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A Tableau Style Proof System for Two Paraconsistent Logics

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Abstract This paper presents a tableau based proof technique that is suitable for proving theorems in the two paraconsistent logics; **LP** and Belnap's 4-valued logic. While coupled tree proof systems exist for both logics the tableau proof system described has several advantages over them. First, it is easier to use and second, it lends itself to first-order and modal extensions of the above logics. Truth signs are used, in a novel way, to represent categories of truth values instead of, as is usual, single truth values.

A logic is explosive if $\alpha, \neg\alpha \vdash \beta$ for arbitrary α and β. A logic is paraconsistent if it is not explosive. It would seem likely that paraconsistent logics would not be amenable to refutation based proof styles, like tableaux. Indeed, the tableau (coupled tree) style proof systems that exist for paraconsistent logics (e.g., Dunn [3], Priest [5]) seem unnatural when compared with the very natural tableau systems for classical logic (Smullyan [9]). Here we present a tableau based proof system for two paraconsistent logics, **LP** (Priest [5], [6], [7], [8]) and Belnap's 4-valued logic [1], [2], which is not only more natural than their existing coupled tree proof systems [3], [5] but both easier to use and easier to extend to first-order and modal versions of these logics.

The logic **LP** (Table 1) is just Kleene's strong 3-valued logic [4] where the middle element denotes, paraconsistently, both true and false. By the use of signed formulas, it is possible to construct a tableau style proof system for **LP** (Figure 1). The truth signs used are **T** for at least true, **F** for at least false, $\bar{\mathbf{T}}$ for definitely not true (exactly false), and $\bar{\mathbf{F}}$ for definitely not false (exactly true). Figure 2 presents the branch closure rules for this logic. Thus to show $\gamma_1, \gamma_2, \ldots, \gamma_n \vdash \delta$ we create a tableau branch where for each γ_i we add $\mathbf{T}\gamma_i$ to the branch and we add $\bar{\mathbf{T}}\delta$ to the end of the branch. Then we apply the decomposition rules to each branch until either it becomes closed (contains two signed formulas $\Lambda_1\alpha$ and $\Lambda_2\alpha$ where Λ_1 and Λ_2 are opposite in the sense of Figure 2) or is open (a rule has been applied to each of the nonatomic signed formulas on a branch and

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		$\neg\alpha$	$\alpha \wedge \beta$			$\alpha \vee \beta$			$\alpha \supset \beta$			$\alpha \equiv \beta$		
	β		**f**	**t**	$\top$	**f**	**t**	$\top$	**f**	**t**	$\top$	**f**	**t**	$\top$
	f	**t**	**f**	**f**	**f**	**f**	**t**	$\top$	**t**	**t**	**t**	**t**	**f**	$\top$
α	**t**	**f**	**f**	**t**	$\top$	**t**	**t**	**t**	**f**	**t**	$\top$	**f**	**t**	$\top$
	$\top$	$\top$	**f**	$\top$	$\top$	$\top$	**t**	$\top$	$\top$	**t**	$\top$	$\top$	$\top$	$\top$

Table 1. Truth tables for the operators of **LP**, where **t** denotes just true, **f** just false, and $\top$ both true and false. The designated truth values are **t** and $\top$.

still that branch is not closed). Figure 3 shows an example of both a closed and an open tableau. Because of the finite nature of the formulas and associated rules, a branch will always become either closed or open after a finite number of steps.

Now we will prove the soundness and completeness of the proof system for **LP**.

$$\begin{array}{c}\mathbf{T}\neg\alpha\\ \hline \mathbf{F}\alpha\end{array} \qquad \begin{array}{c}\mathbf{F}\neg\mathbf{q}\\ \hline \mathbf{T}\alpha\end{array} \qquad \begin{array}{c}\bar{\mathbf{T}}\neg\alpha\\ \hline \bar{\mathbf{F}}\alpha\end{array} \qquad \begin{array}{c}\bar{\mathbf{F}}\neg\alpha\\ \hline \bar{\mathbf{T}}\alpha\end{array}$$

$$\begin{array}{c}\mathbf{T}\alpha\wedge\beta\\ \hline \mathbf{T}\alpha\\ \mathbf{T}\beta\end{array} \qquad \begin{array}{c|c}\multicolumn{2}{c}{\mathbf{F}\alpha\wedge\beta}\\ \hline \mathbf{F}\alpha & \mathbf{F}\beta\end{array} \qquad \begin{array}{c|c}\multicolumn{2}{c}{\bar{\mathbf{T}}\alpha\wedge\beta}\\ \hline \bar{\mathbf{T}}\alpha & \bar{\mathbf{T}}\beta\end{array} \qquad \begin{array}{c}\bar{\mathbf{F}}\alpha\wedge\beta\\ \hline \bar{\mathbf{F}}\alpha\\ \bar{\mathbf{F}}\beta\end{array}$$

$$\begin{array}{c|c}\multicolumn{2}{c}{\mathbf{T}\alpha\vee\beta}\\ \hline \mathbf{T}\alpha & \mathbf{T}\beta\end{array} \qquad \begin{array}{c}\mathbf{F}\alpha\vee\beta\\ \hline \mathbf{F}\alpha\\ \mathbf{F}\beta\end{array} \qquad \begin{array}{c}\bar{\mathbf{T}}\alpha\vee\beta\\ \hline \bar{\mathbf{T}}\alpha\\ \bar{\mathbf{T}}\beta\end{array} \qquad \begin{array}{c|c}\multicolumn{2}{c}{\bar{\mathbf{F}}\alpha\vee\beta}\\ \hline \bar{\mathbf{F}}\alpha & \bar{\mathbf{F}}\beta\end{array}$$

$$\begin{array}{c|c}\multicolumn{2}{c}{\mathbf{T}\alpha\supset\beta}\\ \hline \mathbf{F}\alpha & \mathbf{T}\beta\end{array} \qquad \begin{array}{c}\mathbf{F}\alpha\supset\beta\\ \hline \mathbf{T}\alpha\\ \mathbf{F}\beta\end{array} \qquad \begin{array}{c}\bar{\mathbf{T}}\alpha\supset\beta\\ \hline \bar{\mathbf{F}}\alpha\\ \bar{\mathbf{T}}\beta\end{array} \qquad \begin{array}{c|c}\multicolumn{2}{c}{\bar{\mathbf{F}}\alpha\supset\beta}\\ \hline \bar{\mathbf{T}}\alpha & \bar{\mathbf{F}}\beta\end{array}$$

$$\begin{array}{c|c}\multicolumn{2}{c}{\mathbf{T}\alpha\equiv\beta}\\ \hline \mathbf{F}\alpha & \mathbf{T}\alpha\\ \mathbf{F}\beta & \mathbf{T}\beta\end{array} \qquad \begin{array}{c|c}\multicolumn{2}{c}{\mathbf{F}\alpha\equiv\beta}\\ \hline \mathbf{T}\alpha & \mathbf{F}\alpha\\ \mathbf{F}\beta & \mathbf{T}\beta\end{array} \qquad \begin{array}{c|c}\multicolumn{2}{c}{\bar{\mathbf{T}}\alpha\equiv\beta}\\ \hline \bar{\mathbf{F}}\alpha & \bar{\mathbf{T}}\alpha\\ \bar{\mathbf{T}}\beta & \bar{\mathbf{F}}\beta\end{array} \qquad \begin{array}{c|c}\multicolumn{2}{c}{\bar{\mathbf{F}}\alpha\equiv\beta}\\ \hline \bar{\mathbf{T}}\alpha & \bar{\mathbf{F}}\alpha\\ \bar{\mathbf{T}}\beta & \bar{\mathbf{F}}\beta\end{array}$$

Figure 1. Tableau decomposition rules for **LP**. The truth signs **T**, **F**, $\bar{\mathbf{T}}$, and $\bar{\mathbf{F}}$ are taken to mean at least true, at least false, not true and not false, respectively.

$\bar{\mathbf{T}}\alpha$	$\mathbf{T}\alpha$
$\bar{\mathbf{T}}\alpha$	$\bar{\mathbf{F}}\alpha$
$\mathbf{F}\alpha$	$\bar{\mathbf{F}}\alpha$

Figure 2. The closure conditions for **LP**. Thus, since $\bar{\mathbf{T}}$ and $\mathbf{T}$ are opposite truth signs, if both $\bar{\mathbf{T}}\alpha$ and $\mathbf{T}\alpha$ appear on a branch then that branch is closed.

Definition 1 A *true value* is one of the set $\{\mathbf{t}, \mathbf{f}, \top\}$.

Definition 2 A *truth sign* is one of the set $\{\mathbf{T}, \mathbf{F}, \bar{\mathbf{T}}, \bar{\mathbf{F}}\}$.

Definition 3 A *signed formula* is a formula together with a truth sign (e.g., $\mathbf{T}P$ means P is at least true).

Definition 4 A *valuation* (ν) is a function that maps the set of formulas to the set of truth values such that Table 1 is preserved. For example, $\nu(\alpha \wedge \beta) = \mathbf{t}$ iff $\nu(\alpha) = \mathbf{t}$ and $\nu(\beta) = \mathbf{t}$.

Definition 5 The signed formula $\mathbf{T}\alpha$ is said to *hold* in a valuation ν iff $\nu(\alpha) \in \{\mathbf{t}, \top\}$. Similarly, the signed formulas $\mathbf{F}\alpha$, $\bar{\mathbf{T}}\alpha$, and $\bar{\mathbf{F}}\alpha$ hold in a valuation ν iff $\nu(\alpha) \in \{\mathbf{f}, \top\}$, $\nu(\alpha) \in \{\mathbf{f}\}$, and $\nu(\alpha) \in \{\mathbf{t}\}$, respectively.

Definition 6 A signed formula $\mathcal{T}$ is *satisfiable* iff a valuation ν exists such that $\mathcal{T}$ holds in ν. Similarly, a set of formulas is *satisfiable* if all the formulas in it hold in a valuation ν.

Definition 7 A tableau branch is *satisfiable* iff a valuation ν exists such that each signed formula on the branch holds in ν.

Definition 8 A tableau is *satisfiable* iff at least one branch on it is satisfiable.

Definition 9 A pair of truth signs are *opposite* iff either one is $\bar{\mathbf{T}}$ and the other $\mathbf{T}$, or one is $\bar{\mathbf{T}}$ and the other $\bar{\mathbf{F}}$, or one is $\mathbf{F}$ and the other $\bar{\mathbf{F}}$. Similarly, two signed formulas are opposite iff their formulas are identical and their truth signs are opposite.

Figure 3. Example proof of P, Q ⊢ P ∧ Q (left) and open tableau for ¬P ∨ Q, P ⊢ Q (right).

Definition 10 A tableau is *closed* iff each branch on it contains at least two opposite signed formulas.

Definition 11 A tableau is *open* iff it contains a branch such that (1) no decomposition rule of Figure 1 can be applied to a signed formula on that branch to yield a signed formula not already on that branch and (2) no two signed formulas on that branch are opposite.

Lemma 1 *If a tableau $\mathcal{T}$ is satisfiable then it will also be satisfiable after the application of any of the tableau formulation rules of Figure 1.*

Proof: Let $\mathcal{T}'$ be the new tableau. Let $\mathcal{B}^*$ be the branch the rule is applied to. Since $\mathcal{T}$ is satisfiable, it must have a branch $\mathcal{B}$ that is satisfiable. We have two cases:

1. $\mathcal{B}^* \neq \mathcal{B}$. Since no rule was applied to $\mathcal{B}$, $\mathcal{B}$ will still be a branch of $\mathcal{T}'$ and thus $\mathcal{T}'$ will be satisfiable.
2. $\mathcal{B}^* = \mathcal{B}$. We show that for at least one branch produced by a rule all the new signed formulas hold. Let $\mathcal{F}$ be the signed formula the rule is applied to. Since $\mathcal{B}$ is satisfiable, $\mathcal{F}$ must hold for some valuation ν. We have twenty cases; for brevity, we will show only two:
   (a) $\mathcal{F} = \mathbf{T}\alpha \wedge \beta$. The formulas added to the branch will be $\mathbf{T}\alpha$ and $\mathbf{T}\beta$. Since $\mathbf{T}\alpha \wedge \beta$ holds, we know that $\nu(\alpha \wedge \beta) \in \{\mathbf{t}, \top\}$ and hence that $\nu(\alpha) \in \{\mathbf{t}, \top\}$ and $\nu(\beta) \in \{\mathbf{t}, \top\}$. Thus $\mathbf{T}\alpha$ and $\mathbf{T}\beta$ must also hold and thus $\mathcal{T}'$ will be satisfiable.
   (b) $\mathcal{F} = \mathbf{T}\alpha \supset \beta$. The formulas added to the two new branches will be $\mathbf{F}\alpha$ and $\mathbf{T}\beta$. We show that at least one of them will hold. Since $\mathbf{T}\alpha \supset \beta$ holds, we know that either $\nu(\alpha) \in \{\mathbf{f}, \top\}$ or $\nu(\beta) \in \{\mathbf{t}, \top\}$. Thus either $\mathbf{F}\alpha$ or $\mathbf{T}\beta$ must hold and thus $\mathcal{T}'$ will be satisfiable.

Lemma 2 *If there is a closed tableau for a set of signed formulas Γ then Γ is not satisfiable.*

Proof: Assume that Γ is satisfiable and a closed tableau exists for Γ. We show a contradiction. Let $\mathcal{T}$ be a tableau consisting of the signed formulas of Γ on a single branch. Since Γ is satisfiable, $\mathcal{T}$ will be satisfiable. By Lemma 1 every new tableau we construct from $\mathcal{T}$ will also be satisfiable, including the final closed tableau. But examination of the closure conditions for **LP** (Figure 2) shows that no closed tableau can be satisfiable. Thus we have a contradiction.

Theorem 1 *The above tableau system is sound.*

Proof: We show that if $\Gamma = \{\mathbf{T}\gamma_1, \mathbf{T}\gamma_2, \ldots, \mathbf{T}\gamma_n, \bar{\mathbf{T}}\delta\}$ has a tableau proof then $\gamma_1, \gamma_2, \ldots, \gamma_n \vdash \delta$. From Lemma 2 if Γ leads to a closed tableau then Γ is not satisfiable and hence $\gamma_1, \gamma_2, \ldots, \gamma_n \vdash \delta$ is a theorem.

Definition 12 A *signed Hintikka set* $\mathbf{H}$ is a set of signed formulas such that:
1. For any propositional letter ρ it is neither the case that both $\mathbf{T}\rho \in \mathbf{H}$ and $\bar{\mathbf{T}}\rho \in \mathbf{H}$, nor that both $\mathbf{F}\rho \in \mathbf{H}$ and $\bar{\mathbf{F}}\rho \in \mathbf{H}$, nor that both $\bar{\mathbf{T}}\rho \in \mathbf{H}$ and $\bar{\mathbf{F}}\rho \in \mathbf{H}$.
2. If $\mathbf{T}\neg\alpha \in \mathbf{H}$ then $\mathbf{F}\alpha \in \mathbf{H}$.
3. etc. (according to Table 1).

For example, $\{\mathbf{T}P \vee (Q \wedge R), \mathbf{T}P\}$ and $\{\mathbf{T}P \vee (Q \wedge R), \mathbf{T}Q \wedge R, \mathbf{T}Q, \mathbf{T}R\}$ are both Hintikka sets. It's worth noting that a signed Hintikka set can never contain both $\mathbf{T}\alpha$ and $\bar{\mathbf{T}}\alpha$, or both $\mathbf{F}\alpha$ and $\bar{\mathbf{F}}\alpha$, or both $\bar{\mathbf{T}}\alpha$ and $\bar{\mathbf{F}}\alpha$.

Lemma 3 *Every signed Hintikka set is satisfiable.*

Proof: Let $\mathbf{H}$ be a signed Hintikka set. We show that a valuation ν can always be constructed from $\mathbf{H}$ in such a way that every signed formula in $\mathbf{H}$ holds. Let ν be constructed as follows:

1. If for some propositional letter ρ, $\bar{\mathbf{T}}\rho \in \mathbf{H}$ then $\nu(\rho) = \mathbf{f}$.
2. If for some propositional letter ρ, $\bar{\mathbf{F}}\rho \in \mathbf{H}$ then $\nu(\rho) = \mathbf{t}$.
3. If for some propositional letter ρ, neither $\bar{\mathbf{T}}\rho \in \mathbf{H}$ nor $\bar{\mathbf{F}}\rho \in \mathbf{H}$ then $\nu(\rho) = \top$.
4. If $\nu(\alpha) = \mathbf{f}$ then $\nu(\neg\alpha) = \mathbf{t}$.
5. etc. (according to Table 1).

From Condition 1 of Definition 12 it follows that ν is well defined over predicate letters. By structural induction it follows that every signed formula in $\mathbf{H}$ holds in ν.

Lemma 4 *Every open branch $\mathcal{B}$ of a tableau has a corresponding signed Hintikka set containing all the signed formulas of $\mathcal{B}$.*

Proof: Since $\mathcal{B}$ is open, none of the closure conditions of Figure 2 apply for any proposition ρ and thus Condition 1 of Definition 12 must be satisfied. The other conditions follow by structural induction over signed formulas.

Theorem 2 *The above tableau system is complete.*

Proof: We need to show that if $\gamma_1, \gamma_2, \ldots, \gamma_n \vdash \delta$ then $\Gamma = \{\mathbf{T}\gamma_1, \mathbf{T}\gamma_2, \ldots, \mathbf{T}\gamma_n, \bar{\mathbf{T}}\delta\}$ produces a closed tableau. We shall show the contrapositive; i.e., if Γ produces an open tableau then $\gamma_1, \gamma_2, \ldots, \gamma_n \nvdash \delta$. Assume Γ produces an open tableau. Then it must have an open branch $\mathcal{B}$ and, by Lemma 4, $\mathcal{B}$ will have a corresponding Hintikka set $\mathbf{H}$. Thus, by Lemma 3, that set must be satisfiable and hence Γ must be satisfiable; thus $\gamma_1, \gamma_2, \ldots, \gamma_n \nvdash \delta$.

First order and modal extensions to **LP** can be handled in the usual ways. More interesting is the fact that, with a minor change, the above system also applies to Belnap's 4-valued logic (Table 2) where the original logic has been extended to include material implication $\supset$ (but not $\equiv$). Since truth value gaps are present in this logic, $\bar{\mathbf{T}}\alpha$ and $\bar{\mathbf{F}}\alpha$ can no longer be a valid closure condition. For convenience, the new closure conditions can be found in Figure 4. The sys-

$\bar{\mathbf{T}}\alpha$	$\mathbf{T}\alpha$
$\bar{\mathbf{F}}\alpha$	$\mathbf{F}\alpha$

Figure 4. The closure conditions for Belnap's 4-valued logic. Thus, since $\bar{\mathbf{T}}$ and $\mathbf{T}$ are opposite truth signs, if both $\bar{\mathbf{T}}\alpha$ and $\mathbf{T}\alpha$ appear on a branch then that branch is closed.

		$\neg\alpha$	$\alpha \wedge \beta$				$\alpha \vee \beta$				$\alpha \supset \beta$			
	β		$\bot$	**f**	**t**	$\top$	$\bot$	**f**	**t**	$\top$	$\bot$	**f**	**t**	$\top$
α	$\bot$	$\bot$	$\bot$	**f**	$\bot$	**f**	$\bot$	$\bot$	**t**	**t**	$\bot$	$\bot$	**t**	**t**
	f	**t**	**f**	**f**	**f**	**f**	$\bot$	**f**	**t**	$\top$	**t**	**t**	**t**	**t**
	t	**f**	$\bot$	**f**	**t**	$\top$	**t**	**t**	**t**	**t**	$\bot$	**f**	**t**	$\top$
	$\top$	$\top$	**f**	**f**	$\top$	$\top$	**t**	$\top$	**t**	$\top$	**t**	$\top$	**t**	$\top$

Table 2. Truth tables for the operators of Belnap's 4-valued logic, where **t** denotes just true, **f** just false, $\top$ both true and false, and $\bot$ neither true nor false. The designated truth values are **t** and $\top$.

tem is both consistent and complete with the proofs following the earlier proofs very closely.

In conclusion, the tableau system presented above is interesting in two ways. First, it provides a simple proof system for two paraconsistent logics. And, second, it demonstrates a novel use of truth signs—instead of, as is usual, having a one to one correspondence between truth signs and truth values the proof system uses truth signs to represent categories of truth values.

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