}, 2\nu + \frac{1}{2}; \mu + \nu + 1; 1 - a\right),$$

$|a-1| < 1, \quad \Re(\nu) > -\frac{1}{4}, \quad \Re(\nu - \mu) < \frac{1}{2}$

$$20. \int_0^\infty \operatorname{erfc}(x) M_{\mu, \nu}(ax^2) e^{-1/2(a-2)x^2} x^{2\nu} dx$$

$$= \frac{\Gamma(2\nu+1)\Gamma\left(2\nu + \frac{3}{2}\right)\Gamma(\mu - \nu) a^{\nu+1/2}}{2\pi\Gamma\left(\mu + \nu + \frac{3}{2}\right)} {}_2F_1\left(\mu + \nu + \frac{1}{2}, \frac{3}{2} + 2\nu; \mu + \nu + \frac{3}{2}; 1 - a\right),$$

$|a-1| < 1, \quad \Re(\mu) > \quad \Re(\nu) > -\frac{1}{2}$

$$21. \int_0^\infty \operatorname{erfc}(x) M_{\lambda, \mu}(ax^2) x^\mu \exp(ax^2/2) dx$$

$$= \frac{\Gamma\left(\mu + \frac{1}{2}p + \frac{3}{2}\right) a^{\mu+1/2}}{2\sqrt{\pi}\left(\mu + \frac{1}{2}p + 1\right)} {}_3F_2\left(\lambda + \mu + \frac{1}{2}, \mu + \frac{p}{2} + \frac{3}{2}, \mu + \frac{p}{2} + 1; 2\mu + 1, \mu + \frac{p}{2} + 2, -a\right),$$

$\mu + \frac{1}{2}p + 1 > 0.$

### 5. Appendix. Some Relevant Integrals Involving Elementary Functions

$$(A1) \int z^n e^{az} dz = e^{az} \sum_{k=0}^n (-1)^k \frac{n!}{(n-k)!} \frac{z^{n-k}}{a^{k+1}}$$

$$(A2) \int e^{-a^2 z^2 + bz} dz = \frac{\sqrt{\pi}}{2a} \exp\left(\frac{b^2}{4a^2}\right) \operatorname{erf}\left(az - \frac{b}{2a}\right)$$

$$(A3) \int_0^{\infty} e^{-a^2x^2+bx} dx = \frac{\sqrt{\pi}}{2a} \exp\left(\frac{b^2}{4a^2}\right) \left[1 + \operatorname{erf}\left(\frac{b}{2a}\right)\right]$$

$$(A4) \int z e^{-a^2z^2+bz} dz = \frac{1}{2a^2} \exp\left(\frac{b^2}{4a^2}\right) \left[\frac{b\sqrt{\pi}}{2a} \operatorname{erf}\left(az - \frac{b}{2a}\right) - e^{-(az-b/2a)^2}\right]$$

$$(A5) \int_0^{\infty} x e^{-a^2x^2+bx} dx = \frac{1}{2a^2} \left[\frac{\sqrt{\pi}b}{2a} e^{b^2/4a^2} \operatorname{erfc}\left(-\frac{b}{2a}\right) + 1\right]$$

$$(A6) \int_0^{\infty} x^n e^{-a^2x^2+bx} dx = (2a^2)^{-(n+1)/2} n! \exp\left(\frac{b^2}{8a^2}\right) D_{-(n+1)}\left(-\frac{b}{a\sqrt{2}}\right)$$

$$(A7) \int z^n e^{-a^2z^2+bz} dz = a^{-n-1} \exp\left(\frac{b^2}{4a^2}\right) \sum_{k=0}^n \frac{n!}{k!(n-k)!} \left(\frac{b}{2a}\right)^{n-k} \int u^k e^{-u^2} du, \quad u = az - \frac{b}{2a}$$

$$(A8) \int u^k e^{-u^2} du = -\frac{e^{-u^2}}{2} \sum_{j=0}^{r-1} \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k+1}{2}-j\right)} u^{k-2j-1} + \frac{1}{2} (1-s) \Gamma\left(r+\frac{1}{2}\right) \operatorname{erf}(u),$$

$$k = 2r - s, \quad s = 0 \text{ or } 1$$

$$(A9) (n-1) \int z^{-n} e^{-a^2z^2} dz = -z^{-n+1} e^{-a^2z^2} - 2a^2 \int z^{-n+2} e^{-a^2z^2} dz$$

$$(A10) \int z^{-n} e^{-a^2z^2} dz = \frac{e^{-a^2z^2}}{2a^2 \Gamma\left(\frac{n+1}{2}\right)} \sum_{k=1}^{n/2} (-1)^k \Gamma\left(\frac{n+1}{2}-k\right) a^{2k} z^{2k-n-1} \\ + \frac{(-1)^{n/2} \pi a^{n-1}}{2 \Gamma\left(\frac{n+1}{2}\right)} \operatorname{erf}(az), \quad n \text{ even positive integer} \\ = \frac{e^{-a^2z^2}}{2a^2 \Gamma\left(\frac{n+1}{2}\right)} \sum_{k=1}^{(n-1)/2} (-1)^k \Gamma\left(\frac{n+1}{2}-k\right) a^{2k} z^{2k-n-1} \\ + \frac{(-1)^{(n-1)/2}}{2 \left(\frac{n-1}{2}\right)!} Ei(-a^2z^2), \quad n > 1 \text{ odd positive integer}$$

$$(A11) \int f(z) e^{-(a^2z^2+2bz)} dz = \frac{1}{a} e^{b^2/a^2} \int f\left(\frac{au-b}{a^2}\right) e^{-u^2} du, \quad \text{where } u = a\left(z + \frac{b}{a^2}\right)$$

$$(A12) \int (z-a)^n e^{-c^2(z-b)^2} dz = \sum_{k=0}^n \frac{n!}{k!(n-k)!} \frac{(b-a)^{n-k}}{c^{k+1}} \int u^k e^{-u^2} du, \quad \text{where } u = c(z-b)$$

$$(A13) B_1(p, \alpha, u) \equiv \int u^p e^{-\alpha u} \ln u du = -u^{p+1} \left[ \sum_{j=0}^{\infty} \frac{(-\alpha u)^j}{j!(p+j+1)^2} - \ln u \sum_{j=0}^{\infty} \frac{(-\alpha u)^j}{j!(p+j+1)} \right], \quad p > -1$$

$$(A14) \alpha B_1(p, \alpha, u) = p B_1(p-1, \alpha, u) + \frac{1}{\alpha^p} \gamma(p, \alpha u)$$

$$(A15) \int_u^{\infty} \ln x e^{-ax} \frac{dx}{x} = \frac{1}{2} \{(\gamma + \ln au)^2 + \zeta(2) + 2 \ln u E_1(au)\} + \sum_{k=0}^{\infty} \frac{(-au)^{k+1}}{k!(k+1)^3}$$

$$(A16) \int_0^{\infty} x^p e^{-ax} \ln x dx = \frac{\Gamma(p+1)}{a^{p+1}} [\psi(p+1) - \ln a]$$

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