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A Takayama-type extension theorem

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Abstract

We prove a theorem on the extension of holomorphic sections of powers of adjoint bundles from submanifolds of complex codimension 1 having non-trivial normal bundle. The first such result, due to Takayama, considers the case where the canonical bundle is twisted by a line bundle that is a sum of a big and nef line bundle and a $\mathbb{Q}$-divisor that has Kawamata log terminal singularities on the submanifold from which extension occurs. In this paper we weaken the positivity assumptions on the twisting line bundle to what we believe to be the minimal positivity hypotheses. The main new idea is an L^2 extension theorem of Ohsawa–Takegoshi type, in which twisted canonical sections are extended from submanifolds with non-trivial normal bundle.

1. Introduction

Let X be a compact complex algebraic manifold, $E \to X$ a holomorphic line bundle and $Z \subset X$ a smooth codimension 1 submanifold. The main goal of this paper is to establish sufficient conditions for extending sections of the pluri-adjoint bundles $m(K_Z + E|Z)$ from Z to X.

Such an extension theorem was established by Takayama [Tak06, Theorem 4.1], in the situation where E, seen as a $\mathbb{Q}$-divisor, is a sum of a big and nef $\mathbb{Q}$-divisor and a $\mathbb{Q}$-divisor whose restriction to Z is Kawamata log terminal. (The precise statement is Theorem 2.1 below.) In this paper, we wish to weaken the positivity hypotheses on E.

The following theorem is our main result.

Theorem 1. *Let X be a projective algebraic manifold and $Z \subset X$ a smooth complex submanifold of codimension 1. Denote by $T \in H^0(X, Z)$ the canonical holomorphic section whose zero divisor is Z. Let $E, B \to X$ be holomorphic line bundles and assume that there exist singular Hermitian metrics $e^{-\varphi_Z}$, $e^{-\varphi_E}$ and $e^{-\varphi_B}$ for the line bundle associated to Z, E and B, respectively, with the following properties.*

(R) *The metrics $e^{-\varphi_Z}$, $e^{-\varphi_E}$ and $e^{-\varphi_B}$ restrict to singular Hermitian metrics on Z.*

(B) *The metric $e^{-\varphi_Z}$ satisfies the uniform bound*

$$\sup_X |T|^2 e^{-\varphi_Z} < +\infty.$$

(P) *There is an integer $\mu > 0$ such that*

$$\sqrt{-1}\,\partial\bar\partial(\varphi_E + \varphi_B) \geqslant 0 \quad \text{and} \quad \mu\sqrt{-1}\,\partial\bar\partial(\varphi_E + \varphi_B) \geqslant \sqrt{-1}\,\partial\bar\partial\varphi_Z,$$
$$\sqrt{-1}\,\partial\bar\partial(m\varphi_E + \varphi_B) \geqslant 0 \quad \text{and} \quad \mu\sqrt{-1}\,\partial\bar\partial(m\varphi_E + \varphi_B) \geqslant \sqrt{-1}\,\partial\bar\partial\varphi_Z.$$

(T) *The multiplier ideal of $(\varphi_Z + \varphi_E)|Z$ is trivial: $\mathscr{I}(e^{-(\varphi_Z+\varphi_E)}|Z) = \mathcal{O}_Z$.*

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Then every element of $H^0(Z, \mathcal{O}_Z(m(K_Z+E|Z)+B|Z)\otimes \mathscr{I}(e^{-(\varphi_Z+\varphi_E+\varphi_B)}|Z))$ extends to a global holomorphic section in $H^0(X, m(K_X+Z+E)+B)$.

Remark. When we say that a section $s\in H^0(Z, K_Z+E|Z)$ extends to a section S in $H^0(X, K_X+Z+E)$, we mean that

$$S|Z = s\wedge dT.$$

Remark. Recall that a singular Hermitian metric is a Hermitian metric for a holomorphic line bundle such that, if the metric is represented locally by $e^{-\varphi}$, then φ is $L^1_{\ell oc}$. In most situations, people deal with such metrics only when φ is plurisubharmonic. Such φ are locally bounded above. Here we have a metric in the picture that need not be plurisubharmonic, namely $e^{-\varphi_Z}$. Thus we add to our definition of singular Hermitian metrics the additional requirement that the local potentials φ be uniformly bounded above on their domain of definition.

An immediate corollary of Theorem 1 is the following result.

COROLLARY 1.1. *Let the notation and hypotheses of Theorem 1 hold. In addition, assume that the metric $e^{-(\varphi_Z+\varphi_E+\varphi_B)}|Z$ is locally integrable on Z. Then the natural restriction map*

$$H^0(X, m(K_X+Z+E)+B)\to H^0(Z, m(K_Z+E|Z)+B|Z)$$

is surjective. In particular, if $B=\mathcal{O}_X$ (in which case $\varphi_B\equiv 0$ and thus condition (P) of Theorem 1 simplifies to $\sqrt{-1}\,\partial\bar\partial\varphi_E\geqslant 0$ and $\mu\sqrt{-1}\,\partial\bar\partial\varphi_E\geqslant\varphi_Z$), then

$$H^0(X, m(K_X+Z+E))\to H^0(Z, m(K_Z+E|Z))$$

is surjective.

In § 2 we deduce Takayama's theorem from Corollary 1.1 as well as a more general algebro-geometric extension theorem (Theorems 2.1 and 2.3, respectively).

A typical way to extend sections on line bundles in algebraic geometry is through the use of vanishing theorems. In the case of Takayama's theorem, it is the Nadel vanishing theorem that provides a key step in the proof. In order to use Nadel's theorem, Takayama must assume that the divisor E above is big and nef. From the analytic perspective, Takayama's method requires that E support a singular Hermitian metric having strictly positive curvature current. It would be desirable to remove the strict positivity assumption.

The positivity assumptions in Theorem 1 are in some sense minimal, and in particular are insufficient to support the use of vanishing theorems. A technique for dealing with such a situation was initiated by Siu in [Siu02], and we are going to use a similar approach. Siu's idea was to use an L^2-extension technique that does not require strict positivity. This technique first arose in the work of Ohsawa and Takegoshi, in the context of the extension of holomorphic functions, square integrable with respect to a plurisubharmonic weight, from hyperplanes to a pseudoconvex domain in $\mathbb{C}^n$ (see [OT87]). Since that time there have been several extensions of the result. The simplest general exposition was finally given by Siu [Siu02], and used as one of two fundamental tools in establishing the celebrated deformation invariance of plurigenera. Extensions to more singular settings were established by McNeal and the present author in [MV07]. Unfortunately, to the best of the author's knowledge, none of these theorems can be used to prove our main result. The reason is that, so far, the L^2-extension theorems that have been used in extending pluricanonical sections are tailored to the case where the normal bundle of Z is trivial. In this setting, special properties of the canonical bundle allow one to obtain an extension theorem that requires only the non-negativity of the twisting line bundle.

In this paper we treat the situation in which the normal bundle of Z is not trivial. In view of the adjunction formula, which is a formula for local extension of canonical sections, we are forced to deal with the possible negative contribution from the curvature of the conormal bundle of the divisor Z. While there are results in the literature (see, e.g., [Man93, Dem01]) that handle the case in which the normal bundle of Z is non-trivial, these results are not nicely compatible with the condition (P) of Theorem 1. However, the new methods developed by McNeal and the present author can be adapted to the situation of non-trivial normal bundle so as to handle a condition like (P).

Let us be slightly more precise. Suppose that we are attempting to extend an H-twisted canonical section s on Z such that

$$\int_Z |s|^2 e^{-\kappa} < +\infty \quad \text{and} \quad s \wedge dT \in \mathscr{I}(e^{-(\varphi_Z+\kappa)}|Z)$$

for some singular Hermitian metric $e^{-\kappa}$. Here T is the canonical holomorphic section for the line bundle associated to Z, whose zero divisor is Z. The need for a metric $e^{-\varphi_Z}$ and the multiplier ideal requirement arises from the adjunction formula. It turns out that the technique we use requires the following curvature condition:

(C) $\sqrt{-1}\,\partial\bar{\partial}\kappa \geqslant 0$ and for some positive integer μ, $\mu\sqrt{-1}\,\partial\bar{\partial}\kappa \geqslant \sqrt{-1}\,\partial\bar{\partial}\varphi_Z$.

Under such a condition, we establish in § 3 an L^2 extension theorem, namely Theorem 2, which we consider to be the main new contribution of the present paper.

Ultimately, the extension theorem is used in the following way. Suppose that we are given a singular metric $e^{-\varphi_Z}$ and a global section T of the line bundle associated to the divisor Z, metrics $e^{-\varphi_E}$ and $e^{-\varphi_B}$ for the line bundles $E \to X$ and $B \to X$, respectively, and a section $s \in H^0(Z, m(K_Z + E|Z) + B|Z)$ such that

$$\int_Z |s|^2 \omega^{-n(m-1)} e^{-((m-1)\gamma_E+\varphi_E+\varphi_B)} < +\infty \quad \text{and} \quad s \wedge dT \in \mathscr{I}(e^{-(\varphi_Z+\varphi_E+\varphi_B)}|Z).$$

Then we seek to construct a singular metric $e^{-\psi}$ such that

$$\int_Z |s|^2 e^{-\psi} < +\infty \quad \text{and} \quad s \wedge dT \in \mathscr{I}(e^{-(\varphi_Z+\psi)}|Z).$$

If this metric satisfies condition (C), then the extension theorem will give an extension of the section s to a holomorphic section S of $m(K_X + Z + E) + B$ over X such that $S|Z = s \wedge dT$.

Siu's idea is to use the extension theorem not only to extend s, but also to construct $e^{-\psi}$. However, to get the construction off the ground, one needs to twist certain line bundles by a sufficiently positive line bundle. Then a limiting process is used to eliminate this positive line bundle, through the method of taking powers and roots.

Our construction of the metric $e^{-\psi}$ is substantially shortened by the use of a new method of Paun introduced in [Pau05]. Paun has eliminated the need for an effective global generation of multiplier ideal sheaves, which was a long and difficult part of Siu's approach.

Finally we should mention the paper [Cla07] of Claudon, which at the time of writing of this paper was very recent, but preceded the present article. Claudon handles the case where the normal bundle of Z is trivial. For this purpose, one can use the version of Ohsawa–Takegoshi due to Siu in [Siu02].

2. Algebro-geometric corollaries of Theorem 1

In this section we derive corollaries of Theorem 1 that can be phrased in terms of more algebro-geometric properties of E and Z.

2.1 Takayama's theorem

As mentioned in the introduction, Theorem 1 was motivated by the desire to generalize the following theorem of Takayama.

THEOREM 2.1 [Tak06, Theorem 4.1]. *Let X be a complex projective manifold, $Z \subset X$ a complex submanifold of codimension 1 and E an integral divisor on X. Assume that $E \sim_{\mathbb{Q}} A + D$ for some big and nef Q-divisor A and some effective $\mathbb{Q}$-divisor D such that Z is in A-general position and $Z \not\subset \mathrm{Support}(D)$ and the pair $(Z, D|Z)$ is klt. Then the natural restriction map*

$$H^0(X, m(K_X + Z + E)) \to H^0(Z, m(K_Z + E|Z))$$

is surjective.

Remark. Recall that if D is a $\mathbb{Q}$-divisor on Z, then:

(i) the multiplier ideal $\mathscr{I}(D)$ is the multiplier ideal for the metric $e^{-\log|s_D|^2}$, where s_D is the multi-section with $\mathbb{Q}$-divisor D; and

(ii) one says that the pair (X, D) is *klt* (Kawamata log terminal) if $\mathscr{I}(D) = \mathcal{O}_X$.

(We say that s is a multi-section of a $\mathbb{Q}$-divisor D if there is an integer $m > 0$ such that mD is a $\mathbb{Z}$-divisor and $s^m \in H^0(X, mD)$.)

Theorem 2.1 is a corollary of Theorem 1. To see this, we argue as follows. It is not hard to see (cf. [Tak06, § 4]) that we may assume without loss of generality that A is an ample $\mathbb{Q}$-divisor. Thus, we can construct a singular metric $e^{-\varphi_E}$ of positive curvature for E as follows: take a multiple mA that is very ample and let s_D be the canonical multi-section of D whose $\mathbb{Q}$-divisor is D. Then we set

$$\varphi_E = \log|s_D|^2 + \log\left(\sum_{j=1}^{N} |s_j|^{2/m}\right),$$

where $s_1, \ldots, s_N$ is a basis for $H^0(X, mA)$. Since Z is not contained in the support of D, $e^{-\varphi_E}$ restricts to Z as a well-defined singular metric. Fix any smooth metric $e^{-\varphi_Z}$ for Z and let $B = \mathcal{O}_X$ and $\varphi_B \equiv 0$. Then evidently hypotheses (P) and (T) of Theorem 1 are satisfied. Moreover, the multiplier ideal $\mathscr{I}(e^{-(\varphi_E + \varphi_Z)}|Z)$ is supported away from Z because $(Z, D|Z)$ is klt. Thus, Theorem 2.1 follows.

2.2 More general conditions on E

In Theorem 2.1, we would like to remove the hypothesis that E is big. In some sense, this is achieved in Theorem 1. (Indeed, the desire to handle the case where E is not necessarily big forms the initial impetus for the present article.) However, as we mentioned, we would like to state a result that uses more intrinsic properties of the divisors, rather than a result that includes a choice of metrics.

DEFINITION 2.2. Let L be an integral divisor on X. For each integer $k > 0$, fix bases

$$s_1^{(k)}, \ldots, s_{N_k^L}^{(k)} \in H^0(X, kL).$$

Then define

$$\psi_L = \log \sum_{k=1}^{\infty} \varepsilon_k \left(\sum_{j=1}^{N_k^L} |s_j^{(k)}|^2\right)^{2/k},$$

where $\varepsilon_k > 0$ are small enough to make the sum converge. We extend the definition to $\mathbb{Q}$-divisors L by setting

$$\psi_L = \frac{1}{m}\psi_{mL},$$

where $m > 0$ is the smallest integer such that mL is an integral divisor.

Remark. Metrics of the form $e^{-\psi_L}$ have the following property: for every integer $m > 0$ such that mL is integral, the natural inclusion

$$H^0(X, mL \otimes \mathscr{I}(e^{-m\psi_L})) \to H^0(X, mL)$$

is an isomorphism. Indeed, if for all $m > 0$, $H^0(X, mL) = \{0\}$, there is nothing to prove. On the other hand, if $\sigma = \sum c^j s_j^{(m)} \in H^0(X, mL)$ then

$$|\sigma|^2 e^{-m\psi_L} \lesssim \frac{|\sigma|^2}{|s_1^{(m)}|^2 + \cdots + |s_{N_m}^{(m)}|^2} \leqslant \sum |c^j|^2$$

is bounded and thus integrable. Thus, in some sense, the metrics $e^{-\psi_L}$ have minimal singularities.

Recall that the set theoretic base locus $\mathrm{Bs}(|L|)$ of an integral divisor L is the common zero locus of all holomorphic sections of the line bundle associated to L. For a more thorough discussion of this and many other matters in algebraic geometry, as well as a more algebro-geometric approach to multiplier ideals, see [Laz04].

Theorem 2.3. *Let X be a projective algebraic manifold, $Z \subset X$ a smooth divisor and E an integral divisor on X. Assume that one can write $E \sim_{\mathbb{Q}} E_1 + E_2$ such that the following properties hold.*

($\mathrm{P_a}$) *For some $\mu \in \mathbb{N}$ such that μE_1 is integral,*

$$\mathrm{Bs}(|\mu E_1|) \cup \mathrm{Bs}(|\mu E_1 - Z|) = \emptyset.$$

($\mathrm{T_a}$) *The singular metric $e^{-\psi_{E_2}}$ restricts to a singular metric on Z, and*

$$\mathscr{I}(e^{-\psi_{E_2}}|Z) = \mathcal{O}_Z.$$

Then the restriction map

$$H^0(X, m(K_X + Z + E)) \to H^0(Z, m(K_Z + E|Z))$$

is surjective.

Proof of Theorem 2.3 from Theorem 1. Let

$$\zeta = \psi_{\mu E_1} \quad \text{and} \quad \eta = \psi_{\mu E_1 - Z}.$$

By hypothesis ($\mathrm{P_a}$), the curvature currents of the singular metrics $e^{-\zeta}$ and $e^{-\eta}$ for μE_1 and $\mu E_1 - Z$, respectively, are non-negative and smooth. Let

$$\varphi_Z := \zeta - \eta, \quad \varphi_{E_1} = \frac{1}{\mu}\zeta \quad \text{and} \quad \varphi_E := \varphi_{E_1} + \psi_{E_2}.$$

Note the following.

(i) The metrics φ_{E_1}, φ_E and $\mu\varphi_E - \varphi_Z = \mu\psi_{E_2} + \eta$ have non-negative curvature currents.

(ii) The metrics φ_Z and ψ_{E_1} are smooth, and thus $\mathscr{I}(e^{-(\varphi_Z+\varphi_E)}|Z) = \mathscr{I}(e^{-\psi_{E_2}}|Z) = \mathcal{O}_Z$ by hypothesis ($\mathrm{T_a}$).

Now take $B = \mathcal{O}_X$, $\varphi_B \equiv 0$. The metrics $e^{-\varphi_Z}$, $e^{-\varphi_E}$ and $e^{-\varphi_B}$ satisfy the hypotheses of Theorem 1, and moreover

$$\mathscr{I}(e^{-(\varphi_Z+\varphi_E+\varphi_B)}|Z) = \mathcal{O}_Z.$$

We thus obtain Theorem 2.3. □

Remark. If one can write $E \sim_{\mathbb{Q}} E_1 + E_2$ where E_1 is an ample $\mathbb{Q}$-divisor and $(Z, E_2|Z)$ is klt, then properties $(\mathrm{P_a})$ and $(\mathrm{T_a})$ certainly hold. We thus recover Theorem 2.1.

Remark. A natural way in which the hypotheses of Theorem 2.3 might arise is the following. Suppose that X and Y are projective manifolds, $\dim_{\mathbb{C}} Y > \dim_{\mathbb{C}} X$, and $\pi : Y \to X$ is a holomorphic map whose fibers have constant dimension. Let Z and E be divisors on X such that (X, Z, E) satisfy the hypotheses of Takayama's theorem (Theorem 2.1). Then the hypotheses of Theorem 2.3 hold for π^*E and π^*Z on Y.

3. Extension with L^2 estimates

In this section we discuss the extension of twisted canonical sections from a codimension 1 submanifold. We begin with the local extension, which is the adjunction formula, and then pass to L^2 extension in the presence of certain minimal positivity hypotheses. If the positivity hypotheses are made strict, we obtain a proof of L^2 extension by older methods.

3.1 Restriction of the canonical bundle

Let Z be a smooth complex hypersurface in a Kähler manifold Y and let $T \in H^0(Y, Z)$ be a section whose zero divisor is Z. Since the canonical bundle is the determinant of the cotangent bundle, one has

$$K_Y|Z = K_Z + N_Z^*,$$

where N_Z^* denotes the conormal bundle of Z in Y, i.e. the annihilator of T_Z. The adjunction formula, which amounts to saying that dT is a nowhere zero section of $N_Z^* + Z$ over Z, tells us that the line bundle associated to the divisor Z restricts, along Z, to the normal bundle N_Z of Z in Y. Equivalently,

$$(K_Y + Z)|Z = K_Z.$$

More locally, if s is a local section of K_Z, then $s \wedge dT$ is a local section of $(K_Y + Z)|Z$.

3.2 Ohsawa–Takegoshi-type L^2-extension theorem

3.2.1 *Statement of the theorem.* The setting of the theorem is the following. Let Y be a Kähler manifold of complex dimension n. Assume that there exists an analytic hypersurface $V \subset Y$ such that $Y - V$ is Stein. Thus, there are relatively compact subsets $\Omega_j \subset\subset Y - V$ such that

$$\Omega_j \subset\subset \Omega_{j+1} \quad \text{and} \quad \bigcup_j \Omega_j = Y - V.$$

Examples of such manifolds are Stein manifolds (where V is empty) and projective algebraic manifolds (where one can take V to be the intersection of Y with a projective hyperplane in some projective space in which Y is embedded).

Fix a smooth hypersurface $Z \subset Y$ such that $Z \not\subset V$.

Theorem 2. *Suppose that we are given a holomorphic line bundle $H \to Y$ with a singular Hermitian metric $e^{-\kappa}$ and a singular Hermitian metric $e^{-\varphi_Z}$ for the line bundle associated to the divisor Z, such that the following properties hold.*

(i) *The restriction $e^{-(\kappa+\varphi_Z)}|Z$ is a singular metric for $Z + H$.*

(ii) *There is a global holomorphic section $T \in H^0(Y, Z)$ such that*

$$Z = \{T = 0\} \quad \textit{and} \quad \sup_Y |T|^2 e^{-\varphi_Z} = 1.$$

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(iii) *We have* $\sqrt{-1}\,\partial\bar\partial\kappa \geqslant 0$ *and there is an integer* $\mu > 0$ *such that* $\mu\sqrt{-1}\,\partial\bar\partial\kappa \geqslant \sqrt{-1}\,\partial\bar\partial\varphi_Z$.

Then for every $s \in H^0(Z, K_Z + H)$ *such that*

$$\int_Z |s|^2 e^{-\kappa} < +\infty \quad \text{and} \quad s \wedge dT \in \mathscr{I}(e^{-(\varphi_Z+\kappa)}|Z),$$

there exists a section $S \in H^0(Y, K_Y + Z + H)$ *such that*

$$S|Z = s \wedge dT \quad \text{and} \quad \int_Y |S|^2 e^{-(\varphi_Z+\kappa)} \leqslant 40\pi\mu \int_Z |s|^2 e^{-\kappa}.$$

3.2.2 *The twisted basic estimate.* Let Ω be a smoothly bounded pseudoconvex domain in our Kähler manifold Y. Let τ and A be positive smooth functions on Ω, and let $e^{-\psi}$ be a singular metric for $E + Z$ over Ω. The following lemma is well known (see, e.g., [MV07]).

Lemma 3.1. *For any* $(n, 1)$*-form* u *in the domain of the adjoint* $\bar\partial^*_\psi$ *of* $\bar\partial$*, the following inequality holds.*

$$\begin{aligned}\int_\Omega (\tau + A)|\bar\partial^*_\psi u|^2 e^{-\psi} &+ \int_\Omega \tau |\bar\partial u|^2 e^{-\psi} \\ &\geqslant \int_\Omega \Big(\tau\sqrt{-1}\,\partial\bar\partial\psi - \sqrt{-1}\,\partial\bar\partial\tau - \frac{1}{A}\sqrt{-1}\,\partial\tau\wedge\bar\partial\tau\Big)(u, u) e^{-\psi}. \qquad (1)\end{aligned}$$

3.2.3 *Choices for* τ*,* A *and* ψ*, and an a priori estimate.* We follow the approach of [MV07].

First, we set

$$\tau = a + h(a) \quad \text{and} \quad A = \frac{(1 + h'(a))^2}{-h''(a)},$$

where

$$h(x) = 2 - x + \log(2e^{x-1} - 1)$$

and $a\colon \Omega \to [1, \infty)$ is a function to be chosen shortly. Observe that for $x \geqslant 1$,

$$h'(x) = \frac{1}{2e^{x-1} - 1} \in (0, 1) \quad \text{and} \quad h''(x) = \frac{-2e^{x-1}}{(2e^{x-1} - 1)^2} < 0,$$

and thus since $1 + \log r \geqslant 1/r$ when $r \geqslant 1$,

$$\tau \geqslant 1 + h'(a).$$

Moreover, $A > 0$, which is necessary in our choice of A. We also take this opportunity to note that

$$A = 2e^{a-1}.$$

With these choices of τ and A, we have

$$\begin{aligned}-\partial_\alpha\partial_{\bar\beta}\tau - \frac{\partial_\alpha\tau\overline{\partial_\beta\tau}}{A} &= -\partial_\alpha((1 + h'(a))\partial_{\bar\beta} a) - \frac{(1 + h'(a))^2\partial_\alpha a\overline{\partial_\beta a}}{A} \\ &= (1 + h'(a))(-\partial_\alpha\partial_{\bar\beta} a). \qquad (2)\end{aligned}$$

Our next task is to construct the function a. To this end, define

$$v = \log|T|^2 - \varphi_Z.$$

We note that $v \leqslant 0$. Fix a constant $\gamma > 1$. We define the function a to be

$$a = a_\varepsilon := \gamma - \frac{1}{\mu}\log(e^v + \varepsilon^2),$$

where $\varepsilon > 0$ is chosen to be so small that $a \geqslant 1$. Later we will let ε go to 0 and $\gamma \to 1$.

We calculate that

$$
\begin{aligned}
-\sqrt{-1}\,\partial\bar{\partial}a &= \frac{\sqrt{-1}}{\mu}\partial\bar{\partial}\log(e^v+\varepsilon^2)\\
&= \frac{e^v}{\mu(e^v+\varepsilon^2)}\sqrt{-1}\,\partial\bar{\partial}v + \frac{4\varepsilon^2\sqrt{-1}\,\partial(e^{v/2})\wedge\bar{\partial}(e^{v/2})}{\mu((e^{v/2})^2+\varepsilon^2)^2}\\
&= -\frac{1}{\mu}\frac{e^v}{(e^v+\varepsilon^2)}\sqrt{-1}\,\partial\bar{\partial}\varphi_Z + \frac{4\varepsilon^2\sqrt{-1}\,\partial(e^{v/2})\wedge\bar{\partial}(e^{v/2})}{\mu((e^{v/2})^2+\varepsilon^2)^2}.
\end{aligned}
$$

In the last equality we have used the fact that

$$\sqrt{-1}\,\partial\bar{\partial}v = \pi[Z] - \sqrt{-1}\,\partial\bar{\partial}\varphi_Z,$$

where $[Z]$ is the current of integration over Z. The term involving the current of integration vanishes because $e^v|Z \equiv 0$.

It remains to choose the metric $e^{-\psi}$. We take

$$\psi = \kappa + \log|T|^2.$$

Then

$$
\begin{aligned}
&\tau\sqrt{-1}\,\partial\bar{\partial}\psi - \sqrt{-1}\,\partial\bar{\partial}\tau - \frac{\sqrt{-1}\,\partial\tau\wedge\bar{\partial}\tau}{A}\\
&\quad= \tau\sqrt{-1}\,\partial\bar{\partial}\kappa + \pi[Z] + (1+h'(a))\bigg(-\frac{e^v}{\mu(e^v+\varepsilon^2)}\sqrt{-1}\,\partial\bar{\partial}\varphi_Z + \frac{4\varepsilon^2\sqrt{-1}\,\partial(e^{v/2})\wedge\bar{\partial}(e^{v/2})}{\mu((e^{v/2})^2+\varepsilon^2)^2}\bigg)\\
&\quad\geqslant \frac{4\varepsilon^2\sqrt{-1}\,\partial(e^{v/2})\wedge\bar{\partial}(e^{v/2})}{\mu((e^{v/2})^2+\varepsilon^2)^2}.
\end{aligned}
$$

The inequality follows from assumption (i) and the fact that $\tau \geqslant 1+h'(a) \geqslant 1$. Combining with (1), we obtain the following lemma.

Lemma 3.2. *Let $T = \bar{\partial}\circ\sqrt{\tau+A}$ and $S = \sqrt{\tau}\bar{\partial}$. Then for any $(n,1)$-form u in the domain of the adjoint T^*, the following inequality holds:*

$$\int_\Omega |\langle u, \bar{\partial}(e^{v/2})\rangle|^2 \frac{4\varepsilon^2}{\mu(e^v+\varepsilon^2)^2}e^{-\psi} \leqslant (\|T^*u\|^2_\psi + \|Su\|^2_\psi).$$

Remark. Note that the metric $e^{-\psi}$ is non-negatively curved, and thus by standard results (see, for example, [Hor90]) the smooth forms lying in the union of the domains of T^* and S are dense in that union.

3.2.4 *A smooth extension and its holomorphic correction.* Since Ω is Stein, we can extend $s\wedge dT$ to a $Z+H$-valued holomorphic n-form $\tilde{s}$ on Ω. By extending to a Stein neighborhood of Ω (which exists by hypothesis) we may also assume that

$$\int_\Omega |\tilde{s}|^2 e^{-(\varphi_Z+\kappa)} < +\infty.$$

(Here we have used the local integrability of $|s\wedge dT|^2 e^{-(\varphi_Z+\kappa)}$ on Z.) Of course, we have no better estimate on this $\tilde{s}$. In particular, the estimate could degenerate as Ω grows.

In order to tame the growth of this extension $\tilde{s}$, we first modify it to a smooth extension. To this end, let $\delta > 0$ and let $\chi \in \mathcal{C}^\infty_0([0,1))$ be a cutoff function with values in $[0,1]$ such that

$$\chi \equiv 1 \text{ on } [0,\delta] \quad\text{and}\quad |\chi'| \leqslant 1+\delta.$$

We write

$$\chi_\varepsilon := \chi\left(\frac{e^v}{\varepsilon^2}\right).$$

We distinguish the smooth $(n,1)$-form with values in $Z+H$

$$\alpha_\varepsilon := \bar\partial \chi_\varepsilon \tilde{s}.$$

Then one has the estimate

$$\begin{aligned}
|(u,\alpha_\varepsilon)_\psi|^2 &\leqslant \left(\int_\Omega |\langle u,\alpha_\varepsilon\rangle| e^{-\psi}\right)^2 \\
&= \left(\int_\Omega \left|\left\langle u, \chi'\left(\frac{e^v}{\varepsilon^2}\right)\tilde{s}\wedge \frac{2e^{v/2}\bar\partial(e^{v/2})}{\varepsilon^2}\right\rangle\right| e^{-\psi}\right)^2 \\
&\leqslant \mu\int_\Omega \left|\frac{\tilde{s}}{\varepsilon^2}\chi'\left(\frac{e^v}{\varepsilon^2}\right)\right|^2 \frac{(e^v+\varepsilon^2)^2}{\varepsilon^2} e^{-(\psi-v)} \int_\Omega |\langle u,\bar\partial(e^{v/2})\rangle|^2 \frac{4\varepsilon^2}{\mu(e^v+\varepsilon^2)^2} e^{-\psi} \\
&\leqslant C_\varepsilon(\|T^*u\|_\psi^2 + \|Su\|_\psi^2)
\end{aligned}$$

where the last inequality follows from Lemma 3.2, and

$$C_\varepsilon := \frac{4\mu(1+\delta)^2}{\varepsilon^2}\int_{e^v\leqslant\varepsilon^2} |\tilde{s}|^2 e^{-(\varphi_Z+\kappa)} \xrightarrow{\varepsilon\to 0} 8\pi\mu(1+\delta)^2\int_Z |s|^2 e^{-\kappa}.$$

As a result of this estimate, we obtain the following theorem.

THEOREM 3.3. *There exists a smooth n-form β_ε such that*

$$T\beta_\varepsilon = \alpha_\varepsilon \quad \text{and} \quad \int_\Omega |\beta_\varepsilon|^2 e^{-\psi} \leqslant C_\varepsilon.$$

In particular,

$$\beta_\varepsilon | Z \equiv 0.$$

Proof. The inequality

$$|(u,\alpha_\varepsilon)| \leqslant C_\varepsilon(\|T^*u\|_\psi^2 + \|Su\|_\psi^2)$$

implies the continuity of the linear functional

$$\ell\colon T^*u \mapsto (u,\alpha_\varepsilon)_\psi$$

on Image$(T^*|\text{Kernel}(S))$. By letting $\ell\equiv 0$ on Image$(T^*|\text{Kernel}(S))^\perp$, we can assume, without increasing the norm, that ℓ is defined on the whole Hilbert space. The Riesz representation theorem then gives a solution to $T\beta_\varepsilon = \alpha_\varepsilon$ with the stated norm inequality on β_ε. The smoothness of β_ε follows from elliptic regularity.

It remains only to show that $\beta_\varepsilon|Z\equiv 0$. However, one may notice that ψ is at least as singular as $\log|T|^2$, and thus $e^{-\psi}$ is not locally integrable at any point of Z. The desired vanishing of β_ε follows. □

Conclusion of the proof of Theorem 2. We note first that

$$\begin{aligned}
\lim_{\varepsilon\to 0}\sup_\Omega(e^v(\tau+A)) &= \lim_{\varepsilon\to 0}\sup_\Omega e^{\mu(\gamma-a)}(2+\log(2e^{a-1}-1)+2e^{a-1}) \\
&\leqslant \sup_{x\geqslant 1} e^{\mu(\gamma-x)}(3+2e^{x-1}) \\
&= 5e^{\mu(\gamma-1)}.
\end{aligned}$$

Next let

$$S_\varepsilon := \chi_\varepsilon\tilde{s} - \sqrt{\tau+A}\,\beta_\varepsilon.$$

Then S_ε is a holomorphic section, $S_\varepsilon|Z = \tilde{s}|Z = s \wedge dT$ and we have the estimate

$$\int_\Omega |S_\varepsilon|^2 e^{-(\varphi_Z+\kappa)} \leqslant 5e^{\mu(\gamma-1)}(1+o(1))8\pi\mu(1+\delta)^2 \int_Z |s|^2 e^{-\kappa} \quad \text{as } \varepsilon \to 0.$$

Here we have used the estimate obtained in Theorem 3.3. The term $o(1)$ comes in because $\chi_\varepsilon \tilde{s}$ is a smooth, uniformly bounded section whose support approaches a set of measure zero, and thus the integral of the square norm of $\chi_\varepsilon \tilde{s}$ converges to zero.

Now, by the sub-mean value property, uniform L^2-estimates with weights $e^{-\xi}$ with ξ locally bounded above implies locally uniform sup-norm estimates. It follows from Montel's theorem that S_ε converges to a holomorphic section S. Evidently

$$S|Z \cap \Omega = s \wedge dT$$

and

$$\int_\Omega |S|^2 e^{-(\varphi_Z+\kappa)} \leqslant 5e^{\gamma-1} \times 8\pi\mu(1+\delta)^2 \int_Z |s|^2 e^{-\kappa}.$$

Finally, by the uniformity of all estimates, we may let $\delta \to 0$, $\gamma \to 1$ and then $\Omega \to Y$. The proof is complete. □

3.3 Extension by classical methods in the case of strictly positive curvature

When we wish to extend canonical sections with values in a line bundle equipped with a singular metric of strictly positive curvature, the Ohsawa–Takegoshi extension technique is not needed. Instead one can appeal to an older method called the Hörmander–Bombieri–Skoda technique of solving the $\bar\partial$-equation with singular weighted-L^2 estimates. Of course, this result is insufficient for the proof of our main theorem, but it can be used for instance to prove Takayama's theorem (Theorem 2.1).

Remark. The result and method of proof of this section are of independent interest, but are not germane to the main theorem of the paper. The reader who is interested only in the proof of the main theorem should skip ahead to § 4.

Let Y be a compact complex algebraic manifold and $Z \subset Y$ a smooth complex hypersurface.

Theorem 3. *Suppose that we are given a holomorphic line bundle $H \to Y$ with a singular Hermitian metric $e^{-\kappa}$ and a smooth Hermitian metric $e^{-\varphi_Z}$ for the line bundle associated to the divisor Z, such that the following properties hold.*

(i) *The restriction $e^{-(\kappa+\varphi_Z)}|Z$ is a singular metric for $Z + H$.*

(ii) *There is a global holomorphic section $T \in H^0(Y, Z)$ such that*

$$\sup_Y |T|^2 e^{-\varphi_Z} = 1.$$

(iii) *There is a constant $c > 0$ such that $\sqrt{-1}\,\partial\bar\partial\kappa \geqslant c$.*

Then there exists a constant C depending on Z and Y but not on H, with the following property. For every $s \in H^0(Z, K_Z + H)$ such that

$$\int_Z |s|^2 e^{-\kappa} < +\infty \quad \text{and} \quad s \wedge dT \in \mathscr{I}(e^{-(\varphi_Z+\kappa)}) \cdot \mathcal{O}_Z$$

there exists a section $S \in H^0(Y, K_Y + Z + H)$ such that

$$S|Z = s \wedge dT \quad \text{and} \quad \int_Y |S|^2 e^{-(\varphi_Z+\kappa)} \leqslant \frac{C}{c} \int_Z |s|^2 e^{-\kappa}.$$

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Remark. We emphasize that the method of proof we give here relies heavily on the assumption (iii) of strict positivity. This result is insufficient for the proof of our main theorem.

The approach we take to prove Theorem 3 is as follows. We first extend the section s locally, with uniform estimates. The local extensions need not agree, so we correct their discrepancy by solving a Cousin I problem with good L^2 bounds.

3.3.1 *Local extension.*

Lemma 3.4. *Let U be a smoothly bounded open set in Y such that:*

(a) *U is biholomorphic to the unit ball; and*

(b) *$Z \cap U$ is the hyperplane $z^n = 0$ in the coordinates on the ball U.*

Let $U' \subset\subset U$ be the ball of radius $1/2$. Then for every section $s \in H^0(U \cap Z, K_Z + H)$ such that

$$\int_{Z\cap U} |s|^2 e^{-\kappa} < +\infty \quad \text{and} \quad s \wedge dT \in \mathscr{I}(e^{-(\varphi_Z+\kappa)}|Z \cap U'),$$

there exists a section $\tilde{s} \in H^0(U', K_X + Z + H)$ such that

$$\tilde{s}|(U' \cap Z) = s_o \wedge dT \quad \text{and} \quad \int_{U'} |\tilde{s}|^2 e^{-(\varphi_Z+\kappa)} \leqslant C_o \int_{Z\cap U'} |s|^2 e^{-\kappa}.$$

The constant C_o is independent of s.

One could deduce a stronger result directly from the classical Ohsawa–Takegoshi extension theorem in [OT87], but Lemma 3.4 is far more elementary. We leave the proof as an exercise to the reader.

3.3.2 *An L^2 Cousin I problem.* Let us cover Y by a finite number of coordinate unit balls U_j, such that the concentric balls V_j of radius $1/2$ also cover Y. Suppose that we have sections $\tilde{s}_j \in H^0(V_j, K_X + Z + H)$ such that

$$\tilde{s}_j|V_j \cap Z = s \wedge dT|V_j \cap Z.$$

(If $V_j \cap Z = \emptyset$, then the restriction condition is vacuously satisfied.) Consider the cocycle

$$G_{ij} = \tilde{s}_i - \tilde{s}_j \quad \text{on } U_i \cap U_j.$$

Note that

$$\int_{V_i\cap V_j} |G_{ij}|^2 e^{-(\varphi_Z+\kappa)} \leqslant C \int_{V_j\cup V_j\cap Z} |s|^2 e^{-\kappa} \quad \text{and} \quad G_{ij}|U_i \cap U_j \cap Z = 0.$$

Thus, since $dT|Z \neq 0$, we see that

$$G_{ij}/T = f_{ij} \in H^0(U_i \cap U_j, K_Y + H)$$

and

$$\int_{V_i\cap V_j} |G_{ij}|^2 e^{-(\kappa+\log|T|^2)} = \int_{V_i\cap V_j} |f_{ij}|^2 e^{-\kappa}.$$

We seek holomorphic sections $g_j \in H^0(V_j, K_Y + Z + H)$ such that

$$\int_{V_j} |g_j|^2 e^{-(\varphi_Z+\kappa)} \leqslant K_1 \int_Z |s|^2 e^{-(\varphi_Z+\kappa)},$$
$$g_i - g_j = G_{ij} \quad \text{and} \quad g_i|V_j \cap Z = 0.$$

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If such sections are found, then the section $s \in H^0(Y, K_Y + Z + H)$ defined by

$$s = s_i - g_i \quad \text{on } V_i$$

gives the proof of Theorem 3.

3.3.3 *Solution of the Cousin I problem.* We use the convention that the constant C may change from line to line. We leave it to the reader to check that all uses of constants in the estimates do not depend on the section s, but only on properties of Y and Z.

Let $\{\chi_j\}$ be a partition of unity subordinate to the open cover $\{V_j\}$. Consider the smooth sections

$$h_j = \sum_k G_{jk}\chi_k.$$

Then

$$h_i - h_j = \sum_k (G_{ik} + G_{kj})\chi_k = \sum_k G_{ij}\chi_k = G_{ij}.$$

It follows that the differential $(0,1)$-form α with values in $K_Y + Z + H$ defined by

$$\alpha = \bar\partial h_j \quad \text{on } U_j$$

is well-defined. Moreover,

$$\int_Y |\alpha|^2 e^{-(\kappa + \log|T|^2)} \leqslant C \sum_{i,j} \int_{V_i \cap V_j} |f_{ij}|^2 e^{-\kappa}.$$

Recalling that V_j is the ball of radius $1/2$, let $W_j \subset V_j$ denote the spherical shells of inner radius $1/4$ and outer radius $1/2$. Then

$$\begin{aligned}
\int_{V_i \cap V_j} |f_{ij}|^2 e^{-\kappa} &\leqslant C \int_{W_i \cap W_j} |f_{ij}|^2 e^{-\kappa} \\
&\leqslant C \int_{W_j \cap W_j} |T|^2 |f_{ij}|^2 e^{-(\varphi_Z + \kappa)} \\
&\leqslant C \int_{V_i \cap V_j} |G_{ij}|^2 e^{-(\varphi_Z + \kappa)} \\
&\leqslant C \int_{V_i \cup V_j} |s|^2 e^{-\kappa}.
\end{aligned}$$

(The first inequality follows from an appropriate application of the sub-mean value property.) Thus,

$$\int_Y |\alpha|^2 e^{-(\kappa + \log|T|^2)} \leqslant C \int_Z |s|^2 e^{-\kappa}.$$

By Hörmander's theorem, there is a smooth section u such that

$$\bar\partial u = \alpha \quad \text{and} \quad \int_Y |u|^2 e^{-(\varphi_Z + \kappa)} \leqslant \int_Y |u|^2 e^{-(\kappa + \log|T|^2)} \leqslant \frac{C}{c} \int_Z |s|^2 e^{-\kappa},$$

where $\sqrt{-1}\,\partial\bar\partial\kappa > c\omega$, and in the first inequality we have used $|T|^2 e^{-\varphi_Z} \leqslant 1$. Observe that the second inequality forces u to vanish along Z.

We let $g_i = h_i - u$. Then g_i are holomorphic, vanish along $V_j \cap Z$, and satisfy the estimate

$$\int_{V_i} |g_i|^2 e^{-(\varphi_Z + \kappa)} \leqslant \frac{C}{c} \int_Z |s|^2 e^{-(\varphi_Z + \kappa)}.$$

The proof of Theorem 3 is complete. ☐

4. Proof of Theorem 1

For the rest of the paper, we normalize our canonical section T of the line bundle associated to Z, so that

$$\sup_X |T|^2 e^{-\varphi_Z} = 1.$$

Remark. Let $s \in H^0(Z, \mathcal{O}_Z(m(K_Z + E|Z) + B|Z) \otimes \mathscr{I}(e^{-(\varphi_B+\varphi_B+\varphi_Z)}|Z))$ be the section to be extended. We note that, in fact,

$$\int_Z |s|^2 \omega^{-(n-1)(m-1)} e^{-(m-1)\gamma_E} e^{-(\varphi_E+\varphi_B)} < +\infty,$$

because

$$e^{-(\varphi_E+\varphi_B)} \leqslant \Big(\sup_Z e^{\varphi_Z-\gamma_Z}\Big) e^{\gamma_Z} e^{-(\varphi_E+\varphi_B+\varphi_Z)} \leqslant C e^{\gamma_Z} e^{-(\varphi_E+\varphi_B+\varphi_Z)}.$$

The last inequality follows because, by our convention, the local potentials of singular metrics are locally uniformly bounded above.

4.1 Paun's induction

Fix a holomorphic line bundle $A \to X$ that is sufficiently positive so as to have the following property:

(GG) for each $0 \leqslant p \leqslant m-1$ the global sections $H^0(X, p(K_X + Z + E) + A)$ generate the sheaf $\mathcal{O}_X(p(K_X + Z + E) + A)$.

Let us fix bases

$$\{\tilde{\sigma}_j^{(p)}; 1 \leqslant j \leqslant N_p\}$$

of $H^0(X, p(K_X + Z + E) + A)$. We let $\sigma_j^{(p)} \in H^0(Z, p(K_Z + E|Z) + A|Z)$ be such that

$$\tilde{\sigma}_j^{(p)}|Z = \sigma_j^{(p)} \wedge (dT)^{\otimes p}.$$

We also fix smooth metrics $e^{-\gamma_Z}$, $e^{-\gamma_E}$ and $e^{-\gamma_B}$ for $Z \to X$, $E \to X$ and $B \to X$, respectively.

Proposition 4.1. *There exist a constant $C < +\infty$ and sections*

$$\{\tilde{\sigma}_j^{(km+p)} \in H^0(X, (km+p)(K_X + Z + E) + kB + A); 1 \leqslant j \leqslant N_p\}_{0 \leqslant p \leqslant m-1, k=0,1,2,\dots}$$

with the following properties.

(a) *We have* $\tilde{\sigma}_j^{(mk+p)}|Z = s^{\otimes k} \otimes \sigma_j^{(p)} \wedge (dT)^{(km+p)}$

(b) *If* $k \geqslant 1$,

$$\int_X \frac{\sum_{j=1}^{N_0} |\tilde{\sigma}_j^{(mk)}|^2 e^{-(\gamma_Z+\gamma_E+\gamma_B)}}{\sum_{j=1}^{N_{m-1}} |\tilde{\sigma}_j^{(mk-1)}|^2} \leqslant C.$$

(c) *For* $1 \leqslant p \leqslant m-1$,

$$\int_X \frac{\sum_{j=1}^{N_p} |\tilde{\sigma}_j^{(mk+p)}|^2 e^{-(\gamma_Z+\gamma_E)}}{\sum_{j=1}^{N_{p-1}} |\tilde{\sigma}_j^{(mk+p-1)}|^2} \leqslant C.$$

Proof. (Double induction on k and p.) Fix a constant $\widehat{C}$ such that

$$\sup_X \frac{\sum_{j=1}^{N_0} |\tilde{\sigma}_j^{(0)}|^2 \omega^{n(m-1)} e^{(m-1)(\gamma_Z+\gamma_E)}}{\sum_{j=1}^{N_{m-1}} |\tilde{\sigma}_j^{(m-1)}|^2} \leqslant \widehat{C}$$

and

$$\sup_Z \frac{\sum_{j=1}^{N_0} |\sigma_j^{(0)}|^2 \omega^{(n-1)(m-1)} e^{(m-1)\gamma_E}}{\sum_{j=1}^{N_{m-1}} |\sigma_j^{(m-1)}|^2} \leqslant \widehat{C},$$

and for all $0 \leqslant p \leqslant m-2$,

$$\sup_X \frac{\sum_{j=1}^{N_{p+1}} |\tilde{\sigma}_j^{(p+1)}|^2 \omega^{-n} e^{-(\gamma_Z+\gamma_E)}}{\sum_{j=1}^{N_p} |\tilde{\sigma}_j^{(p)}|^2} \leqslant \widehat{C},$$

and

$$\sup_Z \frac{\sum_{j=1}^{N_{p+1}} |\sigma_j^{(p+1)}|^2 \omega^{-(n-1)} e^{-\gamma_E}}{\sum_{j=1}^{N_p} |\sigma_j^{(p)}|^2} \leqslant \widehat{C}.$$

($k=0$.) As far as extension there is nothing to prove. Note that

$$\int_X \frac{\sum_{j=1}^{N_p} |\tilde{\sigma}_j^{(p)}|^2 e^{-(\gamma_Z+\gamma_E)}}{\sum_{j=1}^{N_{p-1}} |\tilde{\sigma}_j^{(p-1)}|^2} \leqslant \widehat{C} \int_X \omega^n.$$

($k \geqslant 1$.) Assume the result has been proved for $k-1$.

($p=0$.) Consider the sections $s^{\otimes k} \otimes \sigma_j^{(0)}$, and define the semi-positively curved metric

$$\psi_{k,0} := \log \sum_{j=1}^{N_{m-1}} |\tilde{\sigma}_j^{(km-1)}|^2$$

for the line bundle $(mk-1)(K_X+Z+E)+(k-1)B+A$. Observe that locally,

$$\begin{aligned} |(s \wedge dT^m)^k \otimes \sigma_j^{(0)}|^2 e^{-(\varphi_Z+\psi_{k,0}+\varphi_E+\varphi_B)} &= |s \wedge dT^m|^2 \frac{|\sigma_j^{(0)}|^2 e^{-(\varphi_Z+\varphi_E+\varphi_B)}}{\sum_{j=1}^{N_{m-1}} |\sigma_j^{(m-1)}|^2} \\ &\lesssim |s|^2 e^{-(\varphi_Z+\varphi_E+\varphi_B)}. \end{aligned}$$

Moreover, we have

$$\mu\sqrt{-1}\,\partial\bar{\partial}(\psi_{k,0}+\varphi_E+\varphi_B) \geqslant \max(\sqrt{-1}\,\partial\bar{\partial}\varphi_Z, 0).$$

Finally,

$$\int_Z |s^k \otimes \sigma_j^{(0)}|^2 e^{-(\psi_{k,0}+\varphi_E+\varphi_B)} = \int_Z |s|^2 \frac{|\sigma_j^{(0)}|^2 e^{(m-1)\gamma_E} e^{-((m-1)\gamma_E+\varphi_E+\varphi_B)}}{\sum_{j=1}^{N_{m-1}} |\sigma_j^{(m-1)}|^2} < +\infty.$$

In view of the remark at the beginning of § 4, we may thus apply Theorem 2 to obtain sections

$$\tilde{\sigma}_j^{(km)} \in H^0(X, mk(K_X+Z+E)+kB+A), \quad 1 \leqslant j \leqslant N_0$$

such that

$$\tilde{\sigma}_j^{(km)}|Z = s^{\otimes k} \otimes \sigma_j^{(0)} \wedge (dT)^{\otimes km}, \quad 1 \leqslant j \leqslant N_0,$$

and

$$\int_X |\tilde{\sigma}_j^{(km)}|^2 e^{-(\psi_{k,0}+\varphi_Z+\varphi_E+\varphi_B)} \leqslant 40\pi\mu \int_Z |s|^2 \frac{|\sigma_j^{(0)}|^2 e^{-(\varphi_E+\varphi_B)}}{\sum_{j=1}^{N_{m-1}} |\sigma_j^{(m-1)}|^2}.$$

Summing over j, we obtain

$$\begin{aligned}
&\int_X \frac{\sum_{j=1}^{N_o} |\tilde{\sigma}_j^{(km)}|^2 e^{-(\gamma_Z+\gamma_E+\gamma_B)}}{\sum_{j=1}^{N_{m-1}} |\tilde{\sigma}_j^{(km-1)}|^2} \\
&\quad \leqslant \sup_X e^{\varphi_Z+\varphi_E+\varphi_B-\gamma_Z-\gamma_E-\gamma_B} \int_X \frac{\sum_{j=1}^{N_o} |\tilde{\sigma}_j^{(km)}|^2 e^{-(\varphi_Z+\varphi_E+\varphi_B)}}{\sum_{j=1}^{N_{m-1}} |\tilde{\sigma}_j^{(km-1)}|^2} \\
&\quad \leqslant 40\pi \sup_X e^{\varphi_Z+\varphi_E+\varphi_B-\gamma_Z-\gamma_E-\gamma_B} \int_Z |s|^2 \frac{\sum_{j=1}^{N_0} |\sigma_j^{(0)}|^2 e^{-(\varphi_E+\varphi_B)}}{\sum_{j=1}^{N_{m-1}} |\sigma_j^{(m-1)}|^2} e^{-\kappa} \\
&\quad \leqslant 40\pi \widehat{C} \sup_X e^{\varphi_Z+\varphi_E+\varphi_B-\gamma_Z-\gamma_E-\gamma_B} \int_Z |s|^2 \omega^{-(n-1)(m-1)} e^{-((m-1)\gamma_E+\varphi_E+\varphi_B)}.
\end{aligned}$$

($1 \leqslant p \leqslant m-1$.) Assume that we have obtained the sections $\tilde{\sigma}_j^{(km+p-1)}$, $1 \leqslant j \leqslant N_{p-1}$. Consider the non-negatively curved singular metric

$$\psi_{k,p} := \log \sum_{j=1}^{N_{p-1}} |\tilde{\sigma}_j^{(mk+p-1)}|^2$$

for $(km+p-1)(K_X+Z+E)+kB+A$. We have

$$|s^k \otimes \sigma_j^{(p)}|^2 e^{-(\varphi_Z+\psi_{k,p}+\varphi_E)} = \frac{|\sigma_j^{(p)}|^2 e^{-(\varphi_Z+\varphi_E)}}{\sum_{j=1}^{N_{p-1}} |\sigma_j^{(p-1)}|^2} \lesssim e^{-(\varphi_Z+\varphi_E)},$$

which is locally integrable by the hypothesis (T). Next,

$$\begin{aligned}
\int_Z |s^k \otimes \sigma_j^{(p)}|^2 e^{-(\psi_{k,p}+\varphi_E)} &= \int_Z \frac{|\sigma_j^{(p)}|^2 e^{-\varphi_E}}{\sum_{j=1}^{N_{p-1}} |\sigma_j^{(p-1)}|^2} \\
&\leqslant C^\star \int_Z e^{\gamma_Z} \frac{|\sigma_j^{(p)}|^2 e^{-(\varphi_Z+\varphi_E)}}{\sum_{j=1}^{N_{p-1}} |\sigma_j^{(p-1)}|^2} < +\infty,
\end{aligned}$$

where

$$C^\star := \sup_Z e^{\varphi_Z-\gamma_Z}.$$

By Theorem 2 there exist sections

$$\tilde{\sigma}_j^{(km+p)} \in H^0(X, (mk+p)(K_X+Z+E)+kB+A), \quad 1 \leqslant j \leqslant N_0$$

such that

$$\tilde{\sigma}_j^{(km+p)}|Z = s^{\otimes k} \otimes \sigma_j^{(p)} \wedge (dT)^{\otimes km+p}, \quad 1 \leqslant j \leqslant N_p,$$

and

$$\int_X |\tilde{\sigma}_j^{(km+p)}|^2 e^{-(\psi_{k,p}+\varphi_Z+\varphi_E)} \leqslant 40\pi\mu \int_Z \frac{|\sigma_j^{(p)}|^2 e^{-\varphi_E}}{\sum_{j=1}^{N_{p-1}} |\sigma_j^{(p-1)}|^2}.$$

Summing over j, we obtain

$$\int_X \frac{\sum_{j=1}^{N_p} |\tilde{\sigma}_j^{(km+p)}|^2 e^{-(\gamma_Z+\gamma_E)}}{\sum_{j=1}^{N_{p-1}} |\tilde{\sigma}_j^{(km+p-1)}|^2} \leqslant 40\pi\mu \sup_X e^{\varphi_Z+\varphi_E-\gamma_Z-\gamma_E} \widehat{C} \int_Z e^{-\varphi_E} \omega^{n-1}.$$

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Letting C be the maximum of the numbers

$$\begin{aligned}
&\widehat{C}\int_X \omega^n,\\
&40\pi\widehat{C}\sup_X e^{\varphi_Z+\varphi_E+\varphi_B-\gamma_Z-\gamma_E-\gamma_B}\int_Z |s|^2\omega^{-n-1(m-1)}e^{-((m-1)\gamma_E+\varphi_E+\varphi_B)}\\
\text{and}\quad &40\pi\mu\sup_X e^{\varphi_Z+\varphi_E-\gamma_Z-\gamma_E}\widehat{C}\int_Z e^{-\varphi_E}\omega^{n-1}
\end{aligned}$$

completes the proof. □

4.2 Siu's construction of the metric

Fix a smooth metric $e^{-\psi}$ for $A \to X$. Consider the functions

$$\lambda_N := \log\sum_{j=1}^{N_p} |\tilde{\sigma}_j^{(km+p)}|^2\omega^{-n(mk+p)}e^{-(km(\gamma_Z+\gamma_E)+k\gamma_B+\psi)},$$

where $N = km + p$. We then have the following lemma.

Lemma 4.2. *For any non-empty open subset $V \subset X$ and any smooth function $f : \overline{V} \to \mathbb{R}_+$,*

$$\frac{1}{\int_V f\omega^n}\int_V (\lambda_N - \lambda_{N-1})f\omega^n \leqslant \log\biggl(\frac{C\sup_V f}{\int_V f\omega^n}\biggr).$$

Proof. Observe that by Proposition 4.1, there exists a constant C such that for any open subset $V \subset X$,

$$\int_V (e^{\lambda_N-\lambda_{N-1}})f\omega^n \leqslant C\sup_V f.$$

The lemma follows from an application of (the concave version of) Jensen's inequality to the concave function log. □

Consider the function

$$\Lambda_k = \frac{1}{k}\lambda_{mk}.$$

Note that Λ_k is locally the sum of a plurisubharmonic function and a smooth function. By applying Lemma 4.2 and using the telescoping property, we see that for any open set $V \subset X$ and any smooth function $f : \overline{V} \to \mathbb{R}_+$,

$$\frac{1}{\int_V f\omega^n}\int_V \Lambda_k f\omega^n \leqslant m\log\biggl(\frac{C\sup_V f}{\int_V f\omega^n}\biggr). \tag{3}$$

Proposition 4.3. *There exists a constant C_o such that*

$$\Lambda_k(x) \leqslant C_o, \quad x \in X.$$

Proof. Let us cover X by coordinate charts $V_1, \ldots, V_N$ such that for each j there is a biholomorphic map F_j from V_j to the ball $B(0,2)$ of radius 2 centered at the origin in $\mathbb{C}^n$, and such that if $U_j = F_j^{-1}(B(0,1))$, then $U_1, \ldots, U_N$ is also an open cover. Let $W_j = V_j \setminus F_j^{-1}(B(0,3/2))$.

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Now, on each V_j, Λ_k is the sum of a plurisubharmonic function and a smooth function. Say $\Lambda_k = h + g$ on V_j, where h is plurisubharmonic and g is smooth. Then for constant A_j we have

$$\begin{aligned}\sup_{U_j} \Lambda_k &\leqslant \sup_{U_j} g + \sup_{U_j} h \\ &\leqslant \sup_{U_j} g + A_j \int_{W_j} h \cdot F_{j*}\, dV \\ &\leqslant \sup_{U_j} g - A_j \int_{W_j} g \cdot F_{j*}\, dV + A_j \int_{W_j} \Lambda_k \cdot F_{j*}\, dV.\end{aligned}$$

Let

$$C_j := \sup_{U_j} g - A_j \int_{W_j} g \cdot F_{j*}\, dV$$

and define the smooth function f_j by

$$f_j \omega^n = F_{j*}\, dV.$$

Then by (3) applied with $V = W_j$ and $f = f_j$, we have

$$\sup_{U_j} \Lambda_k \leqslant C_j + mA_j \log\left(\frac{C \sup_{W_j} f_j}{\int_{W_j} f_j \omega^n}\right) \int_{W_j} f_j \omega^n.$$

Letting

$$C_o := \max_{1 \leqslant j \leqslant N} \left\{ C_j + mA_j \log\left(\frac{C \sup_{W_j} f_j}{\int_{W_j} f_j \omega^n}\right) \int_{W_j} f_j \omega^n \right\}$$

completes the proof. □

As the upper regularization of the lim sup of a uniformly bounded sequence of plurisubharmonic functions is plurisubharmonic (see, e.g., [Hor90, Theorem 1.6.2]), we essentially have the following corollary.

Corollary 4.4. *The function*

$$\Lambda(x) := \limsup_{y \to x}\, \limsup_{k \to \infty} \Lambda_k(y)$$

is locally the sum of a plurisubharmonic function and a smooth function.

Proof. One need only observe that the function Λ_k is obtained from a singular metric on the line bundle $m(K_X + Z + E) + B$ (this singular metric $e^{-\kappa_k}$ is described below) by multiplying by a fixed smooth metric of the dual line bundle. □

Consider the singular Hermitian metric $e^{-\kappa}$ for $m(K_X + Z + E) + B$ defined by

$$e^{-\kappa} = e^{-\Lambda} \omega^{-nm} e^{-(m(\gamma_Z + \gamma_E) + \gamma_B)}.$$

This singular metric is given by the formula

$$e^{-\kappa(x)} = \exp\Big(-\limsup_{y \to x}\, \limsup_{k \to \infty} \kappa_k(y)\Big),$$

where

$$e^{-\kappa_k} = e^{-\Lambda_k} \omega^{-nm} e^{-(m(\gamma_Z + \gamma_E) + \gamma_B)}.$$

The curvature of $e^{-\kappa_k}$ is thus

$$\begin{aligned}\sqrt{-1}\,\partial\bar{\partial}\kappa_k &= \frac{\sqrt{-1}}{k}\partial\bar{\partial}\log\sum_{j=1}^{N_0}|\tilde{\sigma}_j^{(mk)}|^2 - \frac{1}{k}\sqrt{-1}\,\partial\bar{\partial}\psi \\ &\geqslant -\frac{1}{k}\sqrt{-1}\,\partial\bar{\partial}\psi.\end{aligned}$$

We claim next that the curvature of $e^{-\kappa}$ is non-negative. To see this, it suffices to work locally. Then we have that the functions

$$\kappa_k + \frac{1}{k}\psi$$

are plurisubharmonic. However,

$$\limsup_{y\to x}\ \limsup_{k\to\infty}\kappa_k + \frac{1}{k}\psi = \limsup_{y\to x}\ \limsup_{k\to\infty}\kappa_k = \kappa.$$

It follows that κ is plurisubharmonic, as desired.

4.3 Conclusion of the proof

Note that, after identifying K_Z with $(K_X+Z)|Z$ by dividing by dT,

$$\kappa_k|Z = \log|s|^2 + \frac{1}{k}\log\sum_{j=1}^{N_0}|\sigma_j^{(0)}|^2.$$

Thus, we obtain

$$e^{-\kappa}|Z = \frac{1}{|s|^2}.$$

It follows that

$$\begin{aligned}&\int_Z |s|^2\exp\biggl[-\frac{(m-1)\kappa + m\varphi_E + \varphi_B}{m}\biggr] \\ &\quad= \int_Z |s|^{2/m}\exp\biggl[-\frac{((m-1)+1)\varphi_E + \varphi_B}{m}\biggr] \\ &\quad= \int_Z |s|^{2/m}\exp\biggl[-\frac{(m-1)(\varphi_E-\gamma_E) + (m-1)\gamma_E + \varphi_E + \varphi_B}{m}\biggr] \\ &\quad\leqslant \biggl(\int_Z e^{\gamma_E-\varphi_E}\omega^{n-1}\biggr)^{(m-1)/m}\biggl(\int_Z |s|^2\omega^{-(n-1)(m-1)}e^{-(\varphi_E+(m-1)\gamma_E+\varphi_B)}\biggr)^{1/m} \\ &\quad< +\infty,\end{aligned}$$

where the first inequality is a consequence of Hölder's inequality. Next, working locally on Z and identifying sections and metric with functions, we have

$$\begin{aligned}&|s\wedge dT|^2\exp\biggl[-\biggl(\varphi_Z + \frac{(m-1)\kappa + m\varphi_E + \varphi_B}{m}\biggr)\biggr] \\ &\qquad\sim |s|^{2/m}\exp\biggl[-\frac{\varphi_Z+\varphi_E+\varphi_B}{m}\biggr]\exp\biggl[-\frac{m-1}{m}(\varphi_Z+\varphi_E)\biggr].\end{aligned}$$

Now, by another application of Hölder's Inequality, we have (locally on Z) that

$$\begin{aligned}&\int |s|^{2/m}\exp\biggl[-\frac{\varphi_Z+\varphi_E+\varphi_B}{m}\biggr]\exp\biggl[-\frac{m-1}{m}(\varphi_Z+\varphi_E)\biggr] \\ &\qquad\leqslant \biggl(\int |s|^2 e^{-(\varphi_Z+\varphi_E+\varphi_B)}\biggr)^{1/m}\times\biggl(\int e^{-(\varphi_Z+\varphi_E)}\biggr)^{(m-1)/m},\end{aligned}$$

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and thus we obtain the local integrability of

$$|s|^{2/m} \exp\left[-\frac{\varphi_Z + \varphi_E + \varphi_B}{m}\right] \exp\left[-\frac{m-1}{m}(\varphi_Z + \varphi_E)\right].$$

Finally,

$$\mu m\sqrt{-1}\,\partial\bar{\partial}\left(\frac{m-1}{m}\kappa + \frac{m\varphi_E + \varphi_B}{m}\right) \geqslant \mu\sqrt{-1}\,\partial\bar{\partial}(m\varphi_E + \varphi_B) \geqslant \max(\sqrt{-1}\,\partial\bar{\partial}\varphi_Z, 0).$$

An application of Theorem 2 completes the proof of Theorem 1.

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