A Taste of Jordan Algebras

Kevin McCrimmon

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Kevin McCrimmon
Department of Mathematics
University of Virginia
Kerchof Hall, Cabell Drive
Charlottesville, VA 22904-4137
USA
kmm4m@virginia.edu

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Dedicated to the memory of Jake and Florie

To Jake (the Doktor-Vater) for his mathematical influence on my research, and to Florie (the Doktor-Mutter) for helping me (and all Jake's students) to get to know him as a warm human being. Future histories of mathematics should take into account the role of Doktor-Mutters in the fostering of mathematics.

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Preface

On several occasions I and colleagues have found ourselves teaching a one-semester course for students at the second year of graduate study in mathematics who want to gain a general perspective on Jordan algebras, their structure, and their role in mathematics, or want to gain direct experience with nonassociative algebra. These students typically have a solid grounding in first–year graduate algebra and the Artin–Wedderburn theory of associative algebras, and a few have been introduced to Lie algebras (perhaps even Cayley algebras, in an offhand way), but otherwise they have not seen any nonassociative algebras. Most of them will not go on to do research in nonassociative algebra, so the course is not primarily meant to be a training or breeding ground for research, though the instructor often hopes that one or two will be motivated to pursue the subject further.

This text is meant to serve as an accompaniment to such a course. It is designed first and foremost to be *read* by students on their own without assistance by a teacher. It is a direct mathematical conversation between the author and a reader whose mind (as far as nonassociative algebra goes) is a tabula rasa. In keeping with the tone of a private conversation, I give more heuristic and explanatory comment than is usual in graduate texts at this level (pep talks, philosophical pronouncements on the proper way to think about certain concepts, historical anecdotes, mention of some mathematicians who have contributed to our understanding of Jordan algebras, etc.), and employ a few English words which do not standardly appear in mathematical works. I have tried to capture to colloquial tone and rhythm of an oral presentation, and have not felt bound (to my copy editor's chagrin) to always adhere to the formal "rules" of English grammar.

This book tells the story of one aspect of Jordan structure theory: the origin of the theory in an attempt by quantum physicists to find algebraic systems more general than hermitian matrices, and ending with the surprising proof by Efim Zel'manov that there is really only one such system, the 27-dimensional Albert algebra, much too small to accommodate quantum mechanics. I try to give students a feeling for the historical development of the subject, so they will realize that mathematics is an *ongoing process*, but I have not tried to write a history of Jordan theory; I mention people only as they contribute directly to the particular story I am telling, so I have had to leave out many important colleagues. I also try to give students a sense of how the subject of Jordan structures has become intertwined with other aspects of mathematics, so students will realize that mathematics *does not develop in isolation*; I describe some applications outside algebra which have been enriched by Jordan theory, and which have influenced in turn purely algebraic developments, but I have not tried to give a compendium of applications, and I have had to leave out many important ones.

It is important for the reader to develop a visceral intuitive feeling for the living subject. The reader should see isomorphisms as cloning maps, isotopes as subtle rearrangements of an algebra's DNA, radicals as pathogens to be isolated and removed by radical surgery, annihilators as biological agents for killing off elements, Peircers as mathematical enzymes ("Jordan-ase") which break an algebra down into its Peirce spaces. Like Charlie Brown's kite-eating trees, Jordan theory has Zel'manov's tetrad-eating ideals (though we shall stay clear of these carnivores in our book). The reader must think of both mathematicians and abstract ideas as active participants in the theory. Just as the mathematicians have proper names that need to be recognized, so too the results need to be appreciated and remembered. To this end, I have christened all statements (theorems, examples, definitions, etc.) and basic equations with a proper name (using capital letters as with ordinary proper names). Instead of saying "by Lemma 21.2.1(1), which of course you will remember," I say "by Nuclear Slipping 21.2.1(1)," hoping to trigger long-repressed memories of a formula for how nuclear elements of alternative algebras slip in and out of associators. The reader should get on a first-name basis with these characters in our story, and be able to comfortably use locutions like "Nuclear Slipping says that such-and-such holds".

While I wind up doing most of the talking, there is some room in Parts II and III for the reader to participate (and stay mathematically fit) by doing exercises. The Exercises give slight extensions, or alternate proofs, of results in the text, and are placed immediately after the results; they give practice in proving variations on the previous mathematical theme. At the end of each chapter I gather a few problems and questions. The Problems usually take the form "Prove that something-or-other"; they involve deeper investigations or lengthier digressions than exercises, and develop more extensive proof skills on a new theme. The Questions are more open-ended, taking the form "What can you say about something-or-other" without giving a hint as to which way the answer goes; they develop proof skills in uncharted territories, in composing a mathematical theme from scratch (most valuable for budding researchers). Hints are given at the back of the book for the starred exercises, problems, and questions (though these should be consulted only after a good-faith effort to prove them).

The Introduction *A Colloquial Survey of Jordan Theory* is in the nature of an extended colloquium talk, a brief survey of the life and times of Jordan algebras, to provide appreciation of the role Jordan algebras play on the broader stage of mathematics. It is divided into eight sections: the origin of the species, the genus of related Jordan structures, and links to six other areas of mathematics (Lie algebras, differential geometry, Riemannian symmetric spaces, bounded symmetric domains, functional analysis, and projective geometry). Since the students at this level cannot be assumed to be familiar with all these areas, the description has to be a bit loose; readers can glean from this part just enough respect and appreciation to sanction and legitimate their investment in reading further. There are no direct references to this material in the rest of the book.

Part I *A Historical Survey of Jordan Structure Theory* is designed to provide an overview of Jordan structure theory in its historical context. It gives a general historical survey, divided chronologically into eight chapters, from the origins in quantum mechanics in 1934 to Efim Zel'manov's breathtaking description of arbitrary simple algebras in 1983 (which later played a role in his Fields Medal work on the Burnside Problem). I give precise definitions and examples, but no proofs, except in the last chapter where I give brief sketches of Zel'manov's revolutionary proof techniques. In keeping with its nature as a survey, I have not included any exercises.

In contrast to the Introduction, the definitions and results in the Survey will be recur in Parts II and III when material from the Survey is being restated. These restatements not only make Part II and Part III fully independent units, capable of serving as course texts, but the repetition itself helps solidify the material in students' minds. All statements (theorems, examples, definitions, etc.) and named equations throughout the book have been christened with a proper name, and readers should try to remember statements by their verbal mnemonic tags. When material from the Survey is being repeated, this will be quietly noted in a footnote. I have been careful to try to keep the same name as in the Survey. Hopefully the name itself will trigger memory of the result, but a numerical tag is included in the reference to help locate the result when the mnemonic tag has not been memorable enough. For the purpose of navigating back to the tagged location, each chapter in the Survey is divided into numbered sections.

Part II *The Classical Theory* and Part III *Zel'manov's Exceptional Theorem* are designed to provide direct experience with nonassociativity, and either one (in conjunction with Part I) could serve as a basis for a one-semester course. Throughout, I stick to linear Jordan algebras over rings of scalars containing $\frac{1}{2}$, but give major emphasis to the quadratic point of view.

The Classical Part gives a development of Jacobson's classical structure theory for nondegenerate Jordan algebras with capacity, in complete detail and with full proofs. It is suitable for a one-semester course aiming to introduce students to the methods and techniques of nonassociative algebra. The (sometimes arcane) details of Peirce decompositions, Peirce relations, and coordinatization theorems are the key tools leading to Jacobson's Classical Structure Theory for Jordan algebras with capacity. The assumption of nondegeneracy allows me to avoid a lengthy discussion of radicals and the passage from a general algebra to a semisimple one.

Zel'manov's Part gives a full treatment of his Exceptional Theorem, that the only simple i-exceptional Jordan algebras are the Albert algebras, closing the historical search for an exceptional setting for quantum mechanics. This part is much more concerned with understanding and translating to the Jordan setting some classical ideas of associative theory, including primitivity; it is suitable for a one-semester course aiming to introduce students to the modern methods of Jordan algebras. The ultrafilter argument, that if primitive systems come in only a finite number of flavors then a prime system must come in one of those pure flavors, is covered in full detail; ultrafilters provide a useful tool that many students at this level are unacquainted with. Surprisingly, though the focus is entirely on prime and simple algebras, along the way we need to introduce and characterize several different radicals. Due to their associative heritage, the techniques in this Part seem more intuitive and less remote than the minutiae of Peirce decompositions.

The book contains five appendices. The first three establish important results whose technical proofs would have disrupted the narrative flow of the main body of the text. We have made free use of these results in Parts II and III, but their proofs are long, combinatorial or computational, and do not contribute ideas and methods of proof which are important for the mainstream of our story. These are digressions from the main path, and should be consulted only after gaining a global picture. A hypertext version of this book would have links to the appendices which could only be opened after the main body of text had been perused at least once. Appendix A *Cohn's Special Theorems* establishes the useful Shirshov–Cohn Theorem which allows us to prove results involving only two elements entirely within an associative context. Appendix B *Macdonald's Theorem* establishes Macdonald's Theorem that likewise reduces verification of operator identities in two variables to an associative setting. Appendix C *Jordan Algebras of Degree* 3 gives detailed proofs that the constructions of cubic factors in Section II.4 do indeed produce Jordan algebras. I have made the treatment of strict simplicity in Part II and prime dichotomy in Part III independent of the Density Theorem, but their proofs could have been streamlined using this powerful result; in Appendix D *The Jacobson–Bourbaki Density Theorem* I give a proof of this theorem. A fifth Appendix E *Hints* gives hints to selected exercises and problems (indicated by an asterisk).

In addition to the appendices, I include several indexes. The first index is a very brief *Index of Collateral Reading*, listing several standard reference books in Jordan theory and a few articles mentioned specifically in the text. In keeping with the book's purpose as a textbook, I do not attempt a detailed bibliography of monographs and research articles; students wishing to pursue a topic of research in more detail will be guided by an advisor to the relevant literature.

A *Pronouncing Index of Names* lists the mathematicians mentioned in the book, and gives references to places where their work is mentioned (but not to every occurrence of a theorem named after them). In addition, it gives a phonetic description of how to pronounce their name correctly — a goal more to strive for than to achieve. (In preparing the guide I have learned to my chagrin that I have been mispronouncing my colleagues' names for years: Let not the sins of the father pass on to the children!)

An *Index of Notations* contains symbols other than words which are defined in the text, with a helpful but brief description of their meaning, and a reference to the location of their formal introduction. An *Index of Named Statements* provides an alphabetical list of the given names of all statements or equations, with a reference to the location of their statement, but does restate them or list all references to them in the text. All other boldface terms are collected in a final *Index of Definitions*, where again reference is given only to their page of definition.

I have dedicated the book to the memory of Nathan and Florie Jacobson, both of whom passed away during this book's long gestation period. They had an enormous influence on my mathematical development. I am greatly indebted to my colleague Kurt Meyberg, who carefully read through Part II and made many suggestions which vastly improved the exposition. I am also deeply indebted to my colleague Wilhelm Kaup, who patiently corrected many of my misconceptions about the role of Jordan theory in differential geometry, improving the exposition in Part I and removing flagrant errors. My colleague John Faulkner helped improve my discussion of applications to projective geometries. I would also like to thank generations of graduate students at Virginia who read and commented upon the text, especially my students Jim Bowling, Bernard Fulgham, Dan King, and Matt Neal. Several colleagues helped correct my pronunciation of the names of foreign mathematicians. Finally, I wish to thank David Kramer for his careful and illuminating copyediting of the manuscript, and Michael Koy of Springer-Verlag for his patience as editor.

Charlottesville, Virginia, USA *Kevin McCrimmon*

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Standard Notation

General Typographical Conventions

• Rings of *scalars* (unital, commutative, associative rings) are indicated by capital Greek letters Φ, Ω. Individual scalars are denoted by lowercase Greek letters: $\alpha, \beta, \gamma, \ldots$. Our algebraic systems will be algebras or modules over a fixed ring of scalars Φ which usually contains an element $\frac{1}{2}$.

• Mere *sets* are indicated by italic capital letters X, Y, Z at the end of the alphabet, index sets also by I, J, S. The *cardinality* of the set X is denoted by $|X|$. The *power set* $\mathcal{P}(X)$ is the set of all subsets of the set X.

• Modules and linear *spaces* are denoted by italic capital letters: A, B, C, J, V, $W, \ldots$. The *zero subspace* will be denoted by boldface $\mathbf{0}$ to distinguish it from the vector 0 and the scalar 0. This signals a subtle and not-too-important distinction between the set $\mathbf{0} = \{0\}$ consisting of a single element zero, and the *element* itself.

• *Algebraic systems* are denoted by letters in small caps: general linear algebras by $\mathrm{A, B, C}$, ideals by $\mathrm{I, J, K}$. *Associative algebras* are indicated by D when they appear as coordinates for Jordan algebras. *Jordan algebras* are indicated by $\mathrm{J}, \mathrm{J}_i, \mathrm{J}'$, etc.

• Maps or *functions* between sets or spaces are denoted by italic lowercase letters $f, g, h, \ldots$; *morphisms* between algebraic systems often by lowercase Greek letters $\varphi, \sigma, \tau, \rho$, sometimes uppercase italic letters T, S. The *restriction* of a mapping $f : X \to Y$ to a subset $U \subseteq X$ is denoted by $f|_U$. The *range* $f(X)$ of some function on a set X will always be a set, while the *value* $f(x)$ will be an element. The range or *image* of a map f is denoted by $\mathrm{Im}(f)$, while the *kernel* of a (linear) map is denoted by $\mathrm{Ker}(f)$. *Injective* maps are denoted by $\hookrightarrow$, *surjective* maps by $\twoheadrightarrow$; the notation $T \leftrightarrow S$ is used to indicate that the operators T, S *commute.*

• *Functors* and functorial constructions are denoted by script capital letters $\mathcal{F}$, $\mathcal{G}$, or by abbreviations beginning with a script capital (e.g., $\mathcal{D}er(\mathrm{A})$, $\mathcal{A}ut(\mathrm{A})$).

• Blackboard bold is used for the standard systems $\mathbb{N}$ (natural numbers $1, 2, \ldots$), the even-more-natural numbers $\mathbb{I}$ ($0, 1, 2, \ldots$ used as indices or cardinals), $\mathbb{Z}$ (the ring of integers), the fields $\mathbb{Q}$ (rational numbers), $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), the real division rings $\mathbb{H}$ (Hamilton's quaternions), $\mathbb{K}$ (Cayley's [nonassociative] octonions), $\mathbb{O}$ (split octonions), and $\mathbb{A}$ (the Albert algebra, a formally real exceptional Jordan algebra).

Specific Typographical Notation

• The *identity map* on a set X is denoted by $\mathbb{1}_X$, the $n \times n$ *identity matrix* by $\mathbb{1}_{nn}$. The *projection* of a set on a quotient set is denoted by $\pi : X \to X/\sim$; the *coset* or *equivalence class* of $x \in X$ is denoted by $\pi(x) = \bar{x} = [x]$.

• *Cartesian products* of sets are denoted by $X \times Y$. *Module direct sums* are indicated by $V \oplus W$. *Algebra direct sums, where multiplication as well as addition is performed componentwise, are written* $\mathrm{A} \boxplus \mathrm{B}$ *to distinguish them from mere module direct sums.*

• *Subsets* are denoted by $X \subseteq Y$, with *strict inclusion* denoted by $X \subset Y$. Subspaces of linear spaces are denoted by $\mathrm{B} \leq \mathrm{A}$, with *strict inclusion* $\mathrm{B} < \mathrm{A}$. Two-sided, left, and *right ideals* are indicated by $\triangleleft, \triangleleft_\ell, \triangleleft_r$, e.g., $\mathrm{B} \triangleleft \mathrm{A}$.

• *Isomorphic* algebraic systems are denoted by $\mathrm{A} \cong \mathrm{B}$. The linear transformations from Φ-modules V to W are denoted by $\mathcal{H}om_\Phi(V, W)$, the linear operators on V by $\mathcal{E}nd_\Phi(V)$. [We usually suppress the Φ.]

• *Involutions* on algebras are indicated by a star $*$. $\mathcal{H}(\mathrm{A}, *)$ denotes the *hermitian elements* $x^* = x$ of an algebra A under an involution $*$, $\mathcal{S}kew(\mathrm{A}, *)$ the *skew elements* $x^* = -x$. Involutions on coordinate algebras of matrix algebras are often denoted by a bar, $x \mapsto \bar{x}$, while reversal involutions on algebras generated by hermitian elements (for example, the free algebra $\mathcal{F}[X]$ on a set X) are denoted by ρ.

• *Products* in algebras are denoted by $x \cdot y$ or just xy (especially for associative products); the special symbol $x \bullet y$ is used for the bilinear product in Jordan algebras. The *left* and *right multiplication operators* by an element x in a linear algebra are denoted by L_x, R_x. The *quadratic* and *trilinear products* in Jordan algebras are denoted by $U_x y$ and $\{x, y, z\}$, with operators $U_x, V_{x,y}$.

• *Unit elements* of algebras are denoted by 1. We will speak of a *unit element* and *unital algebras* rather than identity element and algebras with identity; we will reserve the term *identity* for *identical relations* or *laws* (such as the Jordan identity or associative law). $\widehat{\mathrm{A}}$ will denote the *formal unital hull*, the Φ-algebra $\Phi\hat{1} \oplus \mathrm{A}$ obtained by formal adjunction of a unit element.

• $n \times n$ *matrices* and hermitian matrices are denoted by $\mathcal{M}_n$ and $\mathcal{H}_n$. Matrices are denoted by $X = (x_{ij})$; their *traces* and *determinants* are denoted by $\mathrm{tr}(X)$ and $\det(X)$ respectively, and the *transpose* is indicated by X^{tr}. The *standard matrix units* (with 1 in the ij-entry and 0's elsewhere) are denoted by E_{ij}. If the matrix entries come from a ring with involution, the *standard involution* (the adjoint or conjugate transpose) on matrices has $X^* = (\overline{X})^{tr} = \big(\overline{x_{ji}}\big)$. A *diagonal* $n \times n$ matrix with $\gamma_1, \ldots, \gamma_n$ down the diagonal will be denoted by $\Gamma = diag(\gamma_1, \ldots, \gamma_n)$; a matrix involution of the form $X \mapsto \Gamma X^* \Gamma^{-1}$ will be called a *canonical involution.*

• We use := to mean *equals by definition.* This is used whenever a term is first defined by means of a formula. It is occasionally used thereafter to remind the reader that the equality does not follow from any fact or calculation, but directly from the definition.

• An open box □ in Parts II and III indicates the end of a *proof*; if it immediately follows the statement of a result, it means the result follows immediately from previous results and there will be no further proof given. In the Introduction and Part I no proofs are ever given, and we use □ only in Chapter 8 (describing Zel'manov's exceptional methods) to indicate the end of a *proof sketch.*

Labeling Conventions

• *Terms* or *names of theorems or equations* being defined or stated for the first time are given in **boldface** type. Ordinary terms are given without capital letters. Most of the important equations, rules, and laws are given proper names (with capital letters), and have the same status as named theorems.

• In the Introduction and the Historical Survey (Part I), statements are named but not numbered. In Parts II and III, statements are numbered consecutively within each section. References of the form II.1.2.3 mean Part II, Chapter 1, Section 2, Statement 3. References within a single Part omit the initial Part designation, e.g., 1.2.3. Some theorems or definitions have several sub-parts, indicated by parenthesized numbers (1),(2),(3) etc. To guide the reader to the specific location within the item I use the notation II.1.2.3(1), so the parenthesized numbers are not statement counters. All statements and named equations have a proper name attached to them, as in "Nuclear Slipping Formula 21.2.1(1)," and I hope the reader will try to remember statements by their verbal mnemonic tags; the numerical tag is there only to look up the reference when the mnemonic tag is not sufficient.

• In the course of a proof, the first time a result is referred to its full name is given (e.g., "Peirce Associativity 9.1.3"), but thereafter it is referred to only by its first name without the numerical tag (e.g., "Peirce Associativity"). This first-name usage sometimes carries over to an immediately following proof which continues the same train of thought. We seldom include a result's middle name (e.g., "Peirce Associativity Proposition 9.1.3").

• *Exercises* are given in footnotesize type, in order to distract as little as possible from the flow of the narrative. They follow immediately after a numbered result, and usually provide variations on the theme of that result; they are given the same number as the result (with suffixes A,B, etc. if there are several exercises on the same theme), and are titled in small caps (less distractive than boldface). *Problems* and *Questions* are given at the end of each chapter, and are each numbered consecutively in that chapter; they are titled in small caps, but are given in normal size type. Some exercises and problems carry an asterisk, indicating that a hint appears in Appendix E.

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Part V Indexes

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A Colloquial Survey of Jordan Theory

In this Survey I want to sketch what Jordan algebras are, and why people might want to study them. On a philosophical level, in the rapidly expanding universe of mathematics it is important for you to have a feeling for the overall connectedness of things, to remember the common origins and the continuing ties that bind distant areas of mathematics together. On a practical level, before you plunge into a detailed study of Jordan algebras in the rest of the book, it is important for you to have an overall picture of their "Sitz im Leben" — to be aware of the larger Jordan family of algebraic systems (algebras, triples, pairs, and superalgebras), and to have some appreciation of the important role this family plays in the mathematical world at large.

Jordan algebras were created to illuminate a particular aspect of physics, quantum-mechanical observables, but turned out to have illuminating connections with many areas of mathematics. Jordan systems arise naturally as "coordinates" for Lie algebras having a grading into 3 parts. The physical investigation turned up one unexpected system, an "exceptional" 27-dimensional simple Jordan algebra, and it was soon recognized that this exceptional Jordan algebra could help us understand the five exceptional Lie algebras.

Later came surprising applications to differential geometry, first to certain symmetric spaces, the self-dual homogeneous cones in real n-space, and then a deep connection with bounded symmetric domains in complex n-space. In these cases the algebraic structure of the Jordan system encodes the basic geometric information for the associated space or domain. Once more the exceptional geometric spaces turned out to be connected with the exceptional Jordan algebra.

Another surprising application of the exceptional Jordan algebra was to octonion planes in projective geometry; once these planes were realized in terms of the exceptional Jordan algebra, it became possible to describe their automorphisms.

This presentation is meant for a general graduate-level mathematical audience (graduate students rather than experts in Lie algebras, differential geometry, etc.). As in any colloquium, the audience is not expected to be acquainted with all the mathematical subjects discussed. There are many other applications (to differential equations, genetics, probability, statistics), but I won't try to present a compendium of applications here — this survey can only convey a taste of how Jordan algebras come to play a role in several important subjects. It will suffice if the reader comes to understand that Jordan algebras are not an eccentric axiomatic system, but a mathematical structure which arises naturally and usefully in a wide range of mathematical settings.

0.1 Origin of the Species

Jordan algebras arose from the search for an "exceptional" setting for quantum mechanics. In the usual interpretation of quantum mechanics (the "Copenhagen model"), the physical observables are represented by self-adjoint or Hermitian matrices (or operators on Hilbert space). The basic operations on matrices or operators are multiplication by a complex scalar, addition, multiplication of matrices (composition of operators), and forming the complex conjugate transpose matrix (adjoint operator). But these underlying matrix operations are not "observable": the scalar multiple of a hermitian matrix is not again hermitian unless the scalar is real, the product is not hermitian unless the factors happen to commute, and the adjoint is just the identity map on hermitian matrices.

In 1932 the physicist Pascual Jordan proposed a program to *discover a new algebraic setting for quantum mechanics*, which would be freed from dependence on an invisible all-determining metaphysical matrix structure, yet would enjoy all the same algebraic benefits as the highly successful Copenhagen model. He wished to study the intrinsic algebraic properties of hermitian matrices, to capture these properties in formal algebraic properties, and then to see what other possible non-matrix systems satisfied these axioms.

Jordan Algebras

The first step in analyzing the algebraic properties of hermitian matrices or operators was to decide what the basic *observable operations* were. There are many possible ways of combining hermitian matrices to get another hermitian matrix, but after some empirical experimentation Jordan decided that they could all be expressed in terms of *quasi-multiplication*

$$x \bullet y := \tfrac{1}{2}(xy + yx)$$

(we now call this symmetric bilinear product the *Jordan product*). Thus in addition to its observable linear structure as a real vector space, the model carried a basic observable product, quasi-multiplication.

The next step in the empirical investigation of the algebraic properties enjoyed by the model was to decide what crucial formal *axioms* or *laws* the operations on hermitian matrices obey. Jordan thought the key law governing quasi-multiplication, besides its obvious commutativity, was

$$x^2 \bullet (y \bullet x) = (x^2 \bullet y) \bullet x$$

(we now call this equation of degree four in two variables the *Jordan identity*, in the sense of *identical relation* satisfied by all elements). Quasi-multiplication satisfied the additional "positivity" condition that a sum of squares never vanishes, which (in analogy with the recently-invented formally real fields) was called *formal reality*. The outcome of all this experimentation was a distillation of the algebraic essence of quantum mechanics into an axiomatically defined algebraic system.

Jordan Definition. *A* Jordan algebra *consists of a real vector space equipped with a bilinear product $x \bullet y$ satisfying the commutative law and the Jordan identity:*

$$x \bullet y = y \bullet x, \qquad (x^2 \bullet y) \bullet x = x^2 \bullet (y \bullet x).$$

A Jordan algebra is formally real *if*

$$x_1^2 + \cdots + x_n^2 = 0 \implies x_1 = \cdots = x_n = 0.$$

Any associative algebra A over $\mathbb{R}$ gives rise to a Jordan algebra A^+ under quasi-multiplication: the product $x \bullet y := \frac{1}{2}(xy + yx)$ is clearly commutative, and satisfies the Jordan identity since

$$\begin{aligned} 4(x^2 \bullet y) \bullet x &= (x^2y + yx^2)x + x(x^2y + yx^2) \\ &= x^2yx + yx^3 + x^3y + xyx^2 \\ &= x^2(yx + xy) + (yx + xy)x^2 = 4x^2 \bullet (y \bullet x). \end{aligned}$$

A Jordan algebra is called *special* if it can be realized as a Jordan subalgebra of some A^+. For example, if A carries an involution $*$ then the subspace of hermitian elements $x^* = x$ is also closed under the Jordan product, since if $x^* = x, y^* = y$ then $(x \bullet y)^* = y^* \bullet x^* = y \bullet x = x \bullet y$, and therefore forms a special Jordan algebra $\mathcal{H}(\mathrm{A}, *)$. These hermitian algebras are the archetypes of all Jordan algebras.

It is easy to check that the hermitian matrices over the reals, complexes, and quaternions form special Jordan algebras that are formally real. One obtains another special formally real Jordan algebra (which we now call a *spin factor* $\mathcal{JS}pin_n$) on the space $\mathbb{R}1 \oplus \mathbb{R}^n$ for $n \geq 2$, by making 1 act as unit and defining the product of vectors v, w in $\mathbb{R}^n$ to be given by the dot or inner product

$$v \bullet w := \langle v, w \rangle 1.$$

In a special Jordan algebra the algebraic structure is derived from an ambient associative structure xy via quasi-multiplication. What the physicists were looking for, of course, were Jordan algebras where there is no invisible structure xy governing the visible structure $x \bullet y$ from behind the scenes. A

Jordan algebra is called *exceptional* if it is not special, i.e., does not result from quasi-multiplication.

The Jordan Classification

Having settled on the basic axioms for his systems, it remained to find exceptional Jordan algebras. Jordan hoped that by studying *finite-dimensional* algebras he could find families of simple exceptional algebras E_n parameterized by natural numbers n, so that letting n go to infinity would provide a suitable infinite-dimensional exceptional home for quantum mechanics. In a fundamental 1934 paper, Jordan, John von Neumann, and Eugene Wigner showed that every finite-dimensional formally real Jordan algebra is a direct sum of a finite number of simple ideals, and that there are only five basic types of simple building blocks: four types of hermitian $n \times n$ matrix algebras $\mathcal{H}_n(\mathrm{C})$ corresponding to the four real division algebras C [the reals, complexes, quaternions, and the *octonions* or *Cayley algebra* $\mathbb{K}$] of dimensions $1, 2, 4, 8$ (but for the octonions only $n \leq 3$ is allowed), together with the spin factors.

There were two surprises in this list, two new structures which met the Jordan axioms but weren't themselves hermitian matrices: the spin factors and $\mathcal{H}_3(\mathbb{K})$. While the spin factor was not one of the invited guests, it was related to the guest of honor: it can be realized as a certain subspace of all hermitian $2^n \times 2^n$ real matrices, so it too is special. The other uninvited guest, $\mathcal{H}_3(\mathbb{K})$, was quite a different creature. It did not *seem* to be special, since its coordinates came from the *not-associative* coordinate Cayley algebra $\mathbb{K}$, and A.A. Albert showed that it is indeed an exceptional Jordan algebra of dimension 27 (we now call such 27-dimensional exceptional algebras *Albert algebras*, and denote $\mathcal{H}_3(\mathbb{K})$ by $\mathbb{A}$).

The Physical End of Exceptional Algebras

These results were deeply disappointing to physicists: there was only one exceptional algebra in this list, for $n = 3$; for $n > 3$ the algebra $\mathcal{H}_n(\mathbb{K})$ is not a Jordan algebra at all, and for $n = 2$ it is isomorphic to $\mathcal{JS}pin_9$, and for $n = 1$ it is just $\mathbb{R}^+$. This lone exceptional algebra $\mathcal{H}_3(\mathbb{K})$ was too tiny to provide a home for quantum mechanics, and too isolated to give a clue as to the possible existence of infinite-dimensional exceptional algebras. Half a century later the brilliant young Novosibirsk mathematician Efim Zel'manov quashed all remaining hopes for such an exceptional system. In 1979 he showed that even in infinite dimensions there are no simple exceptional Jordan algebras other than Albert algebras: as it is written,

> *... and there is no new thing under the sun*
> *especially in the way of exceptional Jordan algebras;*
> *unto mortals the Albert algebra alone is given.*

In 1983 Zel'manov proved the astounding theorem that any simple Jordan algebra, of arbitrary dimension, is either (1) an algebra of Hermitian elements $\mathcal{H}(\mathrm{A}, *)$ for a $*$-simple associative algebra with involution, (2) an algebra of spin type determined by a nondegenerate quadratic form, or (3) an Albert algebra of dimension 27 over its center. This brought an end to the search for an exceptional setting for quantum mechanics: it is an ineluctable fact of mathematical nature that simple algebraic systems obeying the basic laws of Jordan *must* (outside of dimension 27) have an invisible associative support behind them.

Special Identities

While physicists abandoned the poor orphan child of their theory, the Albert algebra, algebraists adopted it and moved to new territories. This orphan turned out to have many surprising and important connections with diverse branches of mathematics. Actually, the child should never have been conceived in the first place: it does *not* obey all the algebraic properties of the Copenhagen model, and so was in fact unsuitable as a home for quantum mechanics, not superficially due to its finite-dimensionality, but genetically because of its unsuitable algebraic structure. In 1963 Jacobson's student C.M. Glennie discovered two identities satisfied by hermitian matrices (indeed, by all special Jordan algebras) but *not* satisfied by the Albert algebra. Such identities are called *special identities* (or s-identities) since they are satisfied by all special algebras but not all Jordan algebras, and so serve to separate the special from the non-special.

Jordan can be excused for missing these identities, of degree 8 and 9 in 3 variables, since they cannot even be intelligibly expressed without using the (then new-fangled) *quadratic Jordan product* and *Jordan triple product*

$$U_x(y) := 2x \bullet (x \bullet y) - x^2 \bullet y,$$
$$\{x, y, z\} := 2\big(x \bullet (y \bullet z) + (x \bullet y) \bullet z - (x \bullet z) \bullet y\big)$$

(corresponding to xyx and $xyz + zyx$ in special Jordan algebras). In 1958 I.G. Macdonald had established that this U-operator satisfied a very simple identity, which Jacobson called the *Fundamental Formula*:

$$U_{U_x(y)} = U_x U_y U_x.$$

In terms of these products, Glennie's identities take the none-too-memorable forms

$$G_8 : \quad H_8(x, y, z) = H_8(y, x, z), \qquad G_9 : \quad H_9(x, y, z) = H_9(y, x, z),$$

where

$$H_8(x, y, z) := \{U_x U_y(z), z, x \bullet y\} - U_x U_y U_z(x \bullet y),$$
$$H_9(x, y, z) := 2U_x(z) \bullet U_{y,x} U_z(y^2) - U_x U_z U_{x,y} U_y(z).$$

Observe that G_8, G_9 vanish in special algebras since H_8, H_9 reduce to the symmetric 8 and 9-tads $\{x,y,z,y,x,z,x,y\}, \{x,z,x,x,z,y,y,z,y\}$ respectively [by an *n-tad* we mean the symmetric associative product $\{x_1,\ldots,x_n\} := x_1\cdots x_n + x_n\cdots x_1$].

In 1987 Armin Thedy came out of right field (or rather, right alternative algebras) carrying an operator s-identity of degree 10 in 3 variables that finally a mortal could remember:

$$T_{10}: \; U_{U_{[x,y]}(z)} = U_{[x,y]}U_zU_{[x,y]} \quad \big(U_{[x,y]} := 4U_{x\bullet y} - 2(U_xU_y + U_yU_x)\big).$$

This is just a "fundamental formula" or "structural condition" for the U-operator of the "commutator" $[x,y]$. Notice that this vanishes on all special algebras because $U_{[x,y]}$ really *is* the map $z \mapsto (xy+yx)z(xy+yx) - 2(xyzyx + yxzxy) = [x,y]z[x,y]$, which clearly satisfies the fundamental formula. Of course, there is no such thing as a commutator in a Jordan algebra (in special Jordan algebras $\mathrm{J} \subseteq \mathrm{A}^+$ the commutators do exist in the associative envelope A), but these spiritual entities still manifest their presence by acting on the Jordan algebra. In 1999 Ivan Shestakov discovered that Glennie's identities could be rewritten in a very memorable form using commutators, $[[x,y]^3, z^2] = \{z, [[x,y]^3, z]\}$ and $[[x,y]^3, z^3] = \{z^2, [[x,y]^3, z]\} + U_z\big([[x,y]^3, z]\big)$.

The quick modern proof that the Albert algebra is exceptional is to show that for *judicious* choice of x, y, z some polynomial $G_8, G_9, Ш_8, Ш_9$, or T_{10} does not vanish on $\mathcal{H}_3(\mathbb{K})$.

Thus it was serendipitous that the Albert algebra was allowed on the mathematical stage in the first place. Many algebraic structures are famous for 15 minutes and then disappear from the action, but others go on to feature in a variety of settings (keep on ticking, like the bunny) and prove to be an enduring part of the mathematical landscape. So it was with the Albert algebra. This Survey will describe some of the places where Jordan algebras, especially the Albert algebra, have played a starring role.

0.2 The Jordan River

The stream of Jordan theory originates in Jordan algebras, but soon divides into several algebraic structures (quadratic Jordan algebras, Jordan triples, Jordan pairs, and Jordan superalgebras). All these more general systems take their basic genetic structure from the parental algebras, but require their own special treatment and analysis, and result in new fields of application. Although this book is entirely concerned with algebras, any student of Jordan algebras must be aware of the extended family of Jordan systems to which they belong.

Quadratic Jordan Algebras

In their mature roles, Jordan algebras appear not just wearing a Jordan product, but sporting a powerful quadratic product as well. It took audiences some getting used to this new product, since (unlike quasi-multiplication, Lie bracket, dot products, or any of the familiar algebraic products) it is *not bilinear*: its polarized version is a triple product *trilinear* in three variables, but it is itself a binary product *quadratic* in one variable and linear in the other.

Over the years algebraists had developed a comprehensive theory of finite-dimensional Jordan algebras over arbitrary fields of characteristic different from 2. But it was clear that quasi-multiplication, with its reliance on a scalar $\frac{1}{2}$, was not sufficient for a theory of Jordan algebras in characteristic 2, or over arbitrary rings of scalars. In particular, there was no clear notion of Jordan *rings* (where the ring of scalars was the integers). For example, arithmetic investigations led naturally to hermitian matrices over the integers, and residue class fields led naturally to characteristic 2.

In the 1960s several lines of investigation revealed the crucial importance of the quadratic product $U_x(y)$ and the associated triple product $\{x, y, z\}$ in Jordan theory. Kantor, and Koecher and his students, showed that the triple product arose naturally in connections with Lie algebras and differential geometry. Nathan Jacobson and his students showed how these products facilitated many purely algebraic constructions.

The *U-operator* of "two-sided multiplication" U_x by x has the somewhat messy form $U_x = 2L_x^2 - L_{x^2}$ in terms of left multiplications $L_x(y) := x \bullet y$ in the algebra, and the important *V-operator* $V_{x,y}(z) := \{x, y, z\} := (U_{x+z} - U_x - U_z)(y)$ of "left multiplication" by x, y in the Jordan triple product has the even-messier form $V_{x,y} = 2\big(L_{x\bullet y} + [L_x, L_y]\big)$. In Jordan algebras with a unit element $1 \bullet x = x$ we have $U_1(x) = x$ and $x^2 = U_x(1)$. The *brace product* is the linearization of the square, $\{x, y\} := U_{x,y}(1) = \{x, 1, y\} = 2x \bullet y$, and we can recover the linear product from the quadratic product. We will see that the brace product and its multiplication operator $V_x(y) := \{x, y\}$ are in many situations more natural than the bullet product $x \bullet y$ and L_x.

The crucial property of the quadratic product was the so-called *Fundamental Formula* $U_{U_x(y)} = U_x U_y U_x$; this came up in several situations, and seemed to play a role in the Jordan story like that of the associative law $L_{xy} = L_x L_y$ in the associative story. After a period of experimentation (much like Jordan's original investigation of quasi-multiplication), it was found that the entire theory of unital Jordan algebras could be based on the U-operator. A *unital quadratic Jordan algebra* is a space together with a distinguished element 1 and a product $U_x(y)$ linear in y and *quadratic* in x, which is *unital* and satisfies the *Commuting Formula* and the *Fundamental Formula*

$$U_1 = \mathbb{1}_J, \qquad U_x V_{y,x} = V_{x,y} U_x, \qquad U_{U_x(y)} = U_x U_y U_x.$$

These are analogous to the axioms for the bilinear product $x \bullet y$ of old-fashioned unital linear Jordan algebras

$$L_1 = \mathbb{1}_{\mathrm{J}}, \qquad L_x = R_x, \qquad L_{x^2} L_x = L_x L_{x^2}$$

in terms of left and right multiplications L_x, R_x. Like Zeus shoving aside Kronos, the quadratic product has largely usurped governance of the Jordan domain (though to this day pockets of the theory retain the old faith in quasi-multiplication).

Moral: *The story of Jordan algebras is not the story of a nonassociative product $x \bullet y$, it is the story of a quadratic product $U_x(y)$ which is about as associative as it can be.*

Jordan Triples

The first Jordan stream to branch off from algebras was *Jordan triple systems*, whose algebraic study was initiated by Max Koecher's student Kurt Meyberg in 1969 in the process of generalizing the Tits–Kantor–Koecher construction of Lie algebras (to which we will return when we discuss applications to Lie theory). Jordan triples are basically Jordan algebras with the unit thrown away, so there is no square or bilinear product, only a Jordan triple product $\{x, y, z\}$ which is *symmetric* and satisfies the 5-*linear Jordan identity*

$$\{x, y, z\} = \{z, y, x\},$$
$$\{x, y, \{u, v, w\}\} = \{\{x, y, u\}, v, w\} - \{u, \{y, x, v\}, w\} + \{u, v, \{x, y, w\}\}.$$

A space with such a triple product is called a *linear Jordan triple.* This theory worked only in the presence of a scalar $\frac{1}{6}$; in 1972 Meyberg gave axioms for *quadratic Jordan triple systems* that worked smoothly for arbitrary scalars. These were based on a quadratic product $P_x(y)$ (the operators U, V are usually denoted by P, L in triples) satisfying the *Shifting Formula* , *Commuting Formula* , and the *Fundamental Formula*

$$L_{P_x(y),y} = L_{x,P_y(x)}, \qquad P_x L_{y,x} = L_{x,y} P_x, \qquad P_{P_x(y)} = P_x P_y P_x.$$

Any Jordan algebra J gives rise to a Jordan triple J^t by applying the forgetful functor, throwing away the unit and square and setting $P_x(y) := U_x(y)$, $L_{x,y} := V_{x,y}$. In particular, any associative algebra A gives rise to a Jordan triple A^t via $P_x(y) := xyx$, $\{x, y, z\} := xyz + zyx$. A triple is *special* if it can be imbedded as a sub-triple of some A^t, otherwise it is *exceptional.* An important example of a Jordan triple which doesn't come from a bilinear product consists of the *rectangular matrices* $\mathcal{M}_{pq}(\mathbb{R})$ under $xy^{tr}z + zy^{tr}x$; if $p \neq q$ there is no natural way to multiply two $p \times q$ matrices to get a third $p \times q$ matrix. Taking rectangular 1×2 matrices $\mathcal{M}_{12}(\mathbb{K}) = \mathbb{K}E_{11} + \mathbb{K}E_{12}$ over a Cayley algebra gives an exceptional 16-dimensional *bi-Cayley triple* (so called because it is obtained by gluing together two copies of the Cayley algebra).

Experience has shown that the archetype for Jordan triple systems is A^{t*} obtained from an associative algebra A *with involution* $*$ via

$$P_x(y) := xy^*x, \qquad \{x,y,z\} := xy^*z + zy^*x.$$

The presence of the involution $*$ on the middle factor warns us to expect reversals in that position. It turns out that a triple is special iff it can be imbedded as a sub-triple of some A^{t*}, so that either model A^{t+} or A^{t*} can be used to define speciality.

Jordan Pairs

The second stream to branch off from algebras flows from the same source, the Tits–Kantor–Koecher construction of Lie algebras. Building on an offhand remark of Meyberg that the entire TKK construction would work for "verbundene Paare," two independent spaces J^+, J^- acting *on each other* like Jordan triples, a full-grown theory of Jordan pairs sprang from Ottmar Loos's mind in 1974. *Linear Jordan pairs* are pairs $\mathrm{V} = (V^+, V^-)$ of spaces with trilinear products $\{x^+, u^-, y^+\} \in V^+, \{u^-, x^+, v^-\} \in V^-$ (but incestuous triple products containing adjacent terms from the same space are strictly forbidden!) such that both products are symmetric and satisfy the 5-linear identity. *Quadratic Jordan pairs* have quadratic products $Q_{x^\varepsilon}(u^{-\varepsilon}) \in V^\varepsilon$ $(\varepsilon = \pm)$ satisfying the three quadratic Jordan triple axioms (the operators P, L are usually denoted by Q, D in Jordan pairs).

Every Jordan triple J can be *doubled* to produce a Jordan pair $\mathrm{V(J)} = (J,J)$, $V^\varepsilon := J$ under $Q_{x^\varepsilon}(y^{-\varepsilon}) := P_x(y)$. The double of rectangular matrices $M_{pq}(F)$ could be more naturally viewed as a pair $(M_{pq}(F), M_{qp}(F))$. More generally, for any two vector spaces V, W over a field F we have a "rectangular" pair $(Hom_F(V,W),\ Hom_F(W,V))$ of different spaces under products xux, uxu making no reference to a transpose. This also provides an example to show that pairs are more than doubled triples. In finite dimensions all semisimple Jordan pairs have the form $\mathrm{V} = (\mathrm{J},\mathrm{J})$ for a Jordan triple system J; in particular, V^+ and V^- have the same dimension, but this is quite accidental and ceases to be true in infinite dimensions. Indeed, for a vector space W of infinite dimension d the rectangular pair $\mathrm{V} := (Hom_F(W,F),$ $Hom_F(F,W)) \cong (W^*, W)$ has $\dim(W^*) = \mid F \mid^d \geq 2^d > d = \dim(W)$.

The perspective of Jordan pairs has clarified many aspects of the theory of Jordan triples and algebras.

Jordan Superalgebras

The third main branching of the Jordan River leads to *Jordan superalgebras* introduced by KKK (Victor Kac, Issai Kantor, Irving Kaplansky). Once more this river springs from a physical source: Jordan superalgebras are dual to the *Lie superalgebras* invented by physicists to provide a formalism to encompass *supersymmetry,* handling bosons and fermions in one algebraic system. A Lie superalgebra is a $\mathbb{Z}_2$-graded algebra $\mathrm{L} = \mathrm{L}_0 \oplus \mathrm{L}_1$ where L_0 is a Lie algebra and L_1 an L_0-module with a "Jordan-like" product into L_0. Dually, a Jordan superalgebra is a $\mathbb{Z}_2$-graded algebra $\mathrm{J} = \mathrm{J}_0 \oplus \mathrm{J}_1$ where J_0 is a Jordan algebra and J_1 a J_0-bimodule with a "Lie-like" product into J_0. For example, any $\mathbb{Z}_2$-graded associative algebra $\mathrm{A} = \mathrm{A}_0 \oplus \mathrm{A}_1$ becomes a Lie superalgebra under the *graded Lie product*

$$[x_i, y_j] = x_i y_j - (-1)^{ij} y_j x_i$$

(reducing to the Lie bracket $xy - yx$ if at least one factor is even, but to the Jordan brace $xy + yx$ if both i, j are odd), and dually becomes a Jordan superalgebra under the *graded Jordan brace*

$$\{x_i, y_j\} = x_i y_j + (-1)^{ij} y_j x_i$$

(reducing to the Jordan brace $xy + yx$ if at least one factor is even, but to the Lie bracket $xy - yx$ if both factors are odd). Jordan superalgebras shed light on Pchelinstev Monsters, a strange class of prime Jordan algebras which seem genetically unrelated to all normal Jordan algebras.

We have indicated how these branches of the Jordan river had their origin in an outside impetus. We now want to indicate how the branches, in turn, have influenced and enriched various areas of mathematics.

0.3 Links with Lie Algebras and Groups

The Jordan river flows parallel to the Lie river with its extended family of Lie systems (algebras, triples, and superalgebras), and an informed view of Jordan algebras must also reflect awareness of the Lie connections. Historically, the first connection of Jordan algebras to another area of mathematics was to the theory of Lie algebras, and the remarkably fruitful interplay between Jordan and Lie theory continues to generate interest in Jordan algebras.

Jordan algebras played a role in Zel'manov's celebrated solution of the Restricted Burnside Problem, for which he was awarded a Fields Medal in 1994. That problem, about finiteness of finitely-generated torsion groups, could be reduced to a problem in certain Lie p-rings, which in turn could be illuminated by a natural Jordan algebra structure of characteristic p. The most difficult case, where the characteristic was $p = 2$, could be most clearly settled making use of a quadratic Jordan structure. Lie algebras in characteristic 2 are weak, pitiable things; *linear* Jordan algebras aren't much better, since the Jordan product $xy + yx$ is indistinguishable from the Lie bracket $xy - yx$ in that case. The crucial extra product xyx of the *quadratic* Jordan theory was the key that turned the tide of battle.

Lies

Recall that a *Lie algebra* is a linear algebra with product $[x, y]$ which is *anticommutative* and satisfies the *Jacobi Identity*

$$[x, y] = -[y, x], \qquad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Just as any associative algebra A becomes a linear Jordan algebra A^+ under the *anticommutator* or *Jordan brace* $\{x, y\} = xy + yx$, it also becomes a Lie algebra A^- under the *commutator* or *Lie bracket* $[x, y] := xy - yx$. In contrast to the Jordan case, *all* Lie algebras (at least those that are free as modules, e.g., algebras over a field) are *special* in the sense of arising as a commutator-closed subspace of some A^-. A^- contains many subspaces which are closed under the Lie bracket but not under the ordinary product; any such subspace produces a Lie subalgebra which need not be of the form E^- for any associative subalgebra E of A.

There are three main ways of singling out such subspaces. The first is by means of the trace: If A is the algebra of $n \times n$ matrices over a field, or more abstractly all linear transformations on a finite-dimensional vector space, then the subspace of elements of *trace zero* is closed under brackets since the Lie bracket $T_1T_2 - T_2T_1$ of *any two transformations* has trace zero, due to symmetry $\mathrm{tr}(T_1T_2) = \mathrm{tr}(T_2T_1)$ of the trace function.

The second way is by means of an involution: In general, for any involution $*$ on an associative algebra A the subspace $\mathcal{S}kew(\mathrm{A}, *)$ of skew elements $x^* = -x$ is closed under brackets since $[x_1, x_2]^* = [x_2^*, x_1^*] = [-x_2, -x_1] = [x_2, x_1] = -[x_1, x_2]$. If a finite-dimensional vector space V carries a nondegenerate symmetric or skew bilinear form, the process $\langle T(v), w\rangle = \langle v, T^*(w)\rangle$ of moving a transformation from one side of the bilinear form to the other induces an *adjoint involution* on the linear transformations $\mathrm{A} = \mathcal{E}nd_F(V)$, and the *skew transformations* T satisfying $\langle T(v), w\rangle = -\langle v, T(w)\rangle$ form a Lie subalgebra.

The third method is by means of derivations: In the associative algebra $\mathrm{A} = \mathcal{E}nd(\mathrm{C})$ of all linear transformations on an *arbitrary* linear algebra C (not necessarily associative or commutative), the subspace $\mathcal{D}er(\mathrm{A})$ of *derivations* of A (linear transformations satisfying the *product rule* $D(xy) = D(x)y + xD(y)$) is closed under brackets, due to symmetry in D_1, D_2 of

$$\begin{aligned}(D_1D_2)(xy) &- (D_1D_2)(x)y - x(D_1D_2)(y)\\ &= D_1\big(D_2(x)y + xD_2(y)\big) - D_1\big(D_2(x)\big)y - xD_1\big(D_2(y)\big)\\ &= D_2(x)D_1(y) + D_1(x)D_2(y).\end{aligned}$$

The exceptional Jordan algebra (the Albert algebra), the exceptional composition algebra (the Cayley or octonion algebra), and the five exceptional Lie algebras are enduring features of the mathematical landscape, and they are all genetically related. Algebraists first became interested in the newborn

Albert algebra through its unexpected connections with exceptional Lie algebras and groups. The four great classes of simple Lie algebras (respectively groups) A_n, B_n, C_n, D_n consist of matrices of trace 0 (respectively determinant 1) or skew $T^* = -T$ (respectively isometric $T^* = T^{-1}$) with respect to a nondegenerate symmetric or skew-symmetric bilinear form. The five exceptional Lie algebras and groups G_2, F_4, E_6, E_7, E_8 appearing mysteriously in the nineteenth-century Cartan–Killing classification were originally defined in terms of a multiplication table over an algebraically closed field. When these were found in the 1930s, '40s, and '50s to be describable in an intrinsic coordinate-free manner using the Albert algebra $\mathbb{A}$ and Cayley algebra $\mathbb{K}$, it became possible to study them over general fields. The Lie algebra (resp. group) G_2 of dimension 14 arises as the derivation algebra (resp. automorphism group) of $\mathbb{K}$; F_4 of dimension 52 arises as the derivation algebra $\mathcal{D}er$ (resp. automorphism group $\mathcal{A}ut$) of $\mathbb{A}$; E_6 arises by reducing the structure algebra $\mathcal{S}trl(\mathbb{A}) := L(\mathbb{A}) + \mathcal{D}er(\mathbb{A})$ (resp. structure group $\mathcal{S}trg(\mathbb{A}) := U(\mathbb{A})\mathcal{A}ut(\mathbb{A})$) of $\mathbb{A}$ to get $\mathcal{S}trl_0(\mathbb{A}) := L(\mathbb{A}_0) + \mathcal{D}er(\mathbb{A})$ of dimension $(27-1)+52 = 78$ (the subscript 0 indicates trace zero elements); E_7 arises from the Tits–Kantor–Koecher construction $\mathcal{TKK}(\mathbb{A}) := \mathbb{A} \oplus \mathcal{S}trl(\mathbb{A}) \oplus \mathbb{A}$ (resp. $\mathcal{TKK}$ group) of $\mathbb{A}$ of dimension $27 + 79 + 27 = 133$, while E_8 of dimension 248 arises in a more complicated manner from $\mathbb{A}$ and $\mathbb{K}$ by a process due to Jacques Tits. We first turn to this construction.

The Freudenthal–Tits Magic Square

Jacques Tits discovered in 1966 a general construction of a Lie algebra $\mathcal{FT}(\mathrm{C}, \mathrm{J})$, starting from a composition algebra C and a Jordan algebra J of "degree 3," which produces E_8 when J is the Albert algebra and C the Cayley algebra. Varying the possible ingredients leads to a square arrangement that had been noticed earlier by Hans Freudenthal:

The Freudenthal–Tits Magic Square: $\mathcal{FT}(\mathrm{C}, \mathrm{J})$

$\mathrm{C} \setminus \mathrm{J}$	$\mathbb{R}$	$\mathcal{H}_3(\mathbb{R})$	$\mathcal{H}_3(\mathbb{C})$	$\mathcal{H}_3(\mathbb{H})$	$\mathcal{H}_3(\mathbb{K})$
$\mathbb{R}$	0	A_1	A_2	C_3	F_4
$\mathbb{C}$	0	A_2	$A_2 \oplus A_2$	A_5	E_6
$\mathbb{H}$	A_1	C_3	A_5	A_6	E_7
$\mathbb{K}$	G_2	F_4	E_6	E_7	E_8

Some have doubted whether this is square, but no one has ever doubted that it is magic. The ingredients for this Lie recipe are a composition algebra C ($\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{K}$ of dimension $1, 2, 4, 8$) and a Jordan algebra J (either $\mathbb{R}$ or $\mathcal{H}_3(\mathrm{D})$ $[\mathrm{D} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{K}]$ of dimension either 1 or $6, 9, 15, 27$). The recipe creates a space $\mathcal{FT}(\mathrm{C}, \mathrm{J}) := \mathcal{D}er(\mathrm{C}) \oplus (\mathrm{C}_0 \otimes \mathrm{J}_0) \oplus \mathcal{D}er(\mathrm{J})$ (again the subscript 0 indicates trace zero elements) with complicated Lie product requiring $\frac{1}{12}$. For example, the lower right corner of the table produces a Lie algebra of dimension $14 + (7 \times 26) + 52 = 248$, which is precisely the dimension of E_8.

The Tits–Kantor–Koecher Construction

In the late 1960s Issai Kantor and Max Koecher independently showed how to build a Lie algebra $\mathcal{TKK}(\mathrm{J}) := L_1 \oplus L_0 \oplus L_{-1}$ with "short 3-grading" $[L_i, L_j] \subseteq L_{i+j}$ by taking $L_{\pm 1}$ to be two copies of any Jordan algebra J glued together by the *inner structure Lie algebra* $L_0 = Instrl(\mathrm{J}) := V_{J,J}$ spanned by the V-operators (note that the 5-linear elemental identity $\{x, y, \{u, v, w\}\} = \{\{x, y, u\}, v, w\} - \{u, \{y, x, v\}, w\} + \{u, v, \{x, y, w\}\}$ becomes the operator identity $[V_{x,y}, V_{u,v}] = V_{\{x,y,u\},v} - V_{u,\{y,x,v\}}$ acting on the element w, showing that the inner structure algebra is closed under the Lie bracket). Thus we have a space

$$\mathrm{L} := \mathcal{TKK}(\mathrm{J}) := J_1 \oplus Instrl(\mathrm{J}) \oplus J_{-1}$$

(where the J_i are copies of the module J under cloning maps $x \mapsto x_i$) with an anticommutative bracket determined by

$$\begin{aligned} &[T, x_1] := T(x)_1, \qquad [T, y_{-1}] := -T^*(y)_{-1}, \quad [x_i, y_i] := 0 \quad (i = \pm 1), \\ &[x_1, y_{-1}] := V_{x,y}, \qquad [T_1, T_2] := T_1 T_2 - T_2 T_1 \end{aligned}$$

for $T \in Instrl(\mathrm{J}), x, y \in J$; the natural involution $*$ on $Instrl(\mathrm{J})$ determined by $V_{x,y}^* = V_{y,x}$ extends to an involution $(x, T, y) \mapsto (y, T^*, x)$ on all of L. Later it was noticed that this was a special case of a 1953 construction by Jacques Tits of a Lie algebra $\mathrm{L}' = (\mathrm{J} \otimes \mathrm{L}_0) \oplus Inder(\mathrm{J})$ built out of a Jordan algebra J and a simple 3-dimensional Lie algebra L_0 of type A_1 with bracket determined by

$$\begin{aligned} &[D, x \otimes \ell] := D(x) \otimes \ell, \qquad [D_1, D_2] := D_1 D_2 - D_2 D_1, \\ &[x \otimes \ell, y \otimes m] := (x \bullet y) \otimes [\ell, m] + \tfrac{1}{8}\kappa(\ell, m) D_{x,y} \end{aligned}$$

for *inner derivations* D spanned by all $D_{x,y} := V_{x,y} - V_{y,x} = [V_x, V_y],\ x, y \in \mathrm{J},\ \ell, m \in \mathrm{L}_0,\ \kappa$ the Killing form $\mathrm{tr}(\mathrm{ad}(x)\mathrm{ad}(y))$ on L_0. Since always $V_{\mathrm{J},\mathrm{J}} = V_{\mathrm{J}} \oplus Inder(\mathrm{J})$, in the special case where $\mathrm{L}_0 = \{e, f, h\}$ is a *split* $s\ell_2$, the map $(x \otimes e + y \otimes f + z \otimes h) \oplus D \mapsto x_1 \oplus (V_z + D) \oplus y_{-1}$ is an isomorphism sending e, f, h to $1_1, 1_{-1}, V_1 = 2\mathbb{1}_{\mathrm{J}}$, and L is a clone of L'.

The Tits–Kantor–Koecher Construction is not only intrinsically important, it is historically important because it gave birth to two streams in Jordan theory. The Jacobi identity for TKK to be a Lie algebra boils down to outer-symmetry and the 5-linear identity for the Jordan triple product. This observation led Meyberg to take these two conditions as the axioms for a new algebraic system, a *Jordan triple system*, and he showed that the Tits–Kantor–Koecher construction $\mathcal{TKK}(\mathrm{J}) := J \oplus \mathcal{S}trl(\mathrm{J}) \oplus J$ produced a graded Lie algebra with reversal involution $x \oplus T \oplus y \mapsto y \oplus T^* \oplus x$ iff J was a linear Jordan triple system. This was the first Jordan stream to branch off the main line.

The second stream branches off from the same source, the $\mathcal{TKK}$ construction. Loos formulated the axioms for Jordan pairs $\mathrm{V} = (V_1, V_{-1})$ (a pair of spaces V_1, V_{-1} acting on each other like Jordan triples), and showed that they are precisely what is needed in the $\mathcal{TKK}$-Construction of Lie algebras: $\mathcal{TKK}(\mathrm{V}) := V_1 \oplus Inder(\mathrm{V}) \oplus V_{-1}$ produces a graded Lie algebra iff $\mathrm{V} = (V_1, V_{-1})$ is a linear Jordan pair. Jordan triples arise precisely from pairs with involution, and Jordan algebras arise from pairs where the grading and involution come from a little $s\ell_2 = \{e, f, h\} = \{1_1, 1_{-1}, 2(\mathbb{1}_\mathrm{J})\}$.

Thus Jordan systems arise naturally as "coordinates" for graded Lie algebras, leading to the dictum of Kantor: *"There are no Jordan algebras, there are only Lie algebras."* Of course, this can be turned around: *nine times out of ten, when you open up a Lie algebra you find a Jordan algebra inside which makes it tick.*

0.4 Links with Differential Geometry

Though mathematical physics gave birth to Jordan algebras and superalgebras, and Lie algebras gave birth to Jordan triples and pairs, differential geometry has had a more pronounced influence on the algebraic development of Jordan theory than any other mathematical discipline. Investigations of the role played by Jordan systems in differential geometry have revealed new perspectives on purely algebraic features of the subject. We now indicate what Jordan algebras were doing in such a strange landscape.

Inverses and Isotopy

For the first application we need to say a few words about inverses in unital Jordan algebras. An element x is *invertible* in J iff the operator U_x is an invertible operator on J, in which case its inverse is the operator $U_x^{-1} = U_{x^{-1}}$ of the *inverse element* $x^{-1} := U_x^{-1}(x)$. In special Jordan algebras $\mathrm{J} \subseteq \mathrm{A}^+$ an element x is invertible iff it is invertible in A and its associative inverse x^{-1} lies in J, in which case x^{-1} is also the Jordan inverse.

An important concept in Jordan algebras is that of *isotopy.* The fundamental tenet of isotopy is the belief that all invertible elements of a Jordan algebra have an equal entitlement to serve as unit element. If u is an invertible element of an associative algebra A, we can form a new associative algebra, the *associative isotope* A_u with new product, unit, and inverse given by

$$x_u y := x u^{-1} y, \qquad 1_u := u, \qquad x^{[-1,u]} := u x^{-1} u.$$

We can do the same thing in any Jordan algebra: the *Jordan isotope* $\mathrm{J}_{[u]}$ has new bullet, quadratic, and triple products

$$x \bullet_{[u]} y := \tfrac{1}{2}\{x, u^{-1}, y\}, \quad U_{x[u]} := U_x U_{u^{-1}}, \quad \{x, y, z\}_{[u]} := \{x, U_{u^{-1}} y, z\}$$

and new unit and inverses

$$1_{[u]} := u, \qquad x^{[-1,u]} = U_u x^{-1}.$$

Thus each invertible u is indeed the unit in its own isotope.

As an example of the power of isotopy, consider the *Hua Identity*

$$(x + xy^{-1}x)^{-1} + (x + y)^{-1} = x^{-1}$$

in associative division algebras, which plays an important role in the study of projective lines. It is not hard to get bogged down trying to verify the identity directly, but for $x = 1$ the "Weak Hua Identity" $(1 + y^{-1})^{-1} + (1 + y)^{-1} = 1$ has a "third-grade proof":

$$\frac{1}{1 + y^{-1}} + \frac{1}{1 + y} = \frac{y}{y + 1} + \frac{1}{1 + y} = \frac{y + 1}{1 + y} = 1.$$

Using the concept of isotopy, we can bootstrap the commutative third-grade proof into a noncommutative graduate-school proof: taking Weak Hua in the isotope A_x gives

$$\left(1_x + y^{[-1,x]}\right)^{[-1,x]} + \left(1_x + y\right)^{[-1,x]} = 1_x,$$

which becomes, in the original algebra, using the above formulas

$$x(x + xy^{-1}x)^{-1}x + x(x + y)^{-1}x = x.$$

Applying U_x^{-1} to both sides to cancel x fore and aft gives us Strong Hua

$$(x + xy^{-1}x)^{-1} + (x + y)^{-1} = x^{-1}.$$

One consequence of the Hua Identity is that the U-product can be built out of translations and inversions. T.A. Springer later used this to give an axiomatic description of Jordan algebras just in terms of the operation of inversion, which we may loosely describe by saying that an algebra is Jordan iff its inverse is *given by the geometric series in all isotopes,*

$$(1 - x)^{[-1,u]} = \sum_{n=0}^{\infty} x^{[n,u]}.$$

This formula also suggests why the inverse might encode all the information about the Jordan algebra: it contains information about all the powers of an element, and from $x^{[2,u]} = U_x u^{-1}$ for $u = 1$ we can recover the square, hence the bullet product.

0.5 Links with the Real World

We begin with situations in which real Jordan algebras are intimately connected with real differentiable manifolds, especially the formally real Jordan algebras with the positive cones.

Riemannian Symmetric Spaces

A Riemannian manifold is a smooth (C^∞) manifold M carrying a *Riemannian metric*,a smoothly-varying positive-definite inner product $\langle\ ,\ \rangle_p$ on the tangent space $T_p(M)$ to the manifold at each point p. For simplicity we will assume that *all our manifolds are connected.* An *isomorphism* f of Riemannian manifolds is a diffeomorphism of smooth manifolds whose differential df is *isometric* on each tangent space, i.e., preserves the inner product $\langle df_p(u), df_p(v)\rangle_{f(p)} = \langle u, v\rangle_p$. Recall that the *differential* df_p lives on the tangent space $T_p(M)$ with values $df_p(v) = \partial_v f|_p$ on v representing the *directional derivative* of f in the direction of the tangent vector v at the point p.

In such manifolds we can talk of geodesics (paths of "locally shortest length"), and at p there is a unique geodesic curve $\gamma_{p,v}(t)$ (defined for t in a neighborhood of 0) passing through p at $t = 0$ with tangent vector v. At each point p there is a neighborhood on which the *geodesic symmetry* $\gamma_{p,v}(t) \mapsto \gamma_{p,v}(-t)$ is an involutive local diffeomorphism [in general not an isometry with respect to the Riemann metric] having p as isolated fixed point. The *exponential map* $\exp_p$ maps a neighborhood of 0 in $T_p(M)$ down onto a neighborhood of p in the manifold via $v \mapsto \gamma_{p,v}(1)$, so that the straight line segments through the origin in the tangent space go into the local geodesics through p. In case M is *complete* (the geodesics $\gamma_{p,v}(t)$ are defined for *all* t), $\exp_p$ maps all of $T_p(M)$ down onto all of M.

A *Riemannian symmetric space* is a Riemannian manifold M having at each point p a *symmetry* s_p (an involutive global isometry of the manifold having p as isolated fixed point, equivalently having differential $-\mathbb{1}$ at p). The existence of such symmetries is a very strong condition on the manifold. For example, it forces M to be complete and the group of all isometries of M to be a real Lie group G acting transitively on M, in particular it forces the manifold to be *real analytic* instead of merely smooth. Then the symmetries s_p are unique and induce the geodesic symmetry about the point p $[s_p(\exp_p(v)) = \exp_p(-v)]$. Note that by uniqueness any isometry $g \in G$ conjugates the symmetry s_p at a point p to the symmetry $g \circ s_p \circ g^{-1}$ at the point $g(p)$. Thus we can rephrase the condition for symmetry for M as (i) there is a symmetry s_{p_0} at one point p_0, (ii) isometries act transitively on M. The subgroup K of G fixing a point p is *compact*, and M is isomorphic to G/K. This leads to the traditional Lie-theoretic approach to symmetric spaces.

In 1969 Loos gave an algebraic formulation of symmetric spaces which clearly revealed a Jordan connection: A symmetric space is a (Hausdorff C^∞)

manifold with a differentiable multiplication $x \cdot y$ whose left multiplications $s_x(y) = x \cdot y$ satisfies the *symmetric axioms*

s_x is an involutive map: $s_x^2 = \mathbb{1}_M$,
s_x satisfies the Fundamental Formula: $s_{x \cdot y} = s_x s_y s_x$,
x is an isolated fixed point of s_x.

Here s_x represents the symmetry at the point $x \in M$. If one fixes a basepoint $c \in M$, the maps $Q_x := s_x s_c$ satisfy the usual Fundamental Formula $Q_{Q_x y} = Q_x Q_y Q_x$. Here $Q_c = \mathbb{1}_M$, and $j = s_c$ is "inversion." For example, in any Lie group the product $x \cdot y := xy^{-1}x$ gives such a multiplication, and for $c = e$ the maps Q, j take the form $Q_x(y) = xyx, j(x) = x^{-1}$ in terms of the group operations.

Self-Dual Cones

Certain symmetric spaces can be described more fruitfully using Jordan algebras. For instance, there is a 1-to-1 correspondence between the self-dual open homogeneous cones in $\mathbb{R}^n$ and n-dimensional formally real Jordan algebras, wherein the geometric structure is intimately connected with the Jordan algebraic structure living in the tangent space.

A subset $\mathcal{C}$ of a real vector space V is *convex* if it is closed under convex combinations $[tx + (1-t)y \in \mathcal{C}$ for all $x, y \in \mathcal{C}$ and $0 \leq t \leq 1]$, and therefore connected. A set $\mathcal{C}$ is a *cone* if it is closed under positive dilations $[tx \in \mathcal{C}$ for all $x \in \mathcal{C}$ and $t > 0]$. An open convex cone is *regular* if it contains no affine lines. The *dual* of a regular open convex cone $\mathcal{C}$ is the cone $\mathcal{C}^* := \{\ell \in V^* \mid \ell(\mathcal{C}) > 0\}$ of functionals which are strictly positive on the original cone. For example, for $0 < \theta < 2\pi$ the wedge of angle θ given by $W_\theta := \{z \in \mathbb{C} \mid -\frac{\theta}{2} < \arg(z) < \frac{\theta}{2}\}$ is an open cone in $\mathbb{R}^2 \cong \mathbb{C}$, which is convex iff $\theta \leq \pi$ and is regular iff $\theta < \pi$.

A positive-definite bilinear form σ on V allows us to identify V^* with V and $\mathcal{C}^*$ with the cone $\{y \in V \mid \sigma(y, \mathcal{C}) > 0\}$ in V, and we say that a cone is *self-dual* with respect to σ if under this identification it coincides with its dual. For example, the dual of the above wedge W_θ with respect to the usual inner product σ on $\mathbb{R}^2$ is $W_\theta^* = W_{\pi-\theta}$, so W_θ is self-dual with respect to σ only for $\theta = \frac{\pi}{2}$.

Formally Real Jordan Algebras

The formally real Jordan algebras investigated by Jordan, von Neumann, and Wigner are precisely the finite-dimensional real unital Jordan algebras such that every element has a unique *spectral decomposition* $x = \lambda_1 e_1 + \cdots + \lambda_r e_r$ for a supplementary family of orthogonal idempotents $1 = e_1 + \cdots + e_r$ and an increasing family $\lambda_1 < \cdots < \lambda_r$ of distinct real eigenvalues. This family, the *spectrum* of x, is uniquely determined as $\mathcal{S}pec(x) := \{\lambda \in \mathbb{C} \mid \lambda 1 - x$ is not invertible in the complexification $J_{\mathbb{C}}\}$. In analogy with Jordan canonical form for matrices (Camille, not Pascual), we can say that the elements all have a *diagonal* Jordan form with *real* eigenvalues. It is easy to see that this spectral reality is equivalent to formal reality $x^2 + y^2 = 0 \Longrightarrow x = y = 0$: if spectra are real and $x^2 = \sum_k \lambda_k^2 e_k$ with spectrum $\{\lambda_k^2\}$ agrees with $-y^2 = \sum_\ell (-\mu_\ell^2) f_\ell$ with spectrum $\{-\mu_\ell^2\}$, then all λ_k, μ_ℓ must be 0 and hence $x = y = 0$; conversely, if some $\mathcal{S}pec(z)$ is not real, containing $\lambda_k = \alpha_k + i\beta_k$ for $\beta_k \neq 0$, then $x^2 + y^2 = 0$ for $x := U_{e_k} z - \alpha_k e_k = i\beta_k e_k,\ y := \beta_k e_k \neq 0$.

The spectral decomposition leads to a *functional calculus*: *Every* real-valued function $f(t)$ defined on a subset $S \subseteq \mathbb{R}$ induces a map on those $x \in \mathrm{J}$ whose spectrum lies wholly in S via $f(x) = \sum_k f(\lambda_k) e_k$. In particular, the invertible elements are those with $\mathcal{S}pec(x) \subseteq \mathbb{R} \setminus \{0\}$, and $f(t) = 1/t$ induces inversion $f(x) = x^{-1} = \sum_k \lambda_k^{-1} e_k$. [When f is the discontinuous function defined on all of $S = \mathbb{R}$ by $f(t) = 1/t$ for $t \neq 0$ and $f(0) = 0$, then $f(x)$ is the *generalized inverse* or *pseudo-inverse* of *any* x.] If S is open and $f(t)$ is continuous, smooth, or given by a power series as a function of t, then so is $f(x)$ as a function of x. It is clear that the set of *positive elements* (those x with positive spectrum $\mathcal{S}pec(x) \subseteq \mathbb{R}^+$) coincides with the set of *invertible squares* $x^2 = \sum_k \lambda_k^2 e_k$ and also the *exponentials* $\exp(x) = \sum_{n=0}^\infty \frac{1}{n!} x^n = \sum_k e^{\lambda_k} e_k$, and can (with some effort) be described as the connected component of the identity element in the set J^{-1} of invertible elements. The set of positive elements is called the *positive cone* $\mathcal{C}one(\mathrm{J})$ of the formally real Jordan algebra J.

Positive Cone Theorem. *The positive cone $\mathcal{C} := \mathcal{C}one(\mathrm{J})$ of an n-dimensional formally real Jordan algebra* J *is an open regular convex cone in* $\mathrm{J} \cong \mathbb{R}^n$ *that is self-dual with respect to the positive definite bilinear trace form* $\sigma(x, y) := \mathrm{tr}(V_{x \bullet y}) = \mathrm{tr}(V_{x,y})$. *The linear operators U_x for $x \in \mathcal{C}$ generate a group G of linear transformations acting transitively on $\mathcal{C}$. Since $\mathcal{C}$ is a connected open subset of* J, *it is naturally a smooth manifold, and we may identify the tangent space $T_p(\mathcal{C})$ with* J; *taking $\langle x, y\rangle_p := \sigma(U_p^{-1} x, y)$ at each point gives a G-invariant Riemannian metric on $\mathcal{C}$. Thus the positive cone $\mathcal{C}one(\mathrm{J})$ of a formally real Jordan algebra is in a canonical way a homogeneous Riemannian manifold.*

The inversion map $j : x \mapsto x^{-1}$ induces a diffeomorphism of J *of period* 2 *leaving $\mathcal{C}$ invariant, and having there a unique fixed point* 1 [*the fixed points of the inversion map are the $e - f$ for $e + f = 1$ supplementary orthogonal idempotents, and those with $f \neq 0$ lie in the other connected components of* J^{-1}], *and provides a symmetry of the Riemannian manifold $\mathcal{C}$ at $p = 1$; here the exponential map is the ordinary algebraic exponential* $\exp_1(x) = e^x$ *from* $T_1(M) = \mathrm{J}$ *to* $\mathcal{C}one(\mathrm{J})$, *and negation $x \mapsto -x$ in the tangent space projects to inversion $e^x \mapsto e^{-x} = \left(e^x\right)^{-1}$ on the manifold. The Jordan product and U-operator arise from the inversion symmetry $s_1 = j$ via $dj_x = -U_x^{-1}$ and $d^2 j_1(u, v) = \partial_u \partial_v j|_1 = V_u(v) = 2\ u \bullet v$. If we choose a basis $\{x_k\}$ for* J, *then the Christoffel symbols determining the affine connection at* 1 *are just the multiplication constants for the algebra: $x_i \bullet x_j = \sum_k \Gamma_{ij}^k x_k$.*

Any other point p in $\mathcal{C}one(\mathrm{J})$ can be considered the unit element in its own algebraic system; since $\mathrm{J}_{[p]}$ has the same invertible elements as J, and by choice p lies in the same connected component as e, so $\mathrm{J}_{[p]}$ has the same connected component of the identity: $\mathcal{C}one(\mathrm{J}_{[p]}) = \mathcal{C}one(\mathrm{J})$. Therefore the manifold has a symmetry at the point p given by $x \mapsto x^{[-1,p]}$, the exponential map is $\exp_p(x) = e^{[x,p]}$, and the Christoffel symbols are just the multiplication constants of $\mathrm{J}_{[p]} : x_i \bullet_p x_j = \sum_k \Gamma^k_{ij}[p] x_k$. Thus every point is an isolated fixed point of a symmetry, given by inversion in a Jordan isotope, and the self-dual positive cone $\mathcal{C}one(\mathrm{J})$ becomes a Riemannian symmetric space.

Rather than use isotopy, we noted above that we can use transitivity to create the symmetry at $p = g(1)$ once we have one at 1. The structure group $\mathcal{S}trg(\mathrm{J})$ of the Jordan algebra is a real Lie group leaving the set of invertible elements invariant, and having isotropy group at 1 precisely the automorphism group $\mathcal{A}ut(\mathrm{J})$. Its connected component $G := \mathcal{S}trg(\mathrm{J})^0$ of the identity leaves the cone $\mathcal{C}$ invariant, and acts transitively because already the linear transformations U_c $(c \in \mathcal{C})$ belong to G and every positive $p = \sum_k \lambda_k e_k \in \mathcal{C}$ $(\lambda_k > 0)$ has the form $p = c^2 = U_c(1)$ for positive $c = \sqrt{p} = \sum_k \sqrt{\lambda_k} e_k \in \mathcal{C}$. Thus $\mathcal{C} \cong G/K$ for the isotropy group $K = G \cap \mathcal{A}ut(\mathrm{J})$ of G at the identity (the structural transformations which fix 1 are precisely the automorphisms of J). The group $\mathcal{A}ut(\mathrm{J})$ is compact, since it leaves invariant the positive-definite inner product $\sigma(x,y) := \mathrm{tr}(V_{x,y})$, so K is compact; K is also connected, since a standard argument from simple connectivity of $\mathcal{C}$ shows that $K = \mathcal{A}ut(\mathrm{J})^0$. We get a G-invariant metric on $\mathcal{C}$ via $\langle U_c x, U_c y\rangle_{U_c(1)} := \sigma(x,y)$, so $\langle x,y\rangle_p := \sigma(U_c^{-1}x, U_c^{-1}y) = \sigma(U_c^{-2}x, y) = \sigma(U_p^{-1}x, y)$ for all points $p = U_c(1) \in \mathcal{C}$.

A particular example may make this clearer.

Hermitian Complex Matrix Example. *Let $\mathrm{J} = \mathcal{H}_n(\mathbb{C})$ be the formally real Jordan algebra of dimension n^2 over the reals consisting of all $Z \in \mathcal{M}_n(\mathbb{C})$ with $Z^* = Z$. Then the positive cone $\mathcal{C}one(\mathrm{J})$ consists precisely of the positive-definite matrices (the hermitian matrices whose Jordan form has only positive real eigenvalues). The structure group is generated by the two involutory transformations $Z \mapsto -Z$ and $Z \mapsto \overline{Z} = Z^{tr}$ together with the connected subgroup $G = \mathcal{S}trg(\mathrm{J})^0$ of all $Z \mapsto AZA^*$ for $A \in \mathrm{GL}_n(\mathbb{C})$. The connected component $K = \mathcal{A}ut(\mathrm{J})^0$ of the automorphism group consists of all $Z \mapsto UZU^* = UZU^{-1}$ for unitary $U \in U_n(\mathbb{C})$. The Riemannian metric flows from $\sigma(X,Y) := \mathrm{tr}(V_{X,Y}) = 2n\ \mathrm{tr}(XY) = 2n \sum_{j,k=1}^n x_{jk}\overline{y_{jk}}$ for $X = \big(x_{jk}\big), Y = \big(y_{jk}\big)$, corresponding to a multiple of the Hilbert–Schmidt norm $\|X\| = \big(\sum_{j,k=1}^n |x_{jk}|^2\big)^{1/2}$. The Riemann metric $\langle X,Y\rangle_P$ at a point P is given by the matrix trace $2n\,\mathrm{tr}(P^{-1}XP^{-1}Y)$.*

To Infinity and Beyond: JB-algebras

In 1978 Alfsen, Shultz, and Størmer obtained a Gelfand–Naimark Theorem for "Jordan C^*-algebras" in which they showed that the only exceptional C^*-factors are Albert algebras. In retrospect, this was a harbinger of things to come, but at the time it did not diminish hope among algebraists for infinite-dimensional exceptional algebras.

These Jordan C^*-algebras are a beautiful generalization of formally real Jordan algebras to the infinite-dimensional setting, which combines differential geometry with functional analysis. The infinite-dimensional algebra is kept within limits by imposing a norm topology. A *norm* on a real or complex vector space V is a real-valued function $\|\cdot\| : V \to \mathbb{R}$ which is *homogeneous* $[\,\|\alpha x\| = |\alpha| \cdot \|x\|$ for all scalars $\alpha\,]$, *positive definite* $[\,x \neq 0 \Longrightarrow \|x\| > 0\,]$, and satisfies the *triangle inequality* $[\,\|x+y\| \leq \|x\| + \|y\|\,]$. A *Euclidean* or *hermitian* norm is one that comes from a Euclidean or hermitian inner product $\langle\, ,\, \rangle$ on V via $\|x\| = \sqrt{\langle x, x\rangle}$. Any norm induces a metric topology with neighborhoods of x the $x + B_r$ for B_r the open r-ball $\{y \in V \mid \|y\| < r\}$ (so $x_n \to x$ iff $\lim_{n\to\infty} \|x_n - x\| = 0$). A *Banach space* is a normed linear space which is *complete* in the norm topology. In finite dimensions every Banach space can be re-normed to become a Hilbert space (with norm given by an inner product), but in infinite dimensions this is far from true.

A *Jordan–Banach algebra* or *JB-algebra* is a real Jordan algebra J which is at the same time a Banach space, with the two structures related by

Banach algebra condition: $\|x \bullet y\| \leq \|x\|\,\|y\|$,
C^*-condition: $\|x^2\| = \|x\|^2$,
Positivity condition: $\|x^2\| \leq \|x^2 + y^2\|$.

An associative algebra is called a *Banach algebra* if it has a Banach norm satisfying $\|xy\| \leq \|x\|\,\|y\|$; it is called a real C^*-algebra if in addition it has an isometric involution satisfying $\|xx^*\| = \|x\|^2$ and all $1 + xx^*$ are invertible. Thus JB-algebras are a natural Jordan analogue of the associative C^*-algebras.

Such a norm on a Jordan algebra is *unique* if it exists, and automatically $\|1\| = 1$ if J has a unit element. With some effort it can be shown that any JB-algebra can be imbedded in a unital one $\mathrm{J}' := \mathbb{R} \oplus \mathrm{J}$ using the spectral norm on J', and *we will henceforth assume that all our JB-algebras are unital.* Notice that the C^* and positivity conditions have as immediate consequence the usual formal reality condition $x^2 + y^2 = 0 \Rightarrow x = y = 0$. Conversely, all finite-dimensional formally real Jordan algebras (in particular, the real Albert algebra $\mathcal{H}_3(\mathbb{K})$) carry a unique norm making them JB-algebras, so in finite dimensions JB is the same as formally real. This is false in infinite-dimensions: A formally real Jordan algebra can satisfy the Banach algebra and C^*-conditions but not the crucial Positivity condition. The classical "concrete" JB-algebra is the algebra $\mathcal{H}(B(H), *)$ consisting of all bounded self-adjoint operators on a Hilbert space H with operator norm and the usual Hilbert-space adjoint.

The analogue of special algebras in the JB category are the "concrete" *JC-algebras,* those isomorphic to norm-closed subalgebras of some $\mathcal{H}(B(H),*)$. The Albert algebra $\mathcal{H}_3(\mathbb{K})$ is JB but not JC, since it is not even special.

The celebrated *Gelfand–Naimark Theorem for JB-algebras* of Erik Alfsen, Frederic Shultz, and Erling Størmer asserts that every JB-algebra can be isometrically isomorphically imbedded in some $\mathcal{H}(B(H),*) \oplus \mathcal{C}(X, \mathcal{H}_3(\mathbb{K}))$, a direct sum of the JC-algebra of all hermitian operators on some real Hilbert space, and the purely exceptional algebra of all continuous Albert-algebra-valued functions on some compact topological space X. This has the prophetic consequence that as soon as a JB-algebra satisfies Glennie's Identity G_8 it is immediately a special JC-algebra.

We have a *continuous functional calculus* for JB-algebras: For any element x the smallest norm-closed unital subalgebra containing x is isometrically isomorphic to the commutative associative C^*-algebra $C(\mathcal{S}pec(x))$ of all continuous real-valued functions on the compact *spectrum* $\mathcal{S}pec(x) := \{\lambda \in \mathbb{R} \mid \lambda 1 - x$ is not invertible in J$\}$, under an isomorphism sending x to the identity function $1_{\mathbb{R}}(t) = t$. Note that x is invertible iff $0 \notin \mathcal{S}pec(x)$. An element is *positive* if it has positive spectrum $\mathcal{S}pec(x) \subseteq \mathbb{R}^+$; once more, this is equivalent to being an invertible square, or to being an exponential.

The *positive cone* $\mathcal{C}one(\mathrm{J})$ consists of all positive elements; it is a regular open convex cone, and again the existence of square roots shows that the group G generated by the invertible operators U_c $(c \in \mathcal{C})$ acts transitively on $\mathcal{C}$, so we have a homogeneous cone. Again each point $p \in \mathcal{C}$ is an isolated fixed point of a symmetry $s_p(x) = x^{[-1,p]} = U_p x^{-1}$. Since in infinite dimensions the tangent spaces $T_p(\mathcal{C})$ are merely Banach spaces, not Hilbert spaces, there is in general no G-invariant Riemannian metric to provide the usual concepts of differential geometry (geodesics, curvature, etc.), and these must be defined directly from the symmetric structure. For example, a *geodesic* is a connected 1-dimensional submanifold M of $\mathcal{C}$ which is *symmetric* (or *totally geodesic*) in the sense that it is invariant under the local symmetries, $s_p(M) = M$ for all $p \in M$, and any two distinct points in $\mathcal{C}$ can be joined by a unique geodesic.

The category of JB-algebras is equivalent under complexification to the category of JB^*-algebras. A *JB^*-algebra* $(\mathrm{J}, *)$ is a complex Jordan algebra with a $*$ (a conjugate-linear algebra involution) having at the same time the structure of a complex Banach space, where the algebraic structure satisfies the norm conditions

Banach algebra condition: $\|x \bullet y\| \leq \|x\|\,\|y\|$,
C^*-condition: $\|\{x, x^*, x\}\| = \|x\|^3$,
Isometry condition: $\|x^*\| = \|x\|$.

A complex associative Banach algebra is called a C^*-algebra it has a $\mathbb{C}$-antilinear algebra involution (necessarily isometric) satisfying $(xy)^* = y^*x^*$

and $\|x^*x\| = \|x\|^2$. Thus JB^*-algebras are a natural Jordan analogue of the complex C^*-algebras.

Every complex JB^*-algebra $(\mathrm{J}, *)$ produces a real JB-algebra $\mathcal{H}(\mathrm{J}, *)$ consisting of all self-adjoint elements, and conversely, for every JB-algebra J the natural complexification $\mathrm{J}_{\mathbb{C}} := \mathrm{J} \oplus i\mathrm{J}$ with involution $(x + iy)^* := x - iy$ can (with difficulty) be shown to carry a norm which makes it a JB^*-algebra. We have a similar *Gelfand–Naimark Theorem for JB^* algebras* that every JB^*-algebra can be isometrically isomorphically imbedded in some $B(H) \oplus \mathcal{C}(X, \mathcal{H}_3(\mathbb{K}_{\mathbb{C}}))$, a direct sum of the C^*-algebra of all bounded operators on some complex Hilbert space H and a purely exceptional algebra of all continuous functions on some compact topological space X with values in the complex Albert algebra.

These complex spaces provide a setting for an important bounded symmetric domain associated with the cone.

0.6 Links with the Complex World

Next we discuss connections between complex Jordan structures and complex analytic manifolds These hermitian Jordan algebras are complex vector spaces, but the Jordan triple product is antilinear in the middle variable, so by our conventions they are real rather than complex Jordan triples.

Bounded Symmetric Domains

The complex analogue of a Riemannian manifold is a *hermitian manifold,* a complex analytic manifold M carrying a *hermitian metric*, a smoothly-varying positive-definite hermitian inner product $\langle\ ,\ \rangle_p$ on the tangent space $T_p(M)$ to the manifold at each point p. An *isomorphism* of Hermitian manifolds is a biholomorphic map of analytic manifolds whose differential is isometric on each tangent space. A *hermitian symmetric space* is a (connected) hermitian manifold having at each point p a *symmetry* s_p [an involutive global isomorphism of the manifold having p as isolated fixed point]. We henceforth assume that all our Hermitian manifolds are connected.

These are abstract manifolds, but every Hermitian symmetric space of "noncompact type" [having negative holomorphic sectional curvature] is a *bounded symmetric domain,* a down-to-earth bounded domain in $\mathbb{C}^n$ each point of which is an isolated fixed point of an involutive biholomorphic map of the domain. Initially there is no metric on such a domain, but there is a natural way to introduce one (for instance, the Bergmann metric derived from the Bergmann kernel on a corresponding Hilbert space of holomorphic functions).

In turn, every bounded symmetric domain is biholomorphically equivalent via its Harish–Chandra realization to a *bounded homogeneous circled domain,* a bounded domain containing the origin which is *circled* in the sense that the circling maps $x \mapsto e^{it}x$ are automorphisms of the domain for all real t, and *homogeneous* in the sense that the group of all biholomorphic automorphisms acts transitively on the domain. These domains are automatically convex, and arise as the open unit ball with respect to a certain norm on $\mathbb{C}^n$.

The classical example of an unbounded realization of a bounded symmetric domain is the *upper half-plane* M in $\mathbb{C}$, consisting of all $x+iy$ for real $x, y \in \mathbb{R}$ with $y > 0$. This is the home of the theory of automorphic forms and functions, a subject central to many areas of mathematics. The upper half-plane can be mapped by the Cayley transform $z \mapsto \frac{i-z}{i+z} = \frac{1+iz}{1-iz}$ onto the *open unit disk* Δ (consisting of all $w \in \mathbb{C}$ with $|w| < 1$, i.e., $1 - w\bar{w} > 0$).

This was generalized by Carl Ludwig Siegel to *Siegel's upper half-space* M (consisting of all $X + iY$ for symmetric $X, Y \in \mathcal{M}_n(\mathbb{R})$ with Y positive definite), to provide a home for the Siegel modular forms in the study of functions of several complex variables. Again, this is mapped by the Cayley transform $Z \mapsto (i\mathbb{1}_{nn} - Z)(i\mathbb{1}_{nn} + Z)^{-1} = (\mathbb{1}_{nn} + iZ)(\mathbb{1}_{nn} - iZ)^{-1}$ onto the *generalized unit disk* D consisting of all symmetric $W \in \mathcal{M}_n(\mathbb{C})$ with $\mathbb{1}_{nn} - W\overline{W}$ positive definite.

Max Koecher began his life as an arithmetic complex analyst, but his studies of modular functions led him inexorably to Jordan algebras. He generalized Siegel's upper half-space to the case of an arbitrary formally real Jordan algebra J: *Koecher's upper half-space* $M = \mathit{Half}(\mathrm{J})$ consists of all $Z = X + iY$ for $X, Y \in \mathrm{J}$ with Y in the positive cone $\mathcal{C}one(\mathrm{J})$. These half-spaces or *tube domains* $\mathrm{J} \oplus i\mathcal{C}$ are open and convex in the complexification $\mathrm{J}_{\mathbb{C}} := \mathrm{J} \oplus i\mathrm{J}$. The geometry of the unbounded $\mathit{Half}(\mathrm{J})$ is nicely described in terms of the Jordan algebra $\mathrm{J}_{\mathbb{C}}$: The biholomorphic automorphisms of $\mathit{Half}(\mathrm{J})$ are the linear fractional transformations generated by inversion $Z \mapsto -Z^{-1}$, translations $Z \mapsto Z + A$ ($A \in \mathrm{J}$), and $Z \mapsto \tilde{T}(Z) = T(X) + iT(Y)$ for complex-linear extensions $\tilde{T}$ of automorphisms $T \in \mathcal{A}ut(\mathrm{J})$. These half-spaces are mapped by the Cayley transform $Z \mapsto (1 + iZ)(1 - iZ)^{-1}$ onto the *generalized open unit disk* $D(\mathrm{J})$ consisting of all $Z = X + iY \in \mathrm{J}_{\mathbb{C}} = \mathrm{J} \oplus i\mathrm{J}$ with $\mathbb{1}_{\mathrm{J}} - \frac{1}{2}V_{Z,\overline{Z}}$ positive definite with respect to the trace form $\mathrm{tr}(V_{W,\overline{Z}})$ on the Jordan algebra $\mathrm{J}_{\mathbb{C}}$ (where $\overline{Z} := X - iY$). $D(\mathrm{J})$ can also be characterized as the unit ball of $\mathrm{J}_{\mathbb{C}}$ under a certain spectral norm.

Positive Hermitian Triple Systems

Ottmar Loos showed that there is a natural 1-to-1 correspondence between bounded homogeneous circled domains D in $\mathbb{C}^n$ and the finite-dimensional positive hermitian Jordan triples. A *hermitian Jordan triple* is a complex vector space J with triple product $\{x, y, z\} = L_{x,y}(z)$ which is symmetric and $\mathbb{C}$-linear in the outer variables x, z and conjugate-linear in the middle variable y, satisfying the 5-linear axiom $\{x, y, \{z, w, v\}\} = \{\{x, y, z\}, w, v\} - \{z, \{y, x, w\}, v\} + \{z, w, \{x, y, v\}\}$ for a Jordan triple system. A finite-dimensional hermitian Jordan triple is *positive* if the trace form $\mathrm{tr}(L_{x,y})$ is a positive definite Hermitian scalar product.

Every nonzero element has a unique spectral decomposition $x = \sum \lambda_k e_k$ for nonzero orthogonal *tripotents* $e_k^3 = e_k$ and distinct positive *singular values* $0 < \lambda_1 < \cdots < \lambda_r \in \mathbb{R}$ (called the *singular spectrum* of x); the *spectral norm* is the maximal size $\|x\| := \max_i \lambda_i = \lambda_r$ of the singular values. At first sight it is surprising that every element seems to be "positive," but recall that by conjugate linearity in the middle variable a tripotent can absorb any unitary complex scalar μ to produce an equivalent tripotent $e' = \mu e$, so any complex "eigenvalue" $\zeta = \lambda e^{i\theta} = \lambda\mu$ can be replaced by a real singular value $\lambda :\ \zeta e = \lambda e'$ for the tripotent $e' = \mu e$. The real singular value is determined up to $\pm$, but if we use only an *odd functional calculus* $f(x) = \sum_k f(\lambda_k) e_k$ for *odd* functions f on $\mathbb{R}$ this ambiguity $(-\lambda)(-e) = \lambda e$ is resolved: $f(-\lambda)(-e) = -f(\lambda)(-e) = f(\lambda)e$ is independent of the choice of sign.

The *Bergmann kernel function* $K(x, y)$, the reproducing function for the Hibert space of all holomorphic L^2-functions on the domain, is intimately related to the *Bergmann operator* $B_{x,y} := \mathbb{1}_{\mathrm{J}} - L_{x,y} + P_x P_y$ of the triple by the formula $K(x, y) = \kappa / \det(B_{x,y})$ for a fixed constant κ. [This is the reason for the name and the letter B, although Stefan Bergmann himself had never heard of a Jordan triple system!]

We have a complete algebraic description of all these positive triples:

Hermitian Triple Classification. *Every finite-dimensional positive hermitian triple system is a finite direct sum of simple triples, and there are exactly six classes of simple triples* (*together with a positive involution*)*: four great classes of special triples,*

(1) *rectangular matrices* $\mathcal{M}_{pq}(\mathbb{C})$,

(2) *skew matrices* $Skew_n(\mathbb{C})$,

(3) *symmetric matrices* $Symm_n(\mathbb{C})$,

(4) *spin factors* $\mathcal{J}Spin_n(\mathbb{C})$,

and two sporadic exceptional systems,

(5) *the bi-Cayley triple* $\mathcal{M}_{12}\big(\mathbb{K}_{\mathbb{C}}\big)$ *of dimension* 16,

(6) *the Albert triple* $\mathcal{H}_3(\mathbb{K}_{\mathbb{C}})$ *of dimension* 27

determined by the split octonion algebra $\mathbb{K}_{\mathbb{C}}$ *over the complexes.*

The geometric properties of bounded symmetric domains are beautifully described by the algebraic properties of these triple systems.

Jordan Unit Ball Theorem. *Every positive hermitian Jordan triple system* J *gives rise to a bounded homogeneous convex circled domain* $D(\mathrm{J})$ *that is the open unit ball of* J *as a Banach space under the spectral norm, or equivalently,*

$$D(\mathrm{J}) := \big\{x \in \mathrm{J} \mid \mathbb{1}_{\mathrm{J}} - \tfrac{1}{2} L_{x,x} > 0\big\}.$$

Every bounded symmetric domain D arises in this way: its bounded realization is $D(\mathrm{J})$, where the algebraic triple product can be recovered from the Bergmann metric $\langle\, ,\, \rangle$ and Bergmann kernel K of the domain as the logarithmic derivative of the kernel at 0*:*

$$\langle \{u, v, w\}, z \rangle_0 := d_0^4 K(x, x)(u, v, w, z) := \partial_u \partial_v \partial_w \partial_z \log K(x, x)|_{x=0}.$$

The domain $D := D(\mathrm{J})$ of the triple J *becomes a hermitian symmetric space under the Bergmann metric $\langle x, y \rangle_p := \mathrm{tr}(L_{B_{p,p}^{-1} x, y})$. The automorphisms of the domain D fixing* 0 *are linear and are precisely the algebraic automorphisms of the triple* J*. At the origin the exponential map $\exp_0 : T_0(D) = \mathrm{J} \to D$ is a real analytic diffeomorphism given by the odd function $\exp_0(v) = \tanh(v)$, and the curvature tensor is given by $R(u, v)_0 = L_{v,u} - L_{u,v}$.*

The Shilov boundary of D is the set of maximal tripotents of J*, and coincides with the set of all extremal points of the convex set $\overline{D}$; it can be described algebraically as the set of $z \in \mathrm{J}$ with $B_{z,z} = 0$. The holomorphic boundary components of D are precisely the faces of the convex set $\overline{D}$, which are just the sets $e + \big(D \cap \mathrm{Ker}(B_{e,e})\big)$ for all tripotents e of* J*.*

As an example, in rectangular matrices $\mathcal{M}_{pq}(\mathbb{C})$ for $p \le q$ every X can be written as $X = UDV$ for a unitary $p \times p$ matrix U and a unitary $q \times q$ matrix V and *diagonal real* $p \times q$ matrix D with scalars $d_1 \ge \cdots \ge d_p \ge 0$ down the "main diagonal." These d_i are precisely the singular values of X, and are precisely the nonnegative square roots of the *eigenvalues* of $XX^* = UDD^{tr}U^{-1}$; the spectral norm of X is $\|X\| = d_1$. The condition $\mathbb{1}_{\mathrm{J}} - \frac{1}{2} L_{X,X} > 0$ is equivalent to $1 > d_1$, i.e., the unit ball condition. The triple trace is $\mathrm{tr}(L_{X,Y}) = (p+q)\mathrm{tr}(XY^*)$, so the hermitian inner product is

$$\langle X, Y \rangle_P = (p+q)\ \mathrm{tr}\big((\mathbb{1}_{pp} - PP^*)^{-1} X (\mathbb{1}_{qq} - P^*P)^{-1} Y^*\big).$$

0.7 Links with the Infinitely Complex World

There are many instances in mathematics where we gain a better understanding of the finite-dimensional situation by stepping back to view the subject from the infinite-dimensional perspective. In Jordan structure theory, Zel'manov's general classification of simple algebras reveals the sharp distinction between algebras of Hermitian, Clifford, and Albert type, whereas the original Jordan–von Neumann–Wigner classification included $\mathcal{H}_3(\mathbb{K})$ as an outlying member of the hermitian class, and for a long time the viewpoint of idempotents caused the division algebras (algebras of capacity 1) to be considered in a class by themselves.

So too in differential geometry the infinite perspective has revealed more clearly what is essential and what is accidental in the finite-dimensional situation.

From Bounded Symmetric Domains to Unit Balls

In infinite dimensions not all complex Banach spaces can be renormed as Hilbert spaces, so we cannot hope to put a hermitian norm (coming from an inner product) on tangent spaces. However, it turns out that there is a natural way to introduce a norm. A *bounded symmetric domain* D in a complex Banach space V is a *domain* [a connected open subset] which is *bounded* in norm [$\|D\| \leq r$ for some r] and is *symmetric* in the sense that at each point $p \in D$ there is a *symmetry* s_p [a biholomorphic map $D \to D$ of period 2 having p as isolated fixed point]. Again the existence of symmetries at all points is a strong condition: the symmetries s_p are unique, and the group $G := \mathcal{A}ut(D)$ of biholomorphic automorphisms of D is a *real Banach–Lie group* [locally coordinatized by patches from a fixed real Banach space instead of $\mathbb{R}^n$] acting analytically and transitively on D. Note that any biholomorphic $g \in G$ conjugates a symmetry s_p at a point p to a symmetry $g \circ s_p \circ g^{-1}$ at the point $q = g(p)$. Thus we can rephrase the condition for symmetry for a domain as (i) there is a symmetry s_{p_0} at some particular basepoint, (ii) $\mathcal{A}ut(D)$ acts transitively on D.

Again in infinite dimensions there is no G-invariant Bergmann metric to provide the usual concepts of differential geometry. Instead of a hermitian metric there is a *canonical* G-invariant Banach norm on each tangent space $T_p(D)$, the *Carathéodory tangent norm* $\|v\| := \sup_{f \in \mathcal{F}_p} |df_p(v)|$ taken over the set $\mathcal{F}_p$ of all holomorphic functions of D into the open unit disk which vanish at p. In finite dimensions the existence of a hermitian inner product on $\mathbb{C}^n$ seduces us into forming a Hilbert norm, even though in many ways the Carathéodory norm is more natural (for example, for hermitian operators the Carathéodory norm is the intrinsic operator norm, whereas the Hilbert–Schmidt norm $\|X\|^2 = \sum_{j,k=1}^n |x_{jk}|^2$ is basis-dependent). By G-invariance, for any *basepoint* p_0 of D all tangent spaces can be identified with $T_{p_0}(D)$, which is equivalent to V.

The Harish–Chandra realization of a bounded symmetric domain in $\mathbb{C}^n$ is replaced by a *canonical bounded realization* as a unit ball: A Riemann Mapping–type theorem asserts that every bounded symmetric domain with basepoint p_0 is biholomorphically equivalent (uniquely up to a linear isometry) to the full open unit ball with basepoint 0 of a complex Banach space $V = T_{p_0}(D)$. Here the norm on V grows naturally out of the geometry of D.

Jordan Unit Balls

Note that a unit ball always has the symmetry $x \mapsto -x$ at the basepoint 0, so the above conditions (i),(ii) for D to be symmetric reduce to just transitivity. Thus the Banach unit balls which form symmetric domains are precisely those whose biholomorphic automorphism group $\mathcal{A}ut(D)$ acts transitively. The map $(y, z) \mapsto s_y(z)$ gives a "multiplication" on D, whose linearization gives rise to a *triple product*

$$\{u, v, w\} := -\tfrac{1}{2}\partial_u^z \partial_v^y \partial_w^z \big(s_y(z)\big)|_{(0,0)}$$

uniquely determined by the symmetric structure. This gives a *hermitian Jordan triple product*, a product which is complex linear in u, w and complex anti-linear in v satisfying the 5-linear identity $\{x, y, \{u, v, w\}\} = \{\{x, y, u\}, v, w\} - \{u, \{y, x, v\}, w\} + \{u, v, \{x, y, w\}\}$.

Moreover, it is a *positive hermitian Jordan triple system*. In infinite dimensions there is seldom a trace, so we can no longer use as definition of positivity for a hermitian Jordan triple Loos's condition that the trace $\mathrm{tr}(L_{x,y})$ be positive-definite. Instead, we delete the trace and require that the operators $L_{x,x}$ themselves be *positive definite* in the sense that

$L_{x,x}$ is hermitian: $\exp(itL_{x,x})$ is an isometry for all real t,
C^*-condition for $L_{x,x}$: $\|L_{x,x}\| = \|x\|^2$,
Nonnegative spectrum: $\mathcal{S}pec(L_{x,x}) \geq 0$.

[The usual notion $T^* = T$ of hermitian operators on Hilbert space does not apply to operators on a Banach space; the correct general definition of *hermitian operator* is a bounded complex-linear transformation T on V such that the invertible linear transformation $\exp(itT)$ is an isometry of V for all real t (just as a complex number z is real iff all e^{itz} are points on the unit circle, $|e^{itz}| = 1$).]

Because of the norm conditions, these are also called *JB^*-triples* in analogy with JB^*-algebras. Nontrivial consequences of these axioms are

Banach triple condition: $\|\{x, y, z\}\| \leq \|x\| \, \|y\| \, \|z\|$,
C^*-triple condition: $\|\{z, z, z\}\| = \|z\|^3$.

Any associative C^*-algebra produces a JB^*-triple with triple product $\{x, y, z\} := \frac{1}{2}(xy^*z + zy^*x)$. Thus a JB^*-triple is a Jordan triple analogue of a C^*-algebra. It can be shown that every JB^*-algebra J carries the JB^*-triple structure J^t via $\{x, y, z\}^t := \{x, y^*, z\}$. [Certainly this modification is a hermitian Jordan triple product: it continues to satisfy the Jordan triple axiom since $*$ is a Jordan isomorphism, and it is conjugate-linear in the middle variable due to the involution $*$ (the triple product on the JB-algebra $\mathcal{H}(\mathrm{J}, *)$ is $\mathbb{R}$-linear, and that on its complexification $\mathrm{J} = \mathcal{H}_{\mathbb{C}}$ is $\mathbb{C}$-linear). The C^*-algebra condition immediately implies the C^*-triple condition, but the other JB^*-triple conditions require more work.] The celebrated *Gelfand–Naimark Theorem for JB^* Triples* of Yakov Friedman and Bernard Russo asserts that every such triple imbeds isometrically and isomorphically in a triple $\mathrm{J}^t = B(H)^t \oplus \mathcal{C}(X, \mathcal{H}_3(\mathbb{O}_{\mathbb{C}}))^t$ obtained by "tripling" a JB^*-algebra. (Notice that the triple of rectangular $p \times q$ matrices can be imbedded in the algebra of $(p+q) \times (p+q)$ matrices, and the exceptional 16-dimensional triple imbeds in the tripled Albert algebra.)

The unit ball $D(\mathrm{J})$ of any JB^*-triple automatically has a transitive biholomorphic automorphism group. Thus the open unit ball of a Banach space V is a bounded symmetric domain iff V carries naturally the structure of a JB^*-triple.

Jordan Functor Theorem. *There is a category equivalence between the category of all bounded symmetric domains with base point and the category of all JB^*-triples, given by the functors $(D, p_0) \mapsto (T_{p_0}(D), \{\cdot,\cdot,\cdot\})$ and $\mathrm{J} \mapsto (D(\mathrm{J}), 0)$.*

We have seen that the unit balls of JB^*-algebras $(\mathrm{J}, *)$ are bounded symmetric domains which have an unbounded realization (via the inverse Cayley transform $w \mapsto i(\mathbb{1} - w)(\mathbb{1} + w)^{-1}$) as *tube domains* $\mathcal{H}(\mathrm{J}, *) + i\mathcal{C}$. The geometric class of tube domains can be singled out algebraically from the class of all bounded symmetric domains: they are precisely the domains that come from JB^*-triples which have *invertible elements*.

These examples, from the real and complex world, suggest a general Principle: *Geometric structure is often encoded in algebraic Jordan structure.*

0.8 Links with Projective Geometry

Another example of the serendipitous appearance of Jordan algebras is in the study of projective planes. In 1933 Ruth Moufang used an octonion division algebra to construct a projective plane which satisfied the Harmonic Point Theorem and Little Desargues's Theorem, but not Desargues's Theorem. However, this description did not allow one to describe the automorphisms of the plane, since the usual associative approach via invertible 3×3 matrices breaks down for matrices with nonassociative octonion entries. In 1949 P. Jordan found a way to construct the real octonion plane inside the formally real Albert algebra $\mathbb{A} = \mathcal{H}_3(\mathbb{K})$ of hermitian 3×3 Cayley matrices, using the set of primitive idempotents to coordinatize both the points and the lines. This was rediscovered in 1951 by Hans Freudenthal. In 1959 this was greatly extended by T.A. Springer to reduced Albert algebras $\mathrm{J} = \mathcal{H}_3(\mathrm{O})$ for octonion division algebras over arbitrary fields of characteristic $\neq 2, 3$, obtaining a "Fundamental Theorem of Octonion Planes" describing the automorphism group of the plane in terms of the "semi-linear structure group" of the Jordan ring. In this not-formally-real case the coordinates were the "rank 1" elements (primitive idempotents or nilpotents of index 2). Finally, in 1970 John Faulkner extended the construction to algebras over fields of any characteristic, using the newly-hatched quadratic Jordan algebras. Here the points and lines are inner ideals with inclusion as incidence; structural maps naturally preserve this relation, and so induce geometric isomorphisms.

The Fundamental Theorem of Projective Geometry for octonion planes says that all isomorphisms are obtained in this way. Thus octonion planes find a natural home for their isomorphisms in Albert algebras.

Projective Planes

Recall that an abstract *plane* $\Pi = (\mathcal{P}, \mathcal{L}, I)$ consists of a set of *points* $\mathcal{P}$, a set of *lines* $\mathcal{L}$, and an *incidence relation* $\mathrm{I} \subset \mathcal{P} \times \mathcal{L}$. If $P \,\mathrm{I}\, L$ we say that P *lies on* L and L *lies on* or *goes through* P. A collection of points are *collinear* if they are all incident to a common line, and a collection of lines are *concurrent* if they lie on a common point. A plane is *projective* if it satisfies the three axioms (I) every two distinct points P_1, P_2 are incident to a unique line (denoted by $P_1 \vee P_2$), (II) every two distinct lines L_1, L_2 are incident to a unique point (denoted by $L_1 \wedge L_2$), (III) there exists a 4-*point* (four points, no three of which are collinear). We get a category of projective planes by taking as morphisms the *isomorphisms* $\sigma = (\sigma_{\mathcal{P}}, \sigma_{\mathcal{L}}) : \Pi \to \Pi'$ consisting of bijections $\sigma_{\mathcal{P}} : \mathcal{P} \to \mathcal{P}'$ of points and $\sigma_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}'$ of lines which preserve incidence, $P \,\mathrm{I}\, L \Leftrightarrow \sigma_{\mathcal{P}}(P) \,\mathrm{I}'\, \sigma_{\mathcal{L}}(L)$. (In fact, either of $\sigma_{\mathcal{P}}$ or $\sigma_{\mathcal{L}}$ completely determines the other. Automorphisms σ of a plane are called *collineations* by geometers, since the $\sigma_{\mathcal{P}}$ are precisely the maps of points which preserve collinearity: all lines have the form $L = P_1 \vee P_2$, and $P \,\mathrm{I}\, \big(P_1 \vee P_2\big) \Leftrightarrow \{P, P_1, P_2\}$ are collinear.)

The most important example of a projective plane is the *vector space plane* $\mathcal{P}roj(V)$ determined by a 3-dimensional vector space V over an associative division ring Δ, where the points are the 1-dimensional subspaces P, the lines are the 2-dimensional subspaces L, and incidence is inclusion $P \,\mathrm{I}\, L \Leftrightarrow P \subseteq L$. Here $P_1 \vee P_2 = P_1 + P_2,\ L_1 \wedge L_2 = L_1 \cap L_2$ (which has dimension $\dim(L_1) + \dim(L_2) - \dim(L_1 + L_2) = 2 + 2 - 3 = 1$). If v_1, v_2, v_3 form a basis for V_3, then these together with $v_4 = v_1 + v_2 + v_3$ form a four point (no three are collinear since any three span the whole 3-dimensional space). These planes can also be realized by a construction $\mathcal{P}roj(\Delta)$ directly from the underlying division ring. Every 3-dimensional left vector space V is isomorphic to ${}_\Delta\Delta^3$, with dual space V^* isomorphic to the right vector space Δ^3_Δ under the nondegenerate bilinear pairing $\langle v, w\rangle = \langle(\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3)\rangle = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3$. The points $P = \Delta v = [v]_* \le {}_\Delta\Delta^3$ are coordinatized (up to a left multiple) by a nonzero vector v, the lines L are in 1-to-1 correspondence with their 1-dimensional orthogonal complements $L^\perp \le V^*$ corresponding to a "dual point" $[w]^* = w\Delta \le \Delta^3_\Delta$, and incidence $P \subseteq L = (L^\perp)^\perp$ reduces to *orthogonality* $P \perp L^\perp$, i.e.,

$$[v]_* \,\mathrm{I}\, [w]^* \Leftrightarrow \langle v, w\rangle = 0$$

(which is independent of the representing vectors for the 1-dimensional spaces). We choose particular representatives v, w of the points $\Delta\mathbf{x}$, $\mathbf{y}\Delta$ for nonzero vectors $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$ as follows: if $x_3 \neq 0$ choose $v = \frac{1}{x_3}\mathbf{x} = (x, y, 1)$; if $x_3 = 0 \neq x_1$ choose $v = \frac{1}{x_1}\mathbf{x} = (1, n, 0)$; if $x_3 = x_1 = 0$ choose $v = \frac{1}{x_2}\mathbf{x} = (0, 1, 0)$. Dually, if $y_2 \neq 0$ choose

$w = \frac{-1}{y_2}\mathbf{y} = (m, -1, b)$; if $y_2 = 0 \neq y_1$ choose $w = \frac{-1}{y_1}\mathbf{y} = (-1, 0, a)$; if $y_2 = y_1 = 0$ choose $w = \frac{1}{y_3}\mathbf{y} = (0, 0, 1)$. Then we obtain the following table of incidences $\langle v, w\rangle = 0$:

Incidence Table $[v]_* \,\mathrm{I}\, [w]^*$ for $\mathcal{P}roj(\Delta)$

$[v]_* \setminus [w]^*$	$[(m, -1, b)]^*$	$[(-1, 0, a)]^*$	$[(0, 0, 1)]^*$
$[(x, y, 1)]_*$	$y = xm + b$	$x = a$	never
$[(1, n, 0)]_*$	$n = m$	never	always
$[(0, 1, 0)]_*$	never	always	always

Affine Planes

In affine planes, two distinct lines no longer need to intersect, they may be *parallel* (nonintersecting). A plane is called *affine* if it satisfies the four axioms (I) every two distinct points are incident to a unique line, (II) every two nonparallel lines are incident to a unique point, (II′) for every point P and line L there exists a unique line through P parallel to L (denoted by $P\|L$), (III) there exists a 4-point. We again get a category of affine planes by taking as morphisms the isomorphisms. Parallelism turns out to be an equivalence relation on lines, and we can speak of the *parallel class* $\|(L)$ of a given line L.

Just as in the real affine plane familiar from calculus, every 2-dimensional vector space V gives rise to an affine plane $\mathcal{A}\!f\!f(V)$ with points the vectors of V and the lines the 1-dimensional *affine subspaces* (translates $A = v + W$ of 1-dimensional linear subspaces W), with incidence being membership $v \,\mathrm{I}\, A \Leftrightarrow v \in A$. Two such lines A, A' are parallel iff $W = W'$, and coincide iff $v - v' \in W = W'$. Thus the parallel classes correspond to the 1-dimensional subspaces W.

More concretely, we may represent V as Δ^2 and $\mathcal{A}\!f\!f(\Delta)$ with affine points being (x, y) for $x, y \in \Delta$, affine lines being either vertical lines $[a] = \{(x, y) \mid x = a\}$ or nonvertical lines $[m, b] = \{(x, y) \mid y = xm + b\}$ of "slope" m and "y-intercept" b, and incidence again being membership. The parallel classes here are the vertical class (all $[a]$) together with the slope classes (all $[m, b]$ for a fixed m).

Back and Forth

We can construct affine planes from projective planes and vice versa, indeed the category of affine planes is equivalent to the category of projective planes *with choice of distinguished line at infinity.* Given a pair (Π, L) consisting of a projective plane and a distinguished line, we construct the *affine restriction* $\mathcal{A}\!f\!f(\Pi, L)$ by removing the line L and all points on it, and taking the incidence relation induced by restriction. Notice that two affine lines are parallel iff their intersection (which always exists in Π) does not belong to the affine part of Π, i.e., iff the lines intersect on the "line at infinity" L.

Conversely, from any affine plane Π_a we can construct the *projective completion* $\Pi = \mathcal{P}roj(\Pi_a)$ by adjoining one *ideal line* L_∞ and adjoining one *ideal point* for each parallel class $\|(L)$, with incidence extended by declaring that all ideal points lie on the ideal line, and an ideal point $\|(L)$ lies on an affine line M iff $M\|L$. The pair (Π, L_∞) thus constructed is a projective plane with distinguished line.

These two constructions are functorial, and provide a category equivalence: $\Pi_a \to (\Pi, L_\infty) \to \mathcal{A}f\!f(\Pi, L_\infty) = \Pi_a$ is the identity functor, while $(\Pi, L) \to \Pi_a \to (\mathcal{P}roj(\Pi_a), L_\infty)$ is naturally isomorphic to the identity functor. Thus we can use projective planes (where all lines are created equal) or affine planes (where coordinatization is natural), whichever is most convenient. [Warning: In general, a projective plane looks different when viewed from different lines L and L', $\mathcal{A}f\!f(\Pi, L) \not\cong \mathcal{A}f\!f(\Pi, L')$, since in general there will not be an automorphism of the whole plane sending L to L'.]

For example, it is easy to verify that the projective completion of $\mathcal{A}f\!f(\Delta)$ is isomorphic to $\mathcal{P}roj(\Delta)$, where points are mapped $(x, y) \mapsto [(x, y, 1)]_*$, $(n) := \|[n, b] \mapsto [(1, n, 0)]_*$, $(\infty) := \|[a] \mapsto [(0, 1, 0)]_*$, and lines are mapped $[m, b] \mapsto [(m, -1, b)]^*$, $[a] \mapsto [(-1, 0, a)]^*$, $[\infty] := L_\infty \mapsto [(0, 0, 1)]^*$, since the incidences $(x, y)\mathrm{I}[m, b]$ etc. in $\mathcal{P}roj(\mathcal{A}f\!f(\Delta))$ coincide with the incidences $[v]_*\mathrm{I}[w]^*$ in $\mathcal{P}roj(\Delta)$ by the incidence table for $\mathcal{P}roj(\Delta)$.

Coordinates

In the spirit of Descartes's program of analytic geometry, we can introduce "algebraic coordinates" into any projective plane using a *coordinate system*, an ordered 4-point $\chi = \{X_\infty, Y_\infty, \mathbf{0}, \mathbf{1}\}$. Here we interpret the plane as the completion of an affine plane by a *line at infinity* $L_\infty := X_\infty \vee Y_\infty$, with $\mathbf{0}$ as *origin* and $\mathbf{1}$ as *unit point*, $X := \mathbf{0} \vee X_\infty, Y := \mathbf{0} \vee Y_\infty$ the X, Y *axes,* and $U := \mathbf{0} \vee \mathbf{1}$ the *unit line*. The *coordinate set* consists of the affine points x of U, together with a symbol ∞. We introduce coordinates (*coordinatize* the plane) for the affine points P, points at infinity P_∞, affine lines L, and line at infinity L_∞ via

$$\begin{aligned} P &\mapsto (x, y), & L \not\| Y &\mapsto [m, b], \\ P_\infty \neq Y_\infty &\mapsto (n), & L \| Y &\mapsto [a], \\ P_\infty = Y_\infty &\mapsto (\infty), & L_\infty &\mapsto [\infty], \end{aligned}$$

where the coordinates of points are $x = \pi_X(P) := \big(P\|Y\big) \wedge U$, $y = \pi_Y(P) := \big(P\|X\big) \wedge U$, $n = \pi_Y(1, n) = \pi_Y\big((P_\infty \vee \mathbf{0}) \wedge (\mathbf{1}\|Y)\big)$, and the coordinates of lines are $a = L \wedge U$, $b = \pi_Y(0, b) = \pi_Y\big(L \wedge Y\big)$, $m = \pi_Y(1, m) = \pi_Y\big((\mathbf{0}\|L) \wedge (\mathbf{1}\|Y)\big)$. With this set of points and lines we obtain a projective plane $\mathcal{P}roj(\Pi, \chi)$ isomorphic to Π, where incidence is given by a table similar to that for $\mathcal{P}roj(\Delta)$:

Incidence Table $P\,\mathrm{I}\,L$ for $\mathcal{P}roj\,(\Pi,\chi)$

$P \quad \backslash \quad L$	$[m,b]$	$[a]$	$[\infty]$
(x,y)	$y=T(x,m,b)$	$x=a$	never
(n)	$n=m$	never	always
(∞)	never	always	always

Thus, once a coordinate system has been chosen, the entire incidence structure is encoded algebraically in what is called a *projective ternary system* $\mathcal{T}ern(\Pi,\chi)$, the set of affine coordinates with the ternary product $T(x,m,b)$ and distinguished elements $0,1$ (somewhat optimistically called a *ternary ring* in hopes that the product takes the form $T(x,m,b) = xm+b$ for $a+b=T(a,1,b), xm=T(x,m,0)$). In general, the ternary system depends on the coordinate system (different systems χ,χ' produce nonisomorphic ternary systems $\mathcal{T}ern(\Pi,\chi) \not\cong \mathcal{T}ern(\Pi,\chi')$), and the ternary product cannot be written as $xm+b$; only when the plane has sufficient "symmetry" do we have a true coordinate *ring* with a (commutative, associative) addition and a (nonassociative, noncommutative) bilinear multiplication.

Central Automorphisms

Projective planes are often classified according to how many central automorphisms they possess (following to the motto "the more the merrier"). An automorphism σ is *central* if it has a *center*, a point C which is fixed *linewise* by the automorphism: $\sigma_{\mathcal{P}}(C)=C,\ \sigma_{\mathcal{L}}(L)=L$ for all L incident to C. In this case σ automatically has an *axis* M, a line which is fixed *pointwise*: $\sigma_{\mathcal{L}}(M)=M,\ \sigma_{\mathcal{P}}(P)=P$ for all P incident to M, and if $\sigma\neq 1$ it fixes no points except the center and those on the axis, and fixes no lines except the axis and those on the center. If the center lies on the axis, σ is called a *translation*, otherwise it is called a *dilation*. For example, in the completion of $\mathcal{A}\!f\!f(\Delta)$ translation $\tau(P)=\mathbf{t}+P$ by a fixed nonzero vector is a translation with axis the line at infinity and center the point at infinity corresponding to the parallel class of all lines parallel to $\Delta\mathbf{t}$; the dilation $\delta(P)=\delta P$ by a fixed scalar $\delta\neq 0,1$ is a dilation with center $\mathbf{0}$ the unique affine fixed point, and axis again the line at infinity.

A plane is (C,M)-*transitive* if the subgroup of (C,M)-automorphisms (those with center C and axis M) acts as transitively as it can, namely, transitively on the points Q of any line L through C (except for the fixed points $C, L\wedge M$): any point Q off the center and axis can be moved to any other such point Q' by some (C,M)-automorphism σ *as long as* C,Q,Q' *are collinear*, because σ fixes the line $L=C\vee Q$ and so must send Q to another point Q' on the same line. Geometrically, a plane is (C,M)-transitive iff *Desargues's* (C,M) *Theorem* holds: Whenever two triangles $\triangle,\ \triangle'$ with vertices $P_1P_2P_3,\ P_1'P_2'P_3'$ and sides $L_1L_2L_3,\ L_1'L_2'L_3'$ ($L_k=P_i\vee P_j$ for distinct i,j,k) are in *central perspective* from C [i.e., C,P_i,P_i' are collinear for $i=1,2,3$] with two sides in *axial perspective* from L [i.e., L,L_i,L_i' are concurrent for

$i = 1, 2$], then also the third sides are in perspective from L [L, L_3, L'_3 are concurrent], so the three pairs of sides meet at three points of L.

If a plane is (C, M) transitive for two different centers C on the axis M, then it is (C, M)-transitive for *all* centers C on M, and it is called a *translation plane* with respect to the axis M; this happens iff its ternary product when we coordinatize it using $L_\infty := M$ is *linear*, $T(x, m, b) = x \cdot m + b$, in which case we speak of $\mathcal{T}ern(\Pi, \chi)$ as a *coordinate ring* $(R, +, \cdot, 0, 1)$.

If a plane is a translation plane with respect to *two* distinct axes M, M', then it is a translation plane with respect to *every line* on $C = M \wedge M'$; this happens iff its coordinate ring using $(\infty) = Y_\infty := C, L_\infty = [\infty] := M$ is a *left Moufang division ring*: it is a unital nonassociative ring with all nonzero elements invertible, satisfying the *Left Inverse Property* $x^{-1}(xy) = y$ for all $x \neq 0, y$, equivalently the *left Moufang Law* $\big(x(yx)\big)z = x\big(y(xz)\big)$ [this implies, and in characteristic $\neq 2$ is equivalent to, the *left alternative law* $x^2 z = x(xz)$]. We call such a plane a *left Moufang plane.*

If a plane is a translation plane with respect to *three* nonconcurrent axes M, M', M'', then it is a translation plane with respect to *every line*, so all possible translations exist and the plane satisfies the *Little Desargues's Theorem* (Desargues's (C, M) Theorem for all pairs with C on M). This happens iff its coordinate ring is an *alternative division ring*: it is a unital nonassociative ring with all nonzero elements invertible, satisfying the *Inverse Property* $x^{-1}(xy) = y = (yx)x^{-1}$ for all y, $x \neq 0$, equivalently the *Left and Right Moufang Laws* [where Right Moufang is $z\big((xy)x\big) = \big((zx)y\big)x$], equivalently the *left and right alternative laws* $x^2 z = x(xz), zx^2 = (zx)x$, and is called a translation plane or *Moufang plane.* In this case all coordinatizations produce isomorphic coordinate rings, and we can speak of *the* alternative coordinate ring $\mathbb{D}$ of the plane. In particular, every octonion division algebra $\mathbb{O}$ produces a Moufang projective plane $\mathcal{M}ouf(\mathbb{O})$ coordinatized by all $(x, y), (n), (\infty), [m, b], [a], [\infty]$ for $x, y, n, m, b, a \in \mathbb{O}$; such a plane is called an *octonion plane.*

If a plane is (C, M)-transitive for all centers C and all axes M *not on* C, then it is (C, M)-transitive for all centers C and *all axes* M *whatsoever,* and Desargues's Theorem holds for all (C, M). Such a plane is called *Desarguian.* This happens precisely iff *some* coordinate ring is an *associative division ring* Δ, in which case *all* coordinates are isomorphic to Δ, and the plane is isomorphic to $\mathcal{P}roj(\Delta)$. The coordinate ring is a field $\Delta = \Phi$ iff the plane satisfies Pappus's Theorem (which implies Desargues's).

Thus the left Moufang, Moufang, Desarguian, and Pappian planes Π rich in central automorphisms are just the planes coordinatized by nonassociative division rings that are left Moufang, alternative, associative, or fields. For these algebraic systems we have powerful theorems (some of the first structure theorems proven for algebras of arbitrary dimension).

Kleinfeld–Skornyakov–San Soucie Theorem. *Every left Moufang division ring is alternative.*

Bruck–Kleinfeld Theorem. *Every alternative division ring is either associative or an octonion division algebra.*

A celebrated theorem of Wedderburn asserts that every *finite* associative division ring is *commutative,* hence a *finite field* $GF(p^n)$ for some prime power. This holds even for alternative division rings.

Artin–Zorn Theorem. *Every finite alternative division ring is associative, hence a finite field (equivalently, there are no finite octonion division algebras because a quadratic form of dimension* 8 *over a finite field is isotropic).*

From these algebraic theorems, we obtain geometric theorems for which no known geometric proofs exist.

Moufang Consequences Theorem. (1) *Every left Moufang plane is Moufang.* (2) *Every Moufang plane is either a Desarguian* $\mathcal{P}roj(\Delta)$ *for an associative division ring* Δ*, or an octonion plane* $\mathcal{M}ouf(\mathrm{O})$ *for an octonion division algebra* O. (3) *Every finite left Moufang or Moufang plane is a Pappian plane* $\mathcal{P}roj(GF(p^n))$ *determined by a finite field.*

This is a powerful instance of Descartes's program of bringing algebra to bear on geometric questions.

Albert Algebras and Octonion Planes

We have seen that the affine part of an octonion plane $\mathcal{M}ouf(\mathrm{O})$ has a coordinatization in terms of points (x, y) and lines $[m, b], [a]$ with octonion coordinates, behaving much like the usual Euclidean plane. However, the Fundamental Theorem of Projective Geometry (which says that the isomorphisms of Desarguian planes $\mathcal{P}roj(V)$ come from semilinear isomorphisms of $V \cong \Delta^3$, represented by 3×3 matrices in $\mathcal{M}_3(\Delta)$) breaks down when the coordinates are nonassociative: the associative composition of isomorphisms cannot be faithfully captured by the nonassociative multiplication of octonion matrices in $\mathcal{M}_3(\mathrm{O})$. In order to represent the isomorphisms, we must find a more abstract representation of $\mathcal{M}ouf(\mathrm{O})$. A surprising approach through Albert algebras $\mathcal{H}_3(\mathrm{O})$ was discovered by Jordan and Freudenthal, then refined by Springer and Faulkner.

Let $\mathrm{J} = \mathcal{H}_3(\mathrm{O})$ be a reduced Albert algebra for an octonion division algebra O over a field Φ. As with ordinary 3×3 matrices, J carries a cubic norm form N and quadratic adjoint map $x \mapsto x^\#$, with linearization $(x, y) \mapsto x\#y$ $(N(x) \in \Phi, x^\#, x\#y \in \mathrm{J})$. We construct an octonion plane $\mathcal{P}roj(\mathrm{J})$ with points $\mathcal{P}$ the 1-dimensional inner ideals B [these are precisely the spaces $\mathrm{B} = \Phi b$ determined by *rank-one elements* b with $U_b\mathrm{J} = \Phi b \neq 0$, equivalently $b^\# = 0 \neq b$] and lines $\mathcal{L}$ the 10-dimensional inner ideals C [these are precisely the spaces $c\#\mathrm{J}$ for rank-one c], with inclusion as incidence [$B \,\mathrm{I}\, C \Leftrightarrow B \subseteq C$]. Every octonion plane arises (up to isomorphism) by this construction: $\mathcal{M}ouf(\mathrm{O}) \cong \mathcal{P}roj(\mathcal{H}_3(\mathrm{O}))$.

If J, J' are two such Albert algebras over fields Φ, Φ', then we call a map $T : \mathrm{J} \to \mathrm{J}'$ *structural* if it is a bijective $\mathbb{Z}$-linear map such that there exists a $\mathbb{Z}$-linear bijection $T^* : \mathrm{J}' \to \mathrm{J}$ with

$$U'_{T(x)} = TU_xT^*$$

for all x. Here T is automatically a τ-linear transformation $T(\alpha x) = \alpha^\tau T(x)$ for an isomorphism $\tau : \Phi \to \Phi'$ of the underlying fields, and T^* is uniquely determined as $T^{-1}U'_{T(1)}$. This ring-theoretic structural condition turns out to be equivalent to Jacobson's original cubic norm condition that T be a τ-linear norm similarity, $N'(T(x)) = \nu N(x)^\tau$ for all $x \in \mathrm{J}$. Any such structural T induces an isomorphism $\mathcal{P}roj(T) : \mathcal{P}roj(\mathrm{J}) \to \mathcal{P}roj(\mathrm{J}')$ of projective planes, since it preserves dimension, innerness, and incidence.

In particular, since $U_u : \mathrm{J}^{(u)} \to \mathrm{J}$ is always structural, J and any of its isotopes $\mathrm{J}^{(u)}$ produce isomorphic Moufang planes. This explains why we obtain all the octonion planes from just the standard $\mathcal{H}_3(\mathrm{O})$'s, because the most general reduced Albert algebra is a canonical $\mathcal{H}_3(\mathrm{O}, \Gamma)$, which is isomorphic to a *diagonal isotope* $\mathcal{H}_3(\mathrm{O})^\Gamma$ of a standard Albert algebra.

In Faulkner's original description, points and lines are two copies of the same set: the 10-dimensional $c\#\mathrm{J} = \{x \in \mathrm{J} \mid V_{x,c} = 0\}$ is uniquely determined by the 1-dimensional Φc, so we can take as points and lines all $[b]_*$ and $[c]^*$ for rank-one elements b, c, where incidence becomes $[b]_* \,\mathrm{I}\, [c]^* \Leftrightarrow V_{b,c} = 0$ [analogous to $[v]_* \,\mathrm{I}\, [w]^* \Leftrightarrow \langle v, w\rangle = 0$ in $\mathcal{P}roj(\Delta)$]. A structural T induces an isomorphism of projective planes via $\sigma_{\mathcal{P}}([b]_*) := [T(b)]_*$, $\sigma_{\mathcal{L}}([b]^*) := [T'(b)]^*$ for $T' = (T^*)^{-1}$ because this preserves incidence: $V'_{T(b),T'(c)} = 0 \Leftrightarrow V_{b,c} = 0$ because T automatically satisfies $V'_{T(x),T'(y)} = TV_{x,y}T^{-1}$ [acting on $T(z)$ this is just the linearization $x \mapsto x, z$ of the structural condition].

Fundamental Theorem of Octonion Planes. *The isomorphisms $\mathcal{P}roj(\mathrm{J}) \to \mathcal{P}roj(\mathrm{J}')$ of octonion planes are precisely all $\mathcal{P}roj(T)$ for structural maps $T : \mathrm{J} \to \mathrm{J}'$ of Albert algebras.*

The arguments for this make heavy use of a rich supply of structural transformations on J in the form of Bergmann operators $B_{x,y}$ (known geometrically as *algebraic transvections* $T_{x,-y}$); while the U_x for invertible x are also structural, the $B_{x,y}$ form a more useful family to get from one rank 1 element to another. The methods are completely Jordanic rather than octonionic. Thus the octonion planes as well as their automorphisms find a natural home in the Albert algebra, which has provided shelter for so many exceptional mathematical structures.

Conclusion

This survey has acquainted you with some of the regions this mathematical river passes through, and a few of the people who have contributed to it. Mathematics is full of such rivers, gaining nourishment from the mathematical landscapes they pass through, enriching in turn those regions and others further downstream.

Part I

A Historical Survey of Jordan Structure Theory

Introduction

In the Colloquial Survey we discussed the origin of Jordan algebras in Pascual Jordan's attempt to discover a new algebraic setting for quantum mechanics, freed from dependence on an invisible but all-determining metaphysical matrix structure. In studying the intrinsic algebraic properties of hermitian matrices, he was led to a linear space with commutative product satisfying the Jordan identity, and an axiomatic study of these abstract systems in finite dimensions led back to hermitian matrices plus one tiny new system, the Albert algebra. Later, the work of Efim Zel'manov showed that even in infinite dimensions this is the only simple Jordan algebra which is not governed by an associative algebra lurking in the background.

In this Historical Survey of Jordan Structure Theory (Part I), we tell the story of Jordan algebras in more detail, describing chronologically how our knowledge of the structure of Jordan algebras grew from the physically-inspired investigations of Jordan, von Neumann, and Wigner in 1934 to the inspired insights of Zel'manov in 1983. We include no exercises and give no proofs, though we sketch some of the methods used, comparing the classical methods using idempotents to the ring-theoretic methods of Zel'manov. The goal is to get the "big picture" of Jordan structure theory, to understand its current complete formulation while appreciating the older viewpoints.

In this Part we aim to educate as well as enlighten, and an educated understanding requires that one at least be able to correctly pronounce the names of the distinguished mathematicians starring in the story. To that end I have included warning footnotes, and at the end of the book a page of pronunciations for all foreign names to help readers past the most egregious errors (like Henry Higgins leading a mathematical Liza Doolittle, who talks of the *Silo* theorem for groups as if it were a branch of American agriculture, to *See-loff* [as in Shilov boundary], though perhaps never to *Sÿ-lovf* [as in Norway]). This would all be unnecessary if I were giving these lectures orally.

1

Jordan Algebras in Physical Antiquity: The Search for an Exceptional Setting for Quantum Mechanics

Jordan algebras were conceived and grew to maturity in the landscape of physics. They were born in 1933 in a paper of the physicist Pascual Jordan[1], "Über Verallgemeinerungsmöglichkeiten des Formalismus der Quantenmechanik," which, even without DNA testing, reveals their physical parentage. Just one year later, with the help of John von Neumann and Eugene Wigner in the paper "On an algebraic generalization of the quantum mechanical formalism," they reached adulthood. Students can still benefit from reading this paper, since it was a beacon for papers in nonassociative algebra for the next 50 years (though nowadays we would derive the last 14 pages in a few lines from the theory of composition algebras).

1.1 The Matrix Interpretation of Quantum Mechanics

In the usual interpretation of quantum mechanics (the so-called *Copenhagen interpretation*), the physical observables are represented by Hermitian matrices (or operators on Hilbert space), those which are self-adjoint $x^* = x$. The basic operations on matrices or operators are given in the following table:

	Matrix Operations
λx	multiplication by a complex scalar λ
$x + y$	addition
xy	multiplication of matrices (composition of operators)
x^*	complex conjugate transpose matrix (adjoint operator)

[1] In English, his name and his algebras are *Dgor-dan* as in *Dgudge* or Michael, not *Zhor-dahn* as in canonical form. As a German (despite Pascual!), his real name was (and his algebras still are) *Yor-dahn*.

This formalism is open to the objection that the operations are not "observable," not intrinsic to the physically meaningful part of the system: the scalar multiple λx is not again hermitian unless the scalar λ is real, the product xy is not observable unless x and y commute (or, as the physicists say, x and y are "simultaneously observable"), and the adjoint is invisible (it is the identity map on the observables, though nontrivial on matrices or operators in general).

	Observable Operations
αx	multiplication by a real scalar α
$x + y$	addition
x^n	powers of matrices
x	identity map

Not only was the matrix interpretation *philosophically* unsatisfactory because it derived the observable algebraic structure from an unobservable one, there were *practical* difficulties when one attempted to apply quantum mechanics to relativistic and nuclear phenomena.

1.2 The Jordan Program

In 1932 Jordan proposed a program to *discover a new algebraic setting for quantum mechanics*, which would be freed from dependence on an invisible but all-determining metaphysical matrix structure, yet would enjoy all the same algebraic benefits as the highly successful Copenhagen model. He proposed:

- *To study the intrinsic algebraic properties of hermitian matrices, without reference to the underlying (unobservable) matrix algebra*;
- *To capture the algebraic essence of the physical situation in formal algebraic properties that seemed essential and physically significant*;
- *To consider abstract systems axiomatized by these formal properties and see what other new (non-matrix) systems satisfied the axioms.*

1.3 The Jordan Operations

The first step in analyzing the algebraic properties of hermitian matrices or operators was to decide what the basic *observable operations* were. There are many possible ways of combining hermitian matrices to get another hermitian matrix. The most natural observable operation was that of forming *polynomials*: if x was an observable, one could form an observable $p(x)$ for any real polynomial $p(t)$ with zero constant term; if one experimentally measured the value v of x in a given state, the value associated with $p(x)$ would just be $p(v)$. Breaking the operation of forming polynomials down into its basic ingredients, we have the operations of multiplication αx by a real scalar, addition $x+y$, and raising to a power x^n. By linearizing the quadratic squaring operation x^2 we obtain a symmetric bilinear operation[2] $x \bullet y$, to which Jordan gave the none-too-informative name *quasi-multiplication*:

$$x \bullet y := \tfrac{1}{2}(xy + yx).$$

This is also frequently called the *anticommutator* (especially by physicists), but we will call it simply the **Jordan product**.

1.4 Digression on Linearization

We interrupt our chronological narration for an important announcement about the general process of *linearization* (often called *polarization*, especially in analysis in dealing with quadratic mappings on a complex space). This is an important technique in nonassociative algebras which we will encounter frequently in the rest of the book. Given a homogeneous polynomial $p(x)$ of degree n, the process of linearization is designed to create a *symmetric multilinear* polynomial $p'(x_1, \ldots, x_n)$ in n variables such that the original polynomial arises by specializing all the variables to the same value x : $p'(x, \ldots, x) = p(x)$. For example, the full linearization of the square $x^2 = xx$ is $\frac{1}{2}(x_1x_2 + x_2x_1)$, and the full linearization of the cube $x^3 = xxx$ of degree 3 is $\frac{1}{6}(x_1x_2x_3 + x_1x_3x_2 + x_2x_1x_3 + x_2x_3x_1 + x_3x_1x_2 + x_3x_2x_1)$.

In general, in order to recover the original $p(x)$ of degree n we will have to divide by $n!$. This, of course, never bothers physicists or workers in characteristic 0, but over general scalars this step is highly illegal. In many algebraic investigations, the linearization process is important even in situations where we can't divide by $n!$, such as when the scalars are integers or a field of prime characteristic. Thus we will describe a bare-bones linearization that recovers only $n!p(x)$, but provides crucial information even without division.

Full linearization is usually achieved one step at a time by a series of partial linearizations, in which a polynomial homogeneous of degree n in a particular variable x is replaced by one of degree $n-1$ in x and *linear* in a new variable y. Intuitively, in an expression with n occurrences of x we simply replace each occurrence of x, one at a time, by a y, add up the results [we would have to multiply by $\frac{1}{n}$ to ensure that restoring y to x produces the original polynomial instead of n copies of it]. For example, in xx we can replace the first or the second occurrence of x by y, leading to yx or xy, so the first linearization is $yx + xy$, which is linear in x and y. In xxx we can replace the first, second, or third occurrence of x by a y, so the first linearization produces $yxx + xyx + xxy$, which is quadratic in x and linear in y. [Notice

[2] Many authors denote the product by $x.y$ or $x \cdot y$, but since these symbols are often used for run-of-the-mill products in linear algebras, we will use the bolder, more distinctive bullet $x \bullet y$.

the inconspicuous term xyx buried inside the linearization of the cube; we now know that this little creature actually governs all of Jordan theory!] We repeat this process over and over, reducing variables of degree n by a pair of variables, one of degree $n-1$ and one of degree 1. Once we are down to the case of a polynomial of degree 1 in each of its variables, we have the full linearization.

In most situations we can't naively reach into $p(x)$ and take the x's one at a time: we often have no very explicit expression for p, and must describe linearization in a more intrinsic way. The clearest formulation is to take $p(x+\lambda y)$ for an indeterminate scalar λ and expand this out as a polynomial in λ:

$$p(x+\lambda y) = p(x) + \lambda p_1(x;y) + \lambda^2 p_2(x;y) + \cdots + \lambda^n p(y).$$

Here $p_i(x;y)$ is homogeneous of degree $n-i$ in x and i in y (intuitively, we obtain it by replacing i of the x's in $p(x)$ by y's in all possible ways), and *the linearization is just the coefficient* $p_1(x;y)$ *of* λ [we would have to divide by n to recover p]. If we are working over a field with at least n distinct elements, we don't have to drag in an indeterminate, we simply let λ run through n different scalars and use a Vandermonde method to solve a system of equations to pick out $p_1(x;y)$.

A fancy way of getting the full linearization in one fell swoop would be to replace x by a formal linear combination $\lambda_1 x_1 + \lambda_2 x_2 + \cdots \lambda_n x_n$ of n new variables x_i and n indeterminates λ_i; then the full linearization $p(x_1, \ldots, x_n)$ is precisely the coefficient of $\lambda_1 \lambda_2 \cdots \lambda_n$ in the expansion of $p(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n)$ [again we would have to divide by $n!$ to recover p].

The linearization process is very simple when applied to quadratic mappings $q(x)$ of degree 2, which is the only case we need at this particular point in our story. Here we need to linearize only once, and the process takes place wholly within the original space (there is no need to drag in indeterminate scalars): we take the value on the sum $x+y$ of two elements, and then subtract the pure x and y terms to obtain $q(x,y) := q(x+y) - q(x) - q(y)$. Note that despite the fact that *we will assume throughout the book that we have a scalar* $\frac{1}{2}$, we will *never* divide the expression $q(x,y)$ by 2. Thus for us $q(x,x) = q(2x) - q(x) - q(x) = 4q(x) - 2q(x) = 2q(x)$ does *not* recover the original quadratic map. With a glimpse of quadratic Jordan algebras in our rear-view mirror, we will drive through the Jordan landscape avoiding the scalar $\frac{1}{2}$ as far as possible, only calling upon it in our hour of need.

1.5 Back to the Bullet

Returning to the story of Jordan's empirical investigation of the algebraic properties of hermitian matrices, it seemed to him that all the products could be expressed in terms of the Jordan product $x \bullet y$. For example, it was not hard to see that the powers could be defined from the Jordan product via $x^1 = x, x^{n+1} = x \bullet x^n$, so the power maps could be derived from a single bilinear product. We saw that linearization of the cube involved a product xyx linear in y but quadratic in x; linearizing x to x, z leads to a trilinear product $xyz+zyx$ (now known as the *Jordan triple product*); this too could be expressed in terms of the bilinear Jordan product: $xyz+zyx = 2\big(x\bullet(y\bullet z)+z\bullet(y\bullet x)-(x\bullet z)\bullet y\big)$. Of course, like the important dog in Sherlock Holmes who did *not* bark in the night, the important product that does *not* lead back to hermitian matrices is the associative product xy: the product of two hermitian matrices is again hermitian iff the two matrices commute. Thus in addition to its observable linear structure as a real vector space, the model carries a basic observable product, the Jordan product, out of which more complicated observable products such as powers or Jordan triple products can be built, but it does not carry an associative product. Thus Jordan settled on the Jordan product as the basic algebraic operation.

We now realize that Jordan overlooked several other natural operations on hermitian elements. Note that if x, y, x_i are hermitian matrices or operators, so are the *quadratic product* xyx, the *inverse* x^{-1}, and the *n-tad products* $\{x_1, \ldots, x_n\} := x_1 \cdots x_n + x_n \cdots x_1$. The quadratic product and inverse can be defined using the Jordan product, though this wasn't noticed for another 30 years; later, each of these was used (by McCrimmon and Springer) to provide an alternate axiomatic foundation on which to base the entire Jordan theory. The n-tad for $n = 2$ is just twice the Jordan product, and we have already noted that the 3-tad, or Jordan triple product, can be expressed in terms of the Jordan product. On the other hand, the n-tads for $n \geq 4$ *cannot* be expressed in terms of the Jordan product. In particular, the *tetrads* $\{x_1, x_2, x_3, x_4\} := x_1x_2x_3x_4 + x_4x_3x_2x_1$ were inadvertently excluded from Jordan theory. As we shall see, this oversight allowed two uninvited guests to join the theory, the spin factor and the Albert algebra, who were not closed under tetrads but who had influential friends and applications in many areas of mathematics.

1.6 The Jordan Axioms

The next step in the empirical investigation of the algebraic properties enjoyed by the model was to decide what crucial formal *axioms* or *laws* the operations on hermitian matrices obey.[3] As far as its linear structure went, the operations of addition and scaling by a real number must of course satisfy the familiar vector-space rules. But what conditions to impose on the multiplicative structure was much less clear. The most obvious rule for the operation of forming polynomials in an observable was the rule that if $r(t) = p(q(t))$ is the composite of the polynomials p, q then for all observables x we have $r(x) = p(q(x))$. If we write the powers occurring in the polynomials in terms of the Jordan product, this composition rule is equivalent to *power-associativity*:

[3] Notice the typical terms algebraists use to describe their creations: we say that an algebra *enjoys* a property or *obeys* a law, as if it never had secret longings to be associative or to lead a revolt against the Jordan identity.

$$x^n \bullet x^m = x^{n+m} \qquad \text{(power associative law).}$$

Jordan discovered an important law or *identity* (in the sense of *identical relation* satisfied by all elements) of degree four in two variables satisfied by the Jordan product:

$$x^2 \bullet (y \bullet x) = (x^2 \bullet y) \bullet x$$

(which we now call the *Jordan identity*). For example, fourth-power associativity $x^2 \bullet x^2 = x^4 (:= x^3 \bullet x)$ follows immediately from this Jordan identity by setting $y = x$, and some non-trivial fiddling shows that the Jordan identity is strong enough to imply associativity of *all* powers. Thus Jordan thought he had found the key law governing the Jordan product, besides its obvious commutativity.

The other crucial property of the Jordan product on hermitian matrices is its "positive-definiteness." Recall that a symmetric bilinear form b on a real or complex vector space is called *positive definite* if $b(x, x) > 0$ for all $x \neq 0$ (of course, this does not mean $b(x, y) \geq 0$ for all x, y!!). Notice that for an $n \times n$ complex hermitian matrix $X = (x_{ij})$ $(x_{ji} = \overline{x_{ij}})$, the square has as ith diagonal entry $\sum_{j=1}^n x_{ij}x_{ji} = \sum_{j=1}^n x_{ij}\overline{x_{ij}} = \sum_{j=1}^n |x_{ij}|^2$, so the trace bilinear form $b(X, Y) := \mathrm{tr}(XY)$ has $b(X, X) = \mathrm{tr}(X^2) = \sum_{i=1}^n \sum_{j=1}^n |x_{ij}|^2 > 0$ unless all x_{ij} are 0, i.e., unless $X = 0$. This positivity of squares involves the trace bilinear form, which is not part of the axiomatic framework considered by Jordan, but it has purely algebraic consequences for the Jordan product: a sum of squares can never vanish, since if $\sum_{k=1}^r X_k^2 = 0$ then taking traces gives $\sum_{k=1}^r \mathrm{tr}(X_k^2) = 0$, which forces each of the positive quantities $\mathrm{tr}(X_k^2)$ to vanish individually, and therefore each X_k is 0. This "formal reality" property was familiar to Jordan from the recent Artin–Schreier theory of formally real fields.

It was known from the Wedderburn theory of finite-dimensional associative algebras that once you removed the radical (the largest ideal consisting of nilpotent elements) you obtained a nice semisimple algebra which was a direct sum of simple pieces (all of the form $\mathcal{M}_m(\Delta)$ consisting of all $m \times m$ matrices over a division ring Δ). Formal reality immediately guarantees that there are no nilpotent elements whatsoever, providing instant semisimplicity: if there are nonzero nilpotent elements of *index* $n > 1$ $(x^n = 0 \neq x^{n-1})$ then there are also elements of index 2 $(y^2 = 0 \neq y = x^{n-1})$, and formal reality for sums of length $r = 1$ shows that squares never vanish: $x_1^2 = 0 \Longrightarrow x_1 = 0$.

After a little empirical experimentation, it *seemed* to Jordan that all other laws satisfied by the Jordan product were consequences of *commutativity*, the *Jordan identity*, and *positivity* or *formal reality*. The outcome of all this experimentation was a distillation of the algebraic essence of quantum mechanics into an axiomatic definition of a new algebraic system.

Jordan Definition. *A real* **Jordan algebra** $\mathrm{J} = (V, p)$ *consists of a real vector space* V *equipped with a bilinear product* $p : V \times V \longrightarrow V$ *(usually abbreviated* $p(x, y) = x \bullet y$*) satisfying the* **Commutative Law** *and the* **Jordan Identity**:

$$\text{(JAX1)} \qquad x \bullet y = y \bullet x \qquad \text{(Commutative Law)},$$

$$\text{(JAX2)} \qquad (x^2 \bullet y) \bullet x = x^2 \bullet (y \bullet x) \qquad \text{(Jordan Identity)}.$$

A Jordan algebra is called **Euclidean** (*or* **formally real**) *if it satisfies the formal reality axiom*

$$x_1^2 + \cdots + x_n^2 = 0 \Longrightarrow x_1 = \cdots = x_n = 0.$$

Jordan originally called these *r-number algebras*; the term “Jordan algebra” was first used by A.A. Albert in 1946, and caught on immediately.

We now know that Jordan was wrong in thinking that his axioms had captured the hermitian essence — he had overlooked some algebraic properties of hermitian matrices, so instead he had captured something slightly more general. Firstly, he missed an algebraic operation (the tetrad product) which could not be built from the bullet. Secondly, he missed some laws for the bullet which cannot be derived from the Jordan identity. The first and smallest of these so-called “s-identities” is Glennie’s Identity G_8 of degree 8 in 3 variables discovered in 1963, so Jordan may perhaps be excused for overlooking it! In 1987 Thedy discovered a more transparent s-identity T_{10}, even though it was an operator identity of degree 10 in 3 variables. These overlooked identities will come to play a vital role later in our story. Not only did the Albert algebra not carry a tetrad operation as hermitian matrices do, but even with respect to its Jordan product it was distinguishable from hermitian matrices by its refusal to obey the s-identities. Thus it squeezed through two separate gaps in Jordan’s axioms. As we saw in the Colloquial Survey, a large part of the richness of Jordan theory is due to its exceptional algebras (with their connections to exceptional Lie algebras, and exceptional symmetric domains), and much of the power of Jordan theory is its ability to handle these exceptional objects and hermitian objects in one algebraic framework.

1.7 The First Example: Full Algebras

Let us turn to the Three Basic Examples (Full, Hermitian, and Spin) of these newly-christened Jordan algebras. Like the Three Musketeers, these three will cut a swath through our story, and we will meet them at every new concept. They will be assisted later by the Cubic Examples, which have a quite different lineage. The progenitor of all three basic examples is an associative algebra A. If we take the full algebra under the bullet product, we create a Jordan algebra which has not evolved far from associativity.

Full Example. *Any associative algebra* A *over* $\mathbb{R}$ *can be converted into a Jordan algebra, denoted by* A^+, *by forgetting the associative structure but retaining the Jordan structure*: *the linear space* A, *equipped with the product* $x \bullet y := \frac{1}{2}(xy + yx)$, *is commutative as in* (JAX1)*, and satisfies the Jordan identity* (JAX2).

In the Colloquial Survey we went through the verification of the Jordan identity (JAX2) in any A^+; however, these Jordan algebras are almost never Euclidean. For example, for the algebra $\mathrm{A} = \mathcal{M}_n(\mathrm{D})$ of $n \times n$ matrices over an associative ring D, the algebra $\mathrm{A}^+ = \mathcal{M}_n(\mathrm{D})^+$ is never Euclidean if $n > 1$, since the matrix units E_{ij} for $i \neq j$ square to zero. However, we have seen that even if the full matrix algebra is not Euclidean, the *symmetric* matrices over the reals or the *hermitian* matrices over the complexes are Euclidean.

1.8 The Second Example: Hermitian Algebras

Any subspace of A which is closed under the Jordan product will continue to satisfy the axioms (JAX1) and (JAX2), and hence provide a Jordan algebra. The most natural (and historically most important) method of selecting out a Jordan-closed subspace is by means of an *involution* $*$, a linear anti-isomorphism of an associative algebra of period 2.

Hermitian Example. *If* $*$ *is an involution on* A, *then the space* $\mathcal{H}(\mathrm{A},^*)$ *of hermitian elements* $x^* = x$ *is closed under symmetric products, but not in general under the noncommutative product* xy*:* $\mathcal{H}(\mathrm{A},^*)$ *is a Jordan subalgebra of* A^+, *but not an associative subalgebra of* A.

Indeed, if x and y are symmetric we have $(xy)^* = y^*x^*$ (by definition of anti-isomorphism) $= yx$ (by definition of x, y being hermitian) and dually $(yx)^* = xy$, so xy is not hermitian unless x and y happen to commute, but $(x \bullet y)^* = \frac{1}{2}(xy + yx)^* = \frac{1}{2}((xy)^* + (yx)^*) = \frac{1}{2}(yx + xy) = x \bullet y$ is always hermitian.

A particularly important case is that in which $\mathrm{A} = \mathcal{M}_n(\mathrm{D})$ consists of all $n \times n$ matrices $X = (x_{ij})$ over a unital associative coordinate algebra D with involution $d \mapsto \bar{d}$, and $*$ is the *conjugate transpose involution* $X^* := \overline{X}^{tr}$ on matrices (with ij-entry $\overline{x_{ji}}$).

Jordan Matrix Example. *If* $\mathrm{A} = \mathcal{M}_n(\mathrm{D})$ *is the algebra of all* $n \times n$ *matrices over an associative coordinate* $*$*-algebra* $(\mathrm{D}, -)$, *then the algebra* $\mathcal{H}(\mathrm{A}, *)$ *of hermitian elements under the conjugate transpose involution is the* Jordan matrix algebra $\mathcal{H}_n(\mathrm{D}, -)$ *consisting of all* $n \times n$ *matrices* X *whose entries satisfy* $x_{ji} = \overline{x_{ij}}$.

Such a Jordan matrix algebra will be Euclidean if the coordinate $*$*-algebra* D *is a* Euclidean $*$-algebra *in the sense that* $\sum_r x_r \overline{x_r} = 0 \Longrightarrow$ *all* $x_r = 0$. *In particular, the hermitian matrices* $\mathcal{H}_n(\mathbb{R})$, $\mathcal{H}_n(\mathbb{C})$, $\mathcal{H}_n(\mathbb{H})$ *with entries from the reals* $\mathbb{R}$, *complexes* $\mathbb{C}$, *or quaternions* $\mathbb{H}$, *under their usual involutions, all have this positivity.*

1.9 The Third Example: Spin Factors

The first non-hermitian Jordan algebras were the spin factors discovered by Max Zorn, who noticed that we get a Jordan algebra from $\mathbb{R}^{n+1} = \mathbb{R}1 \oplus \mathbb{R}^n$ if we select the elements in $\mathbb{R}1$ to act as "scalars," and the elements of $\mathbb{R}^n$ to act as "vectors" with bullet product a scalar multiple of 1 given by the dot or inner product. We will use the term *inner product* instead of dot product, both to avoid confusion with the bullet and for future generalization.

Spin Factor Example. *If* $\langle \cdot, \cdot \rangle$ *denotes the usual Euclidean inner product on* $\mathbb{R}^n$*, then* $\mathcal{JS}pin_n = \mathbb{R}1 \oplus \mathbb{R}^n$ *becomes a Euclidean Jordan algebra, a* Jordan spin factor*, if we define* 1 *to act as unit and the bullet product of vectors* $\mathbf{v}, \mathbf{w}$ *in* $\mathbb{R}^n$ *to be given by the inner product,*

$$\mathbf{v} \bullet \mathbf{w} := \langle \mathbf{v}, \mathbf{w} \rangle 1.$$

Indeed, commutativity (JAX1) comes from symmetry of the inner product, the Jordan identity (JAX2) comes from the fact that x^2 is a linear combination of 1 and x, and formal reality comes from $\sum \big(\alpha_i 1 \oplus \mathbf{v}_i\big)^2 = \sum(\alpha_i^2 + < \mathbf{v}_i, \mathbf{v}_i >)1 \oplus \big(2 \sum \alpha_i \mathbf{v}_i\big)$, where the coefficient of 1 can vanish only if all α_i and all $\mathbf{v}_i$ are 0 by positive definiteness of the inner product.

This algebra gets its name from mathematical physics. The *spin group* is a simply connected universal covering group for the group of *rotations* on n-space (the *special orthogonal group*). The spin group has a mathematical realization as certain invertible elements of the *Clifford algebra*, an associative algebra generated by elements v_i (corresponding to an orthonormal basis for n-space) with defining relations

$$v_i^2 = 1, \qquad v_i v_j + v_j v_i = 0 \quad (i \neq j).$$

Any such system of "orthogonal symmetries" v_i in an associative algebra is called a *spin system.* The linear span of 1 and the v_i does not form an *associative* algebra (the Clifford algebra has dimension 2^n, not $n+1$), but it does form a *Jordan* algebra with square $\left(\alpha 1 + \sum_i \alpha_i v_i\right)^2 = \left(\alpha^2 + \sum_i \alpha_i^2\right) 1 + 2\alpha \left(\sum_i \alpha_i v_i\right)$, which is precisely the square in $\mathcal{JS}pin_n$.

Notice that the trace form $t(\alpha 1 \oplus v) := \alpha$ leads to a trace bilinear form $b(x, y) := t(x \bullet y)$ with $b(x, x) = \alpha^2 + \langle v, v \rangle \; > 0$ just as for hermitian matrices, and any time we have such a positive definite bilinear form built out of multiplications we automatically have formal reality.

Another way to see that this algebra is Euclidean is to note that you can (if you're careful!) imbed it in the hermitian $2^n \times 2^n$ real matrices: let W be the real inner product space with 2^n orthonormal basis vectors e_I parameterized by the distinct n-tuples $I = (\varepsilon_1, \ldots, \varepsilon_n)$ for $\varepsilon_k = \pm 1$. Define linear transformations v_k, $k = 1, 2, \ldots n$, to act on the basis vectors by $v_k(e_I) := \varepsilon_1 \cdots \varepsilon_{k-1} e_{I_k}$, scaling them by a factor ± 1 which is the product of the first $k-1$ ε's, and switching the kth index of the n-tuple to its negative $I_k := (\varepsilon_1, \ldots, -\varepsilon_k, \ldots, \varepsilon_n)$. Then by careful calculation we can verify that (1) the v_k are self-adjoint with respect to the inner product, since $\langle v_k(e_I), e_J\rangle = \langle e_I, v_k(e_J)\rangle$ are both zero unless I and J are obtained from each other by negating the kth index, in which case the inner products are both the factor $\varepsilon_1 \cdots \varepsilon_{k-1}$, and (2) the v_k have products $v_k^2 = \mathbb{1}_W,\ v_k \bullet v_j = 0$ for $j \neq k$. Thus the n "orthogonal symmetries" v_k form a spin system in $\mathcal{E}nd(W)$, so the $(n+1)$-dimensional real subspace spanned by $\mathbb{1}_W$ and the orthogonal symmetries v_k forms a Jordan algebra with the same multiplication table as $\mathcal{JS}pin_n$. Thus we may regard the algebra $\mathcal{JS}pin_n$ as some sort of "thin" Jordan subalgebra of a full algebra of hermitian matrices.

1.10 Special and Exceptional

Recall that the whole point of Jordan's investigations was to discover Jordan algebras (in the above sense) which did not result simply from the Jordan product in associative algebras. We call a Jordan algebra *special* if it comes from the Jordan product in an associative algebra, otherwise it is *exceptional.* In a special Jordan algebra the algebraic structure is derived from an ambient associative product xy.

Special Definition. *A Jordan algebra is* **special** *if it can be linearly imbedded in an associative algebra so that the product becomes the Jordan product* $\frac{1}{2}(xy+yx)$, *i.e., if it is isomorphic to some Jordan subalgebra of some Jordan algebra* A^+, *otherwise it is* **exceptional**.

One would expect a variety where every algebra is either special or exceptional to have its roots in Lake Wobegon! The examples $\mathcal{H}(\mathrm{A}, {}^*)$ are clearly special, living inside the associative algebra A. In particular, matrix algebras $\mathcal{H}_n(\mathrm{D}, -)$ are special, living inside $\mathrm{A} = \mathcal{M}_n(\mathrm{D})$. $\mathcal{JS}pin_n$ is also special, since we just noted that it can be imbedded in suitably large hermitian matrices.

1.11 Classification

Having settled, he thought, on the basic axioms for his systems, Jordan set about trying to classify them. The algebraic setting for quantum mechanics would have to be infinite-dimensional, of course, but since even for associative algebras the study of infinite-dimensional algebras was in its infancy, there seemed no hope of obtaining a complete classification of infinite-dimensional Jordan algebras. Instead, it seemed reasonable to study first the finite-dimensional algebras, hoping to find families of simple exceptional algebras E_n parameterized by natural numbers n (e.g., $n \times n$ matrices), so that by letting n go to infinity a suitable home could be found for quantum mechanics. The purely algebraic aspects were too much for Jordan to handle alone, so he called in the mathematical physicist Eugene Wigner and the mathematician John von Neumann.

In their fundamental 1934 paper the J–vN–W triumvirate showed that in finite-dimensions the only simple building blocks are the usual hermitian matrices and the algebra $\mathcal{JSpin}_n$, except for one small algebra of 3×3 matrices whose coordinates come from the nonassociative 8-dimensional algebra $\mathbb{K}$ of Cayley's octonions.

Jordan–von Neumann–Wigner Theorem. *Every finite-dimensional formally real Jordan algebra is a direct sum of a finite number of simple ideals, and there are five basic types of simple building blocks: four types of hermitian matrix algebras corresponding to the four composition division algebras* $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{K}$ *over the reals, together with the spin factors. Every finite-dimensional simple formally real Jordan algebra is isomorphic to one of*:

$$\mathrm{H}_n^1 = \mathcal{H}_n(\mathbb{R}), \quad \mathrm{H}_n^2 = \mathcal{H}_n(\mathbb{C}), \quad \mathrm{H}_n^4 = \mathcal{H}_n(\mathbb{H}), \quad \mathrm{H}_3^8 = \mathcal{H}_3(\mathbb{K}), \ \mathcal{JSpin}_n.$$

The notation was chosen so that H_n^k (called M_n^k in the original paper) denoted the $n\times n$ hermitian matrices over the k-dimensional real composition division algebra. $\mathbb{R}$ and $\mathbb{C}$ are old friends, of course, and $\mathbb{H}$amilton's quaternions $\mathbb{H}$ (with basis $1, i, j, k$) are at least a nodding acquaintance; $\mathbb{K}$ayley's octonions $\mathbb{K} = \mathbb{H}\oplus\mathbb{H}\ell$ may be a stranger — they have basis $\{1, i, j, k\}\cup\{\ell, i\ell, j\ell, k\ell\}$ with nonassociative product $h_1(h_2\ell) = (h_2\ell)h_1^* = (h_2h_1)\ell, (h_1\ell)(h_2\ell) = -(h_2^*h_1)$, and involution $(h\ell)^* = -h\ell$ in terms of the new basic unit ℓ and old elements $h, h_1, h_2 \in \mathbb{H}$. All four of these carry a positive definite quadratic form $Q(\sum \alpha_i x_i) = \sum \alpha_i^2$ (relative to the indicated bases $\{x_i\}$) which "admits composition" in the sense that Q of the product is the product of the Q's, $Q(xy) = Q(x)Q(y)$, and for that reason are called *composition algebras*. A celebrated theorem of Hurwitz asserts that the *only* possible composition algebras over any field are the field (dimension 1), a quadratic extension (dimension 2), a quaternion algebra (dimension 4), and an octonion algebra (dimension 8). By the Jordan Matrix Example we see that the first three algebras H_n^k for $k = 1, 2, 4$ are special, living inside the full associative algebras $\mathcal{M}_n(\mathbb{R})$, $\mathcal{M}_n(\mathbb{C})$, $\mathcal{M}_n(\mathbb{H})$ of $n \times n$ matrices, and after the Spin Factor Example we noted that the fifth example $\mathcal{JSpin}_n$ lives inside large hermitian matrices.

On the other hand, the fourth example $\mathrm{H}_3^8 = \mathcal{H}_3(\mathbb{K})$ did not *seem* to be special, since its coordinates came from the nonassociative algebra $\mathbb{K}$, but the authors were unable to prove its exceptionality (and could prove only with difficulty that it satisfied the Jordan identity). They turned to a bright young algebraist A.A. Albert, who showed that it was indeed exceptional Jordan.

Albert's Exceptional Theorem. *The algebra* $\mathcal{H}_3(\mathbb{K})$ *is an exceptional Jordan algebra of dimension* 27.

It is easy to see that $\mathcal{H}_3(\mathbb{K})$ has dimension 27: in a typical element $\begin{pmatrix} \alpha_{11} & a_{12} & a_{13} \\ \overline{a_{12}} & \alpha_{22} & a_{23} \\ \overline{a_{13}} & \overline{a_{23}} & \alpha_{33} \end{pmatrix}$, each of the 3 independent diagonal entries α_{ii} comes from the 1-dimensional space $\mathbb{R}1$ of symmetric octonions, and each of the 3 independent upper diagonal entries a_{ij} comes from the 8-dimensional space of all octonions [the sub-diagonal entries are then completely determined], leading to 1 + 1 + 1 + 8 + 8 + 8 = 27 independent parameters. In view of Albert's proof of exceptionality, and his later construction of Jordan division algebras which are forms of $\mathcal{H}_3(\mathbb{K})$, these 27-dimensional exceptional algebras are now known as *Albert algebras*, and $\mathcal{H}_3(\mathbb{K})$ is denoted by $\mathbb{A}$.

As we noted in the Colloquial Survey, these results were deeply disappointing to physicists, since the lone exceptional algebra $\mathbb{A}$ was too tiny to provide a home for quantum mechanics, and too isolated to give a clue as to the possible existence of infinite-dimensional exceptional algebras. It was still possible that infinite-dimensional exceptional algebras existed, since there were well-known associative phenomena that appear only in infinite dimensions: in quantum mechanics, the existence of operators p, q on Hilbert space with $[p, q] = \frac{\hbar}{2\pi}\mathbb{1}$ ($\hbar$ = Planck's constant) is possible only in infinite dimensions (in finite dimensions the commutator matrix $[p, q]$ would have trace 0, hence could not be a nonzero multiple $\alpha\mathbb{1}_{nn}$ of the identity matrix $\mathbb{1}_{nn}$, since the trace of $\alpha\mathbb{1}_{nn}$ is $n\alpha \neq 0$). So there remained a faint hope that there might still be an exceptional home for quantum mechanics somewhere.

2

Jordan Algebras in the Algebraic Renaissance: Finite-Dimensional Jordan Algebras over Algebraically Closed Fields

The next stage in the history of Jordan algebras was taken over by algebraists. While the physicists lost interest in the search for an exceptional setting for quantum mechanics (the philosophical objections to the theory paling in comparison to its amazing achievements), the algebraists found unsuspected relations between, on the one hand, the strange exceptional simple Albert algebra of dimension 27 and, on the other hand, the five equally strange exceptional simple Lie groups and algebras of types G_2, F_4, E_6, E_7, E_8 of dimensions 14, 52, 78, 133, 248. While these had been discovered by Wilhelm Killing and Elie Cartan in the 1890s, they were known only through their multiplication tables: there was no concrete representation for them (the way there was for the four great classes A_n, B_n, C_n, D_n discovered by Sophus Lie in the 1870s). During the 1930s Jacobson discovered that the Lie group G_2 could be realized as the automorphism group (and the Lie algebra G_2 as the derivation algebra) of a Cayley algebra, and in the early 1950s Chevalley, Schafer, Freudenthal, and others discovered that the Lie group F_4 could be realized as the automorphism group (and the Lie algebra F_4 as the derivation algebra) of the Albert algebra, that the group E_6 could be realized as the isotopy group (and the algebra E_6 as the structure algebra) of the Albert algebra, and that the algebra E_7 could be realized as the superstructure Lie algebra of the Albert algebra. [E_8 was connected to the Albert algebra in a more complicated manner.]

These unexpected connections between the physicists' orphan child and other important areas of mathematics, spurred algebraists to consider Jordan algebras over more general fields. By the late 1940s the J–vN–W structure theory had been extended by A.A. Albert, F. and N. Jacobson, and others to finite-dimensional Jordan algebras over an arbitrary algebraically closed field of characteristic not 2, with essentially the same cast of characters appearing in the title roles.

2.1 Linear Algebras over General Scalars

We begin our algebraic history by recalling the basic categorical concepts for general nonassociative algebras OVER AN ARBITRARY RING OF SCALARS Φ. When dealing with Jordan algebras we will have to assume that $\frac{1}{2} \in \Phi$, and we will point this out explicitly. An *algebra* is simultaneously a ring and a module over a ring of scalars Φ, such that the ring multiplication interacts correctly with the linear structure.

Linear Algebra Definition. *A* **ring of scalars** *is a unital commutative associative ring* Φ*. A* (*nonassociative*) **linear algebra** *over* Φ (*or* Φ*-algebra, for short*) *is a* Φ*-module* A *equipped with a* Φ*-bilinear product* $\mathrm{A} \times \mathrm{A} \longrightarrow \mathrm{A}$ (*abbreviated by juxtaposition* $(x, y) \mapsto xy$)*. Bilinearity is equivalent to the condition that the product satisfies the left and right distributive laws*

$$x(y + z) = xy + xz, \quad (y + z)x = yx + zx,$$

and that scalars flit in and out of products,

$$(\alpha x)y = x(\alpha y) = \alpha(xy),$$

for all elements x, y, z *in* A*. The algebra is* **unital** *if there exists a* (*two-sided*) **unit element** 1 *satisfying* $1x = x1 = x$ *for all* x*.*

Notice that we do not require associativity of the product nor existence of a unit element in A (though we always demand a unit scalar in Φ). Lack of a unit is easy to repair: we can always enlarge a linear algebra slightly to get a unital algebra.

Unital Hull Definition. *Any linear algebra can be imbedded as an ideal in its* **unital hull**

$$\widehat{\mathrm{A}} := \Phi\hat{1} \oplus \mathrm{A}, \quad (\alpha\hat{1} \oplus x)(\beta\hat{1} \oplus y) := \alpha\beta\hat{1} \oplus (\alpha y + \beta x + xy).$$

A is always an ideal in $\widehat{\mathrm{A}}$ since multiplication by the new elements $\alpha\hat{1}$ are just scalar multiplications; this means that we can often conveniently formulate results inside A making use of its unital hull. For example, in an associative algebra the left ideal $Ax + \Phi x$ generated by an element x can be written succinctly as $\widehat{\mathrm{A}}x$ (the left ideal $\mathrm{A}x$ needn't contain x if A is not already unital).

2.2 Categorical Nonsense

We have the usual notions of morphisms, sub-objects, and quotients for linear algebras.

Morphism Definition. *A* **homomorphism** $\varphi : \mathrm{A} \to \mathrm{A}'$ *is a linear map of Φ-modules which preserves multiplication,*

$$\varphi(xy) = \varphi(x)\varphi(y);$$

an **anti-homomorphism** *is a linear map which reverses multiplication,*

$$\varphi(xy) = \varphi(y)\varphi(x).$$

The kernel $\mathrm{Ker}(\varphi) := \varphi^{-1}(0')$ *is the set of elements mapped into* $0' \in \mathrm{A}'$, *and the* image $\mathrm{Im}(\varphi) := \varphi(\mathrm{A})$ *is the range of the map. An* **isomorphism** *is a bijective homomorphism; we say that* A *is* **isomorphic to** A', *or* A *and* A' **are isomorphic** (*written* $\mathrm{A} \cong \mathrm{A}'$), *if there is an isomorphism of* A *onto* A'. *An* **automorphism** *is an isomorphism of an algebra with itself. We have corresponding notions of* **anti-isomorphism** *and* **anti-automorphism** *for anti-homomorphisms.*

$*$-Algebra Definition. *An* **involution** *is an anti-automorphism of period* 2,

$$\varphi(xy) = \varphi(y)\varphi(x) \quad \textit{and} \quad \varphi(\varphi(x)) = x.$$

We will often be concerned with involutions, since they are a rich source of Jordan algebras. The natural notion of morphism in the category of **$*$-algebras** (*algebras together with a choice of involution*) *is that of* **$*$-homomorphism** $(\mathrm{A}, *) \to (\mathrm{A}', *')$, *which is a homomorphism* $\varphi : \mathrm{A} \to \mathrm{A}'$ *of algebras which preserves the involutions,* $\varphi \circ * = *' \circ \varphi$ (*i.e.,* $\varphi(x^*) = \varphi(x)^{*'}$ *for all* $x \in \mathrm{A}$).

One important involution is the standard involution on a quaternion or octonion algebra.

Ideal Definition. *A* **subalgebra** $\mathrm{B} \leq \mathrm{A}$ *of a linear algebra* A *is a Φ-submodule closed under multiplication:* $\mathrm{BB} \subseteq \mathrm{B}$. *An* **ideal** $B \triangleleft \mathrm{A}$ *of* A *is a Φ-submodule closed under left and right multiplication by* A: $\mathrm{AB} + \mathrm{BA} \subseteq \mathrm{B}$. *If* A *has an involution, a* **$*$-ideal** *is an ideal invariant under the involution:* $\mathrm{B} \triangleleft \mathrm{A}$ *and* $\mathrm{B}^* \subseteq \mathrm{B}$. *We will always use* $\mathbf{0}$ *to denote the zero* submodule, *while ordinary* 0 *will denote the zero* scalar, vector, *or* transformation (*context will decide which is meant*).

Quotient Definition. *Any ideal* $\mathrm{B} \triangleleft \mathrm{A}$ *is the kernel of the* canonical homomorphism $\pi : x \mapsto \overline{x}$ *of* A *onto the* **quotient algebra** $\overline{\mathrm{A}} = \mathrm{A}/\mathrm{B}$ (*consisting of all cosets* $\overline{x} := [x]_{\mathrm{B}} := x + \mathrm{B}$ *with the induced operations* $\alpha\overline{x} := \overline{\alpha x}$, $\overline{x} + \overline{y} := \overline{x + y}$, $\overline{x}\;\overline{y} := \overline{xy}$). *The quotient* A/B *of a $*$-algebra by a $*$-ideal is again a $*$-algebra under the induced involution* $\overline{x}^* := \overline{x^*}$.

We have the usual tripartite theorem relating homomorphisms and quotients.

Fundamental Theorem of Homomorphisms. *For homomorphisms (and similarly for $*$-homomorphisms) we have:*

(I) *If* $\varphi : A \to A'$ *is a homomorphism, then* $\mathrm{Ker}(\varphi) \lhd \mathrm{A}$, $\mathrm{Im}(\varphi) \leq \mathrm{A}'$, *and* $\mathrm{A}/\mathrm{Ker}(\varphi) \cong \mathrm{Im}(\varphi)$ *under the map* $\overline{\varphi}(\overline{x}) := \varphi(x)$.

(II) *There is a* 1*-to-*1 *correspondence between the ideals (respectively subalgebras)* $\overline{\mathrm{C}}$ *of the quotient* $\overline{\mathrm{A}} = \mathrm{A}/\mathrm{B}$ *and those* C *of* A *which contain* B, *given by* $\mathrm{C} \mapsto \pi(\mathrm{C})$ *and* $\overline{\mathrm{C}} \mapsto \pi^{-1}(\overline{\mathrm{C}})$; *for such ideals* C *we have* $\overline{\mathrm{A}}/\overline{\mathrm{C}} \cong \mathrm{A}/\mathrm{C}$.

(III) *If* $\mathrm{B} \lhd \mathrm{A}$, $\mathrm{C} \leq \mathrm{A}$ *then* $\mathrm{C}/(\mathrm{C} \cap \mathrm{B}) \cong (\mathrm{C} + \mathrm{B})/\mathrm{B}$ *under the map* $\varphi([x]_{\mathrm{C}\cap\mathrm{B}}) = [x]_{\mathrm{B}}$.

As usual, we have a way of gluing different algebras together as a direct sum in such a way that the individual pieces don't interfere with each other.

Direct Sum Definition. *The* **direct sum** $\mathrm{A}_1 \boxplus \cdots \boxplus \mathrm{A}_n$ *of a finite number of algebras is the Cartesian product* $\mathrm{A}_1 \times \cdots \times \mathrm{A}_n$ *under the* componentwise operations

$$\begin{aligned} \alpha(x_1, \ldots, x_n) &:= (\alpha x_1, \ldots, \alpha x_n), \\ (x_1, \ldots, x_n) + (y_1, \ldots, y_n) &:= (x_1 + y_1, \ldots, x_n + y_n), \\ (x_1, \ldots, x_n)(y_1, \ldots, y_n) &:= (x_1 y_1, \ldots, x_n y_n). \end{aligned}$$

We will consistently write an algebra *direct sum with* $\boxplus$, *and a mere* module *direct sum with* $\oplus$.

In algebras with finiteness conditions we only need to consider finite direct sums of algebras. Direct sums are the most useful (but rarest) "rigid" decompositions, and are the goal of many structure theories. In the wide-open spaces of the infinite-dimensional world, direct sums (finite or otherwise) do not suffice, and we need to deal with infinite direct products.

Direct Product Definition. *The* **direct product** $\prod_{i\in I} \mathrm{A}_i$ *of an arbitrary family of algebraic systems* A_i *indexed by a set* I *is the Cartesian product* $\mathsf{X}_{i\in I}\mathrm{A}_i$ *under the componentwise operations. We may think of this as all "strings"* $a = \prod a_i$ *or "I-tuples"* $a = (\ldots a_i \ldots)$ *of elements, one from each family member* A_i, *or more rigorously as all maps* $a : I \to \cup_i A_i$ *such that* $a(i) \in \mathrm{A}_i$ *for each* i, *under the pointwise operations.*

The **direct sum** $\boxplus_{i\in I}\mathrm{A}_i$ *is the subalgebra of all tuples with only a finite number of nonzero entries (so they can be represented as a finite sum* $a_{i_1} + \cdots + a_{i_n}$ *of elements* $a_j \in \mathrm{A}_j$*). For each* $i \in I$ *we have* canonical projections π_i *of both the direct product and direct sum onto the* i*th component* A_i.[1]

[1] The direct product and sum are often called the *product* $\prod \mathrm{A}_i$ and *coproduct* $\coprod \mathrm{A}_i$; especially in algebraic topology, these appear as "dual" objects. For finite index sets, the two concepts coincide.

Infinite direct products are complicated objects, more topological or combinatorial than algebraic. They play a crucial role in the "logic" of algebras through the construction of ultraproducts. Semi-direct product decompositions are fairly "loose," but are crucial in the study of radicals.

Subdirect Product Definition. *An algebra is a* **subdirect product** $\mathrm{A} \precapprox \prod_{i \in I} \mathrm{A}_i$ *of algebras (more often and more inaccurately called a* **semidirect sum**) *if there is* (1) *a monomorphism* $\varphi : \mathrm{A} \to \prod_{i \in I} \mathrm{A}_i$ *such that* (2) *for each* $i \in I$ *the canonical projection* $\pi_i(\varphi(\mathrm{A})) = \mathrm{A}_i$ *maps onto all of* A_i. *By the Fundamental Theorem,* (2) *is equivalent to* $\mathrm{A}_i \cong \mathrm{A}/\mathrm{K}_i$ *for an ideal* $\mathrm{K}_i \triangleleft \mathrm{A}$, *and* (1) *is equivalent to* $\bigcap_I \mathrm{K}_i = \mathbf{0}$. *Thus a semi-direct product decomposition of* A *is essentially the same as a "disjoint" family of ideals.*

For example, the integers $\mathbb{Z}$ are a subdirect product of fields $\mathbb{Z}_p$ for any infinite collection of primes p, and even of $\mathbb{Z}_{p^n}$ for a fixed p but infinitely many n.

In a philosophical sense, an algebra A can be recovered from an ideal B and its quotient A/B. The basic building blocks are those algebras which cannot be built up from smaller pieces, i.e., have no smaller ingredients B.

Simple Definition. *An algebra is* **simple** *if it has no proper ideals and is not trivial,* $\mathrm{AA} \neq \mathbf{0}$. *Analogously, a* $*$*-algebra is* $*$-**simple** *if it has no proper* $*$*-ideals and is not trivial. Here a submodule* B *is* proper *if it is not zero or the whole module,* $\mathrm{B} \neq \mathbf{0}, \mathrm{A}$. *An algebra is* **semisimple** *if it is a finite direct sum of simple algebras.*

2.3 Commutators and Associators

We can reformulate the algebra conditions in terms of the **left and right multiplication operators** L_x and R_x by the element x, defined by

$$L_x(y) := xy =: R_y(x).$$

Bilinearity of the product just means the map $L : x \mapsto L_x$ (or equivalently the map $R : y \mapsto R_y$) is a linear mapping from the Φ-module A into the Φ-module $\mathcal{E}nd_\Phi(\mathrm{A})$ of Φ-linear transformations on A.

The product (and the algebra) is **commutative** if $xy = yx$ for all x, y, and **skew** if $xy = -yx$ for all x, y; in terms of operators, commutativity means that $L_x = R_x$ for all x, and skewness means that $L_x = -R_x$ for all x, so in either case we can dispense with the right multiplications and work only with the L_x. In working with the commutative law, it is convenient to introduce the **commutator**

$$[x, y] := xy - yx,$$

which measures how far two elements are from commuting: x and y commute iff their commutator is zero. In these terms the commutative law is $[x, y] = 0$, so an algebra is commutative iff all commutators vanish.

The product (and the algebra) is **associative** if $(xy)z = x(yz)$ for all x, y, z, in which case we drop all parentheses and write the product as xyz. We can interpret associativity in three ways as an operator identity, depending on which of x, y, z we treat as the variable: on z it says that $L_{xy} = L_x L_y$, i.e., that L is a homomorphism of A into $\mathcal{E}nd_\Phi(\mathrm{A})$; on x it says that $R_z R_y = R_{yz}$, i.e., that R is an anti-homomorphism; on y it says that $R_z L_x = L_x R_z$, i.e., that all left multiplications L_x commute with all right multiplications R_z.[2] It is similarly convenient to introduce the **associator**

$$[x, y, z] := (xy)z - x(yz),$$

which measures how far three elements are from associating: x, y, z associate iff their associator is zero. In these terms an algebra is associative iff all its associators vanish, and the Jordan identity becomes $[x^2, y, x] = 0$.

Nonassociativity can never be repaired, it is an incurable illness. Instead, we can focus on the parts of an algebra which do behave associatively. The **nucleus** $\mathcal{N}uc(\mathrm{A})$ of a linear algebra A is the part which "associates" with all other elements, the elements n which hop blithely over parentheses:

$$\mathcal{N}uc(\mathrm{A}):\ (nx)y = n(xy),\ (xn)y = x(ny),\ (xy)n = x(yn)$$

for all x, y in A. In terms of associators, nuclear elements are those which vanish when put into an associator,

$$\mathcal{N}uc(\mathrm{A}) := \{n \in \mathrm{A} \mid [n, \mathrm{A}, \mathrm{A}] = [\mathrm{A}, n, \mathrm{A}] = [\mathrm{A}, \mathrm{A}, n] = \mathbf{0}\}.$$

Nuclear elements will play a role in several situations (such as forming nuclear isotopes, or considering involutions whose hermitian elements are all nuclear). The associative ring theorist Jerry Martindale offers this advice for proving theorems about nonassociative algebras: never multiply more than two elements together at a time. We can extend this secret for success even further: when multiplying n elements together, make sure that at least n–2 of them belong to the nucleus!

Another useful general concept is that of the **center** $\mathcal{C}ent(\mathrm{A})$, the set of elements c which both commute and associate, and therefore act like scalars:

$$\mathcal{C}ent(\mathrm{A}):\ cx = xc,\ c(xy) = (cx)y = x(cy),$$

[2] Most algebraists of yore were right-handed, i.e., they wrote their maps on the right: a linear transformation T on V had values xT, the matrix of T with respect to an ordered basis was built up row by row, and composition $S \circ T$ meant first do S and then T. For them, the natural multiplication was $R_y, xR_y = xy$. Modern algebraists are all raised as left-handers, writing maps on the left ($f(x)$ instead of xf), as learned in the calculus cradle, and building matrices column by column. Whichever hand you use, in dealing with modules over noncommutative rings of scalars it is important to *keep the scalars on the opposite side of the vectors from the operators*, so linear maps have either $T(x\alpha) = (Tx)\alpha$ or $(\alpha x)T = \alpha(xT)$. Since the dual V^* of a left (resp. right) vector space V over a noncommutative division algebra Δ is a *right* (resp. *left*) vector space over Δ, it is important to be ambidextrous, writing a linear map as $T(x)$ on V, but its adjoint as $(x^*)T^*$ on the dual.

or in terms of associators and commutators

$$\mathcal{C}ent(\mathrm{A}) := \{c \in \mathcal{N}uc(\mathrm{A}) \mid [c, \mathrm{A}] = \mathbf{0}\}.$$

Any unital algebra may be considered as an algebra over its center, which is a ring of scalars over Φ: we simply replace the original scalars by the center with scalar multiplication $c \cdot x := cx$. If A is unital then $\Phi 1 \subseteq \mathcal{C}ent(\mathrm{A})$, and the original scalar action is preserved in the form $\alpha x = (\alpha 1) \cdot x$. In most cases the center forms the "natural" scalars for the algebra; a unital Φ-algebra is **central** if its center is precisely $\Phi 1$. Central-simple algebras (those which are central and simple) are crucial building-blocks of a structure theory.

2.4 Lie and Jordan Algebras

In defining Jordan algebras over general scalars, the theory always required the existence of a scalar $\frac{1}{2}$ (ruling out characteristic 2) to make sense of its basic examples, the special algebras under the Jordan product. Outside this restriction, the structure theory worked smoothly and uniformly in all characteristics.

Jordan Algebra Definition. *If Φ is a commutative associative ring of scalars containing $\frac{1}{2}$, a* **Jordan algebra** *over Φ is a linear algebra* J *equipped with a commutative product $p(x, y) = x \bullet y$ which satisfies the Jordan identity. In terms of commutators and associators these can be expressed as*

$$\text{(JAX1)} \qquad [x, y] = 0 \qquad \text{(Commutative Law).}$$

$$\text{(JAX2)} \qquad [x^2, y, x] = 0 \qquad \text{(Jordan Identity).}$$

The product is usually denoted by $x \bullet y$ rather than by mere juxtaposition. In operator terms, the axioms can be expressed as saying that left and right multiplications coincide, and left multiplication by x^2 commutes with left multiplication by x:

$$(\text{JAX1}^{op}) \qquad L_x = R_x, \qquad (\text{JAX2}^{op}) \qquad [L_{x^2}, L_x] = 0.$$

Lie algebras can be defined over general rings, though in practice pathologies crop up as soon as you leave characteristic 0 for characteristic p (and by the time you reach characteristic 2 almost nothing remains of the structure theory).

Lie Algebra Definition. *A* **Lie algebra**[3] *over any ring of scalars Φ is a linear algebra* L *equipped with an anti-commutative product, universally denoted by brackets $p(x, y) := [x, y]$, satisfying the Jacobi identity*

$$\text{(LAX1)} \qquad [x, y] = -[y, x] \qquad \text{(Anti-commutative Law),}$$

$$\text{(LAX2)} \qquad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \qquad \text{(Jacobi Identity).}$$

[3] "Lee" as in Sophus or Sara or Robert E., not "Lye."

We can write these axioms too as illuminating operator identities:

$$(\text{LAX1}^{op}) \quad L_x = -R_x, \qquad (\text{LAX2}^{op}) \quad L_{[x,y]} = [L_x, L_y],$$

so that L is a homomorphism $\mathrm{L} \to \mathcal{E}nd_\Phi(\mathrm{L})^-$ of Lie algebras (called the *adjoint representation,* with the left multiplication map called the *adjoint map* $Ad(x) := L_x$). The use of the bracket for the product conflicts with the usual notation for the commutator, which would be $[x,y] - [y,x] = 2[x,y]$, but this shows that there is no point in using commutators in Lie algebras to measure commutativity: the bracket says it all.

2.5 The Three Basic Examples Revisited

The creation of the plus and minus algebras A^+, A^- makes sense for arbitrary linear algebras, and these produce Jordan and Lie algebras when A is associative. These are the first (and most important) examples of Jordan and Lie algebras.

Full Example. *If* A *is any linear algebra with product* xy *over a ring of scalars* Φ *containing* $\frac{1}{2}$, *the* plus algebra A^+ *denotes the linear* Φ*-algebra with commutative "Jordan product"*

$$\mathrm{A}^+ : \qquad x \bullet y := \tfrac{1}{2}(xy + yx).$$

If A *is an associative* Φ*-algebra, then* A^+ *is a Jordan* Φ*-algebra.*

Just as everyone should show, *once and only once in his or her life,* that every associative algebra A gives rise to a Lie algebra A^- by verifying directly the anti-commutativity and Jacobi identity for the commutator product, so should everyone show that A also gives rise to a Jordan algebra A^+ by verifying directly the commutativity and Jordan identity for the anti-commutator product.

The previous notions of speciality and exceptionality also make sense in general.

Special Definition. *A Jordan algebra is* **special** *if it can be imbedded in an algebra* A^+ *for* A *associative* (*i.e., if it is isomorphic to a subalgebra of some* A^+), *otherwise it is* **exceptional**. *We usually think of special algebras as* living inside *associative algebras.*

As before, the most important examples of special Jordan or Lie subalgebras are the algebras of hermitian or skew elements of an associative algebra with involution.

Hermitian Example. *If a linear algebra* A *has an involution* $*$, *then* $\mathcal{H}(\mathrm{A}, *)$ *denotes the* hermitian elements $x^* = x$. *It is easy to see that if* A *is an associative* Φ*-algebra with involution, then* $\mathcal{H}(\mathrm{A}, *)$ *is a Jordan* Φ*-subalgebra of* A^+.

The third basic example of a special Jordan algebra is a spin factor, which has no natural Lie analogue.

Spin Factor Example. *We define a linear* Φ*-algebra structure* $\mathcal{J}Spin_n(\Phi)$ *on* $\Phi 1 \oplus \Phi^n$ *over an arbitrary ring of scalars* Φ *by having* 1 *act as unit element and defining the product of vectors* $\mathbf{v}, \mathbf{w} \in \Phi^n$ *to be the scalar multiple of* 1 *given by the dot product* $\langle \mathbf{v}, \mathbf{w} \rangle$ *(for column vectors this is* $\mathbf{v}^{tr}\mathbf{w}$*),*

$$\mathbf{v} \bullet \mathbf{w} := \langle \mathbf{v}, \mathbf{w} \rangle 1,$$

so the global expression for the product is

$$(\alpha 1 \oplus \mathbf{v}) \bullet (\beta 1 \oplus \mathbf{w}) := \big(\alpha\beta + \langle \mathbf{v}, \mathbf{w} \rangle\big) 1 \oplus \big(\alpha \mathbf{w} + \beta \mathbf{v}\big).$$

Spin factors over general scalars are Jordan algebras just as they were over the reals, by symmetry of the dot product and the fact that L_{x^2} is a linear combination of $L_x, \mathbb{1}_J$, and again they can be imbedded in hermitian $2^n \times 2^n$ matrices over Φ.

2.6 Jordan Matrix Algebras with Associative Coordinates

An important special case of a Hermitian Jordan algebra $\mathcal{H}(A, *)$ is that where the linear algebra $A = \mathcal{M}_n(D)$ is the algebra of $n \times n$ matrices over a *coordinate algebra* $(D, -)$ (a unital linear algebra with involution $d \mapsto \bar{d}$). These are especially useful since one can give an explicit "multiplication table" for hermitian matrices in terms of the coordinates of the matrices, and the properties of $\mathcal{H}$ closely reflect those of D.

Hermitian Matrix Example. *For an arbitrary linear* $*$*-algebra* D *with involution* $\overline{}$*, the conjugate transpose mapping* $X^* := \overline{X}^{tr}$ $\big(\overline{X} := (\overline{x_{ij}})\big)$ *is an involution on the linear algebra* $\mathcal{M}_n(D)$ *of all* $n \times n$ *matrices with entries from* D *under the usual matrix product* XY*. The* Φ*-module* $\mathcal{H}_n(D, -)$ *of all hermitian matrices* $X^* = X$ *with respect to this involution is closed under the Jordan product* $X \bullet Y = \frac{1}{2}(XY + YX)$*.*[4]

Using the multiplication table one can show why the exceptional Jordan matrix algebras in the Jordan–von Neumann–Wigner Theorem stop at $n = 3$: in order to produce a Jordan matrix algebra, the coordinates must be alternative if $n = 3$ and associative if $n \geq 4$.[5]

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[4] If we used a single symbol $\mathcal{D} = (D, -)$ for a $*$-algebra, the hermitian example would take the form $\mathcal{H}_n(\mathcal{D})$. Though this notation more clearly reveals that $\mathcal{H}_n$ is a functor from the categories of associative $*$-algebras to Jordan algebras, we will almost always include the involution in the notation. The one exception is for composition algebras with their standard involution: we write $\mathcal{H}_n(C)$ when C is $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{K}$, or an octonion algebra O.

[5] In fact, any "respectable" Jordan algebras of "degree" 4 or more (whether or not they have the specific form of matrix algebras) must be special.

Associative Coordinates Theorem. *If the hermitian matrix algebra* $\mathcal{H}_n(\mathrm{D}, -)$ *for* $n \geq 4$ *and* $\frac{1}{2} \in \Phi$ *is a Jordan algebra under the product* $X \bullet Y = \frac{1}{2}(XY + YX)$*, then* D *must be associative and* $\mathcal{H}_n(\mathrm{D}, -)$ *is a special Jordan algebra.*

2.7 Jordan Matrix Algebras with Alternative Coordinates

When $n = 3$ we can even allow D to be slightly nonassociative: the coordinate algebra must be alternative.

Alternative Algebra Definition. *A linear algebra* D *is* **alternative** *if it satisfies the* **Left and Right Alternative Laws**

$$\text{(AltAX1)} \qquad x^2 y = x(xy) \qquad \text{(Left Alternative Law)},$$
$$\text{(AltAX2)} \qquad yx^2 = (yx)x \qquad \text{(Right Alternative Law)}$$

for all x, y *in* D*. An alternative algebra is automatically* **flexible***,*

$$\text{(AltAX3)} \qquad (xy)x = x(yx) \qquad \text{(Flexible Law)}.$$

In terms of associators or operators these identities may be expressed as

$$[x, x, y] = [y, x, x] = [x, y, x] = 0, \qquad \textit{or}$$
$$L_{x^2} = (L_x)^2, \quad R_{x^2} = (R_x)^2, \quad L_x R_x = R_x L_x.$$

From the associator conditions we see that alternativity is equivalent to the associator $[x, y, z]$ being an *alternating* multilinear function of its arguments (in the sense that it vanishes if any two of its variables are equal). Perhaps it would be better to call the algebras *alternating* instead of *alternative*. Notice that the nuclearity conditions can be written in terms of associators as $[n, x, y] = [x, n, y] = [x, y, n] = 0$, so by alternation nuclearity reduces to $[n, x, y] = 0$ in alternative algebras.

It is not hard to see that for a matrix algebra $\mathcal{H}_3(\mathrm{D}, -)$ to be a Jordan algebra it is necessary that the coordinate algebra D be alternative and that the diagonal coordinates, the hermitian elements $\mathcal{H}(\mathrm{D}, -)$, lie in the nucleus. The converse is true, but painful to prove. Note that in the octonions the hermitian elements do even better: they are *scalars* lying in $\Phi 1$.

Alternative Coordinates Theorem. *The hermitian matrix algebra* $\mathcal{H}_3(\mathrm{D}, -)$ *over* Φ *containing* $\frac{1}{2}$ *is a Jordan algebra iff the* $*$*-algebra* D *is alternative with nuclear involution, i.e., its hermitian elements are contained in the nucleus,*

$$[\mathcal{H}(\mathrm{D}, -), \mathrm{D}, \mathrm{D}] = \mathbf{0}.$$

2.8 The n-Squares Problem

Historically, the first nonassociative algebra, the Cayley numbers (progenitor of the theory of alternative algebras), arose in the context of the number-theoretic problem of quadratic forms permitting composition. We will show how this number-theoretic question can be transformed into one concerning certain algebraic systems, the composition algebras, and then how a precise description of these algebras leads to precisely one nonassociative coordinate algebra suitable for constructing Jordan algebras, the 8-dimensional octonion algebra with scalar involution.

It was known to Diophantus that sums of two squares could be *composed,* i.e., that the product of two such terms could be written as another sum of two squares: $(x_0^2 + x_1^2)(y_0^2 + y_1^2) = (x_0y_0 - x_1y_1)^2 + (x_0y_1 + x_1y_0)^2$. Indian mathematicians were aware that this could be generalized to other "binary" (two-variable) quadratic forms, yielding a "two-square formula"

$$(x_0^2 + \lambda x_1^2)(y_0^2 + \lambda y_1^2) = (x_0y_0 - \lambda x_1y_1)^2 + \lambda(x_0y_1 + x_1y_0)^2 = z_0^2 + \lambda z_1^2.$$

In 1748 Euler used an extension of this to "quaternary" (4-variable) quadratic forms $x_0^2 + x_1^2 + x_2^2 + x_3^2$, and in 1770 Lagrange used a general "4-square formula":

$$\begin{aligned}(x_0^2 + \lambda x_1^2 + \mu x_2^2 + \lambda\mu x_3^2) \times (y_0^2 + \lambda y_1^2 + \mu y_2^2 + \lambda\mu y_3^2)\\ = z_0^2 + \lambda z_1^2 + \mu z_2^2 + \lambda\mu z_3^2\end{aligned}$$

for z_i defined by

$$\begin{aligned}z_0 &:= x_0y_0 - \lambda x_1y_1 - \mu x_2y_2 - \lambda\mu x_3y_3,\\ z_1 &:= x_0y_1 + x_1y_0 + \mu x_2y_3 - \mu x_3y_2,\\ z_2 &:= x_0y_2 - \lambda x_1y_3 + x_2y_0 + \lambda x_3y_1,\\ z_3 &:= x_0y_3 + x_1y_2 - x_2y_1 + x_3y_0.\end{aligned}$$

In 1845 an "8-square formula" was discovered by Cayley; J.T. Graves claimed to have discovered this earlier, and in fact C.F. Degan had already noted a more general formula in 1818:

$$\begin{aligned}&(x_0^2 + \lambda x_1^2 + \mu x_2^2 + \lambda\mu x_3^2 + \nu x_4^2 + \lambda\nu x_5^2 + \mu\nu x_6^2 + \lambda\mu\nu x_7^2)\\ &\quad\times (y_0^2 + \lambda y_1^2 + \mu y_2^2 + \lambda\mu y_3^2 + \nu y_4^2 + \lambda\nu y_5^2 + \mu\nu y_6^2 + \lambda\mu\nu y_7^2)\\ &\quad\quad= (z_0^2 + \lambda z_1^2 + \mu z_2^2 + \lambda\mu z_3^2 + \nu z_4^2 + \lambda\nu z_5^2 + \mu\nu z_6^2 + \lambda\mu\nu z_7^2)\end{aligned}$$

for z_i defined by

$$\begin{aligned}z_0 &:= x_0y_0 - \lambda x_1y_1 - \mu x_2y_2 - \lambda\mu x_3y_3 - \nu x_4y_4 - \lambda\nu x_5y_5 - \mu\nu x_6y_6 - \lambda\mu\nu x_7y_7,\\ z_1 &:= x_0y_1 + x_1y_0 + \mu x_2y_3 - \mu x_3y_2 + \nu x_4y_5 - \nu x_5y_4 - \mu\nu x_6y_7 + \mu\nu x_7y_6,\\ z_2 &:= x_0y_2 - \lambda x_1y_3 + x_2y_0 + \lambda x_3y_1 + \nu x_4y_6 + \lambda\nu x_5y_7 - \nu x_6y_4 - \lambda\nu x_7y_5,\\ z_3 &:= x_0y_3 + x_1y_2 - x_2y_1 + x_3y_0 + \nu x_4y_7 - \nu x_5y_6 + \nu x_6y_5 - \nu x_7y_4,\\ z_4 &:= x_0y_4 - \lambda x_1y_5 - \mu x_2y_6 - \lambda\mu x_3y_7 + x_4y_0 + \lambda x_5y_1 + \mu x_6y_2 + \lambda\mu x_7y_3,\\ z_5 &:= x_0y_5 + x_1y_4 - \mu x_2y_7 + \mu x_3y_6 - x_4y_1 + x_5y_0 - \mu x_6y_3 + \mu x_7y_2,\\ z_6 &:= x_0y_6 + \lambda x_1y_7 + x_2y_4 - \lambda x_3y_5 - x_4y_2 + \lambda x_5y_3 + x_6y_0 - \lambda x_7y_1,\\ z_7 &:= x_0y_7 - x_1y_6 + x_2y_5 + x_3y_4 - x_4y_3 - x_5y_2 + x_6y_1 + x_7y_0.\end{aligned}$$

This is clearly not the sort of formula you stumble upon during a casual mathematical stroll. Indeed, this is too cumbersome to tackle directly, with its mysterious distribution of plus and minus signs and assorted scalars.

2.9 Forms Permitting Composition

A more concise and conceptual approach is needed. If we interpret the variables as coordinates of a vector $x = (x_0, \dots, x_7)$ in an 8-dimensional vector space, then the expression $x_0^2+\lambda x_1^2+\mu x_2^2+\lambda\mu x_3^2+\nu x_4^2+\lambda\nu x_5^2+\mu\nu x_6^2+\lambda\mu\nu x_7^2$ defines a quadratic norm form $N(x)$ on this space. The 8-square formula asserts that this quadratic form *permits* (or admits) *composition* in the sense that $N(x)N(y) = N(z)$, where the "composite" $z = (z_0, \dots, z_7)$ is automatically a bilinear function of x and y (i.e., each of its coordinates z_i is a bilinear function of the x_j and y_k). We may think of $z = x\cdot y$ as some sort of "product" of x and y. This product is linear in x and y, but it need not be commutative or associative. Thus the existence of an n-squares formula is equivalent to the existence of an n-dimensional algebra with product $x \cdot y$ and distinguished basis $e_0, \dots, e_{n-1}$ such that $N(x) = N(x_0e_0 + \cdots + x_{n-1}e_{n-1}) = \sum_{i=0}^{n} \lambda_i x_i^2$ permits composition $N(x)N(y) = N(x \cdot y)$ (in the classical case all $\lambda_i = 1$, and this is a "pure" sum of squares). The element $e_0 = (1, 0, \dots, 0)$ (having $x_0 = 1$, all other $x_i = 0$) acts as unit element: $e_0 \cdot y = y$, $x \cdot e_0 = x$. When the quadratic form is anisotropic ($N(x) = 0 \Longrightarrow x = 0$) the algebra is a "division algebra": it has no divisors of zero, $x, y \neq 0 \Longrightarrow x \cdot y \neq 0$, so in the finite-dimensional case the *injectivity* of left and right multiplications makes them *bijections*.

The algebra behind the 2-square formula is just the *complex numbers* $\mathbb{C}$: $z = x_0 1+x_1 i$ with basis $1, i$ over the reals, where 1 acts as identity and $i^2 = -1$ and $N(z) = x_0^2 + x_1^2 = z\overline{z}$ is the ordinary norm squared (where $\overline{z} = x_0 1 - x_1 i$ is the ordinary complex conjugate). This interpretation was well known to Gauss. The 4-squares formula led Hamilton to the *quaternions* $\mathbb{H}$ consisting of all $x = x_0 1 + x_1 i + x_2 j + x_3 k$, where the formula for $x \cdot y$ means that the basis elements $1, i, j, k$ satisfy the now-familiar rules

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

Clearly, this algebra is no longer commutative. Again $N(x) = x\overline{x}$ is the ordinary norm squared (where $\overline{x} = x_0 1 - x_1 i - x_2 j - x_3 k$ is the ordinary quaternion conjugate).

Clifford and Hamilton invented 8-dimensional algebras (biquaternions), which were merely the direct sum $\mathbb{H} \boxplus \mathbb{H}$ of two quaternion algebras. Because of the presence of zero divisors, these algebras were of minor interest. Cayley was the first to use the 8-square formula to create an 8-dimensional division algebra $\mathbb{K}$ of *octonions* or *Cayley numbers*. By 1847 he recognized that this algebra was not commutative or associative, with basis $e_0, \ldots, e_7 = 1, i, j, k, \ell, i\ell, j\ell, k\ell$ with multiplication table

$$e_0 e_i = e_i e_0 = e_i, \quad e_i^2 = -1, \quad e_i e_j = -e_j e_i = e_k$$
$$\text{for} \quad ijk = 123, 145, 624, 653, 725, 734, 176.$$

A subsequent flood of (false!!) higher-dimensional algebras carried names such as quadrinions, quines, pluquaternions, nonions, tettarions, plutonions. Ireland especially seemed a factory for such counterfeit division algebras. In 1878 Frobenius showed that the only *associative* division algebras over the reals (permitting composition or not) are $\mathbb{R}, \mathbb{C}, \mathbb{H}$ of dimensions 1, 2, 4. In 1898 Hurwitz proved via group representations that the only quadratic forms permitting composition over the reals are the standard ones of dimension 1, 2, 4, 8; A.A. Albert later gave an algebra-theoretic proof over a general field of scalars (with an addition by Irving Kaplansky to include characteristic 2 and non-unital algebras). Only recently was it established that the only *finite-dimensional* real nonassociative division algebras have dimensions 1, 2, 4, 8; the algebras themselves were not classified, and the proof was topological rather than algebraic.

2.10 Composition Algebras

The most important alternative algebras with nuclear involutions are the composition algebras. A **composition algebra** is a unital algebra having a nondegenerate quadratic norm form N which **permits composition**,

$$Q(1) = 1, \qquad Q(xy) = Q(x)Q(y).$$

In general, a quadratic form Q on a Φ-module V is **nondegenerate** if all nonzero elements in the module contribute to the values of the form. The slackers (the set of elements which contribute nothing) are gathered in the **radical**

$$\mathcal{R}ad(Q) := \{z \in V \mid Q(z) = Q(z, V) = 0\},$$

so nondegeneracy means that $\mathcal{R}ad(Q) = \mathbf{0}$.[6]

Even better than nondegeneracy is anisotropy. A vector x is **isotropic** if it has “zero weight” $Q(x) = 0$, and **anisotropic** if $Q(x) \neq 0$. A form is isotropic if it has nonzero isotropic vectors, and anisotropic if it has none:

$$Q \textit{ anisotropic} \quad \textit{iff} \quad Q(x) = 0 \Longleftrightarrow x = 0.$$

For example, the positive definite norm form $Q(x) = x \cdot x$ on any Euclidean space is anisotropic. Clearly, any anisotropic form is nondegenerate.

[6] Since $Q(z) = \frac{1}{2}Q(z, z)$, when $\frac{1}{2} \in \Phi$ the radical of the quadratic form reduces to the usual radical $\mathcal{R}ad(Q(\cdot, \cdot)) := \{z \in V \mid Q(z, V) = 0\}$ of the associated bilinear form $Q(\cdot, \cdot)$ (the vectors which are “orthogonal to everybody”). But in characteristic 2 there is an important difference between the radical of the quadratic form and the “bilinear radical” of its associated bilinear form.

2.11 The Cayley–Dickson Construction and Process

The famous Hurwitz Theorem of 1898 states that over the real numbers composition algebras can exist only in dimensions $1, 2, 4$, and 8. In 1958 Nathan Jacobson gave a beautiful "bootstrap" method, showing clearly how all composition algebras are generated internally, by repeated "doubling" (of the module, the multiplication, the involution, and the norm) starting from any composition subalgebra. As its name suggests, the Cayley–Dickson doubling process is due to A.A. Albert.

Cayley–Dickson Definition. *The* **Cayley–Dickson Construction** *builds a new $*$-algebra out of an old one together with a choice of scalar. If* A *is a unital linear algebra with involution* $a \mapsto \bar{a}$ *whose norms satisfy* $a\bar{a} = n(a)1$ *for scalars* $n(a) \in \Phi$, *and* μ *is an invertible scalar in* Φ, *then the Cayley–Dickson algebra*

$$\mathcal{KD}(\mathrm{A}, \mu) = \mathrm{A} \oplus \mathrm{A}m$$

is obtained by doubling the module A (*adjoining a formal copy* $\mathrm{A}m$) *and defining a product, scalar involution, and norm by the* **Cayley–Dickson Recipe***:*

$$\begin{aligned}(a \oplus bm)(c \oplus dm) &= (ac + \mu\bar{d}b) \oplus (da + b\bar{c})m,\\ (a \oplus bm)^* &= \bar{a} \oplus -bm,\\ N(a \oplus bm) &= n(a) - \mu n(b).\end{aligned}$$

The **Cayley–Dickson Process** *consists of iterating the Cayley–Dickson Construction over and over again. Over a field* Φ *the Process iterates the Construction starting from the* 1*-dimensional* $\mathrm{A}_0 = \Phi$ (*the* **scalars**) *with trivial involution and nondegenerate norm* $N(\alpha) = \alpha^2$ *to get a* 2*-dimensional commutative* **binarion algebra** $\mathrm{A}_1 = \mathcal{KD}(\mathrm{A}_0, \mu_1) = \Phi \oplus \Phi i$ $(i^2 = \mu_1 1)$ *with nontrivial involution,*[7] *then a* 4*-dimensional noncommutative* **quaternion algebra** $\mathrm{A}_2 = \mathcal{KD}(\mathrm{A}_1, \mu_2) = \mathrm{A}_1 \oplus \mathrm{A}_1 j$ $(j^2 = \mu_2 1)$, *and finally an* 8*-dimensional nonassociative* **octonion algebra** $\mathrm{A}_3 = \mathcal{KD}(\mathrm{A}_2, \mu_3) = \mathrm{A}_2 \oplus \mathrm{A}_2 \ell$ $(\ell^2 = \mu_3 1)$, *all with nondegenerate norms.*

Thus octonion algebras are obtained by gluing two copies of a quaternion algebra together by the Cayley–Dickson Recipe. If the Cayley–Dickson doubling process is carried beyond dimension 8, the resulting algebras no longer permit composition and are no longer alternative (so cannot be used in constructing Jordan matrix algebras). Jacobson's Bootstrap Theorem shows that over a field Φ the algebras with involution obtained from the Cayley–Dickson Process are precisely the *composition algebras with standard involution* over Φ: every composition algebra arises by this construction. If we take $\mathrm{A}_0 = \Phi = \mathbb{R}$

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[7] In characteristic 2, starting from Φ the construction produces larger and larger algebras with trivial involution and possibly degenerate norm; to get out of the rut, one must construct *by hand* the binarion algebra $\mathrm{A}_1 := \Phi 1 + \Phi v$ where v $[\approx \frac{1}{2}(1+i)]$ has $v^2 := v - \nu 1$, $v^* = 1 - v$, with nondegenerate norm $N(\alpha + \beta v) := \alpha^2 + \alpha\beta + \nu\beta^2$.

the reals and $\mu_1 = \mu_2 = \mu_3 = -1$ in the Cayley–Dickson Process, then A_1 is the complex numbers $\mathbb{C}$, A_2 is Hamilton's quaternions $\mathbb{H}$ (the Hamiltonions), and A_3 is Cayley's octonions $\mathbb{K}$ (the Caylions, Cayley numbers, or Cayley algebra), precisely as in the Jordan–von Neumann–Wigner Theorem.

Notice that we are adopting the convention that the dimension 4 composition algebras will all be called (generalized) *quaternion* algebras (as is standard in noncommutative ring theory) and denoted by Q; by analogy, the dimension 8 composition algebras will be called (generalized) *octonion* algebras, and denoted by O (even though this looks dangerously like zero), and the dimension 2 composition algebras will all be called *binarion* algebras and denoted by B. In the alternative literature the octonion algebras are called *Cayley algebras*, but we will reserve the term *Cayley* for the unique 8-dimensional real division algebra $\mathbb{K}$ (*the* Cayley algebra), just as Hamilton's quaternions are the unique 4-dimensional real division algebra $\mathbb{H}$. There is no generally accepted term for the 2-dimensional composition algebras, but that won't stop us from calling them binarions. If the 1-dimensional scalars insist on having a high-falutin' name too, we can call them *unarions.*

Notice that a composition algebra C consists of a unital algebra *plus a choice of norm form* N, and therefore always carries a *standard involution* $\bar{x} = N(x,1)1 - x$. Thus a composition algebra is always a $*$-algebra (and the $*$ determines the norm, $N(x)1 = x\bar{x}$).

2.12 Split Composition Algebras

We will often be concerned with *split unarions, binarions, quaternions,* and *octonions.* Over an algebraically closed field the composition algebras are all "split." This is an imprecise metaphysical term, meaning roughly that the system is "completely isotropic," as far removed from an "anisotropic" or "division system" as possible, as well as being defined in some simple way over the integers.[8] Each category separately must decide on its own definition of "split." For example, in the theory of finite-dimensional associative algebras we define a *split* simple algebra over Φ to be a matrix algebra $\mathcal{M}_n(\Phi)$ coordinatized by the ground field. The theory of central-simple algebras shows that every simple algebra $\mathcal{M}_n(\Delta)$ coordinatized by a division algebra Δ becomes split in some scalar extension, because of the amazing fact that finite-dimensional division algebras Δ can *be split* (turned into $\mathcal{M}_r(\Omega)$) by tensoring with a *splitting field* Ω; in particular, every division algebra has square dimension $\dim_\Phi(\Delta) = r^2$ over its center! In the theory of quadratic forms, a "split" form would have "maximal Witt index," represented relative to a suitable basis by the matrix consisting of hyperbolic planes $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ down the diagonal, with an additional 1×1 matrix (1) if the dimension is odd, $Q(\sum_{i=1}^n(\alpha_{2i-1}x_{2i-1} \oplus \alpha_{2i}x_{2i}) + \alpha x_{2n+1}) = \sum_{i=1}^n \alpha_{2i-1}\alpha_{2i} + \alpha_{2n+1}^2$.

[8] This has no relation to "split" exact sequences $0 \to A \to B \to C \to 0$, which have to do with the middle term "splitting" as a *semi-direct sum* $B \cong A \oplus C$.

The split composition algebras over an arbitrary ring of scalars (not just a field) are defined as follows.

Split Definition. *The* **split composition algebras** *over a scalar ring* Φ *are defined to be those* $*$*-algebras of dimension* 2^{n-1}, $n = 1, 2, 3, 4$, *isomorphic to the following models*:

SPLIT UNARIONS $\mathcal{U}(\Phi) := \Phi$, *the scalars* Φ *with trivial involution* $\bar{\alpha} := \alpha$ *and norm* $N(\alpha) := \alpha^2$;

SPLIT BINARIONS $\mathcal{B}(\Phi) = \Phi \boxplus \Phi$, *a direct sum of scalars with the standard (exchange) involution* $(\alpha, \beta) \mapsto (\beta, \alpha)$ *and norm* $N(\alpha, \beta) := \alpha\beta$;

SPLIT QUATERNIONS $\mathcal{Q}(\Phi)$ *with standard involution, i.e., the algebra* $\mathcal{M}_2(\Phi)$ *of* 2×2 *matrices with* symplectic involution $\overline{a} = \begin{pmatrix} \beta & -\gamma \\ -\delta & \alpha \end{pmatrix}$ *for* $a = \begin{pmatrix} \alpha & \gamma \\ \delta & \beta \end{pmatrix}$ *and norm* $N(a) := \det(a)$;

SPLIT OCTONIONS $\mathcal{O}(\Phi) = \mathcal{Q}(\Phi) \oplus \mathcal{Q}(\Phi)\ell$ *with standard involution* $\overline{a \oplus b\ell} = \overline{a} - b\ell$ *and norm* $N(a \oplus b\ell) := \det(a) - \det(b)$.

There is (up to isomorphism) a *unique* split composition algebra of given dimension over a given Φ, and the constructions $\Phi \mapsto \mathcal{U}(\Phi), \mathcal{B}(\Phi), \mathcal{Q}(\Phi), \mathcal{O}(\Phi)$ are functors from the category of scalar rings to the category of composition algebras.

Notice that over the reals these split composition algebras are at the opposite extreme from the division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{K}$ occurring in the J–vN–W classification. They are obtained from the Cayley–Dickson process by choosing all the ingredients to be $\mu_i = 1$ instead of $\mu_i = -1$. It is an important fact that composition algebras over a field are either division algebras or split: as soon as the quadratic norm form is the least bit isotropic (some nonzero element has norm zero) then it is split as a quadratic form, and the algebra has proper idempotents and splits entirely:

$$N \textit{ anisotropic} \iff \mathcal{KD} \textit{ division}, \quad N \textit{ isotropic} \iff \mathcal{KD} \textit{ split.}$$

This *dichotomy* for composition algebras, of being entirely anisotropic (division algebra) or entirely isotropic (split), does not hold for quadratic forms in general, or for other algebraic systems. In Jordan algebras there is a *trichotomy*: an algebra can be *anisotropic* ("division algebra"), *reduced* (has nonzero idempotents but coordinate ring a division algebra), or *split* (nonzero idempotents and coordinate ring the ground field). The **split Albert algebra** $\mathcal{A}lb(\Phi)$ over Φ is the 27-dimensional Jordan algebra of 3×3 hermitian matrices over the split octonion algebra (with standard involution),

$$\mathcal{A}lb(\Phi) := \mathcal{H}_3(\mathcal{O}(\Phi)) \qquad \text{(split Albert)}.$$

As we will see in the next section, over an algebraically closed field this is the only exceptional Jordan algebra. But over general fields we can have reduced Albert algebras $\mathcal{H}_3(\mathrm{O})$ for non-split octonion algebras, and (as first shown by Albert) we can even have Albert division algebras (though these can't be represented in the form of 3×3 matrices, which would always have a non-invertible idempotent E_{11}).

2.13 Classification

We now return from our long digression on general linear algebras, and consider the development of Jordan theory during the Algebraic Renaissance, whose crowning achievement was the classification of simple Jordan algebras over an arbitrary algebraically closed field Φ (of characteristic not 2, of course!). As in the J–vN–W Theorem, the classification of simple Jordan algebras proceeds according to "degree," where the *degree* is the maximal number of supplementary orthogonal idempotents (analogous to the matrix units E_{ii}). From another point of view, the degree is the degree of the *generic minimum polynomial* of the algebra, the "generic" polynomial $m_x(\lambda) = \lambda^n - m_1(x)\lambda^{n-1} + \cdots + (-1)^n m_n(x)$ ($m_i : \mathrm{J} \to \Phi$ homogeneous of degree i) of minimal degree satisfied by all x, $m_x(x) = 0$. Degree 1 algebras are just the 1-dimensional Φ^+; the degree 2 algebras are the $\mathcal{JS}pin_n$; the degree n algebras for $n \geq 3$ are all Jordan matrix algebras $\mathcal{H}_n(\mathrm{C})$ where the coordinate $*$-algebras C are precisely the split composition algebras over Φ with their standard involutions. This leads immediately to the basic classification of finite-dimensional Jordan algebras over an algebraically closed field.

Renaissance Structure Theorem. *Consider finite-dimensional Jordan algebras* J *over an algebraically closed field* Φ *of characteristic* $\neq 2$.

• *The radical of* J *is the maximal nilpotent ideal, and the quotient* $\mathrm{J}/\mathcal{R}ad(\mathrm{J})$ *is semisimple.*

• *An algebra is semisimple iff it is a finite direct sum of simple ideals. In this case, the algebra has a unit element, and its simple decomposition is unique: the simple summands are precisely the minimal ideals.*

• *Every simple algebra is automatically central-simple over* Φ.

• *An algebra is simple iff it is isomorphic to exactly one of*:

Ground Field Φ^+ *of degree* 1,

Spin Factor $\mathcal{JS}pin_n(\Phi)$ *of degree* 2, *for* $n \geq 2$,

Hermitian Matrices $\mathcal{H}_n(\mathrm{C}(\Phi))$ *of degree* $n \geq 3$ *coordinatized by a split composition algebra* $\mathrm{C}(\Phi)$ (Split Unarion, Split Binarion, Split Quaternion, or Split Octonion Matrices):

$\mathcal{H}_n(\Phi)$ *for* Φ *the ground field,*

$\mathcal{H}_n(\mathcal{B}(\Phi)) \cong \mathcal{M}_n(\Phi)^+$ *for* $\mathcal{B}(\Phi)$ *the split binarions,*

$\mathcal{H}_n(\mathcal{Q}(\Phi))$ *for* $\mathcal{Q}(\Phi)$ *the split quaternions,*

$\mathcal{A}lb(\Phi) = \mathcal{H}_3(\mathcal{O}(\Phi))$ *for* $\mathcal{O}(\Phi)$ *the split octonions.*

Once more, the only exceptional algebra in the list is the 27-dimensional split Albert algebra. Note that the 1-dimensional algebra $\mathcal{JS}pin_0$ is the same as the ground field; the 2-dimensional $\mathcal{JS}pin_1 \cong \mathcal{B}(\Phi)$ is not simple when Φ is algebraically closed, so only $\mathcal{JS}pin_n$ for $n \geq 2$ contribute new simple algebras.

We are beginning to isolate the Albert algebras conceptually; even though the split Albert algebra and the real Albert algebra discovered by Jordan, von Neumann, and Wigner appear to fit into the family of Jordan matrix algebras, we will see in the next chapter that their non-reduced forms really come via a completely different construction out of a cubic form.

3

Jordan Algebras in the Enlightenment: Finite-Dimensional Jordan Algebras over General Fields

After the structure theory of Jordan algebras over an algebraically closed fields was complete, algebraists naturally turned to algebras over general fields.

3.1 Forms of Algebras

Life over an algebraically closed field Ω is split. When we move to non–algebraically–closed fields Φ we encounter modifications or "twistings" in the split algebras which produce new kinds of simple algebras. There may be several non-isomorphic algebras A over Φ which are not themselves split algebras $\mathrm{S}(\Phi)$, but "become" $\mathrm{S}(\Omega)$ over the algebraic closure when we extend the scalars: $\mathrm{A}_\Omega \cong \mathrm{S}(\Omega)$. We call such A's *forms* of the split algebra S; in some Platonic sense they are incipient S's, and are prevented from revealing their true S-ness only by deficiencies in the scalars: once they are released from the constraining field Φ they can burst forth[1] and reveal their true split personality.

Scalar Extension Definition. *If Ω is a unital commutative associative algebra over Φ (we call it, by abuse of language, an extension of the ring of scalars Φ), the* scalar extension A_Ω *of a linear algebra* A *over* Φ *is defined to be the tensor product as a module with the natural induced multiplication:*

$$\mathrm{A}_\Omega := \Omega \otimes_\Phi \mathrm{A}, \quad (\omega_1 \otimes x_1)(\omega_2 \otimes x_2) := \omega_1\omega_2 \otimes x_1x_2.$$

Thus A_Ω consists of "formal Ω-linear combinations" of elements from A. It is always a linear algebra over Ω, and there is a natural homomorphism $\mathrm{A} \to \mathrm{A}_\Omega$ of Φ-algebras via $x \mapsto 1 \otimes x$. If A or Ω is free as a Φ-module (e.g., if Φ is a field), this natural map is a monomorphism, but in general we can't view either Φ as a subalgebra of Ω, or A as a subalgebra of A_Ω.

[1] The analogy of the alien creature released from its spacesuit in the movie "Independence Day" is apt, though not flattering.

Form Definition. *If* Ω *is an extension of* Φ, *we say that a linear algebra* A *over* Φ *is a* form *of an algebra* A′ *over* Ω *if* $A_\Omega \cong A'$.

The general line of attack on the structure theory over general fields Φ is a "top-down" method: we start from the known simple structures S over the algebraic closure Ω and try to classify all possible forms of a given $S(\Omega)$. In the Jordan case, we need to know which simple Jordan algebras over Φ will grow up to be Albert algebras over Ω, which ones will grow up to be spin factors, and which ones will grow into hermitian algebras. In Lie theory, one says that a Lie algebra L is of *type S* (G_2, F_4, or whatever) if L_Ω is the (unique simple) Lie algebra $S(\Omega)$ ($G_2(\Omega), F_4(\Omega)$, or whatever), so the problem becomes that of classifying simple Lie algebras of a given *type*. The exact classification of all the possible forms of a given type usually depends very delicately on the arithmetic nature of the field (e.g., it is trivial for the complex numbers, easy for the real numbers, but hard for algebraic number fields).

Octonion Example. *The only octonion algebra over an algebraically closed field* Ω *is the split* $\mathcal{O}(\Omega)$. *Over a non–algebraically–closed field* Φ *of characteristic* $\neq 2$, *every octonion algebra* O *can be obtained via the Cayley–Dickson Construction using a triple of nonzero scalars* μ_i *starting from* Φ, $O = \mathcal{KD}(\Phi, \mu_1, \mu_2, \mu_3)$, *but it is a delicate question when two such triples of scalars produce isomorphic octonion algebras* (*or, what turns out to be the same thing, equivalent quadratic norm forms*), *so producing a precise description of the distinct isomorphism classes of octonion algebras* $\mathcal{KD}(\Phi, \mu_1, \mu_2, \mu_3)$ *over* Φ *is difficult. Nevertheless, all* $\mathcal{KD}(\Phi, \mu_1, \mu_2, \mu_3)$ *are forms of* $\mathcal{O}(\Omega)$: *if we pass to the algebraic closure* Ω *of* Φ, *the octonion algebra splits,* $\mathcal{KD}(\Phi, \mu_1, \mu_2, \mu_3)_\Omega \cong \mathcal{O}(\Omega)$.

3.2 Inverses and Isotopes

The most important method of twisting Jordan algebras is to take isotopes by invertible elements. It is helpful to think of passing to an isotope as *changing the unit* of the Jordan algebra.

Linear Jordan Inverse Definition. *An element* x *of a unital Jordan algebra is* **invertible** *if it has an* **inverse** y *satisfying the* **Linear Jordan Inverse Conditions**

$$\text{(LJInv1)} \quad x \bullet y = 1, \qquad \text{(LJInv2)} \quad x^2 \bullet y = x.$$

A unital Jordan algebra is a **division algebra** *if every nonzero element is invertible.*

For elements of associative algebras, Jordan invertibility is exactly the same as ordinary invertibility. We don't even need the entire *algebra* A to be associative, as long as the invertible *element* u is associative (i.e., lies in the nucleus).

Nuclear Inverse Proposition. *If x is a nuclear element of a unital linear algebra over a ring of scalars containing $\frac{1}{2}$, then x has an inverse with respect to the Jordan product iff it has an ordinary inverse with respect to multiplication,*

$$x \bullet y = 1,\ x^2 \bullet y = x \Longleftrightarrow\ xy = yx = 1,$$

in which case the element y is uniquely determined by x, and is again nuclear with inverse x. We write $y = x^{-1}$. In particular, if A *is associative then* A^+ *is a Jordan division algebra iff* A *is an associative division algebra.*

Homotope Proposition. *If u is an arbitrary element of a Jordan algebra* J, *then the* **Jordan u-homotope** $\mathrm{J}^{(u)}$ *is the Jordan algebra with product*

$$x \bullet^{(u)} y := x \bullet (u \bullet y) + (x \bullet u) \bullet y - u \bullet (x \bullet y).$$

If J *is unital and u is invertible, then the u-homotope is again a unital Jordan algebra, with unit*

$$1^{(u)} := u^{-1},$$

and we call it the **Jordan u-isotope** *of* J. *Two Jordan algebras are* isotopic *if one is isomorphic to an isotope of the other.*

Isotopy is an equivalence relation, more general than isomorphism, since we have **Isotope Reflexivity, Transitivity, and Symmetry**:

$$\mathrm{J}^{(1)} = \mathrm{J}, \qquad \left(\mathrm{J}^{(u)}\right)^{(v)} = \mathrm{J}^{(u \bullet_v u)}, \qquad \mathrm{J} = \left(\mathrm{J}^{(u)}\right)^{(u^{-2})}.$$

It is a long and painful process to work with inverses and verify that $\mathrm{J}^{(u)}$ is indeed a Jordan algebra; when we learn to use the U-operators and Jordan triple products in the next chapter, this will become almost a triviality.

3.3 Nuclear Isotopes

The original motivation for homotopes comes from special algebras, where they have a thoroughly natural explanation: we obtain a new bilinear product by sticking a u in the middle of the old associative product.[2] We can even take u-homotopes in nonassociative algebras as long as the element u is nuclear; this will allow us to form isotopes of the exceptional $\mathcal{H}_3(\mathrm{O})$.

[2] Thus the parameter in a homotope is the inserted element, and the new unit is the inverse of this element. In the Colloquial Survey we found it convenient to emphasize the new unit as the parameter in subscripted notation: $\mathrm{J}_{[u]} := \mathrm{J}^{(u^{-1})}$ has inverse $1_{[u]} = (u^{-1})^{-1} = u$. Since homotopes as well as isotopes play a role in Jordan theory, from now on we will stick to the superscript version and consider isotopes as particular cases of homotopes.

Nuclear Isotope Definition. *Let* A *be a unital linear algebra, and let* u *be an invertible element in the nucleus of* A. *Then we obtain a new unital linear algebra, the* **nuclear** u**-isotope** A_u *with the same linear structure but new unit* $1_u = u^{-1}$ *and new multiplication*

$$x_u y := xuy := (xu)y = x(uy).$$

Notice that by nuclearity of u *this product is unambiguous.*

We distinguish the nuclear and Jordan concepts of isotope by using plain subscripts u *for nuclear isotopes, and* parenthesized superscripts (u) *for Jordan isotopes.*

Nuclear Isotope Proposition. *A nuclear isotope is always isomorphic to the original algebra*: $\mathrm{A}_u \cong \mathrm{A}$ *via the isomorphism* $\varphi = L_u$. *Thus if* A *is associative, so is* A_u, *and for elements* u *which happen to be nuclear, the Jordan isotope of the plus algebra of* A *is just the plus algebra of the nuclear isotope, and hence is isomorphic to the original plus algebra*:

$$(\mathrm{A}^+)^{(u)} = (\mathrm{A}_u)^+ \cong \mathrm{A}^+ : x \bullet^{(u)} y = \tfrac{1}{2}(xuy + yux) = x \bullet_u y \ (u \in \mathcal{N}uc(\mathrm{A})).$$

The analogous recipe uxy for a "left isotope" of an associative algebra produces a highly *non-associative, non-unital* algebra; only "middle isotopes" produce associative algebras again.

3.4 Twisted Involutions

As far as the entire linear algebra goes, A_u produces just another copy of A, which is why the concept of isotopy is largely ignored in associative theory. Isotopy *does* produce something new when we consider *algebras with involution*, because $\varphi(x) = ux$ usually does not map $\mathcal{H}(\mathrm{A}, *)$ to itself, and the twisted involution $*_u$ need not be algebraically equivalent to the original involution $*$.

Twisted Hermitian Proposition. *Let* A *be a unital linear algebra with involution* $*$ *over a ring of scalars* Φ *containing* $\frac{1}{2}$, $\mathrm{J} = \mathcal{H}(\mathrm{A}, *) \subseteq \mathrm{A}^+$ *the subalgebra of hermitian elements under the Jordan product, and let* u *be an invertible hermitian element in the nucleus of* A. *Then* $*$ *remains an involution on the linear algebra* A_u *whose hermitian part is just the Jordan isotope* $\mathrm{J}^{(u)}$:

$$\mathcal{H}(\mathrm{A}, *)^{(u)} = \mathcal{H}(\mathrm{A}_u, *).$$

This isotope $\mathrm{J}^{(u)}$ *coincides with* J *as a* Φ*-module, but not necessarily as an algebra under the Jordan product.*

A better way to view the situation is to keep the original algebra structure but form the **isotopic involution** $*_u$, *the* u*-conjugate*

$$x^{*_u} := ux^*u^{-1}.$$

This is again an involution on A, *with new hermitian elements*

$$\mathcal{H}(\mathrm{A}, *_u) = u\mathcal{H}(\mathrm{A}, *).$$

The map L_u *is now a* $*$*-isomorphism* $(\mathrm{A}_u, *) \to (\mathrm{A}, *_u)$ *of* $*$*-algebras, so induces an isomorphism of hermitian elements under the Jordan product*:

$$\mathrm{J}^{(u)} = \mathcal{H}(\mathrm{A}_u, *) \cong \mathcal{H}(\mathrm{A}, *_u) = u\mathcal{H}.$$

Thus $\mathrm{J}^{(u)}$ *is really a copy of the* u*-translate* $u\mathcal{H}$, *not of* $\mathrm{J} = \mathcal{H}$.

Thus the Jordan isotope $\mathcal{H}(\mathrm{A}, *)^{(u)}$ can be viewed either as keeping the involution $*$ but twisting the product via u, or (via the identification map L_u) keeping the product but twisting the involution via u. An easy example where the algebraic structure of $\mathrm{J}^{(u)}$ is not the same as that of J is when $\mathrm{A} = \mathcal{M}_n(\mathbb{R})$, the real $n \times n$ matrices with transpose involution $*$, and $u = \mathrm{diag}(-1, 1, 1, \dots, 1)$. Here the original involution is "positive definite" and the Jordan algebra $\mathrm{J} = \mathcal{H}_n(\mathbb{R})$ is formally real, but the isotope $*_u$ is indefinite and the isotope $\mathrm{J}^{(u)}$ has nilpotent elements ($x = E_{11} + E_{22} - E_{12} - E_{21}$ has $x \bullet_u x = 0$), so $\mathrm{J}^{(u)}$ cannot be algebraically isomorphic to J.

3.5 Twisted Hermitian Matrices

If we consider $\mathrm{A} = \mathcal{M}_n(\mathrm{D})$ with the conjugate transpose involution, any diagonal matrix $u = \Gamma$ with invertible hermitian nuclear elements of D down the diagonal is invertible hermitian nuclear in A and can be used to twist the Jordan matrix algebra. Rather than consider the isotope with twisted product, we prefer to keep within the usual Jordan matrix operations and twist the involution instead.

Twisted Matrix Example. *For an arbitrary linear* $*$*-algebra* $(\mathrm{D}, -)$ *with involution* $d \mapsto \bar{d}$ *and diagonal matrix* $\Gamma = \mathrm{diag}(\gamma_1, \dots, \gamma_n)$ *whose entries* γ_i *are invertible hermitian elements in the nucleus of* D, *the twisted conjugate transpose mapping* $X^{*_\Gamma} = \Gamma \overline{X}^{tr} \Gamma^{-1}$ *is an involution on the algebra* $\mathcal{M}_n(\mathrm{D})$ *of all* $n \times n$ *matrices with entries from* D *under the usual matrix product* XY. *The* Φ*-module* $\mathcal{H}_n(\mathrm{D}, \Gamma) := \mathcal{H}(\mathcal{M}_n(\mathrm{D}), *_\Gamma)$ *of all hermitian matrices* $X^{*_\Gamma} = X$ *with respect to this new involution forms a Jordan algebra under the usual matrix operation* $X \bullet Y = \frac{1}{2}(XY + YX)$. *The Jordan and nuclear isotope* $\mathcal{H}_n(\mathrm{D}, -)^{(\Gamma)} = \mathcal{H}_n(\mathrm{D}, -)_\Gamma \subseteq \mathcal{M}_n(\mathrm{D})_\Gamma$ *is isomorphic under* L_Γ *to this twisted matrix algebra* $\mathcal{H}_n(\mathrm{D}, \Gamma)$.

Note that no parentheses are needed in these products, since the γ_i lie in the nucleus of D.

3.6 Spin Factors

The final Jordan algebras we consider are generalizations of the "split" spin factors $\mathcal{JSpin}_n(\Phi)$ determined by the dot product. Over non–algebraically–closed fields we must consider general symmetric bilinear forms. We do not demand that our modules be finite-dimensional, nor that our forms be non-degenerate. The construction of spin factors works for arbitrary forms on modules. (Functional analysts would only use the term "factor" when the resulting algebra is indecomposable or, in our case, simple.)

Spin Factor Example. *If M is a Φ-module with symmetric bilinear form $\sigma : M \times M \to \Phi$, then we can define a linear Φ-algebra structure $\mathcal{JSpin}(M, \sigma)$ on $\Phi 1 \oplus M$ by having 1 act as unit element and defining the product of vectors x, y in M to be a scalar multiple of the unit, given by the bilinear form*:

$$x \bullet y := \sigma(x, y)1.$$

Since x^2 is a linear combination of 1 and x, L_{x^2} commutes with L_x, and the resulting algebra is a Jordan algebra, a **Jordan spin factor**.

If $M = \Phi^n$ consists of column vectors with the standard dot product $\sigma(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$, then the resulting $\mathcal{JSpin}(M, \sigma)$ is just $\mathcal{JSpin}_n(\Phi)$ as defined previously:

$$\mathcal{JSpin}_n(\Phi) = \mathcal{JSpin}(\Phi^n, \cdot).$$

If we try to twist these algebras, we regret having chosen a particular splitting of the algebra into unit and vectors, with the unit having a multiplication rule all to itself: when we change to a new unit element which is half unit, half vector, the calculations become clumsy. It is better (both practically and conceptually) to give a global description of the operations. At the same time we will pass to a quadratic instead of a bilinear form, because the quadratic norm form explicitly encodes important information about the algebra.

3.7 Quadratic Factors

Since we are working in modules having scalar $\frac{1}{2}$, there is no real difference between *bilinear* forms and *quadratic* forms. Recall that a **quadratic form** Q on a module V is a quadratic mapping from V to Φ, i.e., it is *homogeneous of degree* 2 ($Q(\alpha x) = \alpha^2 Q(x)$ for all $\alpha \in \Phi, x \in V$) and its **linearization**

$$Q(x, y) := Q(x + y) - Q(x) - Q(y)$$

is bilinear in x and y (hence a symmetric bilinear form on V). Note that by definition $Q(x, x) = Q(2x) - Q(x) - Q(x) = 4Q(x) - 2Q(x) = 2Q(x)$, so Q can be recovered from the bilinear form $Q(\cdot, \cdot)$ only with the help of a scalar $\frac{1}{2}$, in which case the correspondences $Q \leftrightarrow \tau$ given by $\tau(x, y) := \frac{1}{2}Q(x, y)$, $Q(x) := \tau(x, x)$ give an isomorphism between the categories of quadratic forms and symmetric bilinear forms. If we wished to study Jordan algebras where $\frac{1}{2}$ is not available, we would *have to* use a description in terms of quadratic rather than bilinear forms.

A **basepoint** for a quadratic form Q is a point c with unit **norm**,[3] $Q(c) = 1$. We can form the associated linear **trace form** $T(x) := Q(x, c)$, which automatically has $T(c) = Q(c, c) = 2$. In the presence of $\frac{1}{2}$ we can always decompose our module as $M = \Phi c \oplus M_0$ for $M_0 := \{x \mid T(x) = 0\}$ the "orthogonal complement" $c^\perp$, but instead of stressing the unit and its splitting, we will emphasize the equality of vectors and the unity of M. It turns out that reformulation in terms of quadratic forms and basepoints is the "proper" way to think of these Jordan algebras.

Quadratic Factor Example. (1) *If Q is a quadratic norm form on a space M with basespoint c over Φ containing $\frac{1}{2}$, we obtain a Jordan algebra* $\mathrm{J} = \mathcal{J}ord(Q, c)$ *on M with unit $1 := c$ and product*

$$x \bullet y := \tfrac{1}{2}\big(T(x)y + T(y)x - Q(x, y)1\big) \qquad (T(x) := Q(x, c)).$$

Taking $y = x$ shows that every element x satsifies the second-degree equation

$$x^2 - T(x)x + Q(x)1 = 0$$

(*we say that $\mathcal{J}ord(Q, c)$ has "degree 2"*). *The norm determines invertibility: an element in $\mathcal{J}ord(Q, c)$ is invertible iff its norm is an invertible scalar,*

$$x \text{ invertible in } \mathcal{J}ord(Q, c) \iff Q(x) \text{ invertible in } \Phi,$$

in which case the inverse is a scalar multiple of $\bar{x} := T(x)1 - x$,[4]

$$x^{-1} = Q(x)^{-1}\bar{x}.$$

---

[3] This use of the term "norm" has nothing to do with the metric concept, such as the *norm* in a Banach space; the trace and norm are analogues of trace and determinant of matrices, in particular are always *polynomial* functions. There is a general notion of "generic norm" for n-dimensional unital power associative algebras A over a field Φ: in the scalar extension A_Ω, for $\Omega := \Phi[t_1, \ldots, t_n]$ the polynomial ring in n indeterminates t_i, the element $x := t_1x_1 + \cdots + t_nx_n$ is a "generic element" of A in the sense that every actual element of A arises from x through specialization $t_i \mapsto \alpha_i$ of the indeterminates to particular scalars in Φ. Then x satisfies a *generic minimum polynomial* resembling the characteristic polynomial for matrices: $x^m - T(x)x^{m-1} + \cdots + (-1)^m N(x)1 = 0$. The constant term $N(x)$ is the *generic norm*; it is a homogeneous polynomial function of $t_1, \ldots, t_n$ of degree m.

[4] This is completely analogous to the recipe $x^{-1} = \det(x)^{-1}\mathrm{adj}(x)$ in 2×2 matrices for the inverse as a scalar multiple of the adjoint. In the case of associative or alternative algebras the norm form, like the determinant, "permits composition" $N(ab) = N(a)N(b)$ for all elements $a, b \in \mathrm{A}$. In the case of a Jordan algebra the norm permits "Jordan composition" $N(2a \bullet (a \bullet b) - a^2 \bullet b) = N(a)N(b)N(a)$ [this bizarre formula will make more sense once we introduce the U-operator $U_a b = 2a \bullet (a \bullet b) - a^2 \bullet b$ in the next chapter and realize that it represents the product aba in special algebras]. The crucial property in both cases is that a is invertible iff its norm $N(a)$ is nonzero (i.e., invertible) in Φ.

(2) *Over a field, the norm and basepoint completely determine the Jordan algebra: if we define two quadratic forms with basepoint to be* equivalent (*written* $(Q, c) \cong (Q', c')$) *if there is a linear isomorphism* $\varphi : M \to M'$ *which preserves norms and basepoint,* $Q'(\varphi(x)) = Q(x)$ *for all* $x \in M$ *and* $\varphi(c) = c'$, *then*

$$\mathcal{J}ord(Q, c) \cong \mathcal{J}ord(Q', c') \Longleftrightarrow (Q, c) \cong (Q', c').$$

(3) *All isotopes are again quadratic factors,*

$$\mathcal{J}ord(Q, c)^{(u)} = \mathcal{J}ord(Q^{(u)}, c^{(u)})$$
$$(\textit{where } Q^{(u)} = Q(u)Q \textit{ and } c^{(u)} = u^{-1}).$$

Thus isotopy just changes basepoint and scales the norm form. It is messier to describe in terms of bilinear forms: in $\mathcal{J}Spin(M, \sigma)^{(u)}$, *the new* $\sigma^{(u)}$ *is related in a complicated manner to the old* σ, *since it lives half on* M *and half off, and we are led ineluctably to the global* Q:

$$\mathcal{J}Spin(M, \sigma)^{(u)} \cong \mathcal{J}Spin(N, \sigma^{(u)})$$
$$\textit{for } N := \{x \mid Q(x, \bar{u}) = 0\}, \;\; \sigma^{(u)}(x, y) := -\tfrac{1}{2} Q(u)Q(x, y).$$

All nondegenerate quadratic forms Q (equivalently, all symmetric bilinear forms σ) of dimension $n + 1$ over an algebraically closed field Ω of characteristic $\neq 2$ are equivalent: they can be represented, relative to a suitable basis (starting with the basepoint c), as the dot product on Ω^{n+1}, $Q(\mathbf{v}) = \sigma(\mathbf{v}, \mathbf{v}) = \mathbf{v}^{tr}\,\mathbf{v}$ for column vectors $\mathbf{v}$. Therefore all the corresponding Jordan algebras $\mathcal{J}ord(Q, c)$ are isomorphic to good old $\mathcal{J}Spin_n(\Omega)$. Over a non-algebraically-closed field Φ, every nondegenerate Q can be represented as a linear combination of squares using a diagonal matrix $\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ for nonzero λ_i, but it is a delicate question as to when two such diagonal matrices produce equivalent quadratic forms. Nevertheless, all quadratic factors $\mathcal{J}ord(Q, c)$ of dimension $n + 1$ over a field Φ are *forms* of a spin factor $\mathcal{J}Spin_n(\Omega)$: $\mathcal{J}ord(Q, c)_\Omega \cong \mathcal{J}Spin_n(\Omega)$.

We consistently talk of Jordan algebras constructed from a bilinear form as *spin factors* $\mathcal{J}Spin(M, \sigma)$, and those constructed from a quadratic form with basepoint as *quadratic factors* $\mathcal{J}ord(Q, c)$.

3.8 Cubic Factors

In contrast to the case of *quadratic* forms with basepoint, only certain very special *cubic* forms with basepoint can be used to construct Jordan algebras. A **cubic form** N on a Φ-module M over Φ is a map $M \to \Phi$ which is homogeneous of degree 3 $[N(\alpha x) = \alpha^3 N(x)]$ and which extends to arbitrary scalar extensions M_Ω by

$$N\Big(\sum_i \omega_i x_i\Big) = \sum_i \omega_i^3 N(x_i) + \sum_{i\neq j} \omega_i^2 \omega_j N(x_i; x_j) + \sum_{i<j<k} \omega_i \omega_j \omega_k N(x_i; x_j; x_k),$$

where the **linearization** $N(x; y)$ is quadratic in x and linear in y, while $N(x; y; z)$ is symmetric and trilinear.

The test for admission to the elite circle of Jordan cubics is the existence of a unit having well-behaved trace and adjoint. A **basepoint** for N is a point $c \in M$ with $N(c) = 1$. We have an associated linear **trace form** $T(x) := N(c; x)$ and a quadratic **spur form**[5] $S(x) := N(x; c)$ whose linearization is just $S(x, y) := S(x + y) - S(x) - S(y) = N(x; y; c)$; automatically

$$N(c) = 1, \qquad S(c) = T(c) = 3.$$

Jordan Cubic Definition. *A finite-dimensional cubic form with basepoint* (N, c) *over a field* Φ *of characteristic* $\neq 2$ *is defined to be a* **Jordan cubic**[6] *if* (1) N *is* **nondegenerate** *at the basepoint* c, *in the sense that the* **trace bilinear form**

$$T(x, y) := T(x)T(y) - S(x, y)$$

is a nondegenerate bilinear form, and (2) *the quadratic* **sharp**(*or* **adjoint**) *map* $\# : M \to M$, *defined uniquely by* $T(x^{\#}, y) = N(x; y)$, *strictly satisfies the* **Adjoint Identity**

$$(x^{\#})^{\#} = N(x)x.$$

In the presence of nondegeneracy, the Adjoint Identity is all we need.

Springer Construction. (1) *From every Jordan cubic form with basepoint we obtain a Jordan algebra* $\mathcal{J}ord(N, c)$ *with unit* $1 := c$ *and product determined from the linearization* $x\#y := (x + y)^{\#} - x^{\#} - y^{\#}$ *of the sharp mapping by the formula*

$$x \bullet y := \tfrac{1}{2}\big(x\#y + T(x)y + T(y)x - S(x, y)1\big).$$

This algebra has degree 3,

$$x^3 - T(x)x^2 + S(x)x - N(x)1 = 0,$$

with sharp mapping

$$x^{\#} = x^2 - T(x)x + S(x)1.$$

(2) *An element is invertible iff its norm is nonzero, in which case the inverse is a multiple of the adjoint* (*just as in matrix algebras*):

$$u \in \mathrm{J} \textit{ is invertible iff } N(u) \neq 0, \textit{ in which case } \quad u^{-1} = N(u)^{-1}u^{\#}.$$

---

[5] I'm hoping that the reader won't notice that "Spur" is just the German word for trace; in English a spur is something sharp, and here $S(x) = T(x^{\#})$ is the trace of the sharp, so "spur" is the best I could think of.

[6] In the literature this is also called an *admissible* cubic, but this gives no clue as to what it is admitted for, whereas the term *Jordan* makes it clear that the cubic is to be used for building Jordan algebras.

(3) *For invertible elements the isotope is obtained* (*as in quadratic factors*) *by scaling the norm and shifting the unit*:

$$\mathcal{J}ord(N,c)^{(u)} = \mathcal{J}ord(N^{(u)}, c^{(u)})$$
$$\text{for} \quad c^{(u)} = u^{-1}, \quad N^{(u)}(x) = N(u)N(x).$$

3.9 Reduced Cubic Factors

Over a field, the quadratic and cubic factors are either *division algebras* [where the norm Q or N is anisotropic], or *reduced* [where the norm is isotropic, equivalently, the algebra has proper idempotents]. Simple reduced $\mathcal{J}ord(N,c)$'s are always isomorphic to Jordan matrix algebras $\mathcal{H}_3(\mathrm{D},\Gamma)$ for a composition algebra D, and are *split* if the coordinate algebra is a split $\mathrm{D} = \mathrm{C}(\Phi)$, in which case we can take all $\gamma_i = 1$, $\Gamma = \mathbb{1}_3$ the identity matrix, and $\mathcal{H}_3(\mathrm{D},\Gamma)$ becomes $\mathcal{H}_3(\mathrm{C}(\Phi))$. We define an **Albert algebra** over a field Φ to be an algebra $\mathcal{J}ord(N,c)$ determined by a Jordan cubic form of dimension 27; these come in three flavors, *Albert division algebras*, *reduced Albert algebras* $\mathcal{H}_3(\mathrm{O},\Gamma)$ for an octonion division algebra O with standard involution, and *split Albert algebras* $\mathcal{H}_3(\mathcal{O}(\Phi)) = \mathcal{A}lb(\Phi)$.

The cubic factor construction was originally introduced by H. Freudenthal for 3×3 hermitian matrices, where we have a very concrete representation of the norm.

Freudenthal Construction. *If* D *is an alternative* $*$*-algebra with involution such that* $\mathcal{H}(\mathrm{D},-) = \Phi 1$, *then for diagonal* $\Gamma = \mathrm{diag}(\gamma_1,\gamma_2,\gamma_3)$ *with entries invertible scalars, the twisted matrix algebra* $\mathcal{H}_3(\mathrm{D},\Gamma)$ *is a cubic factor* $\mathcal{J}ord(N,c)$ *with basepoint* $c := e_1+e_2+e_3$ *and norm given by*

$$N(x) := \alpha_1\alpha_2\alpha_3 - \textstyle\sum_{cyclic}\left(\alpha_i\gamma_j\gamma_k n(a_i)\right) + \gamma_1\gamma_2\gamma_3 t(a_1a_2a_3)$$

summed over all cyclic permutations (i,j,k) *of* $(1,2,3)$ *when the elements* x, y *are given for* $\alpha_i,\beta_i \in \Phi$, $a_i, b_i \in \mathrm{D}$ *by*

$$x = \textstyle\sum_{cyclic}\left(\alpha_i e_i + a_i[jk]\right), \qquad y = \textstyle\sum_{cyclic}\left(\beta_i e_i + b_i[jk]\right).$$

In this case we have

$$T(x) = \textstyle\sum_i \alpha_i, \qquad S(x) = \textstyle\sum_{cyclic}\left(\alpha_j\alpha_k - \gamma_j\gamma_k n(a_i)\right),$$
$$T(x,y) = \textstyle\sum_{cyclic}\left(\alpha_i\beta_i + \gamma_j\gamma_k t(\overline{a_i}b_i)\right).$$

3.10 Classification

Besides twisting, a new phenomenon that arises only over non-algebraically-closed fields is that of division algebras. For associative algebras, the only finite-dimensional division algebra over an algebraically closed field (like the complexes) is the field itself, but over the reals we have $\mathbb{R}$ (of course), as well as $\mathbb{C}$ of dimension 2 (which is not central-simple), but also the central-simple algebra of Hamilton's quaternions $\mathbb{H}$ of dimension 4. In the same way, the only finite-dimensional Jordan division algebra over an algebraically closed field Ω is Ω itself, but over a general field Φ there may be others. We have already noted that A^+ is a Jordan division algebra iff A is an associative division algebra, $\mathcal{J}ord(Q, c)$ is a division algebra iff the quadratic form Q is anisotropic, and analogously $\mathcal{J}ord(N, c)$ is a division algebra iff the cubic form N is anisotropic. We now have all the ingredients to classify finite-dimensional Jordan algebras over an arbitrary field of characteristic $\neq 2$, the crowning achievement of the Age of Enlightenment. If J is finite-dimensional simple over Φ, its center Ω is a finite extension field of Φ, and J is finite-dimensional central-simple over Ω, so it suffices to classify all central-simple algebras.

Enlightenment Structure Theorem. *Consider finite-dimensional Jordan algebras* J *over a field* Φ *of characteristic* $\neq 2$*.*

• *The radical of* J *is the maximal nilpotent ideal, and the quotient* $\mathrm{J}/\mathcal{R}ad(\mathrm{J})$ *is semisimple.*

• *An algebra is semisimple iff it is a finite direct sum of simple ideals. In this case, it has a unit element, and the simple decomposition is unique: the simple summands are precisely the minimal ideals.*

• *Every simple algebra is central-simple over its center, which is a field.*

• *An algebra is central-simple over* Φ *iff it is isomorphic to exactly one of:*

DIVISION TYPE: *a finite-dimensional central Jordan division algebra over* Φ;

QUADRATIC TYPE: $\mathcal{J}ord(Q, c)$ *for an isotropic nondegenerate quadratic form with basepoint of finite dimension* ≥ 3 *over* Φ;

HERMITIAN TYPE: $\mathcal{H}_n(\mathrm{D}, \Gamma)$ *with coordinates* $(\mathrm{D}, -)$ *of exchange, division, or split quaternion type and* $n \geq 3$*, i.e., an algebra of* EXCHANGE, ORTHOGONAL, *or* SYMPLECTIC TYPE *isomorphic respectively to*:

$\mathcal{H}_n(\mathcal{E}x(\Delta)) \cong \mathcal{M}_n(\Delta)^+$ *for a finite-dimensional central associative division algebra* Δ *over* Φ;

$\mathcal{H}_n(\Delta, \Gamma)$ *for a finite-dimensional central associative division algebra* Δ *with involution over* Φ;

$\mathcal{H}_n(\mathcal{Q}(\Phi))$ *for* $\mathcal{Q}(\Phi)$ *the split quaternion algebra over* Φ *with standard involution*;

ALBERT TYPE: $\mathcal{J}ord(N, c) = \mathcal{H}_3(\mathrm{O}, \Gamma)$ *of dimension* 27 *for* O *an octonion algebra over* Φ *with standard involution, only for* $n = 3$.

Once more, the only exceptional algebras in the list are the 27-dimensional Albert algebras and, possibly, some exceptional division algebras. Note that the anisotropic $\mathcal{J}ord(Q,c)$ are division algebras; all $\mathcal{J}ord(Q,c)$ of dimension 1 are Φ^+, while those of dimension 2 are either division algebras or non-simple split binarions $\Phi^+ \boxplus \Phi^+$. Hermitian algebras for $n = 1,2$ are all of degree 2 and hence are of Division or Quadratic Type. A hermitian $\mathcal{H}_n(\mathrm{Q})$ is of Orthogonal or Symplectic Type according as the coordinate quaternion algebra is a division algebra or split. All $\mathcal{J}ord(N,c)$ which are not reduced are division algebras. Thus the listed types of simple algebras are complete and non-overlapping.

At this stage of development there was no way to classify the division algebras, especially to decide whether there were any exceptional ones which were not Albert algebras determined by anisotropic cubic forms. In fact, to this very day *there is no general classification of all finite-dimensional associative division algebras*: there is a general construction (crossed products) which yields all the division algebras over many important fields (including all algebraic number fields), but in 1972 Amitsur gave the first construction of a non–crossed–product division algebra, and there is as yet no general characterization of these.

4

The Classical Theory: Jordan Algebras with Minimum Condition

In the 1960s surprising new connections were found between Euclidean Jordan algebras and homogeneous cones, symmetric spaces, and bounded homogeneous domains in real and complex differential geometry by Max Koecher and his students. These connections demanded a broadening of the concept of Jordan systems to include triples and pairs, which were at the same time arising spontaneously from the mists of Lie algebras through the Tits–Kantor–Koecher construction.[1]

A leading role in these investigations was played by Jordan triple products rather than binary products, especially by the U-operator and its inverse $H_x = U_{x^{-1}}$, which was a geometrically important transformation. The same U-operator was also cropping up in purely algebraic investigations of N. Jacobson and his students, in connection with inverses, isotopies, and generic norms. The U-product led naturally to the notion of inner ideal and algebras with minimum condition on inner ideals, analogues of one-sided ideals and artinian algebras in associative theory.

4.1 U-Operators

The classical theory of Jordan algebras cannot be understood without a clear understanding of Jacobson's U-operators U_x and its **auxiliary operators**, the corresponding Jordan triple product $\{x, y, z\}$ obtained from it by linearization, and the related operators $U_{x,y}$, $V_{x,y}$, V_x:

$$U_x := 2L_x^2 - L_{x^2}, \quad U_{x,z} := \big(U_{x+z} - U_x - U_z\big),$$

$$V_{x,y}(z) := \{x, y, z\} := U_{x,z}(y),$$

$$V_x(y) := \{x, y\} := \{x, 1, y\}, \qquad V_x = V_{x,1} = U_{x,1} = V_{1,x} = 2L_x.$$

[1] To avoid embarrassing American mispronunciations, the three distinguished European mathematicians' names are *zhak teets, iss-eye kahn-tor,* and (approximately) *muks ke(r)-ssher*. The German *ch-* should definitely not be pronounced like the *ch-* in church, nor quite like Scottish *Loch* as in Ness, more like the Scottish *braw bricht moonlicht nicht* (fine bright moonlit night).

Note that $\{x,y,x\} = 2U_xy$. *In your heart, you should always think of the U operator as "outer multiplication"* (simultaneous left and right multiplication), *and the V operators as left plus right multiplications*, leading to a *meta-principle*:

$$U_xy \approx xyx, \quad \{x,y,z\} \approx xyz + zyx, \quad \{x,y\} \approx xy + yx.$$

This is exactly what these operators amount to in special algebras.[2]

Now we check what these operators amount to in our basic examples of Jordan algebras.

Special U Example. *In any special Jordan algebra* $\mathrm{J} \subseteq \mathrm{A}^+$ *the U-operator and its relatives are given by*

$$U_xy = xyx, \qquad V_xy = \{x,y\} = xy + yx,$$
$$V_{x,y}z = U_{x,z}y = \{x,y,z\} = xyz + zyx.$$

Quadratic Factor U Example. *In the Jordan algebra* $\mathcal{J}ord(Q,c)$ *determined by a quadratic form with basepoint, the U-operators take the form*

$$U_xy = Q(x,\bar{y})x - Q(x)\bar{y} \quad \textit{for} \quad \bar{y} = T(y)c - y.$$

Cubic Factor U Example. *In the Jordan algebra* $\mathcal{J}ord(N,c)$ *determined by a Jordan cubic form N with basepoint over* Φ, *the U-operators take the form*

$$U_xy = T(x,y)x - x^{\#}\#y.$$

4.2 The Quadratic Program

Jacobson conjectured, and in 1958 I.G. Macdonald first proved, the **Fundamental Formula**

$$U_{U_xy} = U_xU_yU_x$$

for arbitrary Jordan algebras. Analytic and geometric considerations led Max Koecher to these same operators and the same formula (culminating in his book *Jordan-Algebren* with Hel Braun in 1966). Several notions which had been cumbersome using the L-operators became easy to clarify with U-operators. These operators will appear on almost every page in the rest of this book.

[2] WARNING: In the linear theory it is more common to use $\frac{1}{2}\{xyz\}$ as the triple product, so that $\{x,y,x\} = U_xy$; you must always be careful to determine which convention is being used. We will always use the convention that makes no reference to $\frac{1}{2}$. Thus our triple product in special algebras becomes the 3-tad $\{x_1,x_2,x_3\} = x_1x_2x_3 + x_3x_2x_1$.

The emphasis began to switch from the Jordan product $x \bullet y = \frac{1}{2}(xy + yx)$, which was inherently limited to characteristic $\neq 2$, to the product xyx, which was ring-theoretic in nature and made sense for arbitrary scalars. If a description of Jordan algebras in quadratic terms could be obtained, it would not only fill in the gap of characteristic 2, but would also open the way to a study of *arithmetic properties* of Jordan *rings* (where the ring of scalars was the integers $\mathbb{Z}$, which certainly did not contain $\frac{1}{2}$). In analogy with Jordan's search for Jordan axioms, it was natural to seek a quadratic axiomatization which:

- *Agreed with the usual one over scalars containing* $\frac{1}{2}$;
- *Admitted all four basic examples (Full, Hermitian, Quadratic, and Cubic) of simple algebras in characteristic* 2;
- *Admitted essentially nothing new in the way of simple algebras.*

It was well-known that in the theory of Lie algebras the passage from the classical characteristic 0 theory to characteristic p produces a raft of new simple algebras (only classified in characteristics $p > 7$ during the 1980s). In contrast, Jordan algebras behave exactly the same in characteristic $p \neq 2$ as they do in characteristic 0, and it was hoped that this would continue to characteristic 2 as well, making the theory completely uniform.

4.3 The Quadratic Axioms

A student of Emil Artin, Hans-Peter Lorenzen, attempted in 1965 an axiomatization based entirely on the Fundamental Formula, but was not quite able to get a satisfactory theory. It turned out that, in the presence of a unit element, one other axiom is needed. The final form, which I gave in 1967, goes as follows.[3]

Quadratic Jordan Definition. *A* **unital quadratic Jordan algebra** J *consists of a* Φ*-module on which a product* $U_x y$ *is defined which is linear in the variable* y *and quadratic in* x *(i.e.,* $U : x \mapsto U_x$ *is a quadratic mapping of* J *into* $\mathcal{E}nd_\Phi(\mathrm{J})$*), together with a choice of* unit element 1, *such that the following operator identities hold* strictly:

$$\text{(QJAX1)} \qquad U_1 = \mathbb{1}_{\mathrm{J}},$$

$$\text{(QJAX2)} \qquad V_{x,y}U_x = U_x V_{y,x} \qquad (V_{x,y}z := \{x, y, z\} := U_{x,z}y),$$

$$\text{(QJAX3)} \qquad U_{U_x y} = U_x U_y U_x.$$

[3] In a 1966 announcement I used in place of (QJAX2) the simpler axiom (QJAX2)$'$: $V_{x,x} = V_{x^2}$ or $\{x, x, y\} = \{x^2, y\}$, but in proving that homotopes remained Jordan it became clear that one could get the job done quicker by assuming (QJAX2) as the axiom. [Note that multiplying (QJAX2) on the right by U_y leads immediately to $V_x^{(y)} U_x^{(y)} = U_x^{(y)} V_x^{(y)}$, which is just the Jordan identity in the y-homotope.] Further evidence that this is the "correct" axiomatization is that Kurt Meyberg showed in 1972 that quadratic Jordan triples, and Ottmar Loos in 1975 that quadratic Jordan pairs, could be axiomatized by (QJAX2), (QJAX3), and (QJAX4): $V_{U_x y, y} = V_{x, U_y x}$ [whereas (QJAX2$'$) made sense only in algebras].

The strictness condition means that these identities hold not only for all elements x, y in J, but that they also continue to hold in all scalar extensions J_Ω; it suffices if they hold in the polynomial extension $J_{\Phi[t]} = J[t]$ for an indeterminate t. This turns out to be equivalent to the condition that all formal linearizations of these identities remain valid on J itself. (Notice that (QJAX2) is of degree 3, and (QJAX3) of degree 4, in x, hence do not automatically linearize [unless there are sufficiently many invertible scalars, e.g., a field with at least four elements], but they do automatically linearize in y, since they are respectively linear and quadratic in y.)

Nonunital Jordan algebras could be intrinsically axiomatized in terms of two products, $U_x y$ and x^2 (which would result from applying U_x to the absent unit element). However, the resulting axioms are too messy to remember, and it is much easier to define non-unital algebras as those whose *unital hull*, under $U_{\alpha 1 \oplus x}(\beta 1 \oplus y) := \alpha^2\beta 1 \oplus [\alpha^2 y + 2\alpha\beta x + \alpha\{x, y\} + \beta x^2 + U_x y]$, satisfies the three easy-to-grasp identities (QJAX1)–(QJAX3).[4]

4.4 Justification

We have already indicated why these axioms meet the first criterion of the program, that quadratic and linear Jordan algebras are the same thing in the presence of $\frac{1}{2}$. It is not hard to show that these axioms also meet the second criterion, that the U-operators of the four basic examples *do satisfy* these axioms (though the cubic factors provide some tough slogging). For example, in special algebras $J \subseteq A^+$ the operations xyx and $xyz + zyx$ satisfy

$$\text{(QJAX1)} \qquad 1z1 = z,$$

$$\text{(QJAX2)} \qquad xy(xzx) + (xzx)yx = x(yxz + zxy)x,$$

$$\text{(QJAX3)} \qquad (xyx)z(xyx) = x\Big(y(xzx)y\Big)x,$$

and the same remains true in the polynomial extension since the extension remains special, $J[t] \subseteq A[t]^+$.

With considerable effort it can be shown that the third criterion is met, that these are (up to some characteristic 2 "wrinkles") the *only* simple unital quadratic Jordan algebras. Thus the quadratic approach provides a uniform way of describing Jordan algebras in all characteristics.

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[4] For any masochists in the audience, the requisite identities for non-unital algebras are (QJ1) $V_{x,x} = V_{x^2}$, (QJ2) $U_x V_x = V_x U_x$, (QJ3) $U_x(x^2) = (x^2)^2$, (QJ4) $U_{x^2} = U_x^2$, (QJ5) $U_x U_y(x^2) = (U_x y)^2$, (QJ6) $U_{U_x(y)} = U_x U_y U_x$. For some purposes it is easier to work with axioms involving only one element x, replacing (QJ5-6) by (CJ5′) $(x^2)^3 = (x^3)^2$, (CJ6′) $U_{x^3} = U_x^3$.

4.5 Inverses

One of the first concepts to be simplified using U-operators was the notion of inverses. In fact, inverses and U-operators were introduced and the Fundamental Formula conjectured by Jacobson in 1956 to fight his way through algebras of "degree 1," but until the Fundamental Formula was approved for use following I.G. Macdonald's proof in 1958, it was fighting with one hand tied behind one's back.

Quadratic Inverse Proposition. *The following conditions on an element x of a unital Jordan algebra are equivalent:*

(1) *x has a Jordan inverse y: $x \bullet y = 1,\ x^2 \bullet y = x$;*
(2) *x has a quadratic Jordan inverse y: $U_x y = x,\ U_x(y^2) = 1$;*
(3) *the U-operator U_x is an invertible operator.*

In this case the inverse y is unique; if we denote it by x^{-1}, we have $x^{-1} = \big(U_x\big)^{-1} x$, and it satisfies $U_{x^{-1}} = \big(U_x\big)^{-1}$.

In general, the operator L_x is not invertible if x is, and even when it is invertible we do not generally have $L_{x^{-1}} = L_x^{-1}$. For example, the real quaternion algebra is a division algebra (both as an associative algebra $\mathbb{H}$ and as a Jordan algebra $\mathbb{H}^+$), yet the invertible elements i and j satisfy $i \bullet j = 0$, so they are "divisors of zero" with respect to the Jordan product, and the operators L_i and L_j are not invertible. More generally, in an algebra $\mathcal{J}ord(Q, c)$ determined by a quadratic form with basepoint, two invertible elements x, y ($Q(x), Q(y)$ invertible in Φ) may well have $x \bullet y = 0$ (if they are orthogonal and traceless, $Q(x, y) = T(x) = T(y) = 0$).

Let's check what inverses amount to in our basic examples of Jordan algebras, and exhibit the basic examples of Jordan division algebras.

Special Inverse Example. *In any special Jordan algebra $\mathrm{J} \subseteq \mathrm{A}^+$, an element $x \in \mathrm{J}$ is invertible in J iff it is invertible in A and the associative inverse $x^{-1} \in \mathrm{A}$ falls in J.*

If D is an associative algebra, then D^+ is a Jordan division algebra iff D is an associative division algebra. If D has an involution, the inverse of a hermitian element is again hermitian, so $\mathcal{H}(\mathrm{D}, ^*)$ is a Jordan division algebra if (but not only if) D is an associative division algebra.

Quadratic and Cubic Factor Inverse Examples. *In a quadratic or cubic factor $\mathcal{J}ord(Q, c)$ or $\mathcal{J}ord(N, c)$ determined by a quadratic form Q or cubic form N over Φ, x is invertible iff $Q(x)$ or $N(x)$ is an invertible scalar (over a field Φ this just means that $Q(x) \neq 0$ or $N(x) \neq 0$).*[5]

[5] This result holds in any Jordan algebra with a "generic norm N," the analogue of the determinant. See footnote [3] back in Section 3.7.

Anisotropic forms beget Jordan division algebras: *$\mathcal{J}ord(Q,c)$ or $\mathcal{J}ord(N,c)$ determined by a quadratic or cubic form over a field is a division algebra iff Q or N is anisotropic* ($Q(x)=0$ *or* $N(x)=0$ *implies that* $x=0$).

4.6 Isotopes

Another concept that is considerably simplified by adopting the U-viewpoint is that of isotopes; recall from the Homotope Definition in Section 3.2 that for any element u the homotope $\mathrm{J}^{(u)}$ was the Jordan algebra with product

$$x \bullet_u y = x \bullet (u \bullet y) + (x \bullet u) \bullet y - u \bullet (x \bullet y).$$

Though the left multiplication operator is messy to describe, $L_x^{(u)} = L_{x\bullet u} + [L_x, L_u] = \frac{1}{2}V_{x,u}$, the U-operator and square have crisp formulations.

Quadratic Homotope Definition. (1) *If u is any element of a Jordan algebra* J, *the u-**homotope** $\mathrm{J}^{(u)}$ has shifted products*

$$\mathrm{J}^{(u)}: \quad x^{(2,u)} = U_x u, \quad x \bullet_u y = \tfrac{1}{2}\{xuy\} = \tfrac{1}{2}U_{x,y}u, \quad U_x^{(u)} = U_x U_u.$$

The homotope is again a Jordan algebra, as can easily be checked using the quadratic axioms for a Jordan algebra. If $\mathrm{J} \subseteq \mathrm{A}^+$ *is special, so is any homotope,* $\mathrm{J}^{(u)} \subseteq \big(\mathrm{A}_u\big)^+$.

(2) *The homotope is unital iff the original algebra is unital and u invertible, in which case the unit is* $1^{(u)} = u^{-1}$, *and we speak of the u-**isotope**.*

Homotopy is *reflexive* and *transitive*,

$$\mathrm{J}^{(1)} = \mathrm{J}, \qquad \big(\mathrm{J}^{(u)}\big)^{(v)} = \mathrm{J}^{(U_u v)},$$

but unlike isotopy is not in general *symmetric*, therefore not an equivalence relation: if u is not invertible we cannot recover J from $\mathrm{J}^{(u)}$, and information is lost in the passage to the homotope.

4.7 Inner Ideals

The greatest single advantage of looking at things from the U rather than the non-U point of view is that it leads naturally to one-sided ideals. Linear Jordan algebras, or any linear algebra with a commutative or anticommutative product, will have no notion of one-sided ideal: every left or right ideal is automatically a two-sided ideal. The quadratic product xyx doesn't have a *left* and *right* side, it has an *in*side and an *out*side: we multiply x on the inside by y, and y on the outside by x. Just as a left ideal in a linear algebra A is a submodule B invariant under multiplication by A on the left, $\widehat{\mathrm{A}}\mathrm{B} \subseteq \mathrm{B}$, and a right ideal is invariant under multiplication on the right, $\mathrm{B}\widehat{\mathrm{A}} \subseteq \mathrm{B}$, it is natural to define an inner ideal to be invariant under multiplication on the inside. The introduction of inner ideals is one of the most important steps toward the modern structure theories of Jacobson and Zel'manov.

Inner Ideal Definition. *A submodule* $\mathrm{B} \subseteq \mathrm{J}$ *is called an* **inner ideal** *if it is closed under multiplication on the inside by* $\widehat{\mathrm{J}}$*:* $U_{\mathrm{B}}\widehat{\mathrm{J}} \subseteq \mathrm{B}$ (*or, equivalently,* $U_{\mathrm{B}}\mathrm{J} \subseteq \mathrm{B}$ *and* $\mathrm{B}^2 \subseteq \mathrm{B}$).

Principal Inner Example. *The Fundamental Formula shows that any element* b *determines an inner ideal*

$$(b] := U_b\widehat{\mathrm{J}} = \Phi b^2 + U_b\mathrm{J},$$

called the **principal inner ideal** *determined by* b.

To get a feel for this concept, let's look at the inner ideals in each of the basic examples of Jordan algebras.

Full Inner Example. *In the Jordan algebra* A^+ *for associative* A, *any left or right ideal* L *or* R *of* A *is an inner ideal, hence also their intersection* $\mathrm{L} \cap \mathrm{R}$, *as well as any submodule* $a\mathrm{A}b$.

Hermitian Inner Example. *In the Jordan algebra* $\mathcal{H}(\mathrm{A}, *)$ *for an associative* $*$*-algebra* A, *any submodule* $a\mathcal{H}a^*$ *for* a *in* A *is an inner ideal.*

Quadratic Factor Inner Example. *In a quadratic factor* $\mathcal{J}ord(Q, c)$ *over* Φ, *any* totally isotropic Φ*-submodule* B (*a* Φ*-submodule consisting entirely of isotropic vectors, i.e., on which the quadratic form is totally trivial,* $Q(\mathrm{B}) = 0$) *forms an inner ideal:* $U_b\mathrm{J} = Q(b, \bar{\mathrm{J}})b - Q(b)\bar{\mathrm{J}} = Q(b, \bar{\mathrm{J}})b \subseteq \Phi b \subseteq \mathrm{B}$. *Notice that if* B *is a totally isotropic submodule, so are all submodules of* B, *hence all submodules of* B *are again inner ideals.*

If Q *is nondegenerate over a field* Φ, *these totally isotropic submodules are the only inner ideals other than* J. *Indeed, the principal inner ideals are* $(b] = \mathrm{J}$ *if* $Q(b) \neq 0$ [*since then* b *is invertible*], *and* $(b] = \Phi b$ *if* $Q(b) = 0$ [*since if* $b \neq 0$ *then by nondegeneracy* $Q(b) = 0 \neq Q(b, \bar{\mathrm{J}})$ *implies that* $Q(b, \bar{\mathrm{J}}) = \Phi$ *since* Φ *a field, so* $U_b\mathrm{J} = Q(b, \bar{\mathrm{J}})b = \Phi b$].

Cubic Inner Example. *Analogously, in a cubic factor* $\mathcal{J}ord(N, c)$ *any* sharpless Φ*-submodule* $\mathrm{B} \subseteq \mathrm{J}$ (*i.e., a submodule on which the sharp mapping vanishes,* $\mathrm{B}^\# = \mathbf{0}$) *forms an inner ideal*: $U_b\mathrm{J} = T(b, \mathrm{J})b - b^\#\#\mathrm{J} = T(b, \mathrm{J})b - 0 \subseteq \Phi b \subseteq \mathrm{B}$.

4.8 Nondegeneracy

Another concept which requires U-operators for its formulation is the crucial "semisimplicity" concept for Jordan algebras, discovered by Jacobson: a Jordan algebra is **nondegenerate** if it has no nonzero trivial elements, where an element $z \in \mathrm{J}$ is **trivial** if its U-operator is trivial on the unital hull (equivalently, its principal inner ideal vanishes):

$$(z] = U_z\widehat{\mathrm{J}} = \mathbf{0}, \qquad \text{i.e.,} \qquad U_z\mathrm{J} = \mathbf{0},\ z^2 = 0.$$

Notice that we never have nonzero elements with L_z trivial on the unital hull.

Nondegeneracy, the absence of *trivial elements,* is the useful Jordan version of the associative concept of semiprimeness, the absence of *trivial ideals* $\mathrm{BB} = \mathbf{0}$. For associative algebras, trivial elements z are the same as trivial ideals $\mathrm{B} = \widehat{\mathrm{A}}z\widehat{\mathrm{A}}$, since $z\widehat{\mathrm{A}}z = \mathbf{0} \iff \mathrm{BB} = (\widehat{\mathrm{A}}z\widehat{\mathrm{A}})(\widehat{\mathrm{A}}z\widehat{\mathrm{A}}) = \mathbf{0}$. A major difficulty in Jordan theory is that there is no convenient characterization of the Jordan ideal generated by a single element z. Because element-conditions are much easier to work with than ideal-conditions, the element-condition of nondegeneracy has proven much more useful than semiprimeness.[6]

We now examine what triviality means in our basic examples of Jordan algebras.

Full Trivial Example. *An element z of A^+ is trivial iff it generates a trivial two-sided ideal $\mathrm{B} = \widehat{\mathrm{A}}z\widehat{\mathrm{A}}$. In particular, the Jordan algebra A^+ is nondegenerate iff the associative algebra A is semiprime.*

Hermitian Trivial Example. *If $\mathcal{H}(\mathrm{A}, *)$ has trivial elements, then so does A. In particular, if A is semiprime with involution then $\mathcal{H}(\mathrm{A}, *)$ is nondegenerate.*

Quadratic Factor Trivial Example. *An element of a quadratic factor $\mathcal{J}ord(Q, c)$ determined by a quadratic form with basepoint over a field is trivial iff it belongs to $\mathcal{R}ad(Q)$. In particular, $\mathcal{J}ord(Q, c)$ is nondegenerate iff the quadratic form Q is nondegenerate, $\mathcal{R}ad(Q) = \mathbf{0}$.*

Cubic Trivial Example. *A cubic factor $\mathcal{J}ord(N, c)$ determined by a Jordan cubic form with basepoint over a field is always nondegenerate.*

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[6] In the words of an alert copy editor, ideal-conditions are less than ideal!

4.9 Radical Remarks

In associative theory there are many different radicals designed to remove different sorts of pathology and create different sorts of "niceness." The more difficult a "pathology" is to remove, the larger the corresponding radical will have to be. The usual "niceness" condition on an associative algebra A is semiprimeness (no trivial ideals, equivalently, no nilpotent ideals), which is equivalent to the vanishing of the *obstacle to semiprimeness,* the *prime radical* $\mathcal{P}rime(\mathrm{A})$ (also called the *semiprime* or *Baer radical*). This obstacle is the smallest ideal R such that A/R is semiprime, and is the intersection of all *prime ideals* (those P such that A/P is a "prime algebra" in the sense of having no orthogonal ideals; for $\mathrm{A} = \mathbb{Z}$, A/P has no orthogonal ideals iff $\mathrm{P} = p\mathbb{Z}$ for a prime p). For commutative associative rings, *prime* means integral domain, *semiprime* means no nilpotent elements $x^2 = 0$, and *simple* means field; every semiprime commutative ring is a subdirect product of prime rings, and every prime ring imbeds in a simple ring (its field of fractions). It is often helpful to think of the relationship of prime to simple for general rings as analogous to that between a domain and its field of fractions. In noncommutative associative algebras, the matrix algebras $\mathcal{M}_n(\mathrm{D})$ are prime if D is a domain, and they imbed in a simple "ring of central quotients" $\mathcal{M}_n(\mathrm{F})$ for F the field of fractions of D.

The more restrictive notion of semiprimitivity (nonexistence of "quasi-invertible" ideals) is equivalent to the vanishing of the obstacle to semiprimitivity, the larger **Jacobson radical** $\mathcal{R}ad(\mathrm{A})$ (also called the *primitive*, *semiprimitive*, or simply *the* radical). This is the smallest ideal R such that A/R is semiprimitive, and is the intersection of all *primitive ideals* (those Q such that A/Q is a "primitive algebra," i.e., has a faithful irreducible representation).[7] For artinian algebras these two important radicals coincide.

There are analogous radicals for Jordan algebras.

Prime Algebra Definition. *Two ideals* $\mathrm{I}, \mathrm{K} \lhd \mathrm{J}$ *in a Jordan algebra are* orthogonal *if* $U_{\mathrm{I}}\mathrm{K} = \mathbf{0}$. *A Jordan algebra is* **prime** *if it has no orthogonal ideals,*

$$U_{\mathrm{I}}\mathrm{K} = \mathbf{0} \Longrightarrow \mathrm{I} = \mathbf{0} \ \ or \ \ \mathrm{K} = \mathbf{0} \qquad (\mathrm{J}\ \text{prime}),$$

[7] Because it is the *quotients* A/P, A/Q that are prime or primitive, it can be confusing to call the ideals P, Q prime or primitive (they are not prime or primitive as algebras in their own right), and it would be more natural to call them *co-prime* and *co-primitive* ideals. An algebra turns out to be *semi*prime iff it is a *subdirect product* of prime algebras, and *semi*primitive iff it is a *subdirect product* of primitive algebras. In general, for any property $\mathcal{P}$ of algebras, *semi–*$\mathcal{P}$ means "subdirect product of $\mathcal{P}$-algebras." The only exception is *semisimple*, which has become fossilized in its finite-dimensional meaning "direct sum of simples." The terminology for radicals gets even more confusing when the radical is named for the *bad* property to be removed (such as *nil* or *degenerate*) rather than the *good* property to be created (*nil-freedom* or *nondegeneracy*).

and is **semiprime** *if it has no self-orthogonal ideals,*

$$U_{\mathrm{I}}\mathrm{I} = \mathbf{0} \Longrightarrow \mathrm{I} = \mathbf{0} \qquad (\mathrm{J}\text{ semiprime}).$$

This latter is equivalent to the absence of trivial ideals $U_{\mathrm{I}}\mathrm{I} = \mathrm{I}^2 = \mathbf{0}$, *or* nilpotent ideals $\mathrm{I}^n = \mathbf{0}$ [*where the* power I^n *is defined as the span of all homogeneous Jordan products of degree* $\geq n$ *when expressed in terms of products* $x \bullet y$ (*thus* $U_x y$ *and* $\{x, y, z\}$ *have degree* 3, $\{x, y\}$ *and* x^2 *have degree* 2)], *or* solvable ideals $\mathrm{I}^{(n)} = \mathbf{0}$ [*where the* derived ideal $\mathrm{I}^{(n)}$ *is defined recursively by* $\mathrm{I}^{(0)} = \mathrm{I},\ \mathrm{I}^{(n+1)} = U_{\mathrm{I}^{(n)}}\mathrm{I}^{(n)}$].

Prime Radical Definition. *The obstacle to semiprimeness is the* **prime** (*or* semiprime *or* Baer) **radical** $\mathcal{P}rime(\mathrm{J})$, *the smallest ideal whose quotient is semiprime, which is the intersection of all ideals whose quotient is prime*; J *is* **semiprime** *iff* $\mathcal{P}rime(\mathrm{J}) = 0$ *iff* J *is a subdirect product of prime algebras.*

Primitive Radical Definition. *A Jordan algebra is* **semiprimitive** *if it has no* **properly quasi-invertible** *elements* [*elements* z *with* $\hat{1} - z$ *invertible in all homotope hulls* $\widehat{\left(\mathrm{J}^{(u)}\right)}$]. *The obstacle to semiprimitivity is the* **primitive** (*or* semiprimitive *or* Jacobson) **radical** $\mathcal{R}ad(\mathrm{J})$, *the smallest ideal whose quotient is semiprimitive*; J *is semiprimitive iff* $\mathcal{R}ad(\mathrm{J}) = 0$. [*We will see in Part III that Zel'manov discovered the correct Jordan analogue of* primitivity, *such that* $\mathcal{R}ad(\mathrm{J})$ *is the intersection of all ideals whose quotients are primitive, and* J *is semiprimitive iff it is a subdirect product of primitive algebras.*]

The archetypal example of a prime algebra is a simple algebra, and the arche-typal example of a *non*-prime algebra is a direct sum $\mathrm{J}_1 \boxplus \mathrm{J}_2$. The archetypal example of a semiprime algebra is a semisimple algebra (direct sum of simples), and the archetypal example of a *non*-semiprime algebra is a direct sum $\mathrm{N} \boxplus \mathrm{T}$ of a nice algebra and a trivial algebra.

4.10 i-Special and i-Exceptional

It had been known for a long time that while the class of special algebras is closed under the taking of subalgebras and direct products, it is not closed under taking homomorphic images: P.M. Cohn had given an example in 1954 showing that the quotient of the free special Jordan algebra on two variables x, y by the ideal generated by $x^2 - y^2$ is no longer special (cf. Example A.5). By a general result of Garrett Birkhoff, any class $\mathcal{C}$ of algebras closed under subalgebras, direct sums, and homomorphic images forms a **variety** defined by a family $\mathcal{F}$ of *identities* (identical relations): $\mathrm{A} \in \mathcal{C} \Longleftrightarrow \mathrm{A}$ *satisfies* all the identities in $\mathcal{F}$ ($f(a_1, \ldots, a_n) = 0$ for all $f(x_1, \ldots, x_n) \in \mathcal{F}$ and all elements $a_1, \ldots, a_n \in \mathrm{A}$). Thus the class of special algebras was not a variety, but its "varietal closure," the slightly larger class of homomorphic images of special algebras, could be defined by identities. These identities are called the *special-identities* or **s-identities**: they are precisely all Jordan polynomials which vanish on all special algebras (hence automatically on their homomorphic images as well), but not on all Jordan algebras (they are nonzero elements of the free Jordan algebra).

The first thing to say is that there weren't supposed to be any s-identities! Remember that Jordan's goal was to capture the algebraic behavior of hermitian operators in the Jordan axioms. The s-identities are in fact just the algebraic identities involving the Jordan product satisfied by all hermitian matrices (of arbitrary size), and in principle all of these were supposed to have been incorporated into the Jordan axioms! A non-constructive proof of the existence of s-identities was first given by A.A. Albert and Lowell J. Paige in 1959 (when it was far too late to change the Jordan axioms), by showing that there must be nonzero Jordan polynomials $f(x, y, z)$ in three variables which vanish on all special algebras (become zero in the free special algebra on three generators). The first explicit s-identities G_8 and G_9 were discovered by Jacobson's student Charles M. Glennie in 1963: as we remarked in the Colloquial Survey, these could not possibly have been discovered without the newly-minted notions of the Jordan triple product and U-operators, and indeed even in their U-form no mortal other than Glennie has been able to remember them for more than 15 minutes.[8] It is known that there are no s-identities of degree ≤ 7, but to this day we do not know exactly what all the s-identities are, or even whether they are finitely generated.

We call an algebra *identity-special* or **i-special** if it satisfies all s-identities, i.e., belongs to the varietal closure of the special algebras. An algebra is *identity-exceptional* or **i-exceptional** if it is not i-special, i.e., does not satisfy all s-identities. Since the class of i-special algebras is slightly larger than that of special algebras, the class of i-exceptional algebras is slightly smaller than that of exceptional algebras. To be i-exceptional means that not only is the algebra exceptional, it doesn't even *look* special as far as its identities go: we can tell it apart from the special algebras just by examining the identities it satisfies, not by all its possible imbeddings in associative algebras. The arguments of Albert, Paige, and Glennie showed that the Albert algebra is in fact i-exceptional. Notice again that according to Jordan's philosophy the i-exceptional algebras were uninteresting (with respect to the s-identities they didn't behave like hermitian operators), only exceptional-but-i-special algebras could provide an alternative setting for quantum mechanics.

[8] At the first Oberwolfach conference on Jordan algebras in 1967 Charles Glennie sat down and explained to me the procedure he followed to discover G_8 and G_9. After 15 minutes it was clear to me that he had a systematic rational method for *discovering* the identities. On the other hand, 15 minutes after his explanation it was also clear that there was no systematic procedure for *remembering* that procedure. Thedy's identity T_{10}, since it was so compactly expressed in terms of a fictitious commutator acting as though it belonged to the (at that time highly fashionable) structure group, fit smoothly into the human brain's Jordan receptor cells.

It was a happy accident that Jordan didn't know about Glennie's identities when he set up his axioms, or else the Albert algebra might never have been born.

4.11 Artin–Wedderburn–Jacobson Structure Theorem

Inner ideals were first introduced by David M. Topping in his 1965 A.M.S. Memoir on Jordan algebras of self-adjoint operators; he called them *quadratic ideals*, and explicitly motivated them as analogues of one-sided ideals in associative operator algebras. Jacobson was quick to realize the significance of this concept, and in 1966 used it to define artinian Jordan algebras in analogy with artinian associative algebras, and to obtain for them a beautiful Artin–Wedderburn Structure Theorem. My own interest in quadratic Jordan algebras began when I read this paper, which showed that an entire structure theory could be based on the U-operator, thus fulfilling Archimedes' prophecy "Give me the Fundamental Formula and I will move the world."

Later on Jacobson proposed the terms "inner" and "outer ideal," which won immediate acceptance.

Artinian Definition. *A Jordan algebra* J *is* **artinian** *if it has* **minimum condition on inner ideals***: every collection* $\{\mathrm{B}_i\}$ *of inner ideals of* J *has a minimal element* (a B_k *not properly containing any other* B_i *of the collection*). *This is, as usual, equivalent to the* **descending chain condition (d.c.c.)** *on inner ideals: any strictly descending chain* $\mathrm{B}_1 > \mathrm{B}_2 > \cdots$ *of inner ideals must stop after a finite number of terms* (*there is no infinite such chain*).

Artin–Wedderburn–Jacobson Structure Theorem. *Consider Jordan algebras* J *over a ring of scalars* Φ *containing* $\frac{1}{2}$.

• *Degeneracy in a Jordan algebra* J *can be localized in the* **degenerate radical**[9] $\mathcal{D}eg(\mathrm{J})$, *the smallest ideal whose quotient is nondegenerate; in general, this is related to the semiprime and the semiprimitive radicals by* $\mathcal{P}rime(\mathrm{J}) \subseteq \mathcal{D}eg(\mathrm{J}) \subseteq \mathcal{R}ad(\mathrm{J})$, *and when* J *has minimum condition then* $\mathcal{D}eg(\mathrm{J}) = \mathcal{R}ad(\mathrm{J})$.

• *A Jordan algebra is nondegenerate with minimum condition iff it is a finite direct sum of ideals which are simple nondegenerate with minimum condition*; *in this case* J *has a unit, and the decomposition into simple summands is unique.*

• *A Jordan algebra is simple nondegenerate with minimum condition iff it is isomorphic to one of the following*:

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[9] This is sometimes called the McCrimmon radical in the Russian literature; although Jacobson introduced nondegeneracy as the "correct" notion of semisimplicity, he did not explicitly collect it into a radical.

Division Type: *a Jordan division algebra*;

Quadratic Type: *a quadratic factor* $\mathcal{J}ord(Q, c)$ *determined by a nondegenerate quadratic form* Q *with basepoint* c *over a field* Ω, *such that* Q *is not split of dimension* 2 *and has no infinite-dimensional totally isotropic subspaces*;

Hermitian Type: *an algebra* $\mathcal{H}(\mathrm{A}, *)$ *for a* $*$*-simple artinian associative algebra* A;

Albert Type: *an exceptional Albert algebra* $\mathcal{J}ord(N, c)$ *of dimension* 27 *over a field* Ω, *given by a Jordan cubic form* N.

In more detail, there are three algebras of Hermitian Type, the standard Jordan matrix algebras coming from the three standard types of involutions on simple artinian algebras:

Exchange Type: $\mathcal{M}_n(\Delta)^+$ *for an associative division ring* Δ (*when* A *is* $*$*-simple but not simple,* $\mathrm{A} = \mathcal{E}x(\mathrm{B}) = \mathrm{B} \boxplus \mathrm{B}^{op}$ *under the exchange involution for a simple artinian algebra* $\mathrm{B} = \mathcal{M}_n(\Delta)$);

Orthogonal Type: $\mathcal{H}_n(\Delta, \Gamma)$ *for an associative division ring* Δ *with involution* ($\mathrm{A} = \mathcal{M}_n(\Delta)$ *simple artinian with involution of orthogonal type*);

Symplectic Type: $\mathcal{H}_n(\mathrm{Q}, \Gamma)$ *for a quaternion algebra* Q *over a field* Ω *with standard involution* ($\mathrm{A} = \mathcal{M}_n(\mathrm{Q})$ *simple artinian with involution of symplectic type*).

Notice the two caveats in Quadratic Type. First, we must rule out the split 2-dimensional case because it is not simple, merely semisimple: in dimension 2 either the quadratic form is anisotropic, hence the Jordan algebra a division algebra, or the quadratic form is reduced and the Jordan algebra has two orthogonal idempotents, hence $\mathrm{J} = \Phi e_1 \oplus \Phi e_2 \cong \Phi \boxplus \Phi$ splits into a direct sum of two copies of the ground field, and is not simple. As soon as the dimension is ≥ 3 the e_1, e_2 are tied back together by other elements v into one simple algebra. The second, more annoying, caveat concerns the d.c.c., not central simplicity: past dimension 2 the nondegenerate Quadratic Types are always central-simple and have at most two orthogonal idempotents, but in certain infinite-dimensional situations they might still have an infinite descending chain of inner ideals: by the Basic Inner Examples any *totally-isotropic* subspace B (where every vector is isotropic, $Q(B) = 0$) is an inner ideal, and any subspace of B is also a totally isotropic inner ideal, so if Q has an infinite-dimensional totally isotropic B with basis $v_1, v_2, \ldots$, then it will have an infinite shrinking chain of inner ideals $\mathrm{B}_{(k)} = \mathrm{Span}(\{v_k, v_{k+1}, \ldots\})$. Loos has shown that such a J always has d.c.c. on *principal* inner ideals, so it just barely misses being artinian. These poor $\mathcal{J}ord(Q, c)$'s are left outside while their siblings party inside with the Artinian simple algebras. The final classical formulation in the next section will revise the entrance requirements, allowing these to join the party too.

The question of the structure of Jordan division algebras remained open: since they had no proper idempotents $e \neq 0, 1$ and no proper inner ideals, the classical techniques were powerless to make a dent in their structure. The nature of the radical also remained open. From the associative theory one expected $\mathcal{Deg}(\mathrm{J})$ to coincide with $\mathcal{Prime}(\mathrm{J})$ and be nilpotent in algebras with minimum condition, but a proof seemed exasperatingly elusive.

At this point a hypertext version of this book would include a melody from the musical *Oklahoma*:

Everything's up to date in Jordan structure theory;
They've gone about as fer as they can go;
They went and proved a structure theorem for rings with d.c.c.,
About as fer as a theorem ought to go. (*Yes, sir*!)

5

The Final Classical Formulation: Algebras with Capacity

In 1983 Jacobson reformulated his structure theory in terms of algebras with capacity. This proved serendipitous, for it was precisely the algebras with capacity, not the (slightly more restrictive) algebras with minimum condition, that arose naturally in Zel'manov's study of arbitrary infinite-dimensional algebras.

5.1 Algebras with Capacity

The key to this approach lies in *division idempotents*. These also play a starring role in the associative Artin-Wedderburn theory, where they go by the stage name of *completely primitive idempotents*.

Division Idempotent Definition. *An element e of a Jordan algebra* J *is called an* **idempotent** *if $e^2 = e$ (then all its powers are equal to itself, hence the name meaning "same-powered"). In this case the principal inner ideal $(e]$ forms a unital subalgebra U_eJ. A* **division idempotent** *is one such that this subalgebra U_eJ is a Jordan division algebra.*

Orthogonal Idempotent Definition. *Two idempotents e, f in* J *are* **orthogonal***, written $e \perp f$, if $e \bullet f = 0$, in which case their sum $e + f$ is again idempotent; an* **orthogonal family** *$\{e_\alpha\}$ is a family of mutually orthogonal idempotents ($e_\alpha \perp e_\beta$ for all $\alpha \neq \beta$). A finite orthogonal family is* **supplementary** *if the idempotents sum to the unit, $\sum_{i=1}^{n} e_i = 1$.*

Connection Definition. *Two orthogonal idempotents e_i, e_j in a Jordan algebra are* **connected** *if there is a* connecting element *$u_{ij} \in U_{e_i,e_j}$J which is invertible in the subalgebra $U_{e_i+e_j}$J. If the element u_{ij} can be chosen such that $u_{ij}^2 = e_i + e_j$, then we say that u_{ij} is a* strongly connecting element *and e_i, e_j are* **strongly connected***.*

We now examine examples of connected idempotents in our basic examples of Jordan algebras.

Full and Hermitian Connection Example. *If* Δ *is an associative division algebra with involution, then in the Jordan matrix algebra* $\mathrm{J} = \mathcal{M}_n(\Delta)^+$ *or* $\mathrm{J} = \mathcal{H}_n(\Delta, -)$ *the diagonal idempotents* E_{ii} *are supplementary orthogonal division idempotents* (*with* $U_{E_{ii}}(\mathrm{J}) = \Delta E_{ii}$ *or* $\mathcal{H}(\Delta, -)E_{ii}$, *respectively*) *strongly connected by the elements* $u_{ij} = E_{ij} + E_{ji}$.

Twisted Hermitian Connection Example. *In the twisted hermitian matrix algebra* $\mathrm{J} = \mathcal{H}_n(\Delta, \Gamma)$, *the diagonal idempotents* E_{ii} *are again orthogonal division idempotents* (*with* $U_{E_{ii}}(\mathrm{J}) = \gamma_i \mathcal{H}(\Delta)E_{ii}$) *connected by the elements* $u_{ij} = \gamma_i E_{ij} + \gamma_j E_{ji}$ *with* $u_{ij}^2 = \gamma_i\gamma_j(E_{ii} + E_{jj})$. *But in general they* cannot *be strongly connected by any element* $v_{ij} = \gamma_i a E_{ij} + \gamma_j \bar{a} E_{ji}$, *since*

$$v_{ij}^2 = (\gamma_i a \gamma_j \bar{a})E_{ii} + (\gamma_j \bar{a} \gamma_i a)E_{jj};$$

for example, if $\Delta =$ *the reals, complexes, or quaternions with standard involution and* $\gamma_i = 1, \gamma_j = -1$, *then we never have* $\gamma_i a \gamma_j \bar{a} = 1$ (*i.e., never* $a\bar{a} = -1$).

Quadratic Factor Connection Example. *In a quadratic factor* $\mathcal{J}ord(Q, c)$ *of dimension* > 2 *over a field* Φ, *every proper idempotent* e ($T(e) = 1, Q(e) = 0$) *is connected to* $e' := 1 - e$ *by every anisotropic* $u \perp 1, e$ *in the orthogonal space* ($Q(u) \neq 0 = Q(e, u) = T(u)$), *and is strongly connected by such* u *iff* $Q(u) = -1$.

Upper Triangular Connection Example. *If* $\mathrm{J} = \mathrm{A}^+$ *for* $\mathrm{A} = \mathcal{T}_n(\Delta)$ *the upper-triangular* $n \times n$ *matrices over* Δ, *then again the* E_{ii} *are orthogonal division idempotents, but they are not connected:* $U_{E_{ii},E_{jj}}(\mathrm{J}) = \Delta E_{ij}$ *for* $i < j$ *and so consists entirely of nilpotent elements* $u_{ij}^2 = 0$.

Capacity Definition. *A Jordan algebra has* **capacity** n *if it has a unit* 1 *which can be written as a finite sum of* n *orthogonal division idempotents:* $1 = e_1 + \cdots + e_n$; *it has* **connected capacity** *if each pair* e_i, e_j *for* $i \neq j$ *is connected. It has* **finite capacity** (*or simply* **capacity**) *if it has capacity* n *for some* n.

It is not immediately clear that capacity is an invariant, i.e., that all decompositions of 1 into orthogonal sums of division idempotents have the same length. This is true, but it was proven much later by Holger Petersson.

5.2 Classification

To analyze an arbitrary algebra of finite capacity, Jacobson broke it into its simple building blocks, analyzed the simple blocks of capacity 1, 2, $n \geq 3$ in succession, then described the resulting coordinate algebras, finally reaching the pinnacle of the classical structure theory in his 1983 Arkansas Lecture Notes.

Classical Structure Theorem. *Consider Jordan algebras over a ring of scalars containing* $\frac{1}{2}$.

• *A nondegenerate Jordan algebra with minimum condition on inner ideals has finite capacity.*

• *A Jordan algebra is nondegenerate with finite capacity iff it is a finite direct sum of algebras with finite connected capacity.*

• *A Jordan algebra is nondegenerate with finite connected capacity iff it is isomorphic to one of the following (in which case it is simple):*

DIVISION TYPE: *a Jordan division algebra;*

QUADRATIC TYPE: *a quadratic factor* $\mathcal{J}ord(Q, c)$ *determined by a nondegenerate quadratic form* Q *with basepoint* c *over a field* Ω (*not split of dimension* 2);

HERMITIAN TYPE: *an algebra* $\mathcal{H}(\mathrm{A}, *)$ *for a* $*$*-simple artinian associative algebra* A;

ALBERT TYPE: *an exceptional Albert algebra* $\mathcal{J}ord(N, c)$ *of dimension* 27 *over a field* Ω, *determined by a Jordan cubic form* N.

In more detail, the algebras of Hermitian Type are twisted Jordan matrix algebras:

EXCHANGE TYPE: $\mathcal{M}_n(\Delta)^+$ *for an associative division ring* Δ (A *is* $*$-*simple but not simple,* $\mathrm{A} = \mathcal{E}x(\mathrm{B})$ *with exchange involution for a simple artinian algebra* $\mathrm{B} = \mathcal{M}_n(\Delta)$);

ORTHOGONAL TYPE: $\mathcal{H}_n(\Delta, \Gamma)$ *for an associative division ring* Δ *with involution* ($\mathrm{A} = \mathcal{M}_n(\Delta)$ *simple artinian with involution of orthogonal type*);

SYMPLECTIC TYPE: $\mathcal{H}_n(\mathrm{Q}, \Gamma)$ *for a quaternion algebra* Q *over a field* Ω *with standard involution* ($\mathrm{A} = \mathcal{M}_n(\mathrm{Q})$ *simple artinian with involution of symplectic type*).

Note that we have not striven to make the types non-overlapping: the last three may include division algebras of the first type (our final goal will be to divide up the Division type and distribute the pieces among the remaining types).

Finite capacity essentially means "finite-dimensional over a division ring" Δ, which may well be infinite-dimensional over its center, though the quadratic factors $\mathcal{J}ord(Q, c)$ can have arbitrary dimension over Ω right from the start.

Note that all these simple algebras actually have minimum condition on inner ideals, with the lone exception of Quadratic Type: by the basic examples of inner ideals in Section 4.7, $\mathcal{J}ord(Q, c)$ always has minimum condition on *principal* inner ideals [the principal inner ideals are $(b] = \mathrm{J}$ and $(b] = \Omega b]$, but has minimum condition on *all* inner ideals iff it has no infinite-dimensional totally isotropic subspaces.

Irving Kaplansky has argued that imposing the d.c.c. instead of finite-dimensionality is an unnatural act, that the minimum condition does not arise "naturally" in mathematics (in contrast to the maximum condition, which, for example, arises naturally in rings of functions attached to finite-dimensional varieties in algebraic geometry, or in A.M. Goldie's theory of orders in associative rings). In Jordan theory the d.c.c. is additionally unnatural in that it excludes the quadratic factors with infinite-dimensional totally isotropic subspaces. Zel'manov's structure theory shows that, in contrast, imposing *finite capacity* is a natural act: finite capacity grows *automatically* out of the finite degree of any non-vanishing s-identity in a prime Jordan algebra.

6

The Classical Methods: Cherchez les Division Idempotents

Those elusive creatures, the division idempotents, are the atoms of structure theory, held together in a Peirce frame to create the living algebra.

6.1 Peirce Decompositions

We will now give an outline of how one establishes the classical structure theory of nondegenerate algebras with capacity. The method is to find as many idempotents as possible, preferably as tiny as possible, and use their Peirce[1] decompositions to refine the structure. Peirce decompositions were the key tool in the classical approach to the Artin–Wedderburn theory of associative rings. In associative rings, a decomposition $1 = \sum_{i=1}^{n} e_i$ of the unit into mutually orthogonal idempotents produces a decomposition $\mathrm{A} = \bigoplus_{i,j=1}^{n} \mathrm{A}_{ij}$ of the algebra into "chunks" (Peirce spaces) with multiplication rules like those for matrix units, $\mathrm{A}_{ij}\mathrm{A}_{k\ell} \subseteq \delta_{jk}\mathrm{A}_{i\ell}$. We can recover the structure of the entire algebra by analyzing the structure of the individual pieces and how they are reassembled (Humpty–Dumpty-wise) to form A.

Jordan algebras have similar Peirce decompositions, the only wrinkle being that (as in the archetypal example of hermitian matrix algebras) the "off-diagonal" Peirce spaces J_{ij} $(i \neq j)$ behave like the sum $\mathrm{A}_{ij} + \mathrm{A}_{ji}$ of two associative Peirce spaces, so that we have symmetry $\mathrm{J}_{ij} = \mathrm{J}_{ji}$. When $n = 2$ the decomposition is determined by a single idempotent $e = e_1$ with its complement $e' := 1 - e$, and can be described more simply.

Peirce Decomposition Theorem. *An idempotent e in a Jordan algebra* J *determines a* **Peirce decomposition**

$$\mathrm{J} = \mathrm{J}_2 \oplus \mathrm{J}_1 \oplus \mathrm{J}_0 \qquad (\mathrm{J}_i := \mathrm{J}_i(e) := \{x \mid V_e x = ix\})$$

[1] Not pronounced "pierce" (as in ear, or the American president Franklin), but "purse" (as in sow's ear, or the American mathematician Benjamin).

of the algebra into the direct sum of **Peirce subspaces***, where the* **diagonal Peirce subalgebras** $\mathrm{J}_2 = U_e\mathrm{J}, \mathrm{J}_0 = U_{1-e}\mathrm{J}$ *are principal inner ideals.*

More generally, a decomposition $1 = e_1 + \cdots + e_n$ *of the unit of a Jordan algebra* J *into a sum of* n *supplementary orthogonal idempotents leads to a* **Peirce decomposition**

$$\mathrm{J} = \bigoplus_{i \le j} \mathrm{J}_{ij}$$

of the algebra into the direct sum of **Peirce subspaces***, where the* **diagonal Peirce subalgebras** *are the inner ideals*

$$\mathrm{J}_{ii} := U_{e_i}(\mathrm{J}) = \{x \in \mathrm{J} \mid e_i \bullet x = x\},$$

and the **off-diagonal Peirce spaces** *are*

$$\mathrm{J}_{ij} = \mathrm{J}_{ji} := U_{e_i,e_j}(\mathrm{J}) = \{x \in \mathrm{J} \mid e_i \bullet x = e_j \bullet x = \tfrac{1}{2}x\} \quad (i \neq j).$$

Because of the symmetry $\mathrm{J}_{ij} = \mathrm{J}_{ji}$ *the multiplication rules for these subspaces are a bit less uniform to describe, but basically the product of two Peirce spaces vanishes unless two of the indices can be linked, in which case it falls in the Peirce space of the remaining indices*:

$$\begin{aligned}
&\{\mathrm{J}_{ii}, \mathrm{J}_{ik}\} \subseteq \mathrm{J}_{ik},\\
&\{\mathrm{J}_{ij}, \mathrm{J}_{jk}\} \subseteq \mathrm{J}_{ik} \quad (i, j, k \neq),\\
&\{\mathrm{J}_{ij}, \mathrm{J}_{ij}\} \subseteq \mathrm{J}_{ii} + \mathrm{J}_{jj} \quad (i \neq j),\\
&\{\mathrm{J}_{ij}, \mathrm{J}_{k\ell}\} = \mathbf{0} \quad (\{i, j\} \cap \{k, \ell\} = \emptyset).
\end{aligned}$$

6.2 Coordinatization

The classical *Wedderburn Coordinatization Theorem* says that if an associative algebra has a supplementary family of $n \times n$ *associative matrix units* e_{ij} $(\sum_i e_{ii} = 1,\ e_{ij}e_{k\ell} = \delta_{jk}e_{i\ell})$, then it is itself a matrix algebra, $\mathrm{A} \cong \mathcal{M}_n(\mathrm{D})$ coordinatized by $\mathrm{D} = \mathrm{A}_{11}$.

Jacobson found an important analogue for Jordan algebras: a Jordan algebra with enough Jordan matrix units is a Jordan matrix algebra $\mathcal{H}_n(\mathrm{D}, -)$ coordinatized by $\mathrm{D} = \mathrm{J}_{12}$. We noted earlier that one needs $n \geq 3$ in order to recover the product in D. Often in mathematics, in situations with low degrees of freedom we may have many "sporadic" objects, but once we get enough degrees of freedom to maneuver we reach a very stable situation. A good example is projective geometry. Projective n-spaces for $n = 1, 2$ (lines and planes) are bewilderingly varied; one important projective plane, the *Moufang plane* discovered by Ruth Moufang, has points (b, c) coordinatized by the octonions. But as soon as you get past 2, projective n-spaces for $n \geq 3$ automatically satisfy Desargues's Axiom, and any n-space for $n \geq 2$ which satisfies Desargues's Axiom is coordinatized by an associative division ring.

The situation is the same in Jordan algebras: degrees 1 and 2 are complicated (see Osborn's Degree 2 Theorem below), but once we reach degree $n = 3$ things are coordinatized by alternative algebras, and once $n \geq 4$ by associative algebras (recall the Associative and Alternative Coordinate Theorems of Sections 2.6 and 2.7). Here we cannot recover the coordinate algebra from the diagonal Peirce spaces J_{ii} coordinatized only by hermitian elements $\mathcal{H}(\mathrm{D}, -)$, we must use instead the off-diagonal Peirce spaces J_{ij} coordinatized by all of D. When the idempotents are *strongly connected* we get a Jordan matrix algebra, while if they are merely *connected* we obtain an isotope (a twisted matrix algebra $\mathcal{H}_n(\mathrm{D}, \Gamma)$). Note that no "niceness" conditions such as nondegeneracy are needed here: the result is a completely general structural result.

Jacobson Coordinatization Theorem. (1) *If a Jordan algebra* J *has a supplementary family of* $n \geq 3$ *Jordan matrix units* ($1 = e_1 + \cdots + e_n$ *for orthogonal idempotents* e_i *strongly connected to* e_1 *via* u_{1i}), *then* J *is isomorphic to a Jordan matrix algebra* $\mathcal{H}_n(\mathrm{D}, -)$ *under an isomorphism*

$$\mathrm{J} \to \mathcal{H}_n(\mathrm{D}, -) \quad \text{via} \quad e_i \mapsto E_{ii}, \quad u_{1j} \mapsto E_{1j} + E_{j1}.$$

The coordinate $*$*-algebra* D *with unit and involution is given by*

$$\begin{aligned} \mathrm{D} &:= \mathrm{J}_{12} = U_{e_1, e_2} \mathrm{J}, \\ 1 &:= u_{12}, \\ \bar{x} &:= U_{u_{12}}(x), \\ xy &:= \{\{x, \{u_{12}, u_{13}\}\}, \{u_{13}, y\}\} \qquad (x, y \in \mathrm{J}_{12}). \end{aligned}$$

Here D *must be associative if* $n \geq 4$, *and if* $n = 3$ *then* D *must be alternative with hermitian elements in the nucleus.*

(2) *A Jordan algebra whose unit is a sum of* $n \geq 3$ *orthogonal connected idempotents is isomorphic to a twisted Jordan matrix algebra: if the* e_i *are merely connected to* e_1 *via the* u_{1i}, *then there is an isotope* $\mathrm{J}^{(u)}$ *relative to a diagonal element* $u = e_1 + u_{22} + \cdots + u_{nn}$ *which is strongly connected,* $\mathrm{J}^{(u)} \cong \mathcal{H}_n(\mathrm{D}, -)$, *and* $\mathrm{J} = (\mathrm{J}^{(u)})^{(u^{-2})} \cong \mathcal{H}_n(\mathrm{D}, -)^{(\Gamma)} \cong \mathcal{H}_n(\mathrm{D}, \Gamma)$.

Notice that the product on the coordinate algebra D is recovered from *brace* products $\{x, y\} = 2x\bullet y$ in the Jordan algebra J, not *bullet* products. We noted in discussing Jordan matrix algebras that the brace products $\{x, y\} \approx xy + yx$ interact more naturally with the coordinates, and we also observed the necessity of associativity and alternativity. The key in the strongly connected case (1) is to construct the "matrix symmetries," automorphisms φ_π permuting the idempotents $\varphi_\pi(e_i) = e_{\pi(1)}$. These are generated by the "transpositions" $\varphi_{ij} = U_{c_{ij}}$ given by $c_{ij} = 1 - e_i - e_j + u_{ij}$. [The $u_{1j} \in \mathrm{J}_{1j}$ are given, and from these we construct the other $u_{ij} = \{u_{1i}, u_{1j}\} \in \mathrm{J}_{ij}$ with $u_{ij}^2 = e_i + e_j$, $\{u_{ij}, u_{jk}\} = u_{ik}$ for distinct indices i, j, k.] Directly from the Fundamental Formula we see that any element c with $c^2 = 1$ determines an automorphism U_c of J of period 2, and here $u_{ij}^2 = e_i + e_j$ implies that $c_{ij}^2 = 1$.

6.3 The Coordinates

Now we must find what the possible coordinate algebras are for an algebra with capacity. Here $U_{e_1}\mathrm{J} \cong \mathcal{H}(\mathrm{D},-)E_{11}$ must be a division algebra, and D must be nondegenerate if J is. These algebras are completely described by the H-K-O Theorem due to I.N. Herstein, Erwin Kleinfeld, and J. Marshall Osborn. Note that there are no explicit simplicity or finiteness conditions imposed, only nondegeneracy; nevertheless, the only possibilities are associative division rings and *finite-dimensional* composition algebras.

Herstein–Kleinfeld–Osborn Theorem. *A nondegenerate alternative $*$-algebra has all its hermitian elements invertible and nuclear iff it is isomorphic to one of*:

NONCOMMUTATIVE EXCHANGE TYPE: *the exchange algebra $\mathcal{E}x(\Delta)$ of a noncommutative associative division algebra Δ*;

DIVISION TYPE: *an associative division $*$-algebra Δ with non-central involution*;

COMPOSITION TYPE: *a composition $*$-algebra of dimension* $1, 2, 4,$ *or* 8 *over a field Ω (with central standard involution): the ground field (unarion), a quadratic extension (binarion), a quaternion algebra, or an octonion algebra.*

In particular, the algebra is automatically $*$-simple, and is associative unless it is an octonion algebra. We can list the possibilities in another way: the algebra is either

EXCHANGE TYPE: *the direct sum $\Delta \boxplus \Delta^{op}$ of an associative division algebra Δ and its opposite, under the exchange involution*;

DIVISION TYPE: *an associative division algebra Δ with involution*;

SPLIT QUATERNION TYPE: *a split quaternion algebra of dimension* 4 *over its center Ω with standard involution; equivalently,* 2×2 *matrices $\mathcal{M}_2(\Omega)$ under the symplectic involution $x^{sp} := s x^{tr} s^{-1}$ for symplectic $s := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$*;

OCTONION TYPE: *an octonion algebra* O *of dimension* 8 *over its center Ω with standard involution.*

6.4 Minimal Inner Ideals

Our next job is to show that the structure theory for artinian algebras can be subsumed under that for algebras with capacity: a nondegenerate Jordan algebra with minimum condition on inner ideals automatically has a finite capacity. To get a capacity, you first need to catch a division idempotent. Luckily, the nondegeneracy and the minimum condition on inner ideals guarantees lots of division idempotents, because minimal inner ideals come from trivial elements or division idempotents.

Minimal Inner Ideal Theorem. *The minimal inner ideals* B *in a Jordan algebra* J *over* Φ *are precisely the following*:

TRIVIAL TYPE: Φz *for a trivial element* z $(U_z\widehat{\mathrm{J}} = \mathbf{0})$;

IDEMPOTENT TYPE: $(e] = U_e\mathrm{J}$ *for a division idempotent* e;

NILPOTENT TYPE: $(b] = U_b\mathrm{J}$ *for* $b^2 = 0$, *in which case* $\mathrm{B} := (b]$ *has* $\mathrm{B}^2 = U_{\mathrm{B}}\mathrm{B} = \mathbf{0}$, *and* B *is* **paired** *with a minimal inner ideal* $\mathrm{E} = U_e\mathrm{J}$ *of Idempotent Type* ($U_b : \mathrm{E} \to \mathrm{B},\ U_e : \mathrm{B} \to \mathrm{E}$ *are inverse bijections*), *and there is also an isotope* $\mathrm{J}^{(u)}$ *in which* B *itself becomes of Idempotent Type,* $b^{(2,u)} = b$.

We now give examples of these three types of minimal inner ideals in the basic Jordan algebras.

Full and Hermitian Idempotent Example. *In the Jordan matrix algebra* $\mathcal{M}_n(\Delta)^+$ *or* $\mathcal{H}_n(\Delta, -)$ *for any associative division ring* Δ *with involution, the diagonal matrix unit* $e = E_{ii}$ *is a division idempotent, and the principal inner ideal* $(e] = \Delta E_{ii}$ *or* $\mathcal{H}(\Delta, -)E_{ii}$ *is a minimal inner ideal of Idempotent Type.*

Full and Hermitian Nilpotent Example. *In* $\mathcal{M}_n(\Delta)^+$ *the off-diagonal matrix unit* $b = E_{ij}$ *is nilpotent, and its principal inner ideal* $(b] = \Delta E_{ij}$ *is a minimal inner ideal of Nilpotent Type, paired with* $(e]$ *for the division idempotent* $e = \frac{1}{2}(E_{ii} + E_{ij} + E_{ji} + E_{jj})$. *In general, a hermitian algebra* $\mathcal{H}_n(\Delta, -)$ *need not contain inner ideals of Nilpotent Type* [*for example,* $\mathcal{H}_n(\mathbb{R})$ *is formally real and has no nilpotent elements at all*], *but if* Δ *contains an element* γ *of norm* $\gamma\bar{\gamma} = -1$, *then in* $\mathcal{H}_n(\Delta, -)$ *the principal inner ideal* $(b]$ *for* $b = E_{ii} + \gamma E_{ij} + \bar{\gamma}E_{ji} - E_{jj}$ *is minimal of Nilpotent Type* (*paired with* $(e]$ *for the division idempotent* $e = E_{ii}$).

Triangular Trivial Example. *In* A^+ *for* $\mathrm{A} = \mathcal{T}_n(\Delta)$ *the upper triangular* $n \times n$ *matrices over* Δ, *where* Φ *is a subfield of* Δ, *then the off-diagonal matrix units* E_{ij} $(i < j)$ *are trivial elements, and the inner ideal* ΦE_{ij} *is a minimal inner ideal of Trivial Type.*

Division Example. *If* J *is a division algebra, then* J *itself is a minimal inner ideal of Idempotent Type* $(e = 1)$.

Quadratic Factor Example. *In the quadratic factor* $\mathcal{J}ord(Q, c)$ *for a nondegenerate isotropic quadratic form over a field* Φ, *the minimal inner ideals are the* $(b] = \Phi b$ *determined by nonzero isotropic vectors* $(Q(b) = 0)$. *If* $T(b) \neq 0$ *then* b *is a scalar multiple of a division idempotent, and* $(b] = \Phi e = (e]$ *is of Idempotent Type. If* $T(b) = 0$ *then* $(b]$ *is of Nilpotent Type, paired with some* $(e] = \Phi e$. [*Imbed* b *in a hyperbolic pair* $Q(d) = Q(b) = 0, Q(d, b) = 1$; *if* $T(d) = \lambda \neq 0$ *then take* $e = \lambda^{-1}d$, *while if* $T(d) = 0$ *take* $e = d + \frac{1}{4}b + \frac{1}{2}1$.]

Cubic Factor Example. *Similarly, in the cubic factor* $\mathcal{J}ord(N, c)$ *for an isotropic Jordan cubic form, the minimal inner ideals are the* $(b] = \Phi b$ *for nonzero sharpless vectors* $b^\# = 0$*; again, if* $T(b) \neq 0$ *then* b *is a scalar multiple of a division idempotent, and* $(b]$ *is of Idempotent Type, while if* $T(b) = 0$ *then* $(b]$ *is of Nilpotent Type (though it is more complicated to describe an inner ideal of Idempotent Type paired with it).*

6.5 Capacity

Once we have division idempotents, we can assemble them carefully into a capacity.

Capacity Existence Theorem. *A nondegenerate Jordan algebra with minimum condition on inner ideals has a finite capacity.*

The idea is to find a division idempotent e_1 in J, then consider the orthogonal Peirce subalgebra $\mathrm{J}_0(e_1) = U_{\hat{1}-e_1}\mathrm{J}$. This inherits nondegeneracy and minimum condition from J (inner ideals or trivial elements in $\mathrm{J}_0(e)$ are actually inner or trivial in all of J); as long as this Peirce 0-space is nonzero we can repeat the process with J replaced by J_0 to find e_2 in $U_{\hat{1}-e_1}\mathrm{J}$, then find e_3 in $U_{\hat{1}-(e_1+e_2)}\mathrm{J}$, etc. This chain $\mathrm{J} > U_{\hat{1}-e_1}\mathrm{J} > U_{\hat{1}-(e_1+e_2)}\mathrm{J} > \cdots$ of "pseudo-principal" inner ideals (remember that we don't yet know that there is a real element 1 in J) eventually terminates in $U_{\hat{1}-(e_1+\cdots+e_n)}\mathrm{J} = \mathbf{0}$ by the descending chain condition, from which one shows by nondegeneracy that $e_1 + \cdots + e_n$ is a unit for J, and J has capacity n.

6.6 Capacity Classification

Once we have obtained capacity, we can forget about the minimum condition. First we break the algebra up into connected components.

Connected Capacity Theorem. *An algebra with capacity is semi-primitive iff it is nondegenerate:* $\mathcal{R}ad(\mathrm{J}) = \mathcal{D}eg(\mathrm{J})$. *A nondegenerate Jordan algebra with capacity splits into the direct sum* $\mathrm{J} = \mathrm{J}_1 \boxplus \cdots \boxplus \mathrm{J}_n$ *of a finite number of nondegenerate ideals* J_k *having connected capacity.*

The idea here is that if e_i, e_j are *not* connected, then by nondegeneracy we can show they are "totally disconnected" in the sense that the connecting Peirce space $\mathrm{J}_{ij} = \mathbf{0}$ vanishes entirely; connectivity is an equivalence relation among the e_i's, so if we break them into connectivity classes and let the f_k be the class sums, then again $U_{f_j,f_k}\mathrm{J} = \mathbf{0}$ for $j \neq k$, so that $\mathrm{J} = \boxplus_k U_{f_k}\mathrm{J} = \boxplus_k \mathrm{J}_k$ is a direct sum of subalgebras $\mathrm{J}_k = U_{f_k}\mathrm{J}$ (which are then automatically ideals) having unit f_k with connected capacity.

An easy argument using Peirce decompositions shows that nondegeneracy plus connected capacity yields simplicity.

Simple Capacity Theorem. *Any nondegenerate algebra with connected capacity is simple.*

Now we start to analyze the simple pieces according to their capacity. Straight from the definitions we have the following result.

Capacity 1 Theorem. *A Jordan algebra has capacity* 1 *iff it is a division algebra.*

At this stage the classical approach can't say anything more about capacity 1: the whole method is to use idempotents to break the algebra down, and Jordan division algebras have no proper idempotents and cannot be broken down further.

Capacity 2 turns out, surprisingly, to be the technically most difficult part of the structure theory.

Osborn's Capacity Two Theorem. *A Jordan algebra is nondegenerate with connected capacity* 2 *iff it is of* Full 2×2, Hermitian 2×2, *or* Quadratic Factor Type, *i.e., iff it is isomorphic to one of*:

$\mathcal{M}_2(\Delta)^+ \cong \mathcal{H}_2(\mathcal{E}x(\Delta))$ *for a noncommutative associative division algebra* Δ;

$\mathcal{H}_2(\Delta,\Gamma)$ *for an associative division algebra* Δ *with non-central involution*;

$\mathcal{J}ord(Q,c)$ *for a nondegenerate isotropic quadratic form* Q *with basepoint over a field* Ω.

[Note that the quadratic form Q has to be isotropic in the third case, for if it were anisotropic then by Quadratic and Cubic Factor Inverse Examples in Section 4.7, the algebra $\mathcal{J}ord(Q,c)$ would be a division algebra of capacity 1.]

Putting the coordinate algebras D of the H–K–O Theorem into the Coordinatization Theorem gives the algebras of Hermitian Type.

Capacity ≥ 3 Theorem. *A Jordan algebra is nondegenerate with connected capacity $n \geq 3$ iff it is isomorphic to one of the following algebras of $n\times n$ hermitian matrices*:

$\mathcal{H}_n(\mathcal{E}x(\Delta)) \cong \mathcal{M}_n(\Delta)^+$ *for an associative division algebra* Δ;

$\mathcal{H}_n(\Delta,\Gamma)$ *for an associative division $*$-algebra* Δ;

$\mathcal{H}_n(\mathrm{Q},\Gamma)$ *for a quaternion algebra* Q *over a field*;

$\mathcal{H}_3(\mathrm{O},\Gamma)$ *for an octonion algebra* O *over a field.*

Crudely put, the reason that the nice Jordan algebras of degree $n \geq 3$ are what they are is because Jordan algebras are naturally coordinatized by alternative algebras, and the only nice alternative algebras are associative or octonion algebras.

With the attainment of the Classical Structure Theory, Jordan algebraists sang a (slightly) different tune:

Everything's up to date in Jordan structure theory;
They've gone about as fer as they can go;
They went and proved a structure theorem for rings **with capacity**,
About as fer as a theorem ought to go. (*Yes,* **sir**!)

At the very time this tune was reverberating in nonassociative circles throughout the West, a whole new song without idempotents was being composed in far-off Novosibirsk.

7

The Russian Revolution: 1977–1983

The storm which broke over Jordan theory in the late 70s and early 80s began brewing in Novosibirsk, far from Western radar screens.

7.1 The Lull Before the Storm

At the end of the year 1977 the classical theory was essentially complete, but held little promise for a general structure theory. One indication that infinite-dimensional simple exceptional algebras are inherently of "degree 3" was the result that a simple algebra with more than three orthogonal idempotents is necessarily special. A more startling indication was the Gelfand–Naimark Theorem for Jordan Algebras obtained by Erik M. Alfsen, Frederic W. Shultz, and Erling Størmer that a JB-algebra (a Banach space with Jordan product satisfying $\|x \bullet y\| \leq \|x\| \, \|y\|$, $\|x^2\| = \|x\|^2$, and a formal reality condition $\|x^2\| \leq \|x^2 + y^2\|$) is built out of special subalgebras of associative C^*-algebras and Albert algebras. Here the test whether an algebra is special or Albert was whether or not it satisfied Glennie's Identity G_8. Few nonassociative algebraists appreciated the clue this gave — the argument was extremely long and ingenious, and depended on subtle results from functional analysis. The idea was to imbed the C^*-algebra in its Arens double dual, which is a Jordan W^*-algebra with lots of idempotents; speciality comes easily in the presence of idempotents.

Tantalizing as these results were, they and the classical methods relied so unavoidably on finiteness or idempotents that there seemed no point of attack on the structure of infinite-dimensional algebras: they offered no hope for a Life After Idempotents. In particular, the following Frequently Asked Questions on the structure of general Jordan algebras seemed completely intractable:

(FAQ1) *Is the degenerate radical nilpotent in the presence of the minimum condition on inner ideals* (*so* $\mathcal{D}eg(\mathrm{J}) = \mathcal{P}rime(\mathrm{J})$)? *Do the trivial elements generate a locally nilpotent ideal* (*equivalently, does every finite set of trivial* *elements in a Jordan algebra generate a nilpotent subalgebra*)? *Is* $\mathcal{D}eg(\mathrm{J}) \subseteq \mathcal{L}oc(\mathrm{J})$ *for the* locally nilpotent *or* Levitzki *radical* (*the smallest ideal* L *such that* J/L *has no locally nilpotent ideals, an algebra being* locally nilpotent *if every finitely-generated subalgebra is nilpotent*)?

(FAQ2) *Do there exist simple exceptional algebras which are not Albert algebras of disappointing dimension* 27 *over their center*?

(FAQ3) *Do there exist special algebras which are simple or division algebras but are not of the classical types* $\mathcal{J}ord(Q,c)$ *or* $\mathcal{H}(\mathrm{A},*)$?

(FAQ4) *Can one develop a theory of Jordan* PI-algebras (*those strictly satisfying a* polynomial identity, *a free Jordan polynomial which is* monic *in the sense that its image in the free special Jordan algebra has a monic term of highest degree*)? *Is the universal special envelope of a finitely generated Jordan PI-algebra an associative PI-algebra*? *Are the Jordan algebras* $\mathcal{J}ord(Q,c)$ *for infinite-dimensional* Q *the only simple PI-algebras whose envelopes are not PI and not finite-dimensional over their centers*?

(FAQ5) *Is the free Jordan algebra* (*cf. Appendix B*) *on three or more generators like a free associative or Lie algebra* (*a domain which can be imbedded in a division algebra, in which case it would necessarily be exceptional yet infinite-dimensional*), *or is it like the free alternative algebra* (*having zero divisors and trivial elements*)?

(FAQ6) *What are the s-identities which separate special algebras and their homomorphic images from the truly exceptional algebras*? *Are there infinitely many essentially different s-identities, or are they all consequences of* G_8 *or* G_9?

Yet within the next six years all of these FAQs became settled FAQTs.

7.2 The First Tremors

The first warnings of the imminent eruption reached the West in 1978. Rumors from visitors to Novosibirsk,[1] and a brief mention at an Oberwolfach conference attended only by associative ring theorists, claimed that Arkady M. Slin'ko and Efim I. Zel'manov had in 1977 settled the first part of FAQ1: the radical is nilpotent when J has minimum condition. Slin'ko had shown this for special Jordan algebras, and jointly with Zel'manov he extended this to arbitrary algebras. A crucial role was played by the concept of the **Zel'manov annihilator** of a set X in a linear Jordan algebra J,

[1] It is hard for students to believe how difficult mathematical communication was before the opening up of the Soviet Union. It was difficult for many Soviet mathematicians, or even their preprints, to get out to the West. The stories of Zel'manov and his wonderful theorems seemed to belong to the Invisible City of Kitezh, though soon handwritten letters started appearing with tantalizing hints of the methods used. It was not until 1982 that Western Jordan algebraists caught a glimpse of Zel'manov in person, at a conference on (not of) radicals in Eger, Hungary.

$$\mathcal{Z}ann_{\mathrm{J}}(X) = \{z \in \mathrm{J} \mid \{z, X, \widehat{\mathrm{J}}\} = \mathbf{0}\}.$$

This annihilator is always *inner* for any set X, and is an *ideal* if X is an ideal. In a special algebra $\mathrm{J} \subseteq \mathrm{A}^+$ where J generates A as associative algebra, an element z belongs to $\mathcal{Z}ann_{\mathrm{J}}(X)$ iff for all $x \in X$ the element $zx = -xz$ lies in the center of A with square zero, so for semiprime A this reduces to the usual two-sided associative annihilator $\{z \mid zX = Xz = 0\}$.

Annihilation is an order-reversing operation (the bigger a set the smaller its annihilator), so a decreasing chain of inner ideals has an increasing chain of annihilator inner ideals, and vice versa. Zel'manov used this to handle both the minimum and the maximum condition simultaneously. The **nil radical** $\mathcal{N}il(\mathrm{J})$ is the largest nil ideal, equivalently the smallest ideal whose quotient is free of nil ideals, where an ideal is **nil** if all its elements are nilpotent.

Zel'manov's Nilpotence Theorem. *If* J *has minimum or maximum condition on inner ideals inside a nil ideal* N, *then* N *is nilpotent. In particular, if* J *has minimum or maximum condition inside the nil radical* $\mathcal{N}il(\mathrm{J})$ *then*

$$\mathcal{N}il(\mathrm{J}) = \mathcal{L}oc(\mathrm{J}) = \mathcal{D}eg(\mathrm{J}) = \mathcal{P}rime(\mathrm{J}).$$

He then went on to settle the rest of (FAQ1) for completely general algebras.

Zel'manov's Local Nilpotence Theorem. *In any Jordan algebra, the trivial elements generate a locally nilpotent ideal,*

$$\mathcal{D}eg(\mathrm{J}) \subseteq \mathcal{L}oc(\mathrm{J});$$

equivalently, any finite set of trivial elements generates a nilpotent subalgebra.

The methods used were extremely involved and "Lie-theoretic," based on the notion of "thin sandwiches" (the Lie analogue of trivial elements), developed originally by A.I. Kostrikin to attack the Burnside Problem and extended by Zel'manov in his Fields–medal–winning conquest of that problem. At the same time, Zel'manov was able to characterize the radical for PI-algebras.

PI Radical Theorem. *If a Jordan algebra over a field satisfies a polynomial identity, then* $\mathcal{D}eg(\mathrm{J}) = \mathcal{L}oc(\mathrm{J}) = \mathcal{N}il(\mathrm{J})$. *In particular, if* J *is nil of bounded index then it is locally nilpotent:* $\mathrm{J} = \mathcal{L}oc(\mathrm{J})$.

7.3 The Main Quake

In 1979 Zel'manov flabbergasted the Jordan community by proving that there were no new exceptional algebras in infinite dimensions, settling once and for all FAQ2 which had motivated the original investigation of Jordan algebras by physicists in the 1930s, and showing that there is no way to avoid an invisible associative structure behind quantum mechanics.

Zel'manov's Exceptional Theorem. *There are no simple exceptional Jordan algebras but the Albert algebras: any simple exceptional Jordan algebra is an Albert algebra of dimension* 27 *over its center. Indeed, any prime exceptional Jordan algebra is a form of an Albert algebra: its central closure is a simple Albert algebra.*

Equally flabbergasting was his complete classification of Jordan division algebras: there was nothing new under the sun in this region either, answering the division algebra part of (FAQ3).

Zel'manov's Division Theorem. *The Jordan division algebras are precisely those of classical type*:

QUADRATIC TYPE: $\mathcal{J}ord(Q, c)$ *for an anisotropic quadratic form* Q *over a field*;

FULL ASSOCIATIVE TYPE: Δ^+ *for an associative division algebra* Δ;

HERMITIAN TYPE: $\mathcal{H}(\Delta, *)$ *for an associative division algebra* Δ *with involution* $*$;

ALBERT TYPE: $\mathcal{J}ord(N, c)$ *for an anisotropic Jordan cubic form* N *in* 27 *dimensions.*

As a coup de grâce, administered in 1983, he classified all possible simple Jordan algebras in arbitrary dimensions, settling (FAQ2) and (FAQ3).

Zel'manov's Simple Theorem. *The simple Jordan algebras are precisely those of classical type*:

QUADRATIC FACTOR TYPE: $\mathcal{J}ord(Q, c)$ *for a nondegenerate quadratic form* Q *over a field*;

HERMITIAN TYPE: $\mathcal{H}(\mathrm{B}, *)$ *for a* $*$*-simple associative algebra* B *with involution* $*$, (*hence either* A^+ *for a simple* A, *or* $\mathcal{H}(\mathrm{A}, *)$ *for a simple* A *with involution*);

ALBERT TYPE: $\mathcal{J}ord(N, c)$ *for a Jordan cubic form* N *in* 27 *dimensions.*

As if this weren't enough, he actually classified the prime algebras (recall the definition in Section 4.9).

Zel'manov's Prime Theorem. *The prime nondegenerate Jordan algebras are precisely*:

QUADRATIC FACTOR FORMS: *special algebras with central closure a simple quadratic factor* $\mathcal{J}ord(Q, c)$ (Q *a nondegenerate quadratic form with basepoint over a field*);

HERMITIAN FORMS: *special algebras* J *of hermitian elements squeezed between two full hermitian algebras,* $\mathcal{H}(\mathrm{A}, *) \triangleleft \mathrm{J} \subseteq \mathcal{H}(\mathcal{Q}(\mathrm{A}), *)$ *for a* $*$*-prime associative algebra* A *with involution* $*$ *and its Martindale ring of symmetric quotients* $\mathcal{Q}(\mathrm{A})$;

ALBERT FORMS: *exceptional algebras with central closure a simple Albert algebra* $\mathcal{J}ord(N, c)$ (N *a Jordan cubic form in* 27 *dimensions*).

This can be considered the ne plus ultra, or Mother of All Classification Theorems. Notice the final tripartite division of the Jordan landscape into Quadratic, Hermitian, and Albert Types. The original Jordan–von Neumann–Wigner classification considered the Hermitian and Albert types together because they were represented by hermitian matrices, but we now know that this is a misleading feature (due to the "reduced" nature of the Euclidean algebras). The final three types represent genetically different strains of Jordan algebras. They are distinguished among themselves by the sorts of identities they do or do not satisfy: Albert fails to satisfy the s-identities (Glennie's or Thedy's), Hermitian satisfies the s-identities but not Zel'manov's eater identity, and Quadratic satisfies s-identities as well as Zel'manov's eater identity.

The restriction to *nondegenerate* algebras is important; while all simple algebras are automatically nondegenerate, the same is not true of prime algebras. Sergei Pchelintsev was the first to construct prime special Jordan algebras which have trivial elements (and therefore *cannot* be quadratic or Hermitian forms); in his honor, such algebras are now called *Pchelintsev monsters.*

7.4 Aftershocks

Once one had such an immensely powerful tool as this theorem, it could be used to bludgeon most FAQs into submission. We first start with FAQ4.

Zel'manov's PI Theorem. *Each nonzero ideal of a nondegenerate Jordan PI-algebra has nonzero intersection with the center* (*so if the center is a field, the algebra is simple*). *The central closure of a prime nondegenerate PI-algebra is central-simple. Any primitive PI-algebra is simple. Each simple PI-algebra is either finite-dimensional, or a quadratic factor $\mathcal{J}ord(Q, c)$ over its center.*

Next, Ivan Shestakov settled a question raised by Jacobson in analogy with the associative PI theory.

Shestakov's PI Theorem. *If* J *is a special Jordan PI-algebra, then its universal special envelope is locally finite* (*so if* J *is finitely generated, its envelope is an associative PI-algebra*).

The example of an infinite-dimensional simple $\mathcal{J}ord(Q, c)$, which as a degree 2 algebra satisfies lots of polynomial identities yet has the infinite-dimensional simple Clifford algebra $\mathcal{C}liff(Q, c)$ as its universal special envelope, shows that we cannot expect the envelope to be globally finite.

The structure theory has surprising consequences for the free algebra, settling FAQ5.

Free Consequences Theorem. *The free Jordan algebra on three or more generators has trivial elements.*

Indeed, note that the free Jordan algebra $\mathcal{FJ}[X]$ on $|X| \geq 3$ generators (cf. Appendix B) is i-exceptional: if it were i-special, all its homomorphic images would be too, but instead it has the i-exceptional Albert algebra as a homomorphic image, since the Albert algebra can be generated by three elements $\begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix}, \begin{pmatrix} 0&-i&j\\ i&0&-\ell\\ j&\ell&0 \end{pmatrix}$. The free algebra $\mathcal{FJ}[X]$ is not itself an Albert form: the "standard Jordan identity"

$$SJ_4(x,y,z) := S_4\big(V_{x^3,y}, V_{x^2,y}, V_{x^1,y}, V_{1,y}\big)(z)$$

built out of the alternating "standard associative identity"

$$S_4(x_1,x_2,x_3,x_4) := \textstyle\sum_\pi (-1)^\pi x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}x_{\pi(4)}$$

summed over all permutations on four letters, vanishes identically on any Albert form $\mathrm{J} \subseteq \mathcal{H}_3(\mathcal{O}(\Omega))$ but not on $\mathcal{FJ}[x,y,z]$. [This $p(x,y,z)$ vanishes on J, indeed any degree 3 algebra where each element x has x^3 an Ω-linear combination of $x^2, x, 1$, because the alternating form $S_4(x_1,x_2,x_3,x_4)$ vanishes whenever the elements x_1,x_2,x_3,x_4 are linearly dependent; $p(x,y,z)$ doesn't vanish on $\mathcal{FJ}[x,y,z]$ because it doesn't even vanish on the homomorphic image $\mathcal{FA}[x,y,z]^+$ for $\mathcal{FA}[x,y,z]$ the free associative algebra on 3 generators (if we order the generators $x > y > z$ then $SJ_4(x,y,z)$ has monic lexicographically leading term $x^3yx^2yxyyz \in \mathcal{FA}[x,y,z]$).]

Thus by Zel'manov's Prime Theorem $\mathrm{FJ}[X]$ is not prime nondegenerate: either it is degenerate (has trivial elements), or is not prime (has orthogonal ideals). In fact, Yuri Medvedev explicitly exhibited trivial elements.

These *radical identities* are the most degenerate kind of s-identities possible: they are nonzero Jordan expressions which produce only trivial elements in any Jordan algebra, and vanish identically in special algebras.

These results also essentially answer FAQ6: just as Alfsen–Shultz–Størmer indicated, the Glennie Identity G_8 is the "only" s-identity, or put another way, all s-identities which survive the Albert algebra are "equivalent."

i-Speciality Theorem. *All semiprimitive i-special algebras are special, so for primitive algebras i-speciality is equivalent to speciality. If f is any particular s-identity which does not vanish on the Albert algebra (such as G_8, G_9, or T_{11}), then a semiprimitive algebra will be special as soon as it satisfies the one particular identity f: all such f do equally well at separating special algebras from exceptional algebras. As a consequence, the ideal of all s-identities in the free algebra on an infinite number of variables is quasi-invertible modulo the endvariant-ideal generated by the single identity f.*

An *endvariant* ideal of A is one invariant under all algebra endomorphisms of A. In a free algebra $\mathcal{FJ}[X]$, invariance under all homomorphisms $\mathcal{FJ}[X] \to \mathcal{FJ}[X]$ is equivalent to invariance under all substitutions $p(x_1,\dots,x_n) \mapsto$

$p(y_1, \ldots, y_n)$ for $y_i = y_i(x_1, \ldots x_n) \in \mathcal{FJ}[X]$.[2] The reason for the *equivalence* is that a semiprimitive algebra is a subdirect product of primitive (hence prime nondegenerate) algebras, which by Zel'manov's Prime Theorem are either special or Albert forms; if the semiprimitive algebra is i-special it must satisfy f, and as soon as it satisfies f all its primitive factors do likewise, so (by hypothesis on f) none of these factors can be an Albert algebra, and the original algebra is actually special as a subdirect product of special algebras. The reason for the *consequence* is that if $\overline{\mathrm{K}}$ denotes the quasi-invertible closure of the endvariant ideal K generated by f in the free Jordan algebra $\mathrm{FJ}[X]$ (so $\overline{\mathrm{K}}/\mathrm{K}$ is the Jacobson radical of $\mathrm{FJ}[X]/\mathrm{K}$), then by construction the quotient $\mathrm{FJ}[X]/\overline{\mathrm{K}}$ is semiprimitive and *satisfies* f (here it is crucial that K is the *endvariant ideal*, not just the *ordinary ideal*, generated by $f(x_1, \ldots, x_n)$: this guarantees not only that $f(x_1, \ldots, x_n)$ is 0 in the quotient, since it lies in K, but that any substitution $f(y_1, \ldots, y_n)$ also falls in K and hence vanishes in the quotient). By the first part of the theorem the quotient is special. But then all s-identities vanish on the special algebra $\mathrm{FJ}[X]/\overline{\mathrm{K}}$, so all their values on $\mathrm{FJ}[X]$ fall into $\overline{\mathrm{K}}$: $\overline{\mathrm{K}}$ contains all s-identities, so the ideal generated by all s-identities is contained in $\overline{\mathrm{K}}$, and hence it too is quasi-invertible mod K.

[2] In a free algebra such ideals are usually called *T-ideals*, for no apparent reason. One could similarly use *autvariant* for automorphism-invariant ideals. In group theory autvariant subgroups are called *characteristic*, and endvariant subgroups are variously called *fully characteristic, fully invariant*, or (in free groups) *verbal*.

8

Zel'manov's Exceptional Methods

Here we will sketch the methods used by Zel'manov in his proof of his Exceptional Theorem. This first third of his trilogy was completed in 1979. Zel'manov in fact classified the primitive i-exceptional algebras, then waved a few magic wands to obtain the classification of prime and simple i-exceptional algebras. His methods involved ordinary associative algebra concepts and results which could be transferred, although seldom in an obvious way, to Jordan algebras. (The second and third parts of his trilogy involved the notion of tetrad eater, whose origins go back to the theory of alternative algebras).

We warn readers to fasten their seatbelts before entering this chapter, because the going gets rougher: the arguments are intrinsically more difficult, and we get closer to these turbulent arguments by giving sketches of how the main results are proved.

8.1 I-Finiteness

First we characterize abstractly those algebras that are rich, but not too rich, in idempotents. On the way to showing that simple i-exceptional algebras are 27-dimensional, *generation* of idempotents comes from algebraicness, while *finiteness* of families of orthogonal idempotents comes from the finite-degree of a non-vanishing s-identity.

I-Genic Definition. *An algebra* J *is* **I-genic** (**idempotent-generating**) *if every non-nilpotent element b generates a nonzero idempotent $e \in (b] = U_b\widehat{\mathrm{J}}$.*

Every non-nilpotent algebraic element in a [not necessarily unital] power-associative algebra over a field generates an idempotent, providing the most important source of I-genic algebras. Recall that an element b is **algebraic** over Φ if it satisfies a *monic* polynomial $p(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda^1$ with zero constant term, i.e., some power b^n can be expressed as a Φ-linear combination of lower powers, so the [non-unital] subalgebra $\Phi_0[b]$ generated by b is a finite-dimensional commutative associative algebra. An algebra is **algebraic** if each of its elements is. Algebraicness is thus a condition of "local" finiteness at each individual element.

Algebraic I Proposition. *Every algebraic algebra over a field is I-genic.*

Indeed, if b is not nilpotent, the subalgebra $\Phi_0[b]$ is not nil, and hence by associative theory contains an idempotent $e \in \Phi_0[b]$, so $e = U_e(e) \in U_b(\mathrm{J}) \subseteq (b]$.

I-Finite Proposition. J *is said to be* **I-finite** (**idempotent-finite**) *if it has no infinite orthogonal family $e_1, e_2, \ldots$ of nonzero idempotents. I-finiteness is equivalent to the* **a.c.c. on idempotents**:

(1) *There is no infinite strictly increasing chain of idempotents,*

$$f_1 < f_2 < \cdots,$$

where $e > f$ means that $f \in \mathrm{J}_2(e)$ lives in the Peirce subalgebra governed by e, in which case $g := e - f$ is an idempotent and $e = f + g$ for orthogonal idempotents $f, g \in \mathrm{J}_2(e)$. The a.c.c. always implies (*and for unital algebras, is* equivalent *to*) *the* **d.c.c. on idempotents**:

(2) *There is no infinite strictly decreasing chain of idempotents*

$$g_1 > g_2 > \cdots.$$

In general, the d.c.c. is equivalent to the condition that J *have no infinite orthogonal family $\{e_i\}$ of nonzero idempotents which is* bounded above *in the sense that there is an idempotent g with all $e_i < g$. Note that if* J *is unital then all families of idempotents are bounded by* 1, *so the two chain conditions are equivalent: $f_i \uparrow \iff (1 - f_i) \downarrow$.*

Since an element $x = \alpha_1 e_1 + \cdots + \alpha_n e_n$ has $\alpha_1, \ldots, \alpha_n$ as its eigenvalues, a global bound on the number of eigenvalues an individual element can have forces a global bound on the number of mutually orthogonal idempotents the algebra can have. In other words, **Eigenvalues Bound Idempotents**:

(3) *If all elements $x \in \mathrm{J}$ have at most N eigenvalues in an infinite field, then* J *has at most N nonzero mutually orthogonal idempotents, hence is I-finite.*

PROOF SKETCH: The reason for the a.c.c. is that a strictly ascending chain would give rise to an orthogonal family of nonzero idempotents $e_{i+1} = f_{i+1} - f_i \in \mathrm{J}_0(f_i) \cap \mathrm{J}_2(f_{i+1}) \subseteq \mathrm{J}_0(e_j)$ for all $j < i+1$, and conversely, any mutually orthogonal family of nonzero idempotents gives rise to a strictly increasing chain $f_1 < f_2 < \cdots$ for $f_1 = e_1,\ f_{i+1} = f_i + e_{i+1} = e_1 + \cdots + e_{i+1}$. For the d.c.c., a decreasing chain $g_1 > g_2 > \cdots$ would give rise to an orthogonal family $e_{i+1} = g_i - g_{i+1} \in \mathrm{J}_2(g_i) \cap \mathrm{J}_0(g_{i+1}) \subseteq J_0(e_j)$ for all $j > i+1$ bounded by $g = g_1$; conversely, an orthogonal family of e_i bounded by g would give rise to a descending chain $g > g - e_1 > g - (e_1 + e_2) > \cdots > g - (e_1 + \cdots + e_n) > \cdots$. For eigenvalues bounding idempotents, if $e_1, \ldots, e_n$ are nonzero orthogonal idempotents, then for distinct $\lambda_i \in \Phi$ (which exist if Φ is infinite) the element $x = \sum_{i=1}^n \lambda_i e_i$ has eigenvalues $\lambda_1, \ldots, \lambda_n$, forcing $n \leq N$. □

As soon as we have the right number of idempotents we have a capacity.

I-Finite Capacity Theorem. *A semiprimitive algebra which is I-genic and I-finite necessarily has a capacity.*

PROOF SKETCH: It is not hard to guess how the proof goes. From the d.c.c. on idempotents we get minimal idempotents, which (after a struggle) turn out to be division idempotents. Once we get division idempotents, we build an orthogonal family of them reaching up to 1: from the a.c.c. we get an idempotent maximal among all finite sums of division idempotents, and it turns out (after a longer struggle) that this must be 1, so we have reached our capacity. Indeed, this maximal e has Peirce subalgebra $\mathrm{J}_0(e)$ containing by maximality no further division idempotents, hence by d.c.c. no idempotents at all, so by I-generation it must be entirely nil. We then show that radicality of $\mathrm{J}_0(e)$ would infect J $[\mathrm{J}_0(e) \subseteq \mathcal{R}ad(\mathrm{J})]$, so semiprimitivity of J forces $\mathrm{J}_0(e)$ to be zero, and once $\mathrm{J}_0(e) = \mathbf{0}$ mere nondegeneracy forces $\mathrm{J}_1(e) = \mathbf{0}$ too (after another struggle), and e is the unit for $\mathrm{J} = \mathrm{J}_2(e)$. □

8.2 Absorbers

A key new concept having associative roots is that of *outer absorber* of an *inner* ideal, analogous to the *right absorber* $r(\mathrm{L}) = \{z \in \mathrm{L} \mid z\mathrm{A} \subseteq \mathrm{L}\}$ of a *left* ideal L in an associative algebra A. In the associative case the absorber is an ideal; in the Jordan case the *square* of the absorber (the *quadratic absorber*) is *close* to an ideal.

Absorbers Theorem. (1) *The* **linear absorber** $\ell a(\mathrm{B})$ *of an inner ideal* B *in a Jordan algebra* J *absorbs* linear multiplication *by* J *into* B:

$$\ell a(\mathrm{B}) := \{z \in \mathrm{B} \mid V_{\mathrm{J}} z \subseteq \mathrm{B}\} = \{z \in \mathrm{B} \mid V_z \mathrm{J} \subseteq \mathrm{B}\}.$$

The **quadratic absorber** $qa(\mathrm{B})$ *absorbs* quadratic multiplications *by* J *into* B, *and coincides with the* second linear absorber:

$$qa(\mathrm{B}) := \{z \in \mathrm{B} \mid V_{\mathrm{J},\widehat{\mathrm{J}}} z + U_{\mathrm{J}} z \subseteq \mathrm{B}\} = \ell a(\ell a(\mathrm{B})).$$

The linear and quadratic absorbers of an inner ideal B *in a Jordan algebra* J *are again inner ideals in* J, *and ideals in* B.

(2) *The linear absorber also absorbs s-identities, and the quadratic absorber absorbs cubes of s-identities*: *we have* **Specializer Absorption**

$$i\text{-}\mathcal{S}pecializer(\mathrm{B}) \subseteq \ell a(\mathrm{B}), \quad i\text{-}\mathcal{S}pecializer(\mathrm{B})^3 \subseteq qa(\mathrm{B}).$$

When J *is nondegenerate, an absorberless* B *is i-special* (*strictly satisfies all s-identities*):

$$qa(\mathrm{B}) = \mathbf{0} \quad \Longrightarrow \quad i\text{-}\mathcal{S}pecializer(\mathrm{B}) = \mathbf{0} \qquad (\mathrm{J}\ \text{nondegenerate}).$$

Here *i-Specializer*(B) denotes the **i-specializer** of B, the smallest ideal of B whose quotient is i-special, consisting precisely of all values attained on B by all s-identities [the Jordan polynomials that were supposed to vanish if B were special].[1] The beautiful idea behind Specializer Absorption is that for an inner ideal B the Jordan algebra $B/\ell a(B)$ is manifestly *special*, since the map $b \mapsto V_b = 2L_b$ is a *Jordan homomorphism* of B into the special Jordan algebra $\mathcal{E}nd_\Phi(J/B)^+$ (note that J/B is merely a Φ-module, without any particular algebraic structure) with kernel precisely $\ell a(B)$, and therefore induces an imbedding of $B/\ell a(B)$ in $\mathcal{E}nd_\Phi(J/B)^+$, thanks to the Specialization Formulas $V_{b^2} = V_b^2 - 2U_b,\ V_{U_b c} = V_b V_c V_b - U_{b,c} V_b - U_b V_c$ together with $U_b J + U_{b,c} J \subseteq B$ for $b, c \in B$ *by definition of innerness.*

For example, if $B = U_e J = J_2(e)$ is the Peirce inner ideal determined by an idempotent in a Jordan algebra, the linear and quadratic absorbers are $\ell a(B) = qa(B) = \{z \in B | \{z, J_1(e)\} = \mathbf{0}\}$, which is just the kernel of the *Peirce specialization* $x \mapsto V_x$ of B in $\mathcal{E}nd_\Phi(J_1(e))^+$, so $B/\ell a(B) \cong V(J_2(e))|_{J_1(e)}$ is a Jordan algebra of linear transformations on $J_1(e)$.

The associative absorber is an ideal, and the Jordan quadratic absorber is only a few steps removed from being an ideal: the key (and difficult) result is the following theorem.

Zel'manov's Absorber Nilness Theorem. *The ideal in* J *generated by the quadratic absorber* $qa(B)$ *of an inner ideal* B *is nil mod* $qa(B)$ (*hence nil mod* B).

8.3 Modular Inner Ideals

In his work with the radical Zel'manov had already introduced the notion of modular inner ideal and primitive Jordan algebra. In the associative case a left ideal L is *modular* if it has a *modulus* c, an element which acts like a *right unit* ("modulus" in the older literature) for A modulo L: $ac \equiv a$ modulo L for all $a \in A$, i.e., $A(\hat{1} - c) \subseteq L$. If A is unital then *all* left ideals are modular with modulus $c = 1$. The concept of modularity was invented for the Jacobson radical in non-unital algebras: in the unital case $\mathcal{R}ad(A)$ is the intersection of all maximal left ideals, in the non-unital case it is the intersection of all maximal *modular* left ideals. Any translate $c + b$ ($b \in L$) is another modulus, and as soon as L contains one of its moduli then it must be all of A.

It turns out that the obvious Jordan condition $U_{\hat{1}-c} J \subseteq B$ for an inner ideal B isn't quite enough to get an analogous theory.

[1] We denote the set of special identities in variables X by *i-Specializer*(X). Note that *i-Specializer*(J) is not the "i-special part" of J; on the contrary, it is the "anti-i-special part," the obstacle whose *removal* creates i-speciality. To make it clear that we are *creating* i-speciality, we call the obstacle the i-special**izer**.

Modular Definition. *An inner ideal* B *in a Jordan algebra* J *is* **modular** *with* **modulus** c *if*

$$\text{(Mod 1) } U_{\hat{1}-c}\mathrm{J} \subseteq \mathrm{B}, \quad \text{(Mod 2) } c - c^2 \in \mathrm{B}, \quad \text{(Mod 3) } \{\hat{1} - c, \widehat{\mathrm{J}}, \mathrm{B}\} \subseteq \mathrm{B}.$$

(Mod 3) *can be expanded to* (Mod 3a) $\{\hat{1} - c, \mathrm{J}, \mathrm{B}\} \subseteq \mathrm{B}$ *and* (Mod 3b) $\{c, \mathrm{B}\} \subseteq \mathrm{B}$)*; it guarantees that any* translate $c + b$ *for* $b \in \mathrm{B}$ *of a modulus* c *is again a modulus;* (Mod 2) *then guarantees that any* power c^n *of a modulus remains a modulus, since it is merely a translate* $c - c^n \in \mathrm{B}$ *for all* $n \geq 1$.

Modulus Exclusion. *The* **Modulus Exclusion Property** *states that a proper inner ideal cannot contain its modulus*:

$$\textit{If } \mathrm{B} \textit{ has modulus } c \in \mathrm{B}, \textit{ then } \mathrm{B} = \mathrm{J}.$$

The **Strong Modulus Exclusion Property** *even forbids the ideal generated by the quadratic absorber of a proper inner ideal from containing the modulus*:

$$\textit{If } \mathrm{B} \textit{ has modulus } c \in \mathcal{I}_{\mathrm{J}}(qa(\mathrm{B})), \textit{ then } \mathrm{B} = \mathrm{J}.$$

Proof Sketch: The reason for Modulus Exclusion is that then $\mathbb{1}_{\mathrm{J}} = U_{(\hat{1}-c)+c} = U_{\hat{1}-c} + U_{\hat{1}-c,c} + U_c$ maps J into B by (Mod 1), (Mod 3) and innerness. Strong Modulus Exclusion holds because $c \in \mathcal{I}_{\mathrm{J}}(qa(\mathrm{B}))$ implies that some power c^n is in B by Absorber Nilness, which remains a modulus for B, so $\mathrm{B} = \mathrm{J}$ by ordinary Modulus Exclusion. □

8.4 Primitivity

An associative algebra A is *primitive* if it has a faithful irreducible representation, or in more concrete terms if there exists a left ideal such that A/L is a faithful irreducible left A-module. *Irreducibility* means that L is *maximal modular*, while *faithfulness* means that the *core* of L (the maximal ideal of A contained in L, which is just its *right absorber* $\{z \mid z\widehat{\mathrm{A}} \subseteq \mathrm{L}\}$) vanishes; this core condition that no nonzero ideal I is contained in L means that $\mathrm{I} + \mathrm{L} > \mathrm{L}$, hence in the presence of maximality means that $\mathrm{I} + \mathrm{L} = \mathrm{A}$, so that L *supplements all nonzero ideals*. Once a modular L_0 has this property, it can always be enlarged to a maximal modular left ideal L which is even more supplementary. In the Jordan case A/L is not going to provide a representation anyway, so there is no need to work hard to get the maximal L (the "irreducible" representation), any supplementary L_0 will do.

Primitive Definition. *A Jordan algebra is* **primitive** *if it has a* **primitizer** P, *a proper modular inner ideal* $\mathrm{P} \neq \mathrm{J}$ *which has the* **Supplementation Property** *that it* supplements *all nonzero ideals*:

$$\mathrm{I} + \mathrm{P} = \mathrm{J} \textit{ for all nonzero ideals } \mathrm{I} \textit{ of } \mathrm{J}.$$

Another way to express this is the **Ideal Modulus Property**:

$$\textit{Every nonzero ideal } \mathrm{I} \textit{ contains a modulus for } \mathrm{P}.$$

Absorberless Primitizer Proposition. *Although quadratic absorber and core do not in general coincide for Jordan algebras, besides a zero core the primitizer has zero absorber, and primitizers are* always i-special *by the* **Absorberless Primitizer Property**:

$$qa(\mathrm{P}) = \mathbf{0}, \textit{ hence } i\text{-}\mathcal{S}pecializer(\mathrm{P}) = \mathbf{0}.$$

PROOF SKETCH: Supplementation implies Ideal because if $i + p = c$ then I contains the translated modulus $i = c - p$, and Ideal implies Supplementation since if I contains a modulus for P then the inner ideal I + P contains its own modulus and must equal J by Modulus Exclusion. For Absorberless Primitizer, note that P proper implies by Strong Modulus Exclusion that $\mathrm{I} = \mathcal{I}_{\mathrm{J}}(qa(\mathrm{P}))$ contains no modulus for P, so by Ideal Modulus it must be $\mathbf{0}$, and $qa(\mathrm{P}) = 0$. □

Even for associative algebras there is a subtle difference between the concepts of simplicity and primitivity. A primitive algebra is one that has a faithful representation as linear transformations acting irreducibly on a module. The archetypal example is the algebra $\mathcal{E}nd_{\Delta}(V)$ of all linear transformations on an infinite-dimensional vector space V over a division ring Δ. This algebra is far from simple; an important proper ideal is the *socle*, consisting of all transformations of finite rank. On the other hand, most of the familiar simple rings are primitive: a simple ring will be primitive as soon as it is semiprimitive (a subdirect product of primitive quotients $\mathcal{P}_\alpha = \mathrm{A}/\mathrm{I}_\alpha$, since in the simple case these nonzero quotients can only be $\mathcal{P}_\alpha = \mathrm{A}$ itself), and semiprimitivity is equivalent to vanishing of the Jacobson radical, so a simple algebra is primitive unless it coincides with its Jacobson radical. But Sasiada's complicated example of a simple radical ring shows that not all simple algebras are primitive.

8.5 The Heart

The next concept is a straightforward translation of a very simple associative notion: the heart of a Jordan algebra is the minimal nonzero ideal, just as in the associative case.

Heart Definition. *The* **heart** *of a Jordan algebra is defined to be its smallest nonzero ideal, if such exists*:

$$\mathbf{0} \neq \heartsuit(\mathrm{J}) = \bigcap\{\mathrm{I} \mid \mathrm{J} \rhd \mathrm{I} \neq \mathbf{0}\}.$$

Heart Principles. *We have a* **Unital Heart Principle** *as well as a* **Capacious Heart Principle**:

- *if* $\heartsuit(\mathrm{J})$ *has a unit element, then* $\mathrm{J} = \heartsuit(\mathrm{J})$ *is simple and all heart,*
- *if* $\heartsuit(\mathrm{J})$ *has a capacity, then* $\mathrm{J} = \heartsuit(\mathrm{J})$ *is also simple with capacity.*

PROOF SKETCH: For Unital Heart, by a Peirce argument an ideal I which has a unit e is automatically a direct summand $\mathrm{J} = \mathrm{I} \oplus \mathrm{J}_0(e)$, since the glue $\mathrm{J}_1(e)$ disappears: $\mathrm{J}_1 = e \bullet \mathrm{J}_1 \subseteq \mathrm{I}$ (ideal) $\subseteq \mathrm{J}_2$ (unit for I) forces $\mathrm{J}_1 = \mathbf{0}$, $\mathrm{I} = \mathrm{J}_2$, $\mathrm{J} = \mathrm{J}_2 \boxplus \mathrm{J}_0$. But if $\mathrm{J}_2 = \mathrm{I} = \heartsuit(\mathrm{J})$ is the heart, it must be contained in the ideal J_0, forcing $\mathrm{J}_0 = \mathbf{0}$ and $\mathrm{J} = \mathrm{I} = \heartsuit(\mathrm{J})$. Capacious Heart follows since algebras with capacity are by definition unital. ☐

Simple algebras are all heart, and Unital Heart shows that unital heartiness leads to simplicity. Examples of heartless algebras are associative matrix algebras like $\mathcal{M}_n(\mathbb{Z})$, $\mathcal{M}_n(\Phi[X])$ coordinatized by UFDs which are not fields. Zel'manov opened up primitive i-exceptional algebras and made the amazing anatomical discovery that they all have hearts.

Primitive Exceptional Heart Theorem. *A primitive i-exceptional Jordan algebra has heart* $\heartsuit(\mathrm{J}) = i\text{-}\mathcal{S}pecializer(\mathrm{J})$ *consisting of all values on* J *of all s-identities.*

PROOF SKETCH: $\mathrm{S} := i\text{-}\mathcal{S}pecializer(\mathrm{J})$ is an ideal, and since J is *i-exceptional*, i.e., *not i-special*, i.e., does not satisfy all s-identities, $\mathrm{S} \neq \mathbf{0}$ is a nonzero ideal. We need to show that each nonzero ideal I contains S; but $i\text{-}\mathcal{S}pecializer(\mathrm{J}) \subseteq \mathrm{I}$ iff $i\text{-}\mathcal{S}pecializer(\mathrm{J}/\mathrm{I}) = \mathbf{0}$ in $\mathrm{J}/\mathrm{I} = (\mathrm{I} + \mathrm{P})/\mathrm{I} \cong \mathrm{P}/\mathrm{P} \cap \mathrm{I}$ by the Complementation Property of the primitizer and the Third Homomorphism Theorem; but $i\text{-}\mathcal{S}pecializer(\mathrm{P}/\mathrm{P} \cap \mathrm{I}) = i\text{-}\mathcal{S}pecializer(\mathrm{P})/\mathrm{I}$ vanishes by the Absorberless Primitizer Principle $i\text{-}\mathcal{S}pecializer(\mathrm{P}) = \mathbf{0}$. ☐

8.6 Spectra

The key to creating finiteness is (as suggested by Eigenvalues bounding Idempotents) to bound spectra. The *spectrum* of any element z is the set of scalars λ for which $\lambda\hat{1} - z$ is not invertible. This depends on the set of allowable scalars Φ and the algebra J in which we are considering z, so we use the notation $\mathcal{S}pec_{\Phi,\mathrm{J}}(z)$, though in everyday speech we omit reference to Φ, J when they are understood by the context.

The ordinary spectrum of an element z is the set of scalars $\lambda \in \Phi$ such that the element $\lambda\hat{1} - z \in \widehat{\mathrm{J}}$ is singular in the everyday sense. Different notions of "singularity" will lead to different notions of spectrum. We will be interested in three different spectra, depending on three slightly different notions of singularity, but for hearty elements all three are almost the same; one of these spectra is always naturally bounded by the degree N of a non-vanishing polynomial, and this is where all the finiteness comes in.

Eigenvalue Definition. *Let z be an element of a Jordan algebra* J *over a field* Φ. *We say that a scalar* $\lambda \in \Phi$ *is an* **eigenvalue** *for* z *if* $\lambda\hat{1} - z$ *is singular in the most flagrant sense that its* U*-operator kills somebody in* J*: it has an* **eigenvector**, *an element* $0 \neq x \in \mathrm{J}$ *with* $U_{\lambda\hat{1}-z}x = 0$, *equivalently, the operator* $U_{\lambda\hat{1}-z}$ *is not injective on* J. $\mathcal{E}ig_{\Phi,\mathrm{J}}(z)$ *denotes the set of eigenvalues in* Φ *of* z *considered as an element of* J:

$$\mathcal{E}ig_{\Phi,\mathrm{J}}(z) := \{\lambda \in \Phi \mid U_{\lambda\hat{1}-z} \text{ not injective on } \mathrm{J}\}.$$

In contrast to non-injectivity of the operator $U_{\lambda\hat{1}-z}$, the three notions of spectra concern non-surjectivity.

Φ–Spectrum Definition. *The* Φ–**spectrum** $\mathcal{S}pec_{\Phi,\mathrm{J}}(z)$ *of an element* $z \in \mathrm{J}$ *is the set of* $\lambda \in \Phi$ *such that the element* $\lambda\hat{1} - z \in \widehat{\mathrm{J}}$ *is singular in the sense that its operator* $U_{\lambda\hat{1}-z}$ *is not surjective on* $\widehat{\mathrm{J}}$ (*equivalently, not surjective on* J), *and thus the inner ideal* $U_{\lambda\hat{1}-z}\mathrm{J}$ *can be distinguished as a set from* J:

$$\begin{aligned}\mathcal{S}pec_{\Phi,\mathrm{J}}(z) &:= \{\lambda \in \Phi \mid U_{\lambda\hat{1}-z} \text{ not invertible on } \mathrm{J}\} \\ &= \{\lambda \in \Phi \mid U_{\lambda\hat{1}-z} \text{ not surjective, } U_{\lambda\hat{1}-z}\mathrm{J} < \mathrm{J}\}.\end{aligned}$$

If J *is non-unital, this spectrum is the set of all* λ *such that* $\lambda\hat{1} - z$ *is not invertible in* $\widehat{\mathrm{J}}$, *while if* J *is unital the spectrum is the set of all* λ *such that* $\lambda 1 - z$ *is not invertible in* J (*since* $U_{\lambda\hat{1}-z} = U_{\lambda 1 - z}$ *on* J).

f-Spectrum Definition. *If* J *is i-exceptional and* $f \in$ *i-*$\mathcal{S}$*pecializer*(X) *an s-identity which does not vanish strictly on* J, *the* f-**spectrum** *is the set* f-$\mathcal{S}pec_{\Phi,\mathrm{J}}(z)$ *of scalars such that* $\lambda\hat{1} - z$ *is singular in the sense that it gives rise to an inner ideal where* f *does vanish strictly; then* $U_{\lambda\hat{1}-z}\mathrm{J}$ *can be distinguished from* J *using* f:

$$f\text{-}\mathcal{S}pec_{\Phi,\mathrm{J}}(z) := \{\lambda \in \Phi \mid f(U_{\lambda\hat{1}-z}\mathrm{J}) \equiv 0\} \quad (\text{while } f(\mathrm{J}) \not\equiv 0).$$

Absorber Spectrum Definition. *The* **absorber spectrum** *of an element* z *is the set* $\mathcal{A}bs\,\mathcal{S}pec_{\Phi,\mathrm{J}}(z)$ *of scalars such that* $\lambda\hat{1} - z$ *is singular in the sense that it gives rise to an absorberless inner ideal, thus* $U_{\lambda\hat{1}-z}\mathrm{J}$ *can be distinguished from* J *using the absorber*:

$$\mathcal{A}bs\,\mathcal{S}pec_{\Phi,\mathrm{J}}(z) := \{\lambda \in \Phi \mid qa(U_{\lambda\hat{1}-z}\mathrm{J}) = \mathbf{0}\} \quad (\text{while } qa(\mathrm{J}) = \mathrm{J}).$$

8.7 Comparing Spectra

These various spectra are closely related, especially for elements of the heart.

Hearty Spectral Relations Proposition. *If f does not vanish strictly on an i-exceptional algebra* J, *where we require $f \in$ i-Specializer$(X)^3$ in general (but only $f \in$ i-Specializer(X) when* J *is nondegenerate), then we have the* **Spectral Relations**

$$\mathcal{A}bs\,\mathcal{S}pec_{\Phi,\mathrm{J}}(z) \subseteq f\text{-}\mathcal{S}pec_{\Phi,\mathrm{J}}(z) \subseteq \mathcal{S}pec_{\Phi,\mathrm{J}}(z).$$

If z is an element of the heart $\heartsuit(\mathrm{J})$, then its absorber spectrum almost coincides with its spectrum: we have the **Hearty Spectral Relations**

$$\mathcal{S}pec_{\Phi,\mathrm{J}}(z) \subseteq \mathcal{A}bs\,\mathcal{S}pec_{\Phi,\mathrm{J}}(z) \cup \{0\} \subseteq \mathcal{S}pec_{\Phi,\mathrm{J}}(z) \cup \{0\} \text{ for } z \in \heartsuit(\mathrm{J}).$$

Proof Sketch: If $0 \neq \lambda \in \mathcal{S}pec(z)$ for $z \in \heartsuit(\mathrm{J})$, then $w := \lambda^{-1}z \in \heartsuit$ and $\mathrm{B} := U_{\hat{1}-w}\mathrm{J} = U_{\lambda\hat{1}-z}\mathrm{J} < \mathrm{J}$ is proper with modulus $c := 2w - w^2 \in \heartsuit$; but then $qa(\mathrm{B})$ must vanish, otherwise $\mathcal{I}_{\mathrm{J}}(qa(\mathrm{B}))$ would contain $\heartsuit$ and hence c, contrary to Strong Modulus Exclusion. □

Zel'manov gave a beautiful combinatorial argument, mixing polynomial identities and inner ideals to show that a non-vanishing polynomial f puts a bound on at least that part of the spectrum where f vanishes strictly, the f-spectrum. This turns out to be the *crucial finiteness condition which dooms exceptional algebras to a 27-dimensional life*: the finite degree of the polynomial puts a finite bound on this spectrum.

f-Spectral Bound Theorem. *If a polynomial f of degree N does* not *vanish strictly on a Jordan algebra* J, *then* J *can contain at most $2N$ inner ideals* B_k *where f* does *vanish strictly and which are relatively prime in the sense that*

$$\mathrm{J} = \textstyle\sum_{i,j} \mathrm{C}_{i,j} \ \textit{for}\ \mathrm{C}_{i,j} = \bigcap_{k \neq i,j} \mathrm{B}_k.$$

In particular, in a Jordan algebra over a field there is always a uniform f- **spectral bound** $2N$ *on the size of f-spectra,*

$$|f\text{-}\mathcal{S}pec_{\Phi,\mathrm{J}}(z)| \leq 2N.$$

If $f \in$ i-Specializer(X)(or in i-Specializer$(X)^3$ if J *is degenerate) and z is an element of the heart $\heartsuit(\mathrm{J})$, then we have the* **Hearty Spectral Size Relations**

$$|\mathcal{S}pec_{\Phi,\mathrm{J}}(z)| \leq |\mathcal{A}bs\,\mathcal{S}pec_{\Phi,\mathrm{J}}(z) \cup \{0\}| \leq |f\text{-}\mathcal{S}pec_{\Phi,\mathrm{J}}(z) \cup \{0\}| \leq 2N + 1.$$

Proof Sketch: Since f does not vanish strictly on J, some linearization f' of f is nonvanishing. We can assume that f is a homogeneous function of N variables of total degree N, so its linearizations $f'(x_1, \ldots, x_N)$ still have total degree N. By relative primeness we have

$$0 \neq f'(\mathrm{J}, \ldots, \mathrm{J}) = f'(\textstyle\sum \mathrm{C}_{ij}, \ldots, \sum \mathrm{C}_{ij}) = \sum f''(\mathrm{C}_{i_1 j_1}, \ldots, \mathrm{C}_{i_N j_N})$$

summed over further linearizations f'' of f', so again some $f''(\mathrm{C}_{i_1j_1},\ldots,\mathrm{C}_{i_Nj_N})$ is not 0. Consider such a collection $\{\mathrm{C}_{i_1j_1},\ldots,\mathrm{C}_{i_Nj_N}\}$: each C_{ij} avoids at most 2 indices i,j (hence lies in B_k for all others), so N of the C_{ij}'s avoid at most $2N$ indices, so *when there are more than* $2N$ *of the* B_k*'s then at least one index* k *is unavoided,* $\mathrm{C}_{i_1j_1},\ldots,\mathrm{C}_{i_Nj_N}$ lie in this B_k, and

$$f''(\mathrm{C}_{i_1j_1},\ldots,\mathrm{C}_{i_Nj_N}) \subseteq f''(\mathrm{B}_k,\ldots,\mathrm{B}_k) = 0$$

by the hypothesis that f *vanishes strictly on each individual* B_k, which contradicts our choice of the collection $\{\mathrm{C}_{i_1j_1},\ldots,\mathrm{C}_{i_Nj_N}\}$. Thus there cannot be more than $2N$ of the B_k's.

In particular, there are at most $2N$ distinct λ's for which the inner ideals $\mathrm{B}_\lambda := U_{\lambda\hat{1}-z}(\mathrm{J})$ satisfy f strictly, since such a family $\{\mathrm{B}_{\lambda_k}\}$ is automatically relatively prime: the scalar polynomials $g_i(t)=\prod_{j\neq i}(\lambda_j - t)$ are relatively prime, so there are polynomials $k_i(t)=g_i(t)h_i(t)$ with $\sum k_i(t)=1$, hence substituting z for t yields $\mathrm{J}=U_{\hat{1}}(\mathrm{J}) = U_{\sum_i k_i(z)}(\mathrm{J}) = \sum_{i,j}\mathrm{J}_{ij}$, where $\mathrm{J}_{ii} := U_{k_i(z)}(\mathrm{J}) \subseteq U_{g_i(z)}(\mathrm{J}) \subseteq \bigcap_{k\neq i} U_{\lambda_k\hat{1}-z}(\mathrm{J}) = \mathrm{C}_{ii}$, and $\mathrm{J}_{ij} = U_{k_i(z),k_j(z)}(\mathrm{J}) \subseteq \bigcap_{k\neq i,j} U_{\lambda_k\hat{1}-z}(\mathrm{J}) = \mathrm{C}_{ij}$, therefore $\mathrm{J}=\sum_{i,j}\mathrm{J}_{ij}\subseteq\sum_{i,j}\mathrm{C}_{ij}$.

The size inequalities come from the Hearty Spectral Relations and the f-Spectral Bound. □

8.8 Big Resolvents

The complement of the spectrum is the **Φ-resolvent** $\mathcal{R}es_{\Phi,\mathrm{J}}(z)$ of z, the set of λ such that $\lambda\hat{1}-z\in\widehat{\mathrm{J}}$ *is* invertible on J. In a division algebra the spectrum of z contains exactly one element λ if $z=\lambda 1\in\Phi 1$, otherwise it is empty, so the resolvent contains all but at most one element of Φ. Good things happen when the resolvent is big enough. The saying "many scalars make light work" indicates that some of life's problems are due to a scalar deficiency, and taking a scalar supplement may make difficulties go away of their own accord.

Big Definition. *If* J *is a Jordan algebra over a field* Φ, *a set of scalars* $\Phi_0\subseteq\Phi$ *will be called* **big** (*relative to* J) *if*

$$\Phi_0 \ \ \textit{is infinite, and} \ \ |\Phi_0| > \dim_\Phi \mathrm{J}.$$

We will be particularly interested in the case where J *is a* **Jordan algebra over a big field** (*in the sense that* Φ *itself is big relative to* J). *Note that every algebra over a field* Φ *can be imbedded in an algebra* J_Ω *over a big field* Ω (*take any infinite* Ω *with* $|\,\Omega\,| > \dim_\Phi \mathrm{J}$, *since* $\dim_\Omega \mathrm{J}_\Omega = \dim_\Phi \mathrm{J}$).

Bigness isn't affected by finite modifications, due to the infinitude of Φ_0, so if the resolvent is big in the above sense then we also have $|\mathcal{R}es_{\Phi,\mathrm{J}}(z)|-1 > \dim_\Phi \widehat{\mathrm{J}}$, which guarantees that there are too many elements $(\lambda\hat{1}-z)^{-1}$ *in the unital hull* $\widehat{\mathrm{J}}$ for *nonzero* λ in the resolvent for them all to be linearly independent. But clearing denominators from a *linear* dependence relation $\sum\alpha_i(\lambda_i\hat{1}-z)^{-1}=0$ yields an *algebraic* dependence relation $p(z)=0$ for z, so z is algebraic over Φ.

Amitsur's Big Resolvent Trick. (1) *If* J *is a Jordan algebra over a field* Φ, *any element* $z \in$ J *which has a big resolvent* (*e.g., if* Φ *is big and* z *has a small spectrum* $|\mathcal{S}pec_{\mathrm{J}}(z)| < |\Phi|$) *is automatically algebraic over* Φ.[2] (2) *If* J *is a Jordan algebra over a big field, then the Jacobson radical is* properly nil (*in the sense that it remains nil in all homotopes*), $\mathcal{R}ad(\mathrm{J}) = \mathcal{P}nil(\mathrm{J})$.

One amazing consequence is that division algebras evaporate in the blazing heat of a big algebraically closed field.

Division Evaporation Theorem. *If* J *is a Jordan division algebra over a big algebraically closed field* Φ, *then* $\mathrm{J} = \Phi 1$.

As Roosevelt said, "We have nothing to fear but Φ itself."

This leads to the main structural result.

Big Primitive Exceptional Theorem. *A primitive i-exceptional Jordan algebra over a big algebraically closed field* Φ *is a simple split Albert algebra* $\mathcal{A}lb(\Phi)$.

Proof Sketch: The heart $\heartsuit(\mathrm{J}) = i\text{-}\mathcal{S}pecializer(\mathrm{J}) \neq \mathbf{0}$ exists by Primitive Exceptional Heart, using *primitivity* and *i-exceptionality*, so there *exists* an $f \in i\text{-}\mathcal{S}pecializer(X)$ of some finite degree N which does not vanish strictly on J, hence by the Hearty Spectral Size Relations there is a *uniform bound* $2N+1$ *on spectra* of *hearty* elements. Once the heart has a global bound on spectra over a *big field* Φ, by Amitsur's Big Resolvent Trick (1) it is *algebraic*. Then it is I-genic, and by Eigenvalues Bound Idempotents spectral boundedness yields I-finiteness, so by I-finite Capacity the heart $\heartsuit(\mathrm{J})$ has capacity. But then by the Capacious Heart Principle $\mathrm{J} = \heartsuit(\mathrm{J})$ is simple with capacity, and by the Classical Structure Theorem the only *i-exceptional* simple algebra it can be is an Albert algebra over its center Ω. Over a *big algebraically closed field* Φ we must have $\Omega = \Phi$ by the Division Evaporation Theorem, and J must be split. $\square$

8.9 Semiprimitive Imbedding

The final step is to analyze the structure of prime algebras. Note that we will go directly from primitive to prime without passing simple: the ultrafilter argument works in the setting of prime nondegenerate algebras, going from *nondegenerate* to *semiprimitive* to *primitive* over big fields. Even if we started with a simple algebra the simplicity would be destroyed in the following passage from nondegenerate to semiprimitive (even in the associative theory there are simple radical rings).

[2] Though it was not quite like the apple dropping on Newton's head, Amitsur attributed the discovery of his Trick to having to teach a course on differential equations, where he realized that spectra and resolvents make sense in algebraic settings.

Semiprimitive Imbedding Theorem. *Every prime nondegenerate Jordan algebra can be imbedded in a semiprimitive Jordan algebra* $\vec{\mathrm{J}} = \prod_{x\in X} \widetilde{\mathrm{J}}_x$ *for primitive algebras* $\widetilde{\mathrm{J}}_x$ *over a big algebraically closed field* Ω, *in such a way that* J *and* $\vec{\mathrm{J}}$ *satisfy exactly the same strict identities* (*in particular,* J *is i-exceptional iff* $\vec{\mathrm{J}}$ *is*).

Proof Sketch: As we will see in Section II.1.6, the centroid Γ of a prime algebra J is always an integral domain acting faithfully on J, so (1) J is imbedded in the algebra of fractions $\mathrm{J}_1 = \Gamma^{-1}\mathrm{J}$ over the field of fractions $\Omega_0 = \Gamma^{-1}\Gamma$. Nondegeneracy of J guarantees [after some hard work] that there are no elements $z \neq 0$ of J which are either (2) *strictly properly nilpotent of bounded index* in the Ω_0-algebra $\mathrm{J}_2 = \mathrm{J}_1[T]$ of polynomials in a countable set of indeterminates T (in the sense that there exists $n_2 = n_2(z)$ with $z^{(n_2,y_2)} = 0$ for all $y_2 \in \mathrm{J}_2$), (3) *properly nilpotent* in the Ω_0-algebra $\mathrm{J}_3 = \mathcal{S}eq(\mathrm{J}_2)$ of all sequences from J_2, with J_2 imbedded as constant sequences (in the sense that for each $y_3 \in \mathrm{J}_3$ there is $n_3 = n_3(z, y_3)$ with $z^{(n_3,y_3)} = 0$), (4) *properly quasi-invertible* in the Ω-algebra $\mathrm{J}_4 = \Omega \otimes_{\Omega_0} \mathrm{J}_3$ for Ω a *big algebraically closed field* with $|\Omega| > \dim_{\Omega_0} \mathrm{J}_3 = \dim_\Omega \mathrm{J}_4$ (in the sense that z is in $\mathcal{R}ad(\mathrm{J}_4) = \mathcal{P}nil(\mathrm{J}_4)$ by Amitsur's Big Resolvent Trick). This guarantees (5) that $\mathrm{J} \cap \mathcal{R}ad(\mathrm{J}_4) = \mathbf{0}$, so J is imbedded in the semiprimitive Ω-algebra $\widetilde{\mathrm{J}} = \mathrm{J}_4/\mathcal{R}ad(\mathrm{J}_4)$. (6) The semiprimitive algebra $\widetilde{\mathrm{J}}$ is imbedded in the direct product $\vec{\mathrm{J}} = \prod_{x\in X} \widetilde{\mathrm{J}}_x$ for primitivie Ω-algebras $\widetilde{\mathrm{J}}_x = \widetilde{\mathrm{J}}/\mathrm{K}_x$ for Ω-ideals K_x in $\widetilde{\mathrm{J}}$ with $\bigcap_{x\in X} \mathrm{K}_x = \mathbf{0}$. Moreover, the scalar extension J_1 inherits all *strict* identities from J and the scalar extension J_2 inherits all *strict* identities from J_1, the direct product J_3 inherits *all* identities from J_2, the scalar extension J_4 inherits all *strict* identities from J_3, the quotient $\widetilde{\mathrm{J}}$ inherits *all* identities from J_4, the quotients $\widetilde{\mathrm{J}}_x$ and their product $\vec{\mathrm{J}}$ inherits *all* identities from $\widetilde{\mathrm{J}}$. Thus $\vec{\mathrm{J}}$ inherits all *strict* identities from J, and conversely the subalgebra J inherits *all* identities from $\vec{\mathrm{J}}$, so they have exactly the *same* strict identities. Note that the algebraically closed field Ω remains big for each $\widetilde{\mathrm{J}}_x$: $|\Omega| > \dim_\Omega \mathrm{J}_4 \geq \dim_\Omega \widetilde{\mathrm{J}} \geq \dim_\Omega \widetilde{\mathrm{J}}_x$. □

8.10 Ultraproducts

We have gone a long way from J up to the direct product $\prod_x \mathrm{J}_x$, and we have to form an ultraproduct to get back down to something resembling J. This will require a basic understanding of filters and ultrafilters.

Filter Definition. *A nonempty collection* $\mathcal{F} \subseteq \mathcal{P}(X)$ *of subsets of a given set* X *is called a* **filter** *on* X *if it is*:

(Filt1) *closed under* finite intersections: $Y_i \in \mathcal{F} \Rightarrow Y_1 \cap \cdots \cap Y_n \in \mathcal{F}$;

(Filt2) *closed under* enlargement: $Y \in \mathcal{F},\ Y \subseteq Z \Rightarrow Z \in \mathcal{F}$;

(Filt3) proper: $\emptyset \notin \mathcal{F}$.

If $\mathcal{F}$ *is a filter on a set* X, *we obtain the* restriction filter $\mathcal{F}|_Y$ *of* $\mathcal{F}$ *to any set* Y *in the filter*:

(RFilt) $\mathcal{F}|_Y := \{Z \in \mathcal{F} \mid Z \subseteq Y\} \quad (Y \in \mathcal{F})$.

Ultrafilter Definition. *An* **ultrafilter** *is a maximal filter; maximality is equivalent to the extremely strong condition*

(UFilt) *for all Y, either Y or its complement $X\backslash Y$ belongs to $\mathcal{F}$.*

By Zorn's Lemma, any filter can be enlarged to an ultrafilter. If $\mathcal{F}$ is an ultrafilter on X, the restriction filter $\mathcal{F}|_Y$ is an ultrafilter on Y for any $Y \in \mathcal{F}$.

Filters on X can be thought of as systems of neighborhoods of a (perhaps ideal) "subset" of X, while ultrafilters should be thought of as neighborhoods of a (perhaps ideal) "point" of X. For example, any nonempty subset Y of X determines a filter consisting of all subsets which contain Y, and this is an ultrafilter iff Y consists of a single point.

To tell the truth, the only reason we are interested in ultrafilters is to use them to tighten direct products.

Ultraproduct Definition. *If* $\mathrm{A} = \prod_{x\in X} \mathrm{A}_x$ *is the direct product of algebraic systems and* $\mathcal{F}$ *is any filter on* X, *the quotient* $\mathrm{A}/\mathcal{F}$ *is the quotient* $\mathrm{A}/\equiv_{\mathcal{F}}$ *of* A *by the congruence*

$$a \equiv_{\mathcal{F}} b \text{ iff } a \text{ agrees with } b \text{ on some } Y \in \mathcal{F} \quad (a(x) = b(x) \text{ for all } x \in Y).$$

For any element a in the direct product $\mathrm{A} = \prod_{x\in X} \mathrm{A}_x$ *of* linear *algebraic systems* (*abelian groups with additional structure*), *we define the* **zero** *and the* **support sets** *of a to be*

$$\mathcal{Z}ero(a) := \{x \in X \mid a(x) = 0\}, \quad \mathcal{S}upp(a) := \{x \in X \mid a(x) \neq 0\}.$$

In these terms, $a \equiv_{\mathcal{F}} b \Leftrightarrow a - b \equiv_{\mathcal{F}} 0$, so $\mathrm{A}/\mathcal{F}$ is the quotient of A *by the ideal*

$$\mathrm{I}(\mathcal{F}) := \{a \in \mathrm{A} \mid a \equiv_{\mathcal{F}} 0\} = \{a \in \mathrm{A} \mid Zer(a) \in \mathcal{F}\}.$$

An **ultraproduct** *is just a quotient $\mathrm{A}/\mathcal{F}$ of a direct product by an ultrafilter on X.*

It is not hard to check that a congruence $\equiv_{\mathcal{F}}$ respects all the algebraic operations on A: if $p(x_1, \ldots, x_n)$ is an n-ary product on all the A_x, and $a_i = b_i$ on $Y_i \in \mathcal{F}$, then $p(a_1, \ldots, a_n) = p(b_1, \ldots, b_n)$ on $Y_1 \cap \cdots \cap Y_n \in \mathcal{F}$.

Intuitively, the ultraproduct consists of "germs" of functions (as in the theory of varieties or manifolds), where we identify two functions if they agree "locally" on some "neighborhood" Y belonging to the ultrafilter. While direct products inherit all identities satisfied by their factors, they fail miserably with other properties (such as being integral domains, or simple). Unlike direct products, ultraproducts preserve "most" algebraic properties.

Basic Ultraproduct Fact. *Any elementary property true of almost all factors* A_x *is inherited by any ultraproduct* $(\prod \mathrm{A}_x)/\mathcal{F}$. *Thus*:

• *Any ultraproduct of division algebras is a division algebra; any ultraproduct of fields* (*or algebraically closed fields*) *is a field* (*or algebraically closed field*).

• *Any ultraproduct of special* (*or i-special algebras*) *is special* (*or i-special*).

• *Any ultraproduct of split Albert algebras over* (*respectively algebraically closed*) *fields is a split Albert algebra over a* (*respectively algebraically closed*) *field.*

• *For any particular* $Y \in \mathcal{F}$ *we can disregard all factors* A_x *coming from outside* Y: $(\prod_{x\in X} \mathrm{A}_x)/\mathcal{F} \cong (\prod_{y\in Y} \mathrm{A}_y)/(\mathcal{F}|_Y)$.

Elementary is here a technical term from mathematical logic. Roughly, it means a property which can be described (using universal quantifiers) in terms of a finite number of elements of the system. For example, algebraic closure of a field requires that each nonconstant polynomial have a root in the field, and this is elementary [for each *fixed* $n > 1$ and *fixed* $\alpha_0, \ldots, \alpha_n$ in $\mathcal{F}$ there *exists* a $\lambda \in \Phi$ with $\sum_{i=0}^n \alpha_i \lambda^i = 0$]. However, simplicity of an algebra makes a requirement on *sets* of elements (ideals), or on existence of a finite number n of elements without any bound on n [for each fixed $a \neq 0$ and b in A there exists an n and a set $c_1, \ldots, c_n; d_1, \ldots, d_n$ of $2n$ elements with $b = \sum_{i=1}^n c_i a d_i$]. The trouble with such a condition is that as x ranges over X the numbers $n(x)$ may tend to infinity, so that there is no *finite* set of elements $c_i(x), d_i(x)$ in the direct product with $b(x) = \sum_{i=1}^n c_i(x)a(x)d_i(x)$ for *all* x.

Finite Dichotomy Principle. *If each factor* A_x *is one of a finite number of types* $\{T_1, \ldots, T_n\}$, *then any ultraproduct* $(\prod_{x\in X} \mathrm{A}_x)/\mathcal{F}$ *is isomorphic to a homogeneous ultraproduct* $(\prod_{x\in X_i} \mathrm{A}_x)/(\mathcal{F}|_{X_i})$ *of factors of type* T_i *for some* $i = 1, 2, \ldots, n$ (*where* $X_i := \{x \in X \mid \mathrm{A}_x$ *has type* $T_i\}$). *In particular, if each* A_x *is i-special or a split Albert algebra over a field, then any ultraproduct* $(\prod_{x\in X} \mathrm{A}_x)/\mathcal{F}$ *is either i-special or it is a split Albert algebra over a field.*

Proof Sketch: By hypothesis $X = X_1 \cup \cdots \cup X_n$ is a *finite* union (for any x the factor A_x has some type T_i, so $x \in X_i$) with complement $(X\backslash X_1) \cap \cdots \cap (X\backslash X_n) = \emptyset \notin \mathcal{F}$ by (Filt3), so by the finite intersection property (Filt1) some $X\backslash X_i$ is not in $\mathcal{F}$, hence by the ultrafilter property (UFilt) $X_i \in \mathcal{F}$, and then by a Basic Fact $(\prod_{x\in X} \mathrm{A}_x)/\mathcal{F} \cong (\prod_{x\in X_i} \mathrm{A}_x)/(\mathcal{F}|_{X_i})$. $\square$

8.11 Prime Dichotomy

Prime algebras always have proper support filters, from which we can create ultrafilters and use these to imbed the original algebra in an ultraproduct.

Prime Example. *If a linear algebraic system is* prime *in the sense that the* product *of two nonzero ideals is again nonzero, then every two nonzero* *ideals have nonzero* intersection. *If* $\mathrm{A}_0 \neq \mathbf{0}$ *is a prime nonzero subsystem of* $\mathrm{A} = \prod_{x \in X} \mathrm{A}_x$, *the* **support filter** *of* A_0 *is that generated by the supports of all the nonzero elements of* A_0,

$$\mathcal{F}(\mathrm{A}_0) := \{Z \subseteq X | Z \supseteq \mathcal{S}upp(a_0) \text{ for some } 0 \neq a_0 \in \mathrm{A}_0\}.$$

A_0 *remains imbedded in the ultraproduct* $(\prod \mathrm{A}_x)/\mathcal{F}$ *for any ultrafilter* $\mathcal{F}$ *containing* $\mathcal{F}(\mathrm{A}_0)$.

Proof Sketch: Here $\mathcal{F}(\mathrm{A}_0)$ is a nonempty collection [since $\mathrm{A}_0 \neq \mathbf{0}$] of nonempty subsets [since $a_0 \neq 0$ implies that $\mathcal{S}upp(a_0) \neq \emptyset$] as in (Filt3), which is clearly closed under enlargement as in (Filt2), and is closed under intersections as in (Filt1), since if $a_1, a_2 \neq 0$ in A_0 then *by primeness of* A_0 we have that $\mathrm{I}_{\mathrm{A}_0}(a_1) \cap \mathrm{I}_{\mathrm{A}_0}(a_2) \neq \mathbf{0}$ contains some $a_3 \neq 0$, and $\mathcal{S}upp(a_3) \subseteq \mathcal{S}upp(a_1) \cap \mathcal{S}upp(a_2)$. A_0 remains imbedded since if $a_0 \neq 0$ disappears in $\mathrm{A}/\mathcal{F}$ then $\mathcal{Z}ero(a_0) \in \mathcal{F}$ as well as $\mathcal{S}upp(a_0) \in \mathcal{F}(\mathrm{A}_0) \subseteq \mathcal{F}$, so by (Filt1) $\mathcal{Z}ero(a_0) \cap \mathcal{S}upp(a_0) = \emptyset \in \mathcal{F}$, contrary to (Filt3). □

Now we can put all the pieces together, and wave the ultra wand.

Prime Dichotomy Theorem. *Every prime nondegenerate Jordan algebra of characteristic* $\neq 2$ *is either i-special or a form of a split Albert algebra. Every simple Jordan algebra of characteristic* $\neq 2$ *is either i-special or an Albert algebra* $\mathcal{J}ord(N, c)$.

Proof Sketch: By Imbedding, a *prime nondegenerate* J imbeds in a direct product $\widetilde{\mathrm{J}} = \prod_x \mathrm{J}_x$ for primitive algebras J_x over big algebraically closed fields Ω_x, which by the Big Primitive Exceptional Classification are either i-special or split Albert algebras $\mathcal{A}lb(\Omega_x)$; then by the Prime Example J imbeds in an *ultraproduct* $(\prod_{x \in X} \mathrm{A}_x)/\mathcal{F}$, which by Finite Dichotomy is itself either i-special or a split Albert $\mathcal{A}lb(\Omega)$ over a field Ω. In the second case we claim that $\mathrm{J}_1 = \Omega \mathrm{J}$ is *all* of $\mathcal{A}lb(\Omega)$, so J is indeed a form of a split Albert algebra. Otherwise, J_1 would have dimension < 27 over Ω, as would the semisimple algebra $\mathrm{J}_2 = \mathrm{J}_1/\mathcal{R}ad(\mathrm{J}_1)$ and all of its simple summands; then by the finite-dimensional theory these summands must be *special*, so J_2 is too, yet J remains imbedded in J_2 since $\mathrm{J} \cap \mathcal{R}ad(\mathrm{J}_1) = \mathbf{0}$ [a nondegenerate $x \in \mathrm{J} \cap \mathcal{R}ad(\mathrm{J}_1)$ would lead, by the d.c.c. in the finite-dimensional J_1, to a von Neumann regular element y in $U_x \mathrm{J}_1 \subseteq \mathcal{R}ad(\mathrm{J}_1)$, contrary to the fact that the radical contains no regular elements], and this would contradict the assumed i-exceptionality of J.

If J is *simple* i-exceptional, we will show that it is already 27-*dimensional over its centroid* Γ, which is a field, and therefore again by the finite-dimensional theory $\mathrm{J} = \mathcal{J}ord(N, c)$. Now J is also prime and nondegenerate, so applying the prime case (taking as scalars $\Phi = \Gamma$) gives $\Omega \mathrm{J} = \mathcal{A}lb(\Omega)$ with a natural epimorphism $\mathrm{J}_\Omega = \Omega \otimes_\Gamma \mathrm{J} \longrightarrow \Omega \mathrm{J} = \mathcal{A}lb(\Omega)$. *In characteristic* $\neq 2$ the scalar extension $\mathrm{J}_\Omega = \Omega \otimes_\Gamma \mathrm{J}$ remains simple over Ω [central simple linear algebras are *strictly simple* in the sense that all scalar extensions by a field remain simple, cf. Strict Simplicity II.1.7], so this epimorphism must be an isomorphism, and $\dim_\Gamma(\mathrm{J}) = \dim_\Omega(\mathrm{J}_\Omega) = \dim_\Omega(\mathcal{A}lb(\Omega)) = 27$. □

Thus the magical wand has banished forever the possibility of a prime i-exceptional algebraic setting for quantum mechanics.

Part II

The Classical Theory

Introduction

In Part II *The Classical Theory* I give a self-contained treatment of the Classical Structure Theory for Jordan algebras with capacity, as surveyed in Chapters 5 and 6 of Part I the *Historical Survey*. I will repeat many of the definitions and results of the *Survey*, but giving detailed proofs. I give subliminal references, in footnotes at the beginning of each section, to their previous incarnation (e.g., I.2.3 means Part I, Chapter 2, Section 3). To confirm a reader's sense of *déjà vu*, I have been careful to use exactly the same name as in the *Survey*. All statements (definitions, lemmas, propositions, theorems, examples) in *The Classical Theory* are numbered consecutively within sections; cross-references within Part II omit the Part designation (e.g., 2.3.4 means current Part, Chapter 2, Section 3, 4th numbered statement).

Throughout this part we will work with **linear Jordan algebras** over a fixed (unital, commutative, associative) **ring of scalars Φ containing** $\frac{1}{2}$. All spaces and algebras will be assumed to be Φ-modules, all ideals and subalgebras will be assumed to be Φ-submodules, and all maps will be assumed to be Φ-linear. This ring of scalars will remain rather dormant throughout our discussion: except for the scalar $\frac{1}{2}$, we almost never care what the scalars are, and we have few occasions to change scalars (except to aid the discussion of "strictness" of identities and radicals). Our approach to the structure theory is "ring-theoretic" and "intrinsic," rather than "linear" and "formal": we analyze the structure of the algebras directly over the given ring of scalars, rather than first analyzing finite-dimensional algebras over an algebraically closed field and then analyzing their possible forms over general fields.

We assume that anyone who voluntarily enters *The Classical Theory*, with its detailed treatment of results, does so with the intention of learning at least some of the classical methods and techniques of nonassociative algebra, perhaps with an eye to conducting their own research in the area. To this end, we include a series of *exercises*, *problems*, and *questions*. The exercises are included in the body of the text, and are meant to provide the reader with routine practice in the concepts and techniques of a particular result (often involving alternate proofs). The problems are listed at the end of each chapter, and are of broader scope (often involving results and concepts which are extensions of those in the text, or which go in a completely different direction), and are meant to provide general practice in creating proofs. The questions, at the very end of each chapter, provide even more useful practice for a budding research mathematician: they involve studying (sometimes even formulating) a concept or problem without any hint of what sort of an answer to expect. For some of these exercises, problems, questions (indicated by an asterisk) there are hints given at the end of the book, but these should be consulted as a last resort — the point of the problems is the *experience of creating a proof*, not the proof itself.

First Phase: Back to Basics

In this phase we review the foundations of our subject, the various categories of algebraic structures that we will encounter. We assume that the reader already has a basic acquaintance with associative rings.

In Chapter 1 we introduce linear Jordan algebras and all their categorical paraphernalia. We establish the important theorem that a linear algebra is strictly simple (remains simple under all field extensions) iff it is centroid-simple (the centroid is just the ground field). In Chapter 2 we introduce alternative algebras, especially the composition algebras, giving a full treatment of the Cayley–Dickson doubling process and Hurwitz's Theorem classifying all composition algebras.

In Chapter 3 we give the three most important special Jordan algebras: the full plus-algebras, the hermitian algebras, and the quadratic factors. In Chapter 4 we construct, more laboriously, the general cubic factors determined by sharped cubic forms with basepoint. This construction subsumes the Springer Construction from Jordan cubic forms, the Freudenthal Construction of 3×3 hermitian matrices with entries in composition algebras (including the reduced exceptional Albert algebras), and the Tits Constructions from degree–3 associative algebras. Many of the technical calculations are swept into Appendix C.

In Chapter 5 we introduce two basic labor-saving devices, the Macdonald and Shirshov–Cohn Principles, which allow us to prove that certain formulas will be valid in all Jordan algebras as soon as we verify that they are valid in all associative algebras. Once more, the laborious combinatorial verification of these labor-saving devices is banished to Appendices A and B. Making free use of these powerful devices, we quickly get the five fundamental formulas involving the U-operator, especially *the* Fundamental Formula $U_{U_x y} = U_x U_y U_x$, which are the basic tools used in dealing with Jordan algebras.

With these tools it is a breeze to introduce the basic nondegeneracy condition for our algebras (the absence of elements with trivial U-operator), the crucial concept of inner ideals (spaces closed under inner multiplication by the whole algebra), and in Chapters 6 and 7 the concepts of invertibility (an element is invertible iff its U-operator is an invertible operator) and isotopy (shifting the U-operator from U_x to $U_x U_u$). This closes out the first phase of our theory.

1

The Category of Jordan Algebras

In this chapter we introduce the hero of our story, the Jordan algebra, and in the next chapter his trusty sidekick, the alternative algebra. As we will see in Chapter 15, alternative algebras arise naturally as coordinates of Jordan algebras with three or more connected orthogonal idempotents.

1.1 Categories

We will often state results informally in the language of categories and functors. This is a useful language that you should be fluent in at the colloquial level. Recall that a **category** $\mathcal{C}$ consists of a collection of **objects** and a set $\mathcal{M}or(X,Y)$ of **morphisms** for each pair of objects X, Y, such that these morphisms can be patched together in a monoid-like way: we have a "composition" $\mathcal{M}or(Y,Z) \times \mathcal{M}or(X,Y) \xrightarrow{\circ} \mathcal{M}or(X,Z)$ which is "associative" and has a "unit" $\mathbb{1}_X$ for each object X: $(h \circ g) \circ f = h \circ (g \circ f)$ and $f \circ \mathbb{1}_X = \mathbb{1}_Y \circ f = f$ for all $f \in \mathcal{M}or(X,Y)$, $g \in \mathcal{M}or(Y,Z)$, $h \in \mathcal{M}or(Z,W)$.

In distinction to mappings encountered in set-theoretic aspects of calculus and elsewhere, the choice of the **codomain** or, more intuitively, **target** Y for $f \in \mathcal{M}or(X,Y)$ is just as important as its **domain** X [and must be carefully distinguished from the *range* or *image*, the set of values assumed if f happens to be a concrete set-theoretic mapping], so that composition $f \circ g$ is not defined unless the domain of f and the codomain of g match up. An analogous situation occurs in multiplication of an $m \times n$ matrix A and $p \times q$ matrix B, which is not defined unless the middle indices $n = p$ coincide, whereas at a vector-space level the corresponding maps $\alpha : W \to Z$, $\beta : X \to Y$ can be composed as long as $\mathrm{Im}(\beta) \subseteq W$. While we seldom need to make this distinction, it is crucial in algebraic topology and in abstract categorical settings.

You should always *think* intuitively of the objects as sets with some sort of *structure*, and morphisms as the *set-theoretic mappings which preserve that structure* (giving rise to the usual categories of sets, monoids, groups, rings,

Φ-modules, linear Φ-algebras, etc.). However, it is important that category is *really* an *abstract* notion. A close parallel is the case of projective geometry, where we *think* of a line as a set of points, but we adopt an abstract definition where points and lines are undefined concepts, which makes available the powerful principle of duality between points and lines.

A **functor** $\mathcal{F}$: $\mathcal{C} \to \mathcal{C}'$ from one category to another takes objects to objects and morphisms to morphisms in an *arrow-preserving* manner: each map $A \xrightarrow{f} B$ of $\mathcal{C}$-objects induces a map $\mathcal{F}(A) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(B)$ of $\mathcal{C}'$-objects in a "homomorphic" manner, $\mathcal{F}(\mathbb{1}_X) = \mathbb{1}_{\mathcal{F}(X)}$, $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$. We will always think of a functor as a *construction* of $\mathcal{C}'$-objects out of $\mathcal{C}$-objects, whose recipe involves only ingredients from $\mathcal{C}$, so that a morphism of $\mathcal{C}$-objects preserves all the ingredients and therefore induces a morphism of the constructed $\mathcal{C}'$-objects.[1]

Standard examples of functors are "forgetful functors" which forget part of the structure (e.g., from algebraic objects to the underlying sets), and constructions of group algebras, polynomial rings, tensor algebras, homology groups, etc. We always have a trivial *identity functor* $\mathbb{1}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, which does nothing to objects or morphisms, and if we have functors $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$, $\mathcal{G} : \mathcal{C}' \to \mathcal{C}''$, then we can form the obvious *composite functor* $\mathcal{G} \circ \mathcal{F} : \mathcal{C} \to \mathcal{C}''$ sending objects $X \mapsto \mathcal{G}(\mathcal{F}(X))$ and morphisms $f \mapsto \mathcal{G}(\mathcal{F}(f))$. Intuitively, the composite functor constructs new objects in a two-stage process. We will frequently encounter functors which "commute," $\mathcal{F} \circ \mathcal{G} = \mathcal{G} \circ \mathcal{F}$, which means that it doesn't matter what order we perform the two constructions, the end result is the same.

1.2 The Category of Linear Algebras

There are certain concepts[2] which apply quite generally to all linear algebras, associative or not, though for algebraic systems with quadratic operations (such as quadratic Jordan algebras) they must be modified.

[1] A reader with a basic functorial literacy will observe that we deal strictly with *covariant functors,* never the dual notion of a *contravariant functor* $\mathcal{F}^*$, which acts in an *arrow-reversing* manner on maps $A \xrightarrow{f} B$ to produce $B \xleftarrow{\mathcal{F}^*(f)} A$ in an "anti-homomorphic" manner, $\mathcal{F}^*(g \circ f) = \mathcal{F}^*(f) \circ \mathcal{F}^*(g)$. Important examples of contravariant functors are the *duality* functor $V \longrightarrow V^*$ from vector spaces to vector spaces, and *cohomology* from topological spaces to groups (which Hilton and Wylie cogently, but hopelessly, argued should be called "contra-homology").

[2] Most of these basic categorical notions were introduced in Chapter 2 of the *Historical Survey*: in I.2.1 the definition of linear algebra, in I.2.2 the definitions of morphism, ideal, simple, quotient, direct sum, direct product, subdirect product, and in I.3.1 the definition of scalar extension.

Linear Algebra Definition 1.2.1 *The* **category of linear Φ-algebras** *has as objects the linear Φ-algebras and as morphisms the Φ-algebra homomorphisms. Here a* **linear Φ-algebra** *is a Φ-module* A *equipped with a bilinear multiplication or product* $\mathrm{A} \times \mathrm{A} \to \mathrm{A}$*, which we will write as* $(x, y) \mapsto x \cdot y$.[3] *Bilinearity is equivalent to the left distributive law* $(x + y) \cdot z = x \cdot z + y \cdot z$*, the right distributive law* $z \cdot (x + y) = z \cdot x + z \cdot y$*, plus the scalar condition* $\alpha(x \cdot y) = (\alpha x) \cdot y = x \cdot (\alpha y)$*. A particularly important product derived from the given one is the* **square** $x^2 := x \cdot x$*.*

Morphism Definition 1.2.2 *A* **homomorphism** $\varphi : \mathrm{A} \to \mathrm{A}'$ *between two such linear algebras is a Φ-linear map of modules which preserves multiplication,*

$$\varphi(x \cdot y) = \varphi(x) \cdot' \varphi(y).$$

In dealing with commutative algebras in the presence of $\frac{1}{2}$*, it suffices if* φ *preserves squares*:

$$\varphi \text{ homomorphism} \iff \varphi(x^2) = \varphi(x)^2 \quad (\mathrm{A} \text{ commutative}, \tfrac{1}{2} \in \Phi),$$

since then linearization gives $2\varphi(x \cdot y) = \varphi\big((x + y)^2 - x^2 - y^2\big) = \varphi(x + y)^2 - \varphi(x)^2 - \varphi(y)^2 = 2\varphi(x) \cdot' \varphi(y)$*. We have the usual notions of monomorphism and epimorphism. An* **isomorphism** *is a homomorphism that has an inverse, equivalently, that is bijective as a map of sets*; *an* **automorphism** $\varphi : \mathrm{A} \to \mathrm{A}$ *is an isomorphism of an algebra with itself. The set of automorphisms* φ *of* A *under composition forms the* **automorphism group** $\mathcal{A}ut(\mathrm{A})$*.*

The infinitesimal analogue of an automorphism is a derivation: *a* **derivation** δ *of* A *is a linear transformation* $\delta : \mathrm{A} \to \mathrm{A}$ *satisfying the* **product rule**

$$\delta(xy) = \delta(x)y + x\delta(y).$$

The set of derivations of any linear algebra A *is denoted by* $\mathcal{D}er(\mathrm{A})$*.*

Exercise 1.2.2 (1) Verify that $\mathcal{A}ut(\mathrm{A})$ is indeed a group of linear transformations, and $\mathcal{D}er(\mathrm{A})$ a Lie algebra of transformations (i.e., is closed under the Lie bracket $[D, E] = DE - ED$). (2) Show that δ is a derivation of A iff $\varphi := \mathbb{1}_{\mathrm{A}} + \varepsilon\delta$ is an automorphism of the **algebra of dual numbers** $\mathrm{A}[\varepsilon] := \mathrm{A} \otimes_\Phi \Phi[\varepsilon]$ (with defining relation $\varepsilon^2 = 0$). Thinking of ε as an "infinitesimal," this means that δ is the "infinitesimal part" of an automorphism. (3) Show that δ is a derivation of A iff $\exp(\delta) := \sum_{n=0}^{\infty} t^n \delta^n / n!$ is an automorphism of the formal power series algebra $\mathrm{A}[[t]]$ whenever this makes sense (e.g., for algebras over the rationals, or over scalars of chacteristic p when $\delta^p = 0$ is nilpotent of index $\leq p$).

[3] The product in linear algebras is usually denoted simply by juxtaposition, xy, and we will use this notation for products in associative algebras, or in alternative algebras we are using as "coordinates" for a Jordan algebra. The symbol $x \cdot y$ is a generic one-symbol-fits-all-algebras notation, but has the advantage over juxtaposition that it at least provides a place to hang a tag labeling the product, such as $x \cdot_A y$ or $x \cdot' y$ as below, reminding us that the second product is different from the first.

Ideal Definition 1.2.3 *A* **subalgebra** $\mathrm{B} \leq \mathrm{A}$ *of a linear algebra* A *is a* Φ*-submodule*[4] *closed under multiplication*: $\mathrm{B} \cdot \mathrm{B} \subseteq \mathrm{B}$. *An* **ideal** $\mathrm{K} \lhd \mathrm{A}$ *is a* Φ*-submodule closed under multiplication by* A: $\mathrm{A} \cdot \mathrm{K} + \mathrm{K} \cdot \mathrm{A} \subseteq \mathrm{K}$. *A* **left** (*respectively* **right**) **ideal** $\mathrm{B} \lhd_\ell \mathrm{A}$ (*respectively* $\mathrm{B} \lhd_r \mathrm{A}$) *is closed under left* (*respectively right*) *multiplication by* A, $\mathrm{A} \cdot \mathrm{B} \subseteq \mathrm{B}$ (*respectively* $\mathrm{B} \cdot \mathrm{A} \subseteq \mathrm{B}$).

Simple Definition 1.2.4 *A* **proper** *ideal is one different from the* **improper** *ideals* $\mathrm{A}, \mathbf{0}$. *A linear algebra is* **simple** *if it has no proper ideals and is* **non-trivial** (*i.e.*, $\mathrm{A} \cdot \mathrm{A} \neq \mathbf{0}$).

Quotient Definition 1.2.5 *Any ideal determines a* **quotient algebra** $\overline{\mathrm{A}} = \mathrm{A}/\mathrm{I}$ *and a* **canonical projection** $\pi\colon \mathrm{A} \to \overline{\mathrm{A}}$ *in the usual manner.*

Direct Sum Definition 1.2.6 *The* **direct product** $\prod_{i \in I} \mathrm{A}_i$ *of a family of algebras* A_i *is the Cartesian product under the componentwise operations, and the* **direct sum** $\boxplus_{i \in I} \mathrm{A}_i$ *is the subalgebra of all tuples with only a finite number of nonzero entries. The* **canonical projection** *onto the ith component* A_i *is denoted by* π_i. *We will usually deal with finite sums, and write algebra direct sums as* $\mathrm{A}_1 \boxplus \cdots \boxplus A_n$ *to distinguish them from module direct sums.*

An algebra is a **subdirect product** $\mathrm{A} \precapprox \prod_{i \in I} \mathrm{A}_i$ *of algebras if it is imbedded in the direct product in such a way that for each* $i \in I$ *the canonical projection* $\pi_i(\mathrm{A}) = \mathrm{A}_i$ *maps onto all of* A_i, *equivalently,* $\mathrm{A}_i \cong \mathrm{A}/\mathrm{K}_i$ *for ideals* $\mathrm{K}_i \lhd \mathrm{A}$ *with* $\bigcap_{i \in I} \mathrm{K}_i = \mathbf{0}$.

There is a standard procedure for extending the set of allowable scalars, which is frequently used to pass to an algebraically closed field where the existence of roots of equations simplifies the structure theory.

Scalar Extension Definition 1.2.7 *A* **ring of scalars** Φ *is a unital, commutative, associative ring. An* **extension** Ω *of* Φ *is a ring of scalars which is a* Φ*-algebra*; *note that it need not contain* Φ [*the image* $\Phi 1 \subset \Omega$ *need not be a faithful copy of* Φ.]

For any linear Φ*-algebra* A *and extension* Ω *of* Φ, *the* **scalar extension** A_Ω *is the tensor product* $\mathrm{A}_\Omega := \Omega \otimes_\Phi \mathrm{A}$ *with its natural* Ω*-linear structure and with the natural induced product*

$$(\alpha \otimes x) \cdot (\beta \otimes y) := \alpha\beta \otimes x \cdot y.$$

We obtain a **scalar extension functor** *from the category of* Φ*-algebras to the category of* Ω*-algebras, sending objects* A *to* A_Ω *and* Φ*-morphisms* $\varphi : \mathrm{A} \to \mathrm{A}'$ *to* Ω*-morphisms* $\varphi_\Omega = \mathbb{1}_\Omega \otimes \varphi : \mathrm{A}_\Omega \to \mathrm{A}'_\Omega$ *by*

$$\varphi_\Omega(\alpha \otimes x) = \alpha \otimes \varphi(x).$$

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[4] We have restrained our natural impulse to call these (linear) *subspaces,* since some authors reserve the term *space* for vector spaces over a field.

Although we have defined the product and the map φ_Ω only for the basic tensors $x \otimes y$ and $\alpha \otimes x$, the fact that the recipes are Φ-bilinear guarantees that it extends to a well-defined product on all sums by the universal property of tensor products.[5] Intuitively, A_Ω consists of all formal Ω-linear combinations of elements of A, multiplied in the natural way.

EXERCISE 1.2.7 If $\Phi = \mathbb{Z}$, $\Omega = \mathbb{Z}_n$ (integers mod n), $A = \mathbb{Q}$ (rationals), show that $A_\Omega = \mathbf{0}$, so $\Phi \not\subseteq \Omega$ and $A \not\subseteq A_\Omega$.

If we work in a "variety" of algebras, defined as all algebras satisfying a collection of identical relations $f_\alpha(x, y, \dots) = g_\alpha(x, y, \dots)$ (such as the associative law, commutative law, Jacobi identity, Jordan identity, alternative laws), then a scalar extension will stay within the variety only if the original algebra satisfies all "linearizations" of the defining identities. Satisfaction is automatic for homogeneous identities which are linear or quadratic in all variables, but for higher identities (such as the Jordan identity of degree 3 in x) it is automatic only if the original scalars are "big enough." (We will see that existence of $\frac{1}{2}$ is enough to guarantee that cubic identities extend.) For nonlinear identities, such as the Boolean law $x^2 = x$, scalar extensions almost never inherit the identity; indeed, if we take Ω to be the polynomial ring $\Phi[t]$ in one indeterminate, then replacing x by tx in an identical relation $f(x, y, \dots) = g(x, y, \dots)$ leads to $\sum_i t^i f^{(i)}(x; y, \dots) = \sum_i t^i g^{(i)}(x; y, \dots)$, and this relation holds in $A_{\Phi[t]}$ for all $x, y, \dots \in A$ iff (identifying coefficients of t^i on both sides) $f^{(i)}(x; y, \dots) = g^{(i)}(x; y, \dots)$ holds in A for *each homogeneous component* $f^{(i)}, g^{(i)}$ of f, g of degree i in x.

1.3 The Category of Unital Algebras

The modern fashion is to demand that algebras have unit elements (this requirement is of course waived for Lie algebras!). Once one adopts such a convention, the unit element is considered part of the structure, so one is compelled to demand that homomorphisms preserve units, and (somewhat reluctantly) that all subalgebras $B \leq A$ contain the unit: $1 \in B$. In particular, *ideals do not count as subalgebras in this category.*

[5] Somewhat rashly, we assume that the reader is familiar with the most basic properties of tensor products over general rings of scalars, in particular that the elements of a tensor product $X \otimes Y$ are *finite sums* of basic tensors $x \otimes y$ (the latter are called "indecomposable tensors" because they aren't broken down into a *sum* of terms), and that *it does not always suffice to check a property of the tensor product only on these basic generators.* The purpose in life of the tensor product of Φ-modules $X \otimes_\Phi Y$ is to convert *bilinear* maps $X \times Y \xrightarrow{f} Z$ into *linear* maps $X \otimes Y \xrightarrow{\tilde{f}} Z$ factoring through the canonical bilinear $X \times Y \longrightarrow X \otimes Y$, and its Universal Property is that it always succeeds in life: to define a linear map on the tensor product *it does always suffice to define it on the basic generators $x \otimes y$, as long as this definition is bilinear as a function of x and y.*

Unital Algebra Definition 1.3.1 *A* **unital algebra** *is one that has a* **unit element** 1*, an element which is neutral with respect to multiplication*:

$$1 \cdot x = x = x \cdot 1$$

for all elements x in A*. As always, the unit element is unique once it exists. A* **unital homomorphism** *is a homomorphism $\varphi : \mathrm{A} \to \mathrm{A}'$ of linear algebras which preserves the unit element,*

$$\varphi(1) = 1'.$$

The **category of unital linear Φ-algebras** *has as its objects the unital linear Φ-algebras, and as its morphisms the unital homomorphisms.*[6]

In dealing with unital algebras one must be careful which category one is working in. If A and A′ are unital linear algebras and $\varphi : \mathrm{A} \to \mathrm{A}'$ is a homomorphism in the category of linear algebras, then in the category of unital algebras it need not be a homomorphism, and its image need not be a subalgebra. For example, we cannot think of the 2×2 matrices as a unital subring of the 3×3 matrices, nor is the natural imbedding of 2×2 matrices as the northwest corner of 3×3 matrices a homomorphism from the unital point of view.

Exercise 1.3.1* (1) Verify that the northwest corner imbedding $\mathcal{M}_2(\Phi) \hookrightarrow \mathcal{M}_3(\Phi)$ is always a homomorphism of associative algebras. (2) Use traces to show that if Φ is a field, there is no homomorphism $\mathcal{M}_2(\Phi) \to \mathcal{M}_3(\Phi)$ of unital associative algebras.

Notice that in this category an algebra A is simple iff it has no proper ideals and is nonzero, because nonzero guarantees nontrivial: $\mathrm{AA} \supseteq \mathrm{A}1 = \mathrm{A} \neq 0$.

1.4 Unitalization

A unit element is a nice thing to have around, and by federal law each algebra is entitled to a unit: as in associative algebras, there is a standard way of formally adjoining a unit element if you don't have one already,[7] although this right to a free unit should not be abused by algebras which already have one.

[6] Note that again the notation 1 for the unit is a generic one-term-fits-all-algebras notation; if we wish to be pedantic, or clear ("Will the real unit please stand up?"), we write 1_A to make clear *whose* unit it is. Note in the homomorphism condition how 1′ reminds us that this is not the same as the previous 1. The unit element is often called the *identity element.* We will try to avoid this term, since we often talk about an algebra "with an identity" in the sense of "identical relation" (the Jacobi identity, the Jordan identity, etc.). Of course, in commutative associative rings the term "unit" also is ambiguous, usually meaning "invertible element" (the group of units, etc.), but already in noncommutative ring theory the term is not used this way, and we prefer this lesser of two ambiguities. It is a good idea to think of the unit as a *neutral element* for multiplication, just as 0 is the neutral element for addition.

[7] cf. the definition of unital hull in Section I.2.1.

Unital Hull Definition 1.4.1 *In general a* **unital hull** *of a linear* Φ*-algebra* A *is any extension* $\mathrm{A}^1 = \Phi 1 + \mathrm{A} \supseteq \mathrm{A}$ *obtained by adjoining a unit element. Such a hull always contains* A *as an ideal* (*hence a non-unital subalgebra*). *Any linear algebra is imbedded in its* **(formal) unital hull**, *consisting of the module* $\Phi \oplus \mathrm{A}$ *with product extending that of* A *and having* $(1,0)$ *act as unit,*

$$\widehat{\mathrm{A}}: \quad (\alpha, x)\cdot(\beta, y) = (\alpha\beta, \alpha y + \beta x + x \cdot y).$$

We call $\widehat{\mathrm{A}}$ **the** *unital hull and denote the elements* $(1,0), (0,a)$ *simply by* $\hat{1}, a$, *so* $\Phi\hat{1}$ *is a faithful copy of* Φ, *and we write the product as*

$$(\alpha\hat{1} \oplus x)\cdot(\beta\hat{1} \oplus y) = \alpha\beta\hat{1} \oplus (\alpha y + \beta x + x \cdot y).$$

Notice that this unitalization process depends on what we are counting as the ring of scalars, so we could denote it by $\widehat{\mathrm{A}}^{\Phi}$ to explicitly include the scalars in the notation — it would remind us exactly what we are tacking on to the original A. If A is also an algebra over a larger Ω, then it would be natural to form $\widehat{\mathrm{A}}^{\Omega}$, so the unital version would also be an Ω-algebra. But we will usually stick to the one-size-fits-all hat for the unital hull, trusting the context to make clear which unitalization we are performing.

We obtain a **unitalization functor** from the category of linear Φ-algebras to unital Φ-algebras, sending objects A to $\widehat{\mathrm{A}}$, and morphisms $\varphi : \mathrm{A} \to \mathrm{A}'$ to the unital morphism $\widehat{\varphi} : \widehat{\mathrm{A}} \to \widehat{\mathrm{A}'}$ defined as the natural unital extension $\widehat{\varphi}(\alpha\hat{1} \oplus x) = \alpha\hat{1} \oplus \varphi(x)$. It is easy to verify that this functor commutes[8] with scalar extensions: $\widehat{(\mathrm{A}_\Omega)}^{\Omega} = (\widehat{\mathrm{A}}^{\Phi})_\Omega$.

Notice that you can have too much of a good thing: if A already has a unit 1, tacking on a formal unit $\hat{1}$ will demote the old unit to a mere idempotent, and the new algebra will have a copy of Φ attached as an excrescence rather than an organic outgrowth:

$$\widehat{\mathrm{A}} = \mathrm{A} \boxplus \Phi e \qquad (e := \hat{1} - 1, \ \ 1 \textit{ unit for } \mathrm{A}).$$

Indeed, orthogonality $e\mathrm{A} = 0 = \mathrm{A}e$ is built in, since $\hat{1}$ acts as unit by decree, and 1 already acted as unit on A, so the two competing claimants for unit cancel each other out; we have $(e)^2 = \hat{1} - 2 + 1^2 = \hat{1} - 2 + 1 = e$, so Φe is another faithful copy of Φ. See Problem 1.3 for a tighter unitalization.

[8] The alert reader — so annoying to an author — will notice that we don't really have two functors $\mathcal{U}, \Omega$ which *commute*, but rather a whole family of unitalization functors $\mathcal{U}_\Phi$ (one for each category of Φ-algebras) and two scalar extension functors $\Omega_\Phi, \Omega_\Phi^{(1)}$ (one for linear algebras and one for unital algebras), which *intertwine*: $\Omega_\Phi^{(1)} \circ \mathcal{U}_\Phi = \mathcal{U}_\Omega \circ \Omega_\Phi$. By abuse of language we will continue to talk of these functors or constructions as commuting.

1.5 The Category of Algebras with Involution

The story of Jordan algebras is to some extent the saga of self-adjoint elements in algebras with involutions, just as the story of Lie algebras is the story of skew elements.[9]

$*$-Algebra Definition 1.5.1 *A linear* **$*$-algebra** (**star**-*algebra, or* **algebra with involution**) $(\mathrm{A}, *)$ *consists of a linear algebra* A *together with an* **involution** $*$, *an anti-automorphism of period* 2: *a linear mapping of* A *to itself satisfying*

$$(x \cdot y)^* = y^* \cdot x^*, \qquad (x^*)^* = x.$$

A **$*$-homomorphism** $(\mathrm{A}, *) \to (\mathrm{A}', *')$ *of $*$-algebras is an algebra homomorphism φ which preserves involutions,*

$$\varphi(x^*) = (\varphi(x))^{*\prime}.$$

In any $*$-algebra we denote by $\mathcal{H}(\mathrm{A}, *)$ the space of **hermitian** *elements $x^* = x$, and by $\mathcal{S}kew(\mathrm{A}, *)$ the space of* **skew** *elements $x^* = -x$. Any $*$-homomorphism will preserve symmetric or skew elements.*

There are two general recipes, **norm** $n(x)$ *and* **trace** $t(x)$, *for constructing a hermitian element out of an arbitrary element x:*

$$n(x) := x \cdot x^*, \qquad t(x) := x + x^*.$$

Any $*$-homomorphism will preserve norms and traces. In the presence of $\frac{1}{2}$, every hermitian element $x = x^*$ is a trace, $x = \frac{1}{2}(x + x^*) = t(\frac{1}{2}x)$, but in characteristic 2 *situations the traces $t(\mathrm{A})$ may form a proper subset of $\mathcal{H}(\mathrm{A}, *)$.*

*The category of Φ-$*$-algebras has as objects the $*$-algebras and as morphisms the $*$-homomorphisms. Their kernels are precisely the $*$-***ideals** (*ideals* $\mathrm{I} \triangleleft \mathrm{A}$ *invariant under the involution, hence satisfying* $\mathrm{I}^* = \mathrm{I}$). *An algebra is* **$*$-simple** *if it is not trivial and has no proper $*$-ideals.*

Exercise 1.5.1A Show that if $x^* = \epsilon x, y^* = \eta y$ $(\epsilon, \eta = \pm 1)$ for an involution $*$ in a nonassociative algebra A, then $\{x, y\}^* = \epsilon\eta\{x, y\}$ and $[x, y]^* = -\epsilon\eta[x, y]$, so in particular $\mathcal{H}(\mathrm{A}, *)$ is always closed under $\{x, y\}$ and $\mathcal{S}kew(\mathrm{A}, *)$ under $[x, y]$ (though for general nonassociative A these won't be Jordan or Lie algebras).

Exercise 1.5.1B (1) Show that a *trivial* linear algebra (one with trivial multiplication where all products are zero) has no proper ideals iff it is a 1-dimensional vector space with trivial multiplication over a field $\Omega = \Phi/M$ for a maximal ideal M of Φ. (2) Show that a *trivial* linear $*$-algebra has no proper $*$-ideals iff it is a 1-dimensional vector space with trivial multiplication and trivial involution ($* = \pm \mathbb{1}_{\mathrm{A}}$) over a field $\Omega = \Phi/M$ for a maximal ideal M of Φ.

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[9] cf. the definition of $*$-Algebra in Section I.2.2

Though not written into the constitution, there is a little-known federal program to endow needy algebras with a $*$, using the useful concept of *opposite algebra.*

Opposite Algebra Definition 1.5.2 *The* **opposite algebra** A^{op} *of any linear algebra* A *is just the module* A *with the multiplication turned around*:

$$\mathrm{A}^{op} := \mathrm{A} \ \textit{under} \ x \cdot^{op} y := y \cdot x.$$

In these terms an anti-homomorphism $\mathrm{A} \to \mathrm{A}'$ *is nothing but a homomorphism into* (*or out of*) *the opposite algebra* $\mathrm{A} \to \mathrm{A}'^{op}$ (*or* $\mathrm{A}^{op} \to \mathrm{A}'$).

EXERCISE 1.5.2 Verify the assertion that an anti-homomorphism in nothing but a homomorphism into or out of the opposite algebra.

Exchange Involution Proposition 1.5.3 *Every linear algebra* A *can be imbedded as a subalgebra of a $*$-algebra, its* **exchange algebra** *with* **exchange involution**

$$\mathcal{E}x(\mathrm{A}) := (\mathrm{A} \boxplus \mathrm{A}^{op}, ex), \quad \textit{where} \quad ex(a,b) := (b,a).$$

The hermitian and skew elements under the exchange involution are isomorphic to A *as modules, but not as linear algebras*:

$$\mathcal{H}(\mathcal{E}x(\mathrm{A})) = \{(a,a) \mid a \in \mathrm{A}\},$$
$$\mathcal{S}kew(\mathcal{E}x(\mathrm{A})) = \{(a,-a) \mid a \in \mathrm{A}\}.$$

PROOF. The map $a \mapsto (a,0)$ certainly imbeds A as a non-$*$-invariant subalgebra of $\mathcal{E}x(\mathrm{A})$, and the hermitian and skew elements are as advertised. The map $a \mapsto (a,a)$ is an isomorphism of A^+ with $\mathcal{H}(\mathcal{E}x(\mathrm{A}))$ under the bullet products since $(a,a) \bullet (b,b) = \frac{1}{2}(ab+ba, ba+ab) = (a \bullet b, b \bullet a) = (a \bullet b, a \bullet b)$, and analogously for the Lie brackets $[(a,-a),(b,-b)] = ([a,b],[-b,-a]) = ([a,b],-[a,b])$. $\square$

In this category we again have unitalization and scalar extension functors, which commute with the **exchange functor** from algebras to $*$-algebras sending objects $\mathrm{A} \mapsto \mathcal{E}x(\mathrm{A})$ and morphisms $\varphi \mapsto \mathcal{E}x(\varphi)$, where the $*$-homomorphism is defined by $\mathcal{E}x(\varphi)(a,b) := (\varphi(a), \varphi(b))$.

The notion of $*$-simplicity is not far removed from ordinary simplicity: the only $*$-simple algebra which isn't already simple is an exchange clone of a simple algebra.

$*$-Simple Theorem 1.5.4 *If* $(\mathrm{A},*)$ *is a $*$-simple linear algebra, then either* (1) A *is simple, or* (2) A *is the direct sum* $\mathrm{A} = \mathrm{B} \boxplus \mathrm{B}^*$ *of an ideal* B *and its star, in which case* $(\mathrm{A},*) \cong \mathcal{E}x(\mathrm{B}) = (\mathrm{B} \boxplus \mathrm{B}^{op}, ex)$ *for a simple algebra* B *with exchange involution* $ex(b,c) = (c,b)$. *In the latter case* $\mathcal{H}(\mathrm{A},*) \cong \mathrm{B}$, *so if the nonzero hermitian elements all associate and have inverses, then* B *is an associative division algebra.*

PROOF. If A is NOT simple it has a proper ideal B; since $B \cap B^* < A$ and $B + B^* > \mathbf{0}$ and both are $*$-ideals, by $*$-simplicity they must be $\mathbf{0}$ and A respectively: $B \cap B^* = \mathbf{0}$, $B + B^* = A$, so $(A, *) = (B \boxplus B^*, *) \cong (B \boxplus B^{op}, ex)$ [since $B^* \cong B^{op}$ for any involution], and $* \cong ex$ because $(x \boxplus y^*)^* = y \boxplus x^*$. Here B is simple as an algebra in its own right: if $K \lhd B$ were a proper ideal, then $K \boxplus K^* \lhd A$ would be a proper $*$-ideal, contrary to $*$-simplicity. If the nonzero hermitian elements (b, b) of $(B \boxplus B^{op}, ex)$ all associate (or are invertible), then all nonzero b in B associate (or are invertible), i.e., B is associative (or a division algebra). □

1.6 Nucleus, Center, and Centroid

We recall the concepts of nucleus and center in general,[10] which are most conveniently defined using the concepts of **commutator** and **associator**, defined in any linear algebra by

$$[x, y] := x \cdot y - y \cdot x, \quad [x, y, z] := (x \cdot y) \cdot z - x \cdot (y \cdot z).$$

The commutator is a bilinear mapping from $A \times A$ to A, measuring how far the two elements are from commuting with each other; the associator is a trilinear mapping measuring how far the three elements are from associating (in the given order) with each other.

Nucleus and Center Definition 1.6.1 *The* **nucleus** *of a linear algebra is the gregarious part of the algebra, the part that associates with everyone, consisting of the elements associating in all possible ways with all other elements*:

$$\mathcal{N}uc(A) := \{n \in A \mid [n, A, A] = [A, n, A] = [A, A, n] = \mathbf{0}\}.$$

The **center** *of any linear algebra is the "scalar" part of the algebra which both commutes and associates with everyone, i.e., those nuclear elements commuting with all other elements*:

$$\mathcal{C}ent(A) := \{c \in \mathcal{N}uc(A) \mid [c, A] = \mathbf{0}\}.$$

A unital linear algebra can always be considered as an algebra over its center $\Omega := \mathcal{C}ent(A) \supseteq \Phi 1$. *A unital* Φ*-algebra is* **central** *if its center is precisely the scalar multiples* $\Phi 1$. **Central-simple** *algebras* (*those that are both central and simple*) *are the basic building-blocks of finite-dimensional structure theory.*

In the category of $*$*-algebras, the natural notion of center is the* $*$-**center**, *defined to be the set of central elements fixed by* $*$, $\mathcal{C}ent(A, *) := \{c \in \mathcal{C}ent(A) \mid c^* = c\}$.

[10] cf. the definitions of commutator, associator, nucleus, and center in Section I.2.3.

Exercise 1.6.1A (1) Show that $\mathcal{N}uc(\mathrm{A})$ is always an associative subalgebra of A, and $\mathcal{C}ent(\mathrm{A})$ a commutative associative subalgebra of A; show that both are invariant (as sets, not necessarily pointwise) under all automorphisms and involutions of A. (2) If A has unit, show that $\alpha 1$ lies in the center for all $\alpha \in \Phi$. (3) Show that if A is unital, then its center $\Omega = \mathcal{C}ent(\mathrm{A})$ is a ring of scalars, and A can be considered as a linear algebra over Ω. (4) Show that if A is a simple unital linear algebra, then its center is a field.

Exercise 1.6.1*B Show that in the definition of center, the condition $[\mathrm{A}, c, \mathrm{A}] = 0$ can be omitted: it follows from the other associator conditions in the presence of commutativity.

Another important concept for arbitrary linear algebras is that of the *centroid*, the natural "scalar multiplications" for an algebra.

Centroid Definition 1.6.2 *The* **centroid** $\Gamma(\mathrm{A})$ *of a linear* Φ*-algebra* A *is the set of all linear transformations* $T \in \mathcal{E}nd_\Phi(\mathrm{A})$ *which act as scalars with respect to multiplication,*

$$\text{(CT)} \qquad T(xy) = T(x)y = xT(y) \text{ for all } x, y \in \mathrm{A},$$

equivalently, which commute with all (left and right) multiplications,

$$\text{(CT}'\text{)} \qquad TL_x = L_xT, \quad TR_y = R_yT \text{ for all } x, y \in \mathrm{A},$$

or yet again which centralize the **multiplication algebra** $\mathcal{M}ult(\mathrm{A}) = \langle L_\mathrm{A} \cup R_\mathrm{A} \rangle$ *generated by all left and right multiplications,*

$$\text{(CT}''\text{)} \qquad \Gamma(\mathrm{A}) = \mathcal{C}ent_{\mathcal{E}nd(\mathrm{A})}(\mathcal{M}ult(\mathrm{A})),$$

where for any subset $\mathcal{S}$ *of* $\mathcal{E}nd_\Phi(\mathrm{A})$ *the* **centralizer** *or* **commuting ring** *is* $\mathcal{C}ent_{\mathcal{E}nd(\mathrm{A})}(\mathcal{S}) := \{T \in \mathcal{E}nd_\Phi(\mathrm{A}) \mid TS = ST \text{ for all } S \in \mathcal{S}\}$. *We say that a* Φ*-algebra is* **centroidal** *if* $\Gamma(\mathrm{A}) = \Phi$, *i.e.,* $\Gamma(\mathrm{A}) = \Phi \mathbb{1}_\mathrm{A}$ *and the natural homomorphism* $\Phi \to \Phi \mathbb{1}_\mathrm{A}$ *(via* $\alpha \mapsto \alpha \mathbb{1}_\mathrm{A}$*) is an isomorphism. An algebra is* **centroid-simple** *if it is simple and centroidal.*

Notice that if A is trivial (all products vanish) then $L_x = R_y = 0$, and $\Gamma(\mathrm{A}) = \mathcal{E}nd(\mathrm{A})$ is highly noncommutative. In order for Γ to form a commutative ring of scalars, we need some mild nondegeneracy conditions on A.

Centroid Theorem 1.6.3 (1) *The centroid* $\Gamma(\mathrm{A})$ *of a linear* Φ*-algebra* A *is always an inverse-closed unital associative* Φ*-subalgebra of* $\mathcal{E}nd_\Phi(\mathrm{A})$. *If* $\mathrm{A}^2 = \mathrm{A}$, *or if* $\mathrm{A}^\perp = 0$ *(e.g., if* A *is simple or semiprime or unital), then* $\Gamma(\mathrm{A})$ *is a unital commutative ring of scalars* Ω, *and* A *is in a natural way an* Ω*-algebra via* $\omega \cdot x = \omega(x)$.

(2) *If* A *is unital, the centroid is essentially the same as the center: the center is just the centroidal action on* 1, *and the centroid is just the central multiplications:*

$$\mathcal{C}ent(\mathrm{A}) = \Gamma(\mathrm{A})(1), \qquad \Gamma(\mathrm{A}) = L_{\mathcal{C}ent(\mathrm{A})} = R_{\mathcal{C}ent(\mathrm{A})};$$

and the maps $T \mapsto T(1)$ *and* $c \mapsto L_c = R_c$ *are inverse isomorphisms between the unital commutative rings* $\Gamma(\mathrm{A})$ *and* $\mathcal{C}ent(\mathrm{A})$*.*

(3) *If* A *is simple, then* $\Gamma(\mathrm{A})$ *is a field* Ω*, and* A *becomes a centroid-simple algebra over* Ω*. If* A *is merely prime* (*has no nonzero ideals* I, J *with* $\mathrm{I} \cdot \mathrm{J} = \mathbf{0}$), *then* $\Gamma(\mathrm{A})$ *is an integral domain acting faithfully on* A*.*

PROOF. (1) The centralizer $\mathcal{C}ent_{\mathcal{E}nd(\mathrm{A})}(\mathcal{S})$ of any set $\mathcal{S}$ is an inverse-closed unital subalgebra of $\mathcal{E}nd_\Phi(\mathrm{A})$, i.e., is closed under $\mathbb{1}_\mathrm{A}, \alpha T, T_1 + T_2, T_1 T_2$, and T^{-1}. [Note that whenever a centralizing T has an inverse $T^{-1} \in \mathcal{E}nd(\mathrm{A})$, this inverse is also centralizing, $T^{-1}S = T^{-1}S(TT^{-1}) = T^{-1}(ST)T^{-1} = T^{-1}(TS)T^{-1} = ST^{-1}$ for all $S \in \mathcal{S}$.] The centroid $\Gamma(\mathrm{A})$ is just the particular case where $\mathcal{S}$ is $L_\mathrm{A} \cup R_\mathrm{A}$ or $\mathcal{M}ult(\mathrm{A})$.

The kernel of $[T_1, T_2]$ always contains A^2 (so if $\mathrm{A}^2 = \mathrm{A}$ then $[T_1, T_2] = 0$ and we have commutativity) because of the following *Hiding Trick*:

$$T_1 T_2(xy) = T_1(T_2(x)y) = T_2(x)T_1(y) = T_2(xT_1(y)) = T_2 T_1(xy).$$

The range of $[T_1, T_2]$ is always contained in $\mathrm{A}^\perp$ (so if $\mathrm{A}^\perp = \mathbf{0}$ then $[T_1, T_2](\mathrm{A}) = \mathbf{0}$ and again we have commutativity $[T_1, T_2] = 0$) because by the above hiding trick

$$\mathbf{0} = [T_1, T_2](\mathrm{AA}) = [T_1, T_2](\mathrm{A}) \cdot \mathrm{A} = \mathrm{A} \cdot [T_1, T_2](\mathrm{A}) = \mathbf{0}.$$

(2) If A is unital, we claim that any centroidal T has the form $T = L_c = R_c$ for a central element $c = T(1)$. It has the form $T = L_c$ because $T(x) = T(1x) = T(1)x = L_c x$, and similarly $T = R_c$ because $T(x) = T(x1) = xT(1) = R_c x$. The element $c = T(1)$ commutes with any x because $[c, x] = [T(1), x] = T([1, x]) = 0$). It also associates with any x, y because $[c, x, y] = [T(1), x, y] = T([1, x, y]) = 0$ (similarly, $[x, c, y] = [x, y, c] = 0$). Therefore it commutes and associates, hence is central.

Conversely, if $c \in \mathcal{C}ent(\mathrm{A})$ is a central element, then the linear operator $T = L_c = R_c$ is centroidal by $T(xy) = c(xy) = (cx)y = T(x)y$ and dually $T(xy) = (xy)c = x(yc) = xT(y)$. These correspondences between centroidal T and central c are inverses since $T \mapsto T(1) = c \mapsto L_c = T$ and $c \mapsto L_c = T \mapsto T(1) = L_c(1) = c$. These maps are actually homomorphisms of unital commutative associative rings since $T_1 T_2(1) = T_1 T_2(1 \cdot 1) = T_1(1) T_2(1)$ and $L_{c_1 c_2} = L_{c_1} L_{c_2}$.

(3) If A is simple or prime, then $\mathrm{A}^\perp = 0$, and Γ is commutative by (1). The kernel $\mathrm{Ker}(T)$ and image $\mathrm{Im}(T)$ of a centroidal T are always ideals [invariant under any multiplication $M \in \mathcal{M}ult(\mathrm{A})$, since $T(x) = 0$ implies that $T(M(x)) = M(T(x)) = 0$ and $M(T(\mathrm{A})) = T(M(\mathrm{A})) \subseteq T(\mathrm{A})$]. If $T \neq 0$ then $\mathrm{Ker}(T) \neq \mathrm{A}$ and $\mathrm{Im}(T) \neq \mathbf{0}$, so when A is simple $\mathrm{Ker}(T) = \mathbf{0}$ [T is injective] and $\mathrm{Im}(T) = \mathrm{A}$ [T is surjective], and any nonzero T is bijective. Therefore T has an inverse $T^{-1} \in \mathcal{E}nd(\mathrm{A})$, which by inverse-closure is also centroidal, so in a simple algebra the centroid is a field.

If A is merely prime, then $\Gamma(\mathrm{A})$ is a domain, since if nonzero centroidal T_i kill each other their nonzero image ideals $\mathrm{Im}(T_i)$ are orthogonal, $T_1(\mathrm{A})T_2(\mathrm{A}) = T_1T_2(\mathrm{AA}) = \mathbf{0}$, a contradiction. Each nonzero T is faithful (injective) on A, since its kernel is orthogonal to its nonzero image: $T(\mathrm{A})\cdot \mathrm{Ker}(T) = \mathrm{A}\cdot T(\mathrm{Ker}(T)) = \mathbf{0}$. □

In the unital case centroid and center are pretty indistinguishable, and especially in the theory of finite-dimensional algebras ones talks of *central-simple* algebras, but in general the centroid and centroid-simplicity are the natural concepts. Unital algebras always have a nonzero center $\mathcal{C}ent(\mathrm{A}) \supseteq \Phi 1$, but non-unital algebras (even simple ones) often have no center at all. For example, the algebra $\mathcal{M}_\infty(\Delta)$ of all $\infty\times\infty$ matrices having only a finite number of nonzero entries from an associative division algebra Δ is simple, with centroid $\Omega = \mathcal{C}ent(\Delta)\mathbb{1}_{\mathrm{A}}$, but has no central elements: if the matrix $C = \sum \gamma_{ij}E_{ij}$ were central, then it would be diagonal with all diagonal entries equal, contrary to finiteness $[\gamma_{ij}E_{ij} = E_{ii}CE_{jj} = E_{ii}E_{jj}C = 0$ if $i \neq j$, $\gamma_{ii}E_{ii} = E_{ij}E_{ji}CE_{ii} = E_{ij}CE_{ji}E_{ii} = E_{ij}(\gamma_{jj}E_{jj})E_{ji} = \gamma_{jj}E_{ij}E_{jj}E_{ji} = \gamma_{jj}E_{ii}]$.

Exercise 1.6.3* Let M be an arbitrary left R-module for an associative Φ-algebra R (not necessarily unital or commutative). (1) Show that if $\mathcal{S} \subseteq \mathcal{E}nd_\Phi(M)$ is any set of Φ-linear transformations on M, then $\mathcal{C}_M(\mathcal{S}) = \{T \in \mathcal{E}nd_\Phi(M) \mid TS = ST \text{ for all } S \in \mathcal{S}\}$ is a unital subalgebra of $\mathcal{E}nd_\Phi(M)$ which is *inverse-closed* and *quasi-inverse-closed*: if $T \in \mathcal{C}_M(\mathcal{S})$ is invertible or quasi-invertible, then its inverse or quasi-inverse again belongs to $\mathcal{C}_M(\mathcal{S})$. (2) Prove *Schur's Lemma*: if M has no proper $\mathcal{S}$-invariant subspaces, then $\mathcal{C}_M(\mathcal{S})$ is a division algebra. Deduce the usual version of Schur's Lemma: if M is an irreducible R-module ($\mathrm{R}\cdot M \neq \mathbf{0}$ and M has no proper R-submodules), then $\mathcal{C}_M(\mathrm{R})$ is a division ring. (3) If A is a linear algebra and $\mathrm{R} = \mathcal{M}ult(\mathrm{A})$, show that $\mathcal{C}_M(\mathrm{R}) = \Gamma(\mathrm{A})$, and A is simple iff $M = \mathrm{A}$ is an irreducible R-module.

1.7 Strict Simplicity

A Φ-algebra A is **strictly simple** over the field Φ if all its scalar extensions $\Omega \otimes_\Phi \mathrm{A}$ remain simple ($\Omega \supseteq \Phi$ a field extension). This useful invariance of simplicity under scalar extensions turns out to be equivalent to centroid-simplicity.

Strict Simplicity Theorem 1.7.1 *A simple linear Φ-algebra* A *is centroid-simple over a field Φ iff it is strictly simple.*

PROOF. If a simple A is *not* central, then $\Gamma(\mathrm{A}) = \Omega > \Phi$, and the scalar extension $\widetilde{\mathrm{A}} = \Omega \otimes_\Phi \mathrm{A}$ is not simple because it has a non-injective homomorphism $\varphi : \widetilde{\mathrm{A}} \to \mathrm{A}$ of Ω-algebras via $\varphi(\omega \otimes a) = \omega(a)$. Indeed, this does define an map on the tensor product by the universal property, since the expression is Φ-bilinear $[\varphi(\alpha\omega \otimes a) = \alpha\omega(a) = \omega(\alpha a) = \varphi(\omega \otimes \alpha a)]$. The map is Ω-linear since $\varphi(\omega_0 \cdot \omega \otimes a) = \varphi(\omega_0\omega \otimes a) = (\omega_0\omega)(a) = \omega_0(\omega(a)) = \omega_0\varphi(\omega \otimes a)$. It is an algebra homomorphism since $\varphi((\omega_1 \otimes a_1)(\omega_2 \otimes a_2)) = \varphi(\omega_1\omega_2 \otimes a_1a_2) = (\omega_1\omega_2)(a_1a_2) = (\omega_1(a_1))(\omega_2(a_2)) = \varphi(\omega_1 \otimes a_1)\varphi(\omega_2 \otimes a_2)$. Choosing a basis $\{\omega_i\}$ for Ω as a vector space of dimension > 1 over Φ, we know that $\Omega \otimes \mathrm{A} = \bigoplus_i \omega_i \otimes \mathrm{A}$, so by independence $\omega_1 \otimes \omega_2(a) - \omega_2 \otimes \omega_1(a)$ is nonzero if $a \neq 0$, yet it lies in the kernel of φ since $\omega_1(\omega_2(a)) - \omega_2(\omega_1(a)) = 0$.

More difficult is the converse, that if A *is* centroid-simple, then $\widetilde{\mathrm{A}}$ is also simple. Certainly $\widetilde{\mathrm{A}}^2 = \Omega\Omega \otimes \mathrm{AA} \neq \mathbf{0}$; we must show that that $\widetilde{\mathrm{A}}$ has no proper ideals. We claim that it will suffices to show that any nonzero ideal $\widetilde{\mathrm{I}} \triangleleft \widetilde{\mathrm{A}}$ contains an "irreducible tensor,"

$$(1) \qquad 0 \neq \omega \otimes x \in \widetilde{\mathrm{I}} \qquad (0 \neq \omega \in \Omega,\ 0 \neq x \in \mathrm{A}).$$

From such an element $\omega \otimes x \neq 0$ we can create the entire algebra $\widetilde{\mathrm{A}}$ in two easy steps. First, we use x to create A: $\mathcal{M}ult(\mathrm{A})(x)$ is an ideal containing $0 \neq x = \mathbb{1}_{\mathrm{A}}(x)$, so the ideal must be all of A by simplicity, and we have

$$(2) \qquad x \neq 0 \text{ in } \mathrm{A} \Longrightarrow \mathcal{M}ult(\mathrm{A})(x) = \mathrm{A}.$$

Second, we use the multiplication promised by (2) and the scalar ω^{-1} to create all "monomials" $\widetilde{\omega} \otimes a$: by (2) there is an M with $M(x) = a$, hence $\widetilde{\omega} \otimes a = (\widetilde{\omega}\omega^{-1} \otimes M)(\omega \otimes x) \in \mathcal{M}ult(\widetilde{\mathrm{A}})(\widetilde{\mathrm{I}}) \subseteq \widetilde{\mathrm{I}}$, and we have

$$(3) \qquad 0 \neq \omega \otimes x \in \widetilde{\mathrm{I}} \Longrightarrow \widetilde{\omega} \otimes a \in \widetilde{\mathrm{I}}.$$

Adding these all up shows that $\widetilde{\Omega} \otimes \mathrm{A} \subseteq \widetilde{\mathrm{I}}$, and thus that $\widetilde{\mathrm{I}} = \widetilde{\mathrm{A}}$.

Thus, in order to prove that the only nonzero ideal $\widetilde{\mathrm{I}}$ is $\widetilde{\mathrm{I}} = \widetilde{\mathrm{A}}$, it will suffice to prove (1). If we are willing to bring in the big gun, the Jacobson Density Theorem (cf. Appendix D), we can smash the obstacle (1) to smithereens. By simplicity $\mathcal{M}ult(\mathrm{A})$ acts irreducibly on A with centralizer $\Gamma(\mathrm{A}) = \Phi$ by centroid-simplicity, so if $\{x_\sigma\}_{\sigma\in S}$ is a basis for A over Φ then by tensor product facts $\widetilde{\mathrm{A}} = \bigoplus_{\sigma\in S} \Omega \otimes x_\sigma$. If $0 \neq \sum_{i=1}^n \omega_i \otimes x_i \in \widetilde{\mathrm{I}}$ is an element of minimal length n, then by the Density Theorem there is $M \in \mathcal{M}ult(\mathrm{A})$ with $M(x_1) = x_1$, $M(x_i) = 0$ for $i \neq 1$, so instantly $(1 \otimes M)(\sum \omega_i \otimes x_i) = \sum \omega_i \otimes M(x_i) = \omega_1 \otimes x_1 \neq 0$ lies in $\widetilde{\mathrm{I}}$ as desired in (1).

We can also chip away at the obstacle (1) one term at a time using elements of the centroid. The most efficient method is another *minimal length argument.* Let $\{\omega_\sigma\}_{\sigma\in S}$ be a basis for Ω over Φ, so by tensor product facts $\widetilde{\mathrm{A}} = \bigoplus_\sigma \omega_\sigma \otimes \mathrm{A}$. Assume that $0 \neq \sum \omega_i \otimes x_i \in \widetilde{\mathrm{I}}$ is an expression of minimal length. This minimality guarantees a *domino effect*: if $N \in \mathcal{M}ult(\mathrm{A})$ kills x_1 it kills every x_i, since $(1 \otimes N)(\sum \omega_i \otimes x_i) = \sum \omega_i \otimes N(x_i) \in \widetilde{\mathrm{I}}$ is of shorter length if $N(x_1) = 0$, therefore by minimality must be zero, which by independence of the ω_i implies that each $N(x_i) = 0$. By (2) we can write any $a \in \mathrm{A}$ as $a = M(x_1)$ for some $M \in \mathcal{M}ult(\mathrm{A})$; we use this M to define *centroidal* linear transformations T_i by $T_i(a) = M(x_i)$. These are well-defined by dominos:

$a = M(x_1) = M'(x_1) \Longrightarrow N = M - M'$ has $N(x_1) = 0$, hence $N(x_i) = 0$ and $M(x_i) = M'(x_i)$ is independent of the choice of the M which produces a. Then clearly the T_i are linear: if M, M' produce a, a', then $\alpha M + \alpha' M'$ produces $\alpha a + \alpha' a'$ and $T_i(\alpha a + \alpha' a') = (\alpha M + \alpha' M')(x_i) = \alpha T_i(a) + \alpha' T_i(a')$. The T_i commute with all multiplications $P \in \mathcal{M}ult(\mathrm{A})$: PM produces $P(a)$, so $T_i(P(a)) = (PM)(x_i) = P(M(x_i)) = P(T_i(a))$. Thus $T_i = \gamma_i \in \Gamma = \Phi$ by centroid-simplicity. Here $M = \mathbb{1}_{\mathrm{A}}$ produces $a = x_1$, so $T_i(x_1) = T_i(a) = M(x_i) = \mathbb{1}_{\mathrm{A}}(x_i) = x_i$ and $\gamma_i x_1 = x_i$ is a scalar multiple of x_1. But then our minimal length expression takes the form $0 \neq \sum \omega_i \otimes \gamma_i x_1 = \sum \omega_i \gamma_i \otimes x_1$ [all γ_i lie in Φ and we are tensoring over Φ] $= (\sum \omega_i \gamma_i) \otimes x_1 = \omega \otimes x_1 \in \widetilde{\mathrm{I}}$ as desired in (1).

Thus either way we have our irreducible tensor (1), which suffices to show that $\widetilde{\mathrm{I}} = \widetilde{\mathrm{A}}$ and thus the simplicity of the scalar extension $\widetilde{\mathrm{A}} = \Omega \otimes \mathrm{A}$. □

EXERCISE 1.7.1* In unital algebras the minimum length argument is usually run through the center. Let A be central simple over $\Gamma(\mathrm{A}) = \Phi$, and $0 \neq \sum \omega_i \otimes x_i \in \widetilde{\mathrm{I}}$ be an expression of minimal length ($\{\omega_\sigma\}$ a basis for Ω over Φ). Use (2) of Strict Simplicity to get a multiplication $M \in \mathcal{M}ult(\mathrm{A})$ such that $M(x_1) = 1$ (but no assumption about its effect on the other $M(x_i)$). Apply $1 \otimes M$ to the relation to normalize it so that $x_1 = 1$. Then apply $L_a - R_a, L_{ab} - L_a L_b, [L_a, R_b], R_{ab} - R_b R_a$ to the relation to conclude that $[a, M(x_i)] = [a, b, M(x_i)] = [a, M(x_i), b] = [M(x_i)), a, b] = 0$ for all $a, b \in \mathrm{A}$ and conclude that each $M(x_i) = \gamma_i 1 \in \Phi 1 = \mathcal{C}ent(\mathrm{A})$. Conclude that the relation was $0 \neq \omega \otimes 1 \in \widetilde{\mathrm{I}}$ as in (1) of the above proof, re-establishing Strict Simplicity.

1.8 The Category of Jordan Algebras

The Jordan axioms are succinctly stated in terms of associators.[11]

Jordan Algebra Definition 1.8.1 *A* **Jordan algebra** *over a ring of scalars Φ containing $\frac{1}{2}$ is a linear Φ-algebra* J *equipped with a commutative bilinear product, denoted by $x \bullet y$, which satisfies the Jordan identity*:

(JAX1)	$[x, y] = 0$	(**Commutative Law**),
(JAX2)	$[x^2, y, x] = 0$	(**Jordan Identity**).

A Jordan algebra is **unital** *if it has a* **unit element** 1 *in the usual sense. A* **homomorphism** *of Jordan algebras is just an ordinary homomorphism of linear algebras, $\varphi(x \bullet y) = \varphi(x) \bullet' \varphi(y)$.*

The **category of Jordan Φ-algebras** *consists of all Jordan Φ-algebras with homomorphisms, while the* **category of unital Jordan Φ-algebras** *consists of all unital Jordan Φ-algebras with unital homomorphisms.*

[11] The categorical notions of this section have already been introduced in the *Historical Survey*: direct sums, direct products, and subdirect sums in Section I.2.2, Jordan algebras in I.2.4, the auxiliary operators in I.4.1, and inner ideals in I.4.7.

Exercise 1.8.1A (1) Show that the variety of Jordan algebras is closed under homomorphisms: if $\varphi : \mathrm{J} \to \mathrm{A}$ is a homomorphism of a Jordan algebra J into a linear algebra A, then the image $\varphi(\mathrm{J})$ is a Jordan subalgebra of A. (2) Show that the variety of Jordan algebras is closed under quotients: if J is Jordan, so is any of its quotient algebras J/I. (3) Show that the variety of Jordan algebras is closed under direct products: if J_i are Jordan algebras, so is their direct product $\prod_i \mathrm{J}_i$.

Exercise 1.8.1B Just as everyone should (as mentioned in the *Colloquial Survey*) verify directly, *once and only once in their life,* anti-commutativity $[x, y] = -[y, x]$ and the Jacobi identity $[x, [y, z]]+[y, [z, x]]+[z, [x, y]] = 0$ for $[x, y] = xy-yx$ in associative algebras, showing that every associative algebra A gives rise to a Lie algebra A^- under the commutator product $p(x, y) := [x, y]$, so should everyone show that A^+ under the anti-commutator product $p(x, y) := x \bullet y = \frac{1}{2}(xy + yx)$ gives rise to a Jordan algebra, by verifying directly commutativity $x \bullet y = y \bullet x$ and the Jordan identity $(x^2 \bullet y) \bullet x = x^2 \bullet (y \bullet x)$. (No peeking at Full Example 3.1.1!!)

Exercise 1.8.1C (1) Show that $[x, y, z] = 0$ holds in any linear algebra whenever $x = e$ is a *left unit* ($ea = a$ for all elements $a \in \mathrm{A}$). (2) The identity $[x, y, x] = 0$, i.e., $(xy)x = x(yx)$, is called the *flexible law*; show that every commutative linear algebra is automatically flexible, i.e., that $[x, y, z] = 0$ holds whenever $z = x$.

Auxiliary Products Definition 1.8.2 *From the basic bullet product we can construct several important* **auxiliary products** (*squares, brace products, U-products, triple products, and V-products*):[12]

$$\begin{aligned} x^2 &:= x \bullet x, \qquad \{x, z\} = 2x \bullet z \\ U_x y &:= 2x \bullet (x \bullet y) - x^2 \bullet y \\ \{x, y, z\} &:= U_{x,z} y := \big(U_{x+z} - U_x - U_z\big) y \\ &= 2\big(x \bullet (z \bullet y) + z \bullet (x \bullet y) - (x \bullet z) \bullet y\big) \\ V_x(z) &= \{x, z\}, \qquad V_{x,y}(z) = \{x, y, z\}. \end{aligned}$$

In unital Jordan algebras, setting x or y or z equal to 1 in the above shows that the unit interacts with the auxiliary products by

$$\begin{aligned} U_x 1 &= x^2, & U_1 y &= y, \\ \{x, 1\} &= 2x, & \{x, y, 1\} &= \{x, 1, y\} = \{x, y\}. \end{aligned}$$

Homomorphisms automatically preserve all auxiliary products built out of the basic product.

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[12] In the Jordan literature the brace product $\{x, y\}$, obtained by linearizing the square, is often denoted by a *circle*: $x \circ y = (x+y)^2 - x^2 - y^2$. Jimmie McShane once remarked that it is a strange theory where the *linearization* of the *square* is a *circle*! We use the brace notation in conformity with our general usage for *n-tads* $\{x_1, \ldots, x_n\} := x_1 \cdots x_n + x_n \cdots x_1$, and also since it resembles the brackets of the Lie product.

Ideal Definition 1.8.3 *We have the usual notions of* **subalgebra** $\mathrm{B} \leq \mathrm{J}$ *and* **ideal** $\mathrm{I} \triangleleft \mathrm{J}$ *as for any linear algebra,*

$$\mathrm{B} \bullet \mathrm{B} \subseteq \mathrm{B}, \qquad \mathrm{J} \bullet \mathrm{I} \subseteq \mathrm{I}.$$

Subalgebras and ideals remain closed under all auxiliary Jordan products

$$\mathrm{B}^2 + U_{\mathrm{B}}(\mathrm{B}) + \{\mathrm{B}, \mathrm{B}\} + \{\mathrm{B}, \mathrm{B}, \mathrm{B}\} \subseteq \mathrm{B},$$

$$\mathrm{I}^2 + U_{\mathrm{J}}(\mathrm{I}) + U_{\mathrm{I}}(\mathrm{J}) + \{\mathrm{J}, \mathrm{I}\} + \{\mathrm{J}, \mathrm{J}, \mathrm{I}\} + \{\mathrm{J}, \mathrm{I}, \mathrm{J}\} \subseteq \mathrm{I}.$$

Any ideal determines a **quotient algebra** $\overline{\mathrm{J}} = \mathrm{J}/\mathrm{I}$, *which is again a Jordan algebra.*

In addition, we have a new concept of "one-sided" ideal: an **inner ideal** *of a Jordan algebra* J *is a* Φ*-submodule* B *closed under inner multiplication by* $\widehat{\mathrm{J}}$:

$$U_{\mathrm{B}}(\widehat{\mathrm{J}}) = \mathrm{B}^2 + U_{\mathrm{B}}(\mathrm{J}) \subseteq \mathrm{B}.$$

In particular, inner ideals are always subalgebras $\mathrm{B} \bullet \mathrm{B} \subseteq \mathrm{B}$. *In the unital case a* Φ*-submodule* B *is an inner ideal iff it is a* Φ*-submodule closed under inner multiplication* $U_{\mathrm{B}}(\mathrm{J}) \subseteq \mathrm{B}$.[13]

Product Proposition 1.8.4 *Jordan algebras are closed under subalgebras and the usual notions of products: if* J_i, $i \in I$ *is a family of Jordan algebras, then the direct product* $\prod_{i \in I} \mathrm{J}_i$, *the direct sum* $\boxplus_{i \in I} \mathrm{J}_i$, *and any subdirect product* $\widetilde{\mathrm{J}} \precapprox \prod_{i \in I} \mathrm{J}_i$, *remains a Jordan algebra.*

PROOF. Subalgebras always inherit all the identical relations satisfied by the parent algebra, and the direct product inherits all the identical relations satisfied by each individual factor (since the operations are componentwise). Since the direct sum and subdirect sum are isomorphic to subalgebras of the direct product, we see all these linear algebras inherit the commutative law and the Jordan identity, and therefore remain Jordan algebras. □

We obtain a scalar extension functor from the category of Jordan Φ-algebras to Jordan Ω-algebras, and a unitalization functor $\mathrm{J} \mapsto \widehat{\mathrm{J}}$ from the category of linear Jordan Φ-algebras to unital Jordan Φ-algebras, and these constructions commute as usual.

Linearization Proposition 1.8.5 (1) *Because we are assuming a scalar* $\frac{1}{2}$, *a Jordan algebra automatically satifies the linearizations of the Jordan identity*:

$$\text{(JAX2}'\text{)} \qquad [x^2, y, z] + 2[x \bullet z, y, x] = 0,$$

$$\text{(JAX2}''\text{)} \qquad [x \bullet z, y, w] + [z \bullet w, y, x] + [w \bullet x, y, z] = 0,$$

for all elements x, y, z, w *in* J.

[13] In the literature inner ideals are often defined as what we would call *weak inner ideals* with only $U_{\mathrm{B}}\mathrm{J} \subseteq \mathrm{B}$. We prefer to demand that inner ideals be *subalgebras* as well. Of course, the two definitions agree for unital algebras.

(2) *Any Jordan algebra strictly satisfies the Jordan identities in the sense that all scalar extensions continue to satisfy them, and consequently every scalar extension of a Jordan algebra remains Jordan.*

(3) *Every Jordan algebra can be imbedded in a unital one*: *the unital hull* $\widehat{\mathrm{J}}$ *of a Jordan algebra* J *is again a Jordan algebra having* J *as an ideal.*

PROOF. (1) To linearize the Jordan identity (JAX2) of degree 3 in the variable x, we replace x by $x + \lambda z$ for an arbitrary scalar λ and subtract off the constant and λ^3 terms: since (JAX2) holds for $x = x, z, x+\lambda z$ individually, we must have

$$\begin{aligned} 0 &= [(x + \lambda z)^2, y, x + \lambda z] - [x^2, y, x] - \lambda^3 [z^2, y, z] \\ &= \lambda\Big([x^2, y, z] + 2[x \bullet z, y, x]\Big) + \lambda^2\Big([z^2, y, x] + 2[z \bullet x, y, z]\Big) \\ &= \lambda f(x; y; z) + \lambda^2 f(z; y; x) \end{aligned}$$

for $f(x; y; z)$ the left side of (JAX2)$'$. This relation must hold for all values of the scalar λ. Setting $\lambda = \pm 1$ gives $0 = \pm f(x; y; z) + f(z; y; x)$, so that subtracting gives $0 = 2f(x; y; z)$, and hence the existence of $\frac{1}{2}$ (or just the absence of 2-torsion) guarantees that $f(x; y; z) = 0$ as in (JAX2)$'$. The identity (JAX2)$'$ is quadratic in x (linear in z, y), therefore automatically linearizes: replacing x by $x = x, w, x + \lambda w$ and subtracting the pure x and pure w terms gives $0 = f(x + \lambda w; y; z) - f(x; y; z) - \lambda^2 f(w; y; z) = 2\lambda\left([x \bullet w, y, z] + [x \bullet z, y, w] + [w \bullet z, y, x]\right)$, so for $\lambda = \frac{1}{2}$ we get (JAX2)$''$.

(2) It is easy to check that any scalar extension J_Ω of a Jordan algebra J remains Jordan; it certainly remains commutative as in (JAX1), and for general elements $\tilde{x} = \sum_i \alpha_i \otimes x_i$, $\tilde{y} = \sum_j \beta_j \otimes y_j$ of $\widetilde{\mathrm{J}} = \mathrm{J}_\Omega$ we have

$$\begin{aligned} [\tilde{x}^2, \tilde{y}, \tilde{x}] = &\sum_{j;i} \alpha_i^3 \beta_j \otimes [x_i^2, y_j, x_i] \\ &+ \sum_{j;i\neq k} \alpha_i^2 \alpha_k \beta_j \otimes \Big([x_i^2, y_j, x_k] + 2[x_i \bullet x_k, y_j, x_i]\Big) \\ &+ \sum_{j;i,k,\ell\neq} 2\alpha_i \alpha_k \alpha_\ell \beta_j \otimes \Big([x_i \bullet x_k, y_j, x_\ell] \\ &\qquad + [x_k \bullet x_\ell, y_j, x_i] + [x_\ell \bullet x_i, y_j, x_k]\Big), \end{aligned}$$

which vanishes since each individual term vanishes by (JAX2), (JAX2)$'$, (JAX2)$''$ applied in J.

(3) Clearly J is an ideal in $\widehat{\mathrm{J}}$ as in Unital Hull Definition 1.1.5, and the product on $\widehat{\mathrm{J}}$ is commutative with unit; it satisfies the Jordan identity (JAX2) since for any elements $\hat{x}, \hat{y} \in \widehat{\mathrm{J}}$ we have $[\hat{x}^2, \hat{y}, \hat{x}] = [(\alpha\hat{1} \oplus x)^2, \beta\hat{1} \oplus y, \alpha\hat{1} \oplus x] = [2\alpha x + x^2, y, x]$ [the unit associates with everything] $= 2\alpha[x, y, x] + [x^2, y, x] = 0 + 0 = 0$ because the original J satisfies (JAX2) and (JAX1), and any commutative algebra is flexible: $[x, y, x] = (x \cdot y) \cdot x - x \cdot (y \cdot x) = x \cdot (x \cdot y) - x \cdot (x \cdot y) = 0$. □

Exercise 1.8.5* (1) Let u, v be two elements of a Φ-module M. Show that if $\lambda u + \lambda^2 v = 0$ for *all* $\lambda \in \Phi$, then $u = v = 0$ if there is at least one scalar λ in Φ such that $\mu := \lambda - \lambda^2$ is *cancelable* from u (in the sense $\mu u = 0 \Longrightarrow u = 0$). In particular, conclude that this happens if Φ is a field with at least three elements, or any ring containing $\frac{1}{2}$. (2) Letting $a = f(x; y; z), b = f(z; y; x)$ as in the above proof of Linearization, conclude that (JAX2) automatically implies (JAX2)$'$. (3) Trivially, if a map g from one abelian group M to another N vanishes on M, so does its polarization $g(u, v) := g(u + v) - g(u) - g(v)$, since it is a sum of values of g. Use this to verify that (JAX2)$'$ *always* implies (JAX2)$''$ over any Φ.

1.9 Problems for Chapter 1

Problem 1.1* I.L. Kantor could have said (but didn't), "There are no objects, there are only morphisms." A category theorist could say (and has) "There are no objects, there are only arrows." Show that we can define a category to be a class $\mathcal{M}$ of "abstract morphisms" or "arrows," together with a partial binary operation from a subclass of $\mathcal{M} \times \mathcal{M}$ to $\mathcal{M}$ satisfying the four axioms: (1) (partial associativity) if either of $(fg)h$ or $f(gh)$ is defined, then so is the other and they are equal; (2) (identities) for each arrow f there are identities e_L, e_R such that $e_L f = f = f e_R$ [where $e \in \mathcal{M}$ is an *identity* if whenever ef is defined it equals f, and whenever ge is defined it equals g]; (3) (composition) if e is an identity and both fe, eg are defined, then so is fg; (4) (smallness) if e, e' are identities then the class $e\mathcal{M}e'$ (the class of all $f = (ef)e' = e(fe')$ having e as left unit and e' as right unit) is actually a set. To make composition more aesthetic, consider the morphisms to be arrows $\leftarrow_f$, so that the left (or head) identity represents $\mathbb{1}_X$ for the codomain X, and the right (or tail) identity represents $\mathbb{1}_Y$ for the domain Y, and the arrow belongs to $\mathcal{M}or(X, Y) = \mathcal{M}or(X \leftarrow Y)$. Show that categories in this arrow-theoretic sense are equivalent to categories in the object–morphism sense. Note that using *left* arrows makes composition more aesthetic: $\mathcal{M}or(X, Y)\mathcal{M}or(Y, Z) \subseteq \mathcal{M}or(X, Z)$.

Problem 1.2 (1) Show that any *variety* of linear algebras, defined by a set of identities, is closed under homomorphisms, subalgebras, and direct products. (2) A celebrated theorem of G. Birkhoff says, conversely, that every class $\mathcal{C}$ of linear algebras (actually, algebraic systems in general) which is closed under homomorphisms, subalgebras, and direct products is a variety. Try your hand at proving this; the keys are the existence of free algebras $\mathcal{FC}[X]$ in the class for any set X, and that the class consists precisely of the homomorphic images of free algebras, so the class is defined by identities (the kernel of the canonical map $\mathcal{F}A[X] \to \mathcal{FC}[x]$ for any countably infinite set X).

Problem 1.3* We noticed that if the algebra is already unital, unitalization introduces a supernumerary unit. In most applications any unital hull $\Phi 1 + \mathrm{A}$ would do equally well, we need the convenience of a unit element only to express things in the original algebra more concisely. But in structural studies it is important to be more careful in the choice of hull; philosophically the tight hulls are the "correct" ones to choose. An extension $\widetilde{\mathrm{A}} \supseteq \mathrm{A}$ is a *tight extension* if all the nonzero ideals of the extension have nonzero intersection with the original algebra, i.e., there are no *disjoint* ideals $0 \neq \widetilde{\mathrm{I}} \lhd \widetilde{\mathrm{A}}$, $\widetilde{\mathrm{I}} \cap \mathrm{A} = \mathbf{0}$. (Of course, two subspaces can *never* be disjoint, they always share the zero element 0; nevertheless, by abuse of language we call them disjoint if they are nonzero and have nothing but this one lowly element in common.) (1) Show that the formal unital hull has the universal property that every homomorphism of A into a unital algebra B extends uniquely to a unital homomorphism $\widehat{\mathrm{A}} \to \mathrm{A}$. Show that the unital hulls $\mathrm{A}^1 = \Phi 1 + \mathrm{A}$ of A are, up to isomorphism, precisely all quotients $\widehat{\mathrm{A}}/\mathrm{I}'$ for disjoint ideals I'. (2) Show that there is a handy tightening process to remedy any looseness in a given extension, namely, dividing out a *maximal* disjoint ideal: show that $\widehat{\mathrm{A}}/\mathrm{I}'$ is a tight extension iff I' is a maximal disjoint ideal. (3) A linear algebra is *robust* if it has zero *annihilator* $\mathcal{A}nn_{\mathrm{A}}(\mathrm{A}) := \{z \in \mathrm{A} \mid z\mathrm{A} = \mathrm{A}z = 0\}$. Show that any unital algebra, simple algebra, prime algebra, or semiprime algebra is robust. [An algebra is *prime* if it has no orthogonal ideals ($0 \neq \mathrm{I}, \mathrm{K} \lhd \mathrm{A}$ with $\mathrm{IK} = \mathbf{0}$), and *semiprime* if it has no self-orthogonal ideals ($0 \neq \mathrm{I} \lhd \mathrm{A}$ with $\mathrm{II} = \mathbf{0}$).] (4) Show that in a robust algebra there is a *unique* maximal disjoint ideal of the formal unital hull, namely $\mathrm{M}' := \mathcal{A}nn_{\widehat{\mathrm{A}}}(\mathrm{A}) := \{\hat{z} \in \widehat{\mathrm{A}} \mid \hat{z}\mathrm{A} = \mathrm{A}\hat{z} = 0\}$, and hence there is a unique tightening $\widetilde{\mathrm{A}}$. Thus in the robust case we may speak without ambiguity of *the* tight unital hull.

Problem 1.4* (1) If A already has unit 1, it is robust; show that the maximal disjoint ideal is $\mathrm{M}' = \Phi(\hat{1} - 1)$, and find its tight unital hull $\widetilde{\mathrm{A}}$. (2) The algebra $\mathrm{A} = 2\mathbb{Z}$ of even integers is robust (even a domain). Find its tight unital hull.

Problem 1.5* The tight unital hull is algebraically "closer" to the original algebra: it inherits many important properties. (1) Show that the formal unital hull $\widehat{\mathrm{A}}$ of a simple algebra is never simple, and the tight unital hull $\widetilde{\mathrm{A}}$ is simple only if the original algebra is already simple and unital. (2) Show that the tight unital hull of a prime algebra is prime, but the formal unital hull is prime iff $\mathrm{M}' = 0$ and $\widehat{\mathrm{A}} = \widetilde{\mathrm{A}}$. Give a prime example of infinite matrices where this fails: $\mathrm{A} = \mathcal{M}_\infty(\Phi) + \Omega 1_\infty$ for Ω an ideal in an integral domain Φ, with $\mathrm{M}' = \Omega(\hat{1} \oplus -1_\infty)$. (3) Show that the tight unital hull of a semiprime algebra is always semiprime. Show that if Φ acts faithfully on A ($\mathcal{A}nn_\Phi(\mathrm{A}) := \{\beta \in \Phi \mid \beta\mathrm{A} = \mathbf{0}\}$ vanishes, so nonzero scalars act nontrivially) and A is semiprime, then its formal hull remains semiprime. Show this needn't hold if the action of Φ on A is unfaithful. (4) Show that if A is robust and Φ a field, then $\mathrm{M}' = \mathbf{0}$ and the tight unitalization coincides with the formal one. (5) Show that tightening a robust algebra always induces faithful scalar action: if $\alpha \in \mathcal{A}nn_\Phi(\mathrm{A})$ then $\Phi\alpha \oplus 0$ is an ideal of $\widehat{\mathrm{A}}$ contained in all maximal M's, so $\alpha\widetilde{\mathrm{A}} = \mathbf{0}$, so both A and its tight hull $\widetilde{\mathrm{A}}$ are algebras over the faithful ring of scalars $\Phi' := \Phi/\mathcal{A}nn_\Phi(\mathrm{A})$.

Problem 1.6* (1) Show that $\mathcal{M}_{np}(\Phi) \cong \mathcal{M}_n(\Phi) \otimes \mathcal{M}_p(\Phi)$ as associative algebras, so there is always a homomorphism $\mathcal{M}_n(\Phi) \to \mathcal{M}_{nm}(\Phi)$ of unital associative algebras for any Φ. (2) Show that if Φ is a field there is a homomorphism $\mathcal{M}_n(\Phi) \to \mathcal{M}_m(\Phi)$ of associative algebras iff $n \leq m$, and there is a homomorphism $\mathcal{M}_n(\Phi) \to \mathcal{M}_m(\Phi)$ of *unital* associative algebras iff *n divides m*. (3) Show that the construction of the tight unital hull is not functorial: if $\varphi : \mathrm{A} \to \mathrm{A}'$ is a non-unital homomorphism, there need not be *any* unital homomorphism $\widetilde{\mathrm{A}} \to \widetilde{\mathrm{A}}'$ (much less one extending φ).

Problem 1.7 (1) By "universal nonsense," in any linear algebra an automorphism takes nucleus into nucleus and center into center. Show directly from the definition that the same is true of any derivation: $\delta(\mathcal{N}uc(\mathrm{A})) \subseteq \mathcal{N}uc(\mathrm{A})$ and $\delta(\mathcal{C}ent(\mathrm{A})) \subseteq \mathcal{C}ent(\mathrm{A})$. (2) Show the same using "infinitesimal nonsense": show that in the algebra of dual numbers we have $\mathcal{N}uc(\mathrm{A}[\varepsilon]) = \mathcal{N}uc(A)[\varepsilon]$ and $\varphi = \mathbb{1}_{\mathrm{A}} + \varepsilon\delta$ is an automorphism, and use these to show again that δ preserves the nucleus and center.

Question 1.1* Define the unital hull to be A itself if A is already unital, and the formal unital hull $\widehat{\mathrm{A}}$ otherwise. Does this yield a functor from algebras to unital algebras? Can you see any advantages or disadvantages to adopting this intermediate definition (between the formal and the tight extension)?

Question 1.2* Under what conditions is the scalar extension B_Ω of a Boolean algebra (one with $x^2 = x$ for all x) again Boolean?

Question 1.3* In the Ideal Definition 8.3 we defined the concept of an *inner ideal* in a Jordan algebra. What would an *outer ideal* be? Why didn't we at least mention this concept? Give interesting examples of outer ideals in the quadratic Jordan algebras $\mathcal{H}_n(\mathbb{Z})$ and $\mathcal{H}_n(\Omega)$ for an imperfect field $\Omega^2 < \Omega$ of characteristic 2. These are precisely the new "wrinkles" which create some modifications of the usual simple algebras for quadratic Jordan algebras. Since we are sticking to the linear theory, we won't mention outer ideals again.

2

The Category of Alternative Algebras

The natural coordinates for Jordan algebras are alternative algebras with a nuclear involution. We will develop the basic facts about the variety of alternative algebras in general, and will construct in detail the most important example: the 8-dimensional octonion algebra, a composition algebra obtained by gluing together two copies of a quaternion algebra in a twisted way by the Cayley–Dickson Construction. Octonion algebras with their standard involution coordinatize the simple exceptional Albert algebras. We will prove the celebrated Hurwitz Theorem, that composition algebras exist only in dimensions 1, 2, 4, or 8.

2.1 The Category of Alternative Algebras

Since alternative algebras are second cousins to associative algebras, we write the product by mere juxtaposition.[1]

Alternative Algebra Definition 2.1.1 *A linear algebra* D *is* **alternative** *if it satisfies the* **left** *and* **right alternative laws**

(AltAX1)	$x^2y = x(xy)$	(Left Alternative Law),
(AltAX2)	$yx^2 = (yx)x$	(Right Alternative Law)

for all x, y in D*. An alternative algebra is automatically* **flexible***,*

(AltAX3)	$(xy)x = x(yx)$	(Flexible Law).

In terms of associators or operators, these identities become

$$[x, x, y] = [y, x, x] = [x, y, x] = 0, \qquad \textit{or}$$

$$L_{x^2} = (L_x)^2,\ R_{x^2} = (R_x)^2,\ L_xR_x = R_xL_x.$$

[1] cf. the definition of alternative algebra in Section I.2.7.

The **category of alternative algebras** *consists of all alternative algebras and ordinary homomorphisms. The defining identities are quadratic in x, so they automatically linearize and all scalar extensions of alternative algebras are again alternative: the scalar extension functor stays within the category of alternative algebras.*

We have a corresponding category of unital alternative algebras, and a unitalization functor: any extension $\widetilde{\mathrm{D}} = \Phi 1 + \mathrm{D}$ of an alternative algebra D *will again be alternative, since for any two of its elements $\tilde{x}, \tilde{y}$ we have $[\tilde{x}, \tilde{x}, \tilde{y}] = [\alpha 1 + x, \alpha 1 + x, \beta 1 + y] = [x, x, y]$ (1 is nuclear) $= 0$ (D is alternative), dually $[\tilde{y}, \tilde{x}, \tilde{x}] = [y, x, x] = 0$.*

We will often write D for alternative algebras to distinguish them from mere linear algebras A. Associative algebras are obviously alternative, and rather surprisingly, it turns out that nice alternative algebras come in only two basic flavors: *associative* and *octonion*.[2]

EXERCISE 2.1.1 Linearize the alternative laws in x to see that $[x_1, x_2, x_3]$ is skew in its first and last two variables; since the transpositions (12) and (23) generate the symmetric group S_3, conclude that $[x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}] = (-1)^\pi [x_1, x_2, x_3]$ for any permutation π of $\{1, 2, 3\}$. Show that the associator is an *alternating* function of its variables (vanishes if any two of its arguments are equal), in particular is flexible.

2.2 Nuclear Involutions

Properties of the nucleus and center in simple alternative algebras are the main key to their classification.

Nucleus and Center Definition 2.2.1 *Because of the skew-symmetry of the associator in alternative algebras, the Nucleus and Center Definition* 1.6.1 *for linear algebras simplifies to*

$$\mathcal{N}uc(\mathrm{D}) = \{n \in \mathrm{D} \mid [n, \mathrm{D}, \mathrm{D}] = \mathbf{0}\} \qquad \text{(D alternative)},$$

$$\mathcal{C}ent(\mathrm{D}) = \{c \in \mathrm{D} \mid [c, \mathrm{D}] = [c, \mathrm{D}, \mathrm{D}] = \mathbf{0}\} \qquad \text{(D alternative)}.$$

A unital alternative algebra can always be considered as an alternative algebra over its center $\Omega := \mathcal{C}ent(\mathrm{A}) \supseteq \Phi 1$.

For alternative $*$-algebras, the natural notion of center is the $*$-**center**, *the set of central elements fixed by $*$.*

[2] The celebrated Bruck–Kleinfeld Theorem of 1950 proved that the only alternative division algebras that are not associative are octonion algebras, and in 1953 Erwin Kleinfeld proved that all simple alternative algebras are associative or octonion. This was the second algebraic structure theorem which required no finiteness conditions (the first was Hurwitz's Theorem); much later Efim Zel'manov would provide *Jordan* algebras with such a theory, but even today there is no general classification of all simple *associative* algebras.

We have seen that quaternion and octonion algebras are central: their center is just $\Phi 1$, so center and $*$-center coincide.

We will return in Chapter 21 for more detailed information about the alternative nucleus, in order to pin down in the Herstein–Kleinfeld–Osborn Theorem the particular alternative algebras which coordinatize Jordan algebras with capacity. To coordinatize Jordan algebras in general we need alternative algebras with involutions that are suitably "associative," namely nuclear.

Nuclear Involution Definition 2.2.2 *A* **nuclear** *or* **central** *or* **scalar involution** *on* A *is an involution* $*$ *whose self-adjoint elements lie in the nucleus or center or scalar multiples of* 1, *respectively*:

$$\mathcal{H}(\mathrm{A},*) \subseteq \mathcal{N}uc(\mathrm{A}) \qquad (*\ nuclear),$$

$$\mathcal{H}(\mathrm{A},*) \subseteq \mathcal{C}ent(\mathrm{A}) \qquad (*\ central),$$

$$\mathcal{H}(\mathrm{A},*) \subseteq \Phi 1 \qquad (*\ scalar).$$

For example, all involutions of an associative algebra are nuclear, and the standard involution on a quaternion or octonion algebra is scalar.

As we noted in the $*$*-Algebra Definition* 1.5.1, *when* $\frac{1}{2} \in \Phi$, *then all hermitian elements are traces,* $\mathcal{H}(\mathrm{A},*) = t(\mathrm{A})$, *and when* A *is unital, then all traces are built from norms,* $t(x) = n(x+1) - n(x) - n(1)$. *The only coordinate algebras we deign to consort with in this book are unital and contain* $\frac{1}{2}$, *and for these algebras* nice norms are enough: *if all norms* xx^* *fall in the nucleus, center, or scalar multiples of* 1, *respectively, then all the hermitian elements do too, and the involution is nuclear, central, or scalar, respectively.*

2.3 Composition Algebras

The path leading up to the octonions had a long history.[3] The final stage in our modern understanding of the structure of octonions was Jacobson's 1958 proof of the Hurwitz Theorem, showing how the Cayley–Dickson doubling process is genetically programmed into composition algebras, providing an internal bootstrap operation which inexorably builds up from the 1-dimensional scalars, to a 2-dimensional quadratic extension, to a 4-dimensional quaternion algebra, and finally to an 8-dimensional octonion algebra, at which point the process stops: the construction cannot go beyond the octonions and still permit composition.

We begin by formalizing, over an arbitrary ring of scalars, the concept of composition algebra.

[3] Sketched in the *Historical Survey*, Sections 2.8-2.12.

Forms Permitting Composition 2.3.1 (1) *A quadratic form* $Q : \mathrm{C} \to \Phi$ *with basepoint* c *is* **unital** *if* $Q(c) = 1$. *Then the* **trace** T *is defined by* $T(x) := Q(x, c)$, *and the* **standard trace involution** *is the linear map*

$$\overline{x} := T(x)c - x.$$

The standard involution is involutory and preserves traces and norms: we have the **Trace Involution Properties**

$$\overline{c} = c, \qquad T(\overline{x}) = T(x), \qquad Q(\overline{x}) = Q(x), \qquad \overline{\overline{x}} = x$$

since $T(c) = Q(c, c) = 2Q(c) = 2,\ \overline{c} = T(c)c - c = c,\ T(\overline{x}) = T(x)T(c) - T(x) = T(x),\ Q(\overline{x}) = T(x)^2 Q(c) - T(x)Q(x, c) + Q(x) = Q(x),\ \overline{\overline{x}} = T(\overline{x})c - \overline{x} = T(x)c - \overline{x} = x$.

(2) *A quadratic form* Q *is* **nondegenerate** *if it has zero* **radical** $\mathcal{R}ad(Q) := \{z \mid Q(z) = Q(z, \mathrm{C}) = 0\}$; *here we can omit the condition that* $Q(z) = 0$ *since* $Q(z) = \frac{1}{2}Q(z, z)$ *by our assumption that* $\frac{1}{2} \in \Phi$, *so nondegeneracy of* Q *amounts to nondegeneracy of the linearized bilinear form* $Q(x, y) := Q(x + y) - Q(x) - Q(y)$,

$$Q(z, \mathrm{C}) = \mathbf{0} \Longrightarrow z = 0.$$

(3) *A* **composition algebra** *is a unital linear algebra* C *which carries a nondegenerate quadratic form* Q **permitting composition**, *in the sense that it is* **Unital** *and satisfies the* **Composition Law**

$$Q(1) = 1, \qquad Q(xy) = Q(x)Q(y)$$

for all elements $x, y \in \mathrm{C}$. *We refer to* Q *as the* **norm**.

In the presence of nondegeneracy, the Composition Law has serious algebraic consequences.

Composition Consequences 2.3.2 (1) *Any composition algebra* C *satisfies the* **Left** *and* **Right Adjoint Formulas**

$$Q(xy, z) = Q(y, \overline{x}z), \qquad Q(yx, z) = Q(y, z\overline{x}).$$

In the language of adjoints with respect to a bilinear form, these say that $L_x^* = L_{\overline{x}}, R_x^* = R_{\overline{x}}$. *A composition algebra also satisfies the* **Left** *and* **Right Kirmse Identities**

$$\overline{x}(xy) = x(\overline{x}y) = Q(x)y, \qquad (y\overline{x})x = (yx)\overline{x} = Q(x)y;$$

in operator terms, these say that $L_x L_{\overline{x}} = L_{\overline{x}} L_x = Q(x)\mathbb{1}_{\mathrm{C}} = R_x R_{\overline{x}} = R_{\overline{x}} R_x$.

(2) *A composition algebra is always an alternative algebra satisfying the* **Degree–2 Identity**

$$x^2 - T(x)x + Q(x)1 = 0$$

for all elements x, *and its standard trace involution is a scalar algebra involution which is isometric with respect to the norm form*:

$$\overline{\overline{x}} = x, \qquad \overline{xy} = \overline{y}\,\overline{x}, \qquad T(x)1 = x + \overline{x}, \qquad Q(x)1 = x\overline{x} = \overline{x}x,$$

$$\overline{1} = 1, \quad T(\overline{x}) = T(x), \quad Q(\overline{x}) = Q(x), \quad Q(x, y) = T(x\overline{y}) = T(\overline{x}y).$$

PROOF. (1) We linearize $y \mapsto y, z$ and $x \mapsto x, 1$ in the Composition Law $Q(xy) = Q(x)Q(y)$ to obtain first $Q(xy, xz) = Q(x)Q(y, z)$ then $Q(xy, z) + Q(y, xz) = T(x)Q(y, z)$, so $Q(xy, z) = Q(y, T(x)z - xz) = Q(y, \overline{x}z)$, giving the Left Adjoint Formula. Right Adjoint follows by duality (or from the involution, once we know that it is an algebra anti-automorphism in (2)). The first part of Left Kirmse follows from nondegeneracy and the fact that for all z we have $Q(z, (\overline{x}(xy) - Q(x)y)) = Q(xz, xy) - Q(x)Q(z, y) = 0$ for all z by Left Adjoint and the linearized Composition Law; the second follows by replacing x by $\overline{x}$ [since the standard involution is involutory and isometric by the above], and dually for Right Kirmse.

(2) Applying either Kirmse Identity to $y = 1$ yields $x\overline{x} = \overline{x}x = Q(x)1$, which is equivalent to the Degree–2 Identity since $x^2 - T(x)x + Q(x)1 = [x - T(x)1]x + Q(x)1 = -\overline{x}x + Q(x)1$. Then in operator form Left Kirmse is equivalent to left alternativity, $L_{\overline{x}}L_x - Q(x)\mathbb{1}_C = L_{T(x)1-x}L_x - Q(x)L_1 = L_{T(x)x-Q(x)1} - L_x^2 = L_{x^2} - L_x^2$. Dually for right alternativity.

(3) The Adjoint Formulas and nondegeneracy imply that the trace involution is an algebra involution, since for all z we use a Hiding Trick to see that $Q(z, [\overline{xy} - \overline{y}\,\overline{x}]) = Q((xy)z, 1) - Q(zx, \overline{y})$ [by Left Adjoint for xy and Right Adjoint for x] $= Q(xy, \overline{z}) - Q(x, \overline{z}\,\overline{y})$ [by Right and Left Adjoint for z] $= 0$ [by Right Adjoint for y]. The remaining properties follow from the Trace Involution Properties 2.3.1(1). □

2.4 Split Composition Algebras

The easiest composition algebras to understand are the "split" ones.[4] As the Barbie doll said, "Gee, octonion algebra is hard!," so we will go into some detail about a concrete and elementary approach to the split octonion algebra, representing it as the Zorn vector–matrix algebra 2×2 matrices whose diagonal entries are scalars in Φ and whose off-diagonal entries are row vectors in Φ^3 (note that the dimensions add up: $1 + 3 + 3 + 1 = 8$).

Split Composition Definition 2.4.1 *The* **split** *composition algebras of dimensions* $1, 2, 4, 8$ *over a ring of scalars* Φ *are the following algebras with norm, trace, and involution*:

• SPLIT UNARIONS $\mathcal{U}(\Phi) := \Phi$, *the scalars* Φ *with trivial involution* $\bar{\alpha} := \alpha$, *trace* $T(\alpha) := 2\alpha$, *and norm* $N(\alpha) := \alpha^2$*;*

• SPLIT BINARIONS $\mathcal{B}(\Phi) := \Phi \boxplus \Phi$, *a direct sum of two copies of the scalars with the exchange involution* $\overline{(\alpha \oplus \beta)} := (\beta, \alpha)$, *trace* $T(\alpha \oplus \beta) := \alpha + \beta$, *and norm* $N(\alpha \oplus \beta) := \alpha\beta$*;*

• SPLIT QUATERNIONS $\mathcal{Q}(\Phi) := \mathcal{M}_2(\Phi)$, *the algebra of* 2×2 *matrices with the symplectic involution* $\overline{a} := sa^{tr}s^{tr} = \begin{pmatrix} \beta & -\gamma \\ -\delta & \alpha \end{pmatrix}$ *for* $a = \begin{pmatrix} \alpha & \gamma \\ \delta & \beta \end{pmatrix}$ *and*

[4] cf. the treatment of split algebras in Section I.2.12.

$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, *the matrix trace* $T(a) := \mathrm{tr}(a) = \alpha + \beta$, *and the determinant norm* $N(a) := \det(a) = \alpha\beta - \gamma\delta$*;*

• Split Octonions $\mathcal{O}(\Phi) := \mathcal{Q}(\Phi) \oplus \mathcal{Q}(\Phi)\ell$ *with standard involution* $\overline{a \oplus b\ell} = \overline{a} - b\ell$, *trace* $T(a \oplus b) = \mathrm{tr}(a)$, *and norm* $N(a \oplus b\ell) := \det(a) - \det(b)$.

This description of the split octonion algebra is based on its construction from *two* copies of 2×2 scalar matrices (the split quaternions) via the Cayley–Dickson process. An alternate description represents the octonions as *one* copy of 2×2 matrices, but with more complicated entries.

Zorn Vector-Matrix Example 2.4.2 *The* **Zorn vector–matrix algebra** *consists of all* 2×2 *matrices with scalar entries* $\alpha, \beta \in \Phi$ *on the diagonal and vector entries* $\mathbf{x} = (\alpha_1, \alpha_2, \alpha_3), \mathbf{y} = (\beta_1, \beta_2, \beta_3) \in \Phi^3$ *off the diagonal*:

$$\mathcal{Z}orn(\Phi) := \{A = \begin{pmatrix} \alpha & \mathbf{x} \\ \mathbf{y} & \beta \end{pmatrix} \mid \alpha, \beta \in \Phi,\ \mathbf{x}, \mathbf{y} \in \Phi^3\},$$

with norm, involution, and product

$$N(A) := \alpha\beta - \mathbf{x} \cdot \mathbf{y}, \qquad A^* := \begin{pmatrix} \beta & -\mathbf{x} \\ -\mathbf{y} & \alpha \end{pmatrix},$$

$$A_1 A_2 := \begin{pmatrix} \alpha_1\alpha_2 + \mathbf{x}_1 \cdot \mathbf{y}_2 & \alpha_1\mathbf{x}_2 + \mathbf{x}_1\beta_2 - \mathbf{y}_1 \times \mathbf{y}_2 \\ \mathbf{y}_1\alpha_2 + \beta_1\mathbf{y}_2 + \mathbf{x}_1 \times \mathbf{x}_2 & \beta_1\beta_2 + \mathbf{y}_1 \cdot \mathbf{x}_2 \end{pmatrix}$$

given in terms of the usual dot and cross products

$$\mathbf{x} \cdot \mathbf{y} := \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3,$$
$$\mathbf{x} \times \mathbf{y} := (\alpha_2\beta_3 - \alpha_3\beta_2, \alpha_3\beta_1 - \alpha_1\beta_3, \alpha_1\beta_2 - \alpha_2\beta_1).$$

Most of the Zorn vector–matrix product is just what one would expect from formally multiplying such matrices, keeping in mind that the diagonal entries must be scalars (hence the dot product $\mathbf{x}_1 \cdot \mathbf{y}_2$ in the 11-entry). We somewhat artificially allow scalar multiplication from the right and left, $\gamma\mathbf{x} := (\gamma\alpha_1, \gamma\alpha_2, \gamma\alpha_3) = \mathbf{x}\gamma$, to show more clearly the products with terms from A_1 on the left, A_2 on the right). What is *not* expected is the cross-product term $\mathbf{x}_1 \times \mathbf{x}_2$ in the 21-entry. From the usual associative matrix product we would expect $\begin{pmatrix} 0 & \mathbf{x}_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathbf{x}_2 \\ 0 & 0 \end{pmatrix}$ to vanish, but instead up pops $\begin{pmatrix} 0 & 0 \\ \mathbf{x}_1 \times \mathbf{x}_2 & 0 \end{pmatrix}$ in the opposite corner (and the 12-entry in the product of their transposes has a minus sign, to make things even more complicated!) Recall that the cross product is defined *only* on 3-dimensional space, which explains why this construction works only in dimension 8: the restriction of cross products to dimension 3 parallels the restriction of nonassociative composition algebras to dimension 8.

It is clear in Split Unarion, Binarion, and Quaternion Examples that the split algebra is associative and N permits composition. In the concrete representation of the Split Octonion Example as a Zorn vector–matrix algebra, it is easy to verify the alternative laws using nothing more than freshman vector analysis: using the rules $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$, $\mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}$, $\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) =$

$(\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z}$ the associator of three elements $A_i = \left(\begin{smallmatrix} \alpha_i & \mathbf{x}_i \\ \mathbf{y}_i & \beta_i \end{smallmatrix}\right)$ turns out to be

$$[A_1, A_2, A_3] = \begin{pmatrix} \alpha & \mathbf{x} \\ \mathbf{y} & \beta \end{pmatrix}$$

for

$$\alpha(A_1, A_2, A_3) := -\mathbf{x}_1 \cdot (\mathbf{x}_2 \times \mathbf{x}_3) - \mathbf{y}_3 \cdot (\mathbf{y}_1 \times \mathbf{y}_2),$$
$$\mathbf{x}(A_1, A_2, A_3) := \textstyle\sum_{cyclic}(\mathbf{x}_i \cdot \mathbf{y}_j - \mathbf{y}_i \cdot \mathbf{x}_j)\mathbf{x}_k + \sum_{cyclic}(\alpha_i - \beta_i)\mathbf{y}_j \times \mathbf{x}_k.$$

Alternativity of the algebra reduces to alternativity of these four coordinate functions. Here α is an alternating function of A_1, A_2, A_3, since the vector triple product

$$\mathbf{z} \cdot (\mathbf{z}' \times \mathbf{z}'') = \det \begin{bmatrix} \mathbf{z} \\ \mathbf{z}' \\ \mathbf{z}'' \end{bmatrix} = \det \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1' & \gamma_2' & \gamma_3' \\ \gamma_1'' & \gamma_2'' & \gamma_3'' \end{pmatrix}$$

is certainly an alternating function of its rows, and $\mathbf{x}$ is an alternating function of A_1, A_2, A_3, since its two sums are of the following forms:

$$(1) \qquad \textstyle\sum_{cyclic}(\mathbf{z}_i \cdot \mathbf{w}_j - \mathbf{z}_j \cdot \mathbf{w}_i)\mathbf{z}_k = (\mathbf{z}_1 \cdot \mathbf{w}_2 - \mathbf{z}_2 \cdot \mathbf{w}_1)\mathbf{z}_3$$
$$+(\mathbf{z}_2 \cdot \mathbf{w}_3 - \mathbf{z}_3 \cdot \mathbf{w}_2)\mathbf{z}_1 + (\mathbf{z}_3 \cdot \mathbf{w}_1 - \mathbf{z}_1 \cdot \mathbf{w}_3)\mathbf{z}_2,$$
$$(2) \qquad \textstyle\sum_{cyclic} \gamma_i \mathbf{z}_j \times \mathbf{z}_k = \gamma_1 \mathbf{z}_2 \times \mathbf{z}_3 + \gamma_2 \mathbf{z}_3 \times \mathbf{z}_1 + \gamma_3 \mathbf{z}_1 \times \mathbf{z}_2,$$

which are alternating functions of the pairs $(\mathbf{z}, \mathbf{w})$ and $(\gamma, \mathbf{z})$ respectively, because, for example, if $(\mathbf{z}_1, \mathbf{w}_1) = (\mathbf{z}_2, \mathbf{w}_2)$ then the sum (1) becomes $(\mathbf{z}_1 \cdot \mathbf{w}_1 - \mathbf{z}_1 \cdot \mathbf{w}_1)\mathbf{z}_3 + (\mathbf{z}_1 \cdot \mathbf{w}_3 - \mathbf{z}_3 \cdot \mathbf{w}_1)\mathbf{z}_1 + (\mathbf{z}_3 \cdot \mathbf{w}_1 - \mathbf{z}_1 \cdot \mathbf{w}_3)\mathbf{z}_1 = \mathbf{0}$, while if $(\gamma_1, \mathbf{z}_1) = (\gamma_2, \mathbf{z}_2)$ then (2) becomes $\gamma_1(\mathbf{z}_1 \times \mathbf{z}_3 + \mathbf{z}_3 \times \mathbf{z}_1) + \gamma_3(\mathbf{z}_1 \times \mathbf{z}_1) = \mathbf{0}$ because $\mathbf{z} \times \mathbf{w}$ is alternating.

An analogous computation shows that $\beta, \mathbf{y}$ are alternating functions of A_1, A_2, A_3, so $[A_1, A_2, A_3]$ alternates. We can also show this using "symmetry" in the indices 1, 2: we have two algebra involutions on $\mathcal{Z}orn(\Phi)$, the usual *transpose involution* given by the diagonal flip and the *standard involution* given by an anti-diagonal flip, $A^{tr} := \left(\begin{smallmatrix} \alpha & \mathbf{y} \\ \mathbf{x} & \beta \end{smallmatrix}\right)$, $\overline{A} := \left(\begin{smallmatrix} \beta & -\mathbf{x} \\ -\mathbf{y} & \alpha \end{smallmatrix}\right)$. The former interchanges the 12 and 21 entries, the latter the 11 and 22 entries, and any involution σ has $\sigma([A_1, A_2, A_3]) = -[\sigma(A_3), \sigma(A_2), \sigma(A_1)]$, so

$$\beta(A_1, A_2, A_3) = -\alpha(\overline{A_3}, \overline{A_2}, \overline{A_1}), \quad \mathbf{y}(A_1, A_2, A_3) = -\mathbf{x}(A_3^{tr}, A_2^{tr}, A_1^{tr})$$

are also alternating.

Thus $\mathcal{Z}orn(\Phi)$ is alternative, but it is not associative since the associator $\left[\left(\begin{smallmatrix} 0 & \mathbf{x} \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & \mathbf{y} \\ 0 & 0 \end{smallmatrix}\right)\right] = -\left(\begin{smallmatrix} 0 & 0 \\ \mathbf{x} \times \mathbf{y} & 0 \end{smallmatrix}\right)$ doesn't vanish identically (for $\mathbf{x} = \mathbf{e}_1, \mathbf{y} = \mathbf{e}_2$ we get $\mathbf{x} \times \mathbf{y} = \mathbf{e}_3$).

The Cayley–Dickson construction in the next section builds an octonion algebra from a distinguished quaternion subalgebra; in the Zorn matrix representation there are three separate split quaternion algebras $H_i = \left(\begin{smallmatrix} \Phi & \Phi\mathbf{e}_i \\ \Phi\mathbf{e}_i & \Phi \end{smallmatrix}\right)$, sharing a common diagonal. Notice that in this representation the norm and trace are the "natural" ones for these 2×2 matrices,

$$\begin{aligned} A\overline{A} &= (\alpha\beta - \mathbf{x}\cdot\mathbf{y})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = N(A)1, \\ A + \overline{A} &= \qquad (\alpha+\beta)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = T(A)1. \end{aligned}$$

We can also prove the composition property $N(AB) = N(A)N(B)$ of the norm directly from vector analysis facts, or we can deduce it from alternativity and $N(A)1 = A\overline{A}$ (see the exercises below).

Exercise 2.4.2A Establish the vector formula $(\mathbf{x}_1 \cdot \mathbf{y}_2)(\mathbf{y}_1 \cdot \mathbf{x}_2) + (\mathbf{x}_1 \cdot \mathbf{x}_2) \cdot (\mathbf{y}_1 \cdot \mathbf{y}_2) = (\mathbf{x}_1 \cdot \mathbf{y}_1)(\mathbf{x}_2 \cdot \mathbf{y}_2)$ for any four vectors in Φ^3, and use this to compute norm composition $N(A_1A_2) = N(A_1)N(A_2)$ in the Zorn vector–matrix algebra.

Exercise 2.4.2B In the Zorn vector–matrix algebra, (1) show directly that $\overline{A} = T(A)1 - A$, (2) $A^2 - T(A)A + N(A)1 = 0$, (3) $A\overline{A} = N(A)1$; (4) deduce (3) directly from (1) and (2). (5) Use (1) and alternativity of the Zorn algebra to show that $(AB)\overline{B} = A(B\overline{B})$, $(AB)(BA) = (A(BB))A$, $(AB)(\overline{B}A) = (A(B\overline{B}))A$; then use (3) and (2) to deduce $N(AB)1 = N(A)N(B)1$.

Over a field, a composition algebra whose norm splits the slightest bit (admits at least one nontrivial isotropic vector) splits completely into one of the above algebras, yielding a *unique* split composition algebra of each dimension over a given Φ. The constructions $\Phi \to \mathcal{B}(\Phi), \mathcal{Q}(\Phi), \mathcal{O}(\Phi)$ are functors from the category of scalar rings to the category of composition algebras.

2.5 The Cayley–Dickson Construction

The Cayley–Dickson construction starts with an arbitrary unital algebra A with involution, and doubles it to get a larger and more complicated unital algebra with involution.[5] We will see in the next section that all composition algebras can be obtained by iterating this doubling process.

Cayley–Dickson Construction 2.5.1 *Starting from a unital linear algebra* A *with involution over an arbitrary ring of scalars* Φ*, and an* invertible scalar $\mu \in \Phi$*, we form the direct sum of two copies of* A *and define a product and involution by*

$$(a_1, b_1) \cdot (a_2, b_2) := (a_1a_2 + \mu\overline{b_2}b_1, b_2a_1 + b_1\overline{a_2}), \quad (a, b)^* := (\overline{a}, -b).$$

If we set $m := (0, 1)$*, we can identify* $(a, 0)$ *with the original element* a *in* A*, and* $(0, b) = (b, 0)\cdot(0, 1)$ *with* $bm \in \mathrm{A}m$*, so* $(a, b) = a + bm$*. In this formulation we have an algebra*

[5] cf. the discussion of the Cayley–Dickson process in Section II.2.10.

$$\begin{aligned}\mathcal{KD}(\mathrm{A},\mu) &:= \mathrm{A} \oplus \mathrm{A}m,\\ (a_1 \oplus b_1 m)\cdot(a_2 \oplus b_2 m) &:= \big(a_1 a_2 + \mu \overline{b_2} b_1\big) \oplus \big(b_2 a_1 + b_1 \overline{a_2}\big)m,\\ (a \oplus bm)^* &:= \overline{a} - bm.\end{aligned}$$

The Cayley–Dickson product is less than memorable, but we can break it down into bite-sized pieces which are more easily digested. Besides the fact that A *is imbedded as a subalgebra with its usual multiplication and involution, we have the* **Cayley–Dickson Product Rules**:

$$\begin{aligned}&(\mathrm{KD0}) & am &= m\bar{a},\\ &(\mathrm{KD1}) & ab &= ab,\\ &(\mathrm{KD2}) & a(bm) &= (ba)m,\\ &(\mathrm{KD3}) & (am)b &= (a\bar{b})m,\\ &(\mathrm{KD4}) & (am)(bm) &= \mu\bar{b}a.\end{aligned}$$

PROOF. The new algebra is clearly unital with new unit $1 = (1,0)$, and clearly $*$ is a linear map of period 2; it is an anti-homomorphism of algebras since

$$\begin{aligned}\big((a_1,b_1)\cdot(a_2,b_2)\big)^* &= (\overline{a_1 a_2 + \mu\overline{b_2}b_1}, -(b_2 a_1 + b_1\overline{a_2}))\\ &= (\overline{a_2}\,\overline{a_1} + \mu\overline{b_1}b_2, -b_1\overline{a_2} - b_2\overline{\overline{a_1}})\\ &= (\overline{a_2}, -b_2)\cdot(\overline{a_1}, -b_1) = (a_2,b_2)^*\cdot(a_1,b_1)^*.\end{aligned}$$

The Cayley–Dickson Product Rules follow by examining the products of "homogeneous" elements from A and Am. □

EXERCISE 2.5.1* (1) Show that shifting the scalar μ in the Cayley–Dickson construction by a square produces an isomorphic algebra: $\mathcal{KD}(\mathrm{A},\mu)\cong\mathcal{KD}(\mathrm{A},\alpha^2\mu)$ for any invertible $\alpha \in \Phi$.

We call the resulting algebra the **Cayley–Dickson algebra** $\mathcal{KD}(\mathrm{A},\mu)$[6] obtained from the unital $*$-algebra A and the invertible scalar μ via the **Cayley–Dickson construction** or **doubling process**. The eminently-forgetable product formula is best remembered through the individual Cayley–Dickson Product Rules. Notice the following helpful mnemonic devices: whenever you move an element $b \in \mathrm{A}$ past m it gets conjugated, and when you multiply bm from the left by a or am you slip the a (unconjugated) in *behind* the b.

Let us observe how much algebraic structure gets inherited and how much gets lost each time we perform the Cayley–Dickson process.

[6] Pronounced "Kay-Dee," not "See-Dee."

$\mathcal{KD}$ **Inheritance Theorem** 2.5.2 *If* A *is a unital linear* $*$*-algebra, we have the following conditions in order that the Cayley–Dickson algebra inherit a property from its parent*:

• Scalar Involution: $\mathcal{KD}(\mathrm{A},\mu)$ *always has a nontrivial involution, which is scalar iff the original was*: *If the original involution on* A *is a scalar involution with* $a\overline{a} = n(a)1,\ a+\overline{a} = t(a)$ *for a quadratic norm form* n *and linear trace* t, *then the algebra* $\mathcal{KD}(\mathrm{A},\mu)$ *will again have a scalar involution with new norm and trace*

$$N(a \oplus bm) := n(a) - \mu n(b), \qquad T(a \oplus bm) := t(a).$$

• Commutative: $\mathcal{KD}(\mathrm{A},\mu)$ *is commutative iff* A *is commutative with trivial involution*;

• Associative: $\mathcal{KD}(\mathrm{A},\mu)$ *is associative iff* A *is commutative and associative*;

• Alternative: $\mathcal{KD}(\mathrm{A},\mu)$ *is alternative only if* A *is associative with central involution.*

proof. Clearly $\mathcal{KD}(\mathrm{A},\mu)$ always has a nontrivial involution, since the doubling process adjoins a skew part. If A has a scalar involution with $a+\overline{a} = t(a)1$, then $\overline{a} = t(a)1 - a$ commutes with a, and we have $\overline{a}a = a\overline{a} = n(a)1$. Then the new trace is $(a \oplus bm) + \overline{(a \oplus bm)} = (a \oplus bm) + (\overline{a} \oplus -bm) = (a+\overline{a}) \oplus 0 = t(a)1$, and using the Cayley–Dickson multiplication rule the new norm is $(a \oplus bm)\overline{(a \oplus bm)} = (a \oplus bm)(\overline{a} \oplus (-b)m) = \big(a\overline{a} - \mu\overline{b}b\big) \oplus \big(-ba + b\overline{\overline{a}}\big)m = \big(n(a) - \mu n(b)\big)1$.

For commutativity of $\mathcal{KD}(\mathrm{A},\mu)$, commutativity of the subalgebra A is clearly *necessary*, as is triviality of the involution by $am - ma = (a - \overline{a})m$; these also *suffice* to make the Cayley–Dickson product $\big(a_1a_2 + \mu b_1b_2\big) \oplus \big(a_1b_2 + b_1a_2\big)m$ symmetric in the indices 1 and 2.

For associativity of $\mathcal{KD}(\mathrm{A},\mu)$, associativity of the subalgebra A is clearly *necessary*, as is commutativity by $[a,b,m] = (ab)m - a(bm) = (ab - ba)m$. To see that these conditions *suffice* to make the Cayley–Dickson associator vanish, we use them to compute an arbitrary associator in $\mathcal{KD}$:

$$\begin{aligned}
&[a_1 + b_1m, a_2 + b_2m,\ a_3 + b_3m] \\
&\qquad = \big((a_1a_2 + \mu\overline{b_2}b_1)a_3 + \mu\overline{b_3}(b_1\overline{a_2} + b_2a_1) \\
&\qquad\qquad - a_1(a_2a_3 + \mu\overline{b_3}b_2) - \mu(\overline{b_2\overline{a_3} + b_3a_2})b_1\big) \\
&\qquad + \big((b_1\overline{a_2} + b_2a_1)\overline{a_3} + b_3(a_1a_2 + \mu\overline{b_2}b_1) \\
&\qquad\qquad - (b_2\overline{a_3} + b_3a_2)a_1 - b_1(\overline{a_2a_3 + \mu\overline{b_3}b_2})\big)m,
\end{aligned}$$

which by commutativity and associativity becomes

$$\begin{aligned}
&\big(a_1a_2a_3 + \mu b_1\overline{b_2}a_3 + \mu a_1b_2\,\overline{b_3} + \mu b_1\overline{a_2}\,\overline{b_3}\\
&\qquad - a_1a_2a_3 - \mu a_1b_2\overline{b_3} - \mu b_1\overline{a_2}\,\overline{b_3} - \mu b_1\overline{b_2}a_3\big)\\
&+ \big(b_1\overline{a_2}\,\overline{a_3} + a_1b_2\overline{a_3} + a_1a_2b_3 + \mu b_1\overline{b_2}b_3\\
&\qquad - a_1b_2\overline{a_3} - a_1a_2b_3 - b_1\overline{a_2}\,\overline{a_3} - \mu b_1\overline{b_2}b_3\big)m\\
&= 0.
\end{aligned}$$

Finally, for alternativity of $\mathcal{KD}(\mathrm{A},\mu)$, to see that associativity of A is *necessary* we compute $[c, am, \bar{b}] + [am, c, \bar{b}] = \big((ac)b - (ab)c + (a\bar{c})b - a(\overline{c\bar{b}})\big)m = \big((a\,t(c)1)b - [a,b,c] - a(b\,t(c)1)\big)m = -[a,b,c]m$; from this, if linearized left alternativity holds in $\mathcal{KD}$, then all associators in A vanish, and A is associative. A central involution is also necessary: $[am, am, bm] = \big(\mu b(\bar{a}a) - a(\overline{\mu\bar{b}a})\big)m = \mu\big(b(\bar{a}a) - (a\bar{a})b\big)m$ vanishes iff $a\bar{a} = \bar{a}a$ [setting $b = 1$] commutes with all b, and once all norms are central, so are all traces and (thanks to $\frac{1}{2}$) all self-adjoint elements. We will leave it as an exercise to show that associativity plus central involution are sufficient to guarantee alternativity. □

Exercise 2.5.2 Show in full detail that $\mathcal{KD}(\mathrm{A},\mu)$ is alternative if A is associative with central involution.

One-Sided Simplicity Theorem 2.5.3 *When a simple associative noncommutative algebra* A *with central involution undergoes the Cayley–Dickson process, it loses its associativity and all of its one-sided ideals*: $\mathcal{KD}(\mathrm{A},\mu)$ *has no proper one-sided ideals whatsoever. In particular, an octonion algebra over a field has no proper one-sided ideals, hence is always simple.*

PROOF. When A is not commutative, we know that $\mathcal{KD}(\mathrm{A},\mu) = \mathrm{A} \oplus \mathrm{A}m$ is not associative by $\mathcal{KD}$ Inheritance 2.5.2 of associativity, and we claim that it has no proper left ideals (then, thanks to the involution $*$, it has no proper right ideals either) when A is simple. So suppose B is a nonzero left ideal of $\mathcal{KD}$. Then it contains a nonzero element $x = a + bm$, and we can arrange it so that $a \neq 0$ [if $a = 0$ then $x = bm$ for $b \neq 0$, and B contains $x' = mx = m(bm) = a'$ with $a' = -\mu\bar{b} \neq 0$]. By noncommutativity we can choose a nonzero commutator $[u,v]$ in A; then B also contains $L_{\mathrm{A}}L_{\mathrm{A}m}L_m(L_uL_v - L_{vu})L_{\mathrm{A}}(a + bm) = L_{\mathrm{A}}L_{\mathrm{A}m}L_m(L_uL_v - L_{vu})(\mathrm{A}a + b\mathrm{A}m) = L_{\mathrm{A}}L_{\mathrm{A}m}L_m([u,v]\mathrm{A}a + b\mathrm{A}(vu - vu)m) = L_{\mathrm{A}}L_{\mathrm{A}m}(\overline{[u,v]\mathrm{A}a}m) = L_{\mathrm{A}}(\mu[u,v]\mathrm{A}a\mathrm{A}) = \mu\mathrm{A}[u,v]\mathrm{A}a\mathrm{A} = \mathrm{A}$ [by *simplicity* of A and $\mu, [u,v], a \neq 0$]. Therefore B contains A, hence also $L_m\mathrm{A} = \overline{\mathrm{A}}m = \mathrm{A}m$, and therefore $\mathrm{B} = \mathrm{A} + \mathrm{A}m = \mathcal{KD}(\mathrm{A},\mu)$. Thus as soon as B is nonzero it must be the entire algebra. □

This is an amazing result, because the only unital *associative* algebras having no proper one-sided ideals are the division algebras, but split octonion algebras are far removed from division algebras. It turns out that one-sided ideals are not very useful in the theory of alternative algebras: they are hard to construct, and when they do appear they often turn out to be two-sided. As in Jordan algebras, in alternative algebras the proper analogue of one-sided ideals are the *inner ideals* $\mathrm{B} \subseteq \mathrm{A}$ with $U_{\mathrm{B}}\widehat{\mathrm{A}} \subseteq \mathrm{B}$, i.e., b^2, $bab \in \mathrm{B}$ for all $b \in \mathrm{B}, a \in \mathrm{A}$: an alternative algebra is a division algebra iff it has no proper inner ideals (see Problem 2.2 at the end of this Chapter).

2.6 The Hurwitz Theorem

We will show why the Cayley–Dickson doubling process with its twisted multiplication takes place naturally inside a composition algebra, and continues until it exhausts the entire algebra.

Jacobson Necessity Theorem 2.6.1 *If* A *is a proper finite-dimensional unital subalgebra of a composition algebra* C *over a field* Φ*, such that the norm form* N *is nondegenerate on* A*, then there exist elements* $m \in \mathrm{A}^\perp$ *with* $N(m) = -\mu \neq 0$*, and for any such element* $\mathrm{A} + \mathrm{A}m$ *is a subalgebra of* C *which is necessarily isomorphic and isometric to* $\mathcal{KD}(\mathrm{A}, \mu)$*.*

PROOF. Because $N(\cdot,\cdot)$ is a *nondegenerate* bilinear form on the *finite-dimensional* space A, we have a decomposition $\mathrm{C} = \mathrm{A} \oplus \mathrm{A}^\perp$.[7] Here $\mathrm{A}^\perp \neq 0$ since A is *proper*, so by nondegeneracy $\mathbf{0} \neq N(\mathrm{A}^\perp, \mathrm{C}) = N(\mathrm{A}^\perp, \mathrm{A} + \mathrm{A}^\perp) = N(\mathrm{A}^\perp, \mathrm{A}^\perp)$, and N is not identically zero on $\mathrm{A}^\perp$.

Thus we may find an anisotropic vector orthogonal to A,

$$(1) \qquad m \in \mathrm{A}^\perp, \quad N(m) =: -\mu \neq 0.$$

The amazing thing is that we can choose any vector we like, *and then the doubling process starts automatically.* We claim that the space $\mathrm{K} := \mathrm{A} + \mathrm{A}m$ has the structure of $\mathcal{KD}(\mathrm{A}, \mu)$.

The first thing is to establish directness

$$\mathrm{K} := \mathrm{A} \oplus \mathrm{A}m,$$

i.e., $\mathrm{A} \cap \mathrm{A}m = \mathbf{0}$, which will follow from the assumed nondegeneracy $\mathrm{A} \cap \mathrm{A}^\perp = \mathbf{0}$ of A because

$$(2) \qquad \mathrm{A}m \subseteq \mathrm{A}^\perp$$

from $N(am, \mathrm{A}) = N(m, \bar{a}\mathrm{A})$ [by Left Adjoint 2.3.2(1)] $\subseteq N(m, \mathrm{A}) = 0$ since A is assumed to be a *subalgebra* which by construction is orthogonal to m.

In particular (2) implies that $N(a+bm) = N(a)+N(a, bm)+N(b)N(m) = N(a) - \mu N(b)$ [recalling the definition (1) of μ] and $T(am) = N(1, am) \subseteq$

[7] Recall this basic linear algebra fact: each vector $a \in \mathrm{A}$ determines a linear functional $N(a, \cdot)$ on A, and by nonsingularity this map is injective *into* the dual space of A, hence by finite-dimensionality *onto* the dual space; then every $c \in \mathrm{C}$ likewise produces a linear functional $N(c, \cdot)$ on A, but this must have already been claimed by some $a \in \mathrm{A}$, so $c' = c - a$ has $N(c', \cdot) = 0$ and $c' \in \mathrm{A}^\perp$, yielding the decomposition $c = a + c'$.

$N(\mathrm{A}, \mathrm{A}m) = 0$ (it is crucial that A be unital), so the involution and norm are the same in K and $\mathcal{KD}(\mathrm{A}, \mu)$:

$$(3) \qquad N(a + bm) = N(a) - \mu N(b), \quad \overline{a + bm} = \overline{a} - bm.$$

It remains to prove that the products are the same. But *the Cayley–Dickson Product Rules* (KD0)–(KD4) in 2.5.1 *are forced upon us,* and the blame rests squarely on Left and Right Kirmse 2.3.2(1). Notice first the role of the scalar μ: it represents the *square*, not the *norm*, of the element m:

$$(4) \qquad m^2 = \mu 1,$$

since $m^2 = -m\overline{m}$ [by involution (3)] $= -N(m)1$ [by either Kirmse on $y = 1$] $= +\mu 1$ [by the definition (1)]. Then (KD4) follows from $\mu\overline{b}a - (am)(bm) = \overline{b}((am)m) + \overline{(am)}(bm)$ [by (4), right alternativity, and involution (3)] $= N(b, am)m = 0$ [by linearized Left Kirmse and (2)]; (KD3) follows from $(a\overline{b})m - (am)b = (a\overline{b})m + (a\overline{m})b$ [using the involution (3)] $= aN(b, m) = 0$ [by linearized Right Kirmse and (2)]; and (KD2) follows from $a(bm) - (ba)m = -a(\overline{bm}) + (ba)\overline{m}$ [from (3)] $= -(1a)\overline{bm} - (1(bm))\overline{a} + (bm)\overline{a} + (ba)\overline{m} = -1N(a, bm) + bN(a, m) = 0$ [by linearized Right Kirmse and (2)]. (KD0) follows directly from the involution (3), $m\,\overline{a} = -\overline{m}\,\overline{a} = -(\overline{am}) = am$, and (KD1) is trivial.

Thus there is no escape from the strange Cayley–Dickson Product Rules, and $\mathrm{K} \cong \mathcal{KD}(\mathrm{A}, \mu)$. □

Jacobson Necessity can start at any level, not just at the 1-dimensional level $\Phi 1$: from *any* proper nondegenerate composition subalgebra $\mathrm{B}_1 = \mathrm{B}$ of C we can apply the $\mathcal{KD}$ process to form $\mathrm{B}_2 = \mathcal{KD}(\mathrm{B}_1, \mu_2)$ and continue till we exhaust C.

Exercise 2.6.1A* (1) Show that the quaternion subalgebra Q in an 8-dimensional octonion algebra $\mathrm{O} = \mathcal{KD}(\mathrm{Q}, \mu)$ over a field Φ is not uniquely determined: if Q' is *any* 4-dimensional quaternion subalgebra of O then $\mathrm{O} = \mathcal{KD}(\mathrm{Q}', \mu')$ for some μ'. (2) Show that the quaternion subalgebras of O need not all be isomorphic: the real split octonion algebra can be built both from the non-split Hamilton's quaternion algebra $\mathbb{H}$ and from the split quaternion algebra $\mathcal{M}_2(\mathbb{R})$.

Exercise 2.6.1B* Show that shifting the scalar μ in the Cayley–Dickson construction by an invertible square (cf. Exercise 2.5.1) not only produces an *isomorphic* algebra, inside $\mathrm{C} = \mathcal{KD}(\mathrm{A}, \mu)$ it produces the *same* subalgebra: if $\mathrm{C} = \mathrm{A} + \mathrm{A}m = \mathcal{KD}(\mathrm{A}, \mu)$ for a nondegenerate composition subalgebra A, then for any invertible $\alpha \in \Phi$ we also have $\mathrm{C} = \mathrm{A} + \mathrm{A}m' = \mathcal{KD}(\mathrm{A}, \alpha^2\mu)$ for a suitable m'.

Now we are equipped to prove Hurwitz's Theorem, completely describing all composition algebras.

Hurwitz's Theorem 2.6.2 *Any composition algebra* C *over a field* Φ *of characteristic* $\neq 2$ *has finite dimension* 2^n *for* $n = 0, 1, 2, 3$, *and is one of the following*:

(C_0) *The ground field* $C_0 = \Phi 1$ *of dimension* 1, *commutative associative with trivial involution*;

(C_1) *A binarion algebra* $C_1 = \mathcal{KD}(C_0, \mu_1)$ *of dimension* 2, *commutative associative with nontrivial involution*;

(C_2) *A quaternion algebra* $C_2 = \mathcal{KD}(C_1, \mu_2)$ *of dimension* 4, *noncommutative associative*;

(C_3) *An octonion algebra* $C_3 = \mathcal{KD}(C_2, \mu_3)$ *of dimension* 8, *noncommutative nonassociative but alternative.*

PROOF. Start with the subalgebra $C_0 = \Phi 1$ of dimension 1; this is a unital subalgebra, with nondegenerate norm *because the characteristic isn't* 2 (in characteristic 2 the norm $N(\alpha 1, \beta 1) = \alpha\beta T(1) = 2\alpha\beta$ vanishes identically). If C_0 is the whole algebra, we are done. C_0 is commutative associative with trivial involution.

If $C_0 < C$, by Jacobson Necessity we can choose $i \perp C_0$ with $N(i) = -\mu_1 \neq 0$, and obtain a subalgebra $C_1 = \mathcal{KD}(C_0, \mu_1)$ of dimension 2. If C_1 is the whole algebra, we are done. C_1 is still commutative and associative, but with nontrivial involution, by $\mathcal{KD}$ Inheritance 2.5.2, and is called a (generalized) binarion algebra.

If $C_1 < C$, by Jacobson Necessity we can choose $j \perp C_1$ with $N(j) = -\mu_2 \neq 0$, and obtain a subalgebra $C_2 = \mathcal{KD}(C_1, \mu_2)$ of dimension 4. If C_2 is the whole algebra, we are done. C_2 is noncommutative associative by the $\mathcal{KD}$ Inheritance Theorem, and is called a (generalized) quaternion algebra.

If $C_2 < C$, by Jacobson Necessity we can choose $\ell \perp C_2$ with $N(\ell) = -\mu_3 \neq 0$, and obtain a subalgebra $C_3 = \mathcal{KD}(C_2, \mu_3)$ of dimension 8. If C_3 is the whole algebra, we are done. C_3 is nonassociative by $\mathcal{KD}$ Inheritance [but of course is still alternative by the Composition Consequences 2.3.2(2)], and is called a (generalized) octonion algebra.

If $C_3 < C$, by Jacobson Necessity we can choose $m \perp C_3$ with $N(m) = -\mu_4 \neq 0$, and obtain a subalgebra $C_4 = \mathcal{KD}(C_3, \mu_4)$ of dimension 16. But this is no longer alternative by $\mathcal{KD}$ Inheritance, and no longer permits composition.

Thus C_3 cannot be proper, it must be the entire algebra, and we stop at dimension 8. $\square$

Notice that we never assumed that the composition algebra was finite-dimensional: finiteness (with bound 8) is a fact of nature for nonassociative composition algebras. Similarly, finiteness (with bound 27) is a fact of nature for exceptional Jordan algebras.

2.7 Problems for Chapter 2

Problem 2.1 Alternative algebras have a smooth notion of inverses. An element x in a unital alternative algebra is *invertible* if it has an *inverse* y, an element satisfying the usual inverse condition $xy = yx = 1$. An alternative algebra is a *division algebra* if all its nonzero elements are invertible. Show that $y = x^{-1}$ is uniquely determined, and left and right multiplication operators L_x, R_x are invertible operators: $L_x^{-1} = L_{x^{-1}}$, $R_x^{-1} = R_{x^{-1}}$. We will return to this in Problem 21.2.

Problem 2.2 (1) Show that in any alternative algebra A, an element x determines principal inner ideals Ax and xA (which are seldom one-sided ideals), and any two elements determine an inner ideal xAy. (2) Show that a unital alternative algebra is a division algebra iff it has no proper inner ideals. (3) Extend this result to non-unital algebras (as usual, you will have to explicitly exclude a "one-dimensional" trivial algebra).

3

Three Special Examples

We repeat, in more detail, the discussion in the *Historical Survey* of the 3 most important examples of special Jordan algebras.

3.1 Full Type

In any linear algebra we can always introduce a "Jordan product," though the resulting structure is usually not a Jordan algebra. The progenitor of all special examples is the Jordan algebra obtained by "Jordanifying" an associative algebra. We can take any associative algebra A, pin a $^+$ to it, and utter the words "I hereby dub thee Jordan".[1]

Full Example 3.1.1 (1) *If* A *is any linear algebra with product* xy, A^+ *denotes the linear space* A *under the Jordan product*

$$\mathrm{A}^+\colon\ x \bullet y := \tfrac{1}{2}(xy + yx).$$

This will be unital if A *is unital, and Jordan if* A *is associative:any associative* Φ*-algebra* A *may be turned into a Jordan* Φ*-algebra* A^+ *by replacing its associative product by the Jordan product.*

(2) *When* A *is associative, the auxiliary Jordan products are given by*

$$x^2 = xx, \quad \{x, z\} = xz + zx, \quad U_x y = xyx, \quad \{x, y, z\} = xyz + zyx.$$

(3) *In the associative case, any associative homomorphism or anti-homomorphism* $\mathrm{A} \to \mathrm{A}'$ *is at the same time a Jordan homomorphism* $\mathrm{A}^+ \to (\mathrm{A}')^+$*; any associative subalgebra or ideal of* A *is a Jordan subalgebra or ideal of* A^+*. If* A *is unital, the analogous homomorphism and subalgebra results hold in the categories of unital associative and Jordan algebras.*

---

[1] cf. the treatment of the Full Example and the definition of speciality in Section I.2.5, its U-operator in I.4.1, and its inner ideals in I.4.7.

(4) A^+ *is unital iff* A *is unital* (*in which case the units coincide*):

$$e \text{ Jordan unit for } A^+ \iff e \text{ associative unit for } A.$$

(5) *Any left ideal* L *or right ideal* R *of* A *is an inner ideal of* A^+, *as is any intersection* $L \cap R$; *for any elements* $x, y \in A$ *the submodules* xA, Ay, xAy *are inner ideals of* A^+.

(6) A^+ *is simple as a Jordan algebra iff* A *is simple as an associative algebra. If* A *is semiprime, every nonzero Jordan ideal* $I \triangleleft A^+$ *contains a nonzero associative ideal* $B \triangleleft A$.

PROOF. Throughout it will be easier to work with the brace $\{x, y\} = 2x \bullet y$ and avoid all the fractions $\frac{1}{2}$ in the bullet.

(1) A^+ satisfies the Jordan axioms, since commutativity (JAX1) is clear, and the Jordan identity (JAX2) holds because

$$\begin{aligned}\{\{x^2, y\}, x\} &= (xxy + yxx)x + x(xxy + yxx)\\ &= x^2yx + yx^3 + x^3y + xyx^2\\ &= x^3y + x^2yx + xyx^2 + yx^3\\ &= x^2(xy + yx) + (xy + yx)x^2\\ &= \{x^2, \{y, x\}\}.\end{aligned}$$

A "better proof" is to interpret the Jordan identity as the operator identity $[V_{x^2}, V_x] = 0$ for $V_x = L_x + R_x$ in terms of the left and right multiplications in the associative algebra A. The associative law $[x, y, z] = 0$ can be interpreted three ways as operator identities (acting on z, x, y, respectively) $L_{xy} - L_x L_y = 0$ (i.e., the *left regular representation* $x \mapsto L_x$ is a homomorphism), $R_z R_y - R_{yz} = 0$ (the *right regular representation* $x \mapsto R_x$ is an anti-homomorphism), $R_z L_x - L_x R_z = 0$ (all L's commute with all R's), so $[V_{x^2}, V_x] = [L_{x^2} + R_{x^2}, L_x + R_x] = [L_{x^2}, L_x] + [R_{x^2}, R_x] = L_{[x^2,x]} + R_{[x,x^2]} = 0$.

(2) The description of the auxiliary products is a straightforward verification, the first two being trivial, the fourth a linearization of the third, and the (crucial) third formula follows from $2U_x y = \{x, \{x, y\}\} - \{x^2, y\} = x(xy + yx) + (xy + yx)x - (xxy + yxx) = 2xyx$.

(3) Maps preserving or reversing the associative product xy certainly preserve the symmetric Jordan brace $\{x, y\}$: an anti-homomorphisms has $\varphi(\{x, y\}) = \varphi(xy) + \varphi(yx) = \varphi(y)\varphi(x) + \varphi(x)\varphi(y) = \varphi(x)\varphi(y) + \varphi(y)\varphi(x) = \{\varphi(x), \varphi(y)\}$. Subspaces closed under associative products clearly remain closed under Jordan products built out of the associative ones. (4) Any associative unit certainly also acts as unit for the derived Jordan product; conversely, if e is a Jordan unit, then in particular it is an idempotent $e^2 = e$, with $eye = y$ by (2) above, so multiplying on the left by e gives $e(y) = e(eye) = eye = y$, and dually $y = ye$, so e is an associative unit.

(5) One-sided ideals and submodules $B = xAy$ are closed under $B\widehat{A}B$ (which is even stronger than being inner).

(6) For A^+ to be simple, it is certainly necessary that A be simple: any proper associative ideal would be a proper Jordan ideal. The converse is true, but not so obvious, since Jordan ideals need not be associative ideals. It will be enough to prove the containment assertion: if A is simple it is certainly semiprime, and then any proper Jordan $\mathbf{0} < I < A^+$ would contain a proper associative ideal $\mathbf{0} < B \subseteq I < A^+$, contrary to simplicity. But for any nonzero element $b \in I$ we have $b\widehat{A}b \neq \mathbf{0}$ by semiprimeness of A, and the associative ideal $B := \widehat{A}b\widehat{A}b\widehat{A} \neq \mathbf{0}$ is contained entirely within I because it is spanned by elements $xbybz$ for $x, y, z \in \widehat{A}$, which can all be expressed in terms of Jordan products involving I: $\widehat{A}b\widehat{A}b \subseteq I$ since $xbyb = (xb)yb + by(xb) - b(yx)b = \{xb, y, b\} - U_b(yx) \in \{\widehat{A}, \widehat{A}, I\} - U_I\widehat{A} \subseteq I$, therefore $xbybz = \{xbyb, z\} - (zx)byb \in \{I, A\} - I \subseteq I$. □

Exercise 3.1.1A Extend the "better" proof of the Jordan identity in (1) above to show that if C is a commutative subset of A^+ (i.e., $[c_1, c_2] = 0$ for all $c_1, c_2 \in C$), then the elements of C Jordan-operator commute: $[V_C, V_C] = 0$.

Exercise 3.1.1B Repeat the argument of (4) above to show that if $J \subseteq A^+$ is a special Jordan algebra with unit e, then $ex = xe = x$ for all $x \in J$, so that J lives in the Peirce subalgebra eAe of A where e reigns as unit.

Exercise 3.1.1C If A is an associative algebra on which the Jordan product vanishes entirely, all $x \bullet y = 0$, show that the associative product is "alternating" ($x^2 = 0, xy = -yx$ for all x, y), and hence $2A^3 = 0$ ($2xyz = 0$ for all x, y, z).

Exercise 3.1.1D* Show that A^+ is a Jordan algebra if A is merely a *left alternative algebra* (satisfying the left alternative law $L_{x^2} = L_x^2$, but not necessarily flexibility or right alternativity), indeed show that it is even a special Jordan algebra by showing that the left regular representation $x \mapsto L_x$ imbeds A^+ in $\mathcal{E}^+$ for $\mathcal{E}$, the associative algebra $\mathcal{E}nd_\Phi(\widehat{A})$ of endomorphisms of the unital hull of A.

This construction gives us **plus functors** from the categories of associative [respectively unital associative] Φ-algebras to the category of Jordan [respectively unital Jordan] Φ-algebras; it is easy to see that these intertwine with the formal unitalization functor $[(\widehat{A})^+ = \widehat{(A^+)}]$ and both commute with scalar extension functors $[(A_\Omega)^+ = (A^+)_\Omega]$.

The offspring of the full algebras A^+ are all the special algebras, those that result from associative algebras under "Jordan multiplication"; these were precisely the Jordan algebras the physicists originally sought to copy yet avoid.

Special Definition 3.1.2 *A Jordan algebra is* **special** *if it can be imbedded in (i.e., is isomorphic to a subalgebra of) an algebra* A^+ *for some associative* A*, otherwise it is* **exceptional**. *A* **specialization** *of a Jordan algebra* J *in an associative algebra* A *is a homomorphism* $\sigma : J \to A^+$ *of Jordan algebras. Thus a Jordan algebra is special iff it has an injective specialization.*

By means of some set-theoretic sleight of hand, every special algebra actually is a *subset* of an algebra A^+, so we usually think of a special algebra as *living inside* an associative algebra, which is then called an *associative envelope.*[2]

The full algebra A^+ is "too nice" (too close to being associative); more important roles in the theory are played by certain of its subalgebras (the hermitian and spin types), together with the Albert algebra, in just the same way that the brothers Zeus and Poseidon play more of a role in Greek mythology than their father Kronos. We now turn, in the next two sections, to consider these two offspring. (DNA testing reveals that the third brother, Hades, is exceptional and not a son of Kronos at all; we examine his cubic origins in the next chapter.)

3.2 Hermitian Type

The second important class, indeed the archetypal example for all of Jordan theory, is the class of algebras of hermitian type.[3] Here the Jordan subalgebra is selected from the full associative algebra by means of an involution as in $*$-Algebra Definition 1.5.1.

Hermitian Definition 3.2.1 *If* $(A, *)$ *is a* $*$*-algebra then* $\mathcal{H}(A, *)$ *denotes the set of* hermitian *elements* $x \in A$ *with* $x^* = x$, *and* $\mathcal{S}kew(A, *)$ *denotes the set of skew-hermitian elements* $x^* = -x$. *If* C *is a composition algebra with its standard involution, we just write* $\mathcal{H}(C)$ *or* $\mathcal{S}kew(C)$.

In the case of complex matrices (or operators on complex Hilbert space) with the usual conjugate-transpose (or adjoint) involution, these are just the usual hermitian matrices or operators. In the case of real matrices or operators, we get just the symmetric matrices or operators. *For complex matrices there is a big difference between hermitian and symmetric*: the Jordan $\mathbb{R}$-algebra of *hermitian* matrices is *formally real*, $X_1^2 + \cdots + X_r^2 = 0$ for $X_i^* = X_i$ implies that $X_1 = \cdots = X_r = 0$ (in particular, there are no nilpotent elements), whereas the Jordan $\mathbb{C}$-algebra of *symmetric* complex matrices has nilpotent elements $X^2 = 0, X = X^{tr} \neq 0$: witness the matrix $\left(\begin{smallmatrix} 1 & i \\ i & -1 \end{smallmatrix}\right)$.

[2] The set-theoretic operation cuts out the image $\varphi(J)$ with a scalpel and replaces it by the set J, then sews a new product into the envelope using the old recipe. A Jordan algebra can have lots of envelopes, some better than others.

[3] The Hermitian algebras were introduced in Section I.2.5 and I.2.6, and their inner ideals described in I.4.7.

The term "hermitian"[4] is more correctly used for involutions "of the second kind," which are "conjugate-linear," $(\alpha x)^* = \bar{\alpha}\, x^*$ for a nontrivial involution $\alpha \mapsto \bar{\alpha}$ on the underlying scalars, while the term *symmetric* is usually used for involutions "of the first kind," which are linear with respect to the scalars, $(\alpha x)^* = \alpha x^*$. We will allow the trivial involution, so that "hermitian" includes "symmetric," and WE WILL HENCEFORTH SPEAK OF HERMITIAN ELEMENTS AND NEVER AGAIN MENTION SYMMETRIC ELEMENTS! At the same time, we work entirely in the category of modules over a fixed ring of scalars Φ, so *all involutions are Φ-linear.* This just means that we must restrict the allowable scalars. For example, the hermitian complex matrices are a real vector space, but not a complex one (even though the matrix entries themselves are complex).

Hermitian Example 3.2.2 (1) *If* $(\mathrm{A}, *)$ *is an associative $*$-algebra, then* $\mathcal{H}(\mathrm{A}, *)$ *forms a Jordan subalgebra of* A^+ *(which is unital if* A *is), and any $*$-homomorphism or anti-homomorphism* $(\mathrm{A}, *) \to (\mathrm{A}', *')$ *restricts to a Jordan homomorphism* $\mathcal{H}(\mathrm{A}, *) \to \mathcal{H}(\mathrm{A}', *')$.

(2) *If* B *is any subalgebra and* I *any ideal of* A, *then* $\mathcal{H}(\mathrm{B}, *)$ *is a Jordan subalgebra and* $\mathcal{H}(\mathrm{I}, *)$ *a Jordan ideal of* $\mathcal{H}(\mathrm{A}, *)$. *For any element* $x \in \mathrm{A}$ *the module* $x\mathcal{H}(\mathrm{A}, *)x^*$ *is an inner ideal of* $\mathcal{H}(\mathrm{A}, *)$.

(3) *If* A *is a $*$-simple associative algebra and* $\frac{1}{2} \in \Phi$, *then* $\mathcal{H}(A, *)$ *is a simple Jordan algebra; every nonzero Jordan ideal* $\mathrm{I} \triangleleft \mathcal{H}(\mathrm{A}, *)$ *contains* $\mathcal{H}(\mathrm{B}, *)$ *for a $*$-ideal* $\mathrm{B} \triangleleft \mathrm{A}$. *Conversely, if* A *is unital and semiprime, then* $\mathcal{H}(A, *)$ *simple implies that* A *must be $*$-simple.*

PROOF. (1) The anti-automorphism $*$ is a Jordan automorphism of A^+ by Full Example 3.1.1, so the set $\mathcal{H}$ of fixedpoints forms a Jordan subalgebra (unital if A is), since this is true of the set of elements fixed under *any* Jordan automorphism. Any $*$-homomorphism or anti-homomorphism gives, as noted in 3.1.1(3), a homomorphism $\mathrm{A}^+ \to (\mathrm{A}')^+$ which preserves hermitian elements: $x^* = x \in \mathcal{H} \to \varphi(x)^{*'} = \varphi(x^*) = \varphi(x) \in \mathcal{H}'$.

(2) $\mathcal{H}(\mathrm{C}, *) = \mathrm{C} \cap \mathcal{H}(\mathrm{A}, *)$ remains a subalgebra or ideal in $\mathcal{H}(\mathrm{A}, *)$, and $x\mathcal{H}x^*$ satisfies $(xhx^*)\widehat{\mathcal{H}}(xhx^*) = x\big(hx^*\widehat{\mathcal{H}}xh\big)x^* \subseteq x\mathcal{H}x^*$.

(3) To see that $*$-simplicity is sufficient, note that it implies semiprimeness, which in turns implies that Jordan ideals I can't be trivial:

$$0 \neq b \in \mathrm{I} \implies U_b\mathcal{H} \neq \mathbf{0} \implies U_b\mathrm{I} \neq \mathbf{0}.$$

For the first implication, $b \neq 0$ implies $bab \neq 0$ for some $a \in \mathrm{A}$ by semiprimeness, then again some $0 \neq (bab)a'(bab) = b[t(aba') - (a')^*ba^*]bab = U_bt(aba') \cdot ab - b(a')^* \cdot U_b(a^*ba)$, so $b' = U_bh \neq 0$ for either $h = t(aba')$ or

[4] Not named in honor of some mathematical hermit, but in honor of the French mathematician Charles Hermite. Thus the adjective is pronounced more properly "hair-meet-tee-un" rather than the "hurr-mish-un," but Jacobson is the only mathematician I have ever heard give it this more correct pronunciation. Of course, the most correct pronunciation would be "air-meet-tee-un," but no American goes that far.

$h = a^*ba$. For the second implication, we apply this to $b' \in \mathrm{I}$ to get some $0 \neq U_{b'}h' = (bhb)h'(bhb) = U_b\{h, b, h'\} \cdot hb - bh' \cdot U_bU_hb$, so $0 \neq U_bc$ for either $c = \{h, b, h'\} \in \{\mathcal{H}, \mathrm{I}, \mathcal{H}\} \subseteq \mathrm{I}$ or $c = U_hb \in U_{\mathcal{H}}\mathrm{I} \subseteq \mathrm{I}$.

To conclude that $\mathcal{H}$ is simple, it is enough to prove the containment assertion, since then a proper I would lead to a $*$-ideal B which is neither $\mathbf{0}$ nor A because $\mathbf{0} < \mathcal{H}(\mathrm{B}, *) \subseteq \mathrm{I} < \mathcal{H}(\mathrm{A}, *)$. Starting from $bcb \neq 0$ for some $b, c \in \mathrm{I}$ we generate an associative $*$-ideal $\mathrm{B} := \widehat{\mathrm{A}}bcb\widehat{\mathrm{A}} \neq \mathbf{0}$; we claim that $\mathcal{H}(\mathrm{B}, *) \subseteq \mathrm{I}$. Because of $\frac{1}{2}$, $\mathcal{H}(\mathrm{B}, *) = t(\mathrm{B})$ is spanned by all traces $t(xbcby^*)$ for $x, y \in \widehat{\mathrm{A}}$; by linearization, it suffices if all $xbcbx^*$ fall in I. But if we set $h := xb + bx^*, k = cx + x^*c \in \mathcal{H}$, we have

$$xbcbx^* = (h - bx^*)c(h - xb) = hch + b(x^*cx)b - t(hcxb)$$

where $hch = U_hc \in U_{\mathcal{H}}\mathrm{I} \subseteq \mathrm{I},\ b(x^*cx)b = U_b(x^*cx) \in U_{\mathrm{I}}\mathcal{H} \subseteq \mathrm{I}$, and $t(hcxb) = t(h[k - x^*c]b) = t(hkb) - t(hx^*cb)$ for $t(hkb) = \{h, k, b\} \in \{\mathcal{H}, \mathcal{H}, \mathrm{I}\} \subseteq \mathrm{I}$ and $t(hx^*cb) = t(xbx^*cb) + t(bx^*x^*cb) = \{xbx^*, c, b\} + U_bt(x^*x^*c) \in \{\mathcal{H}, \mathrm{I}, \mathrm{I}\} + U_{\mathrm{I}}\mathcal{H} \subseteq \mathrm{I}$. Thus all the pieces fall into I, and we have containment.

To see that $*$-simplicity is necessary, note that if B were a proper $*$-ideal then $\mathcal{H}(\mathrm{B}, *)$ would be a proper ideal of $\mathcal{H} = \mathcal{H}(A, *)$: it is certainly an ideal, and by *unitality* it can't be all of $\mathcal{H}$ $[1 \in \mathcal{H} = \mathcal{H}(\mathrm{B}, *) \subseteq \mathrm{B}$ would imply $\mathrm{B} = \mathrm{A}]$, and by *semiprimeness* it can't be zero [else $b + b^* = b^*b = 0 \implies bab = (ba + a^*b^*)b - a^*b^*b = \big((ba) + (ba)^*\big)b - a^*\big(b^*b\big) = 0$ for all $a \in \mathrm{A}$, contrary to semiprimeness]. □

Exercise 3.2.2A* Let Φ be an arbitrary ring of scalars. (1) Show that if I is a skew $*$-ideal of an associative $*$-algebra A, in the sense that $\mathcal{H}(\mathrm{I}, *) = 0$, then all $z \in \mathrm{I}$ are skew with $z^2 = 0$ and $zx = -x^*z$ for all $x \in \mathrm{A}$; conclude that each $\mathrm{Z} := \widehat{\mathrm{A}}z\widehat{\mathrm{A}}$ is a trivial $*$-ideal $\mathrm{Z}^2 = 0$. (2) Conclude that if A is semiprime then $\mathrm{I} \neq 0 \implies \mathcal{H}(\mathrm{I}, *) \neq 0$. (3) Show that from any $*$-algebra A we can construct a standard counterexample, a larger $*$-algebra A' with an ideal N having nothing to do with $\mathcal{H}:\ \mathrm{N} \cap \mathcal{H}(\mathrm{A}', *) = 0$. Use this to exhibit a nonzero $*$-ideal in (the non-semiprime!) A' with $\mathcal{H}(\mathrm{I}, *) = 0$.

Exercise 3.2.2B Under certain conditions, a unit for $\mathcal{H}(\mathrm{A}, *)$ must serve as unit for all of the associative algebra A. (1) Show that if A has involution and $h \in \widehat{\mathcal{H}}(A, *),\ x, a \in \mathrm{A}$, we have $xahx = t(xa)hx - t(a^*x^*hx) + (xhx^*)a \in \mathcal{H}\mathrm{A}$ in terms of the trace $t(x) = x + x^*$; conclude that $h\mathcal{H} = 0 \Longrightarrow hx\mathrm{A}hx = 0$. If A is semiprime, conclude that $h\mathcal{H} = 0 \Longrightarrow h\mathrm{A} = 0$ (and even $h = 0$ if h lies in $\mathcal{H}$ instead of just in $\widehat{\mathcal{H}}$). (2) Show that if A is semiprime and $\mathcal{H}(\mathrm{A}, *)$ has Jordan unit e, then e must be the associative unit of A [use the result of Exercise 3.1.1B to see that $ek = k$ for all $k \in \mathcal{H}$, then apply (1) to $h = \hat{1} - e \in \widehat{\mathcal{H}}$]. (3) Use the standard counterexample of the previous Exercise 3.2.2A to show that this can fail if A is not semiprime.

We can describe the situation by saying that we have a **hermitian functor** from the category of associative $*$-algebras (with morphisms the $*$-homomorphisms) to the category of Jordan algebras, sending $(\mathrm{A}, *) \mapsto \mathcal{H}(\mathrm{A}, *)$ and $*$-morphisms $\varphi \mapsto \varphi|_{\mathcal{H}}$. This functor commutes with unitalization and scalar extensions. Somewhat surprisingly, the Exchange Involution Proposition 1.5.3 shows that the hermitian functor gobbles up its parent, the full functor.

Full is Hermitian Proposition 3.2.3 A^+ *is isomorphic to the algebra of hermitian elements of its exchange algebra under the exchange involution*:

$$\mathrm{A}^+ \cong \mathcal{H}(\mathcal{E}x(\mathrm{A}), ex). \qquad \square$$

While $\mathcal{H}(\mathrm{A}, *)$ initially seems to arise as a particular Jordan *subalgebra* of A^+, at the same A^+ arises as a particular *instance* of $\mathcal{H}$. You should always keep $\mathcal{H}(\mathrm{A}, *)$ in mind as the "very model of a modern Jordan algebra"; historically it provided the motivation for Jordan algebras, and it still serves as *the* archetype for all Jordan algebras.

The most important hermitian algebras with finiteness conditions are the hermitian matrix algebras. For $n = 3$ (but for no larger matrix algebras) we can even allow the coordinates to be alternative and still get a Jordan algebra. To see this we have to get "down and dirty" inside the matrices themselves, examining in detail how the Jordan matrix products are built from coordinate building blocks.

Hermitian Matrix Example 3.2.4 (1) *For an arbitrary linear $*$-algebra* D*, the standard conjugate-transpose involution* $X^* := \overline{X}^{tr}$ *is an involution on the linear algebra* $\mathcal{M}_n(\mathrm{D})$ *of all* $n \times n$ *matrices with entries from* D *under the usual matrix product* XY*. The* **hermitian matrix algebra** $\mathcal{H}_n(\mathrm{D}, -) := \mathcal{H}(\mathcal{M}_n(\mathrm{D}), *)$ *consists of the module* $\mathcal{H}_n(\mathrm{D}, -)$ *of all hermitian matrices* $X^* = X$ *under the Jordan bullet or brace product* $X \bullet Y = \frac{1}{2}(XY + YX)$ *or* $\{X, Y\} = XY + YX = 2X \bullet Y$*. In* **Jacobson box notation** *in terms of the usual* $n \times n$ *matrix units* E_{ij} $(1 \leq i, j \leq n)$*, the space* $\mathcal{H}_n(\mathrm{D}, -)$ *is spanned by the basic hermitian elements*

$$\begin{aligned} \delta[ii] &:= \delta E_{ii} && (\delta = \bar{\delta} \in \mathcal{H}(\mathrm{D}, -) \text{ hermitian}), \\ d[ij] &:= dE_{ij} + \bar{d}E_{ji} && (d \in \mathrm{D} \text{ arbitrary }), \\ d[ij] &= \bar{d}[ji] && (\text{symmetry relation}). \end{aligned}$$

(2) *For the square* $X^2 = XX$ *and brace product* $\{X, Y\} = XY + YX$ *we have the multiplication rules* (*for distinct indices* i, j, k) *consisting of* **Four Basic Brace Products**

$$\begin{aligned} \delta[ii]^2 &= \delta^2[ii], & \{\delta[ii], \gamma[ii]\} &= (\delta\gamma + \gamma\delta)[ii], \\ d[ij]^2 &= d\bar{d}[ii] + \bar{d}d[jj], & \{d[ij], b[ij]\} &= (d\bar{b}{+}b\bar{d})[ii]{+}(\bar{d}b{+}\bar{b}d)[jj], \\ \{\delta[ii], d[ij]\} &= \delta d[ij], & \{d[ij], \delta[jj]\} &= d\delta[ij], \\ \{d[ij], b[jk]\} &= db[ik], \end{aligned}$$

and a **Basic Brace Orthogonality** *rule*

$$\{d[ij], b[k\ell]\} = 0 \qquad \text{if} \qquad \{i, j\} \cap \{k, \ell\} = \emptyset.$$

(3) *When the coordinate algebra* D *is associative, or alternative with hermitian elements in the nucleus* (*so that any two elements and their bars lie in an associative subalgebra*), *the* **Three Basic** U**-Products** *for* $U_x y = \frac{1}{2}(\{x,\{x,y\}\} - \{x^2,y\})$ *are*

$$
\begin{aligned}
U_{\delta[ii]}\beta[ii] &= \delta\beta\delta[ii], \\
U_{d[ij]}b[ij] &= d\bar{b}d[ij] \qquad && (i \neq j), \\
U_{d[ij]}\beta[jj] &= d\beta\bar{d}[ii] \qquad && (i \neq j),
\end{aligned}
$$

with **Basic** U**-Orthogonality**

$$U_{d[ij]}b[k\ell] = 0 \quad \textit{if} \qquad \{k,\ell\} \not\subseteq \{i,j\}$$

together with their linearizations in d.

(4) *If* D *is associative, for Jordan triple products with the outer factors from distinct spaces we have* **Three Basic Triple Products**

$$
\begin{aligned}
\{d[ij], b[ji], \gamma[ii]\} &= (db\gamma + \gamma\bar{b}\bar{d})[ii] \qquad && (i \neq j), \\
\{d[ij], b[ji], c[ik]\} &= dbc[ik] \qquad && (k \neq i,j), \\
\{d[ij], b[jk], c[k\ell]\} &= dbc[i\ell] \qquad && (i,j,k,\ell \textit{ distinct}),
\end{aligned}
$$

with **Basic Triple Orthogonality**

$$\{d[ij], b[k\ell], c[pq]\} = 0 \quad (\textit{if the indices can't be connected}). \qquad \square$$

Throughout the book we will continually encounter these same Basic Products.[5] Notice how braces work more smoothly than bullets: the bullet products $\{ii,ij\},\{ij,jk\}$ would involve a messy factor $\frac{1}{2}$. When D is an associative $*$-algebra then $(\mathcal{M}_n(\mathrm{D}),*)$ is again an associative $*$-algebra, and $\mathcal{H}_n(\mathrm{D},-)$ is a *special* Jordan algebra of hermitian type. We will see in the Jordan Coordinates Theorem 14.1.1 that D *must* be associative when $n \geq 4$ in order to produce a Jordan algebra, but when $n = 3$ it suffices if D is alternative with nuclear involution. This will not be special if D is not associative; from an octonion algebra with standard involution we obtain a reduced *exceptional* Albert algebra by this hermitian matrix recipe.

Exercise 3.2.4A (1) For general linear algebras, show that the U-products (3) above become unmanageable,

$$
\begin{aligned}
U_{c[ii]}b[ii] &= \tfrac{1}{2}\left((cb)c + c(bc) + [b,c,c] - [c,c,b]\right)[ii], \\
U_{d[ij]}b[ij] &= \tfrac{1}{2}\left((d\bar{b})d + d(\bar{b}d) + [b,\bar{d},d] - [d,\bar{d},b]\right)[ij], \\
U_{d[ij]}b[jj] &= \tfrac{1}{2}\left((db)\bar{d} + d(b\bar{d})\right)[ii] + \tfrac{1}{2}\left([b,\bar{d},d] - [\bar{d},d,b]\right)[jj],
\end{aligned}
$$

and the triple products (4) with distinct indices become

$$\{d[ij], b[jk], c[k\ell]\} = \tfrac{1}{2}\left(d(bc) + (db)c\right)[i\ell],$$

[5] To keep the mnemonics simple, we think of these as four basic brace products, though they are actually two squares and two braces (plus two braces which are linearizations of a square).

while those with repeated indices become

$$\begin{aligned}
\{d[ij], b[ji], c[ik]\} &= \tfrac{1}{2}\left(d(bc) + (db)c + [\bar{b}, \bar{d}, c]\right)[ik],\\
\{d[ij], b[jk], c[ki]\} &= \tfrac{1}{2}\left(d(bc) + (db)c + (\bar{c}\bar{b})\bar{d} + \bar{c}(\bar{b}\bar{d})\right)[ii]\\
&\quad + \tfrac{1}{2}\left([b, c, d] - [\bar{d}, \bar{c}, \bar{b}]\right)[jj] + \tfrac{1}{2}\left([\bar{b}, \bar{d}, \bar{c}] - [c, d, b]\right)[kk],\\
\{d[ij], b[ji], c[ii]\} &= \tfrac{1}{2}\left(d(bc) + (db)c + (c\bar{b})\bar{d} + c(\bar{b}\bar{d}) + [\bar{b}, \bar{d}, c] - [c, d, b]\right)[ii]\\
&\quad + \tfrac{1}{2}\left([b, c, d] - [\bar{d}, c, \bar{b}]\right)[jj].
\end{aligned}$$

(Yes, Virginia, there are jj, kk-components!) (2) If D is alternative with nuclear involution (all hermitian elements in the nucleus), show that the U-products and triple products become manageable again:

$$\begin{aligned}
U_{c[ii]}b[ii] &= cbc[ii], & \{d[ij], b[jk], c[k\ell]\} &= \tfrac{1}{2}\left(d(bc) + (db)c\right)[i\ell],\\
U_{d[ij]}b[ij] &= d\bar{b}d[ij], & \{d[ij], b[ji], c[ik]\} &= d(bc)[ik],\\
U_{d[ij]}b[jj] &= db\bar{d}[ii], & \{d[ij], b[jk], c[ki]\} &= \mathrm{tr}(dbc)[ii],\\
& & \{d[ij], b[ji], c[ii]\} &= \mathrm{tr}(dbc)[ii].
\end{aligned}$$

using $\mathrm{tr}([\mathrm{D}, \mathrm{D}, \mathrm{D}]) = 0$ for $\mathrm{tr}(b) := b + \bar{b}$ (this trace is usually denoted by $t(b)$, cf. 1.5.1).

EXERCISE 3.2.4B Show that the correspondences $\mathrm{D} \mapsto \mathcal{H}_n(\mathrm{D}, -)$, $\varphi \mapsto \mathcal{H}_n(\varphi)$ (given by $d[ij] \mapsto \varphi(d)[ij]$) is a functor from the category of unital associative $*$-algebras to the category of unital Jordan algebras.

3.3 Quadratic Form Type

The other great class of special Jordan algebras is the class of algebras coming from quadratic forms with basepoint.[6] If we think of a quadratic form as a *norm*, a *basepoint* is just an element with norm 1.

Quadratic Factor Example 3.3.1 *If $Q : M \to \Phi$ is a unital quadratic form on a Φ-module M with basepoint c, then we can define a Jordan Φ-algebra structure on $\mathcal{J}ord(Q, c)$ on M by*

$$x \bullet y := \tfrac{1}{2}\big(T(x)y + T(y)x - Q(x, y)c\big).$$

Here the unit is $1 = c$, and every element satisfies the **Degree–2 Identity**

$$x^2 - T(x)x + Q(x)c = 0.$$

The standard trace involution is an algebra involution,

$$\overline{x \bullet y} = \bar{x} \bullet \bar{y}.$$

The auxiliary U-product is given in terms of the trace involution by

$$U_x y = Q(x, \bar{y})x - Q(x)\bar{y},$$

and the norm form Q **permits Jordan composition** *with U,*

$$Q(c) = 1, \qquad Q(U_x y) = Q(x)Q(y)Q(x).$$

[6] Spin factors were introduced in Section I.2.6 and I.3.6, quadratic factors in I.3.7, and their U-operators in I.4.1.

If Q is a nondegenerate quadratic form over a field, then $\mathcal{J}ord(Q,c)$ is a simple Jordan algebra unless Q is isotropic of dimension 2.

PROOF. The element c acts as unit by setting $y = c$ in the definition of the product, since $T(c) = 2, Q(x,c) = T(x)$ by the Trace Involution Properties 2.3.1(1), and the Degree–2 Identity follows by setting $y = x$ in the definition of the product since $Q(x,x) = 2Q(x)$. The fact that $\mathcal{J}ord(Q,c)$ is Jordan then follows from a very general lemma:

Degree–2 Lemma 3.3.2 *Any unital commutative linear algebra in which every element x satisfies a degree–2 equation,*

$$x^2 - \alpha x + \beta 1 = 0 \qquad (\textit{for some } \alpha, \beta \in \Phi \textit{ depending on } x),$$

is a Jordan algebra.

PROOF. Commutativity (JAX1) is assumed, and the Jordan identity (JAX2) holds since the degree–2 equation implies that we have $[x^2, y, x] = [\alpha x - \beta 1, y, x] = \alpha[x, y, x]$ (1 is always nuclear) $= 0$ [since commutative algebras are always flexible, $(xy)x = x(xy) = x(yx)$]. □

Since the standard involution preserves traces T and norms Q and fixes c by the Trace Involution Properties, it preserves the product $x \bullet y$ built out of those ingredients.

To obtain the expression for the U-operator we use the fact that c is the unit to compute

$$\begin{aligned} U_x y &= 2x \bullet (x \bullet y) - x^2 \bullet y \\ &= x \bullet \Big(T(x)y + T(y)x - Q(x,y)c\Big) - \Big(T(x)x - Q(x)c\Big) \bullet y \\ &= \Big(T(x)x \bullet y + T(y)x^2 - Q(x,y)x\Big) - \Big(T(x)x \bullet y - Q(x)y\Big) \\ &= T(y)\Big(T(x)x - Q(x)c\Big) - Q(x,y)x + Q(x)y \\ &= \Big(T(x)T(y) - Q(x,y)\Big)x - Q(x)\Big(T(y)c - y\Big) \\ &= Q(x,\bar{y})x - Q(x)\bar{y} \end{aligned}$$

[by $Q(x,c) = T(x)$ and the definition of $\bar{y}$]. For Q permitting composition with U, we use our newly-hatched formula for the U-operator to compute

$$\begin{aligned} Q(U_x y) &= Q\left(Q(x,\bar{y})x - Q(x)\bar{y}\right) \\ &= Q(x,\bar{y})^2 Q(x) - Q(x,\bar{y})Q(x)Q(x,\bar{y}) + Q(x)^2 Q(\bar{y}) \\ &= Q(x)^2 Q(y). \end{aligned}$$

[by Trace Involution again].

To see the simplicity assertion, note that a proper ideal I can never contain the unit, over a field it can't contain a nonzero scalar multiple of the unit, so it must be totally isotropic: if $b \in \mathrm{I}$ then from the definition of the product we see $Q(b)1 = T(b)b - b^2 \in \mathrm{I}$ forces $Q(b) = 0$. Similarly, I can never have traceless elements: if $T(b) = 0$, by nondegeneracy some x has $Q(x,b) = 1$ and therefore $1 = Q(x,b)1 = T(x)b + T(b)x - \{x,b\} = T(x)b - \{x,b\} \in \mathrm{I}$, a contradiction. Thus T must be an injective linear transformation from I to the field Φ, showing immediately that I can only be one-dimensional. Normalizing b so that $T(b) = 1$, from $x - Q(x,b)1 = T(b)x - Q(x,b)1 = \{x,b\} - T(x)b \in \mathrm{I}$ we see all x lie in $\Phi 1 + \mathrm{I} = \Phi 1 + \Phi b$ and J is 2-dimensional and isotropic.[7] □

We can form a category of *quadratic forms with basepoint* by taking as objects all (Q,c) and as morphisms $(Q,c) \to (Q',c')$ the Φ-linear isometries φ which preserve basepoint, $Q'(\varphi(x)) = Q(x)$ and $\varphi(c) = c'$. Since such a φ preserves the traces $T'(\varphi(x)) = Q'(\varphi(x),c') = Q'(\varphi(x),\varphi(c)) = Q(x,c) = T(x)$ and hence all the ingredients of the Jordan product, it is a homomorphism of Jordan algebras. In these terms we can formulate our result by saying that we have a **quadratic Jordanification functor** from the category of Φ-quadratic forms with basepoint to unital Jordan Φ-algebras, given on objects by $(Q,c) \to \mathcal{J}ord(Q,c)$ and trivially on morphisms by $\varphi \to \varphi$. This functor commutes with scalar extensions.

EXERCISE 3.3.2* (1) Show that over a field Φ the category of quadratic factors (with morphisms the isomorphisms) is equivalent to the category of quadratic forms with basepoint (with morphisms the isometries preserving basepoints), by proving that a linear map φ is an isomorphism of unital Jordan algebras $\mathcal{J}ord(Q,c) \to \mathcal{J}ord(Q',c')$ iff it is an isometry of quadratic forms-with-basepoint $(Q,c) \to (Q',c')$. (2) Prove that any *homomorphism* φ whose image is not entirely contained in $\Phi c'$ is a (not necessarily bijective) isometry of forms, and in this case $\mathrm{Ker}(\varphi) \subseteq \mathcal{R}ad(Q)$. (3) Show that a linear map φ with $\varphi(c) = 1$ is a homomorphism of $\mathcal{J}ord(Q,c)$ into the 1-dimensional quadratic factor Φ^+ iff the quadratic form Q takes the form $Q(\alpha c + x_0) = \alpha^2 - \varphi(x_0)^2$ relative the decomposition $\mathrm{J} = \Phi c \oplus \mathrm{J}_0$ for $\mathrm{J}_0 = \{x_0 \mid T(x_0) = 0\}$ the subspace of trace zero elements.

Spin Factor Example 3.3.3 *Another construction starts from a "unit-less" Φ-module M with symmetric bilinear form σ, first forms* $\mathrm{J} := \Phi 1 \oplus M$ *by externally adjoining a unit, and then defines a product on* J *by having* 1 *act as unit element and having the product of "vectors"* $v, w \in M$ *be a scalar multiple of* 1 *determined by the bilinear form*:

$$\mathcal{J}Spin(M,\sigma)\colon \mathrm{J} = \Phi 1 \oplus M, \quad 1 \bullet x = x, \quad v \bullet w = \sigma(v,w)1.$$

It is easy to verify directly that the resulting algebra is a unital Jordan algebra, but this also follows because

$$\mathcal{J}Spin(M,\sigma) = \mathcal{J}ord(Q,c)$$

$$\textit{for} \quad c := 1 \oplus 0, \quad Q(\alpha 1 \oplus v) := \alpha^2 - \sigma(v,v), \quad T(\alpha 1 \oplus v) := 2\alpha.$$

[7] We will see later that in the non-simple case we have $\mathrm{I} = \Phi e$, $\mathrm{J} = \Phi e \boxplus \Phi(1-e)$ for an idempotent e, and $\mathcal{J}ord(Q,c)$ is precisely a split binarion algebra.

In fact, since we are dealing entirely with rings of scalars containing $\frac{1}{2}$, *this construction is perfectly general: every quadratic factor* $\mathcal{J}ord(Q, c)$ *is naturally isomorphic to the spin factor* $\mathcal{J}Spin(M, \sigma)$ *for* $\sigma(v, w) = -\frac{1}{2}Q(v, w)$ *the negative of the restriction of* $\frac{1}{2}Q(\cdot, \cdot)$ *to* $M = c^{\perp}$, *since we have the natural decomposition* $\mathrm{J} = \Phi c \oplus M$.

The special case where σ *is the ordinary dot product on* $M = \Phi^n$ *is denoted by* $\mathcal{J}Spin_n(\Phi) := \mathcal{J}Spin(\Phi^n, \cdot)$. □

BEWARE: The restriction of the global Q to M is the NEGATIVE of the quadratic form $q(v) = \sigma(v, v)$. The scalar $q(v)$ comes from the coefficient of 1 in $v^2 = q(v)1$, and the Q (the generic norm) comes from the coefficient of 1 in the degree–2 equation, which you should think of as

$$Q(x) \sim x\bar{x},$$

so that for the skew elements $v \in M$ it reduces to the *negative* of the square. This is the same situation as in the composition algebras of the last chapter, where the skew elements $x = i, j, k, \ell$ with *square* $x^2 = -1$ have *norm* $Q(x) = +1$. The bilinear form version in terms of σ on M is more useful in dealing with bullet products, and the global quadratic version in terms of Q on J is more useful in dealing with U-operators; one needs to be ambidextrous in this regard.

We can form a category of symmetric Φ-bilinear forms by taking as objects all σ and as morphisms $\sigma \to \sigma'$ all Φ-linear *isometries*. In these terms we can formulate 3.3.3 by saying that we have a **spin functor** from symmetric Φ-bilinear forms to unital Jordan Φ-algebras given by $\sigma \mapsto \mathcal{J}Spin(M, \sigma)$ and $\varphi \mapsto \mathcal{J}Spin(\varphi)$, where the Jordan homomorphism is defined by extending φ unitally to J, $\mathcal{J}Spin(\varphi)(\alpha 1 \oplus v) := \alpha 1 \oplus \varphi(v)$. This functor again commutes with scalar extensions.

Since $\frac{1}{2} \in \Phi$ we have a category equivalence between the category of symmetric Φ-bilinear forms and the category of Φ-quadratic forms with basepoint, given by the *pointing correspondence* P sending $\sigma \mapsto (Q, 1)$ on $\mathrm{J} := \Phi 1 \oplus M$, $\varphi \mapsto \mathcal{J}Spin(\varphi)$ as above, with inverse T sending $(Q, c) \mapsto \sigma := -\frac{1}{2}Q(\cdot, \cdot)|_{c^{\perp}}$, $\varphi \mapsto \varphi|_{c^{\perp}}$. [Warning: while TP is the identity functor, PT is only naturally isomorphic to the identity: $PT(Q, c) = (Q, 1)$ has had its basepoint surgically removed and replaced by a formal unit 1.] The Spin Factor Example says that under this equivalence it doesn't matter which you start with, a symmetric bilinear form σ or a quadratic form with basepoint (Q, c), the resulting Jordan algebra will be the same.

EXERCISE 3.3.3 (1) Show directly that we can create a Jordan algebra $\mathcal{J}ord(\sigma', c)$ out of any global bilinear symmetric form σ' and choice of basepoint $c(\sigma'(c,c) = 1)$ on a module M' over Φ, by taking as unit element $1 = c$ and as product $x \bullet y := -\sigma(x,c)y + -\sigma(c,y)x + \sigma(x,y)c$ (cf. Quadratic Factor Example 3.3.1). (2) Show that every spin factor $\mathcal{JS}pin(M,\sigma)$ of a bilinear form arises in this way: $\mathcal{JS}pin(M,\sigma) = \mathcal{J}ord(\sigma', c)$ for M' the unital hull $\widehat{M} = \Phi\hat{1} \oplus M$ with basepoint $c = \hat{1}$ the formal unit adjoined to M, and with global bilinear form $\sigma'(x,y) := \alpha\beta + \sigma(v,w)$ for $x = \alpha\hat{1} \oplus v,\ y = \beta\hat{1} \oplus w$. (3) Conversely, show that every $\mathcal{J}ord(\sigma', c)$ is isomorphic to $\mathcal{JS}pin(M,\sigma)$ for the bilinear form σ which is the negative of σ' restricted to the orthogonal complement of the basepoint: $\mathcal{J}ord(\sigma', c) \cong \mathcal{JS}pin(M,\sigma)$ for $M = \{x \in M' \mid \sigma'(x,c) = 0\},\ \sigma = -\sigma'|_M$. Thus both constructions produce the same Jordan algebras. (4) Show that bilinear forms with basepoint and quadratic forms with basepoint produce exactly the same algebras: $\mathcal{J}ord(\sigma', c) = \mathcal{J}ord(Q,c)$ for $Q(x) := \sigma'(x,x)$, and $\mathcal{J}ord(Q,c) = \mathcal{J}ord(\sigma', c)$ for $\sigma'(x,y) := \frac{1}{2}Q(x,y)$.

3.4 Reduced Spin Factors

There is a way to build a more specialized sort of Jordan algebra out of a quadratic or bilinear form, an algebra which has idempotents and is thus "reduced," the first step towards being "split." These will be the degree–2 algebras which occur in the final classical structure theory of this Classical Part in Chapter 23, and thus are now for us the most important members of the quadratic factor family.

We will see in Chapter 8, when we discuss idempotents and Peirce decompositions, that $\mathcal{JS}pin(M,\sigma)$ has a proper idempotent $e \neq 0, 1$ iff the quadratic form is "unit-valued": $q(v) := \sigma(v,v) = 1$ for some v. In our "reduced" construction, instead of just adjoining a unit we externally adjoin *two* idempotents e_1, e_2; in effect, we are adjoining a unit $1 = e_1 + e_2$ together with an element $v = e_1 - e_2$ where $q(v)$ will automatically be 1. We define the product by having e_i act as half-unit element on M and having the product of vectors w, z in M be a scalar multiple of 1 determined by the bilinear form as usual:

$$\mathrm{J} := \Phi e_1 \oplus M \oplus \Phi e_2, \qquad\qquad 1 := e_1 + e_2,$$
$$e_i \bullet e_j := \delta_{ij} e_i, \quad e_i \bullet w := \tfrac{1}{2} w, \qquad\qquad w \bullet z := \sigma(w,z)1.$$

By making identifications $e_1 = (1,0,0),\ e_2 = (0,0,1),\ w = (0,w,0)$ we can describe this more formally in terms of ordered triples. At the same time, because of the primary role played by q, in our reduced construction *we henceforth depose the bilinear form σ and replace it everywhere by the quadratic form $q(x) = \sigma(x,x)$.*

Reduced Spin Example 3.4.1 *The* **reduced spin factor of a quadratic form** *q on a Φ-module M consists of the direct product module*[8] $\mathcal{R}ed\mathcal{S}pin(q) := \Phi \times M \times \Phi$ *with square*

$$(\alpha, w, \beta)^2 := (\alpha^2 + q(w), (\alpha + \beta)w, \beta^2 + q(w)),$$

hence bullet products

$$(\alpha, w, \beta) \bullet (\gamma, z, \delta) := \tfrac{1}{2}(2\alpha\gamma + q(w, z), (\alpha + \beta)z + (\gamma + \delta)w, 2\beta\delta + q(w, z)).$$

*It is again easy to verify directly from the Degree–*2 *Lemma* 3.3.2 *that this is a unital Jordan algebra, but this also follows since it is a spin construction:*

$$\mathcal{J}\mathcal{S}pin(M, \sigma) = \Phi 1 \oplus M \subseteq \mathcal{J}\mathcal{S}pin(M', \sigma') = \Phi 1 \oplus \Phi v \oplus M$$
$$= \mathcal{R}ed\mathcal{S}pin(q) = \Phi e_1 \oplus \Phi e_2 \oplus M$$

for

$$1 := (1, 0, 1),\ v := (1, 0, -1), \quad e_1 := (1, 0, 0),\ e_2 := (0, 0, 1),$$
$$\sigma'(\alpha v \oplus w, \beta v \oplus z) := \alpha\beta + \sigma(w, z) \text{ on } M' := \Phi v \oplus M,\ \sigma(w, z) = \tfrac{1}{2} q(w, z).$$

Like any spin factor, it can also be described as in Quadratic Factor 3.3.1 *in terms of a global quadratic form:*

$$\mathcal{R}ed\mathcal{S}pin(q) = \mathcal{J}ord(Q, c)$$

for

$$Q((\alpha, w, \beta)) := \alpha\beta - q(w), \qquad c := (1, 0, 1),$$
$$T(\alpha, w, \beta) := \alpha + \beta, \qquad \overline{(\alpha, w, \beta)} := (\beta, -w, \alpha).$$

Thus the quadratic form is obtained by adjoining a "hyperbolic plane" $Q(\alpha e_1 \oplus \beta e_2) = \alpha\beta$ to the negative of q on M. Here the U-operator is given explicitly by

$$U_{(\alpha, w, \beta)}(\gamma, z, \delta) := (\varepsilon, y, \eta)$$

for

$$\varepsilon := [\alpha^2\gamma + \alpha q(w, z) + q(w)\delta],$$
$$\eta := [\beta^2\delta + \beta q(w, z) + q(w)\gamma],$$
$$y := [\alpha\gamma + \beta\delta + q(w, z)]w + [\alpha\beta - q(w)]z.$$

PROOF. That the ordinary construction for σ is contained in the reduced construction for q, and that this coincides with the ordinary construction for σ', follow since (1) everyone has 1 as unit and product on M determined by σ, and (2) in the latter two the element v behaves the same [the same product

[8] The reader might ask why we are not "up front" about the two external copies of Φ we adjoin, parameterizing the module as $\Phi \times \Phi \times M$; the answer is that we have an ulterior motive — we are softening you up for the Peirce decomposition in Chapter 8 with respect to $e = (1, 0, 0)$, which has $(\Phi, 0, 0), (0, M, 0), (0, 0, \Phi)$ as the Peirce spaces $\mathrm{J}_2, \mathrm{J}_1, \mathrm{J}_0$, respectively.

$v \bullet v = (1,0,-1)^2 = (1,0,1) = \sigma'(v,v)1$ with itself, and the same products $v \bullet w = (1,0,-1) \bullet (0,w,0) = (0,0,0) = \sigma'(v,w)1$ with elements $w \in M$].

The Quadratic Factor Example 3.3.2 has quadratic form given by $Q(x) = \tilde{\alpha}^2 - \sigma'(\tilde{v},\tilde{v}) = \tilde{\alpha}^2 - \tilde{\beta}^2 - q(w)$ when $\tilde{v} = \tilde{\beta}v \oplus w$. Here in our case $x = (\alpha, w, \beta) = \tilde{\alpha}1 \oplus \tilde{\beta}v \oplus w$, where the scalars $\tilde{\alpha}, \tilde{\beta}$ are determined by $\tilde{\alpha}(1,0,1) + \tilde{\beta}(1,0,-1) = \alpha(1,0,0) + \beta(0,0,1)$, i.e., $\tilde{\alpha} + \tilde{\beta} = \alpha,\ \tilde{\alpha} - \tilde{\beta} = \beta$ with solution $\tilde{\alpha} = \frac{1}{2}(\alpha+\beta)$, $\tilde{\beta} = \frac{1}{2}(\alpha-\beta)$. Therefore $\tilde{\alpha}^2 - \tilde{\beta}^2 = \frac{1}{4}[(\alpha^2+2\alpha\beta+\beta^2) - (\alpha^2-2\alpha\beta+\beta^2)] = \frac{1}{4}[4\alpha\beta] = \alpha\beta$ and $\sigma(w,w) = q(w)$ as required.

The U-operator in the Quadratic Factor Example is $U_x y = Q(x,\bar{y})x - Q(x)\bar{y}$, which becomes

$$[\alpha\gamma + \beta\delta + q(w,z)](\alpha, w, \beta) - [\alpha\beta - q(w)](\delta, -z, \gamma)$$

for $x = (\alpha, w, \beta)$, $Q(x) = [\alpha\beta - q(w)]$, $y = (\gamma, z, \delta)$, $\bar{y} = (\delta, -z, \gamma)$, $Q(x,\bar{y}) = Q\big((\alpha, w, \beta), (\delta, -z, \gamma)\big) = [\alpha\gamma + \delta\beta - q(w,-z)] = [\alpha\gamma + \beta\delta + q(w,z)]$, where the three coordinates reduce to

$$(1)\ [\alpha\gamma + \beta\delta + q(w,z)]\alpha - [\alpha\beta - q(w)]\delta = [\alpha^2\gamma + \alpha q(w,z) + q(w)\delta],$$

$$(2)\ [\alpha\gamma + \beta\delta + q(w,z)]w - [\alpha\beta - q(w)](-z)$$
$$= [\alpha\gamma + \beta\delta + q(w,z)]w + [\alpha\beta - q(w)]z,$$

$$(3)\ [\alpha\gamma + \beta\delta + q(w,z)]\beta - [\alpha\beta - q(w)]\gamma = [\beta^2\delta + \beta q(w,z) + q(w)\gamma],$$

which are just ε, y, η, as required. □

3.5 Problems for Chapter 3

PROBLEM 3.1 If A^+ is simple then A must certainly be simple too (associative ideals being Jordan ideals). A famous result of I.N. Herstein (cf. the Full Example 3.1.1(6)) says that the converse is true: if A is simple, then so is A^+. Here the ring of scalars Φ can be arbitrary, we don't need $\frac{1}{2}$ (as long as we treat A^+ as a *quadratic* Jordan algebra). Associative rings are equal-characteristic employers; except for Lie or involution results, they are completely oblivious to characteristic.

(1) Verify the following Jordan expressions for associative products: (1) $zbza = \{z, b, za\} - z(ab)z$, (2) $azzb = \{a, z, zb\} - zbza$, (3) $2zaz = \{z, \{z, a\}\} - \{z^2, z\}$, (4) $zazbz = \{zazb, z\} - z^2azb$. (2) Use these to show that if $\mathrm{I} \triangleleft \mathrm{A}^+$ is a Jordan ideal, then for any $z \in \mathrm{I}$ we have $z\widehat{\mathrm{A}}z\widehat{\mathrm{A}} + \widehat{\mathrm{A}}z^2\widehat{\mathrm{A}} \subseteq \mathrm{I}$. Conclude that if *some* $z^2 \neq 0$ then $\widehat{\mathrm{A}}z^2\widehat{\mathrm{A}}$ is a nonzero associative ideal of A contained in I, while if $z^2 = 0$ for *all* $z \in \mathrm{I}$ then $2z\widehat{A}z = z\widehat{A}z\widehat{A}z = 0$ and any $\mathrm{Z} := \widehat{\mathrm{A}}z\widehat{\mathrm{A}}$ is an associative ideal (not necessarily contained in I) with $2\mathrm{Z}^2 = \mathrm{Z}^3 = \mathbf{0}$. (3) Show from these that if A is semiprime then every nonzero Jordan ideal contains a nonzero associative ideal as in the Full Example 3.1.1(6). (4) An associative algebra is *prime* if it has no orthogonal ideals, i.e., $\mathrm{BC} = \mathbf{0}$ for ideals $\mathrm{B}, \mathrm{C} \triangleleft \mathrm{A}$ implies $\mathrm{B} = \mathbf{0}$ or $\mathrm{C} = \mathbf{0}$. A Jordan algebra is *prime* if it has no U-orthogonal ideals, i.e., $U_{\mathrm{I}}\mathrm{K} = \mathbf{0}$ for $\mathrm{I}, \mathrm{K} \triangleleft \mathrm{J}$ implies $\mathrm{I} = \mathbf{0}$ or $\mathrm{K} = \mathbf{0}$. Show that in a semiprime A, two associative ideals are associatively orthogonal, $\mathrm{BC} = 0$, iff they are Jordan orthogonal, $U_{\mathrm{B}}(\mathrm{C}) = \mathbf{0}$. Use this and (3) to prove that A^+ is prime as a Jordan algebra iff A is prime as an associative algebra. (5) Use the results and techniques of Hermitian Example 3.2.2(3) to show that if A is a $*$-prime associative algebra (no orthogonal $*$-ideals), *and we have a scalar* $\frac{1}{2} \in \Phi$, then $\mathcal{H}(\mathrm{A}, *)$ is a prime Jordan algebra.

PROBLEM 3.2* A famous theorem of L.K. Hua (with origins in projective geometry) asserts that any additive inverse-preserving map $\mathrm{D} \to \mathrm{D}'$ of associative division rings must be a Jordan homomorphism, and then that every Jordan homomorphism is either an associative homomorphism or anti-homomorphism. Again, this result (phrased in terms of quadratic Jordan algebras) holds for an arbitrary ring of scalars. This was generalized by Jacobson and Rickart to domains: if $\varphi : \mathrm{A}^+ \to \mathrm{D}^+$ is a Jordan homomorphism into an associative *domain* D which is a Jordan homomorphism, then φ must be either an associative homomorphism or anti-homomorphism.

(1) Let $\Delta(x, y) := \varphi(xy) - \varphi(x)\varphi(y)$ measure deviation from associative homomorphicity, and $\Delta^*(x, y) := \varphi(xy) - \varphi(y)\varphi(x)$ measure deviation from associative anti-homomorphicity; show that $\Delta(x, y)\Delta^*(x, y) = 0$ for all x, y when φ is a Jordan homomorphism. (Before the advent of Jordan triple products this was more cumbersome to prove!) Conclude that if D is a domain then for each pair $x, y \in \mathrm{A}$ either $\Delta(x, y) = 0$ or $\Delta^*(x, y) = 0$. (2) For any x define $\Delta_x := \{y \in \mathrm{A} \mid \Delta(x, y) = 0\}$ and $\Delta^*_x := \{y \in \mathrm{A} \mid \Delta^*(x, y) = 0\}$.

Conclude that $A = \Delta_x \cup \Delta_x^*$. (3) Using the fact that an additive group cannot be the union of two proper subgroups, conclude that for any x we have $\Delta_x = A$ or $\Delta_x^* = A$. (4) Define $\Delta := \{x \in A \mid \Delta_x = A$, i.e., $\Delta(x, A) = 0\}$ to consist of the "homomorphic elements," and $\Delta^* := \{x \in A \mid \Delta_x^* = A$, i.e., $\Delta^*(x, A) = 0\}$ to consist of the "anti-homomorphic elements." Conclude that $A = \Delta \cup \Delta^*$, and conclude again that $\Delta = A$ or $\Delta^* = A$, i.e., $\Delta(A, A) = 0$ or $\Delta^*(A, A) = 0$; conclude that φ is either an associative homomorphism or anti-homomorphism.

PROBLEM 3.3 A symmetric bilinear form σ on an n-dimensional vector space V is given, relative to any basis $v_1, \ldots, v_n$, by a symmetric matrix S: $\sigma(v, w) = \sum_{i,j=1}^n \alpha_i \beta_j s_{ij}$ ($v = \sum_i \alpha_i v_i$, $w = \sum_j \beta_j v_j$, $s_{ij} = s_{ji}$), and this matrix can always be diagonalized: there is a basis $u_1, \ldots, u_n$ with respect to which $S = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$. (1) Show that the λ_i's can be adjusted by squares in Φ (the matrix of σ with respect to the scaled basis $\alpha_1 u_1, \ldots, \alpha_n u_n$ is $diag(\alpha_1^2 \lambda_1, \alpha_2^2 \lambda_2, \ldots, \alpha_n^2 \lambda_n)$), so if every element in Φ has a square root in Φ then the matrix can be reduced to a diagonal matrix of 1's and 0's. Conclude that over an algebraically closed field Φ all nondegenerate symmetric bilinear forms on V are isometrically isomorphic, so up to isomorphism there is only one nondegenerate algebra $J = \mathcal{JS}pin(V, \sigma) \cong \mathcal{JS}pin_n(\Phi)$ of dimension $n+1$ (V of dimension n). (2) Use your knowledge of Inertia to show that over the real numbers $\mathbb{R}$ we can replace the λ's by ± 1 or 0, and the number of 1s, -1s, and 0s form a complete set of invariants for the bilinear form. Conclude that there are exactly $n+1$ inequivalent nondegenerate symmetric bilinear forms on an n-dimensional real vector space, and exactly $n+1$ nonisomorphic spin factors $J = \mathcal{JS}pin(V, \sigma)$ of dimension $n+1$, but all of these have $J_{\mathbb{C}} \cong \mathcal{JS}pin_n(\mathbb{C})$ over the complex numbers.

PROBLEM 3.4* The J-vN-W classification of simple Euclidean Jordan algebras includes $\mathcal{JS}pin_n(\mathbb{R}) = \mathbb{R}1 \oplus \mathbb{R}^n$ under the positive definite inner (dot) product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$. The standard hermitian inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n \xi_i \overline{\eta_i}$ [$\mathbf{x} = (\xi_1, \ldots, \xi_n)$, $\mathbf{y} = (\eta_1, \ldots, \eta_n)$] on $\mathbb{C}^n$ is also a positive definite hermitian form, $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ if $\mathbf{x} \neq 0$. Show nevertheless that $\mathcal{JS}pin_n(\mathbb{C}) = \mathbb{C}1 \oplus \mathbb{C}^n$ with $\mathbf{x} \bullet \mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle 1$ is *not* Euclidean: even when $n = 1$, there are nonzero elements u, v of $\mathcal{JS}pin_n(\mathbb{C})$ with $u^2 + v^2 = 0$.

PROBLEM 3.5 A derivation of a linear algebra A is a linear transformation which satisfies the product rule. (1) Show that $\delta(1) = 0$ if A is unital, and $\delta(x^2) = x\delta(x) + \delta(x)x$; show that this latter suffices for δ to be a derivation if A is commutative and $\frac{1}{2} \in \Phi$. (2) Show that δ is a derivation iff it normalizes the left multiplication operators $[\delta, L_x] = L_{\delta(x)}$ (dually iff $[\delta, R_y] = R_{\delta(y)}$). Show that $T(1) = 0, [T, L_x] = L_y$ for some y in a unital A implies that $y = T(x)$. (3) Show that the normalizer $\{T \mid [T, \mathcal{S}] \subseteq \mathcal{S}\}$ of any set $\mathcal{S}$ of operators forms a Lie algebra of linear transformations; conclude anew that $\mathcal{D}er(A)$ is a Lie algebra. (4) In an associative algebra show that $\delta(z) := [x, z] := xz - zx$ is a derivation for each x (called the *inner* or *adjoint* derivation $\mathrm{ad}(x)$).

Problem 3.6 (1) Show that any automorphism or derivation of an associative algebra A remains an automorphism or derivation of the Jordan algebra A^+. (2) If $J \subseteq A^+$ is special, show that any associative automorphism or derivation which leaves J invariant is a Jordan automorphism or derivation. In particular, though the $\mathrm{ad}(x)$ for $x \in J$ need not leave J invariant, the maps $D_{x,y} = \mathrm{ad}([x,y]) = [V_x, V_y]$ must, and hence form inner derivations. If $J = \mathcal{H}(A, *)$ is hermitian, show that any $ad(z)$ for skew $z^* = -z \in A$ induces a derivation of J. (3) Show that in any Jordan algebra the map $D_{x,y} = 4[L_x, L_y] = [V_x, V_y]$ is an "inner" derivation for each pair of elements x, y.

Problem 3.7* (1) Show that any $D \in \mathcal{E}nd(J)$ which satisfies $D(c) = 0$, $Q(D(x), x) = 0$ for all x is a derivation of $\mathcal{J}ord(Q, c)$. If Φ is a field, or has no nilpotent elements, show that these two conditions are necessary as well as sufficient for D to be a derivation. (2) Deduce the derivation result from the corresponding automorphism result over the ring of dual numbers: show that D is a derivation of J iff $T = \mathbb{1}_J + \varepsilon D$ is an automorphism of $J[\varepsilon] = J \otimes_\Phi \Phi[\varepsilon]$ $(\varepsilon^2 = 0)$.

Question 3.1 Decide on a definition of *derivations* and *inner derivations* of unital quadratic Jordan algebras which reduces to the previous one for linear Jordan algebras. Show that derivations can be defined operator-wise in terms of their interactions with multiplication operators (find formulas for $[D, V_x], [D, U_x], [D, V_{x,y}]$). Show that derivations form a Lie algebra of endomorphisms, i.e., is closed under the Lie bracket $[D, E] := DE - ED$, with the inner derivations as an ideal.

4

Jordan Algebras of Cubic Forms

In this chapter we describe in detail the Jordan algebras that result from cubic forms with basepoint.[1] Unlike the case of quadratic forms, which always spawn Jordan algebras, only certain very special cubic forms with basepoint give rise to Jordan algebras. Not all algebras resulting from a cubic form are exceptional, but the special ones can be obtained by the previous special constructions, so we are most interested in the 27-dimensional exceptional Albert algebras. These are *forms* of the reduced algebra of 3×3 hermitian matrices $\mathcal{H}_3(\mathrm{O})$ with octonion entries, but are not themselves always reduced (they may not have any proper idempotents), indeed they may be division algebras. Clearly, reduced Albert matrix algebras are not enough – we must handle algebras which do not come packaged as matrices. But even for $\mathcal{H}_3(\mathrm{O})$ it has gradually become clear that we must use the cubic norm form and the adjoint to understand inverses, isotopes, and inner ideals.

Unlike Jordan algebras coming from quadratic forms, those born of cubic forms have a painful delivery. Establishing the Jordan identity and the properties of the Jordan norm requires many intermediate identities, and we will banish most of these technical details to Appendix C, proving only the important case of a nondegenerate form on a finite-dimensional vector space over a field. However, unlike the other inmate of the Appendix, Macdonald's Theorem (whose details are primarily combinatorial), the cubic form details are important features of the cubic landscape for those who intend to work with cubic factors (either for their own sake, or for their relations with algebraic groups, Lie algebras, or geometry). But for general students at this stage in their training, the mass of details would obscure rather than illuminate the development.

[1] These algebras were introduced in the *Historical Survey* in Section I.3.8, but we sketched only the finite-dimensional Springer and Jordan Cubic Constructions over a field, where the norm was nondegenerate (so the sharp mapping was already determined by the norm). In this section we consider the general case over arbitrary scalars.

4.1 Cubic Maps

Although quadratic maps are intrinsically defined by their values (and quadratic relations can always be linearized), cubic and higher degree maps require a "scheme-theoretic" treatment: to truly understand the map, especially its internal workings, we need to know the values of the linearizations on the original module, equivalently we need to know the values of the map over all extensions. To extend a cubic map $f : X \to Y$ of Φ-modules from the scalars from Φ to the polynomial ring $\Phi[t_1, \ldots, t_n]$ we need to form values

$$f(\sum_i t_i x_i) = \sum_i t_i^3 f(x_i) + \sum_{i \neq j} t_i^2 t_j f(x_i; x_j) + \sum_{i<j<k} t_i t_j t_k f(x_i, x_j, x_k).$$

Here the *first linearization* $f(x; y)$ is quadratic in x and linear in y; it appears as the coefficient of t in the expansion

$$f(x + ty) = f(x) + tf(x; y) + t^2 f(y; x) + t^3 f(y)$$

(setting $x_1 = x, x_2 = y, t_1 = 1, t_2 = t$ in the general expansion), and is nothing more than the directional derivative $f(x; y) = \partial_y f|_x$ of f in the direction y, evaluated at x. For cubic maps, since $f(x; y)$ is quadratic in x, it *automatically* linearizes to a trilinear map

$$f(x, y, z) := f(x + z; y) - f(x; y) - f(z; y)$$

which is the full linearization of the general map. Indeed, it is the coefficient of s in $f(x + sz; y)$, so is the coefficient of st in $f([x + sz] + ty) = f(1x + ty + sz)$, which by the general expansion (setting $x_1 = x, x_2 = y, x_3 = z, t_1 = 1, t_2 = t, t_3 = s$) is $f(x, y, z)$. It is important that this map is *symmetric* in all 3 variables (not just x, z): the coefficient $f(x_i, x_j, x_k)$ of $t_i t_j t_k$ for distinct indices i, j, k does not depend on the particular ordering of the indices.[2]

When $\frac{1}{3} \in \Phi$ we can recover f from its linearization since $f(x; x) = 3f(x)$, and if $\frac{1}{6} \in \Phi$ we can recover it from its complete linearization since $f(x, x, x) = 6f(x)$, using **Euler's Equation** for homogeneous maps:

$$\partial_x F|_x = nF(x) \qquad (F \text{ homogeneous of degree } n).$$

In our cubic case, $f(x; x)$ is the coefficient of t in $f(x + tx) = f((1 + t)x) = (1 + t)^3 f(x)$ [by definition of f being homogenous of degree 3] $= (1 + 3t + 3t^2 + t^3)f(x)$, so $f(x; x) = 3f(x)$.

Over general rings of scalars we require these "implicitly" defined linearizations (which appear only over the polynomial extension) to be given as part of the definition of f. When there are suitably many scalars these

[2] There is a well-developed functional calculus, featuring most of your favorite theorems from calculus, and even logarithmic derivatives (but no logarithms, please!), valid for polynomial maps between modules over an arbitrary ring of scalars. The complete linearization can be described as $f(x, y, z) = \partial_x \partial_y \partial_z f|_c$ (for any c, even $c = 0$, since the third derivative of a cubic is a constant), and it is symmetric by the *equality of mixed partials*.

linearizations are completely determined by the values of f on X itself, without need of passing to a scalar extension. For example, just as the existence of $\frac{1}{2}$ guaranteed in the Linearization Proposition 1.8.5 that the Jordan identity of degree 3 in x automatically linearizes, so it guarantees now that cubic maps linearize: $f(x;y) = 2[f(x;y) + f(y;x)] - \frac{1}{2}[2f(x;y) + 4f(y;x)] = 2[f(x+y) - f(x) - f(y)] - \frac{1}{2}[f(x+2y) - f(x) - 8f(y)]$ is expressible in terms of values of f on X.

In general, for a homogeneous map of degree n it suffices if Φ contains $n-1$ invertible elements $\lambda_1, \ldots, \lambda_{n-1}$ whose differences $\lambda_i - \lambda_j$ are also invertible (for example, a field with at least n distinct elements). In that case the linearizations $f_i(x;y)$ in $f(x+ty) = \sum_{i=0}^{n} t^i f_i(x;y)$ can be recovered from the explicit values $f(x+\lambda_j y)$ by solving a system of $n-1$ equations involving λ_j (for $j = 1, \ldots, n-1$) in $n-1$ unknowns $f_i(x;y)$ (for $i = 1, \ldots, n-1$, noting that $f_0(x;y) = f(x)$ and $f_n(x;y) = f(y)$ are already known):

$$\sum_{i=1}^{n-1} \lambda_j^i f_i(x;y) = f(x+\lambda_j y) - f(x) - \lambda_j^n f(y),$$

and this system can be solved (even though the unknowns $f_i(x;y)$ are not scalars, but lie in a general module Y) since the matrix of scalar coefficients

$$\begin{bmatrix} \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{n-1} \\ \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{n-1} \\ \vdots & \vdots & \ldots & \vdots \\ \lambda_{n-1} & \lambda_{n-1}^2 & \ldots & \lambda_{n-1}^{n-1} \end{bmatrix}$$

has invertible Vandermonde determinant $\lambda_1\lambda_2 \ldots \lambda_{n-1} \prod_{i<j}(\lambda_i - \lambda_j)$ by our hypothesis about the scalars. In particular, for $n = 3$ we need two invertible scalars with invertible difference, and we can take $\lambda_1 = 1, \lambda_2 = -1, \lambda_1 - \lambda_2 = 2$ as long as 2 is invertible, i.e., $\frac{1}{2} \in \Phi$ as before.

Exercise 4.1.0* (1) Show directly, without Vandermonde, that the linearizations $N(x;y)$, $N(y;x)$ of a cubic form N are intrinsically determined by its values on the original X when we have an invertible scalar λ with $\lambda, 1-\lambda$ also invertible: find expressions for them as combinations of $N(x), N(y)$, $N(x+y)$, $N(x+\lambda y)$. (2) Conclude that the linearizations of a cubic form are determined by its intrinsic values whenever Φ is any field with at least 3 elements, i.e., anyone except $\mathbb{Z}_2$.

4.2 The General Construction

Here we describe in detail the general construction of Jordan algebras over a general ring of scalars from a suitable triple $(N, \#, c)$, consisting of a cubic norm form with a basepoint and a sharp mapping. This construction was not discussed in the *Historical Survey.* In the next three sections we give particular examples of this construction.

In this section we will usually use the term *formula* for scalar equalities, and the term *identity* for vector equalities (identical relations among elements of an algebra).

Sharped Cubic Definition 4.2.1 *As with quadratic forms, a* **basepoint** *for a cubic form on a* Φ*-module* X *is just a point where the form assumes the value* 1*,*

$$N(c) = 1.$$

(1) *Given a basepoint, we can introduce the trace linear and bilinear forms,*[3] *and the spur quadratic and bilinear forms* S[4] *of the cubic form* N *as follows*:

$$\begin{aligned}
T(x) &:= N(c;x) && (\textbf{Linear Trace Form}),\\
T(x,y) &:= T(x)T(y) - N(c,x,y) && (\textbf{Bilinear Trace Form}),\\
S(x) &:= N(x;c) && (\textbf{Quadratic Spur Form}),\\
S(x,y) &:= N(x,y,c) && (\textbf{Bilinear Spur Form}),\\
N(c) &= 1, \quad T(c) = S(c) = 3 && (\textbf{Unit Values}).
\end{aligned}$$

By symmetry $T(c,y) = T(c)T(y) - N(c,c,y) = 3T(y) - 2N(c;y)$ *and* $S(x,y) = N(x,y,c) = N(c,x,y)$*, so the linear trace, bilinear trace, and bilinear spur forms are related by*

$$\begin{aligned}
&T(c,y) = T(y) && (c\textbf{–Trace Formula}),\\
&S(x,y) = T(x)T(y) - T(x,y) && (\textbf{Spur–Trace Formula}).
\end{aligned}$$

(2) *A* **sharp mapping** *for a cubic form* N *with basepoint* c *is a quadratic map* $\# : X \to X$ *strictly satisfying three relations*:

$$\begin{aligned}
T(x^{\#}, y) &= N(x;y) && (\textbf{Trace–Sharp Formula}),\\
x^{\#\#} &= N(x)x && (\textbf{Adjoint Identity}),\\
c\#y &= T(y)c - y && (c\textbf{–Sharp Identity})
\end{aligned}$$

in terms of the symmetric bilinear **sharp product**

$$x\#y := (x+y)^{\#} - x^{\#} - y^{\#}$$

[3] This can be suggestively written in Hessian form as $T(x,y) = -\partial_x \partial_y \log(N)|_c$. Such bilinear forms play a great role in symmetric spaces, and in the approach to Jordan algebras of Hel Braun and Max Koecher in their book *Jordan Algebren.*

[4] In the characteristic polynomial for $n \times n$ matrices, only the first and last coefficients have standard names (trace and determinant, respectively); in our cubic case the second coefficient, the term $S(x)$, plays an important-enough role to justify giving it a name of its own. I have already apologized once for the choice of name (in footnote 5 in *Historical Survey* Section I.8).

obtained by linearizing the sharp mapping. In this case the spur S can be reduced (setting $y = c$ in Trace–Sharp) to the trace of the sharp,

$$S(x) = T(x^{\#}), \quad S(x,y) = T(x\#y) \qquad \textbf{(Spur Formulas)}.$$

(3) *A* **sharped cubic form** $(N, \#, c)$ *consists of a sharp mapping* $\#$ *for a cubic form* N *with basepoint* c.

We are now ready to describe the general construction. Note how the U-operator arises naturally from the trace and the sharp.

Sharped Cubic Construction 4.2.2 (1) *Any sharped cubic form* $(N, \#, c)$ *gives rise to a unital Jordan algebra* $\mathcal{J}ord(N, \#, c)$ *with unit element* $1 = c$ *and bullet product*

$$x \bullet y := \tfrac{1}{2}\big(x\#y + T(x)y + T(y)x - S(x,y)c\big).$$

Here the square and bilinear product can be expressed in terms of the sharp map and sharp product, and vice versa, by the **Sharp Expressions**:

$$\begin{aligned} x^2 &= x^{\#} + T(x)x - S(x)c, \\ x^{\#} &= x^2 - T(x)x + S(x)c, \\ x\#y &= \{x,y\} - T(x)y - T(y)x + S(x,y)c. \end{aligned}$$

(2) *All elements satisfy the* **Degree–3 Identity**

$$x^3 - T(x)x^2 + S(x)x - N(x)c = 0,$$

and the auxiliary products (the U-operator and the triple product) are given by

$$U_x y = T(x,y)x - x^{\#}\#y, \quad \{x,y,z\} = T(x,y)z + T(z,y)x - (x\#z)\#y.$$

As always, the square and bilinear product can be recovered from U:

$$x^2 = U_x c, \quad \{x,y\} := U_{x,y}c, \quad x \bullet y := \tfrac{1}{2}U_{x,y}c.$$

(3) *The sharp mapping always permits composition with the U-operator: we have the* **Sharp Composition Law**

$$(U_x y)^{\#} = U_{x^{\#}} y^{\#}.$$

If Φ is a faithful ring of scalars [$\alpha \mathrm{J} = 0 \Longrightarrow \alpha = 0$ in Φ, e.g., if Φ is a field or has no 3-torsion], then N is a Jordan norm permitting composition with the U-operator and the sharp: we have the **Norm Composition Laws**

$$N(c) = 1, \quad N(U_x y) = N(x)^2 N(y), \quad N(x^{\#}) = N(x)^2. \quad \square$$

(Remember that the gory verifications have been cast into the outer darkness of Appendix C!) The only fly in this particular ointment is the necessity of requiring faithful scalars in order to get composition. Problem 4.5 at the end of the chapter gives an unfaithful characteristic 3 example of a sharped cubic whose norm doesn't permit composition. Nevertheless, these results will suffice to give us the basic properties of inverses in cubic factors in Chapter 6.

4.3 The Jordan Cubic Construction

One important case of the Sharped Cubic Construction (which will actually suffice for our structure theory) is the case when we have a *finite-dimensional vector space over a field* Φ.[5]

Jordan Cubic Definition 4.3.1 *A cubic form N with basepoint c on a finite-dimensional vector space over a field Φ is* **nondegenerate** *if the bilinear trace T is* nondegenerate *bilinear form. Nondegeneracy and finite-dimensionality mean that we may identify linear functionals with vectors, so in particular for fixed x the linear functional $y \mapsto N(x;y)$ is given by a unique vector $x^\#$, and in this context we automatically obtain a quadratic* **adjoint** *map satisfying the Trace–Sharp Formula*

$$T(x^\#, y) := N(x;y).$$

A **Jordan cubic form** *is a nondegenerate cubic form with basepoint whose induced sharp strictly satisfies the Adjoint Identity*

$$x^{\#\#} = N(x)x.$$

Springer Construction 4.3.2 *Every Jordan cubic form produces a Jordan algebra $\mathcal{J}ord(N,c)$ with unit c and U-operator $U_x y = T(x,y)x - x^\# \# y$ for the adjoint $\#$ defined by $T(x^\#, y) = N(x;y)$.* $\square$

The reason for this is that Jordan cubics are always sharped.

Jordan Cubic Construction 4.3.3 *Every Jordan cubic form is a sharped cubic form, and hence produces a Jordan algebra $\mathcal{J}ord(N,\#,c) = \mathcal{J}ord(N,c)$.*

PROOF. We verify the three sharp axioms 4.2.1(2) of the general Cubic Construction: the Trace–Sharp Formula is satisfied by definition, the Adjoint Identity holds strictly by hypothesis, and all that is left is the c–Sharp Identity. We have

$$(1) \qquad T(x\#y, z) = T(x, y\#z) = N(x,y,z) \quad \textbf{(Sharp Symmetry)}$$

since linearizing Trace–Sharp gives the symmetric function $T(x\#y, z) = N(x,y,z)$. From this the c–Sharp Identity $c\#y = T(y)c - y$ follows using *nondegeneracy* of the trace bilinear form, since for all z we have $T(T(y)c - y, z) =$

[5] This case was discussed in the *Historical Survey* in Section I.3.8. In Jordan structure theory this construction suffices to give the Albert algebras in all characteristics. It does not uniformly give all the degree 3 algebras: in characteristic 2 the little 6-dimensional degree 3 algebra $\mathcal{H}_3(\Phi)$ has degenerate trace form (the radical consists of all matrices with zeroes down their main diagonal). The general construction is really needed just for this one special case.

$T(y)T(z) - T(y,z)$ [by c–Trace Formula 4.2.1(1)] $= N(c,y,z)$ [by definition of the bilinear trace] $= T(c\#y, z)$ [by Sharp Symmetry (1)]. Thus $(N, \#, c)$ is a sharped cubic. □

A Little Reassuring Argument. For those who do not want to plow through Appendix C, but nevertheless want reassurance that these cubic constructions do indeed produce degree–3 Jordan algebras, we will give a brute-force direct proof that $\mathcal{J}ord(N,c)$ is Jordan (depending heavily on nondegeneracy and our trusty $\frac{1}{2}$).

We begin by proving the Degree–3 Identity 4.2.2(1). We have seen that the Sharp Expressions 4.2.2(1) hold by definition of the bullet, and, using these, the Degree–3 Identity $0 = x \bullet [x^2 - T(x)x + S(x)c] - N(x)c = x \bullet x^\# - N(x)c$ becomes

$$x^\# \bullet x = N(x)c. \tag{2}$$

To prove this we will heavily use "associativity" in the trace bilinear form:

$$T(x \bullet y, z) = T(x, y \bullet z) \quad (\textbf{Bullet Symmetry}), \tag{3}$$

since $2T(x \bullet y, z) = T(x\#y + T(x)y + T(y)x - S(x,y)c, z) = T(x\#y, z) + T(x)T(y,z) + T(y)T(x,z) - [T(x)T(y) - T(x,y)]T(z)$ [by c-Trace and Spur–Trace 4.2.1(1)] is symmetric in x, y, z by Sharp Symmetry (1). We also use

$$x^\#\#(x\#y) = N(x)y + T(x^\#, y)x \quad (\textbf{Adjoint}' \textbf{ Identity}), \tag{4}$$

obtained by linearizing the Adjoint Identity 4.2.1(2). From these we compute $2T(x^\# \bullet x, y)$, using Bullet Symmetry (3) above to move x to the other side to avoid having to deal with $S(x^\#, x)$:

$$\begin{aligned}
&2T(x^\# \bullet x, y) = 2T(x^\#, x \bullet y) \\
&= T(x^\#, [x\#y + T(x)y + T(y)x - S(x,y)c]) && \text{[by definition of bullet]} \\
&= [T(x^\#)T(x\#y) - T(x^\#\#(x\#y))] + T(x)T(x^\#, y) + T(y)T(x^\#, x) \\
&\qquad -S(x,y)S(x) && \text{[by Spur–Trace and Spur Formula 4.2.1(2)]} \\
&= [S(x)S(x,y) - N(x)T(y) - T(x^\#, y)T(x)] + T(x)T(x^\#, y) \\
&\qquad +T(y)3N(x) - S(x)S(x,y) && \text{[by Spur, Adjoint}' \text{ (4) above, Spur–Trace, Euler]} \\
&= 2N(x)T(y) \;=\; 2T(N(x)c, y)
\end{aligned}$$

for all vectors y, so by half and nondegeneracy we obtain The Degree–3 Identity (2) above.

Finally, we verify the Jordan identity. By nondegeneracy it can be written as $T\big((x \bullet (x^2 \bullet y) - x^2 \bullet (x \bullet y)\big), z) = 0$ for all x, y, z, hence (using Bullet Symmetry twice) $T(x \bullet (x^2 \bullet y), z) - T(y, x \bullet (x^2 \bullet z)) = 0$, which is just the condition that

$$T(x \bullet (x^2 \bullet y), z) \textit{ is symmetric in } y \textit{ and } z.$$

But linearizing the Degree–3 Identity yields $x^2 \bullet y = -2(x \bullet y) \bullet x + 2T(x)x \bullet y + T(y)x^2 - S(x,y)x - S(x)y + T(x^\#, y)c$ [using Trace–Sharp], so multiplying by x and using Spur–Trace and the Sharp Expression gives $x \bullet (x^2 \bullet y) = \big[-2L_x^3 + 2T(x)L_x^2 - S(x)L_x\big]y + T(y)x^3 + \big[T(x,y) - T(x)T(y)\big]x^2 + T([x^2 - T(x)x + S(x)c], y)x = f(L_x)y + T(y)\big[x^3 - T(x)x^2 + S(x)x\big] + T(x,y)x^2 + T(x^2, y)x - T(x)T(x,y)x$. Taking traces against z and using the Degree–3 Identity and c–Trace, we get

$$\begin{aligned}
T(x \bullet (x^2 \bullet y), z) &= T(f(L_x)y, z) + T(y)N(x)T(z) \\
&\quad + \big(T(x,y)T(x^2, z) + T(x^2, y)T(x,z)\big) - T(x)T(x,y)T(x,z),
\end{aligned}$$

which is symmetric in y, z [using Bullet Symmetry to see that any operator polynomial $f(L_x)$ can be moved from one side of $T(y,z)$ to the other]. Thus we have a degree–3 Jordan algebra $\mathcal{J}ord(N,c)$ (though it still takes some work to get the expression 4.2.2(2) for the U-operator). □

EXERCISE 4.3.3A* (1) Derive $c^\# = c$ for any sharped cubic when $\frac{1}{2} \in \Phi$. (2) Derive it without assuming $\frac{1}{2}$, using nondegeneracy instead (showing that both sides have the same trace against any z).

EXERCISE 4.3.3B* In a Jordan cubic $\mathcal{J}ord(N, c)$, show that the adjoint of the operator $V_{x,y}$ with respect to the trace bilinear form is $V^*_{x,y} = V_{y,x}$, i.e., $T(V_{x,y}z, w) = T(z, V_{y,x}w)$.

EXERCISE 4.3.3C Establish the following relations between an element x and its sharp in $\mathcal{J}ord(N, c)$: (1) $x\#x^\# = [T(x)S(x) - N(x)]c - T(x)x^\# - S(x)x$, (2) $T(x\#x^\#, y) = S(x)S(x, y) - N(x)T(y) - T(x)T(x^\#, y)$, (3) $S(x, x^\#) = T(x\#x^\#) = S(x)T(x) - 3N(x)$, (4) $x\#(x^\#\#y) = N(x)y + T(x, y)x^\#$ (Dual Adjoint′ Identity).

EXERCISE 4.3.3D* Don't assume any longer that the trace is nondegenerate, but assume that the sharp cubic N satisfies the above Dual Adjoint′ Identity. (1) Show that this implies that $x\#U_xy = T(x, y)x^\# - N(x)y$. (2) By Exercise 4.3.3.B above we have $T(V_{x,y}z, w) = T(z, V_{y,x}w)$; use this and (1) to prove the UV-Commuting Formula $\{U_xy, z, x\} = U_x\{y, x, z\}$.

4.4 The Freudenthal Construction

Our first, and most important, example of a sharped cubic form comes from the algebra of 3×3 hermitian matrices over an alternative algebra with central involution.[6]

Freudenthal Construction 4.4.1 *If* D *is a unital alternative algebra over* Φ *with nuclear involution, then the hermitian matrix algebra* $\mathcal{H}_3(\mathrm{D}, -)$ *is a unital Jordan algebra; if the involution is scalar, then* $\mathcal{H}_3(\mathrm{D}, -)$ *is a cubic factor* $\mathcal{J}ord(N, \#, c)$ *whose Jordan structure is determined by the basepoint* c*, cubic form* N*, trace* T*, and sharp* $\#$ *defined as follows:*

$$\begin{aligned}
c &:= 1 = e_1 + e_2 + e_3,\\
N(x) &:= \alpha_1\alpha_2\alpha_3 - \sum_k \alpha_k n(a_k) + t(a_1a_2a_3),\\
T(x) &= \sum_k \alpha_k, \quad T(x, y) = \sum_k \alpha_k\beta_k + \sum_k t(\overline{a_k}b_k),\\
x^\# &:= \sum_k \big(\alpha_i\alpha_j - n(a_k)\big)e_k + \sum_k \big(\overline{a_ia_j} - \alpha_k a_k\big)[ij],\\
x\#y &:= \sum_k \big(\alpha_i\beta_j + \beta_i\alpha_j - n(a_k, b_k)\big)e_k\\
&\quad + \sum_k \big(\,\overline{(a_ib_j + b_ia_j)} - \alpha_k b_k - \beta_k a_k\big)[ij].
\end{aligned}$$

for elements $x = \sum_{i=1}^3 \alpha_i e_i + \sum_{i=1}^3 a_i[jk]$, $y = \sum_{i=1}^3 \beta_i e_i + \sum_{i=1}^3 b_i[jk]$ *with* $\alpha_i, \beta_i \in \Phi$, $a_i, b_i \in \mathrm{D}$ *in Jacobson box notation* ($e_i := E_{ii}$, $d[jk] := dE_{jk} + \bar{d}E_{kj}$), *where* (ijk) *is always a cyclic permutation of* (123). □

[6] The Freudenthal Construction was introduced briefly in I.3.9.

We will relegate this proof to Appendix C as well; it is an exercise in alternative algebras. Once we have established the Sharped Cubic Construction 4.2.2 and verified that the above Freudenthal norm is a Jordan cubic, we will know that the module $\mathcal{H}_3(\mathrm{D}, -)$ carries the structure of a Jordan algebra $\mathcal{J}ord(N, \#, c)$. It remains to verify that this Jordan structure determined by the sharped cubic form coincides with that of the Hermitian Matrix Example 3.2.4. We just have to check the Four Basic Brace Products and Basic Orthogonality 3.2.4(2) for distinct i, j, k:

$$
\begin{aligned}
&(1) \qquad \alpha[ii]^2 = \alpha^2[ii],\\
&(2) \qquad a[ij]^2 = a\bar{a}[ii] + \bar{a}a[jj] = n(a)(e_i + e_j),\\
&(3) \qquad \{\alpha[ii], b[ij]\} = \alpha b[ij],\\
&(4) \qquad \{a[ij], b[jk]\} = ab[ik] = \overline{ab}[ki],\\
&(5) \qquad \{\alpha[ii], \beta[jj]\} = 0,\\
&(6) \qquad \{\alpha[ii], b[jk]\} = 0.
\end{aligned}
$$

But from $x^2 = x^\# + T(x)x - S(x)c$ we see that (1), (2) hold since

$$
\begin{aligned}
&\alpha[ii]^\# = 0, && T(\alpha[ii]) = \alpha, && S(\alpha[ii]) = 0,\\
&a[ij]^\# = -n(a)e_k, && T(a[ij]) = 0, && S(a[ij]) = -n(a),
\end{aligned}
$$

for $c = e_i + e_j + e_k$. From $\{x, y\} = x\#y + T(x)y + T(y)x - S(x, y)c$ we see that (3), (4), (5), (6) hold since

$$
\begin{aligned}
\alpha[ii]\#b[ij] &= 0, & T(b[ij]) &= 0, & S(\alpha[ii], b[ij]) &= 0,\\
a[ij]\#b[jk] &= \overline{ab}[ki], & T(b[jk]) &= 0, & S(a[ij], b[jk]) &= 0,\\
\alpha[ii]\#\beta[jj] &= \alpha\beta e_k, & T(\beta[jj]) &= \beta, & S(\alpha[ii], \beta[jj]) &= \alpha\beta,\\
\alpha[ii]\#b[jk] &= -\alpha b[jk], & T(\alpha[ii]) &= \alpha, & S(\alpha[ii], b[jk]) &= 0. \quad \square
\end{aligned}
$$

For the octonions with standard involution this produces reduced Albert algebras. The result we really need in this book is the fact that these algebras (hence any of their forms) are Jordan.

Reduced Albert Algebra Theorem 4.4.2 *If* O *is an octonion algebra over a field* Φ, *the reduced Albert algebra* $\mathcal{H}_3(\mathrm{O})$ *is a* 27*-dimensional Jordan algebra over* Φ. *If* $\mathcal{O}(\Phi)$ *is the split octonion algebra over an arbitrary ring of scalars, the split Albert algebra* $\mathcal{A}lb(\Phi) = \mathcal{H}_3(\mathcal{O}(\Phi))$ *is a Jordan algebra free of rank* 27 *as a* Φ*-module.* $\square$

We will verify later that an Albert algebra is exceptional, and over a field is simple.

4.5 The Tits Constructions

Two more examples of sharped norm forms are provided by constructions, due to Jacques Tits, of degree–3 Jordan algebras out of certain degree–3 associative algebras (including all separable associative algebras of degree 3 over a field).[7] These algebras need not be reduced, and when we discuss inverses in Chapter 6 we will give an example where this construction produces a 27-dimensional Albert *division algebra.*

Associative Degree–3 Definition 4.5.1 *Let* A *be a unital associative algebra with a* **cubic norm form** n *with basepoint* 1*; as in the Sharped Cubic Definition* 4.2.1(1) *we introduce*

$$\begin{aligned} t(a) &:= n(1; a), \\ t(a, b) &:= t(a)t(b) - n(1, a, b), \\ s(a) &:= n(a; 1), \\ n(1) &= 1, \qquad t(1) = s(1) = 3, \end{aligned}$$

and we can define an associative adjoint map as in 4.2.2(1) *by*

$$a^{\#} := a^2 - t(a)a + s(a)1.$$

using the pre-existing multiplication on A*. We say that* A *is of* **degree** 3 *over* Φ *if it satisfies the following three axioms* (*corresponding to those in* 4.2.1(2) *and* 4.2.2(2))*. First, the algebra strictly satisfies the associative* **Degree–3 Identity**:

$$\text{(A1)} \qquad a^3 - t(a)a^2 + s(a)a - n(a)1 = 0,$$

equivalently,

$$\text{(A1}'\text{)} \qquad aa^{\#} = a^{\#}a = n(a)1.$$

Secondly, the adjoint satisfies the associative **Trace–Sharp Formula**:

$$\text{(A2)} \qquad t(a^{\#}, b) = n(a; b).$$

Finally, the trace bilinear form comes from the associative product by the **Trace-Product Formula**:

$$\text{(A3)} \qquad t(a, b) = t(ab).$$

These algebras automatically (*see below*) *satisfy the* **Adjoint Identity**:

$$\text{(A4)} \qquad a^{\#\#} = n(a)a.$$

[7] These constructions were not discussed in the *Historical Survey*. They provide an elegant and insightful construction of *all* degree–3 Jordan algebras over fields.

EXERCISE 4.5.1 Show that an algebra of degree 3 satisfies (1) c–Trace 4.2.1(1) $t(a,1) = t(a)$, (2) Spur Formula 4.2.1(2) $s(a) = t(a^{\#})$. (3) From (2) and the definition of the adjoint, compute directly that $(a^{\#})^{\#} = n(a)a + (s(a^{\#}) - t(a)n(a))1$. (4) Show that twice the Spur–Trace Formula 4.2.1(1) holds, $2t(a,b) = 2(t(a)t(b) - s(a,b))$. (5) Use (2),(4),(A1′),(A3) to show that $4s(a^{\#}) = 2s(a^{\#}, a^{\#}) = 4t(a)n(a)$, and deduce that if $\frac{1}{2} \in \Phi$ then a degree–3 algebra satisfies the Adjoint Identity $a^{\#\#} = n(a)a$.

First Tits Construction 4.5.2 *Let n be the cubic norm form on a degree–3 associative algebra A over Φ, and let $\mu \in \Phi$ be an invertible scalar. From these ingredients we define a module $\mathrm{J} = \mathrm{A}_{-1} \oplus \mathrm{A}_0 \oplus \mathrm{A}_1$ to be the direct sum of three copies*[8] *of A, and define a basepoint c, norm N, trace T, and sharp $\#$ on J by*

$$\begin{aligned}
c &:= 0 \oplus 1 \oplus 0 = (0,1,0),\\
N(x) &:= \mu^{-1}n(a_{-1}) + n(a_0) + \mu n(a_1) - t(a_{-1}a_0a_1),\\
&= \sum_{i=-1}^{1} \mu^i n(a_i) - t(a_{-1}a_0a_1),\\
T(x) &:= t(a_0),\\
T(x,y) &:= \sum_{i=-1}^{1} t(a_{-i}, b_i),\\
[x^{\#}]_{-i} &:= \mu^i a_i^{\#} - a_{[i+1]}a_{[i-1]},
\end{aligned}$$

for elements $x = (a_{-1}, a_0, a_1)$, $y = (b_{-1}, b_0, b_1)$, where the indices $[j]$ are read modulo 3, using standard representatives $-1, 0, 1$ only. Then $(N, \#, c)$ is a sharped cubic form, and the algebra $\mathcal{J}ord(\mathrm{A}, \mu) := \mathcal{J}ord(N, \#, c)$ is a Jordan algebra. □

For the second construction, let A be an associative algebra of degree 3 over Ω, and let $*$ be an involution of *second kind* on A, meaning that it is not Ω-linear, only *semi-linear* $(\omega a)^* = \omega^* a^*$ for a nontrivial involution on Ω with fixed ring $\Phi := \mathcal{H}(\Omega, *) < \Omega$. (For example, the conjugate-transpose involution on complex matrices is only real-linear). The involution is *semi-isometric* with respect to a norm form if $n(a^*) = n(a)^*$.

Second Tits Construction 4.5.3 *Let n be the cubic norm of a degree–3 associative algebra A over Ω, with a semi-isometric involution of second kind over Φ. Let $u = u^* \in \mathrm{A}$ be a hermitian element with norm $n(u) = \mu\mu^*$ for an invertible scalar $\mu \in \Omega$. From these ingredients we define a module $\mathrm{J} = \mathcal{H}(\mathrm{A}, *) \oplus \mathrm{A}$ to be the direct sum of a copy of the hermitian elements and a copy of the whole algebra, and define a basepoint c, norm N, trace T, and sharp $\#$ on J by*

[8] The copies are usually labeled $0, 1, 2$, but the symmetry is more clearly seen if we take as labels (mod 3) the powers $0, 1, -1$ of μ.

$$\begin{aligned} c &:= 1 \oplus 0 = (1,0), \\ N(x) &:= n(a_0) + \mu n(a) + \mu^* n(a^*) - t(a_0 a u a^*), \\ T(x) &:= t(a_0), \\ T(x,y) &:= t(a_0, b_0) + t(ua^*, b) + t(au, b^*), \\ x^\# &= (a_0^\# - aua^*, \mu^*(a^*)^\# u^{-1} - a_0 a) \end{aligned}$$

for elements

$$x = (a_0, a), \quad y = (b_0, b) \qquad (a_0, b_0 \in \mathcal{H}(\mathrm{A}, *),\ a, b \in \mathrm{A}).$$

Then $(N, \#, c)$ *is a sharped cubic form, and the algebra* $\mathcal{J}ord(\mathrm{A}, u, \mu, *) := \mathcal{J}ord(N, \#, c)$ *is a Jordan algebra. Indeed, it is precisely the fixed points of the algebra* $\mathcal{J}ord(\mathrm{A}, \mu)$ *over* Ω *obtained by the First Construction relative to the semi-linear involution* $(ua_{-1}, a_0, a_1) \mapsto (ua_{-1}^*, a_0^*, a_1^*)$*, and is always a form of the first*: $\mathcal{J}ord(\mathrm{A}, u, \mu, *)_\Omega \cong \mathcal{J}ord(\mathrm{A}, \mu)$. □

The proofs are entirely associative, but require many of the same detailed properties of the cubic norm form n that were needed for N in the proof of the General Construction, so once more we defer them to Appendix C.

4.6 Problems for Chapter 4

Problem 4.1* Establish the *VU-Commuting Theorem*: Let X be a module over Φ containing $\frac{1}{2}$ which carries a quadratic map $U : X \to \mathcal{E}nd_\Phi(X)$ with *unit* $1 \in X$, $U_1 = \mathbb{1}_X$, and satisfying the *VU-Commuting Identity* $V_{x,y} U_x = U_x V_{y,x}$ where, as usual, $V_{x,y}(z) := \{x, y, z\} := U_{x,z} y := (U_{x+z} - U_x - U_z) y$. Then the U-operator gives birth to a unital Jordan product $x \bullet y := \frac{1}{2}\{x, 1, y\}$ ($x^2 = U_x 1$) whose associated U-operator is just the given one: $2L_x^2 - L_{x^2} = U_x$. In particular, $x^3 = U_x x$.

(1) First establish alternate formulations of the L-operator: $2L_x = V_x := V_{x,1} = V_{1,x} = U_{x,1}$. (2) Show that $U_x V_x = V_x U_x$. Thus V_x commutes with the operator U_x, and we will have the Jordan condition that V_x commutes with V_{x^2} if we can show that U_x is the usual U-operator. (3) Show that $2U_x = V_x^2 - V_{x^2}$.

Problem 4.2* To complete the proof that $\mathcal{J}ord(N, \#, c)$ is Jordan in the Cubic Construction 4.2.2, verify that the operator $U_x y = T(x, y)x - x^\# \# y$ of the Cubic Construction satisfies the Unit Condition and the VU-Commuting Identity. (1) Show that $c^\# = c$, $U_c y = y$. (2) Establish the Adjoint′ Identity 4.3.3(4) $x^\# \# (x \# y) = N(x) y + T(x^\#, y) x$ and the *U-x-Sharp Identity* $U_x(x \# y) = T(x^\#, y) x - N(x) y$. (3) Take the Adjoint′ Identity to the other side of the trace bilinear form to turn it into the *Dual Adjoint′ Identity* (see Ex. 4.3.3C(4)) $x \# (x^\# \# z) = N(x) z + T(x, z) x^\#$. (4) Use these to establish the VU-Commuting Identity.

Problem 4.3 A derivation of a quadratic Jordan algebra is a linear transformation D which satisfies (D1) $D(1) = 0$, (D2) $D(U_x y) = U_{D(x),x}y + U_x D(y)$ for all x, y. (1) Show that (D2) implies (D1) if $\frac{1}{2} \in \Phi$. (2) Show that D is a derivation of a cubic factor $\mathcal{J}ord(N, \#, c)$ if it (C1) kills the basepoint $D(c) = 0$, (C2) is trace-skew $T(D(x), y) + T(x, D(y)) = 0$, and (C3) is a sharp-derivation $D(x^\#) = x \# D(x)$. Here skewness (C2) follows from trace-alternation (C2′) $T(D(x), x) = 0$, and is equivalent to it when $\frac{1}{2} \in \Phi$. (2) Show that if $\frac{1}{3} \in \Phi$ then (C2′), (C3) $\Rightarrow$ (C4): $T(x^\#, D(x)) = 0$ and (C3) $\Rightarrow$ (C5): $T(D(x)) = 0$. (4) Show that (C4) implies (C5), (C2′), so that (C1), (C3), (C4) suffice to guarantee that D is a derivation (and if the trace is nondegenerate, (C1), (C4) alone suffice).

Problem 4.4* (1) Show that an invertible linear transformation φ on $\mathcal{J}ord(N, \#, c)$ is an automorphism if it (A1) fixes the basepoint, $\varphi(c) = c$, (A2) is a trace isometry $T(\varphi(x), \varphi(y)) = T(x, y)$, and (A3) is a sharp-homomorphism $\varphi(x^\#) = \big(\varphi(x)\big)^\#$. (2) Use the Adjoint Identity to show that (A1)–(A3) imply that φ is a norm isometry (A4) $N(\varphi(x)) = N(x)$ if Φ acts faithfully on J (i.e., $\alpha \mathrm{J} = 0 \Rightarrow \alpha = 0$ in Φ, e.g., if $1 \in T(\mathrm{J})$, as when $\frac{1}{3} \in \Phi$ by $T(c) = 3$). (3) Apply this to $\mathbb{1}_{\mathrm{J}} + \varepsilon D$ to deduce that D is a derivation of $\mathcal{J}ord(N, c)$ if (in the notation of Problem 4.3) it satisfies (C1), (C2), (C3) (and if Φ acts faithfully then (C1), (C4) alone suffice).

Problem 4.5* Show that in the Cubic Construction 4.2.2 the norm form need not admit Jordan composition $N(U_x y) = N(x)^2 N(y)$ or $N(x^\#) = N(x)^2$ if Φ is faithless of characteristic 3 [$3\Phi = 0,\ \alpha \mathrm{J} = \mathbf{0}$ for some $\alpha \neq 0$]. (1) Show that such a faithless Φ contains a "dual number" ε with $\varepsilon \mathrm{J} = \mathbf{0},\ \varepsilon^2 = 0$. (2) Let $\mathrm{J} = \mathcal{H}_3(\mathrm{C})$ for a simple composition algebra over a field Φ of characteristic 3, so it results by the Cubic Construction from the norm $(N, \#, c)$ of the Freudenthal Construction 4.4.1. Then J remains a simple Jordan algebra over the ring of dual numbers $\Phi' := \Phi[\varepsilon]$ via $\varepsilon \mathrm{J} = \mathbf{0}$. Show that $(N', \#, c)$ for $N'(x) = N(x) + \varepsilon T(x)^3$ remains a sharped cubic for J over Φ'. (3) Exhibit elements x, y in $\Phi e_1 + \Phi e_2 + \Phi e_3 \subseteq \mathrm{J}$ such that $N'(x^\#) \neq N'(x)^2,\ N'(U_y c) \neq N'(y)^2 N'(c)$.

Question 4.1 Is $c^\# = c$ for any sharped cubic form $(N, \#, c)$ (not assuming nondegeneracy or $\frac{1}{2}$)?

Question 4.2 Develop an alternate "U-theoretic" proof that $\mathcal{J}ord(N, c)$ is a degree–3 Jordan algebra. In these algebras of cubic forms, frequently there are clear relations among U-products which correspond to messy relations among bilinear products, and it is useful to have a U-theoretic formulation of Jordanity.

5

Two Basic Principles

Here we gather the basic identities we need, and more importantly, we state the Two Basic Principles for proving operator identities and making calculations involving only two elements. First the general principles.

5.1 The Macdonald and Shirshov–Cohn Principles

These two principles say that life in a Jordan algebra is pretty much like life in an associative algebra, as long as things don't get too crowded. As soon as four or more elements get together, or three elements behaving non-linearly, nonassociativity breaks out.

Macdonald Principle 5.1.1 *Any Jordan polynomial in three variables which has degree* ≤ 1 *in one variable and vanishes in all* associative *algebras necessarily vanishes on all Jordan algebras, i.e., is an identity for Jordan algebras.* □

The proof is an unilluminating combinatorial argument in the free algebra, and we relegate it to Appendix B. The condition that the polynomial be linear in at least one variable is necessary — recall that Glennie's Identities G_8, G_9, which are of degree ≥ 2 in all variables, vanish on all special algebras but not on the Albert algebra.

Since an operator identity $M = 0$ is equivalent to the element identity $M(z) = 0$ for all z, which is linear in z, the Macdonald Principle for elemental identities leads to a general principle for verifying operator identities $M(x, y) = 0$ for multiplication operators built out of two variables.

Macdonald Multiplication Principle 5.1.2 *Any Jordan multiplication operator in two variables which vanishes on all associative algebras necessarily vanishes on all Jordan algebras, i.e., is an operator identity for Jordan algebras.* □

The proof of the next theorem will also be relegated to an appendix, Appendix A. It does involve a few ideas of interest outside the narrow range of the theorem itself, but these ideas revolve around speciality of quotient algebras, and will not concern us in our work.

Shirshov–Cohn Theorem 5.1.3 *Any linear Jordan algebra generated by two elements is special, indeed, is isomorphic to* $\mathcal{H}(\mathrm{A}, *)$ *for an associative* $*$*-algebra* A *generated by two elements.* □

When we are working with expressions involving only two elements x, y in a Jordan algebra J, we are working in the subalgebra $\Phi[x, y]$; by the Shirshov–Cohn Theorem this algebra is isomorphic to $\mathcal{H}(\mathrm{A}, *)$, and we can carry out our calculations inside A. As we will see, this can be very convenient — it allows us to work with asymmetric expressions such as xy, instead of always having to formulate things in terms of symmetric (Jordan) expressions.

Shirshov–Cohn Principle 5.1.4 *In order to verify that certain relations between elements* x, y *in Jordan algebras always imply certain other relations among elements* $f_1(x, y), \ldots, f_n(x, y)$ *in the subalgebra* $\Phi[x, y]$*, it is sufficient to establish the implication for* x, y *in* $\mathcal{H}(\mathrm{A}, *)$ *for all associative* $*$*-algebras* A*.* □

5.2 Fundamental Formulas

We now gather an initial harvest of results from these principles. The first tells us that micro-locally, inside a subalgebra generated by a single element, the Jordan landscape looks completely associative.

Power Definition 5.2.1 *The* **powers** *of an element* x *in a Jordan algebra* J *are defined recursively by*

$$x^0 := 1 \in \widehat{\mathrm{J}}, \qquad x^1 := x, \qquad x^{n+1} := x \bullet x^n.$$

We say that e *is* **idempotent** *("same-powered") if it equals its square,*

$$e^2 = e \qquad \text{(idempotent)},$$

and **nilpotent** *("zero-powered") if some power vanishes,*

$$z^n = 0 \qquad \text{(nilpotent)}.$$

We say that J *is* **nil** *if all its elements are nilpotent. A linear algebra is* **power-associative** *if the subalgebra* $\Phi[x]$ *generated by a single element is commutative and associative,*

$$x^n \bullet x^m = x^{n+m} \qquad (n, m \geq 0).$$

It is easy to check by induction that in special algebras these Jordan powers reduce to the usual associative ones, from which Macdonald's Principle allows us to conclude that powers behave like associative ones in all Jordan algebras.

Power-Associativity Theorem 5.2.2 (1) *Jordan algebras are always power-associative. If* $\mathrm{J} \subseteq \mathrm{A}^+$ *is special, the Jordan powers of* x *in* J *coincide with its associative powers in* A. *Jordan powers always obey the* **Element Power-Associativity Rules**

$$(x^n)^m = x^{nm}, \qquad U_{x^n} x^m = x^{2n+m},$$
$$\{x^n, x^m\} = 2x^{n+m}, \qquad \{x^n, x^m, x^p\} = 2x^{n+m+p}.$$

If e *is idempotent,* $e^2 = e$*, then all powers coincide:* $e^n = e$ *for all* $n \geq 1$*. If* z *is nilpotent, then eventually all powers vanish:* $z^n = 0$ *guarantees that* $z^m = 0$ *for all* $m \geq n$*.*

(2) *The Jordan multiplication operators with powers are*

$$U_{x^n} = U_x^n, \qquad V_{x^n,x^m} = V_{x^{n+m}}, \qquad V_{x^n} U_{x^m} = U_{x^{n+m},x^m}.$$

Indeed, in general we have the **Operator Power-Associativity Rules**

$$U_{f(x)} U_{g(x)} = U_{(f\cdot g)(x)}, \quad V_{f(x),g(x)} = V_{(f\cdot g)(x)}, \quad V_{f(x)} U_{g(x)} = U_{(f\cdot g)(x),g(x)}$$

for any polynomials $f(x), g(x)$ *in* $\Phi[x]$*.* □

Exercise 5.2.2A (1) Use the linearized Jordan identity (JAX2)$''$ (but not power-associativity!) to show that $L_{x^{n+2}}$ can be expressed in terms of L_x, L_{x^2}, L_{x^n}, $L_{x^{n+1}}$, hence by induction that each left multiplication L_{x^n} by a power of x in a Jordan algebra J can be expressed as a polynomial in the (commuting) operators L_x and L_{x^2}. (2) Conclude that any two powers of x operator-commute, i.e., the commutator $[L_{x^n}, L_{x^m}]$ vanishes; use this to show that a Jordan algebra is power-associative.

Exercise 5.2.2B Find a recursive formula $x^n = T_n(x)x + Q_n(x)1$ in terms of scalar-valued functions (polynomials in $T(x), Q(x)$) for powers in $\mathcal{J}ord(Q, c)$ of Quadratic Factor Example 3.3.1.

Exercise 5.2.2C (1) Show directly (without invoking Macdonald) that if e is idempotent, then all its powers coincide: $e^n = e$ for $n \geq 1$. (2) Show directly that if z is nilpotent, $z^n = 0$, then $z^m = 0$ for all $m \geq n$.

Next we gather together the five most important identities for our future work, expressed in terms of the multiplication operators $V_x := 2L_x,\ 2U_x := V_x^2 - V_{x^2},\ U_{x,y} := U_{x+y} - U_x - U_y,\ V_{x,y}(z) := U_{x,z}(y)$.

Five Fundamental Formulas 5.2.3 *Every Jordan algebra satisfies the following element and operator identities, Firstly, in unital algebras we have*:

$$U_{x,1} = V_{x,1} = V_{1,x} = V_x, \quad U_x(1) = x^2.$$

Secondly, we have five fundamental operator identities:

(FFI)	**Fundamental Formula:**	$U_{U_x y} = U_x U_y U_x,$
(FFII)	**Commuting Formula:**	$V_{x,y}U_x = U_x V_{y,x} = U_{U_x y,x},$
	so for $y = 1$ we have	$V_x U_x = U_x V_x = U_{x^2,x},$
(FFIII)	**Triple Shift Formula:**	$V_{U_x y,y} = V_{x,U_y x},$
	so for $y = 1$	$V_{x^2} = V_{x,x},$
(FFIV)	**Triple Switch Formula:**	$V_{x,y} = V_x V_y - U_{x,y},$
(FFV)	**Fundamental Lie Formula:**	$V_{x,y}U_z + U_z V_{y,x} = U_{V_{x,y}(z),z}.$

The corresponding element identities are:

(FFIe)	**Fundamental Identity:**	$U_{U_x y}z = U_x U_y U_x z,$
(FFIIe)	**Commuting Identity:**	$\{x, y, U_x z\} = U_x\{y, x, z\} = \{U_x y, z, x\},$
	so for $x = 1$	$\{1, y, z\} = \{y, 1, z\} = \{y, z, 1\}$
	and hence	$U_{z,1} = V_{z,1} = V_{1,z} = V_z,$
(FFIIIe)	**Triple Shift Identity:**	$\{U_x y, y, z\} = \{x, U_y x, z\},$
	so for $y = 1$	$\{x^2, z\} = \{x, x, z\}$
(FFIVe)	**Triple Switch Identity:**	$\{x, y, z\} + \{x, z, y\} = \{x, \{y, z\}\},$
(FFVe)′	**5-Linear Identity:**	$\{x, y, \{z, w, u\}\} = \{\{x, y, z\}, w, u\}$
		$-\{z, \{y, x, w\}, u\} + \{z, w, \{x, y, u\}\}.$

We can derive an alternate linearized version of Fundamental (FFI); *from* (FFIII) *we get Zel'manov's specialization formulas*; *and linearization of* (FFV) *produces the* 5*-linear formula*:

(FFI)′ **Alternate Fundamental Formula:**

$$U_{\{x,y,z\}} + U_{U_x U_y(z),z} = U_x U_y U_z + U_z U_y U_x + V_{x,y} U_z V_{y,x},$$

(FFIII)′ **Specialization Formulas:** $V_x^2 - V_{x^2} = 2U_x,$

$$V_x V_y V_x - V_{U_x y} = U_{x,y} V_x + U_x V_y = V_x U_{x,y} + V_y U_x,$$

(FFV)′ **5-Linear Formula:** $[V_{x,y}, V_{z,w}] = V_{V_{x,y}(z),w} - V_{z,V_{y,x}(w)}.$

PROOF. In associative algebras the operators on both sides of (FFI), (FFII), (FFIII), (FFIII)′, (FFIV) applied to z produce $xyzzxyx$, $xyxzx + xzxyx$, $xyxyz + zyxyx$, $xyzx + xzyx + x^2zy + yzx^2 + xzxy + yxzx$, $xyz + zyx$

respectively. Since they are operator identities in 2 variables, they hold in all algebras by Macdonald. [Note that (FFIVe) is just a linearization of the version $y = 1$ of (FFIIIe), so (FFIV) is a consequence of (FFIII).]

The Fundamental Lie Formula (FFV) is not amenable to Macdonald directly, since it involves 4 variables, but it follows from linearizations of (FFII),(FFIII): $\big(V_{x,y}U_z + U_zV_{y,x}\big)(w) = \{U_zw, y, x\} + U_z\{y, x, w\}$ [using symmetry of the triple product] $= \big(- \{U_zy, w, x\} + \{z, \{w, z, y\}, x\}\big) + \big(- U_{z,x}\{y, z, w\} + \{U_zy, w, x\} + \{\{z, y, x\}, w, z\}\big)$ [replacing $x, y, z \mapsto z, y, x$ in the linearized Triple Shift Identity and then linearizing $y \mapsto y, w$ for the first, and replacing $x, y, z \mapsto z, y, w$ in the Commuting Identity and then linearizing $z \mapsto z, z, x$ for the second] $= \{\{x, y, z\}, w, z\} = U_{\{x,y,z\},z}(w)$.

The more useful elemental version of (FFV) is the 5-Linear Identity (FFV$'$e); it follows by letting (FFV) act on w and then linearizing $z \mapsto z, u$. If we interpret (FFV$'$e) as an operator acting on u we get (FFV)$'$.

Likewise, Alternate Fundamental (FFI)$'$ involves four variables and doesn't follow from Macdonald. Since linearization of Fundamental (FFI) (replace x by $x + \lambda z$ and take coefficients of λ^2) yields

$$U_{\{x,y,z\}} + U_{U_xy,U_zy} = U_xU_yU_z + U_zU_yU_x + U_{x,z}U_yU_{x,z},$$

Alternate Fundamental is equivalent to

$$U_{U_xU_y(z),z} - V_{x,y}U_zV_{y,x} = U_{U_xy,U_zy} - U_{x,z}U_yU_{x,z}.$$

By acting on a, in special algebras both sides reduce to the 7-tad $\{x, y, z, a, x, y, z\}$, but Macdonald can only lend moral support. To carry on alone we make repeated use of Fundamental Lie [for $V_{z,a}U_x$ on the first term, $V_{x,a}U_z$ on the second term, $V_{U_zy,a}U_x$ (also Triple Shift (FFIII)) on the third term], and by linearized Triple Shift (FFIII) on the fourth term, to see that on the left side we have

$$\{z, a, U_xU_y(z)\} = \overbrace{\{x, U_yz, \{z, a, x\}\}}^{\alpha} - \overbrace{U_x\{a, z, U_yz\}}^{\beta},$$

$$-V_{x,y}U_z\{a, x, y\} = -V_{x,y}\big(- \overbrace{\{x, a, U_zy\}}^{\gamma} + \overbrace{\{\{x, a, z\}, y, z\}}^{\delta}\big),$$

while on the right side we have

$$\begin{aligned} \{U_zy, a, U_xy\} &= \{x, y, \{U_zy, a, x\}\} - U_x\{a, U_zy, y\}, \\ &= V_{x,y}\big(\overbrace{\{x, a, U_zy\}}^{\gamma}\big) - \overbrace{U_x\{U_yz, z, a\}}^{\beta}, \\ -\{x, U_y\{x, a, z\}, z\} &= \{x, U_yz, \{x, a, z\}\} - \{x, y, \{\{x, a, z\}, y, z\}\}, \\ &= \overbrace{\{x, U_yz, \{x, a, z\}\}}^{\alpha} - V_{x,y}\overbrace{\{\{x, a, z\}, y, z\}}^{\delta}. \end{aligned}$$

Thus both sides sum to the same result $\alpha - \beta + V_{x,y}(\gamma) - V_{x,y}(\delta)$. □

EXERCISE 5.2.3 (1) The first part of (FFIII)$'$ is the definition of the U-operator; deduce the second part by setting $z = \hat{1}$ in (FFIIIe); then linearize $y \mapsto y, z$; the equality of the two negative expressions follows by linearizing Commuting $U_x V_x = V_x U_x$. (2) Show that (FFIII) $\Longrightarrow$ 2(FFIV) by identifying coefficients of $\lambda\mu$ when replacing $x, y \mapsto x+\lambda\hat{1}, y+\mu\hat{1}$ in (FFIII). (3) Show that (FFIII) $\Longrightarrow$ (FFIV) by replacing $y \mapsto \hat{1}$ in (FFIII), then linearizing $x \mapsto x, y$.

We have included the Fundamental Lie and 5-Linear Identities primarily because of their historical role in bonding Lie and Jordan algebras through the structure Lie algebra and TKK-construction. We are exclusively concerned with the structure theory of Jordan algebras in this Part and the next, and will have only a few occasions to use these two identities.

The first four identities are "working tools" that we will use constantly in the rest of the book. As one immediate application, which will be of future use, let us rephrase the condition for a unit in terms of U-operators instead of L-operators.

Idempotent Unit Proposition 5.2.4 *An element e is the unit element of* J *iff it is an idempotent with identity U-operator,*

$$e^2 = e, \qquad U_e = \mathbb{1}_{\mathrm{J}}.$$

PROOF. These certainly are necessary conditions; they are sufficient since they imply $L_e = L_e \mathbb{1}_{\mathrm{J}} = L_e U_e = \frac{1}{2} U_{e^2,e}$ [by Commuting Formula (FFII)] $= \frac{1}{2} U_{e,e}$ [by $e^2 = e$] $= U_e = \mathbb{1}_{\mathrm{J}}$. (We could also use the Shirshov–Cohn Principle 5.1.4 directly: in the associative algebra $\Phi\{e, x\}$ we have $ex = e(exe)$ [by $U_e = \mathbb{1}_{\mathrm{J}}$] $= e^2 xe = exe$ [by $e^2 = e$] $= x$, dually $xe = e$, so also $e \bullet x = x$.) $\square$

It is impossible to overstress the importance of the Fundamental Formula. Historically, it was the instigation for Macdonald's theorem. We will use it twice a day (thrice on Sundays). The entire theory of quadratic Jordan algebras (over general scalar rings where $\frac{1}{2}$ may not be available) is based on this formula. The Commuting Formula (FFII) is needed together with (FFI) to axiomatize unital quadratic Jordan algebras; its special case $y = 1$ is equivalent to the Jordan identity (commutativity of V_x and V_{x^2}). The Triple Shift Formula (FFIII) is needed (with (FFI) and (FFII)) to axiomatize quadratic triple systems (to make up for the lack of a unit). We will use the Triple Switch Formula (FFIV) very frequently to switch factors in a triple product (modulo bilinear products); Jordan triple systems sorely lack this handy tool.

Rather surprisingly, we need almost no further identities in the rest of the book, and those few follow from Macdonald's principle, so we will introduce them only when needed. The one exception is the structurality of the Bergmann operator (see Exercise 5.3.1 below).

5.3 Nondegeneracy

An immediate consequence of the Fundamental Formula is the creation of principal inner ideals, which play an important role as Jordan analogues of left ideals. The trivial elements are those whose principal inner ideal collapses; these are pathological, and our entire structure theory will assume that the algebras are *nondegenerate* in the sense of having no nonzero trivial elements.[1]

Principal Inner Proposition 5.3.1 (1) *The* **open, half-open, closed principal inner ideals** *determined by an element x of a Jordan algebra are*

$$(x) := U_x\mathrm{J} \subseteq (x] := U_x\widehat{\mathrm{J}} = \Phi x^2 + U_x\mathrm{J} \subseteq [x] := \Phi x + U_x\widehat{\mathrm{J}}.$$

We will use $(x]$ as our standard, and simply call it the *principal inner ideal determined by x. Certainly $(x) = (x]$ in unital algebras, and all three coincide if x is regular ($x \in U_x\mathrm{J}$). It is usually not crucial which one we use, because they are closely related*:

$$U_{(x)}\widehat{\mathrm{J}} \subseteq U_{(x]}\widehat{\mathrm{J}} \subseteq (x), \quad U_{[x]}\mathrm{J} \subseteq (x), \quad U_{[x]}\widehat{\mathrm{J}} \subseteq (x].$$

PROOF. These containments and innernesses are immediate consequences of the Fundamental and Commuting Formulas (FFI), (FFII) since the formula $U_{\alpha x+U_x a} = \alpha^2 U_x + \alpha U_{x,U_x a} + U_{U_x a} = \alpha^2 U_x + \alpha V_{x,a}U_x + U_xU_aU_x$ and its dual show that

$$U_{\alpha x+U_x a} = B_{\alpha,x,-a}U_x = U_xB_{\alpha,-a,x} \quad (B_{\alpha,x,y} := \alpha^2 1\!\!1_{\mathrm{J}} - \alpha V_{x,y} + U_xU_y)$$

in terms of the useful **generalized Bergmann operators** $B_{\alpha,x,y}$. This implies that $U_{[x]}\widehat{\mathrm{J}} = U_xB_{\Phi,\widehat{\mathrm{J}},x}\widehat{\mathrm{J}} \subseteq U_x\widehat{\mathrm{J}} = (x]$, and also that $U_{[x]}\mathrm{J} + U_{(x]}\widehat{\mathrm{J}} \subseteq U_xB_{\Phi,\widehat{\mathrm{J}},x}\mathrm{J} + U_xU_{\hat{\mathrm{J}}}U_x\widehat{\mathrm{J}} \subseteq U_x\mathrm{J} = (x)$ [since J is an ideal in its unital hull]. □

In general $(b]$ needn't contain the element b, and the inner ideal generated by b (i.e., the smallest inner ideal of J containing b) is the slightly larger $[b]$. As with intervals on the real line, we can start from the "open principal" inner ideal (b) and adjoining the "endpoint" b^2 to get the "principal" inner ideal $(b]$, then adjoin the other endpoint b to get the "closed principal" inner ideal $[b]$.[2] In unital algebras we always have $(b) = (b]$, but not necessarily $(b) = [b]$. An element is called *von Neumann regular* if $b \in U_b\mathrm{J}$, equivalently $(b) = (b] = [b]$; these elements will play a starring role in Chapter 18.

[1] Nondegeneracy and the Full, Hermitian, Quadratic and Cubic Factor Nondegeneracy Examples were discussed in I.2.8, and principal inner ideals were introduced in I.4.7–8.

[2] We could also call these the *weak, ordinary,* and *strong* principal inner ideals. As explained in the *Historical Survey* Section I.4, we take the ordinary principal $(x]$ as our basic inner ideal. As Goldilocks says, it's not to little, not too big, its just right.

The **Bergmann operators** $B_{x,y} := B_{1,x,y}$ play an important role in differential geometry, determining the Bergmann metric for the bounded symmetric domains built out of Jordan triples.

EXERCISE 5.3.1 Prove the fundamental formula $U_{B_{\alpha,x,y}(z)} = B_{\alpha,y,x} U_z B_{\alpha,x,y}$ for the generalized Bergmann operators $B_{\alpha,x,y} := \alpha^2 \mathbb{1}_J - \alpha V_{x,y} + U_x U_y$. Note that this can't be derived immediately from Macdonald's Principle. We will return to this in Bergmann Structurality Proposition III.1.2.2.

Triviality Definition 5.3.2 *An element z of a Jordan algebra* J *is* **trivial** *if its principal inner ideal vanishes, $(z] = \mathbf{0}$. This just means $U_z \widehat{J} = \mathbf{0}$ (i.e., $U_z(J) = \mathbf{0}$ and $z^2 = 0$). An element w is* **weakly trivial** *if its weak principal inner ideal vanishes, $(w) = U_w(J) = \mathbf{0}$. Thus an element z is trivial iff it is weakly trivial and remains weakly trivial in the unital hull.*[3]

A Jordan algebra is **nondegenerate** *if it contains no nonzero trivial elements, $U_z = 0$ on $\widehat{J}$ implies $z = 0$. At the opposite extreme, a Jordan algebra is* **trivial** *if* all *its elements are trivial, equivalently, all Jordan products are zero*: $J \bullet J = 0$. [*Since our convention requires $z^2 = 0$ for a trivial (as opposed to a weakly trivial) element, the condition that all x are trivial implies that all $x \bullet x = 0$ and so by linearization $2x \bullet y = 0$, and* by appealing to $\frac{1}{2}$ *we see that all $x \bullet y = 0$.*]

It will be useful to us to note that as soon as we rid an algebra of trivial elements the weakly trivial elements leave as well, as if called by the Pied Piper.

Weak Riddance Proposition 5.3.3 *A Jordan algebra* J *is nondegenerate iff it has no nonzero weakly trivial elements, $U_z = 0$ on* J *implies $z = 0$, because if an element is weakly trivial then either it or its square is trivial.*

PROOF. If w is weakly trivial, $U_w(J) = \mathbf{0}$, then either it is already trivial, $U_w(\widehat{J}) = U_w(J) + \Phi w^2 = \mathbf{0}$, or else its square w^2 is nonzero. But this square (whether zero or not) is *always* trivial, $U_{w^2}(\widehat{J}) = U_w U_w(\widehat{J})$ [by Operator Power-Associativity 5.2.2(2) $\subseteq U_w(J) = \mathbf{0}$ [by $U_w \widehat{J} \subseteq J$ and weak triviality of w]. ∎

EXERCISE 5.3.3* (1) Show that the image of a trivial (respectively, weakly trivial) element under structural (respectively, weakly structural) transformation $U_{T(x)} = T U_x T^*$ on $\widehat{J}$ (respectively, on J) (see Problem 7.2), is again trivial (respectively weakly trivial).

[3] Trivial elements are also called *absolute zero divisors*, though *absolute outer zero divisor* would be a better term [An *outer zero divisor* would be an element z which kills some nonzero a from the "outside," $U_z a = 0$; the term "absolute" refers to the fact that it kills absolutely everybody.] Many authors do not demand triviality on the unital hull, using weak triviality. Strong triviality is often more natural; for example, elements of a trivial associative ideal B of A are always strongly trivial.

In the structure theory we will be entirely concerned with nondegenerate algebras. We will often abuse language and say that *a nondegenerate algebra is one that has no trivial elements,* where of course we really mean no *nonzero* trivial elements.

We now describe what nondegeneracy amounts to in our basic examples.

Full and Hermitian Trivial Example 5.3.4 (1) *An element* z *of* A^+ *is trivial iff it generates a trivial two-sided ideal* $\mathrm{B} = \widehat{\mathrm{A}}z\widehat{\mathrm{A}}$. *In particular, the full Jordan algebra* A^+ *is nondegenerate iff the associative algebra* A *is semiprime.*

(2) *If* z *is trivial in* $\mathcal{H}(\mathrm{A}, *)$, *then either* z *or some* $z\hat{a}z$ *is trivial in* A. *In particular, if* A *is semiprime with involution, then* $\mathcal{H}(\mathrm{A}, *)$ *is nondegenerate.*

PROOF. (1) holds because $U_z\widehat{\mathrm{A}^+} = \mathbf{0} \Leftrightarrow z\widehat{\mathrm{A}}z = \mathbf{0} \Leftrightarrow (\widehat{\mathrm{A}}z\widehat{\mathrm{A}})(\widehat{\mathrm{A}}z\widehat{\mathrm{A}}) = \mathbf{0} \Leftrightarrow$ the ideal $\mathrm{B} := \widehat{\mathrm{A}}z\widehat{\mathrm{A}}$ generated by z is a trivial associative ideal, $\mathrm{BB} = \mathbf{0}$.

(2) If $z \in \mathcal{H}$ has $z\mathcal{H}z = \mathbf{0}$, then for all $c \in \mathrm{A}$ we have $z(c + c^*)z = 0 \Rightarrow zcz = -zc^*z$. If z is *not* itself trivial in A, then $w = z\hat{a}z \neq 0$ for some $\hat{a} \in \widehat{\mathrm{A}}$; but then w is trivial in A, since for any $\hat{b} \in \widehat{\mathrm{A}}$ we have $w\hat{b}w = (z\hat{a}z)\hat{b}(z\hat{a}z) = z(\hat{a}z\hat{b})z\hat{a}z = -z\big(\hat{a}z\hat{b}\big)^* z\hat{a}z$ [note that thanks to z, the element $c = \hat{a}z\hat{b}$ lies in A, despite the fictitious elements $\hat{a}, \hat{b} \in \widehat{\mathrm{A}}$] $= -z\big(\hat{b}^* z\hat{a}^*\big)zaz = -z\hat{b}^* z\big(\hat{a}^* z\hat{a}\big)z = 0$ [since $\hat{a}^* z\hat{a} \in \mathcal{H}$ is also an actual element]. This shows that w is trivial. □

In general, if $\mathcal{H}$ *is nondegenerate,* A *needn't itself be semiprime unless it is a "tight cover" of* $\mathcal{H}$ (see Problems 5.4, 5.5 at the end of the chapter).

Nondegeneracy of Jordan Matrix Algebras 3.2.4 reduces to nondegeneracy of their coordinate algebras. For these particular hermitian algebras for $n = 3$ we can even allow non-special algebras with alternative coordinates (cf. the 3×3 Coordinatization Theorem C.1.3 in Appendix C).

Jordan Matrix Nondegeneracy Example 5.3.5 *For* D *a unital alternative algebra with nuclear involution and* $n \geq 2$, $\mathcal{H}_n(\mathrm{D}, -)$ *is nondegenerate iff* D *and* $\mathcal{H}(\mathrm{D}, -)$ *are semiprime in the sense that they have no* D*- or* $\mathcal{H}$*-trivial elements, i.e.,* $d\mathrm{D}d = 0 \Longrightarrow d = 0$ *in* D *and* $\delta\mathcal{H}\delta = 0 \Longrightarrow \delta = 0$ *in* $\mathcal{H}$.

PROOF. For unital alternative D with nuclear involution we noted in the Hermitian Matrix Example 3.2.4(3) that the Three Basic U-Products and Basic U-Orthogonality are

$$U_{\delta[ii]}\beta[ii] = (\delta\beta\delta)[ii], \qquad U_{d[ij]}b[ij] = (d\bar{b}d)[ij],$$
$$U_{d[ij]}\beta[jj] = (d\beta\bar{d})[ii], \qquad U_{d[ij]}b[k\ell] = 0 \text{ if } \{k,\ell\} \not\subseteq \{i,j\}$$

(parentheses are unnecessary since δ, η are nuclear and D is flexible).

If there are no trivial coordinates, there are no trivial matrices $z = \sum_i \delta_{ii}[ii] + \sum_{i<j} d_{ij}[ij]$: by the Fundamental Formula any $U_x z$ would be again trivial, so $U_{1[ii]}z = \delta_{ii}[ii]$ trivial implies $\delta_{ii}\mathcal{H}\delta_{ii}[ii] = U_{\delta[ii]}(\mathcal{H}[ii]) = \mathbf{0}$ and δ_{ii} is $\mathcal{H}$-trivial, so all diagonal entries δ_{ii} are 0, in which case $U_{1[ii]+1[jj]}z = d_{ij}[ij]$

trivial implies all $d_{ij}\bar{c}d_{ij}[ij] = U_{d_{ij}[ij]}\big(c[ij]\big) = 0$ and each d_{ij} is D-trivial, so all off-diagonal entries d_{ij} are 0 too.

Conversely, if there are no trivial matrices there can be no trivial coordinates: if δ is $\mathcal{H}$-trivial then $z = \delta[11]$ is trivial, since by the above Basic U-Orthogonality we have $U_z(\mathrm{J}) = U_z\mathcal{H}[11] = \delta\mathcal{H}\delta[11] = \mathbf{0}$; if d is D-trivial then all $d\eta\bar{d}, \bar{d}\eta d \in \mathcal{H}$ are $\mathcal{H}$-trivial [note that $d\eta\bar{d} = d\eta(d+\bar{d}) - d\eta d = d\eta\tau$, where the trace $\tau := d + \bar{d}$ is in $\mathcal{H}$, and $(d\eta\tau)\mathcal{H}(d\eta\tau) = \big(d(\eta\tau\mathcal{H})d\big)\eta\tau = \mathbf{0}$ by nuclearity of $\eta, \tau, \mathcal{H}$] and so as above produce trivial $z = \delta[11]$; if $d\mathcal{H}\bar{d} = \bar{d}\mathcal{H}d = \mathbf{0}$ then $z = d[12]$ itself is trivial, since by Basic U-Orthogonality again $U_z(\mathrm{J}) = U_{d[12]}(\mathcal{H}[11] + \mathrm{D}[12] + \mathcal{H}[22]) = (\bar{d}\mathcal{H}d)[22] + (d\overline{\mathrm{D}}d)[12] + (d\mathcal{H}\bar{d})[11] = \mathbf{0}$. □

Exercise 5.3.5 Show that the hypothesis that D is unital can be removed in the above example, though the surgical operation is painful enough to require an anesthetic. (1) If z is a trivial matrix, use Basic Triple Product Rules 3.2.4(4) in addition to the rules of U-products to examine the ii-entry of $U_z\mathcal{H}[ii] = \mathbf{0}$ to see δ_{ii} is $\mathcal{H}$-trivial. (2) Once all the diagonal δ_{ii} are removed, examine the ij-entry of $U_z\mathrm{D}[ij] = \mathbf{0}$.

Nondegeneracy in Jordan algebras built out of quadratic and cubic forms is closely related to nondegeneracy of the forms themselves.

Quadratic and Cubic Factor Nondegeneracy Example 5.3.6 *Triviality in a quadratic or cubic factor is located in the radical of the norm form, so the algebra is nondegenerate when its norm is a nondegenerate form.*

Quadratic: *In a quadratic factor $\mathcal{J}ord(Q, c)$ over a field, an element z is trivial iff $z \in \mathcal{R}ad(Q)$. Thus $\mathcal{J}ord(Q, c)$ is nondegenerate iff Q is nondegenerate as quadratic form.*

Spin: *A Spin Factor $\mathcal{JS}pin(M, \sigma)$ over a field is nondegenerate iff the bilinear form σ is nondegenerate.*

Reduced Spin: *A Reduced Spin Factor $\mathcal{R}ed\mathcal{S}pin(q)$ over a field is nondegenerate iff the quadratic form q is nondegenerate.*

Cubic: *A cubic factor $\mathcal{J}ord(N, \#, c)$ over a field is nondegenerate iff there are no nonzero trace- and sharp-trivial elements $T(z, \mathrm{J}) = z^{\#} = 0$. A cubic factor $\mathcal{J}ord(N, c)$ determined by a Jordan cubic form with basepoint over a field is always nondegenerate.*

proof. In a quadratic factor over an arbitrary ring of scalars, recall from Forms Permitting Composition 2.3.1(2) that Q is nondegenerate iff it has zero radical $\mathcal{R}ad(Q) = \{z \mid Q(z) = Q(z, \mathrm{J}) = 0\}$. If z is radical then $Q(z) = Q(z, \mathrm{J}) = 0$, hence by Quadratic Factor Example 3.3.1 $U_z(\mathrm{J}) = Q(z, \bar{\mathrm{J}})z - Q(z)\bar{\mathrm{J}} = 0$ and z is trivial. Conversely, if z is trivial over a *field* then $U_z\bar{z} = Q(z, z)z - Q(z)z = Q(z)z$ vanishes, so [because J is torsion-free as a Φ-module when Φ is a field] the scalar $Q(z)$ is equal to 0; then $U_z\bar{x} = Q(z, x)z$ vanishes, which in turn implies $Q(z, x) = 0$ for any x, and z is radical.

The spin and reduced spin examples then follow, since $\mathcal{J}Spin(M,\sigma) = \mathcal{J}ord(Q,1)$ for $Q = 1 \oplus (-q)$, and Q is nondegenerate iff q is, i.e., iff σ is [since $\frac{1}{2} \in \Phi$]. Similarly, $\mathcal{R}ed\mathcal{S}pin(q) = \mathcal{J}ord(Q,c)$ for $Q = Q_H \oplus (-q)$ (H a hyperbolic plane $\Phi e_1 + \Phi e_2, c = e_1 + e_2$), which is again nondegenerate iff q is.

For cubic factors, the Cubic Construction shows that a trace- and sharp-trivial element $T(z,\mathrm{J}) = z^\# = 0$ is trivial in the U-sense, $U_z x = T(z,x)z - z^\# \# x = 0$, no matter what the ring of scalars is. Conversely, over a field if $z \neq 0$ is trivial then $z^3 = U_z z$, $z^2 = U_z c$ vanish. Then the Degree–3 Identity 4.2.2(2) says that $N(z)c = z^3 - T(z)z^2 + S(z)z = S(z)z$; over a field a nonzero trivial z is certainly not a scalar multiple of the unit c, so we must have $S(z) = 0$. Taking traces of the Sharp Expression 4.2.2(1) $z^\# = z^2 - T(z)z + S(z)c = -T(z)z$ yields $-T(z)^2 = T(z^\#) = S(z) = 0$, so $T(z) = 0$ too, whence the Sharp Expression itself gives sharp-triviality $z^\# = 0$. Then U-triviality $\mathbf{0} = U_z\mathrm{J} = T(z,\mathrm{J})z - z^\#\#\mathrm{J} = T(z,\mathrm{J})z$ implies trace-triviality. Since by definition a Jordan cubic form always has a nondegenerate trace, there are no nonzero trace-trivial elements, so $\mathcal{J}ord(N,c)$ is always nondegenerate. □

Exercise 5.3.6A Extend the above nondegeneracy criterion for quadratic factors from fields to arbitrary rings of scalars without nilpotent elements, by showing that $Q(z)z = 0 \Rightarrow Q(z)^3 = 0$ and $Q(z,x)z = 0 \Rightarrow Q(z,x)^2 = 0$.

Exercise 5.3.6B Extend the above nondegeneracy criterion for cubic factors from fields to arbitrary rings of scalars without nilpotent elements. (1) If $z \in \mathcal{J}ord(N,\#,c)$ is trivial, multiply the Degree–3 Identity by z to get $N(z)z = 0$; conclude that $N(z) = 0$. (2) Use the Degree–3 Identity and (1) to get $S(z)z = 0$, hence $S(z) = 0$. (3) Take traces of the Sharp Expression to conclude that $T(z) = 0$, hence $z^\# = 0$. (4) In the presence of sharp-triviality, show that z is trivial iff it is trace-trivial. (5) Show by example that all bets are off if there are nilpotent scalars: for any nondegenerate cubic factor J, the algebra of dual numbers $\mathrm{J}[\varepsilon] = \mathrm{J} \otimes_\Phi \Phi[\varepsilon]$ ($\varepsilon^2 = 0$) has all elements $\varepsilon\mathrm{J}$ trivial, yet T may be a nondegenerate bilinear form.

5.4 Problems for Chapter 5

Problem 5.1 Macdonald's Theorem still holds for quadratic Jordan algebras, but not the Shirshov–Cohn Theorem: there exist elements with $z^2 = 0$ but $z^3 \neq 0$, so even the subalgebra $\Phi[z]$ generated by a single element need not be special. Zounds! Use only quadratic products, and the rule $U_{x^n} x^m = x^{2n+m}$, to show that in quadratic Jordan algebras we at least have $z^n = 0 \Rightarrow z^m = 0$ for all $m \geq 2n$.

Problem 5.2* (1) Show that ideals inherit nondegeneracy: if a Jordan algebra J is nondegenerate, so is any ideal I in J. (2) Show that any nonzero trivial ideal K ($U_{\mathrm{K}}\widehat{\mathrm{K}} = \mathbf{0}$) contains trivial elements. Conclude that nondegeneracy always implies semiprimeness in Jordan algebras. The converse doesn't seem to hold. Indeed, a major step in Zel'manov's infinite-dimensional theory is the proof that the trivial elements in any Jordan algebra live in an ideal which need not be *globally* nilpotent, but is at least *locally* nilpotent in the sense that every finitely-generated subalgebra is nilpotent.

PROBLEM 5.3* Establish a formula $D(x^n) = \mathcal{M}_{n,x}(D(x))$ for the derivative of a power as a multiple of $D(x)$ for any derivation D of a Jordan algebra.

PROBLEM 5.4* Let $\mathrm{J} \subseteq \mathrm{A}^+$ be *any* special Jordan algebra. (1) If B is a trivial associative ideal in A, show that all elements of $\mathrm{B} \cap \mathrm{J}$ are trivial in J. Conclude that if J is nondegenerate, and A is a *tight cover* of J in the sense that there are no nonzero associative ideal of A which are *disjoint* from (have zero intersection with) J, then A is semiprime. (2) We can easily remedy non-tightness by removing a maximal disjoint ideal K, replacing the envelope A by the equally-good envelope $\overline{\mathrm{A}} = \mathrm{A}/\mathrm{K}$: show that J still faithfully imbeds in $\overline{\mathrm{A}}$ and intersects all its nonzero ideals. (3) The associative direct sum $\mathrm{A} = \mathrm{A}_1 \boxplus \mathrm{A}_2$ is about as far as possible from being a tight cover of its direct summand $\mathrm{J} = \mathrm{A}_1^+$. Give an example to show that nondegeneracy of J cannot force semiprimeness of A, since J has no influence on A_2. Give an example to show that nondegeneracy of $\mathrm{J} = \mathcal{H}(\mathrm{A}, *)$ cannot force nondegeneracy of A.

PROBLEM 5.5* (1) An associative $*$-algebra is called $*$-*semiprime* if it has no trivial $*$-ideals $\mathrm{BB} = \mathbf{0}$. Show that this is the same as having no nilpotent $*$-ideals $\mathrm{B}^n = 0$. Show that A is $*$-semiprime as $*$-algebra iff it is semiprime as an ordinary algebra. (2) An associative algebra is called a $*$-*tight cover* of $\mathrm{J} \subseteq \mathcal{H}(\mathrm{A}, *)$ if there are no disjoint $*$-ideals: every nonzero $*$-ideal of A has nonzero intersection with J. Show that any $*$-tight cover of a nondegenerate Jordan algebra is $*$-*semiprime*. (3) If $\frac{1}{2} \in \Phi$ show that a $*$-ideal is disjoint from $\mathrm{J} = \mathcal{H}(\mathrm{A}, *)$ iff it is skew (contained in $Skew(\mathrm{A})$). Conclude that A is $*$-tight iff it has no nonzero skew ideals. (4) Show that any cover can be $*$-tightened. If $\mathrm{J} = \mathcal{H}(\mathrm{A}, *)$ show that there is a *unique* tightening $\overline{\mathrm{A}}$ and still $\mathrm{J} = \mathcal{H}(\overline{\mathrm{A}}, \overline{*})$.

QUESTION 5.1 (1) Is it true that e is idempotent iff V_e is idempotent? Iff U_e is idempotent? (2) Is it true that z is nilpotent iff V_z is nilpotent? Iff U_z is nilpotent?

QUESTION 5.2 What can you say about an element z that is *strongly trivial* in the sense that its strong or closed principal inner ideal $[z]$ vanishes?

QUESTION 5.3* The black sheep of the principal inner ideal family is the right-open $[x) := \Phi x + U_x \mathrm{J}$. We will never mention him again, and all glory goes to his illustrious half brother $(x]$. Decide whether $[x)$ genetically belongs to the inner ideal family, and decide where it fits in the family lattice (the chain of inclusions in Principal Inner Proposition 5.3.1).

QUESTION 5.4* Is every tight cover of a simple Jordan algebra a simple associative algebra? Is every $*$-tight cover of a simple Jordan algebra $*$-simple?

6

Inverses

The Classical Theory of algebras with capacity depends crucially on the properties of inverses: the diagonal building blocks J_{ii} will be division subalgebras, and they will be connected by off-diagonal invertible elements in the subalgebra $J_{ii} + J_{ij} + J_{jj}$. In general there is no one definition of invertibility that works well for all unital nonassociative algebras.

For associative and alternative algebras the condition $xy = yx = 1$ is sufficient for x to be invertible with a "reasonable" inverse y. But this condition is not restrictive enough for a general notion of inverse in *all* algebras: in a Jordan algebra $\mathcal{JS}pin(M, \sigma) = \Phi 1 \oplus M$ $(v \bullet w = \sigma(v, w)1)$ of a bilinear form σ with $\dim(M) > 1$, *every* $v \neq 0$ has promiscuously many "inverses" w with $\sigma(v, w) = 1$, even if v has $\sigma(v, v) = 0$ and hence squares to 0!

At the other extreme, for coordinatizing a projective plane one needs a coordinate ring which is a "division algebra" in the strong sense that for each $x \neq 0$ the operators L_x and R_x should be invertible transformations (so the equations $xy = a,\ zx = b$ have unique solutions y, z for any given a, b). This is overly restrictive, for A^+ fails this test even for the well-behaved associative division algebra $A = \mathbb{H}$ of real quaternions (think of $i \bullet j = 0$ for the eminently invertible complex number i).

6.1 Jordan Inverses

Jacobson discovered the correct elemental definition of inverse at the same time he discovered U-operators, and these have been inextricably linked ever since their birth.[1]

Inverse Definition 6.1.1 *An element x of a unital Jordan algebra is* **invertible** *if it has an* **inverse** *y (denoted by x^{-1}) satisfying the* **Quadratic Jordan Inverse Conditions**

[1] In the *Historical Survey*, the Linear Inverse Conditions were introduced in I.3.2, Quadratic Inverse Conditions were introduced in I.4.5; the Jordan Inverse Recipe, U–Inverse Formula, Special and Factor Invertibility were discussed in I.3.7–8 and I.4.5.

$$\text{(QJInv1)} \quad U_x y = x, \qquad \text{(QJInv2)} \quad U_x y^2 = 1.$$

An algebra in which every nonzero element is invertible is called a **Jordan division algebra**.

In associative algebras invertibility is reflected in the left and right multiplication operators: an element x in a unital associative algebra is invertible iff L_x is invertible, in which case $L_{x^{-1}} = \big(L_x\big)^{-1}$ (and dually for R_x). Analogously, invertibility in Jordan algebras is reflected in the U-operator.

Invertibility Criterion 6.1.2 *The following are equivalent for an element x of a unital Jordan algebra*:
(1) *the element x is invertible in* J;
(2) *the operator U_x is invertible on* J;
(3) *the operator U_x is surjective on* J;
(4) *the unit* 1 *lies in the range of the operator U_x.*

PROOF. Clearly (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4), and (4) $\Longrightarrow$ (2) since $U_x a = 1 \Longrightarrow \mathbb{1}_J = U_1 = U_x U_a U_x \Longrightarrow U_x$ has left and right inverses $\Longrightarrow U_x$ is invertible. Clearly (1) $\Longrightarrow$ (4) by (QJInv2). Finally, (2) $\Longrightarrow$ (1) for $y = \big(U_x\big)^{-1} x$: we have (QJInv1) $U_x y = x$ by definition, and if we write $1 = U_x a$ then $U_x y^2 = U_x U_y 1 = U_x U_y U_x a = U_{U_x y} a$ [by the Fundamental Formula (FFI)] $= U_x a = 1$, so (QJInv2) holds as well. □

Inverse Formulas 6.1.3 *If x is invertible in a unital Jordan algebra, then its inverse is uniquely determined by the* **Jordan Inverse Recipe**

$$x^{-1} = U_x^{-1} x.$$

*The U–***Inverse Formula** *states that the U-operator of the inverse is the inverse of the original U-operator,*

$$U_{x^{-1}} = U_x^{-1},$$

*and the L–***Inverse Formula** *states that L-multiplication by the inverse is given by a shift of original L-multiplication,*

$$L_{x^{-1}} = U_x^{-1} L_x, \text{ which therefore commutes with } L_x.$$

PROOF. The Inverse Recipe holds by (QJInv1) and the fact that U_x is invertible by Invertibility Criterion (2). The U–Inverse Formula holds by multiplying the relation $U_x U_{x^{-1}} U_x = U_{U_x x^{-1}} = U_x$ [by the Inverse Recipe and the Fundamental Formula] on both sides by U_x^{-1}. The L–Inverse Formula holds since the right side commutes with L_x (since U_x does), and is seen to equal the left side by canceling U_x from $U_x V_{x^{-1}} U_x = U_x (U_{x^{-1},1}) U_x = U_{U_x x^{-1}, U_x 1}$ [by the Fundamental Formula] $= U_{x,x^2} = V_x U_x$ [by Commuting Formula (FFII)]. □

EXERCISE 6.1.3 Assume that x is invertible. (1) Use the L-Inverse Formula $U_x L_{x^{-1}} = L_x$ to show that $L_{x^{-1}} L_x = \mathbb{1}_J$ iff $L_{x^2} = L_x^2$ iff x satisfies the associativity condition $[x, x, y] = 0$ for all y.

A unital associative algebra is a division algebra iff it has no proper one-sided ideals, which is equivalent to having no proper inner ideals. The situation for Jordan algebras is completely analogous.

Division Algebra Criterion 6.1.4 *The following are equivalent for a unital Jordan algebra*:
(1) J *is a division algebra*;
(2) J *has no trivial elements or proper principal inner ideals*;
(3) J *has no proper closed principal inner ideals*;
(4) J *has no proper inner ideals.*

PROOF. Notice that since J is unital, hats are unnecessary, and the principal inner ideal is $(x] = (x) = U_x(\mathrm{J})$. J is a division algebra [as in (1)] $\iff$ all $x \neq 0$ are invertible $\iff$ if $x \neq 0$ then $(x] = \mathrm{J}$ [by Invertibility Criterion 6.1.2(3)] $\iff$ if $x \neq 0$ then $(x] \neq \mathbf{0}$ and $(x] \neq \mathbf{0}$ implies $(x] = \mathrm{J}$ $\iff$ no trivial elements and no proper principal inner ideals [as in (2)] $\iff$ if $x \neq 0$ then $[x] = \mathrm{J}$ [as in (3)] (note in the unital case that $(x) = \mathrm{J} \iff [x] = \mathrm{J}$ since $\Longrightarrow$ is clear, and $\Longleftarrow$ holds by $\mathrm{J} = U_1\mathrm{J} \subseteq U_{[x]}\mathrm{J} \subseteq (x)$ from Principal Inner Proposition 5.3.1).

Clearly (4) $\Longrightarrow$ (3), and (3) $\Longrightarrow$ (4) since any nonzero inner ideal B contains a nonzero x, so $[x] \neq \mathbf{0}$ implies $[x] = \mathrm{J}$ and hence B must be all of J. □

We will see later (in the Surjective Unit Theorem 18.1.4) that this remains true even if J is not unital.

EXERCISE 6.1.4* (1) Show that the ban on trivial elements in (2) is necessary: if $\mathrm{J} = \Phi[\varepsilon]$ for a field Φ and $\varepsilon^2 = 0$, show that every element x is either invertible or trivial, so $(x) = (x]$ is either J or $\mathbf{0}$, yet J is not a division algebra. (2) Show directly that an associative algebra A (not necessarily unital) is a division algebra iff it is not trivial and has no proper inner ideals B (Φ-submodules with $b\widehat{\mathrm{A}}b \subseteq \mathrm{B}$ for all $b \in \mathrm{B}$).

Now we describe inversion in our basic Jordan algebras, beginning with Full and Hermitian algebras.

Special Inverse Example 6.1.5 (1) *If* $\mathrm{J} \subseteq \mathrm{A}^+$ *is a unital Jordan subalgebra of a unital associative algebra* A, *then two elements* x, y *in* J *are Jordan inverses iff they are associative inverses.*

$$\text{(AInv1)} \quad xy = 1, \qquad\qquad \text{(AInv2)} \quad yx = 1.$$

Thus x is invertible in J *iff it is invertible in* A *and its inverse belongs to* J.

(2) Full Invertibility: *If* A *is a unital associative algebra, an element* x *has inverse* y *in the Jordan algebra* A^+ *if and only if it has inverse* y *in the associative algebra* A, *and its Jordan and associative inverses coincide*; A^+ *is a Jordan division algebra iff* A *is an associative division algebra.*

(3) Hermitian Invertibility: *If* A *is a unital associative algebra with involution* $*$, *an element in the Jordan algebra* $\mathcal{H}(A,*)$ *of symmetric elements is invertible if and only if it is invertible in* A, *and* $\mathcal{H}(A,*)$ *is a Jordan division algebra if* A *is an associative division algebra.*

proof. (1) The associative inverse conditions (AInv1)–(AInv2) for x, y in A certainly imply the Jordan inverse conditions (QJInv1)–(QJInv2) in A^+; conversely, (QJInv2) $xy^2x = 1$ implies that x has a left and a right inverse, hence an inverse, and canceling x from (QJInv1) $xyx = x$ shows that $xy = yx = 1$. (2) for $J = A^+$ follows immediately, and in (3) for $J = \mathcal{H}(A,*)$, the inverse $y = x^{-1}$ is hermitian if x is. □

Factor Invertibility Example 6.1.6 *An element* x *of a quadratic factor* $\mathcal{J}ord(Q,c)$ *or cubic factor* $\mathcal{J}ord(N,\#,c)$ *determined by a quadratic form* Q *or cubic form* N *with basepoint, is invertible if and only if its norm* $Q(x)$ *or* $N(x)$ *is invertible in the ring of scalars* Φ [*assuming that the scalars act faithfully in the cubic case*], *in which case the inverse is*

$$x^{-1} = Q(x)^{-1}\bar{x}, \qquad Q(x^{-1}) = Q(x)^{-1},$$
$$x^{-1} = N(x)^{-1}x^{\#}, \qquad N(x^{-1}) = N(x)^{-1}.$$

Over a field $\mathcal{J}ord(Q,c)$ *or* $\mathcal{J}ord(N,\#,c)$ *is a Jordan division algebra if and only if* Q *or* N *is an anisotropic quadratic or cubic form.*

proof. Let n stand for the norm (either Q or N). If x has inverse y, then $1 = n(c) = n(U_xy^2) = n(x)^2n(y)^2$ since the norm permits composition with U by Quadratic Factor Example 3.3.1 or the Cubic Construction 4.2.2(3) [here's where we need the annoying faithfulness hypothesis on the scalars], showing that $n(x)$ is an invertible scalar; then canceling $n(x)$ from $n(x) = n(U_xy) = n(x)n(y)n(x)$ shows that $n(x), n(y)$ are inverses as in the last assertion.

Conversely, if $n(x)$ is invertible we can use it to construct the inverse in the usual way: from the Degree–2 Identity 3.3.1 or Degree–3 Identity 4.2.2(2), $x^d - T(x)x^{d-1} + \cdots + (-1)^d n(x)c = 0$ ($d = 2, 3$), which as usual implies that $x \bullet y = 1$ for $y = (-1)^{d+1}n(x)^{-1}[x^{d-1} - T(x)x^{d-2} + \cdots] \in \Phi[x]$. But by Power-Associativity Theorem 5.2.2 this implies that $U_xy = x \bullet (x \bullet y) = x$, $U_xy^2 = (x \bullet y)^2 = 1$, so $y = x^{-1}$ as in the Inverse Definition 6.1. The inverse reduces to $-Q(x)^{-1}(x - T(x)1) = Q(x)^{-1}\bar{x}$ in quadratic factors and to $N(x)^{-1}(x^2 - T(x)x + S(x)1) = N(x)^{-1}x^{\#}$ [by Sharp Expression 4.2.2(2)] in cubic factors. □

EXERCISE 6.1.6 (1) A "norm" on a Jordan algebra is a polynomial function from J to Φ satisfying $N(1) = 1,\ N(U_x y) = N(x)^2 N(y)$ (but seldom $N(x \bullet y) = N(x)N(y)$, another good reason for using the U-operators instead of the L-operators). Use this to show that if x is invertible so is its norm $N(x)$, and $N(x^{-1}) = N(x)^{-1}$. (2) An "adjoint" is a vector-valued polynomial map on J such that $x^\#$ satisfies $U_x x^\# = N(x)x,\ U_x U_{x^\#} = N(x)^2 \mathbb{1}_J$. Show from this that, conversely, if $N(x)$ is invertible then x is invertible, $x^{-1} = N(x)^{-1} x^\#$ (just as with the adjoint of a matrix).

We now verify that the quadratic definition we have given agrees with the original linear definition II.I.3.2 given by Jacobson.

Linear Inverse Lemma 6.1.7 *An element of an arbitrary unital Jordan algebra has inverse y iff it satisfies the* **Linear Jordan Inverse Conditions**

$$\text{(LJInv1)} \quad x \bullet y = 1, \qquad\qquad \text{(LJInv2)} \quad x^2 \bullet y = x.$$

PROOF. The quadratic and linear invertibility conditions (Q) and (L) are strictly between x and y, so all take place in $\Phi[x, y]$, and by the Shirshov–Cohn Principle 5.1.4 we may assume that we are in some associative A^+. The reason that (Q) and (L) are equivalent is that they are both equivalent to the associative invertibility conditions (A): we have already seen this in Full Invertibility 6.1.5 for (Q); (A) certainly implies (LJInv1)–(LJInv2), $\frac{1}{2}(xy + yx) = 1,\ \frac{1}{2}(x^2y + yx^2) = x$, and conversely, multiplying (LJInv1) $xy + yx = 2$ on the left and right by x yields $x^2y + xyx = 2x = xyx + yx^2$, so $x^2y = yx^2$; hence both equal x by (LJInv2) $x^2y + yx^2 = 2x$; therefore $xy = (yx^2)y = y(x^2y) = yx$, so both equal 1 by (LJInv1), and we have (A). $\square$

EXERCISE 6.1.7* Show that the quadratic inverse conditions (Q1)–(Q2) are equivalent to the old-fashioned linear conditions (L1)–(L2) by direct calculation, without invoking any speciality principle. (1) Show that (Q1)–(Q2) $\Longrightarrow U_x$ invertible. (2) Show that U_x invertible $\Longrightarrow$ (L1)–(L2) for $y = U_x^{-1}(x)$ [by canceling U_x from $U_x(x^n \bullet y) = U_x(x^{n-1})$ for $n = 1, 2$ to obtain (Ln)]. (3) Show that U_x invertible $\Longleftarrow$ (L1)–(L2) [get (Q1) by definition of the U-operator, and (Q2) from $x^2 \bullet y^2 = 1$ using the linearized Jordan identity (JAX2)$'$), hence $x \bullet y^2 = y$].

In associative algebras, the product xy of two invertible elements is again invertible, as is xyx, but *not* $xy + yx$ (as we already noted, the invertible $x = i, y = j$ in the quaternion division algebra $\mathbb{H}$ have $xy + yx = 0$!). Similarly, in Jordan algebras, the product $U_x y$ of invertible elements remains invertible, but $x \bullet y$ need not be. Indeed, in a quadratic factor two traceless orthogonal units, $Q(x) = Q(y) = 1,\ T(x) = T(y) = Q(x, y) = 0$, will be invertible but will have $x \bullet y = 0$.

Invertible Products Proposition 6.1.8 (1) *Invertibility of a U-product amounts to invertibility of each factor, by the* **Product Invertibility Criterion**:

$$U_x y \text{ is invertible} \iff x, y \text{ is invertible},$$
$$\text{in which case } (U_x y)^{-1} = U_{x^{-1}} y^{-1}.$$

(2) *Invertibility of a power amounts to invertibility of the root, by the* **Power Invertibility Criterion**:

$$x^n \text{ is invertible } (n \geq 1) \iff x \text{ is invertible, in which case}$$
$$(x^n)^{-1} = (x^{-1})^n =: x^{-n} \quad \text{has} \quad U_{x^{-n}} = (U_x)^{-n}.$$

(3) *Invertibility of a direct product amounts to invertibility of each component*: *if* $\mathrm{J} = \prod \mathrm{J}_i$ *is a direct product of unital Jordan algebras, then we have the* **Direct Product Invertibility Criterion**:

$$x = \prod x_i \text{ is invertible in } \mathrm{J} \iff \text{each } x_i \text{ is invertible in } \mathrm{J}_i,$$
$$\text{in which case} \quad \left(\prod x_i\right)^{-1} = \prod x_i^{-1}.$$

PROOF. (1) $U_x y$ invertible $\iff U_{U_x y} = U_x U_y U_x$ invertible [by the Invertibility Criterion 6.1.2(1)–(3)] $\iff U_x, U_y$ are invertible [$\Longrightarrow$ since U_x has left and right inverses, hence is invertible, so U_y is too, $\Longleftarrow$ is clear]; then by the Inverse Recipe 6.1.3 $(U_x y)^{-1} = \left(U_{U_x y}\right)^{-1}(U_x y) = (U_x U_y U_x)^{-1}(U_x y)$ [by the Fundamental Formula] $= U_{x^{-1}} U_{y^{-1}} y = U_{x^{-1}} y^{-1}$. (2) follows by an easy induction using (1) [$n = 0, 1$ being trivial]. The second part of (2) also follows immediately from the first without any induction: $U_{x^{-n}} := U_{(x^n)^{-1}} = (U_{x^n})^{-1}$ [by the U–Inverse Formula 6.1.3] $= \left(U_x^n\right)^{-1}$ [by Power-Associativity 5.2.2] $= U_x^{-n}$. The first part can also be established immediately from the Shirshov–Cohn Principle: since this is a matter strictly between x and x^{-1}, it takes place in $\Phi[x, x^{-1}]$, so we may assume an associative ambience; but in A^+ Jordan invertibility reduces to associative invertibility by Special Inverse 6.1.5, where the first part is well known. (3) This is clear from the componentwise operations in the direct product and the elemental definition. □

EXERCISE 6.1.8A Carry out the "easy induction" mentioned in the proof of (2) above.

EXERCISE 6.1.8B Define negative powers of an invertible element x by $x^{-n} := (x^{-1})^n$. (1) Show that we have power-associativity $x^n \bullet x^m = x^{n+m}$, $(x^n)^m = x^{nm}$, $\{x^n, x^m\} = 2x^{n+m}$, $U_{x^n} x^m = x^{2n+m}$, $\{x^n, x^m, x^p\} = 2x^{n+m+p}$ and operator identities $U_{x^n} = U_x^n$, $V_{x^n, x^m} = V_{x^{n+m}}$, $V_{x^n, x^m} U_{x^p} = U_{x^{n+m+p}, x^p}$ for *all* integers n, m, p. (2) Show that $U_{f(x,x^{-1})} U_{g(x,x^{-1})} = U_{(f.g)(x,x^{-1})}$, $V_{f(x,x^{-1})} U_{g(x,x^{-1})} = U_{(f.g)(x,x^{-1}), g(x,x^{-1})}$ for any polynomials f, g in the commutative polynomial ring $\Phi[t, t^{-1}]$.

In the Jacobson Coordinatization a crucial role will be played by "symmetry automorphisms" which permute the rows and columns of a matrix algebra. These are generated by a very important kind of "inner" automorphism. Notice that the usual associative inner automorphism $x \mapsto uxu^{-1}$ will be expressible in Jordan terms as $x \mapsto uxu = U_u x$ if $u = u^{-1}$, i.e., if $u^2 = 1$.

Involution Definition 6.1.9 *An element* u *of order* 2 *in a Jordan algebra* J, $u^2 = 1$, *will be called an* **involution in** J. *The corresponding operator* U_u *will be called the* **involution on** J *determined by* u.

Involution Lemma 6.1.10 *If* u *is an involution in a unital Jordan algebra* J, *then the involution* U_u *is an involutory automorphism of* J, $U_u^2 = \mathbb{1}_J$.

PROOF. Notice that an involution is certainly invertible: by the Fundamental Formula $U_u^2 = U_{u^2} = U_1 = \mathbb{1}_J$ shows [by the Invertibility Criterion, 6.1.2 or Power Invertibility Criterion 6.1.8(2)] that U_u, and hence u, is invertible. In Morphism 1.2.2 we noted that $\varphi = U_u$ will be an automorphism as soon as it preserves squares, and $U_u(x^2) = U_u U_x(1) = U_u U_x(u^2) = (U_u x)^2$ by the Fundamental Formula applied to $z = 1$. [Alternately, since this is a matter strictly between the elements u, x we can invoke the Shirshov–Cohn Principle to work in an associative algebra A, and here $U_u a = uau^{-1}$ is just the usual inner associative automorphism determined by the element $u = u^{-1}$ and so certainly preserves Jordan products.] □

6.2 von Neumann and Nuclear Inverses

In considering "twistings" of Jordan matrix algebras by isotopy, we will encounter nonassociative algebras $A = \mathcal{M}_3(D)$ for certain alternative $*$-algebras D where the hermitian part $\mathcal{H}_3(D, -)$ forms a Jordan subalgebra of A^+ even though the entire A^+ is not itself Jordan. We can understand the isotopic twistings much better from the broad perspective of $\mathcal{M}_3$ than from the restricted perspective of $\mathcal{H}_3$, so it will be worth our while to gather some observations about inverses (and later isotopes) by very tame (namely nuclear) elements. There is a very general notion of inverses pioneered by von Neumann.[2]

von Neumann Inverse Definition 6.2.1 *An element* x *of an arbitrary unital linear algebra* A *is* **vonvertible** (*von Neumann invertible*) *if it has a* **vonverse** (*von Neumann inverse*) y *satisfying the* **Vonverse Condition**

$$xy = 1 = yx \quad \text{where} \quad x, y \textbf{ operator-commute},$$

i.e., their operators L_x, L_y, R_x, R_y all commute.

[2] Nuclear inverses were introduced in I.3.2.

Such a vonverse y is unique: if y' is another vonverse for x then $y' = L_{y'}(1) = L_{y'}L_x(y) = L_xL_{y'}(y)$ [by operator-commutativity of x, y'] $= L_xR_y(y') = R_yL_x(y')$ [by operator-commutativity of x, y] $= R_y(1) = y$.

It is easy to see that for commutative Jordan algebras, x, y are vonverses iff they are Jordan inverses. The element x and its Jordan inverse $y = x^{-1}$ certainly satisfy the vonverse condition by the U–Inverse Formula 6.1.3; conversely, if $x \bullet y = 1$ and L_x, L_y commute then also $x^2 \bullet y = L_yL_x(x) = L_xL_y(x) = L_x(1) = x$, so x, y satisfy (LJInv1)–(LJInv2) and by the Linear Inverse Lemma 6.1.7 are Jordan inverses.

When we pass beyond Jordan algebras, as long as we stick to nuclear elements the notion of inverse works as smoothly as it does for associative algebras.

Nuclear Inverse Proposition 6.2.2 *If a nuclear element u of a unital linear algebra A satisfies $uv = 1 = vu$ for some element $v \in \mathrm{A}$, then necessarily v is nuclear too, and we say that u has a* **nuclear inverse**. *In this case the operators L_u, L_v are inverses and R_u, R_v are inverses, and $v := u^{-1}$ is uniquely determined as $u^{-1} = L_u^{-1}1$ or as $u^{-1} = R_u^{-1}1$. The operators L_u, R_u, $U_u := L_uR_u$, $V_u := L_u + R_u$, $L_{u^{-1}}$, $R_{u^{-1}}$, $U_{u^{-1}} := L_{u^{-1}}R_{u^{-1}}$, $V_{u^{-1}} := L_{u^{-1}} + R_{u^{-1}}$ all commute (though the Jordan operators $V_u, V_{u^{-1}}$ are no longer inverses). In particular, u, v are vonverses.*

PROOF. First we verify that any such v is nuclear, $[v, a, b] = [b, v, a] = [a, b, v] = 0$ for all $a, b \in \mathrm{A}$. The trick for the first two is to write $a = ua'$ [note that $a = 1a = (uv)a = u(va)$ by nuclearity, so we may take $a' = va$]; then $[v, ua', b] = (vua')b - v(ua'b)$ [by nuclearity of u] $= a'b - (vu)a'b = 0$ [by $vu = 1$ and nuclearity of u], similarly $[b, v, ua'] = (bv)ua' - b(vua') = (bvu)a' - ba' = 0$; dually, for the third write $b = b'u$, so that $[a, b'u, v] = (ab'u)v - a(b'uv) = (ab')uv - ab' = 0$ by $uv = 1$ and nuclearity of u.

Now L_u commutes with R_x and satisfies $L_uL_x = L_{ux}$ for *all* elements $x \in \mathrm{A}$ by nuclearity $[u, \mathrm{A}, x] = [u, x, \mathrm{A}] = 0$, so it commutes with R_v and $L_uL_v = L_{uv} = L_1 = \mathbb{1}_{\mathrm{A}}$; dually for v since it is nuclear too, so L_u, L_v are inverse operators. Then $L_u(v) = 1$ implies that v is uniquely determined as $v = L_u^{-1}(1)$, and dually for the right multiplications. Then the left and right multiplications and the Jordan operators live in the subalgebra of operators generated by the commuting $L_u, L_{u^{-1}}, R_u, R_{u^{-1}}$, where everyone commutes. $\square$

6.3 Problems for Chapter 6

PROBLEM 6.1* Let J be a unital Jordan algebra. (1) Show that u is an involution in the sense of the Involution Lemma 6.1.10 iff $u = e - e'$ for supplementary orthogonal idempotents e, e', equivalently iff $u = 2e - 1$ for an idempotent e. [We will see later that $U_u = E_2 - E_1 + E_0$ has a nice description as a "Peirce reflection" in terms of the Peirce projections $E_i(e)$.] (2) Show that $u \longleftrightarrow e$ is a bijection between the set of involutions of J and the set of idempotents of J. (3) Show that a bijective linear transformation T on a unital J is an automorphism iff it satisfies $T(1) = 1$ and is "structural," $U_{T(x)} = TU_xT^*$ for some operator T^* and all x, in which case $T^* = T^{-1}$. (4) If $J = J_1 \boxplus J_2$ is a direct sum of unital Jordan algebras J_i with units e_i, and $v_1 \in J_1$ has $v_1^2 = -e_1$, show that $T := U_{v_1+e_2}$ is structural with $T = T^* = T^{-1}$, yet $T(1) = -e_1 + e_2 \neq 1$, so T is not an automorphism.

PROBLEM 6.2* A *derivation* of a quadratic Jordan algebra is an endomorphism D of J such that $D(1) = 0,\ D(U_xy) = U_{D(x),x}y + U_xD(y)$. Show that $D(x^{-1}) = -U_x^{-1}D(x)$ if x is invertible.

PROBLEM 6.3* Recall that a derivation of a linear algebra A is an endomorphism D satisfying the "product rule" $D(xy) = D(x)y + xD(y)$ for all x, y. (1) Show that if x, y are vonverses, then $D(y) = -U_yD(x)$ where the operator U_y is defined to be $L_y(L_y + R_y) - L_{y^2}$ (this is the correct U-operator for noncommutative Jordan algebras). (2) Show that a derivation necessarily maps the nucleus into the nucleus, $D(\mathcal{N}uc(A)) \subseteq \mathcal{N}uc(A)$; show that if x, y are nuclear inverses, then $D(y) = -yD(x)y$. (3) Show that if x, y satisfy the vonverse conditions in a unital Jordan algebra, then $D(y) = -U_yD(x)$.

PROBLEM 6.4 Show that if x, y are ordinary inverses $xy = yx = 1$ in a unital alternative algebra, then they are von Neumann inverses.

QUESTION 6.1 A *noncommutative Jordan algebra* is a linear algebra A which if flexible $[x, y, x] = 0$ and satisfies the Jordan identity $[x^2, y, x] = 0$ for all elements x, y. It can be shown that the algebra is power-associative, so all powers of x lie in the commutative associative algebra $\Phi[x] \subseteq A$, also all powers L_{x^n}, R_{x^m} lie in the commutative associative $\Phi[L_x, L_{x^2}, R_x, R_{x^2}] \subseteq \mathcal{E}nd(A)$, so x operator-commutes with any polynomial $p(x)$. (1) If an element of a unital noncommutative Jordan algebra is algebraic over a field with minimum polynomial having nonzero constant term, is the element necessarily vonvertible? In general, does $xy = yx = 1$ in a noncommutative Jordan algebra imply that x, y are vonverses?

7

Isotopes

In this chapter we will establish in detail the results on homotopy and isotopy mentioned without proof in the *Historical Survey.*

7.1 Nuclear Isotopes

If u is an invertible element of an associative algebra, or more generally, a nuclear element of an arbitrary linear algebra A having a nuclear inverse as in 6.2.2, then we can form a new linear algebra by translating the unit from 1 to u^{-1}. We can also translate an involution $*$ to its conjugate by a $*$-hermitian element.[1]

Nuclear Isotope Definition 7.1.1 *If u is a nuclear element with nuclear inverse in a unital linear algebra* A*, then we obtain a new linear algebra, the* **nuclear u-isotope** A_u*, by taking the same Φ-module structure but defining a new product*

$$\mathrm{A}_u: \qquad x_u y := xuy \quad (= (xu)y = x(uy)) \quad \textit{for} \quad u \in \mathcal{N}uc(A)).$$

Nuclear Isotope Proposition 7.1.2 *If u is an invertible nuclear element in* A*, then the isotopic algebra A_u is just an isomorphic copy of the original algebra*:

$$\varphi : \mathrm{A}_u \to \mathrm{A} \text{ is an isomorphism for } \varphi = L_u \text{ or } R_u.$$

In particular, it has unit $1_u = u^{-1}$, and is associative iff A *is. Nuclear isotopy is an equivalence relation,*

$$\mathrm{A}_1 = \mathrm{A}, \quad (\mathrm{A}_u)_v = \mathrm{A}_{uvu}, \quad (\mathrm{A}_u)_{u^{-2}} = \mathrm{A} \qquad (u, v \in \mathcal{N}uc(\mathrm{A})).$$

[1] Nuclear isotopes were introduced in I.3.3, twisted hermitian algebras in I.3.4.

PROOF. (1) That $\varphi = L_u$ (dually R_u) is an isomorphism follows since $\varphi = L_u$ is a linear bijection with inverse $\varphi^{-1} = L_{u^{-1}}$ by the Nuclear Inverse Proposition 6.2.2, and φ is an algebra homomorphism since $\varphi(x_u y) = u(x(uy)) = (ux)(uy)$ [by nuclearity of u] $= \varphi(x)\varphi(y)$. In particular, the nuclear isotope has unit $\varphi^{-1}(1) = L_u^{-1}(1) = u^{-1}$, which is also easy to verify directly: $u^{-1}{}_u y := u^{-1}uy = 1y = y$ from nuclearity and $u^{-1}u = 1$, and dually, $y_u u^{-1} = y$ from $uu^{-1} = 1$.

The equivalence relations hold because the algebras have the same underlying Φ-module structure and the same products, $x_1 y = xy$, $x_u v_u y = xuvuy$, $uu^{-2}u = 1$ (parentheses unnecessary by nuclearity of u, v). □

EXERCISE 7.1.2A If u is *any* element of a *commutative* linear algebra A, define the *linear u-homotope* $\mathrm{A}^{(u)}$ to be the linear algebra having the same underlying Φ-module as A, but with new product $x \cdot^{(u)} y := (x \cdot u) \cdot y + x \cdot (u \cdot y) - (x \cdot y) \cdot u$. (If A is Jordan, this is just the usual Jordan homotope, cf. 7.2.1 below.) (1) Show that the linear homotope is again commutative. If A is unital and u is vonvertible (cf. 6.2.1), show that the homotope has new unit u^{-1}, using operator-commutativity of the vonverse. In this case we speak of the *linear u-isotope*. (2) If u is nuclear, show that the linear homotope reduces to the *nuclear homotope* defined as in 7.1.1 by $x \cdot_u y := x \cdot u \cdot y$).

EXERCISE 7.1.2B Let A be *any* (not necessarily commutative) linear algebra. (1) Show that the associator $[x, y, z]^+ := (x \bullet y) \bullet z - x \bullet (y \bullet z)$ in A^+ reduces to associators and commutators $\frac{1}{4}\big([x,y,z]+[x,z,y]+[y,x,z]-[z,y,x]-[y,z,x]-[z,x,y]+[y,[x,z]]\big)$ in A itself. (2) Use this to show that if u is a nuclear element of A then $[u, y, z]^+ = 0 \Leftrightarrow [y, [u, z]] = 0$, $[x, u, z]^+ = 0 \Leftrightarrow [u, [x, z]] = 0$, and show that $x \bullet (u \bullet y) = \frac{1}{2}(xuy + yux) \Leftrightarrow [x, [y, u]] = 0$. (3) Conclude that a nuclear u remains nuclear in A^+ iff $[u, [\mathrm{A}, \mathrm{A}]] = [\mathrm{A}, [u, \mathrm{A}]] = 0$, in which case the linear homotope (as in Exercise 7.1.2A above) of the commutative algebra A^+ is the same as the plus algebra of the nuclear homotope of A: $\big(\mathrm{A}^+\big)^{(u)} = \big(\mathrm{A}_u\big)^+$.

Twisted Hermitian Proposition 7.1.3 *If u is an invertible nuclear element of a linear algebra* A *which is hermitian with respect to an involution* $*$, *then* $*$ *remains an involution on the isotope* A_u. *We also obtain a new "involution," the* **nuclear u-isotope** $*_u$, *by*

$$*_u: \qquad x^{*_u} := ux^*u^{-1}.$$

This isotopic involution $*_u$ is an involution on the original algebra A,

$$(x_u y)^* = (y^*)_u (x^*), \qquad (x^{*_u})^{*_u} = x, \quad (xy)^{*_u} = y^{*_u} x^{*_u},$$

and the two algebras are $*$-isomorphic:

$$\varphi = L_u : \ (\mathrm{A}_u, ^*) \to (\mathrm{A}, *_u) \quad \text{is an isomorphism.}$$

In particular, the translated involution has translated space of hermitian elements

$$\mathcal{H}(\mathrm{A}, *_u) = u\mathcal{H}(\mathrm{A}, *).$$

PROOF. The map $*$ remains an involution on A_u because it is still linear of period 2 and is an anti-automorphism of the isotope because $(x_u y)^* = ((xu)y))^* = y^*(u^*x^*)$ [since $*$ is an involution on A] $= y^*(ux^*)$ [since $u^* = u$] $= y^*_u x^*$. That $*_u$ is an involution on A follows from direct calculation: $x^{*_u *_u} = u\left(ux^*u^{-1}\right)^* u^{-1} = u\left((u^*)^{-1}(x^*)^*u^*\right)u^{-1} = \left(u(u)^{-1}\right)x\left(uu^{-1}\right) = x$ [by nuclearity, $u^* = u$, and $x^{**} = x$] and $(xy)^{*_u} = u(xy)^*u^{-1} = uy^*x^*u^{-1} = (uy^*)u^{-1}u(x^*u^{-1})$ [by nuclearity] $= \left(uy^*u^{-1}\right)\left(ux^*u^{-1}\right)$ [by nuclearity] $= y^{*_u}x^{*_u}$.

We saw that the map $\varphi : (\mathrm{A}_u, *) \to (\mathrm{A}, *_u)$ is an isomorphism of algebras, and it preserves involutions since $\varphi(x^*) = ux^* = (ux^*)(u^*u^{-1})$ [since $u^* = u$ is invertible] $= u\left(x^*(u^*u^{-1})\right)$ [since u is nuclear] $= u\left(x^*u^*\right)u^{-1}$ [since $u^* = u$ is nuclear too!] $= u(ux)^*u^{-1}$ [since $*$ is an involution on A] $= (ux)^{*_u} = (\varphi(x))^{*_u}$. This gives another proof that $*_u$ is an involution on A.

The final translation property follows immediately, because a $*$-isomorphism takes hermitian elements to hermitian elements. □

EXERCISE 7.1.3 (1) Go through the dual argument that for nuclear u the map $\varphi = R_u$ is an algebra isomorphism $\mathrm{A}_u \to \mathrm{A}$ and that if $u^* = u$ then, just as L_u is a $*$-isomorphism of $*$ with the *left u-translate* $x^{*_u} := ux^*u^{-1}$, so is R_u with the *right u-translate* $u^{-1}x^*u = x^{*_{u^{-1}}}$: R_u is an isomorphism $(\mathrm{A}_u, *) \to (\mathrm{A}, *_{u^{-1}})$ of $*$-algebras. (2) If $u^* = -u$ is skew, show that $L_u : (\mathrm{A}_u, -*) \to (\mathrm{A}, *_u)$ is an isomorphism of $*$-algebras, where the negative $-*$ is an involution on A_u (but not on the original A!).

Because it just reproduces the original algebra, the concept of nuclear isotopy plays no direct role in nonassociative theory. However, even in the associative case isotopy is nontrivial for *algebras with involution* and thus in Jordan algebras isotopy can lead to new algebras.

7.2 Jordan Isotopes

For Jordan algebras we have a useful notion of isotope for *all* invertible elements, not just the nuclear ones. Just as with nuclear isotopes, the isotope of a Jordan product consists in sticking a factor u in "the middle" of the Jordan product $x \bullet y$. However, it is not so simple to decide where the "middle" is! In the (usually non-Jordan) linear algebra ${\mathrm{A}_u}^+$, the Jordan product is $x \bullet^{(u)} y := \frac{1}{2}(x_u y + y_u x) = \frac{1}{2}(xuy + yux)$, which suggests that we should stick the u in the middle of the triple product $\frac{1}{2}\{x, u, y\}$. As in associative algebras, this process produces a new Jordan algebra even if u is not invertible, leading to the following definition for homotopes of Jordan algebras.[2]

[2] Linear Jordan homotopes were discussed in I.3.2, quadratic homotopes in I.4.6.

Jordan Homotope Proposition 7.2.1 (1) *If u is an arbitrary element of a Jordan algebra, we obtain a new Jordan algebra, the* u**-homotope** $J^{(u)}$, *defined to have the same underlying Φ-module as* J *but new bullet product*

$$x \bullet^{(u)} y := \tfrac{1}{2}\{x, u, y\}.$$

The auxiliary products are then given by

$$\begin{aligned} x^{(2,u)} &= U_x u, & U_x^{(u)} &= U_x U_u, & V_x^{(u)} &= V_{x,u}, \\ U_{x,z}^{(u)} &= U_{x,z} U_u, & V_{x,y}^{(u)} &= V_{x,U_u y}, \\ \{x, y\}^{(u)} &= \{x, u, y\}, & \{x, y, z\}^{(u)} &= \{x, U_u y, z\}. \end{aligned}$$

(2) *If* J *is unital, the homotope will be again unital iff the element u is invertible, in which case the new unit is*

$$1^{(u)} = u^{-1}.$$

In this case we call $J^{(u)}$ *the* u-**isotope** (*roughly corresponding to the distinction between* iso*morphism and* homo*morphism*).

(3) *An isotope has exactly the same trivial and invertible elements as the original algebra*:

$$\begin{aligned} x \textit{ is trivial in } J^{(u)} &\iff x \textit{ is trivial in } J; \\ x \textit{ is invertible in } J^{(u)} &\iff x \textit{ is invertible in } J, \\ \textit{with } x^{(-1,u)} &= U_u^{-1} x^{-1}. \end{aligned}$$

In particular,

$$J^{(u)} \quad \textit{is a division algebra} \iff J \quad \textit{is a division algebra.}$$

(4) *We say that two unital Jordan algebras* J, J' *are* **isotopic** *if one is isomorphic to an isotope of the other. Isotopy is an equivalence relation among unital Jordan algebras*: *we have* **Isotope Reflexivity, Transitivity**, *and* **Symmetry**

$$J^{(1)} = J, \quad (J^{(u)})^{(v)} = J^{(U_u v)}, \quad (J^{(u)})^{(u^{-2})} = J.$$

PROOF. (1) We first check the auxiliary products: all follow from Macdonald's Principle and linearization, or they can all be established by direct calculation: $x^{(2,u)} = \frac{1}{2}\{x, u, x\} = U_x u$, $V_x^{(u)}(y) = \{x, y\}^{(u)} = \{x, u, y\} = V_{x,u}(y)$ by linearization or twice the bullet, and $2U_x^{(u)} := \big(V_x^{(u)}\big)^2 - V_{x^{(2,u)}}^{(u)} = \big(V_{x,u} V_{x,u} - V_{U_x u, u}\big)$ [using the above form of the square and V] $= U_x U_{u,u}$ [by Macdonald's Principle] $= 2U_x U_u$, hence $\{x, y, z\}^{(u)} = U_{x,z}^{(u)} = U_{x,z} U_u$ too by linearization.

The new bullet is clearly commutative as in (JAX1). To show that $V_x^{(u)}$ commutes with $V_{x^{(2,u)}}$ as in the Jordan identity (JAX2), it suffices to show that it commutes with $2U_x^{(u)} = \big(V_x^{(u)}\big)^2 - V_{x^{(2,u)}}$, and $V_x^{(u)} U_x^{(u)} = V_{x,u} U_x U_u = U_x V_{u,x} U_u = U_x U_u V_{x,u}$ using the Commuting Formula (FFII) twice. Thus the homotope is again a Jordan algebra.

(2) If J is unital and the homotope has a unit v, then $1\!\!1_{\mathrm{J}} = U_v^{(u)} = U_v U_u$ implies that U_v is surjective, hence by the Invertibility Criterion 6.1.2 is invertible, so $U_u = U_v^{-1}$ is also invertible, and u is invertible by the Invertibility Criterion again. In this case we can see directly that u^{-1} is the unit for $\mathrm{J}^{(u)}$: by definition of the Jordan triple product $\frac{1}{2}\{u^{-1}, u, y\} = (u^{-1} \bullet u) \bullet y + u^{-1} \bullet (u \bullet y) - u \bullet (u^{-1} \bullet y) = 1 \bullet y = y$ by operator commutativity in the L–Inverse Formula 6.1.3.

(3) follows from the fact that triviality and invertibility of x is measured by its U-operator, and U_u invertible guarantees that $U_x^{(u)} = U_x U_u$ vanishes or is invertible iff U_x is, where in the latter case $x^{(-1,u)} = \big(U_x^{(u)}\big)^{-1} x = \big(U_x U_u\big)^{-1} x = U_u^{-1} U_x^{-1} x = U_u^{-1} x^{-1}$ by the Inverse Recipe 6.1.3.

Just as in the nuclear case 7.1.2, the equivalence relations (4) follow easily: $\frac{1}{2}\{x, 1, y\} = x \bullet y$, $\frac{1}{2}\{x, v, y\}^{(u)} = \frac{1}{2}\{x, U_u v, y\}$ [using (1)]; hence $\frac{1}{2}\{x, u^{-2}, y\}^{(u)} = \{x, U_u(u^{-2}), y\} = \frac{1}{2}\{x, 1, y\}$ [by Quadratic Inverse Condition (QJInv2)] $= x \bullet y$. From these relations for elemental isotopy it is easy to see that general isotopy is an equivalence relation. □

Exercise 7.2.1A* Show that the specific form of the inverse in $\mathrm{J}^{(u)}$ is $x^{(-1,u)} = U_u^{-1} x^{-1}$ by *verifying* that this satisfies the Quadratic Inverse Conditions (QJInv1)–(QJInv2) in $\mathrm{J}^{(u)}$. Verify that it satisfies the Linear Inverse Conditions, and decide which verification is easier!

Exercise 7.2.1B (1) Show that for *any* linear algebra A and *any* $u \in \mathrm{A}$, in A^+ the Jordan product satisfies $4[(x \bullet u) \bullet y + x \bullet (u \bullet y) - (x \bullet y) \bullet u] = 2\,(x(uy) + y(ux)) + [y, u, x] + [x, u, y] - [y, x, u] - [x, y, u] + [u, y, x] + [u, x, y]$. (2) Define nuclear and Jordan *homotopes* as in 7.1.2, 7.2.1 (omitting unitality and the requirement that u be invertible), and use (1) to prove $\big(\mathrm{A}^+\big)^{(u)} = \big(A_u\big)^+ :\ x \bullet^{(u)} y = \frac{1}{2}(xuy + yux)$ for any nuclear $u \in \mathrm{A}$.

Exercise 7.2.1C Show that in the rare case that $u \in \mathrm{J}$ is *nuclear,* the Jordan homotope reduces to the nuclear homotope: $\mathrm{J}^{(u)} = \mathrm{J}_u$.

To get an idea of what life was like in pioneer days, before we had a clear understanding of the Jordan triple product, you should go back and try to prove the Homotope Proposition without mentioning triple products, using only the bullet product and the definition

$$x \bullet_u y := (x \bullet u) \bullet y + x \bullet (u \bullet y) - (x \bullet y) \bullet u.$$

You will quickly come to appreciate the concept and notation of triple products!

7.3 Quadratic Factor Isotopes

Our first example of Jordan isotopy is for Jordan algebras of quadratic forms.[3] Here the description is easy: the u-isotope simply replaces the original unit

[3] cf. I.3.7.

c by u^{-1}, and then performs a cosmetic change of quadratic form from Q to $Q(u)Q$ to insure that the new ruler has norm 1.

Quadratic Factor Isotopes Example 7.3.1 (1) *The isotopes of the quadratic factor* $\mathcal{J}ord(Q,c)$ *are again quadratic factors, obtained by scaling the quadratic form and shifting the basepoint*:

$$\begin{aligned} \mathcal{J}ord(Q,c)^{(u)} &= \mathcal{J}ord(Q^{(u)},c^{(u)}) & &\textit{for} \\ Q^{(u)}(x) &:= Q(x)Q(u), & c^{(u)} &:= u^{-1} = Q(u)^{-1}\bar{u}, \\ T^{(u)}(x) &= Q(x,\bar{u}), & (\bar{x})^{*(u)} &= Q(u)^{-1}\overline{U_u x}. \end{aligned}$$

(2) *The diagonal isotopes of the reduced spin factor* $\mathcal{R}ed\mathcal{S}pin(q)$ *are again reduced spin factors, obtained by scaling the quadratic form*:

$$\mathcal{R}ed\mathcal{S}pin(q)^{(\mu,0,\nu)} \cong \mathcal{R}ed\mathcal{S}pin(\mu\nu q) \quad \text{via} \quad (\alpha,w,\beta) \mapsto (\alpha\mu,w,\beta\nu).$$

PROOF. For (1), recall the definition of $\mathcal{J}ord(Q,c)$ in the Quadratic Factor Example 3.3.1. Note that $(Q^{(u)},c^{(u)})$ is a quadratic form with basepoint: $Q^{(u)}(c^{(u)}) := Q(u^{-1})Q(u) = 1$ by Factor Invertibility 6.1.6. The trace associated with $(Q^{(u)},c^{(u)})$ is $T^{(u)}(x) := Q^{(u)}(x,c^{(u)}) = Q(x,u^{-1})Q(u) = Q(x,\bar{u})$ [by Factor Invertibility]. The involution is given by $\overline{x}^{(u)} := T^{(u)}(x)c^{(u)} - x = Q(x,\bar{u})Q(u)^{-1}\bar{u} - x = Q(u)^{-1}\big(Q(\overline{\overline{u}},\bar{x})\bar{u} - Q(u)\overline{\overline{x}}\big)$ [since the involution is isometric on Q by Trace Involution Properties 2.3.1(1)] $= Q(u)^{-1}\overline{(Q(u,\bar{x})u - Q(u)\bar{x})} = Q(u)^{-1}\overline{U_u x}$.

By Jordan Homotope 7.2.1, the square in $\mathcal{J}ord(Q,c)^{(u)}$ is $x^{(2,u)} = U_x u = Q(x,\bar{u})x - Q(x)\bar{u} = T^{(u)}(x)x - Q(x)Q(u)Q(u)^{-1}\bar{u} = T^{(u)}(x)x - Q^{(u)}(x)c^{(u)}$, which is the square in $\mathcal{J}ord(Q^{(u)},c^{(u)})$, so these two algebras have the same product and thus coincide.

For (2), recall the definition of $\mathcal{R}ed\mathcal{S}pin(q)$ in Reduced Spin Example 3.4.1. The isotope $\mathrm{J}^{(u)}$ by a diagonal element $u = (\mu,0,\nu) \in \mathcal{R}ed\mathcal{S}pin(q) = (\Phi,M,\Phi)$ has squares given by

$$(\alpha,w,\beta)^{(2,(\mu,0,\nu))} = ([\alpha^2\mu + q(w)\nu],[\alpha\mu+\beta\nu]w,[\beta^2\nu + q(w)\mu]),$$

while similarly squares in $\mathcal{R}ed\mathcal{S}pin(\mu\nu q)$ are given by

$$(\tilde{\alpha},w,\tilde{\beta})^2 = ([\tilde{\alpha}^2 + \mu\nu q(w)],[\tilde{\alpha}+\tilde{\beta}]w,[\tilde{\beta}^2 + \mu\nu q(w)]).$$

Thus the mapping $(\alpha,w,\beta) \mapsto (\alpha\mu,w,\beta\nu) =: (\tilde{\alpha},w,\tilde{\beta})$ preserves squares and hence is an isomorphism: $[\tilde{\alpha}^2 + \mu\nu q(w)] = [\alpha^2\mu + q(w)\nu]\mu$, $[\tilde{\beta}^2 + \mu\nu q(w)] = [\beta^2\nu + q(w)\mu]\nu$, $[\tilde{\alpha}+\tilde{\beta}]w = [\alpha\mu+\beta\nu]w$. □

7.4 Cubic Factor Isotopes

Just as the quadratic isotopes are again quadratic factors, so the cubic isotopes are again cubic factors, though the verification takes longer because the requirements for a Jordan cubic are so stringent.[4]

Cubic Factor Isotopes Example 7.4.1 (1) *For each sharped cubic form* $(N, \#, c)$ *and element* u *with invertible norm* $N(u) \in \Phi$*, we obtain an* **isotopic** *sharped cubic form* $(N^{(u)}, \#^{(u)}, c^{(u)})$ *by scaling the cubic form and shifting the sharp and basepoint :*

$$\begin{aligned} N^{(u)}(x) &:= N(x)N(u), \\ c^{(u)} &:= u^{-1} = N(u)^{-1}u^{\#}, \\ x^{\#(u)} &:= N(u)^{-1}U_{u^\#}x^{\#} = N(u)^{-1}\big(U_u x\big)^{\#}, \\ x\#^{(u)}y &= N(u)^{-1}U_{u^\#}(x\#y) = N(u)^{-1}U_u x \# U_u y, \\ T^{(u)}(x,y) &:= T(x, U_u y), \\ T^{(u)}(y) &:= T(u,y), \\ S^{(u)}(x) &= T(u^{\#}, x^{\#}). \end{aligned}$$

If (N, c) *is a Jordan cubic, so is its isotope* $(N^{(u)}, c^{(u)})$.

(2) *Moreover, the Jordan algebras constructed from these cubic form isotopes are just the Jordan isotopes of the Jordan algebras constructed from the original forms,*

$$\mathcal{J}ord(N^{(u)}, \#^{(u)}, c^{(u)}) = \mathcal{J}ord(N, \#, c)^{(u)},$$
$$\mathcal{J}ord(N^{(u)}, c^{(u)}) = \mathcal{J}ord(N, c)^{(u)}.$$

Hence isotopes of cubic factors $\mathcal{J}ord(N, \#, c)$ *and* $\mathcal{J}ord(N, c)$ *are again cubic factors.*

PROOF. We have $u^{-1} = N(u)^{-1}u^{\#}$ by Factor Invertibility 6.1.6, $U_{u^\#}x^{\#} = (U_u x)^{\#}$ by Sharp Composition 4.2.2(3). The triple $(N^{(u)}, \#^{(u)}, c^{(u)})$ certainly is a cubic form with quadratic map and basepoint: $N^{(u)}(c^{(u)}) = N(u^{-1})N(u) = 1$ by Factor Invertibility.

The forms associated with the isotope are

$$\begin{aligned} N^{(u)}(x;y) &= N(u)T(x^{\#}, y), & N^{(u)}(x, y, c^{(u)}) &= T(u^{\#}\#x, y), \\ T^{(u)}(y) &= T(y, u), & T^{(u)}(x,y) &= T(x, U_u y), \\ S^{(u)}(x) &= T(x^{\#}, u^{\#}). & S^{(u)}(x,y) &= T(x, y\#u^{\#}). \end{aligned}$$

[4] cf. I.3.8.

To see these, we compute $N^{(u)}(x;y) = N(u)N(x;y) = N(u)T(x^\#, y)$ [by Trace–Sharp Formula 4.2.1(2)], the quadratic spur form is $S^{(u)}(x) := N^{(u)}(x;c^{(u)}) = N(u)T(x^\#, N(u)^{-1}u^\#) = T(x^\#, u^\#)$, so by linearization $S^{(u)}(x,y) = N^{(u)}(x,y,c^{(u)}) = T(x\#y, u^\#) = T(y, x\#u^\#)$ [by Sharp Symmetry 4.2.4(1)].

The linear trace is $T^{(u)}(y) := N^{(u)}(c^{(u)};y) = N(u)N(N(u)^{-1}u^\#;y) = N(u)^{-1}T((u^\#)^\#, y)$ [by Trace–Sharp] $= N(u)^{-1}T(N(u)u, y)$ [by Adjoint Identity 4.2.1(2)] $= T(u,y)$.

Combining these, we see the bilinear trace is given by $T^{(u)}(x,y) := T^{(u)}(x)T^{(u)}(y) - S^{(u)}(x,y)$ [by Spur–Trace 4.2.1(1)] $= T(x,u)T(y,u) - T(x, y\#u^\#) = T(x, T(u,y)u - u^\#\#y) = T(x, U_u y)$ by definition 4.2.2(2) of the U-operator in $\mathcal{J}ord(N, \#, c)$.

We now verify the three axioms 4.2.1(2) for a sharp mapping. We first establish U**-Symmetry**

$$T(U_z x, y) = T(x, U_z y),$$

which holds because $T(U_z x, y) = T\big((T(z,x)z - z^\#\#x), y\big) = T(z,x)T(z,y) - T(z^\#, x\#y)$ [by Sharp-Symmetry] is symmetric in x and y.

For the Trace–Sharp Formula for $\#^{(u)}$,

$$N^{(u)}(x;y) = T^{(u)}(x^{\#(u)}, y),$$

we compute $N^{(u)}(x;y) = N(u)T(x^\#, y)$ [by Trace–Sharp for #] $= N(u)T(U_u U_{u^{-1}} x^\#, y) = N(u)T(N(u)^{-2} U_u U_{u^\#} x^\#, y)$ [since $u^{-1} = N(u)^{-1}u^\#$] $= T(N(u)^{-1}U_{u^\#}x^\#, U_u y)$ [using the above U–Symmetry] $= T^{(u)}(N(u)^{-1}U_{u^\#}x^\#, y) = T^{(u)}(x^{\#(u)}, y)$.

For the Adjoint Identity

$$x^{\#(u)\#(u)} = N^{(u)}(x)x \text{ for } \#^{(u)},$$

we compute

$$\begin{aligned}
x^{\#(u)\#(u)} &= N(u)^{-1}\big(U_u\big(N(u)^{-1}U_{u^\#}x^\#\big)\big)^\# \\
&= N(u)^{-1}N(u)^{-2}\big(U_u U_{u^\#}x^\#\big)^\# \\
&= N(u)^{-3}\big(N(u)^2 x^\#\big)^\# && [\text{since } u^\# = N(u)u^{-1}] \\
&= N(u)^{-3}N(u)^4\big(x^{\#\#}\big) = N(u)\big(N(x)x\big) && [\text{by Adjoint for } \#] \\
&= N^{(u)}(x)x.
\end{aligned}$$

Finally, the c–Sharp Identity for $\#^{(u)}$,

$$c^{(u)}\#^{(u)}y = T^{(u)}(y)c^{(u)} - y,$$

follows by canceling U_u from

$$\begin{aligned}
&U_u\big[c^{(u)}\#^{(u)}y - T^{(u)}(y)c^{(u)} + y\big]\\
&\quad= U_u\big[N(u)^{-1}U_{u^\#}\big(N(u)^{-1}u^\#\#y\big) - T(u,y)u^{-1} + y\big]\\
&\quad= N(u)^{-2}U_uU_{u^\#}(u^\#\#y) - T(u,y)u + U_uy\\
&\quad= u^\#\#y - T(u,y)u + U_uy = 0
\end{aligned}$$

using $u^\# = N(u)u^{-1}$ and the definition of the U-operator in $\mathcal{J}ord(N,c)$.

Thus the isotope $(N^{(u)},\ \#^{(u)},\ c^{(u)})$ is again a sharped cubic form, and therefore by Cubic Construction 4.2.2 it produces another Jordan algebra $\mathcal{J}ord(N^{(u)}, \#^{(u)},\ c^{(u)})$ whose U-operator is

$$\begin{aligned}
&T^{(u)}(x,y)x - x^{\#(u)}\#^{(u)}y\\
&\quad= T(x,U_uy)x - N(u)^{-1}U_u\big(N(u)^{-1}U_{u^\#}x^\#\big)\#U_uy\\
&\quad= T(x,U_uy)x - N(u)^{-2}U_uU_{u^\#}x^\#\#U_uy\\
&\quad= T(x,U_uy)x - x^\#\#U_uy \quad [\text{since } u^\# = N(u)u^{-1}]\\
&\quad= U_x(U_uy) = U_x^{(u)}y,
\end{aligned}$$

which is the usual U-operator of the Jordan isotope. Since both algebras have the same unit $c^{(u)} = 1^{(u)} = u^{-1}$ and the same U-operators, they also have the same squares $x^2 = U_x1$ and hence the same bullet products, so the two algebras coincide.

If (N,c) is a Jordan cubic as in 4.2.4, i.e., the bilinear form T is nondegenerate, then invertibility of U_u guarantees nondegeneracy of $T^{(u)}$, and $N^{(u)}(x;y) = T^{(u)}(x^{\#(u)}, y)$ [by the above] for all y shows by nondegeneracy that the adjoint induced by $(N^{(u)}, c^{(u)})$ is just $x^{\#(u)}$. □

7.5 Matrix Isotopes

As we noted in Nuclear Isotopy Proposition 7.1.2, nothing much happens when we take isotopes of a full plus-algebra.[5]

Full Isotope Example 7.5.1 *If u is an invertible element of an associative algebra, then the u-isotope of the Jordan algebra* A^+ *is isomorphic to* A^+ *again*: $\mathrm{A}^{+(u)} = \mathrm{A}_u{}^+ \cong \mathrm{A}^+$.

PROOF. The first equality follows from $x \bullet^{(u)} y = \frac{1}{2}\{x,u,y\}$ [by Jordan Homotope Proposition 7.2.1] $= \frac{1}{2}(xuy + yux)$ [by speciality] $= \frac{1}{2}(x\,{}_u\,y + y\,{}_u\,x)$ [by Nuclear Isotope Definition 7.1.1], and the second isomorphism follows from Nuclear Isotope Proposition 7.1.2, since any isomorphism $\mathrm{A} \to \mathrm{B}$ is also an isomorphism $\mathrm{A}^+ \to \mathrm{B}^+$ of the associated plus-algebras. □

[5] cf. *Historical Survey* I.3.4–5 for hermitian and matrix isotopes.

A more substantial change occurs when we take isotopes of Jordan algebras of hermitian type, because we change the nature of the underlying involution: as we have remarked before, $\mathcal{H}_2(\mathbb{R})$ is formally real, but $\mathcal{H}_2(\mathbb{R})^{(u)}$ for $u = E_{12} + E_{21}$ has nilpotent elements $(E_{11})^{(2,u)} = 0$, so isotopy is in general a broader concept than isomorphism.

Hermitian Isotope Example 7.5.2 *The Jordan isotopes of* $\mathcal{H}(\mathrm{A}, *)$ *for an associative* $*$*-algebra* $(\mathrm{A}, *)$ *are obtained either by changing the algebra structure but keeping the involution, or equivalently by keeping the algebra structure but changing the involution: for invertible* $u = u^*$, $\mathcal{H}(\mathrm{A}, *)^{(u)} = \mathcal{H}(\mathrm{A}_u, *) \cong \mathcal{H}(\mathrm{A}, *_u) = u\mathcal{H}(\mathrm{A}, *)$ *under* $\varphi = L_u$.

PROOF. The first equality follows since both are precisely the Jordan subalgebra of $*$-hermitian elements of $\mathrm{A}^{+(u)} = {\mathrm{A}_u}^+$ by Full Isotope above, and the second isomorphism and equality follow from Twisted Hermitian Proposition 7.1.3. □

The isotopes of $\mathcal{H}(\mathcal{M}_n(\mathrm{D}), *)$ which we will need are nuclear isotopes coming from diagonal matrices Γ.

Twisted Matrix Example 7.5.3 (1) *Let* $\mathcal{M}_n(\mathrm{D})$ *be the linear algebra of* $n \times n$ *matrices over a unital linear* $*$*-algebra* D *under the standard conjugate-transpose involution* $X^* = (\overline{X})^{tr}$. *Let* $\Gamma = diag\{\gamma_1, \dots, \gamma_n\}$ $(\gamma_i = \overline{\gamma_i}$ *invertible in* $\mathcal{H}(\mathrm{D}, -) \cap \mathcal{N}uc(\mathrm{D}))$ *be an invertible hermitian nuclear diagonal matrix. Then the* **twisted matrix algebra** $\mathcal{H}_n(\mathrm{D}, \Gamma) := \mathcal{H}(\mathcal{M}_n(\mathrm{D}), *_\Gamma)$ *consisting of all hermitian matrices of* $\mathcal{M}_n(\mathrm{D})$ *under the shifted involution* $X^{*_\Gamma} := \Gamma X^* \Gamma^{-1}$, *is linearly spanned by the elements in* **Jacobson Γ-box notation** (*where* E_{ij} $(1 \le i, j \le n)$ *are the usual* $n \times n$ *matrix units*):

$$\delta[ii]_\Gamma := \gamma_i \delta E_{ii} \qquad (\delta \in \mathcal{H}(\mathrm{D}, -)),$$
$$d[ij]_\Gamma := \gamma_i d E_{ij} + \gamma_j \bar{d} E_{ji} = \bar{d}[ji]_\Gamma \qquad (d \in \mathrm{D}).$$

(2) *The multiplication rules for distinct indices in the isotope consist of* **Four Basic Twisted Brace Products** *and* **Brace Orthogonality**:

$$\delta[ii]_\Gamma^2 = \delta\gamma_i\delta[ii]_\Gamma,$$
$$d[ij]_\Gamma^2 = d\gamma_j\bar{d}[ii]_\Gamma + \bar{d}\gamma_i d[jj]_\Gamma,$$
$$\{\delta[ii]_\Gamma, d[ik]_\Gamma\} = \delta\gamma_i d[ik]_\Gamma, \ \{d[ik]_\Gamma, \delta[kk]_\Gamma\} = d\gamma_k\delta[ik]_\Gamma,$$
$$\{d[ij]_\Gamma, b[jk]_\Gamma\} = d\gamma_j b[ik]_\Gamma,$$
$$\{d[ij]_\Gamma, b[k\ell]_\Gamma\} = 0 \qquad (\text{if } \{i,j\} \cap \{k,\ell\} = \emptyset).$$

(3) *We have an isomorphism of this twisted algebra with the* Γ*-isotope of the untwisted algebra*:

$$\mathcal{H}_n(\mathrm{D}, -)^{(\Gamma)} \to \mathcal{H}_n(\mathrm{D}, \Gamma) \ \text{ via } \ L_\Gamma : d[ij] \to d[ij]_\Gamma.$$

PROOF. Since it is easy to verify that Γ is nuclear in $\mathcal{M}_n(\mathrm{D})$, (3) follows from Hermitian Isotope Example 7.5.2. That $\mathcal{H}_n(\mathrm{D}, \Gamma)$ is spanned by the elements (1) is easily verified directly, using that $(\gamma_i dE_{ij})^{*\Gamma} = \Gamma(\bar{d}\gamma_i E_{ji})\Gamma^{-1} = \gamma_{jj}\bar{d}\gamma_{ii}\gamma_{ii}^{-1}E_{ji} = \gamma_j\bar{d}E_{ji}$; it also follows from the isomorphism (3) and $\Gamma \cdot d[k\ell] = \Gamma(dE_{k\ell} + \bar{d}E_{\ell k}) = \gamma_k dE_{k\ell} + \gamma_\ell \bar{d}E_{\ell k} = d[k\ell]_\Gamma$. The Brace Product Rules follow immediately from the definitions (remembering that we need only check that the ik-entries of two hermitian elements $a[ik]$ coincide, since then their ki-entries will automatically coincide too). □

We will almost never work directly from the multiplication table of a twisted matrix algebra: we will do any necessary calculations in our coordination theorems in Chapters 12 and 17 in the untwisted matrix algebra, then hop over unpleasant twisted calculations with the help of the magic wand of isotopy.

EXERCISE 7.5.3A Show that $\mathcal{N}uc(\mathcal{M}_n(\mathrm{D})) = \mathcal{M}_n(\mathcal{N}uc(\mathrm{D}))$ for any linear algebra D.

EXERCISE 7.5.3B Show that $\mathcal{H}_n(\mathrm{D},\Gamma)$ can also be coordinatized another way: it is spanned by $\delta[ii]^\Gamma := \delta E_{ii}$ $(\delta \in \gamma_i\mathcal{H}(\mathrm{D}))$, $d[ij]^\Gamma := dE_{ij} + \gamma_j\bar{d}\gamma_i^{-1}E_{ji}$ $(d \in \mathrm{D})$. This coordinatization has the advantage that the parameter in $d[ij]^\Gamma$ precisely labels the ij-entry of the matrix, the coefficient of E_{ij}, but it has the drawback that $\mathcal{H}_n(\mathrm{D},\Gamma)$ is spanned by the $a[ij]^\Gamma$ for all $a \in \mathrm{D}$ (as expected), but by the $\alpha[ii]^\Gamma$ for α in the γ_i-translate $\gamma_i\mathcal{H}(\mathrm{D})$ (so the diagonal entries are coordinatized by different submodules). Verify these assertions, show we have the index-reversing relation $d[ij]^\Gamma = \gamma_j\bar{d}\gamma_i^{-1}[ji]^\Gamma$, work out the multiplication rules (Basic Products and Basic Orthogonality), and give an isomorphism $\mathcal{H}_n(\mathrm{D},-)^{(\Gamma)} \to \mathcal{H}_n(\mathrm{D},\Gamma)$ in these terms.

EXERCISE 7.5.3C Work out the formulas for the 3 Basic U-Products, U-orthogonality, 3 Basic Triple Products, and Basic Triple Orthogonality in a twisted matrix algebra $\mathcal{H}_n(\mathrm{D},\Gamma)$ corresponding to those in the untwisted Hermitian Matrix Example 3.2.4.

For the case $n = 3$ we obtain a twisting of the Freudenthal Construction 4.3.1.

Freudenthal Isotope Theorem 7.5.4 *If* D *is a unital alternative algebra with scalar involution over* Φ, *and* $\Gamma = diag(\gamma_1, \gamma_2, \gamma_3)$ *a diagonal matrix for invertible elements* γ_i *of* Φ, *then the twisted Jordan matrix algebra* $\mathcal{H}_3(\mathrm{D},\Gamma)$ *is a cubic factor determined by the twisted sharped cubic form with basepoint* c, *cubic norm form* N, *trace* T, *and sharp* $\#$ *defined as follows for elements* $x = \sum_{i=1}^3 \alpha_i e_i + \sum_{i=1}^3 a_i[jk]_\Gamma$, $y = \sum_{i=1}^3 \beta_i e_i + \sum_{i=1}^3 b_i[jk]_\Gamma$ *with* $\alpha_i, \beta_i \in \Phi$, $a_i, b_i \in \mathrm{D}$ *in Jacobson* Γ-*box notation* ($e_i := E_{ii} = \gamma_i^{-1}[ii]_\Gamma$, $d[jk]_\Gamma := \gamma_j dE_{jk} + \gamma_k\bar{d}E_{kj}$, *where* (ijk) *is always a cyclic permutation of* (123)):

$$c := e_1 + e_2 + e_3,$$

$$N(x) := \alpha_1\alpha_2\alpha_3 - \sum_i \gamma_i\gamma_j\alpha_k n(a_k) + \gamma_1\gamma_2\gamma_3 t(a_1a_2a_3),$$

$$T(x,y) := \sum_i \big(\alpha_i\beta_i + \gamma_j\gamma_k t(\bar{a}_i b_i)\big), \qquad T(x) := \sum_i \alpha_i,$$

$$x^{\#} := \sum_i \big(\alpha_j\alpha_k - \gamma_j\gamma_k n(a_i)\big)e_i + \sum_i \big(\gamma_i\overline{a_j a_k} - \alpha_i a_i\big)[jk]_\Gamma.$$

PROOF. We could repeat the proof of the Freudenthal Construction, throwing in gammas at appropriate places, to show that the indicated cubic form is sharped and the resulting Jordan algebra has the structure of a Jordan matrix algebra. But we have already been there, done that, so we will instead use isotopy. This has the added advantage of giving us insight into *why* the norm form takes the particular form (with the particular distribution of gammas). We know that $\varphi = L_\Gamma$ is an isomorphism $\mathrm{J}^{(\Gamma)} := \mathcal{H}_3(\mathrm{D},-)^{(\Gamma)} \to \mathcal{H}_3(\mathrm{D},\Gamma) =: \mathrm{J}'$, and by Cubic Factor Isotope 7.4.1 the isotope $\mathrm{J}^{(\Gamma)}$ is a cubic factor with norm $N(\Gamma)N(X)$, so the isomorphic copy J' is also a cubic factor with sharped cubic form $N'(\varphi(X)) := N^{(\Gamma)}(X) = N(\Gamma)N(X)$, trace bilinear form $T'(\varphi(X),\varphi(Y)) := T^{(\Gamma)}(X,Y) = T(U_\Gamma X, Y)$, and sharp mapping $\varphi(X)^{\#'} := \varphi(X^{\#(\Gamma)}) = N(\Gamma)^{-1}\varphi(U_{\Gamma^\#}X^\#)$.

The crucial thing to note is that $x = \sum_i \alpha_i e_i + \sum_i a_i[jk]_\Gamma \in \mathcal{H}_3(\mathrm{D},\Gamma)$ is *not quite in Gamma-box notation*: since $e_i = \gamma_i^{-1}[ii]_\Gamma$, its Gamma-box expression is $x = \sum_i(\alpha_i\gamma_i^{-1})[ii]_\Gamma + \sum_i a_i[jk]_\Gamma$, and thus is the image $x = \varphi(X)$ for $X = \sum_i(\alpha_i\gamma_i^{-1})[ii] + \sum_i a_i[jk] \in \mathcal{H}_3(\mathrm{D},-)$; the presence of these γ_i^{-1} down the diagonal is what causes the norm, trace, and sharp to change form. For the norm we have

$$\begin{aligned} N'(x) &= N(\Gamma)N(X) = \gamma_1\gamma_2\gamma_3 N(X) \\ &= \gamma_1\gamma_2\gamma_3\big[(\alpha_1\gamma_1^{-1})(\alpha_2\gamma_2^{-1})(\alpha_3\gamma_3^{-1}) - \textstyle\sum_i(\alpha_k\gamma_k^{-1})n(a_k) + t(a_1a_2a_3)\big] \\ &= \alpha_1\alpha_2\alpha_3 - \textstyle\sum_i \gamma_i\gamma_j\alpha_k n(a_k) + \gamma_1\gamma_2\gamma_3 t(a_1a_2a_3). \end{aligned}$$

For the trace we use $U_{\sum_i \delta_i e_i} a[jk] = \delta_j\delta_k a[jk]$ to see that

$$\begin{aligned} T'(x,y) &= T^{(\Gamma)}(X,Y) = T(U_\Gamma X, Y) \\ &= \textstyle\sum_i \big[(\gamma_i^2\alpha_i\gamma_i^{-1})(\beta_i\gamma_i^{-1}) + t\big((\overline{\gamma_j\gamma_k a_i})b_i\big)\big] \\ &= \textstyle\sum_i \big[\alpha_i\beta_i + \gamma_j\gamma_k t(\overline{a_i}b_i)\big]. \end{aligned}$$

Finally, for the sharp we use $\Gamma^\# = \sum_i \gamma_j\gamma_k e_i,\ U_{\Gamma^\#}a[jk] = (\gamma_k\gamma_i)(\gamma_i\gamma_j)a[jk]$ to see that

$$\begin{aligned} x^{\#'} &= \varphi\big[X^{\#(\Gamma)}\big] = N(\Gamma)^{-1}\varphi\big[U_{\Gamma^\#}X^\#\big] \\ &= (\gamma_1\gamma_2\gamma_3)^{-1}\varphi\big[\textstyle\sum_i(\gamma_j\gamma_k)^2\big((\alpha_j\gamma_j^{-1})(\alpha_k\gamma_k^{-1}) - n(a_i)\big)[ii] \\ &\qquad + \textstyle\sum_i(\gamma_k\gamma_j\gamma_i^2)\big(\overline{a_j a_k} - (\alpha_i\gamma_i^{-1})a_i\big)[jk]\ \big] \\ &= \textstyle\sum_i \gamma_j\gamma_k\big(\gamma_j^{-1}\gamma_k^{-1}\alpha_j\alpha_k - n(a_i)\big)\gamma_i^{-1}[ii]_\Gamma + \sum_i \gamma_i\big(\overline{a_j a_k} - \gamma_i^{-1}\alpha_i a_i\big)[jk]_\Gamma \\ &= \textstyle\sum_i \big(\alpha_j\alpha_k - \gamma_j\gamma_k n(a_i)\big)e_i + \sum_i \big(\gamma_i\overline{a_j a_k} - \alpha_i a_i\big)[jk]_\Gamma. \end{aligned}$$

Thus the norm, trace, and sharp naturally take the form given in the theorem.
□

Exercise 7.5.4 Even though we have already been there, done that, do it again: repeat the proof of the Freudenthal Construction 4.3.1, throwing in gammas at appropriate places, to show directly that the indicated cubic form is sharped and the resulting Jordan algebra has the structure of a Jordan matrix algebra.

7.6 Problems for Chapter 7

Problem 7.1 For an invertible element u of a unital associative algebra A, let $\hat{u}$ denote the *inner automorphism* or *conjugation* by u, $x \mapsto uxu^{-1}$. (1) Show that the inner automorphisms form a subgroup of the automorphism group of A, with $\hat{u} \circ \hat{v} = \widehat{uv}$, and that $\hat{u} = 1\!\!1_{\mathrm{A}}$ is the identity iff u lies in the center of A. If A has an involution, show that $* \circ \hat{u} \circ * = (\widehat{u^*})^{-1}$. (2) Show from these that $*_u := \hat{u} \circ *$ is an anti-automorphism of A, which has period 2 iff $u^* = \lambda u$ for some element λ of the center of A (in which case λ is necessarily *unitary*, $\lambda\lambda^* = 1$). In particular, conclude that if u is hermitian or skew ($u^* = \pm u$), then the (nuclear) *u-isotope* $*_u$ is again an involution of A. (3) Verify that when u is hermitian the map L_u is a $*$-isomorphism $(\mathrm{A}_u, *_u) \to (\mathrm{A}, *)$, and the $*_u$-hermitian elements are precisely all uh for $*$-hermitian elements h. (4) If u is skew, show that the $*_u$-hermitian elements are precisely all us for $*$-skew elements s. [The fact that isotopes can switch skew elements of one involution into hermitian elements of another is crucial to understanding the "symplectic involution" on $2n \times 2n$ matrices (the adjoint with respect to a nondegenerate skew-hermitian bilinear form on a $2n$-dimensional space, equivalently, the conjugate-transpose map on $n \times n$ matrices over a split quaternion algebra).]

Problem 7.2* (1) Define an invertible linear transformation on a unital Jordan algebra J to be *structural* if there exists an invertible "adjoint" transformation T^* on J satisfying $U_{T(x)} = TU_xT^*$ for all $x \in \mathrm{J}$. Show that the adjoint is uniquely determined by $T^* = T^{-1}U_{T(1)}$. Show that the set of all structural transformations forms a group, the *structure group* $\mathcal{S}trg(\mathrm{J})$ of J, containing all invertible scalar multiplications $\alpha 1\!\!1_{\mathrm{J}}$ (for α invertible in Φ) and all invertible operators U_x (for x invertible in J). Show that the structure group contains the automorphism group as precisely the transformations which do not shift the unit [$T(1) = 1$, hence T is "orthogonal": $T^* = T^{-1}$]. Give an easy example to show that this inclusion is strict: there exist orthogonal T which are not automorphisms. (2) Show that any invertible structural T induces an isomorphism $\mathrm{J}^{(u)} \to \mathrm{J}^{(v)}$ for $v = (T^*)^{-1}(u)$ on *any* isotope $\mathrm{J}^{(u)}$. (3) Show conversely that if $T : \mathrm{J}^{(u)} \to \mathrm{J}^{(v)}$ is an isomorphism on *some* isotope, then T is structural with $T^* = U_uT^{-1}U_v^{-1}$ and $T^*(v) = u$ [note that $T(u^{-1}) = v^{-1}$]. (4) Conclude that the group of all structural transformations is precisely the "autotopy group" of J (the isotopies of J with itself), and that two isotopes $\mathrm{J}^{(u)}, \mathrm{J}^{(v)}$ are isomorphic iff the elements u, v are conjugate under the structure group, so that the isomorphism classes of isotopes correspond to the conjugacy classes of invertible elements under the structure group. (5) Show that if u has an invertible square root in J ($u = v^2$ for some invertible v) then $\mathrm{J}^{(u)} \cong \mathrm{J}$. (6) Conclude that if J is finite-dimensional over an algebraically closed field, then every isotope of J is isomorphic to J. (7) If *every* invertible element has a square root, show that the structure group consists of all $T = U_x A$ for an invertible x and automorphism A, so in a sense the structure group "essentially" just contains the automorphisms and U-operators. We will return to structural transformations in II.18.2.1 and III.1.2.1.

PROBLEM 7.3* Define the *left-u-isotope* ${}_u\mathrm{A}$ of a linear algebra A by an invertible nuclear element u to be the same space but new product $x^u y := uxy$. Of course, when u is central the left isotope coincides with the middle isotope, and therefore is perfectly associative. In general, the factor u acts on the left as a formal symbol akin to a parenthesis, and the left isotope is close to being a "free" nonassociative algebra. What a difference the placement of u makes! Can you necessary and sufficient, or at least useful, conditions on u for ${}_u\mathrm{A}$ to be associative? To have a unit?

QUESTION 7.1* In infinite-dimensional situations it is often unnatural to assume a unit, yet useful to have isotopes. We can introduce a notion of *generalized Jordan isotope* $\mathrm{J}^{(\tilde{u})}$, where $\tilde{u}$ is invertible in some larger algebra $\widetilde{\mathrm{J}} \supseteq \mathrm{J}$, as long as this induces a product back on $\mathrm{J}: \{\mathrm{J}, \tilde{u}, \mathrm{J}\} \subseteq \mathrm{J}$ (e.g., if $\widetilde{\mathrm{J}}$ is the unital hull of J, or more generally as long as $\mathrm{J} \lhd \widetilde{\mathrm{J}}$). How much of the Jordan Homotope Proposition 7.2.1 holds true in this general context? Can you give an example of a generalized isotope which is not a standard isotope?

Second Phase: The Tale of Two Idempotents

It was the best of times, it was the worst of times;
the spring of hope, the winter of despair;
we had everything before us, we had nothing before us;
idempotents in the pot, but only two to go around.

In this phase we will investigate the structure of Jordan algebras having two supplementary orthogonal idempotents. The key tool is the Peirce decomposition of a Jordan algebra with respect to a single idempotent detailed in Chapter 8. The resulting Peirce spaces $\mathrm{J}_i\,(i = 2, 1, 0)$ are eigenspaces for the multiplication operators of the idempotent, equivalently, root spaces for the Peircer torus. They have a multiplication table closely resembling that for matrix multiplication (featuring Peirce Brace Rules, U-Rules, Triple Rules, and Orthogonality Rules), and the Peirce Identity Principle says that multiplication rules involving distinct Peirce spaces will hold in all Jordan algebras as soon as they hold in all associative algebras.

Chapter 9 describes two crucial aspects of the Peirce decomposition: the Peirce specializations σ_2, σ_0 of the diagonal spaces J_2, J_0 on the off-diagonal space J_1 (which commute by Peirce Associativity), and the diagonal-valued Peirce quadratic forms q_2, q_0 (the diagonal projections of the square of an off-diagonal element, satisfying q-properties concerning the interaction of q with products).

Chapter 10 discusses connection involutions, symmetries determined by strong connecting elements which interchange the diagonal spaces. Any invertible off-diagonal element becomes a strong connector (an involutory $v_1^2 = 1$) in some diagonal isotope, which allows us to prove coordinatization theorems for strongly connected Peirce frames only, then deduce the theorems for general connected Peirce frames via the magic wand of isotopy.

Jordan algebras with Peirce frames of length 2 come in two basic flavors: the reduced spin factors determined by a quadratic form, and the hermitian algebras of 2×2 matrices with entries from a symmetrically generated associative $*$-algebra. Our main results in this phase are two general coordinatization theorems: in Chapter 11 we establish that Jordan algebras with Peirce 2-frames satisfying the Spin Peirce relation (that $q_2(x), q_0(x)$ both act the same on J_1) are reduced spin factors $\mathcal{R}ed\mathcal{S}pin(q)$, and in Chapter 12 that those with cyclic 2-frames (where the off-diagonal space is generated from a single v_1 by the action of $\sigma_2(\mathrm{J}_2)$) are hermitian algebras $\mathcal{H}_2(\mathrm{D}, \Gamma)$. In both cases the proof for general connected frames is reduced to the strong case (where the argument is easiest and clearest) by passing to an isotope.

8

Peirce Decomposition

In our analysis of Jordan algebras with capacity we are, by the very nature of capacity, forced to deal with sums of orthogonal idempotents and their associated Peirce decompositions. For a while we can get by with the simplest possible case, that of a single idempotent. The fact that the unit element 1 decomposes into two orthogonal idempotents e, e' makes the identity operator decompose into the sum of *three* orthogonal idempotents (projections), and decompositions of the identity into orthogonal projections precisely correspond to decompositions of the underlying module into a direct sum of submodules.

In associative algebras, the decomposition $\mathbb{1}_{\mathrm{A}} = L_1 = L_{e+e'} = L_e + L_{e'}$ of the left multiplication operator leads to a one-sided Peirce decomposition $\mathrm{A} = e\mathrm{A} \oplus e'\mathrm{A}$ into *two* subspaces, and the decomposition $\mathbb{1}_{\mathrm{A}} = L_1 R_1 = L_e R_e + L_e R_{e'} + L_{e'} R_e + L_{e'} R_{e'}$ leads to a two-sided Peirce decomposition into *four* subspaces $\mathrm{A} = \mathrm{A}_{11} \oplus \mathrm{A}_{10} \oplus \mathrm{A}_{01} \oplus \mathrm{A}_{00}$. In Jordan algebras the decomposition of the quadratic U-operator $\mathbb{1}_{\mathrm{J}} = U_1 = U_{e+e'} = U_e + U_{e,e'} + U_{e'}$ leads to a decomposition into *three* spaces $\mathrm{J} = \mathrm{J}_2 \oplus \mathrm{J}_1 \oplus \mathrm{J}_0$ (where J_2 corresponds to A_{11}, J_0 corresponds to A_{00}, but J_1 corresponds to the sum $\mathrm{A}_{10} \oplus \mathrm{A}_{01}$).

8.1 Peirce Decompositions

We recall the most basic properties of idempotent elements.[1]

Idempotent Definition 8.1.1 *An element e of a Jordan algebra* J *is an* **idempotent** *if $e^2 = e$. A* **proper idempotent** *is an idempotent $e \neq 1, 0$; an algebra is* **reduced** *if it contains a proper idempotent. Two idempotents e, f are* **orthogonal** *if $e \bullet f = 0$.*

[1] Idempotents were introduced in the *Historical Survey* I.5.1 (and formally in II.5.2.1), and the multiplication rules for Peirce spaces were described in I.6.1 as being at the heart of the classical methods.

The reason for the terminology is that an idempotent e is "same-potent," all its powers are itself:

$$e^n = e \qquad \textit{for all} \qquad n \geq 1.$$

This is trivially true for $n = 1$, and if true for n then it is also true for $n+1$ since, by the Power Definition 5.2.1, $e^{n+1} = e\bullet e^n = e\bullet e$ [induction hypothesis] $= e$ [idempotence].

The **complementary idempotent** $e' := \hat{1} - e$ lives only in the unital hull $\widehat{\mathrm{J}}$. It is again an idempotent, and orthogonal to e in $\widehat{\mathrm{J}}$: $(e')^2 = (\hat{1} - e)^2 = \hat{1} - 2e + e^2 = \hat{1} - 2e + e = \hat{1} - e = e'$, and $e \bullet e' = e \bullet (\hat{1} - e) = e - e^2 = 0$. Clearly e and its complement are **supplementary** in the sense that they sum to $\hat{1}$: $e + e' = \hat{1} \in \widehat{\mathrm{J}}$. (If J itself is unital, we replace $\widehat{\mathrm{J}}$ by J and take $e' := 1 - e \in \mathrm{J}$ as the complement.)

Peirce Decomposition Theorem 8.1.2 (1) *The* **Peirce projections** $E_i = E_i(e)$ *determined by* e *are the multiplication operators*

$$E_2 = U_e, \qquad E_1 = U_{e,e'}, \qquad E_0 = U_{e'}.$$

They form a supplementary family of projection operators on J, $\sum_i E_i = \mathbb{1}_{\mathrm{J}}$, $E_iE_j = \delta_{ij}E_i$, *and therefore the space* J *breaks up as the direct sum of the ranges: we have the* **Peirce Decomposition** *of* J *into* **Peirce subspaces**

$$\mathrm{J} = \mathrm{J}_2 \oplus \mathrm{J}_1 \oplus \mathrm{J}_0 \quad \textit{for} \quad \mathrm{J}_i := E_i(\mathrm{J}).$$

(2) *Peirce decompositions are inherited by ideals or by subalgebras containing e: we have* **Peirce Inheritance**

$$\mathrm{K} = \mathrm{K}_2 \oplus \mathrm{K}_1 \oplus \mathrm{K}_0 \quad \textit{for} \quad \mathrm{K}_i = E_i(\mathrm{K}) = \mathrm{K} \cap \mathrm{J}_i$$

$$(\mathrm{K} \lhd \mathrm{J} \quad \textit{or} \quad e \in \mathrm{K} \leq \mathrm{J}).$$

PROOF. (1) Since the E_i are multiplication operators they leave the ideal $\mathrm{J} \lhd \widehat{\mathrm{J}}$ invariant; therefore, if they are supplementary orthogonal projections on the unital hull, they will restrict to such on the original algebra. Because of this, it suffices to work in the unital hull, so *we assume from the start that* J *is unital*. Trivially the E_i are supplementary *operators* since e, e' are supplementary *elements*: $E_2 + E_1 + E_0 = U_e + U_{e,e'} + U_{e'} = U_{e+e'} = U_1 = \mathbb{1}_{\mathrm{J}}$ by the basic property of the unit element. Now $e, e' \in \Phi[e] = \Phi 1 + \Phi e$, and by the Operator Power-Associativity Theorem 5.2.2(2) we know that $U_pU_q = U_{p\bullet q}$, hence by linearization $U_pU_{q,r} = U_{p\bullet q, p\bullet r}$ for any $p, q, r \in \Phi[e]$. In particular, $U_pU_q = U_pU_{q,r} = U_{q,r}U_p = 0$ whenever $p \bullet q = 0$, which immediately yields orthogonality of $E_2 = U_e, E_1 = U_{e,e'}, E_0 = U_{e'}$ by orthogonality $e \bullet e' = 0$. It also implies that E_2, E_0 are projections since idempotent elements $p \bullet p = p$ always produce idempotent U-operators, $U_pU_p = U_{p\bullet p} = U_p$ by the Fundamental Formula (or Operator Power-Associativity again). The complement $E_1 = \mathbb{1}_{\mathrm{J}} - (E_2 + E_0)$ must then be a projection too. [We could also check this directly: further linearization of Operator Power-Associativity shows that $U_{p,q}U_{z,w} = U_{p\bullet z,q\bullet w} + U_{p\bullet w,q\bullet z}$, so $U_{e,e'}U_{e,e'} = U_{e\bullet e,e'\bullet e'} + U_{e\bullet e',e'\bullet e} = U_{e,e'} + 0$.]

Thus we have decomposed the identity operator into supplementary orthogonal projections, and this immediately decomposes the underlying Φ-submodule into the Peirce subspaces $\mathrm{J}_i = E_i(\mathrm{J})$.[2]

(2) Since the Peirce projections E_i are built out of multiplications by 1 and e, they map K into itself: an ideal is invariant under *any* multiplications from J, and a subalgebra is invariant under multiplications from *itself*. Thus the Peirce projections induce by restriction a decomposition $\mathbb{1}_{\mathrm{K}} = E_2|_{\mathrm{K}} + E_1|_{\mathrm{K}} + E_0|_{\mathrm{K}}$ of the identity operator on K and so a decomposition $\mathrm{K} = \mathrm{K}_2 \oplus \mathrm{K}_1 \oplus \mathrm{K}_0$ for $\mathrm{K}_i := E_i(\mathrm{K}) \subseteq \mathrm{K} \cap E_i(\mathrm{J}) = \mathrm{K} \cap \mathrm{J}_i$, and we have equality since if $x \in \mathrm{K} \cap \mathrm{J}_i$ then $x = E_i(x) \in E_i(\mathrm{K})$. □

Exercise 8.1.2* Write an expression for the Peirce projections in terms of the operator L_e, and see once more why the U-formulation is preferable.

We will usually denote the Peirce projections by E_i; if there is any danger of confusion (e.g., if there are two different idempotents running around), then we will use the more explicit notation $E_i(e)$ to indicate which idempotent gives rise to the Peirce decomposition.

Historically, the Peirce decomposition in Jordan algebras was introduced by A.A. Albert as the decomposition of the space into eigenspaces for the left-multiplication operator L_e, which satisfies the equation $(t-1)(t-\frac{1}{2})(t-0) = 0$. This approach breaks down over rings without $\frac{1}{2}$, and in any event is messy: the projections have a simple expression in terms of U-operators, but a complicated one in terms of L_e. The most elegant description of the Peirce spaces is due to Loos (it works in all characteristics, and for Jordan triples and pairs as well), using the important concept of *Peircer* (pronounced purser, not piercer!).

Peircer Definition 8.1.3 *For each $\alpha \in \Phi$ we set $e(\alpha) := e' + \alpha e \in \widehat{\mathrm{J}}$ and define the* **Peircer** *$E(\alpha)$ to be its U-operator,*

$$E(\alpha) := U_{e(\alpha)} = E_0 + \alpha E_1 + \alpha^2 E_2 = \textstyle\sum_{i=0}^{2} \alpha^i E_i.$$

Here $e(1) = e' + e = \hat{1}$ and $e(\alpha) \bullet e(\beta) = e(\alpha\beta)$ in the special subalgebra $\Phi[e] = \Phi e \boxplus \Phi e'$ of $\widehat{\mathrm{J}}$, so from Operator Power-Associativity 5.2.2(2) *the Peircer determines a homomorphism $\alpha \mapsto E(\alpha)$ of multiplicative monoids from Φ into $\mathcal{E}nd_\Phi(\widehat{\mathrm{J}})$: we have a 1-dimensional* **Peircer torus** *of operators on $\widehat{\mathrm{J}}$,*

$$E(1) = E_0 + E_1 + E_2 = \mathbb{1}_{\mathrm{J}'}, \qquad E(\alpha\beta) = E(\alpha)E(\beta).$$

[2] Despite our earlier promise to reserve the term *space* for vector spaces, and speak instead of submodules, in this case we cannot resist the ubiquitous usage *Peirce subspace*.

EXERCISE 8.1.3* (1) Show that the Peircer $E(-1)$ is the *Peirce Reflection* around the diagonal Peirce spaces, $T_e = \mathbb{1}$ on the diagonal $\mathrm{J}_2 + \mathrm{J}_0$ and $T = -\mathbb{1}$ on the off-diagonal J_1, and is an involutory map. (2) Show that the element $u(e) = e(-1) = e' - e = \hat{1} - 2e$ is an involution ($u(e)^2 = \hat{1}$) and hence the operator $T_e = U_{u(e)}$ is an involutory automorphism (cf. the Involution Lemma 6.1.10). (3) In Problem 6.1 you showed (we hope) that there is a 1-to-1 correspondence between involutions $u^2 = 1$ in a unital Jordan algebra, and idempotents $e^2 = e$, given by $u(e) := 1 - 2e,\ e(u) = \frac{1}{2}(u+1)$. Show that e, e' determine the same Peirce Reflection, $U_{u(e)} = U_{u(e')}$; explain why this does not contradict the bijection between idempotents and involutions.

We really want to make use of the Peircer for indeterminates. Let $\Omega := \Phi[t]$ be the algebra of polynomials in the indeterminate t, with the natural operations. In particular, Ω is a free Φ-module with basis of all monomials t^i ($i \geq 0$ in $\mathbb{I}$), so $\mathrm{J} \subseteq \mathrm{J}' := \widehat{\mathrm{J}}_\Omega \cong \widehat{\mathrm{J}}[t]$, which consists of all formal polynomials $\sum_{i=0}^n t^i x_i$ in t with coefficients x_i from $\widehat{\mathrm{J}}$ with the natural operations.

Peirce Eigenspace Laws 8.1.4 *The Peirce subspace* J_i *relative to an idempotent* e *is the intersection of* J *with the eigenspace in* $\mathrm{J}[t]$ *of the Peircer* $E(t)$ *with eigenvalue* t^i *for the indeterminate* $t \in \Phi[t]$, *and also the eigenspace of the left multiplications* V_e (*respectively* L_e) *with eigenvalues* i (*respectively* $\frac{1}{2}i$): *we have the* **Peircer, V-, and L-Eigenspace Laws**

$$\mathrm{J}_i = \{x \in \mathrm{J} \mid E(t)x = t^i x\} = \{x \in \mathrm{J} \mid V_e x = ix\} = \{x \in \mathrm{J} \mid L_e x = \tfrac{i}{2}x\}.$$

PROOF. For the Peircer Eigenspace Law, the characterization of $\mathrm{J}_i = E_i(\mathrm{J})$ as eigenspace of $E(t) = \sum_{i=0}^2 t^i E_i$ with eigenvalue t^i, follows immediately from the fact that t is an indeterminate: if $x = x_2 \oplus x_1 \oplus x_0 \in \mathrm{J},\ x_i = E_i(x)$, then in $\mathrm{J}' = \widehat{\mathrm{J}}[t]$ we have $E(t)(x) = t^2 x_2 + t x_1 + x_0$, which coincides with $t^i x$ iff $x_i = x,\ x_j = 0$ for $j \neq i$ by identifying coefficients of like powers of t on both sides. The second Eigenspace Law, characterizing J_i as the i-eigenspace of $V_e = U_{e,1} = U_{e,1-e} + U_{e,e} = E_1 + 2E_2 = \sum_{i=0}^2 iE_i$ for eigenvalue i, follows from the fact that distinct values $i, j \in \{2, 1, 0\}$ have $j - i$ invertible when $\frac{1}{2} \in \Phi$: $V_e(x) = 2x_2 + x_1 + 0x_0$ coincides with ix iff $\sum_j (j-i)x_j = 0$, which by directness of the Peirce decomposition means that $(j-i)x_j = 0$ for $j \neq i$, i.e., $x_j = 0$ for $j \neq i$, i.e., $x = x_i$. Multiplying by $\frac{1}{2}$ gives the third Eigenvalue Law for $L_e = \frac{1}{2}V_e$. □

The final formulation above in terms of L_e explains why, in the older literature (back in the L-ish days), the Peirce spaces were denoted by $\mathrm{J}_1, \mathrm{J}_{\frac{1}{2}}, \mathrm{J}_0$ instead of by $\mathrm{J}_2, \mathrm{J}_1, \mathrm{J}_0$. Note that the Peircer has the advantage that the eigenvalues $t^i = t^2, t, 1$ are independent over any scalar ring Φ, whereas independence of the eigenvalues $i = 2, 1, 0$ for V_e (even more, for L_e) requires injectivity of all $j - i$, equivalently injectivity of 2.

The Peircers also lead quickly and elegantly to the Peirce decomposition itself. As with a single variable, for $\Omega := \Phi[s, t]$ the algebra of polynomials in *two* independent indeterminates, we have $\mathrm{J} \subseteq \mathrm{J}' := \widehat{\mathrm{J}}_\Omega \cong \widehat{\mathrm{J}}[s, t]$ consisting of all formal polynomials in s, t with coefficients from $\widehat{\mathrm{J}}$. The Peircer Torus yields $\sum_{i,j} s^i t^j E_i E_j = E(s)E(t) = E(st) = \sum_i (st)^i E_i$, so identifying coefficients of $s^i t^j$ on both sides beautifully reveals the Peirce projection condition $E_i E_j = \delta_{ij} E_i$.

8.2 Peirce Multiplication Rules

The underlying philosophy of Peirce decompositions is that they are just big overgrown matrix decompositions, behaving like the decomposition of an associative matrix algebra $\mathcal{M}_2(\mathrm{D})$ into Φ-submodules $\mathrm{D}E_{ij}$ which multiply like the matrix units E_{ij} themselves. The only point to be carefully kept in mind is that in the Jordan case, as in hermitian matrices, the off-diagonal spaces $\mathrm{D}E_{12}, \mathrm{D}E_{21}$ cannot be separated, they are lumped into a single space $\mathrm{J}_1 = \mathrm{D}[12] = \{dE_{12} + \bar{d}E_{21} \mid d \in \mathrm{D}\}$. This symmetry in indices is important to keep in mind when talking about spaces whose indices are "linked" or "connected."

The Peircer is even more effective in establishing the multiplicative properties of Peirce spaces, because it is itself a U-operator and hence interacts smoothly with its fellow U-operators by the Fundamental Formula. In contrast, the L-operator does not interact smoothly with U- or L-operators: there is no pithy formula for $U_{e \bullet x}$ or $L_{e \bullet x}$.

We will need an indeterminate t *and its inverse* t^{-1}, requiring us to extend our horizons from polynomials to Laurent polynomials. Recall that *Laurent polynomials* in t are just polynomials in t and its inverse: $\Omega := \Phi[t, t^{-1}]$ consists of all *finite* sums $\sum_{i=-N}^{M} \alpha_i t^i$, with the natural operations. In particular, Ω is a free Φ-module with basis of all monomials t^j $(j \in \mathbb{Z})$ over Φ, so $\mathrm{J} \subseteq \mathrm{J}' := \widehat{\mathrm{J}}_\Omega \cong \widehat{\mathrm{J}}[t, t^{-1}]$, which consists of all finite formal Laurent polynomials $\sum_{i=-N}^{M} t^i x_i$ in t with coefficients x_i from $\widehat{\mathrm{J}}$, with the natural operations.

Peirce Multiplication Theorem 8.2.1 *Let e be an idempotent in a Jordan algebra* J. *We have the following rules for products of Peirce spaces* $\mathrm{J}_k := \mathrm{J}_k(e)$. *We let k, ℓ, m represent general indices, and i a diagonal index $i = 2, 0$ with complementary diagonal index $j = 2 - i$. We agree that $\mathrm{J}_k = \mathbf{0}$ if $k \neq 0, 1, 2$. For the bilinear, quadratic, and triple products we have the following rules*:

Three Peirce Brace Rules: $\mathrm{J}_i^2 \subseteq \mathrm{J}_i, \qquad \{\mathrm{J}_i, \mathrm{J}_1\} \subseteq \mathrm{J}_1, \qquad \mathrm{J}_1^2 \subseteq \mathrm{J}_2 + \mathrm{J}_0,$

Peirce U-Product Rule: $U_{\mathrm{J}_k} \mathrm{J}_\ell \subseteq \mathrm{J}_{2k-\ell},$

Peirce Triple Product Rule: $\{\mathrm{J}_k, \mathrm{J}_\ell, \mathrm{J}_m\} \subseteq \mathrm{J}_{k-\ell+m},$

Peirce Orthogonality Rules: $\{\mathrm{J}_i, \mathrm{J}_j\} = \{\mathrm{J}_i, \mathrm{J}_j, \mathrm{J}\} = U_{\mathrm{J}_i}(\mathrm{J}_k) = \mathbf{0}$ $(k \neq i)$.

In particular, the **diagonal Peirce spaces** $\mathrm{J}_2, \mathrm{J}_0$ *are inner ideals which annihilate each other whenever they get the chance.*

PROOF. These formulas come easily from the fact that we had the foresight to include a scalar t^{-1} in Ω. Since $E(t) = U_{e(t)}$ we can use the Fundamental Formula to see that $E(t)\,(U_{x_i}y_j) = E(t)U_{x_i}E(t)E(t^{-1})y_j$ [by Peircer Torus 8.3, since t and t^{-1} cancel each other] $= U_{E(t)x_i}E(t^{-1})y_j = U_{t^i x_i}t^{-j}y_j = t^{2i-j}U_{x_i}y_j$, and $U_{x_i}y_j$ lies in the eigenspace J_{2i-j}. In particular, we have the U-Orthogonality: $U_{x_2}y_0 \in \mathrm{J}_4 = \mathbf{0}, U_{x_2}y_1 \in \mathrm{J}_3 = \mathbf{0}, U_{x_0}y_1 \in \mathrm{J}_{-1} = \mathbf{0}, U_{x_0}y_2 \in \mathrm{J}_{-2} = \mathbf{0}$. Similarly, the linearized Fundamental Formula shows that $E(t)\{x_i, y_j, z_k\} = \big(E(t)U_{x_i,z_k}E(t)\big)E(t^{-1})y_j = \{E(t)x_i, E(t^{-1})y_j, E(t)z_k\} = \{t^i x_i, t^{-j}y_j, t^k z_k\} = t^{i-j+k}\,\{x_i, y_j, z_k\}$, and $\{x_i, y_j, z_k\}$ lies in the eigenspace J_{i-j+k}. In particular, we have *most* of Triple Orthogonality: $\{\mathrm{J}_2, \mathrm{J}_0, \mathrm{J}_i\} \subseteq \mathrm{J}_{2-0+i} = \mathbf{0}$ if $i \neq 0$, dually $\{\mathrm{J}_0, \mathrm{J}_2, \mathrm{J}_j\} \subseteq \mathrm{J}_{0-2+j} = \mathbf{0}$ if $j \neq 2$.

But we do not spontaneously obtain the triple orthogonality relations that the products $\{\mathrm{J}_2, \mathrm{J}_0, \mathrm{J}_0\} \subseteq \mathrm{J}_2$, $\{\mathrm{J}_0, \mathrm{J}_2, \mathrm{J}_2\} \subseteq \mathrm{J}_0$ actually vanish. By symmetry $\mathrm{J}_i(e) = \mathrm{J}_j(e')$ in $\widehat{\mathrm{J}}$ it suffices to prove $\{\mathrm{J}_2, \mathrm{J}_0, \mathrm{J}_0\} = \mathbf{0}$. We first turn this into a brace, $\{\mathrm{J}_2, \mathrm{J}_0, \mathrm{J}_0\} = -\{\mathrm{J}_0, \mathrm{J}_2, \mathrm{J}_0\} + \{\{\mathrm{J}_2, \mathrm{J}_0\}, \mathrm{J}_0\} \subseteq -\mathbf{0} + \{\{\mathrm{J}_2, \mathrm{J}_0\}, \mathrm{J}\}$ using the Triple Switching Formula (FFIV), then we finish it off using Brace Orthogonality

$$\{\mathrm{J}_2, \mathrm{J}_0\} = \mathbf{0},$$

which follows because $\{\mathrm{J}_2, \mathrm{J}_0\} = \{\mathrm{J}_2, \mathrm{J}_0, 1\} = \{\mathrm{J}_2, \mathrm{J}_0, e + e'\} = \{\mathrm{J}_2, \mathrm{J}_0, e'\} \subseteq \mathrm{J}_2$, and at the same time $\{\mathrm{J}_2, \mathrm{J}_0\} = \{e + e', \mathrm{J}_2, \mathrm{J}_0\} = \{e, \mathrm{J}_2, \mathrm{J}_0\} \subseteq \mathrm{J}_0$.

The brace rules follow by applying the triple product results in $\widehat{\mathrm{J}}$ where the unit is $1 = e_2 + e_0,\ e_2 = e \in \mathrm{J}_2,\ e_0 = e' = \hat{1} - e \in \mathrm{J}_0 : \mathrm{J}_i^2 = U_{\mathrm{J}_i}(e_i + e_j) \subseteq \mathrm{J}_i + \mathbf{0} = \mathrm{J}_i,\ \{\mathrm{J}_i, \mathrm{J}_1\} = \{\mathrm{J}_i, e_i + e_j, \mathrm{J}_1\} \subseteq \{\mathrm{J}_i, \mathrm{J}_i, \mathrm{J}_1\} + \mathbf{0} \subseteq \mathrm{J}_1,\ \mathrm{J}_1^2 = U_{\mathrm{J}_1}(e_2 + e_0) \subseteq \mathrm{J}_0 + \mathrm{J}_2$. □

EXERCISE 8.2.1 Give an alternate proof of the difficult Peirce relation $\{\mathrm{J}_i,\mathrm{J}_j,\mathrm{J}\} = 0$ for complementary diagonal indices. Use $\{x, x, y\} = \{x^2, y\}$ to show that $\{e_i, e_i, e_j\} = 0$, then use a linearization of the Triple Shift Formula (FFIII) to show that $V_{e_i,e_j} = V_{e_i,\{e_i,e_i,e_j\}} = 0,\ V_{y_j,e_i} = V_{e_j,\{y_j,e_j,e_i\}} = 0,\ V_{x_i,y_j} = V_{e_i,\{y_j,e_i,x_i\}} = 0,\ \{x_i, y_j, z\} = V_{x_i,y_j}(z) = 0$.

8.3 Basic Examples of Peirce Decompositions

We now give examples of idempotents and their Peirce decompositions, beginning with Jordan algebras obtained from an associative algebra A. A full algebra $\mathrm{J} = \mathrm{A}^+$ has exactly the same Peirce decompositions as the associative algebra A. In particular, the Peirce space J_1 breaks up into two pieces $\mathrm{A}_{10}, \mathrm{A}_{01}$. This is atypical for Jordan algebras; the archetypal example is hermitian matrices, where the space A_{10} is inextricably tied to the space A_{01} through the involution.[3]

[3] Associative Peirce decompositions were described in I.6.1 to motivate Jordan Peirce decompositions.

Associative Peirce Decomposition 8.3.1 *If e is an idempotent in an associative algebra* A*, it is well known that if we set* $e' = \hat{1} - e \in \widehat{\mathrm{A}}$*, then any element* $x \in \mathrm{A}$ *has a decomposition* $x = \hat{1}x\hat{1} = (e+e')x(e+e') = exe + exe' + e'xe + e'xe' = exe + (ex - exe) + (xe - exe) + (x - ex - xe + exe)$*. If we set* $e_1 = e, e_0 = e'$*, this becomes* $x = e_1xe_1 + e_1xe_0 + e_0xe_1 + e_0xe_0$*, and we obtain an* **associative Peirce decomposition**

$$\mathrm{A} = \mathrm{A}_{11} \oplus \mathrm{A}_{10} \oplus \mathrm{A}_{01} \oplus \mathrm{A}_{00} \quad (\mathrm{A}_{ij} := e_i \mathrm{A} e_j)$$

relative to e*. Since*

$$(e_i\mathrm{A}e_j)(e_k\mathrm{A}e_\ell) = \delta_{jk}e_i\mathrm{A}e_j\mathrm{A}e_\ell \subseteq \delta_{jk}e_i\mathrm{A}e_\ell,$$

these satisfy the **associative Peirce multiplication rules**

$$\mathrm{A}_{ij}\mathrm{A}_{k\ell} \subseteq \delta_{jk}\mathrm{A}_{i\ell}. \qquad \square$$

Every associative algebra can be turned into a Jordan algebra by the plus functor, but in the process information is lost. The fact that we can recover only the product $xy + yx$, not xy itself, causes the off-diagonal Peirce spaces $\mathrm{A}_{10}, \mathrm{A}_{01}$ to get confused and join together.

Full Peirce Decomposition 8.3.2 *If e is an idempotent in an associative algebra* A*, the associated full Jordan algebra* $\mathrm{J} = \mathrm{A}^+$ *has Jordan Peirce decomposition*

$$\mathrm{A}^+ = \mathrm{A}_2^+ \oplus \mathrm{A}_1^+ \oplus \mathrm{A}_0^+$$

for

$$\mathrm{A}_2^+ = \mathrm{A}_{11}, \quad \mathrm{A}_1^+ = \mathrm{A}_{10} \oplus \mathrm{A}_{01}, \quad \mathrm{A}_0^+ = \mathrm{A}_{00}. \qquad \square$$

Thus the Jordan 1-eigenspace is a mixture of the associative "left" eigenspace A_{10} and "right" eigenspace A_{01}. The Peirce Multiplication Rules 8.2.1 are precisely the rules for multiplying matrices, and one should always think of Peirce rules as matrix decompositions and multiplications. If $\mathrm{A} = \mathcal{M}_2(\mathrm{D})$ is the algebra of 2×2 matrices over an associative algebra D and $e = E_{11}$ the first matrix unit, the Peirce spaces A_{ij} relative to e are just the matrices having all entries 0 except for the ij-entry:

$$\mathrm{A}_{11} = \left(\begin{smallmatrix} \mathrm{D} & 0 \\ 0 & 0 \end{smallmatrix}\right), \quad \mathrm{A}_{10} = \left(\begin{smallmatrix} 0 & \mathrm{D} \\ 0 & 0 \end{smallmatrix}\right), \quad \mathrm{A}_{01} = \left(\begin{smallmatrix} 0 & 0 \\ \mathrm{D} & 0 \end{smallmatrix}\right), \quad \mathrm{A}_{00} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{D} \end{smallmatrix}\right).$$

When the associative algebra has an involution, we can reach inside the full algebra and single out the Jordan subalgebra of hermitian elements. Again the off-diagonal Peirce subspace is a mixture of associative ones, but here there are no regrets – there is no longing for J_1 to split into A_{10} and A_{01}, since these spaces do not contribute to the symmetric part. Instead, we have $\mathrm{J}_1 = \{x_{10} + x_{10}^* \mid x_{10} \in \mathrm{A}_{10}\}$ consisting precisely of the symmetric traces of elements of A_{10} (or dually $\mathrm{A}_{01} = (\mathrm{A}_{10})^*$).

Hermitian Peirce Decomposition 8.3.3 *If the associative algebra* A *has an involution* $*$, *and* $e^* = e$ *is a symmetric idempotent, then the associative Peirce spaces satisfy* $\mathrm{A}_{ij}^* = \mathrm{A}_{ji}$, *the Jordan algebra* $\mathcal{H}(\mathrm{A}, *)$ *contains the idempotent* e, *and the Peirce decomposition is precisely that induced from the Full Decomposition* 8.3.2:

$$\mathcal{H}(\mathrm{A}, *) = \mathrm{J}_2 \oplus \mathrm{J}_1 \oplus \mathrm{J}_0$$

for

$$\mathrm{J}_2 = \mathcal{H}(\mathrm{A}_{11}, *), \qquad \mathrm{J}_0 = \mathcal{H}(\mathrm{A}_{00}, *),$$

$$\mathrm{J}_1 = \mathcal{H}(\mathrm{A}_{10} \oplus \mathrm{A}_{01}, *) = \{x_{10} \oplus x_{10}^* \mid x_{10} \in \mathrm{A}_{10}\}. \qquad \square$$

Our final examples of idempotents and Peirce decompositions concerns Jordan algebras of quadratic and cubic forms.

Quadratic Factor Reduction Criterion 8.3.4 *In the Jordan algebra* $\mathcal{J}ord(Q, c) = \mathcal{J}Spin(M, \sigma)$ *of the Quadratic Factor and Spin Examples* 3.3.1, 3.3.3, *we say that an element* e *has* Trace 1 Type *if*

$$T(e) = 1, \quad Q(e) = 0 \quad (\textit{hence } \bar{e} = 1 - e).$$

This is equivalent to

$$e = \tfrac{1}{2}(1 + v) \textit{ for } v \in M \textit{ with } Q(v) = -1 \qquad (\sigma(v, v) = 1).$$

Over any ring of scalars Φ *such an element is a proper idempotent.*[4] *When* Φ *is a field, these are the only proper idempotents,*

$$e \in \mathcal{J}ord(Q, c) \textit{ proper } \iff e \textit{ has Trace } 1 \textit{ Type } (\Phi \textit{ a field}),$$

and such proper idempotents exist iff Q *is isotropic,*

$$\mathcal{J}ord(Q, c) \textit{ is reduced for nondegenerate } Q \textit{ over a field } \Phi$$

$$\iff Q \textit{ is isotropic, } Q(x) = 0 \textit{ for some } \neq 0.$$

PROOF. An e of Trace 1 Type is clearly idempotent by the Degree–2 Identity in the Quadratic Factor Example 3.3.1, $e^2 = T(e)e - Q(e)1 = e$, and differs from $1, 0$ since $T(1) = 2, T(0) = 0$. Here $T(e) = 1$ iff $e = \frac{1}{2}(1+v)$ where $T(v) = 0$, i.e., $v \in M$, in which case $Q(e) = 0$ iff $0 = Q(2e) = Q(1 + v) = 1 + Q(v)$, i.e., iff $Q(v) = -\sigma(v, v) = -1$.

To see that these are the *only* proper idempotents when Φ is a field, an idempotent e is proper $\iff e \neq 1, 0 \iff e \notin \Phi 1$ [because $0, 1$ are the only idempotents in a field], so from the Degree–2 Identity $0 = e^2 - T(e)e + Q(e)1 = [1 - T(e)]e + Q(e)1 \Longrightarrow 1 - T(e) = 0, Q(e) = 0$.

If a proper e exists then Q is certainly isotropic, $Q(e) = 0$ for $e \neq 0$. The surprising thing is that the annihilation by a nondegenerate Q of *some* nonzero

[4] Notice that we could also characterize the improper idempotents $e = 1$ as Trace 2 Type ($T(e) = 2, Q(e) = 1, \bar{e} = e$) and $e = 0$ as Trace 0 Type ($T(e) = Q(e) = 0, \bar{e} = e$), but we will refrain from doing so.

element automatically creates an idempotent! If $Q(x) = 0, T(x) = \lambda \neq 0$, then $e = \lambda^{-1}x$ has $Q(e) = 0, T(e) = 1$ and hence is a proper idempotent of Trace 1 Type. If $T(x) = 0$, we claim that there must exist a element $x' = U_y x$ with nonzero trace: otherwise $T(U_y x) = 0$ for all y, so linearizing $y \mapsto y, 1$ would give $0 = T(\{y,x\}) = T\big(T(y)x + T(x)y - Q(x,y)1\big)$ [by the formula for the bullet in the Quadratic Factor Example] $= 2T(x)T(y) - Q(x,y)2 = 0 - 2Q(x,y)$, and $Q(x,y) = 0 = Q(x)$ for $x \neq 0$ and all y would contradict *nondegeneracy* of Q. Such an element x' is necessarily nonzero and isotropic by Jordan Composition in the Quadratic Factor Example, $Q(x') = Q(U_y x) = Q(y)^2 Q(x) = 0$. Thus in either case, isotropy spawns an idempotent. $\square$

Exercise 8.3.4 The *improper* idempotents in $\mathcal{J}ord(Q,c)$ can't be characterized solely in terms of their trace and norm: show that $T(x) = 2, Q(x) = 1$ iff x is unipotent ($x = 1 + z$ for nilpotent z), and $T(x) = Q(x) = 0$ iff x is nilpotent, and a perfectly respectable $\mathcal{J}ord(Q,c)$ (e.g., Q nondegenerate over a field) may well have nilpotent elements.

We saw in Reduced Spin 3.4.1 a way to build a reduced quadratic factor out of a quadratic form q by adjoining a pair of supplementary orthogonal idempotents e_1, e_2.

Reduced Spin Peirce Decomposition 8.3.5 *The Peirce decomposition of the reduced spin factor* $\mathcal{R}ed\mathcal{S}pin(q) = \Phi e_1 \oplus M \oplus \Phi e_2$ *of a quadratic form, with respect to the created idempotents* $e = e_i, e' = e_{3-i}$ *is just the natural decomposition*

$$\mathcal{R}ed\mathcal{S}pin(q) = \mathrm{J}_2 \oplus \mathrm{J}_1 \oplus \mathrm{J}_0 \quad \text{for} \quad \mathrm{J}_2 = \Phi e,\ \mathrm{J}_1 = M,\ \mathrm{J}_0 = \Phi e'. \square$$

The situation for idempotents in Cubic Factors is more complicated; the proper idempotents come in two types, Trace 1 and Trace 2.

Cubic Reduction Criterion 8.3.6 *In the cubic factor* $\mathcal{J}ord(N,\#,c)$ *we say that an element* e *has* Trace 1 Type *or* Trace 2 Type *if*

$$\text{Trace 1:}\quad T(e) = 1,\ e^{\#} = 0 \quad (\textit{hence } S(e) = N(e) = 0);$$

$$\text{Trace 2:}\quad T(e) = 2,\ e^{\#} = e' \ (\textit{hence } S(e) = 1, N(e) = 0, (e')^{\#} = 0).$$

Over any ring of scalars Φ *such elements are proper idempotents. An idempotent is of Trace* 1 *Type iff its complement is of Trace* 2 *Type. When* Φ *is a field, these are the only proper idempotents,*

$$e \in \mathcal{J}ord(N,\#,c) \textit{ proper} \iff e \textit{ has Trace } 1 \textit{ or } 2 \textit{ Type} \quad (\Phi \textit{ a field}),$$

and proper idempotents exist iff N *is isotropic,*

$$\mathcal{J}ord(N,\#,c) \textit{ is reduced for nondegenerate } N \textit{ over a field } \Phi$$
$$\iff N \textit{ is isotropic, } N(x) = 0 \textit{ for some } x \neq 0.$$

PROOF. Recall from the Sharped Cubic Definition 4.2.1 and Sharped Cubic Construction 4.2.2 the basic Adjoint Identity, c–Sharp Identity, Spur Formula, Sharp Expression, and Degree–3 Identity:

(1)	$x^{\#\#} = N(x)x$	(Adjoint Identity),
(2)	$1\#x = T(x)1 - x$	(c–Sharp Identity),
(3)	$S(x) = T(x^{\#})$	(Spur Formula),
(4)	$x^{\#} = x^2 - T(x)x + S(x)1$	(Sharp Expression),
(5)	$x^3 - T(x)x^2 + S(x)x - N(x)1 = 0$	(Degree–3 Identity).

From the Sharp Expression (4) we see that $x^2 - x = x^{\#} + [T(x) - 1]x - S(x)1$ will vanish (and x will be idempotent) if either $x^{\#} = 0, T(x) = 1, S(x) = 0$ (as in Trace 1) or $x^{\#} = 1 - x, T(x) = 2, S(x) = 1$ (as in Trace 2), so e of Trace 1 or 2 Type is idempotent over *any* ring of scalars. It will be proper, differing from $1, 0$, since $T(1) = 3, T(0) = 0 \neq 1, 2$ as long as the characteristic isn't 2.

Notice that in Trace 1 Type the first two conditions imply the last two, since $S(e) = T(e^{\#}) = 0$ [by the Spur Formula (3)] and $N(e) = N(e)T(e) = T(N(e)e) = T(e^{\#\#}) = 0$ [by the Adjoint Identity (1)]. Similarly, in Trace 2 Type the first two conditions imply the final three, $S(e) = T(e^{\#}) = T(1-e) = 3 - 2 = 1, (e')^{\#} = (1-e)^{\#} = 1^{\#} - 1\#e + e^{\#} = 1 - (T(e)1 - e) + e'$ [by the c–Sharp Identity (2)] $= 1 - (2 - e) + (1 - e) = 0$, so $2N(e) = N(e)T(e) = T(N(e)e) = T(e^{\#\#}) = T((e')^{\#}) = 0$. Thus in all cases it is the trace and sharp conditions which are basic.

If e has Trace 1 Type then its complement $e' = 1 - e$ has Trace 2 Type (and vice versa). For the trace this follows from $T(1 - e) = 3 - T(e)$. For the sharp, e of Trace 2 Type has $(e')^{\#} = 0$ by assumption, and e of Trace 1 Type has $(e')^{\#} = (1 - e)^{\#} = 1^{\#} - 1\#e + e^{\#} = 1 - (T(e)1 - e) + 0$ [by the c–Sharp Identity (2)] $= 1 - (1 - e) = e = (e')'$.

To see that these are the *only* proper idempotents when Φ is a field, for idempotent e the Degree–3 Identity (5) becomes $0 = e - T(e)e + S(e)e - N(e)1 = [1 - T(e) + S(e)]e - N(e)1$, and $e \neq 1, 0 \iff e \notin \Phi 1 \implies 1 - T(e) + S(e) = N(e) = 0$, in which case the Sharp Expression (4) says that $e^{\#} = e - T(e)e + S(e)1 = [1 - T(e)]e + S(e)1 = [-S(e)]e + S(e)1 = S(e)e'$. If $S(e) = 0$ then $1 - T(e) = 0, e^{\#} = 0$, and we have Trace 1 Type; if $S(e) \neq 0$ then $S(e) = T(e^{\#})$ [by the Spur Formula (3)] $= S(e)T(e') = S(e)[3 - T(e)]$, so $S(e)[T(e) - 2] = 0$ implies that $T(e) = 2$ by canceling $S(e)$, then $S(e) = T(e) - 1 = 1, e^{\#} = e'$ as in Trace 2 Type. □

EXERCISE 8.3.6A Show that some restriction on the scalars is necessary in the Quadratic and Cubic Reduction Examples to be able to conclude that proper idempotents have Trace 1 or 2 type: if $\varepsilon \in \Phi$ is a proper idempotent, then $e = \varepsilon 1$ is a proper idempotent in $\mathcal{J}ord(Q, c)$ with $T(e) = 2\varepsilon, Q(e) = \varepsilon, \bar{e} = e$, and similarly, a proper idempotent in $\mathcal{J}ord(N, \#, c)$ with $T(e) = 3\varepsilon, S(e) = \varepsilon, N(e) = \varepsilon, e^{\#} = \varepsilon 1$.

Exercise 8.3.6B Characterize the improper idempotents in $\mathcal{J}ord(N, \#, c)$ as Trace 3 Type ($T(e) = 3, S(e) = 3, N(e) = 1, e^{\#} = e$) and Trace 0 Type ($T(e) = S(e) = N(e) = 0, e^{\#} = e$). (1) Show that an element x has $T(x) = 3, S(x) = 3, N(x) = 1, x^{\#} = x$ iff $x = 1 + z$ for nilpotent z with $T(z) = S(z) = N(z) = 0, z^{\#} = 2z$, then use the Adjoint Identity $N(z)z = z^{\#\#}$ to conclude that $z = 0$, $x = 1$, and hence that e is of Trace 3 Type iff $e = 1$. (2) Use the Adjoint Identity again to show that $N(x) = 0$, $x^{\#} = x$ iff $x = 0$, and hence that e is of Trace 0 Type iff $e = 0$.

To see explicitly what the Peirce decompositions look like, we examine the reduced case of 3×3 hermitian matrices over a composition algebra.

3×3 **Cubic Peirce Decomposition** 8.3.7 *In a cubic factor* $\mathrm{J} = \mathcal{H}_3(\mathrm{C})$ *for a composition algebra of dimension* 2^n $(n = 0, 1, 2, 3)$ *over a general ring of scalars* Φ (*with norm form* $n(x)1 = x\bar{x} = \bar{x}x \in \Phi 1$) *the idempotent* $e = 1[11]$ *is of Trace* 1 *Type, its complement* $e' = 1[22] + 1[33]$ *is of Trace* 2 *Type, and the Peirce decomposition relative to* e *is*

$$\mathrm{J}_2 = \Phi e = \{all \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ for\ \alpha \in \Phi\},$$

$$\mathrm{J}_1 = \mathrm{C}[12] \oplus \mathrm{C}[13] = \{all \begin{pmatrix} 0 & a & b \\ \bar{a} & 0 & 0 \\ \bar{b} & 0 & 0 \end{pmatrix} \ for\ a, b \in \mathrm{C}\},$$

$$\mathrm{J}_0 = \Phi[22] \oplus \mathrm{C}[23] \oplus \Phi[33] = \{all \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & c \\ 0 & \bar{c} & \beta \end{pmatrix} \ for\ \alpha, \beta \in \Phi, c \in \mathrm{C}\}.$$

Thus J_2 *is just a copy of the scalars of dimension* 1, J_1 *has dimension* 2^{n+1}, *and* $\mathrm{J}_0 \cong \mathcal{H}_2(\mathrm{C}) \cong \mathcal{R}ed\mathcal{S}pin(q)$ *has dimension* $2^n + 2$ *for the quadratic form* $q\left(\begin{smallmatrix} 0 & c \\ \bar{c} & 0 \end{smallmatrix}\right) = n(c)$. □

8.4 Peirce Identity Principle

We can use the Shirshov–Cohn Principle to provide a powerful method for verifying facts about Peirce decompositions in general Jordan algebras.

Peirce Principle 8.4.1 *Any Peirce elements* x_2, x_1, x_0 *from* distinct *Peirce spaces* $\mathrm{J}_i(e)$ *relative to an idempotent* e *lie in a special subalgebra* B *of* J *containing* e. *Therefore, all Jordan behavior of distinct Peirce spaces in associative algebras persists in all Jordan algebras*: *if* f_α, f *are Jordan polynomials, a set of relations* $f_\alpha(e, x_2, x_1, x_0) = 0$ *will imply a relation* $f(e, x_2, x_1, x_0) = 0$ *among Peirce elements* x_i *in all Jordan algebras if it does in all associative algebras, i.e., if the relations* $f_\alpha(e, a_{11}, a_{10} + a_{01}, a_{00}) = 0$ *imply* $f(e, a_{11}, a_{10} + a_{01}, a_{00}) = 0$ *for all Peirce elements* $a_{ij} \in \mathrm{A}_{ij}$ *relative to idempotents* e *in all unital associative algebras* A. *In particular, we have the* **Peirce Identity Principle**: *any* Peirce identity $f(e, x_2, x_1, x_0) = 0$ *for a Jordan polynomial* f *will hold for all Peirce elements* x_i *in all Jordan algebras if* $f(e, a_{11}, a_{10} + a_{01}, a_{00}) = 0$ *holds for all Peirce elements* $a_{ij} \in \mathrm{A}_{ij}$ *in all associative algebras* A.

PROOF. It suffices to establish this in the unital hull, so we may assume from the start that J is unital. Then everything takes place in the unital subalgebra $\mathrm{B} = \Phi[e,x]$ of J for $x = x_2 + x_1 + x_0$, since each x_i can be recovered from x by the Peirce projections ($x_2 = E_2(x) = U_e x$, $x_1 = E_1(x) = U_{e,1-e}x$, $x_0 = E_0(x) = U_{1-e}x$). By the Shirshov–Cohn Theorem $\mathrm{B} \cong \mathcal{H}(\mathrm{A},*)$ is special, so if a Jordan polynomial $f(e,x_2,x_1,x_0)$ vanishes in the special algebra B, it will vanish in J. By the Hermitian Peirce Decomposition 8.3.3 of B, we can write $x_2 = a_{11}, x_1 = a_{10} + a_{01}, x_0 = a_{00}$, therefore $f(e,a_{11},a_{10} + a_{01},a_{00}) = 0$ in A implies that $f(e,x_2,x_1,x_0) = 0$ in B and hence J. $\square$

Let us see how powerful the Principle is by rederiving the Peirce identities. The associative Peirce decomposition 8.3.1 has $\mathrm{A}_{ij}\mathrm{A}_{k\ell} = \delta_{jk}\mathrm{A}_{i\ell}$ with orthogonal Peirce projections $C_{ij}(a) = e_i a e_j$, so the Jordan Peirce projections $E_2 = C_{11}, E_1 = C_{10} + C_{01}, E_0 = C_{00}$ are also supplementary orthogonal projections as in 8.1.2. We can recover all Peirce Brace Rules 8.2.1. For squares: $x_2^2 \in \mathrm{A}_{11}\mathrm{A}_{11} \subseteq \mathrm{A}_{11} = \mathrm{J}_2$; dually $x_0^2 \in \mathrm{J}_0$; and $x_1^2 \in (\mathrm{A}_{10} + \mathrm{A}_{01})^2 = \mathrm{A}_{10}\mathrm{A}_{01} + \mathrm{A}_{01}\mathrm{A}_{10} \subseteq \mathrm{A}_{11} + \mathrm{A}_{00} = \mathrm{J}_2 + \mathrm{J}_0$. For braces: $\{x_2,x_1\} \in \mathrm{A}_{11}(\mathrm{A}_{10} + \mathrm{A}_{01}) + (\mathrm{A}_{10} + \mathrm{A}_{01})\mathrm{A}_{11} \subseteq \mathrm{A}_{10} + \mathrm{A}_{01} = \mathrm{J}_1$; dually $x_0 \bullet x_1 \in \mathrm{J}_1$; and $x_2 \bullet x_0 \in \mathrm{A}_{11}\mathrm{A}_{00} + \mathrm{A}_{00}\mathrm{A}_{11} = \mathbf{0}$.

In Peirce Orthogonality 8.2.1 we get $U_{x_2}(y_0 + z_1) \subseteq \mathrm{A}_{11}(\mathrm{A}_{00} + \mathrm{A}_{10} + \mathrm{A}_{01})\mathrm{A}_{11} = \mathbf{0}$, $\{x_2,y_0,z_1\} \subseteq \mathrm{A}_{11}\mathrm{A}_{00}\mathrm{A} + \mathrm{A}\mathrm{A}_{00}\mathrm{A}_{11} = \mathbf{0}$ [*but not* $\{\mathrm{J}_2,\mathrm{J}_0,\mathrm{J}_0\} = \mathbf{0}$ directly]. For the non-orthogonal U-products in 8.2.1 we get $U_{x_1}y_2 \in (\mathrm{A}_{10} + \mathrm{A}_{01})\mathrm{A}_{11}(\mathrm{A}_{10} + \mathrm{A}_{01}) \subseteq \mathrm{A}_{00} = \mathrm{J}_0$, but *we do not get* $U_{x_k}y_k \in \mathrm{J}_k$ directly since two factors come from the same Peirce space (however, we do get $U_{x_k}x_k \in \mathrm{J}_k$, so by linearizing $U_{x_k}y_k \equiv -\{x_k^2, y_k\} \equiv 0 \bmod \mathrm{J}_k$ from our bilinear knowledge); for trilinear products we can use Triple Switch to reduce to $U_{x_k}y_j$ if one index is repeated, otherwise to $\{x_2,y_0,z_1\} = 0$ if all three indices are distinct.

We will see an Indistinct Example 13.5.2 where the restriction to distinct Peirce spaces is important (though it seems hard to give an example with only a single idempotent).

8.5 Problems for Chapter 8

PROBLEM 8.1 Find the Peirce decompositions of $\mathcal{M}_2(\mathrm{D})^+$ and $\mathcal{H}_2(\mathrm{D},-)$ determined by the hermitian idempotent $e = \frac{1}{2}\begin{pmatrix} 1 & d \\ \bar{d} & 1 \end{pmatrix}$ where $d \in \mathrm{D}$ is *unitary*, i.e., $d\bar{d} = \bar{d}d = 1$.

PROBLEM 8.2 We have hyped the Peircer U-operators as a tool for Peirce decomposition. Part of our bias in their favor is due to the fact that Peircers work well over arbitrary scalars, so in quadratic Jordan algebras, but also in Jordan triples where there is no V_e. Now we will confess, in this out-of-the-way corner, that we can actually get along perfectly well in our linear situation with the L or V operators. (1) Derive the Peirce U-Product and Triple Rules 8.2.1 (from which the Brace Rules follow) for the Peirce spaces $\mathrm{J}_i(e)$ as the i-eigenspaces for the operator V_e: use the Fundamental Lie Formula (FFV) and the 5-Linear (FFVe)$'$ to show that $U_{\mathrm{J}_i}(\mathrm{J}_j) \subseteq \mathrm{J}_{2i-j}$, $\{\mathrm{J}_i,\mathrm{J}_j,\mathrm{J}_k\} \subseteq \mathrm{J}_{i-j+k}$.

9

Off-Diagonal Rules

In this chapter we fix a Peirce decomposition $J = J_2 \oplus J_1 \oplus J_0$ with respect to an idempotent e, and gather detailed information about the product on the Peirce space J_1. From this point on, the behavior of the spaces J_2, J_0 begins to diverge from that of the Peirce space J_1. Motivated by the archetypal example of $e = E_{11}$ in hermitian 2×2 matrices, where by Hermitian Example 8.8 $J_2 = \mathcal{H}(D)[11], J_0 = \mathcal{H}(D)[22]$ are represented by diagonal matrices, and $J_1 = D[12]$ by off-diagonal matrices, we will call J_i ($i = 2, 0$) the **diagonal** Peirce spaces, and J_1 the **off-diagonal** Peirce space. The diagonal Peirce spaces are more "scalar-like" subalgebras, and act nimbly on the more lumbering, vector-like off-diagonal space. To make the distinction between the two species visually clearer, *we will henceforth use letters a_i, b_i, c_i and subscripts $i = 2, 0, j = 2 - i$ to denote diagonal elements, while we use letters x_1, y_1, z_1 and subscript 1 to denote off-diagonal elements.*

9.1 Peirce Specializations

By Special Definition 3.1.2, a **specialization** of a Jordan algebra J is a homomorphism $J \to A^+$ for some associative algebra A; this represents J (perhaps not faithfully) as a special algebra. In the important case $A = \mathcal{E}nd_\Phi(M)$ of the algebra of all linear transformations on a Φ-module M, we speak of a *specialization of* J **on** M; this represents J as a Φ-submodule of linear transformations on M, with the product on J represented by the Jordan product $\frac{1}{2}(T_1T_2 + T_2T_1)$ of operators.

A Peirce decomposition provides an important example of a specialization.[1]

[1] This was mentioned in I.8.2 as motivation for Zel'manov's general specialization $B \longrightarrow \mathcal{E}nd(J/B)^+$ for arbitrary inner ideals B.

Peirce Specialization Proposition 9.1.1 *If* $\mathrm{J} = \mathrm{J}_2 \oplus \mathrm{J}_1 \oplus \mathrm{J}_0$ *is the Peirce decomposition of a Jordan algebra with respect to an idempotent e, the V-operators provide* **Peirce Specializations** *of the diagonal Peirce subalgebras* $\mathrm{J}_i\,(i = 2, 0)$ *on the off-diagonal space* J_1 *via*

$$\sigma_i(a_i) := V_{a_i}|_{\mathrm{J}_1} \qquad (a_i \in \mathrm{J}_i).$$

These satisfy the **Peirce Specialization Rules**:

$$\sigma_i(a_i^2) = \sigma_i(a_i)^2, \qquad \sigma_i(U_{a_i} b_i) = \sigma_i(a_i)\sigma_i(b_i)\sigma_i(a_i),$$
$$\sigma_2(e) = \mathbb{1}_{\mathrm{J}_1(e)}, \qquad \sigma_i(a_i)\sigma_i(b_i) = V_{a_i,b_i}|_{\mathrm{J}_1}.$$

PROOF. The Peirce Multiplication Rules 8.2.1 show that the diagonal spaces act on the off-diagonal space: J_1 is closed under brace multiplication by diagonal elements, $\{\mathrm{J}_i, \mathrm{J}_1\} \subseteq \mathrm{J}_1$ for $i = 2, 0$. Thus we may define linear maps σ_i of J_i into $\mathcal{E}nd_\Phi(\mathrm{J}_1)$ by restricting the action of brace multiplication. These satisfy the specialization rules since the differences $V_{a_i,b_i} - V_{a_i}V_{b_i} = -U_{a_i,b_i}$ [using Triple Switch (FFIV)], $V_{a_i^2} - V_{a_i}^2 = -2U_{a_i}$, and $V_{U_{a_i} b_i} - V_{a_i}V_{b_i,a_i} = -V_{b_i}U_{a_i}$ [by Specialization Formulas (FFIII)′] all vanish on J_1 by Peirce Orthogonality 8.2.1 $U_{\mathrm{J}_i}\mathrm{J}_1 = \mathbf{0}$. The subalgebra J_2 is unital with unit e, and the Peirce specialization σ_2 is unital since $V_e = \mathbb{1}_{\mathrm{J}_1}$ on J_1 by the Eigenspace Laws 8.1.4. □

Notice that it is the V-operators that provide the correct action, not the L-operators. When the element a_i clearly indicates its affiliation with J_i, we will omit the redundant second index and simply write $\sigma(a_i)$. We can consider σ to be a linear map from $\mathrm{J}_2 + \mathrm{J}_0$ into $\mathcal{E}nd_\Phi(\mathrm{J}_1)$, but note that σ is no longer a homomorphism when considered on the direct sum $\mathrm{J}_2 \boxplus \mathrm{J}_0$: we will see in Peirce Associativity 9.1.3 that the images $\sigma(\mathrm{J}_2), \sigma(\mathrm{J}_0)$ commute rather than annihilate each other.

In certain important situations the Peirce specializations will be faithful (i.e., injective), so that the diagonal subalgebras J_i will be special Jordan algebras.

Peirce Injectivity Lemma 9.1.2 (1) *We have a* **Peirce Injectivity Principle**: *both Peirce specializations will be injective if there exists an injective element in the off-diagonal Peirce space,*

$$\textit{if } U_{y_1} \textit{ is injective on } \mathrm{J} \textit{ for some } y_1 \in \mathrm{J}_1(e),$$
$$\textit{then already} \quad \sigma(a_i)y_1 = 0 \Longrightarrow a_i = 0 \quad (i = 2, 0).$$

In particular, this will hold if there exists an invertible element y_1.

(2) *We have a* **Simple Injectivity Principle**: *the Peirce specializations will be injective if* J *is simple*: $\mathrm{Ker}(\sigma_i) \triangleleft \mathrm{J}$ *is an ideal in* J, *so if* J *is simple and* $e \neq 1, 0$ *is proper, then the Peirce specializations* σ_2, σ_0 *are injective.*

(3) *We have a* **Peirce Speciality Principle**: *In a simple Jordan algebra all proper idempotents are special, in the sense that the Peirce subalgebras* $J_i(e)$ $(i = 2, 0,\ e \neq 1, 0)$ *are special.*

PROOF. (1) If $\{a_i, y_1\} = 0$ then $2U_{y_1}(a_i) = \{y_1, \{y_1, a_i\}\} - \{y_1^2, a_i\} \in 0 - \{J_2 + J_0, a_i\} \subseteq J_i$ by the Peirce Brace Rules 8.2.1, yet it also falls in J_{2-i} by the Peirce U-Products, so $2U_{y_1}(a_i)$ is zero, hence $a_i = 0$ by the hypothesized injectivity of U_{y_1} and the existence of $\frac{1}{2} \in \Phi$.

(2) Since σ_i is a homomorphism, its kernel $K_i := \mathrm{Ker}(\sigma_i)$ is an ideal of J_i, $\{K_i, J_i\} \subseteq K_i$; this ideal has $\{K_i, J_1\} = \mathbf{0}$ by definition, and $\{K_i, J_j\} = \mathbf{0}$ by Peirce Orthogonality Rules 8.2.1, so $\{K_i, J\} \subseteq K_i$ and K_i is an ideal in J. If J were simple but $K_i \neq \mathbf{0}$ then $K_i = J$ would imply $J_i = J$; but $J = J_2$ would imply $e = 1$, and $J = J_0$ would imply $e = 0$, contrary to our properness hypothesis.

(3) The J_i are special since they have an injective specialization σ_i. □

At this point it is worth stopping to note a philosophical consequence of this basic fact about Peirce specializations: we have a *Peirce limit to exceptionality.* Namely, if J is exceptional with unit 1 then simplicity cannot continue past 1:

If $J \subseteq \widetilde{J}$ *for simple algebras with units* $1 < \tilde{1}$, *then* J *must be special.*

The reason is that $\tilde{1} > 1$ implies that $e = 1 \neq 0, \tilde{1}$ is proper, so $J \subseteq \widetilde{J}_2(e)$ is special. This reminds us of the *Hurwitz limit to nonassociativity*: if C is a nonassociative composition algebra, then composition cannot continue past C.

Recall that the original investigation by Jordan, von Neumann, and Wigner sought a sequence of simple exceptional finite-dimensional (hence unital) algebras $J_{(1)} \subseteq J_{(2)} \subseteq \cdots J_{(n)} \subseteq \cdots$ such that the "limit" might provide a simple exceptional setting for quantum mechanics. We now see why such a sequence was never found: once one of the terms becomes exceptional the rest of the sequence comes to a grinding halt as far as growth of the unit goes. (An exceptional simple J can be imbedded in an egregiously *non-simple* algebra $J \boxplus J'$ with larger unit, and it can be imbedded in larger simple algebras with the *same* unit, such as scalar extensions J_Ω, but it turns out these are the only unital imbeddings which are possible.)

Another important fact about Peirce products is that the Peirce specializations of the diagonal subspaces on J_1 commute with each other; in terms of bullet or brace products this can be expressed as associativity of the relevant products.

Peirce Associativity Proposition 9.1.3 *The Peirce specializations of* σ_i *commute*: *for all elements* $a_i \in J_i$ *we have the operator relations*

$$\sigma(a_2)\sigma(a_0) = \sigma(a_0)\sigma(a_2) = U_{a_2,a_0};$$

in other words, we have the elemental relations

$$[a_2, x_1, a_0] = 0, \qquad (a_2 \bullet x_1) \bullet a_0 = a_2 \bullet (x_1 \bullet a_0) = \tfrac{1}{4}\{a_2, x_1, a_0\}$$

PROOF. This follows from the Peirce Principle 8.4.1 since it involves only e, x_1, a_2, a_0, and is straightforward to verify in an associative algebra A: for $x_1 = a_{10} + a_{01}, a_2 = a_{11}, a_0 = a_{00}$ we have, by associative Peirce orthogonality, $\{a_{11}, \{a_{00}, a_{10} + a_{01}\}\} = \{a_{11}, a_{00}a_{01} + a_{10}a_{00}\} = a_{11}a_{10}a_{00} + a_{00}a_{01}a_{11} = \{a_{11}, a_{10} + a_{01}, a_{00}\}$ and dually. [We can also argue directly: $\{a_2, \{a_0, x_1\}\} = \{a_2, a_0, x_1\} + \{a_2, x_1, a_0\}$ [using Triple Switch (FFIVe)] $= \{a_2, x_1, a_0\}$ by Peirce Orthogonality in 8.2.1, and dually by interchanging 2 and 0.] ☐

9.2 Peirce Quadratic Forms

The remaining facts we need about Peirce products all concern the components of the square of an off-diagonal element. As we delve deeper into the properties of Peirce decompositions, the results get more technical, but they will be crucial to establishing the structure of simple algebras.

Peirce Quadratic Form Definition 9.2.1 *An off-diagonal element x_1 squares to a diagonal element $x_1^2 = U_{x_1}1 = U_{x_1}e_0 + U_{x_1}e_2 \in \mathrm{J}_2 \oplus \mathrm{J}_0$. We define quadratic maps $q_i : \mathrm{J}_1 \to \mathrm{J}_i$ $(i = 2, 0)$ by projecting the square of an off-diagonal element onto one of the diagonal spaces*:

$$q_i(x_1) := E_i(x_1^2), \qquad q_i(x_1, y_1) := E_i(\{x_1, y_1\}).$$

Though these maps are not usually scalar-valued, the Peirce subalgebras J_i $(i = 2, 0)$ are so much more scalar-like than J itself that, by abuse of language, we will refer to the q_i as the **Peirce quadratic forms** (instead of quadratic *maps*). If $\mathrm{J} = \mathcal{R}ed\mathcal{S}pin(q) = \Phi e_2 \oplus M \oplus \Phi e_0$, then $q_i(w) = q(w)e_i$ where $q(w) \in \Phi$ is a true scalar.[2]

q-Properties Proposition 9.2.2 *Let $x_1, y_1 \in \mathrm{J}_1(e)$ relative to an idempotent $e \in \mathrm{J}$, and let $i = 2, 0$, $j = 2 - i$, $e_2 = e$, $e_0 = \hat{1} - e \in \widehat{\mathrm{J}}$. We have an* **Alternate q Expression** *for the quadratic forms*:

$$q_i(x_1) = U_{x_1}e_j.$$

(1) *Powers can be recovered from the quadratic maps by* **Cube Recovery**:

$$x_1^3 = \{q_i(x_1), x_1\}, \qquad q_i(x_1)^2 = U_{x_1}(q_j(x_1)).$$

[2] BEWARE (cf. the warning after Spin Factors Example 3.3.3): the Peirce quadratic forms are built out of squares, and correspond to the forms $q(v)$ of Reduced Spin Factors 3.4.1, and thus to the NEGATIVE of the global form Q in $\mathcal{J}ord(Q, c)$.

(2) *The operator* U_{x_1} *can be expressed in terms of q's by the* $U1q$ **Rules**:

$$U_{x_1}(y_1) = \{q_i(x_1, y_1), x_1\} - \{q_j(x_1), y_1\},$$
$$U_{x_1}(a_i) = q_j(x_1, a_i \bullet x_1),$$
$$\{x_1, a_i, z_1\} = q_j(\{x_1, a_i\}, z_1) = q_j(x_1, \{a_i, z_1\}).$$

(3) *The q's permit composition with braces by the* q**-Composition Rules**:

$$q_i(\{a_i, x_1\}) = U_{a_i}(q_i(x_1)),$$
$$q_i(\{a_i, x_1\}, x_1) = V_{a_i}(q_i(x_1)),$$
$$q_i(\{a_j, x_1\}) = U_{x_1}(a_j^2).$$

PROOF. All except the formula for $U_{x_1}(y_1)$ and the symmetry of the linearized product in the $U1q$ Rules follow directly from the Peirce Principle 8.4.1 since they involve only e, x_1, a_i, and are straightforward to verify in an associative algebra A from the associative Peirce relations. In detail, we may assume that A is unital to have complete symmetry in the indices i, j, so we need only verify the formulas for $i = 2, j = 0$. For $x_1 = a_{10} + a_{01}$ we have $q_2(x_1) = a_{10}a_{01}, q_0(x_1) = a_{01}a_{10}$, so we obtain the Alternate q Expression $U_{x_1}e_0 = x_1e_0x_1 = a_{10}a_{01} = q_2(x_1)$, Cube Recovery $x_1^3 = (a_{10} + a_{01})^3 = a_{10}a_{01}a_{10} + a_{01}a_{10}a_{01} = (a_{10}a_{01})(a_{10} + a_{01}) + (a_{10} + a_{01})(a_{10}a_{01}) = q_2(x_1)x_1 + x_1q_2(x_1)$, fourth-power Recovery $q_2(x_1)^2 = (a_{10}a_{01})(a_{10}a_{01}) = (a_{01} + a_{10})(a_{01})(a_{10})(a_{01} + a_{10}) = x_1q_0(x_1)x_1$, the second $U1q$-Rule $q_0(x_1, \{a_{11}, x_1\}) = q_0(a_{10} + a_{01}, a_{11}a_{10} + a_{01}a_{11}) = a_{01}(a_{11}a_{10}) + (a_{01}a_{11})a_{10} = 2U_{x_1}a_{11}$; the first q-Composition Rule $q_2(\{a_{11}, x_1\}) = q_2(a_{11}a_{10}+a_{01}a_{11}) = (a_{11}a_{10})(a_{01}a_{11}) = a_{11}(a_{10}a_{01})a_{11} = U_{a_{ii}}q_2(x_1)$, the second by linearizing $a_i \mapsto a_i, e_i$, and the third $q_2(\{a_{00}, x_1\}) = q_2(a_{00}a_{01} + a_{10}a_{00}) = (a_{10}a_{00})(a_{00}a_{01}) = a_{10}(a_{00}a_{00})a_{01} = U_{x_1}(a_{00}^2)$.

For the first $U1q$-Rule, linearize $x_1 \mapsto x_1, y_1$ in Cube Recovery to see that $U_{x_1}(y_1) + \{x_1, x_1, y_1\} = \{q_2(x_1), y_1\} + \{q_2(x_1, y_1), x_1\}$ and hence $U_{x_1}(y_1) = \{q_2(x_1, y_1), x_1\} + \{q_2(x_1) - x_1^2, y_1\}$ [from Triple Shift (FFIIIe) with $y = 1$] $= \{q_2(x_1, y_1), x_1\} - \{q_0(x_1), y_1\}$. Symmetry in the third $U1q$-Rule follows by taking Peirce 0-components of $\{x_1, a_2, z_1\} = \{\{x_1, a_2\}, z_1\} - \{a_2, x_1, z_1\}$ by Triple Switch (FFIVe). □

EXERCISE 9.2.2A* Spurn new-fangled principles and use good-old explicit calculation to establish q-Properties. Show that: (1) $E_i(x_1^2) = U_{x_1}(e_j)$; (2) $\{q_i(x_1), x_1\} = U_{x_1}(x_1)$; (3) $\{x_1, a_i, z_1\} = q_j(\{x_1, a_i\}, z_1)$; (4) $q_i(x_1)^2 = U_{x_1}q_j(x_1)$. (5) Show that $q_i(x_1, \{x_1, a_i\}) = V_{a_i}q_i(x_1)$, then $2U_{a_i}(q_i(x_1)) = 2q_i(\{a_i, x_1\})$.

EXERCISE 9.2.2B (1) Show that the linearization of the quadratic map q_i satisfies $q_i(x_1, y_1) = E_i(\{x_1, y_1\}) = \{x_1, e_j, y_1\} = \{x_1, y_1, e_i\} = \{e_i, x_1, y_1\}$. (2) Give an alternate direct proof of $U1q$: linearize the Triple Shift Formula to show that $V_{e_i, U(x_1)y_1} = V_{q_i(x_1, y_1), x_1} - V_{y_1, q_j(x_1)}$ as operators, then have this act on e_j. (3) Generalize the Fourth Power Rule to show that $(U_{x_1}a_j)^2 = U_{x_1}U_{a_j}q_j(x_1)$. (4) Give an alternate direct proof of q-Composition using $U_{\{a,x\}} + U_{U_a(x), x} = U_aU_x + U_xU_a + V_aU_xV_a$ [linearizing $y \mapsto a, 1$ in the Fundamental Formula] and Peirce Orthogonality to show that $U_{\{a_i, x_1\}}(e_j) = U_{a_i}U_{x_1}(e_j)$. (5) When there exists an invertible element $v_1 \in \mathrm{J}_1$, show that the quadratic forms q_i take on all values: $q_i(\mathrm{J}_1) = q_i(v_1, \{\mathrm{J}_j, v_1\}) = \mathrm{J}_i$ [show that $\mathrm{J}_i = U_{v_1}\mathrm{J}_j$].

EXERCISE 9.2.2C Establish a *q-Flipping Formula* $U_{x_1}\big(q_i(x_1)^k\big) = q_j(x_1)^{k+1}$ for all k (even negative k, if x_1 is invertible in J) when $x_1 \in J_1, i = 2$ or $0,\ j = 2 - i$.

q-Nondegeneracy Condition 9.2.3 (1) *In any Jordan algebra, an element $z \in \mathrm{J}_1$ belonging to both radicals is trivial*:

$$\mathcal{R}ad(q_i) := \{z_1 \in \mathrm{J}_1 \mid q_i(z_1, \mathrm{J}_1) = \mathbf{0},$$

$$q_2(z_1, \mathrm{J}_1) = q_0(z_1, \mathrm{J}_1) = \mathbf{0} \implies z_1 \textit{ is trivial.}$$

(2) *If the algebra* J *is nondegenerate, then the Peirce quadratic forms are both nondegenerate*: $\mathcal{R}ad(q_i) = \mathbf{0}$ *for* $i = 2, 0$.

PROOF. (1) follows immediately from the $U1q$-Rules 9.2.2(2), $U_z(\mathrm{J}_1) = U_z(\widehat{\mathrm{J}}_i) = \mathbf{0}$ for any $z = z_1$ with $q_i(z, \mathrm{J}_1) = \mathbf{0}$ for both $i = 2$ and $i = 0$. We must work a bit harder to establish that (2) holds, but some deft footwork and heavy reliance on nondegeneracy will get us through. The first easy step is to note that $U_z(\widehat{\mathrm{J}}_j) = q_i(z, \widehat{\mathrm{J}}_j \bullet z) \subseteq q_i(z, \mathrm{J}_1) = \mathbf{0}$ for $z \in \mathcal{R}ad(q_i)$. The next step is to use this plus nondegeneracy to get $U_z(\widehat{\mathrm{J}}_i) = \mathbf{0}$: each $b_j = U_z(\hat{a}_i) \in \mathrm{J}_j$ vanishes because it is trivial in the nondegenerate J, $U_{b_j}(\widehat{\mathrm{J}}) = U_{b_j}(\widehat{\mathrm{J}}_j)$ [by Peirce U-Products 8.2.1] $= U_z U_{\hat{a}_i}\Big(U_z(\widehat{\mathrm{J}}_j)\Big)$ [by the Fundamental Formula] $= U_z U_{\hat{a}_i}(\mathbf{0})$ [our first step] $= \mathbf{0}$. The third step is to use the $U1q$-Rule to conclude that $U_z(\mathrm{J}_1) = \mathbf{0}$ from $q_i(z, \mathrm{J}_1) = \mathbf{0}$ [our hypothesis] and $q_j(z) = U_z(\hat{e}_i) = 0$ [our second step]. Thus U_z kills all three parts $\widehat{\mathrm{J}}_j, \widehat{\mathrm{J}}_i, \mathrm{J}_1$ of $\widehat{\mathrm{J}}$, so z itself is trivial, and our fourth and final step is to use nondegeneracy of J yet again to conclude that $z = 0$. □

EXERCISE 9.2.3* Let e be an idempotent in a Jordan algebra with the property that $\mathrm{J}_2(e), \mathrm{J}_0(e)$ contain no nonzero nilpotent elements (e.g., if they both are simple inner ideals). (1) Show that the quadratic forms q_2, q_0 on J_1 have the same radical $\mathcal{R}ad(q_i) = \{z_1 \in \mathrm{J}_1 \mid q_i(z_1, \mathrm{J}_1) = \mathbf{0}\}$. (2) Show that $z_1 \in \mathrm{J}_1$ is trivial in such a J iff $z_1 \in \mathcal{R}ad(q_2) = \mathcal{R}ad(q_0)$. (3) Show that if $z = z_2 + z_1 + z_0$ is trivial in any Jordan algebra with idempotent e, so are z_2 and z_0; conclude that $z_2 = z_0 = 0$ when $\mathrm{J}_2(e), \mathrm{J}_0(e)$ contain no nonzero nilpotent elements. (4) Show that such a J is nondegenerate iff the quadratic maps q_i are nondegenerate (in the sense that $\mathcal{R}ad(q_i) = \mathbf{0}$).

9.3 Problems for Chapter 9

QUESTION 9.1 (i) Work out in detail the Peirce specialization (investigating Peirce injectivity and associativity) and Peirce quadratic forms (including $U1q$ Rules, q-Composition Rules, and radicals) for the case of a hermitian matrix algebra $\mathcal{H}_2(\mathrm{D}, -)$ for an associative algebra D with involution. (ii) Repeat the above for the case of a reduced spin factor $\mathcal{R}ed\mathcal{S}pin(q)$.

10

Peirce Consequences

We now gather a few immediate consequences of the Peirce multiplication rules which will be important for our classification theorems. The reader should try to hang on for one more chapter without novocain, before we come to the root of the matter in the next chapter.

10.1 Diagonal Consequences

For showing that nondegenerate algebras with minimum condition on inner ideals have a capacity, it will be important to know that the diagonal Peirce spaces inherit nondegeneracy and the minimum condition. Once more we fix throughout the chapter an idempotent e and its Peirce decomposition into spaces $\mathrm{J}_k = \mathrm{J}_k(e)$, and refer to J_i $(i = 0, 2)$ as a *diagonal* Peirce space, with *opposite* diagonal space J_j $(j = 2 - i)$. Throughout this section we use the crucial **Diagonal Orthogonality** property that the U-operator of a diagonal element kills all elements from the other Peirce spaces:

$$a_i \in \mathrm{J}_i \Longrightarrow U_{a_i}\widehat{\mathrm{J}}_j = U_{a_i}\widehat{\mathrm{J}}_1 = 0, \quad U_{a_i}\widehat{\mathrm{J}} = U_{a_i}\widehat{\mathrm{J}}_i \subseteq \mathrm{J}_i,$$

directly from the Peirce Othogonality Rules 8.2.1 in the unital hull.

Diagonal Inheritance Proposition 10.1.1 *We have the following local–global properties of the diagonal Peirce spaces* J_i $(i = 0, 2)$:

(1) *An element* z_i *is trivial locally iff it is trivial globally*: $z_i \in \mathrm{J}_i$ *is trivial in* J_i *iff it is trivial in* J; *in particular, if* J *is nondegenerate then so is* J_i.

(2) *A* Φ*-submodule* B_i *is inner locally iff it is inner globally*: $\mathrm{B}_i \subseteq \mathrm{J}_i$ *is an inner ideal of* J_i *iff it is an inner ideal of* J.

(3) *An inner ideal* B_i *is minimal locally iff it is minimal globally*: $\mathrm{B}_i \subseteq \mathrm{J}_i$ *is minimal in* J_i *iff it is minimal in* J; *in particular, if* J *has minimum condition on inner ideals, then so does* J_i.

Thus the diagonal Peirce spaces inherit nondegeneracy and minimum condition on inner ideals from J.

PROOF. (1), (2) follow immediately from Diagonal Orthogonality: $U_{z_i}(\widehat{\mathrm{J}}_i) = \mathbf{0} \iff U_{z_i}(\widehat{\mathrm{J}}) = \mathbf{0}$ and $U_{\mathrm{B}_i}(\widehat{\mathrm{J}}_i) \subseteq \mathrm{B}_i \iff U_{\mathrm{B}_i}(\widehat{\mathrm{J}}) \subseteq \mathrm{B}_i$. Then the lattice of inner ideals of B_i is just that part of the lattice of inner ideals of J falling in B_i, so (3) follows. □

It will also be important to know that when we get down to as small a Peirce 0-space as possible ($\mathrm{J}_0 = \mathbf{0}$), we must have the biggest idempotent possible ($e = 1$). We have the following improvement on the Idempotent Unit Proposition 5.2.4.

Idempotent Unit Theorem 10.1.2 *If e is an idempotent in a nondegenerate Jordan algebra with Peirce space $\mathrm{J}_0 = \mathbf{0}$, then $e = 1$ is the unit of* J.

PROOF. We will prove that the Peirce decomposition 8.1.2(1) reduces to $\mathrm{J} = \mathrm{J}_2$, where e is the unit by the Peirce Eigenspace Laws 8.1.4 (or by the old Proposition 5.2.4). By hypothesis $\mathrm{J}_0 = \mathbf{0}$, so we only need to make J_1 vanish.

This we accomplish by showing that every $z_1 \in \mathrm{J}_1$ is weakly trivial, $U_{z_1}(\mathrm{J}) = \mathbf{0}$, and therefore (in view of Weak Riddance 5.3.3) vanishes by nondegeneracy. When $\mathrm{J}_0 = \mathbf{0}$ we trivially have $U_{z_1}(\mathrm{J}_0) = \mathbf{0}$, and $U_{z_1}(\mathrm{J}_2) \subseteq \mathrm{J}_0$ [by Peirce U-Products 8.2.1] $= \mathbf{0}$, so all that remains is to prove $U_{z_1}(\mathrm{J}_1) = \mathbf{0}$ too. By the $U1q$-Rules 9.2.2(2) it suffices to show that $q_2(\mathrm{J}_1) = q_0(\mathrm{J}_1) = \mathbf{0}$; the second vanishes by hypothesis, and for the first we have $z_2 := q_2(z_1) = z_1^2$ weakly trivial since $U_{z_2}(\mathrm{J}) = U_{z_2}(\mathrm{J}_2)$ [by Peirce Orthogonality in 8.2.1] $= U_{z_1}^2(\mathrm{J}_2)$ [by the Fundamental Formula] $\subseteq U_{z_1}(\mathrm{J}_0)$ [by Peirce U-Products in 8.2.1] $= \mathbf{0}$, so by nondegeneracy all $z_2 = q_2(z_1)$ vanish.

[Note that we can't just quote Alternate q Expression 9.2.2 to conclude that $q_2(\mathrm{J}_1) = U_{\mathrm{J}_1}(\widehat{e_0})$ vanishes, because e_0 is alive and well in $\widehat{\mathrm{J}}$, even if his relatives J_0 are all dead in J. Of course, once *all* $z_1^2 = 0$ we can avoid $U1q$-Rules and see directly that all z_1 are trivial: all $z_1 \bullet y_1 = 0$ by linearization, so by definition of the U-operator $U_{z_1}(\hat{y}_k) = 2z_1 \bullet (z_1 \bullet \hat{y}_k) - z_1^2 \bullet \hat{y}_k = 2z_1 \bullet (z_1 \bullet \hat{y}_k)$ vanishes if $k = 1$ since $\mathrm{J}_1 \bullet \widehat{\mathrm{J}_1} = \mathrm{J}_1 \bullet \mathrm{J}_1 = \mathbf{0}$, and also if $k = 2$ or 0 since then $\mathrm{J}_1 \bullet (\mathrm{J}_1 \bullet \widehat{\mathrm{J}_k}) \subseteq \mathrm{J}_1 \bullet \mathrm{J}_1 = \mathbf{0}$.] □

In our strong coordinatization theorems a crucial technical tool is the fact that "both diagonal Peirce spaces are created equal," thanks to symmetries which interchange the two diagonal Peirce spaces. These symmetries arise as involutions determined by strong connection. We say that e and e' are **connected**[1] if there is an invertible element v_1 in the off-diagonal Peirce space J_1 ($v_1^2 = v_2 + v_0$ for v_i invertible in J_i), and are **strongly connected** if there is an off-diagonal involution v_1 ($v_1^2 = 1$). Recall that by the Involution Lemma 6.1.10, any such involution determines an involutory automorphism U_v of J.

[1] Basic examples of connection were given in I.5.1.

Connection Involution Proposition 10.1.3 (1) *Any off-diagonal involution* $v_1 \in \mathrm{J}_1$, $v_1^2 = 1$, *determines a* **connection involution** $x \mapsto \overline{x} := U_{v_1}(x)$ *on* J *which fixes* v_1 *and interchanges* e *and its complement* e',

$$\overline{e} = e', \qquad \overline{e'} = e, \qquad \overline{v_1} = v_1.$$

(2) *In terms of the* **trace functions**

$$t_i(x_1) := q_i(v_1, x_1) \quad (i = 2, 0, j = 2 - i),$$

the **connection action** *determined by the involution* v_1 *can be expressed on the Peirce spaces by*

$$\overline{a_2} = t_0(a_2 \bullet v_1) \ \text{on } \mathrm{J}_2, \qquad \overline{x_1} = \{t_i(x_1), v_1\} - x_1 \ \text{on } \mathrm{J}_1,$$
$$\overline{a_0} = t_2(a_0 \bullet v_1) \ \text{on } \mathrm{J}_0, \qquad \overline{q_i(x_1)} = q_j(\overline{x_1}).$$

(3) *Its off-diagonal* **connection fixed points** *are just the diagonal multiples of* v_1, *i.e., the fixed set of the involution* $x_1 \mapsto \overline{x_1}$ *on* J_1 *is* $\{\mathrm{J}_2, v_1\} = \{\mathrm{J}_0, v_1\}$:

$$\overline{\{a_i, v_1\}} = \{\overline{a_i}, v_1\} = \{a_i, v_1\}.$$

PROOF. All of these except the action on J_1 follow from the Peirce Principle 8.4.1 by a straightforward calculation in special algebras (where $t_j(a_i \bullet v_1) = a_{ji}a_{ii}a_{ij}$ if $v_1 = a_{ij} + a_{ji}$, $a_i = a_{ii}$). We can also calculate them directly, as follows. For convenience of notation set $v = v_1$, $e_2 = e$, $e_0 = 1 - e = e'$. For (1) $U_v(e_2) = E_0(v^2)$ [by the Alternate q Expression 9.2.2] $= e'$ by involutority, $U_v v = v^3 = v \bullet v^2 = v \bullet 1 = v$.

For the action (2) on the individual spaces, on J_i we have by the $U1q$-Rules 9.2.2(2) that $\overline{a_i} = U_v a_i = q_j(v, a_i \bullet v) = t_j(a_i \bullet v)$ [by definition of trace], and that $\overline{x_1} = U_v(x_1) = \{q_i(v, x_1), v\} - \{q_j(v), x_1\} = \{t_i(x_1), v\} - \{e_j, x_1\}$ [by definition of the trace t_i and involutority] $= \{t_i(x_1), v\} - x_1$. The action on q's follows from $\overline{q_i(x)} = \overline{U_x e_j}$ [by Alternate q Expression] $= U_{\bar{x}} \overline{e_j}$ [by the homomorphism property] $= U_{\bar{x}}(e_i)$ [by (1)] $= q_j(\bar{x})$.

(3) If x_1 is fixed then $2x_1 = x_1 + \overline{x_1} = \{t_i(x_1), v\}$ [by (2)] $\in \{\mathrm{J}_i, v\}$, conversely $\{\mathrm{J}_i, v\}$ is fixed because $\overline{\{a_i, v\}} = U_v V_v(a_i) = U_{v^2, v}(a_i)$ [by Commuting Formula (FFII)] $= V_v(a_i) = \{a_i, v\}$. Similarly, $\{\overline{a_i}, v\} = V_v U_v(a_i) = V_v(a_i)$. □

10.2 Diagonal Isotopes

We will reduce general coordinatization theorems to strong ones by passing to a diagonal isotope where the connection becomes strong. First we have a result about general isotopes.

Diagonal Isotope Lemma 10.2.1 *Let* J *be a unital Jordan algebra. If* $J = J_2 \oplus J_1 \oplus J_0$ *is the Peirce decomposition relative to an idempotent* $e \in J$*, and* $u = u_2 + u_0$ *is a diagonal element with* u_i *invertible in* J_i*, then the diagonal isotope* $J^{(u)}$ *has new unit*

$$1^{(u)} = e_2^{(u)} + e_0^{(u)}$$

for new idempotents

$$e^{(u)} := e_2^{(u)} := u_2^{-1}, \qquad e_0^{(u)} := u_0^{-1}$$

(where the inverses are taken in J_i*), but the new Peirce decomposition in* $J^{(u)}$ *coincides exactly with the old one in* J:

$$J_k^{(u)}(e^{(u)}) = J_k(e) \quad (k = 2, 1, 0).$$

The new Peirce specializations and quadratic forms on the off-diagonal space J_1 *are given in terms of the old by*

$$\sigma_i^{(u)}(a_i) = \sigma_i(a_i)\sigma_i(u_i),$$
$$q_i^{(u)}(x_1) = q_i(x_1, u_j \bullet x_1).$$

PROOF. In $\widetilde{J} := J^{(u)}$ we denote the unit, square, and U-operator by $\tilde{1}, x^{\tilde{2}}, \widetilde{U}$. By the Jordan Homotope Proposition 7.2.1(2) and Invertible Products Proposition 6.1.8(3) we have $\tilde{1} := 1^{(u)} = u^{-1} = u_2^{-1} + u_0^{-1}$ [remember that these denote the inverses in the subalgebras J_i, not in J itself!!], where $\tilde{e} := e_2^{(u)} := u_2^{-1}$ satisfies $\tilde{e}^{\tilde{2}} = U_{u_2^{-1}}(u) = U_{u_2^{-1}}(u_2) = u_2^{-1} = \tilde{e}$. Then the complement $\tilde{e}' := \tilde{1} - \tilde{e} = u_0^{-1} = e_0^{(u)}$ is an idempotent orthogonal to $\tilde{e}$ in $\widetilde{J}$.

The Peirce decomposition in $\widetilde{J}$ has $\widetilde{J}_2 = \tilde{U}_{\tilde{e}}(\widetilde{J}) \subseteq U_{J_2}(J) \subseteq J_2$ by Diagonal Orthogonality; similarly, $\widetilde{J}_0 = \tilde{U}_{\tilde{e}'}(\widetilde{J}) \subseteq J_0$ and $\widetilde{J}_1 = \tilde{U}_{\tilde{e},\tilde{e}'}(\widetilde{J}) \subseteq U_{J_2,J_0}(J) \subseteq J_1$, so one decomposition sits inside the other $\widetilde{J} = \bigoplus \widetilde{J}_i(\tilde{e}) \subseteq \bigoplus J_i(e) = J$, which implies that they must coincide: $\widetilde{J}_i(\tilde{e}) = J_i(e)$ for each i.

The Peirce Quadratic forms 9.2.1 take the indicated form since $x^{\tilde{2}} = U_x(u)$ [by Jordan Homotope (1)] $= U_x(u_2) + U_x(u_0) = q_0(x, u_2 \bullet x) + q_2(x, u_0 \bullet x)$ [by the $U1q$-Rules 9.2.2(2)]. The Peirce specializations have $\widetilde{V}_{a_i} = V_{a_i,u_i}$ [by Jordan Homotope (2)] $= V_{a_i}V_{u_i}$ [by the Peirce Specialization Rules 9.1.1]. $\square$

The isotopes we seek are normalized so that $e_2^{(u)} = e, e_0^{(u)} = q_0(v_1)$ for an invertible element v_1. Amazingly, such a u will turn v_1 into an involution. (Talk about frogs and princes!)

Creating Involutions Proposition 10.2.2 (1) *Any invertible off-diagonal element can be turned into an involution by shifting to a suitable isotope: if $v_1 \in \mathrm{J}_1(e)$ is invertible in* J*, and we set $u := u_2 + u_0$ for $u_2 := e$, $u_0 := q_0(v_1)^{-1}$, then $v_1 \in \mathrm{J}_1(e^{(u)})$ becomes involutory in $\mathrm{J}^{(u)}$,*

$$v_1^{(2,u)} = 1^{(u)} = e_2^{(u)} + e_0^{(u)}$$

for shifted idempotents

$$e^{(u)} := e_2^{(u)} := e, \qquad e_0^{(u)} := q_0(v_1)$$

forming supplementary orthogonal idempotents in $\mathrm{J}^{(u)}$ with the same Peirce decomposition:

$$\mathrm{J}_k^{(u)}(e^{(u)}) = \mathrm{J}_k(e).$$

(2) *The corresponding* **shifted connection involution** $x \mapsto \overline{x}^{(u)} := U_{v_1}^{(u)}(x)$ *in the isotope $\mathrm{J}^{(u)}$ is given by*

$$\overline{a_2}^{(u)} = U_{v_1}(a_2), \qquad \overline{a_0}^{(u)} = U_{v_1}^{-1}(a_0), \qquad \overline{x_1}^{(u)} = U_{v_1}\{q_0(v_1)^{-1}, x_1\}.$$

(3) *The* **shifted Peirce specializations** *in the isotope take the form*

$$\sigma_2^{(u)}(a_2) = \sigma_2(a_2), \qquad \sigma_0^{(u)}(a_0) = \sigma_0(a_0)\sigma_0\big(q_0(v_1)\big)^{-1},$$
$$\sigma_2^{(u)}\big(\overline{q_0(v_1)^k}^{(u)}\big) = \sigma_2^{(u)}\big(U_{v_1}^{-1}q_0(v_1)^k\big) = \sigma_2\big(q_2(v_1)\big)^{k-1}.$$

(4) *The* **shifted Peirce quadratic forms** *take the form*

$$q_0^{(u)}(x_1) = q_0(x_1), \qquad q_2^{(u)}(x_1) = q_2(x_1, q_0(v_1)^{-1} \bullet x_1).$$

(5) *We have a* **Flipping Rule** *for the powers*

$$U_v^{\pm}(v_i^k) = v_j^{k\pm1} \qquad (v_i = q_i(v)).$$

PROOF. For convenience, set $q_i(v) = v_i$, so that $v^2 = v_2 + v_0$; then $v^{2k} = v_2^k + v_0^k$ for all k by orthogonality of $\mathrm{J}_2, \mathrm{J}_0$ [even for negative powers, by Direct Product Invertibility Criterion 6.1.8(3)], so the Flipping Rule (5) follows from $U_v^{\pm}(v_i^k) = E_j\big(U_v^{\pm}(v^{2k})\big) = E_j(v^{2k\pm2}) = v_j^{k\pm1}$. In particular, for $j = k = 0$ we have the alternate description $v_0^{-1} = E_0(v^{-2})$.

(1) By Diagonal Isotope Lemma 10.3.1 for $u = u_2 + u_0,\ u_2 := e,\ u_0 := v_0^{-1}$, we know that $\widetilde{\mathrm{J}} := \mathrm{J}^{(u)}$ has unit $\tilde{1} = \tilde{e}_2 + \tilde{e}_0$ for idempotents $\tilde{e}_2 := u_2^{-1} = e,\ \tilde{e}_0 := u_0^{-1} = v_0$ with the same Peirce spaces $\widetilde{\mathrm{J}}_k = \mathrm{J}_k$. In $\widetilde{\mathrm{J}}$ the emperor's new square is $v^{\tilde{2}} = U_v u$ [by Jordan Homotope 7.2.1] $= U_v\big(v_2^0 + v_0^{-1}\big) = v_0^1 + v_2^0$ [by Flipping (5) for $k = 0, -1$] $= v_0 + e = \tilde{1}$, so v has become an involution.

(2) Once v becomes involutory it is granted an involution $\bar{x} := \widetilde{U}_v(x) = U_v U_u(x)$. On J_2 we have $\overline{a_2} = U_v U_u(a_2) = U_v U_{u_2}(a_2) = U_v U_e(a_2) = U_v(a_2)$. On J_0 we have $\overline{a_0} = U_v U_u(a_0) = U_v U_{u_0}(a_0) = U_v U_{E_0(v^{-2})}(a_0)$ [as we noted

above] $= U_v U_{v^{-2}}(a_0)$ [since $E_2(v^{-2})$ can be ignored on J_0 by Peirce Orthogonality 8.2.1] $= U_v U_v^{-2}(a_0)$ [by the Fundamental Formula (FFI) and the U–Inverse Formula 6.1.3] $= U_v^{-1}(a_0)$. Finally, on J_1 we have $\overline{x_1} = U_v U_u(x_1) = U_v U_{e,u_0}(x_1) = U_v\{v_0^{-1}, x_1\}$. This establishes the shifted connection involution (2).

In (3), the first part follows immediately from Diagonal Isotope, $\sigma_2^{(u)}(a_2) = \sigma_2(a_2)\sigma_2(u_2) = \sigma_2(a_2)\sigma_2(e) = \sigma_2(a_2)$ [by unitality in Peirce Specialization 9.1.1] and $\sigma_0^{(u)}(a_0) = \sigma_0(a_0)\sigma_0(u_0) = \sigma_0(a_0)\sigma_0(v_0^{-1}) = \sigma_0(a_0)\big(\sigma_0(v_0)\big)^{-1}$ [since σ_0 is a homomorphism by Peirce Specialization]. The second part follows from applying the homomorphism $\sigma_2^{(u)} = \sigma_2$ to (5) [for $i = 0$].

(4) also follows immediately from the Diagonal Isotope Lemma, $q_0^{(u)}(x_1) = q_0(x_1, u_2 \bullet x_1) = q_0(x_1, e_2 \bullet x_1) = \frac{1}{2} q_0(x_1, x_1)$ [by the Peirce Eigenspace Law 8.1.4] $= q_0(x_1)$ and $q_2^{(u)}(x_1) = q_2(x_1, u_0 \bullet x_1) = q_2(x_1, v_0^{-1} \bullet x_1)$. □

This result will be a life-saver when we come to our coordinatization theorems: once the connecting element v becomes involutory, the bar involution can be used to simplify many calculations.

10.3 Problems for Chapter 10

Problem 10.1 Work out the details of Connection Involution Proposition 10.1.3 (condition for $x \in J_1$ to be involutory, connection action, connection fixed points) and the Creating Involutions Proposition 10.2.2 (shifted connection involution, Peirce specializations, and quadratic forms) relative to an idempotent $e = 1[11]$ in a hermitian matrix algebra $J = \mathcal{H}_2(D, -)$ for an associative algebra D with involution.

Problem 10.2 Repeat the above problem for the case of a reduced spin factor $\mathcal{R}ed\mathcal{S}pin(q)$.

11

Spin Coordinatization

In this chapter we give a general coordinatization theorem for reduced spin Jordan algebras, without assuming any simplicity. This is an example of a whole family of *coordinatization theorems*, asserting that an algebra will be of a certain form as soon as it has a family of elements of a certain kind. The granddaddy of all coordinatization theorems is the Wedderburn Coordinatization Theorem: if an associative algebra A has a supplementary family of $n \times n$ *associative matrix units* $\{e_{ij}\}$ $(1 \le i, j \le n)$, then it is a *matrix algebra*, $\mathrm{A} \cong \mathcal{M}_n(\mathrm{D})$ with coordinate ring $\mathrm{D} = e_{11}\mathrm{A}e_{11}$ under an isomorphism sending $e_{ij} \mapsto E_{ij}$. We will establish Jordan versions of this for $n \times n$ hermitian matrices ($n = 2$ in Chapter 12, $n \ge 3$ in Chapter 17).

Throughout the chapter we deal with a Peirce decomposition $\mathrm{J} = \mathrm{J}_2 \oplus \mathrm{J}_1 \oplus \mathrm{J}_0$ of a unital Jordan algebra J with respect to an idempotent e. Recall from Connection Involution 10.1.3 that e and $1 - e$ are *connected* if there is an invertible element in J_1, and *strongly connected* if there is an involutory element $v_1^2 = 1$ in J_1 (in which case we denote the *connection involution* simply by a bar, $\bar{x} := U_{v_1} x$). We will consistently use the notation $e_2 := e, e_0 := 1-e$, so our Peirce decomposition is always with respect to e_2. Elements of the diagonal Peirce spaces J_i $(i = 2, 0)$ will be denoted by letters a_i, b_i, while off-diagonal elements will be denoted by letters x_1, y_1, and we have Peirce Specializations 9.1.1 $\sigma_i(a_i) = V_{a_i}|_{\mathrm{J}_1}$ of the diagonal spaces on the off-diagonal space, and the Peirce Quadratic Forms 9.2.1 $q_i(x_1) = E_i(x_1^2) \in \mathrm{J}_i$.

11.1 Spin Frames

For both the spin and hermitian algebras of capacity 2, a family of 2×2 hermitian matrix units is just a 2-frame. Here *frame* suggests a frame for weaving, for hanging a pattern around, and *coordinatization* will consist in draping coordinates around a frame.

2-Frame Definition 11.1.1 *A* **2-frame** $\{e_2, v_1, e_0\}$ *for a unital Jordan algebra* J *consists of two supplementary orthogonal idempotents* e_2, e_0 *connected by an invertible element* $v_1 \in J_1$. *The frame is* **strong** *if* v_1 *strongly connects the idempotents,* $v_1^2 = e_2 + e_0 = 1$; *in this case the connection involution* $x \mapsto \overline{x} := U_{v_1}(x)$ *is, by the Connection Involution Proposition, an automorphism interchanging* e_2 *and* e_0.

We begin by investigating the Peirce relation that sets spin factors off from all others.

Spin Frame Definition 11.1.2 *A Jordan algebra* J *satisfies the* **Spin Peirce Relation** *with respect to a pair of supplementary idempotents* e_2, e_0 *if for all* x_1, y_1 *in* J_1

$$q_2(x_1) \bullet y_1 = q_0(x_1) \bullet y_1.$$

A **Spin frame** $\{e_2, v_1, e_0\}$ *is a 2-frame where* J *satisfies the Spin Peirce Relation with respect to* e_2, e_0.

Exercise 11.1.2 (1) Show that the Spin Peirce Relation is equivalent to the following identity for all $a_i \in J_i$, $x, y \in J_1$ (for $i = 2, 0$): $\{U_x(a_i), y\} = \{q_i(x), a_i, y\}$. (2) Show that it is also equivalent to the identity $\{U_x(a_i), y\} = \{a_i, q_i(x), y\}$.

The idea behind the Spin Peirce Relation is that the Peirce quadratic forms q_2, q_0 agree in their action on J_1, and represent the action of scalars (from some larger ring of scalars Ω) determined by an Ω-valued quadratic form q on J_1. The primary examples of Spin frames come from reduced spin factors; hermitian algebras produce Spin frames only if they are really reduced spin factors in disguise.

Spin Frame Example 11.1.3 *Every reduced spin factor* $\mathcal{R}ed\mathcal{S}pin(q) = \Omega \oplus M \oplus \Omega$, *built as in the Reduced Spin Example* 3.4.1 *from an* Ω*-module* M *with* Ω*-quadratic form* q *over a scalar extension* Ω *of* Φ *via*

$$(\alpha, w, \beta)^2 = \big(\alpha^2 + q(w), (\alpha + \beta)w, \beta^2 + q(w)\big)$$

satisfies the Spin Peirce Relation with respect to the supplementary idempotents $e_2 = (1, 0, 0), e_0 = (0, 0, 1)$, *since the Peirce quadratic forms*

$$q_i(w_1) = q(w)e_i \in \Omega e_i$$

are essentially scalars.

For any invertible $v_1 = (0, v, 0)$ *(i.e.,* $q(v)$ *invertible in* Ω*), we have a* **standard Spin frame** $\{e_2, v_1, e_0\}$; *such a frame is strong iff* $q(v) = 1$, *in which case the quadratic form is unit-valued and the connection involution is* $\overline{(\alpha, w, \beta)} = (\beta, q(w, v)v - w, \alpha)$.

PROOF. We saw in the Reduced Spin Decomposition 8.3.5 that the Peirce decomposition relative to $e = e_2$ is $J_i = \Phi e_i$ $(i = 2, 0)$, $J_1 = M$. The explicit multiplication formula shows that $q_i(w_1) = E_i(w_1^2) = q(w)e_i$, so $q_i(w_1) \bullet v_1 = \frac{1}{2}q(w)v_1$ is truly a scalar action, so $\{e_2, v_1, e_0\}$ is a spin frame, which is strong $(v_1^2 = 1)$ iff $q(v) = 1$, in which case $\overline{(\alpha, w, \beta)} = U_{v_1}(\alpha e_2 + w_1 + \beta e_0) = \alpha e_0 + \big(\{q_2(v_1, w_1), v_1\} - \{q_0(v_1), w_1\}\big) + \beta e_2$ [by the $U1q$ Rule 9.2.2(2)] $= \alpha e_0 + \big(q(v, w)v_1 - q(v)w_1\big) + \beta e_2 = (\beta, q(w, v)v - w, \alpha)$ [since $q(v) = 1$ by strongness]. □

Jordan Matrix Frame Example 11.1.4 *The hermitian matrix algebra* $\mathcal{H}_2(\mathrm{D}, -)$ *as in* 3.2.4 *for an associative* $*$*-algebra* D *satisfies the Spin Peirce Relation with respect to* $e_2 = 1[11]$, $e_0 = 1[22]$ *iff the involution* $-$ *on* D *is central, in which case the algebra may be considered as a reduced spin factor over its* $*$*-center* Ω*, with strong Spin frame* $\{1[11], 1[12], 1[22]\}$.

PROOF. By Basic Brace Products 3.2.4 we have $q_2(x[12]) = x\bar{x}[11]$, $q_0(x[12]) = \bar{x}x[22]$, $2\big(q_2(x[12]) - q_0(x[12])\big) \bullet y[12] = \big((x\bar{x})y - y(\bar{x}x)\big)[12]$. Therefore the Spin Peirce Relation holds iff all norms $x\bar{x} = \bar{x}x$ are central (taking first $y = 1$ and then y arbitrary), so as we noted in $*$-Algebra Definition 1.5.1, $\mathcal{H}(\mathrm{D}, -)$ reduces to the $*$-center Ω. Then $\mathcal{H}_2(\mathrm{D}, -) = \mathcal{H}(\mathrm{D}, -)[11] \oplus \mathrm{D}[12] \oplus \mathcal{H}(\mathrm{D}, -)[22] = \Omega e_2 \oplus M \oplus \Omega e_0 \cong \mathcal{R}ed\mathcal{S}pin(q)$ for $q(w) := w\bar{w}$ on $M := \mathrm{D}$. □

Twisted Matrix Frame Example 11.1.5 *A twisted matrix algebra* $\mathcal{H}_2(\mathrm{D}, \Gamma)$ *for* $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2) = \mathrm{diag}(1, \gamma)$ *as in* 7.5.3 [*normalized so that* $\gamma_1 = 1$] *which satisfies the Spin Peirce Relation with respect to* $e_2 := 1[11]_\Gamma = E_{11}$, $e_0 := \gamma^{-1}[22]_\Gamma = E_{22}$ *must have* Γ *central,* $*_\Gamma = *$*, so the algebra coincides with* $\mathcal{H}_2(\mathrm{D}, -)$ *and is a reduced spin factor over its* $*$*-center.*

PROOF. By the Twisted Matrix basic product rules 7.5.3 we have $q_2(x[12]_\Gamma) = x\gamma\bar{x}[11]_\Gamma$, $q_0(x[12]_\Gamma) = \bar{x}x[22]_\Gamma$, $2\big(q_2(x[12]_\Gamma) - q_0(x[12]_\Gamma)\big) \bullet y[12]_\Gamma = \big((x\gamma\bar{x})y - y\gamma(\bar{x}x)\big)[12]_\Gamma$, and the Spin Peirce Relation holds iff $x\gamma\bar{x} = \gamma\bar{x}x$ is central for all x; here $x = 1$ guarantees that γ is invertible in the center of D, hence Γ is central in $\mathcal{M}_2(\mathrm{D})$, and $*_\Gamma = *$. But then we are back to the untwisted case as above. □

EXERCISE 11.1.5 Go ahead, be a masochist — compute the Spin Peirce Relation directly in twisted matrix algebras $\mathcal{H}_2(\mathrm{D}, \Gamma)$ of 7.5.3 for general (un-normalized) $\Gamma = diag\{\gamma_1, \gamma_2\}$. (1) Show that $q_2(x[12]_\Gamma) = x\gamma_2\bar{x}[11]_\Gamma$, $q_0(x[12]_\Gamma) = \bar{x}\gamma_1 x[22]_\Gamma$, $\{q_2(x[12]_\Gamma) - q_0(x[12]_\Gamma), y[12]_\Gamma\} = \big((x\gamma_2\bar{x}\gamma_1)y - y(\gamma_2\bar{x}\gamma_1 x)\big)[12]_\Gamma$ vanishes iff $x\gamma_2\bar{x}\gamma_1 = \gamma_2\bar{x}\gamma_1 x$ is central for all x. (2) Linearize $x \mapsto x, 1$ in this to show that $\gamma_2 = \lambda\gamma_1^{-1}$ for some central λ, and the condition becomes that all $x\big(\gamma_1^{-1}\bar{x}\gamma_1\big) = \big(\gamma_1^{-1}\bar{x}\gamma_1\big)x$ be central. (3) Conclude that $\overline{}$ is an *isotope* $(\bar{x} = \gamma_1 x^* \gamma_1^{-1})$ of a central involution $*$.

11.2 Diagonal Spin Consequences

Now we gather up in three bite-sized lemmas the technical results about the action of the diagonal Peirce spaces which we will need to carry out our strong spin coordinatization.

Diagonal Commutativity Lemma 11.2.1 (1) *If* J *satisfies the Spin Peirce Relation with respect to a pair of supplementary orthogonal idempotents* e_2, e_0, *then*

$$\sigma_i(q_i(x_1, a_i \bullet x_1)) = \sigma_i(a_i)\sigma_i(q_i(x_1)) = \sigma_i(q_i(x_1))\sigma_i(a_i)$$

for all $a_i \in \mathrm{J}_i$, $x_1 \in \mathrm{J}_1$.

(2) *When* $\{e_2, v_1, e_0\}$ *is a Spin frame, then on* J_1 *we have* **Diagonal Commutativity**: $\Omega = \sigma_2(\mathrm{J}_2) = \sigma_0(\mathrm{J}_0)$ *is a commutative associative subalgebra of* $\mathcal{E}nd_\Phi(\mathrm{J}_1)$ *which is isomorphic to* J_i *via* σ_i.

PROOF. (1) The Spin Peirce Relation 11.1.2 just means that $V_{q_2(x_1)} = V_{q_0(x_1)}$ as operators on J_1, which therefore commute with all V_{a_i} on J_1 by Peirce Associativity 9.1.3, so $V_{a_i}V_{q_i(x)} = V_{q_i(x)}V_{a_i} = V_{a_i} \bullet V_{q_i(x)} = V_{a_i \bullet q_i(x)}$ [by linearizing Peirce Specialization Rules 9.1.1] $= V_{q_i(x, a_i \bullet x)}$ [by the q-Composition Rules 9.2.2(3)].

(2) Peirce Associativity shows that $\Omega_i := \sigma_i(\mathrm{J}_i)$ commutes with $\Omega_j := \sigma_j(\mathrm{J}_j) \supseteq \sigma_j\big(q_j(\mathrm{J}_1)\big) = \sigma_i\big(q_i(\mathrm{J}_1)\big)$ [by the Spin Relation] $\supseteq \sigma_i\big(q_i(v, \mathrm{J}_j \bullet v)\big) = \sigma_i\big(U_v(\mathrm{J}_j)\big)$ [by q-Composition Rules] $= \sigma_i\big(\mathrm{J}_i\big)$ [by invertibility of v] $= \Omega_i$, so $\Omega_2 = \Omega_0 =: \Omega$ is a commuting Φ-*module* of Φ-linear transformations. It is also a *unital Jordan* subalgebra closed under squares by Peirce Specialization Rules, hence *because* $\frac{1}{2}$ *lies in* Φ it is also an *associative* Φ-subalgebra of $\mathcal{E}nd_\Phi(\mathrm{J}_1)$: $\sigma(a_i)\sigma(b_i) = \sigma(a_i) \bullet \sigma(b_i) = \sigma(a_i \bullet b_i)$. Moreover, the surjection $\sigma_i : \mathrm{J}_i \to \Omega_i$ is also injective (hence $\mathrm{J}_i \cong \Omega_i^+ = \Omega$ as algebras) by the Peirce Injectivity Criterion 9.1.2(4) and the invertibility of v. This establishes (2). □

Diagonal Spin Isotope Lemma 11.2.2 *If* J *satisfies the Spin Peirce Relation with respect to* e_2, e_0, *then so does any diagonal isotope* $\mathrm{J}^{(u)}$ $(u = u_2 + u_0)$ *with respect to* $e_2^{(u)} := u_2^{-1}$, $e_0^{(u)} := u_0^{-1}$.

PROOF. In $\mathrm{J}^{(u)}$ we know by the Diagonal Isotope Lemma 10.2.1 that $e_2^{(u)}$ is an idempotent with the same Peirce decomposition as e_2, and we have $V^{(u)}_{q_j^{(u)}(x)} = V_{q_j(x, u_i \bullet x)}V_{u_j}$ [by Diagonal Isotope] $= V_{q_i(x, u_i \bullet x)}V_{u_j}$ [by the Spin Relation] $= V_{q_i(x)}V_{u_i}V_{u_j}$ [by the Diagonal Commutativity Lemma 11.2.1] $= V_{q_i(x)}U_{u_i, u_j}$ [by Peirce Associativity], which by the Spin Relation is symmetric in the indices i and j, so the Spin Relation holds for $e_2^{(u)}, e_0^{(u)}$. □

Spin Bar Lemma 11.2.3 *A strong 2-frame $\{e_2, v_1, e_0\}$ satisfies the Spin Peirce Relation iff the connection involution satisfies the* **Spin Bar Relation**:

(1) $$a_2 \bullet y_1 = \overline{a_2} \bullet y_1.$$

The Spin Bar Relation implies three further relations:

(2) *The Peirce form q_i is J_i-bilinear: $a_i \bullet q_i(x_1, y_1) = q_i(\{a_i, x_1\}, y_1)$;*

(3) *The involution on J_1 is isometric: $q_i(\overline{x_1}) = q_i(x_1)$;*

(4) *The involution exchanges q_2, q_0 : $\overline{q_i(x_1)} = q_j(x_1)$.*

PROOF. By the Connection Involution Lemma 10.1.3 we know that bar is an involution of the Jordan algebra which interchanges e_2, e_0, hence their diagonal Peirce spaces. The Peirce Spin Relation implies the Spin Bar Relation because $\overline{a_2} \bullet y = q_0(a_2 \bullet v, v) \bullet y$ [by the Connection Action 10.1.3(2)] $= q_2(a_2 \bullet v, v) \bullet y$ [by the Spin Relation] $= (a_2 \bullet q_2(v)) \bullet y$ [by q-Composition Rules 9.2.2(3)] $= a_2 \bullet y$ [by strongness $q_2(v) = e_2$].

To see that the weaker Spin Bar implies Spin Peirce will require the three relations (2)–(4), which are of independent interest (in other words, we'll need them in the next theorem!). For (2) we compute $a_i \bullet q_i(x, y) - q_i(\{a_i, x_1\}, y) = q_i(a_i \bullet x, y) + q_i(a_i \bullet y, x) - 2q_i(a_i \bullet x, y)$ [by linearized q-Composition Rules] $= q_i(a_i \bullet y, x) - q_i(a_i \bullet x, y) = q_i(\overline{a_i} \bullet y, x) - q_i(\overline{a_i} \bullet x, y)$ [by Spin Bar] $= 0$ [by $U1q$ Rules 9.2.2(2)]. For (3) we compute $q_i(\bar{x}) = q_i(\{t_i(x), v\} - x)$ [by Connection Action] $= U_{t_i(x)} q_i(v) - q_i\big(\{t_i(x), v\}, x\big) + q_i(x)$ [by q-Composition] $= t_i(x)^2 - t_i(x) \bullet q_i(v, x) + q_i(x)$ [by $q_i(v) = e_i$ and (2)] $= q_i(x)$ [by definition of t_i]. Relations (3) and (4) are equivalent by Connection Action. From this it is immediate that Spin Bar implies Spin Peirce: the bar relation implies that $q_2(x) \bullet y = \overline{q_2(x)} \bullet y = q_0(x) \bullet y$ [by (3)]. □

11.3 Strong Spin Coordinatization

Now we are ready to establish the main result of this section, that Jordan algebras with strong Spin frame can be coordinatized as strongly reduced spin factors.

Strong Spin Coordinatization Theorem 11.3.1 *A Jordan algebra has a strong Spin frame iff it is isomorphic to a reduced spin factor $\mathcal{R}ed\mathcal{S}pin(q)$ over a scalar extension Ω of Φ with standard Spin frame and a unit-valued quadratic form q. Indeed, if $\{e_2, v_1, e_0\}$ is a strong Spin frame, i.e.,*

$$v_1^2 = e_2 + e_0 = 1, \qquad q_2(x_1) \bullet y_1 = q_0(x_1) \bullet y_1,$$

for all $x_1, y_1 \in \mathrm{J}_1$, then the map

$$a_2 \oplus y_1 \oplus b_0 \mapsto (\sigma(a_2), y_1, \sigma(b_0)) = (\sigma(a_2), y_1,\ \sigma(\overline{b_0}))$$

is an isomorphism $\mathrm{J} \xrightarrow{\varphi} \mathcal{R}ed\mathcal{S}pin(q)$ where the coordinate map σ, quadratic form q, and scalar ring Ω are given by

$$\sigma(a_i) := \sigma(\overline{a_i}) := V_{a_i}|_{\mathrm{J}_1} \qquad (\bar{a} := U_{v_1} a),$$
$$q(x_1) := \sigma(q_2(x_1)) = \sigma(q_0(x_1)),$$
$$\Omega := \sigma(\mathrm{J}_2) = \sigma(\mathrm{J}_0) \subseteq \mathcal{E}nd_\Phi(\mathrm{J}_1).$$

PROOF. By Diagonal Commutativity 11.2.1(2), Ω is symmetric in the indices $2, 0$ and forms a commutative associative Φ-subalgebra of $\mathcal{E}nd_\Phi(\mathrm{J}_1)$. Therefore J_1 becomes a left Ω-module under $\omega x_1 = \omega(x_1)$. By the Spin Peirce Relation $\sigma(q_2(x_1)) = \sigma(q_0(x_1))$, so the (un-subscripted) quadratic form q is also symmetric in the indices $2, 0$, and is indeed quadratic over Ω since $q(x, y)$ is Ω-bilinear: it is Φ-bilinear and $q(\omega x, y) = \omega q(x, y)$ for all $x, y \in \mathrm{J}_1$, $\omega = \sigma(a_2) \in \Omega$ by the Spin Bar Lemma 11.2.3(2).

To show that φ is a homomorphism, it suffices to prove $\varphi(x^2) = \big(\varphi(x)\big)^2$ for all elements $x = a_2 \oplus y \oplus b_0$. But by the Peirce relations

$$x^2 = [a_2^2 + q_2(y)] \oplus [2(a_2 + b_0) \bullet y] \oplus [b_0^2 + q_0(y)]$$
$$= [a_2^2 + q_2(y)] \oplus [\sigma\big(a_2 + b_0\big)y] \oplus [b_0^2 + q_0(y)],$$

so

$$\varphi(x^2) = \Big(\sigma\big(a_2^2 + q_2(y)\big),\ \sigma\big(a_2 + b_0\big)y,\ \sigma\big(b_0^2 + q_0(y)\big)\Big).$$

On the other hand, by the rule for multiplication in $\mathcal{R}ed\mathcal{S}pin(q)$ given by the Reduced Spin Example 3.4.1, we have

$$\varphi(x)^2 = \big(\sigma(a_2),\ y,\ \sigma(b_0)\big)^2 = \big(\sigma(a_2)^2 + q(y),\ \sigma(a_2 + b_0)y,\ \sigma(b_0)^2 + q(y)\big),$$

so

$$\varphi(x^2) = \varphi(x)^2$$

follows from $\sigma(c_i^2) = \sigma(c_i)^2$ [by Peirce Specialization 9.1.1] and $\sigma(q_i(y)) = q(y)$ [by definition of q].

Finally, to show that φ is an isomorphism, it suffices to prove that it is a linear bijection on each Peirce space: by Diagonal Commutativity φ is a bijection σ_i of J_i onto Ω for $i = 2, 0$, and trivially φ is a bijection $\mathbb{1}_{\mathrm{J}_1}$ on J_1. We have seen, conversely, in Spin Frame Example 11.1.3 that every $\mathcal{R}ed\mathcal{S}pin(q)$

with unit-valued q has a standard strong Spin frame, so (up to isomorphism) these are precisely all Jordan algebras with such a frame. □

11.4 Spin Coordinatization

Coordinatization is most clearly described for strong Peirce frames. When the frames are not strong, we follow the trail blazed by Jacobson to a nearby isotope which is strongly connected. The strong coordinatization proceeds smoothly because the bar involution is an algebra automorphism and interacts nicely with products, and we have complete symmetry between the indices 2 and 0. For the non-strong case the map U_v is merely a structural transformation, transforming J-products into products in an isotope, and we do not have symmetry in $\mathrm{J}_2, \mathrm{J}_0$. Rather than carry out the calculations in the general case, we utter the magic word "diagonal isotope" to convert the general case into a strong case. Our isotope preserves the idempotent e_2 but shifts e_0, so we base our coordinatization on the space J_2. Notice that the main difference between the present case and the previous strong case is that, since our frame is not strong, we do not have a connection involution, and we do not know that the quadratic form q is *unit-valued* (takes on the value 1); all we know is that it is *invertible-valued* (takes on an invertible value at the connecting element v_1).

Spin Coordinatization Theorem 11.4.1 *A Jordan algebra has a Spin frame iff it is isomorphic to a reduced spin factor $\mathcal{R}ed\mathcal{S}pin(q)$ over a scalar extension Ω of Φ with standard Spin frame and an invertible-valued quadratic form q. Indeed, if $\{e_2, v_1, e_0\}$ is a Spin frame, i.e.,*

$$v_1^2 = v_2 + v_0 \quad \textit{for } v_i \textit{ invertible in } \mathrm{J}_i, \quad q_2(x_1) \bullet y_1 = q_0(x_1) \bullet y_1,$$

for all $x_1, y_1 \in \mathrm{J}_1$, then the map

$$a_2 \oplus y_1 \oplus b_0 \mapsto \big(\sigma(a_2), y_1, \mu\sigma\big(U_v^{-1}(b_0)\big)\big)$$

is an isomorphism $\mathrm{J} \xrightarrow{\varphi} \mathcal{R}ed\mathcal{S}pin(q)$ sending the given Spin frame to a standard Spin frame,

$$\{e_2, v_1, e_0\} \mapsto \{(1,0,0), (0, v_1, 0), (0,0,1)\},$$

where the coordinate map σ, quadratic form q, scalar μ, and scalar ring Ω are given by

$$\begin{aligned} \sigma(a_2) &:= V_{a_2}|_{\mathrm{J}_1}, \\ q(x_1) &:= \mu\sigma\big(U_v^{-1}\big(q_0(x_1)\big)\big), \\ \mu &:= \sigma(v_2) \in \Omega, \\ \Omega &:= \sigma(\mathrm{J}_2) \subseteq \mathcal{E}nd_\Phi(\mathrm{J}_1). \end{aligned}$$

PROOF. By the Creating Involutions Proposition 10.2.2(1), the diagonal isotope $\widetilde{\mathrm{J}} := \mathrm{J}^{(u)}$ for $u := e_2 + v_0^{-1}$ has the same Peirce decomposition as J and still satisfies the Spin Peirce Relation (since all diagonal isotopes do by Diagonal Spin Isotope Lemma 11.2.2), but now has a *strong* Spin frame $\{\tilde{e_2}, v_1, \tilde{e_0}\}$ $(\tilde{e_2} = e_2, \tilde{e_0} = v_0)$. Thus by the Strong Spin Coordinatization Theorem $\widetilde{\mathrm{J}} \cong \mathcal{R}ed\mathcal{S}pin(\tilde{q})$, and therefore $\mathrm{J} = (\widetilde{\mathrm{J}})^{(u^{-2})}$ for diagonal $u^{-2} = e_2 + v_0^2$ [by Isotope Symmetry (4) in the Jordan Homotope Proposition 7.2.1 and the Direct Product Invertibility Criterion 6.1.8(3)] is isomorphic to $\mathcal{R}ed\mathcal{S}pin(\tilde{q})^{(\tilde{u})}$ for $\tilde{u} = (1, 0, \mu)$, which in turn is by Quadratic Factor Isotope 7.3.1(2) a reduced spin Jordan algebra $\mathcal{R}ed\mathcal{S}pin(q)$ (where $q = \mu\tilde{q}$ still takes on an invertible value on Ω). This completes the proof that J is a reduced spin algebra.

If we want more detail about q and the form of the isomorphism, we must argue at greater length. By the Creating Involutions Proposition, $\widetilde{\mathrm{J}} = \mathrm{J}^{(u)}$ has supplementary orthogonal idempotents $\tilde{e_2} = e_2$, $\tilde{e_0} = v_0$ strongly connected by the same old v_1, and with the same old Peirce decomposition; by the Strong Spin Coordinatization Theorem 11.3.1 we have an explicit isomorphism $\widetilde{\mathrm{J}} \to \mathcal{R}ed\mathcal{S}pin(\tilde{q})$ by $\tilde{\varphi}\big(a_2 \oplus y \oplus b_0\big) = \big(\tilde{\sigma}(a_2), y, \tilde{\sigma}(b_0)\big)$. Here the coordinate map, scalar ring, and quadratic form in $\widetilde{\mathrm{J}}$ can be expressed in J as

$$\tilde{\sigma}(a_2) = \sigma(a_2), \qquad \tilde{\sigma}(b_0) = \sigma_2\big(U_v^{-1}(b_0)\big),$$
$$\tilde{\Omega} = \Omega, \qquad \tilde{q}(y) = \sigma_2\big(U_v^{-1}(q_0(y))\big).$$

Indeed, by Creating Involutions (2) we have the shifted connection involution $\overline{b_0} = U_v^{-1}(b_0)$ on $\widetilde{\mathrm{J}}_0$, and by Creating Involutions (3) we have shifted Peirce specialization $\tilde{\sigma}_2(a_2) = \sigma_2(a_2) = \sigma(a_2)$ [by definition], $\tilde{\sigma}_0(b_0) = \tilde{\sigma}_2(\overline{b_0}) = \sigma\big(U_v^{-1}(b_0)\big)$. From this, by Strong Coordinatization the scalar ring is $\tilde{\Omega} = \tilde{\sigma}_2(\widetilde{\mathrm{J}_2}) = \sigma_2(\mathrm{J}_2) = \Omega$ as we defined it. By Creating Involutions (4) we have shifted Peirce quadratic form $\tilde{q}_0 = q_0$, so by Strong Coordinatization the quadratic form $\tilde{q}$ is given by $\tilde{q}(y) = \tilde{\sigma}_0(\tilde{q}_0(y)) = \tilde{\sigma}_0(q_0(y)) = \sigma\big(U_v^{-1}(q_0(y))\big)$ [by the above]. Thus the map $\tilde{\varphi}$ reduces to

$$\tilde{\varphi}: \ a_2 \oplus y_1 \oplus b_0 \mapsto \Big(\sigma(a_2), y_1, \sigma\big(U_v^{-1}(b_0)\big)\Big).$$

Under this mapping $\tilde{\varphi}$ the original (weak) Spin frame is sent as follows in terms of $\mu := \sigma(v_2)$:

$$e_2 \mapsto (1,0,0),\ v \mapsto (0,v,0),\ e_0 \mapsto (0,0,\mu^{-1}),\ u^{-2} \mapsto (1,0,\mu).$$

Indeed, $\sigma(e_2) = 1$, and by Creating Involutions 10.2.2(3) we have flipping $\sigma\big(U_v^{-1}(v_0^k)\big) = \sigma(v_2^{k-1}) = \sigma(v_2)^{k-1}$ [since σ is a homomorphism] $= \mu^{k-1}$ [by definition of μ]. Thus for $k = 0$ we have $\tilde{\sigma}_0(e_0) = \mu^{-1}$, and for $k = 2$ we have $\tilde{\sigma}_0(v_0^2) = \mu$. Therefore the diagonal element which recovers J from $\widetilde{\mathrm{J}}$ is sent to $\tilde{\varphi}(u^{-2}) = (1, 0, \mu)$. The isomorphism $\tilde{\varphi}$ is at the same time an isomorphism

$$\mathrm{J} = \big(\mathrm{J}^{(u)}\big)^{(u^{-2})} = \widetilde{\mathrm{J}}^{(u^{-2})} \xrightarrow{\tilde{\varphi}} \mathcal{R}ed\mathcal{S}pin(\tilde{q})^{(\tilde{u})} \qquad (\tilde{u} = (1, 0, \mu))$$

[using Jordan Homotope (4) and our definitions]. By Quadratic Factor Isotopes 7.3.1(2) the map $(a, y, b) \mapsto (a, y, \mu b)$ is an isomorphism $\big(\mathcal{R}ed\mathcal{S}pin(\tilde{q})\big)^{(\tilde{u})} \to \mathcal{R}ed\mathcal{S}pin(\mu\tilde{q}) = \mathcal{R}ed\mathcal{S}pin(q)$ of Jordan algebras (recall our definition of q as $\mu\tilde{q}$); combining this with the isomorphism $\tilde{\varphi}$ above gives an isomorphism $\mathrm{J} \to \mathcal{R}ed\mathcal{S}pin(q)$ given explicitly as in (2) by $a_2 \oplus y_1 \oplus b_0 \mapsto \big(\sigma(a_2), y_1, \sigma\big(U_v^{-1}(b_0)\big)\big) \mapsto \big(\sigma(a_2), y_1, \mu\sigma\big(U_v^{-1}(b_0)\big)\big)$, sending e_2, v, e_0 to $(1, 0, 0), (0, v, 0)$, $(0, 0, 1)$.

We have seen, conversely, that by Spin Frame Example 11.1.3 every $\mathcal{R}ed\mathcal{S}pin(q)$ with invertible-valued q has a Spin frame, so these are again precisely all Jordan algebras with such a frame. □

EXERCISE 11.4.1* (1) Show that $V_{x^2}U_x^{-1} = U_{x,x^{-1}}$ whenever x is invertible in a unital Jordan algebra. (2) Show that $\mu\sigma_2(U_v^{-1}b_0) = \sigma_2(\mathcal{U}(b_0))$ for $\mathcal{U} = \frac{1}{2}U_{v,v^{-1}}, \mu = \sigma(E_2(v^2))$. (3) Show that the isomorphism and quadratic form in the Coordinatization Theorem can be expressed as $\varphi(a_2 \oplus y_1 \oplus b_0) = \big(\sigma_2(a_2), y_1, \sigma_2(\mathcal{U}(b_0))\big)$, $q(y_1) = \sigma(\mathcal{U}(q_0(y_1)))$.

11.5 Problems for Chapter 11

PROBLEM 11.1 Among the two choices for coordinatizing $\widetilde{\mathrm{J}}_0$ in the Strong Coordinatization Theorem 11.3.1, we used the second, moving J_0 over to J_2 (where $\tilde{\sigma} = \sigma$) via U_v^{-1} before coordinatizing it. This is our first example of asymmetry in the indices 2, 0. Show that we could also have used the direct form $\tilde{\sigma}(b_0) = \sigma(b_0)\sigma(v_0)^{-1}$ from Creating Involutions. (2) Among the three choices for the Peirce quadratic form, we used the expression relating the new q to the old q_0 through U_v^{-1}. Show that we also have a direct expression $\tilde{q}(y) = \tilde{\sigma}_2\big(\tilde{q}_2(y)\big) = \sigma\big(q_2(y, v_0^{-1} \bullet y)\big)$ in terms of the index 2, and a direct expression $\tilde{q}(y) = \tilde{\sigma}_0\big(\tilde{q}_0(y)\big) = \tilde{\sigma}_0\big(q_0(y)\big)\sigma(v_0)^{-1}$ in terms of the index 0.

12

Hermitian Coordinatization

In this chapter we give another general coordinatization theorem, this time for those 2×2 hermitian matrix algebras $\mathcal{H}_2(\mathrm{D}, \Gamma)$ where the associative coordinate algebra D is *symmetrically generated* in the sense that it is generated by its symmetric elements. These hermitian algebras will be separated off, not by an identity, but by the property that their off-diagonal Peirce space J_1 is a cyclic J_2-module generated by an invertible (connecting) element.

General hermitian algebras $\mathcal{H}_2(\mathrm{D}, \Gamma)$ need not have D symmetrically generated. For example, the quaternions $\mathbb{H}$ under the standard involution are not generated by $\mathcal{H}(\mathbb{H}) = \mathbb{R}1$. However, for the simple Jordan algebras we are interested in, the only non-symmetrically generated $\mathcal{H}_2(\mathrm{D}, \Gamma)$'s (where $\mathcal{H}(\mathrm{D}, -)$ is a division algebra *not* generating D) are those where D is a composition algebra, and these algebras are simultaneously spin factors. Our view is that it is reasonable to exclude them from the ranks of algebras of "truly hermitian type" and consign them to quadratic factor type.

12.1 Cyclic Frames

We begin by exploring the structural consequences that flow from the condition that there exist a cyclic 2-frame. As usual, throughout the chapter we fix a Peirce decomposition with respect to an idempotent e, and set $e_2 := e$, $e_0 := 1 - e$; i represents a "diagonal" index $i = 2, 0$, and $j = 2 - 1$ its complementary index; the Peirce specializations of J_i on J_1 are denoted by a subscriptless σ.

Cyclic Peirce Condition Definition 12.1.1 *A Jordan algebra* J *is said to satisfy the* **cyclic Peirce condition** *with respect to a pair of supplementary idempotents* e_2, e_0 *if the off-diagonal Peirce space* J_1 *has a* **cyclic Peirce generator** y_1 *as left* $\mathcal{D}$*-module,*

$$\mathrm{J}_1 = \mathcal{D}(y_1)$$

where the **coordinate algebra** $\mathcal{D}$ *is the subalgebra of* $\mathcal{E}nd_\Phi(\mathrm{J}_1)$ *generated by* $\sigma(\mathrm{J}_2)$. *A* **cyclic 2-frame** $\{e_2, v_1, e_0\}$ *for a unital Jordan algebra* J *consists of a pair of supplementary orthogonal idempotents* e_2, e_0 *and an* invertible *cyclic generator* $v_1 \in \mathrm{J}_1$. *As always, the frame is* **strong** *if* v_1 *strongly connects* e_2 *and* e_0,

$$v_1^2 = e_2 + e_0 = 1.$$

Exercise 12.1.1* Let $\mathcal{D}_i$ denote the subalgebra of $\mathcal{E}nd_\Phi(\mathrm{J}_1)$ generated by $\sigma(\mathrm{J}_i)$, and let v be any invertible element of J_1. (1) Show that $\mathcal{D}_2(v) = \mathcal{D}_0(v)$; conclude that $\{e_2, v, e_0\}$ is a cyclic frame iff $\mathcal{D}_0(v) = \mathrm{J}_1$, so cyclicity is symmetric in the indices 2, 0. (2) Show that $\mathcal{D}_2(v) = \mathcal{D}_2(v^{-1})$; conclude that $\{e_2, v^{-1}, e_0\}$ is a cyclic frame iff $\{e_2, v, e_0\}$ is cyclic.

Twisted Hermitian Cyclicity Example 12.1.2 *Let* J *be the twisted* 2×2 *hermitian algebra* $\mathcal{H}_2(\mathrm{A}, \Gamma)$ *of Twisted Matrix Example* 7.5.3 *for a unital associative* $*$*-algebra* A *and diagonal matrix* $\Gamma = \mathrm{diag}\{1, \gamma_2\}$. *Then* $e_2 = 1[11]_\Gamma = E_{11}$, $e_0 = \gamma_2^{-1}[22]_\Gamma = E_{22}$ *are supplementary orthogonal idempotents, with* $\mathrm{J}_1 = \mathrm{A}[12]_\Gamma$, $\mathrm{J}_2 = \mathcal{H}(\mathrm{A})[11]_\Gamma$. *Under the identification of* J_1 *with* A *and* J_2 *with* $\mathcal{H}(\mathrm{A})$, *the endomorphism* $\sigma_2(b[11]_\Gamma)$ *on* J_1 *corresponds to left multiplication* L_b *by* b *on* A,[1] *so* $\mathcal{D} \subseteq \mathcal{E}nd_\Phi(\mathrm{J}_1)$ *corresponds to all left multiplications* $L_\mathrm{B} \subseteq \mathcal{E}nd_\Phi(\mathrm{A})$ *by elements of the subalgebra* $\mathrm{B} = \langle \mathcal{H}(\mathrm{A})\rangle \subseteq \mathrm{A}$ *generated by the symmetric elements*:

$$\mathcal{D}(a[12]_\Gamma) = (\mathrm{B}a)[12]_\Gamma.$$

In this situation the element $a[12]_\Gamma$ *is a cyclic generator for the space* J_1 *iff*

(1) $a \in \mathrm{A}$ *is left-invertible*: $ba = 1$ *for some* $b \in \mathrm{A}$;

(2) A *is symmetrically generated*: $\mathcal{H}(\mathrm{A})$ *generates* A *as an associative algebra, i.e.,* $\mathrm{B} = \mathrm{A}$.

If a *is in fact invertible, then* $\{1[11]_\Gamma, a[12]_\Gamma, \gamma_2^{-1}[22]_\Gamma\}$ *is a cyclic* 2*-frame. Such a frame is strong iff* γ_2 *is the norm of an invertible element of* A:

(3) *strong if* $\gamma_2^{-1} = \bar{a}a$ *for some invertible* a.

In particular, when A *is symmetrically generated and* $\gamma_2 = 1$ *the standard* 2*-frame* $\{1[11], 1[12], 1[22]\}$ *is a strong cyclic* 2*-frame for* $\mathcal{H}_2(\mathrm{A})$. *But even though* $\mathrm{A} = \mathbb{R}$ *is* (*trivially*) *symmetrically generated under the* (*trivial*) *identity involution, and is cyclically generated by any nonzero* $a[12]_\Gamma$, *when we take* $\gamma_2 = -1$ *the twisted algebra* $\mathcal{H}_2(\mathrm{A}, \Gamma)$ *has no* strong *frame* $\{1[11]_\Gamma, a[12]_\Gamma, \gamma_2^{-1}[22]_\Gamma\}$ *because* $\gamma_2^{-1} = -1 \neq \bar{a}a = a^2$.

Proof. (1)–(2) Since by Basic Brace Products 7.5.3(2) we know that $\sigma(b[11]_\Gamma)(a[12]_\Gamma) = ba[12]_\Gamma = (L_b(a))[12]_\Gamma$, the generators $\sigma(b[11]_\Gamma)$ of $\mathcal{D}$ correspond to the left multiplications L_b, and $\mathcal{D}$ corresponds to L_B. Then $a[12]_\Gamma$ is a cyclic generator for J_1 iff $\mathrm{B}a[12]_\Gamma = \mathrm{A}[12]_\Gamma$, i.e., $\mathrm{B}a = \mathrm{A}$. This certainly

[1] Warning: $\sigma_0(b[22]_\Gamma)$ corresponds to right multiplication $R_{\gamma_2 b}$, not to R_b!

happens if a is left-invertible and $\mathrm{A} = \mathrm{B}$, since $ba = 1$ as in (1) implies that $\mathrm{B}a = \mathrm{A}a \supseteq \mathrm{A}ba = \mathrm{A}1 = \mathrm{A}$; conversely, $\mathrm{B}a = \mathrm{A}$ implies left-invertibility $ba = 1$ for some $b \in \mathrm{B}$, so $\mathrm{A} = \mathrm{B}a = \mathrm{B}a\bar{1} = \mathrm{B}a\bar{a}\bar{b} \subseteq \mathrm{B}\mathcal{H}\overline{\mathrm{B}} \subseteq \mathrm{B}$ and $\mathrm{B} = \mathrm{A}$.

(3) By Basic Brace Products $a[12]_\Gamma^2 = a\gamma_2\bar{a}[11]_\Gamma + \bar{a}a[22]_\Gamma$ equals $1 = 1[11]_\Gamma + \gamma_2^{-1}[22]_\Gamma$ iff $\bar{a}a = \gamma_2^{-1}$, $a\gamma_2\bar{a} = 1$; this forces a to have left and right inverses, therefore to be invertible, and conversely if a is invertible with $\bar{a}a = \gamma_2^{-1}$ then automatically $\bar{a}a\gamma_2 = 1$ implies $a\gamma_2\bar{a} = 1$ too. □

12.2 Diagonal Hermitian Consequences

The next several lemmas gather technical information about the coordinate algebras $\mathcal{D}$ of cyclic 2-frames. Throughout, let J be a unital Jordan algebra with a pair of supplementary orthogonal idempotents e_2, e_0.

Cyclic Diagonal Isotope Lemma 12.2.1 *Diagonal isotopes always inherit the cyclic Peirce condition: If $\{e_2, y_1, e_0\}$ is a cyclic frame for* J, *then any diagonal isotope* $\mathrm{J}^{(u)}$ $(u = u_2 + u_0)$ *has cyclic frame* $\{e_2^{(u)} = u_2^{-1},\ y_1^{(u)} = y_1,\ e_0^{(u)} = u_0^{-1}\}$.

PROOF. In any diagonal isotope $\mathrm{J}^{(u)}$ we know by the Diagonal Isotope Lemma 10.2.1 that $e_2^{(u)}, e_0^{(u)}$ are supplementary idempotents with the same Peirce decomposition as e_2, e_0 and we have $\mathcal{D}^{(u)} = \mathcal{D}$ as subalgebras of $\mathcal{E}nd_\Phi(\mathrm{J}_1)$ because $\sigma(a_2)^{(u)} = \sigma(a_2)\sigma(u_2) \subseteq \mathcal{D}$ implies that $\mathcal{D}^{(u)} \subseteq \mathcal{D}$, and conversely $\sigma(a_2) = (\sigma(a_2)\sigma(u_2))(\sigma(u_2^{-2})\sigma(u_2)) \subseteq \mathcal{D}^{(u)}$ implies that $\mathcal{D} \subseteq \mathcal{D}^{(u)}$. Thus $\mathrm{J}_1 = \mathcal{D}(y)$ is cyclic in J iff $\mathrm{J}_1^{(u)} = \mathcal{D}^{(u)}(y)$ is cyclic in $\mathrm{J}^{(u)}$. □

EXERCISE 12.2.1A* Let v be any invertible element of J_1. Show that for $i = 2, 0, j = 2 - i$ that (1) $U_v V_{a_i} U_v^{-1} = V_{U_v(a_i)} V_{u_j}$ on J_1 $(u_j := U_v^{-1}(e_i))$ (use the linearized Fundamental Formula), (2) $U_v \mathcal{D}_i U_v^{-1} = U_v^{-1}\mathcal{D}_i U_v = \mathcal{D}_j$. (3) Conclude anew (as in Exercise 12.1.1) that $\mathrm{J}_1 = \mathcal{D}_2(v) = \mathcal{D}_0(v)$ when $\{e_2, v_1, e_0\}$ is cyclic.

EXERCISE 12.2.1B* Alternately, establish the result of the previous exercise by showing that (1) $\{d_j \in \mathcal{D}_j \mid d_j(v) \in \mathcal{D}_i(v)\}$ is a subalgebra which contains $\sigma(\mathrm{J}_j) = \sigma(U_v \mathrm{J}_i)$, (2) $\mathcal{D}_j(v) \subseteq \mathcal{D}_i(v)$, hence (3) $\mathcal{D}_j(v) = \mathcal{D}_i(v)$.

Coordinate Map Lemma 12.2.2 *When $\{e_2, v_1, e_0\}$ is a strong cyclic 2-frame then for all elements $a_k \in \mathrm{J}_2$, $\sigma(a) = V_a|_{\mathrm{J}_1}$, with respect to the involution $\bar{a} = U_{v_1}(a)$ we have the following relations*:

(1) bar reversal: $\sigma(a_1)\cdots\sigma(a_n)(v_1) = \sigma(\overline{a_n})\cdots\sigma(\overline{a_1})(v_1)$;

(2) bi-cyclicity: $\mathrm{J}_1 = \mathcal{D}_2(v_1) = \mathcal{D}_0(v_1)$ *is cyclic as a left* J_2 *or* J_0*-module for $\mathcal{D}_i$ the subalgebra of $\mathcal{E}nd_\Phi(\mathrm{J}_1)$ generated by $\sigma(\mathrm{J}_i)$*;

(3) faithfulness: $d \in \mathcal{D}$ *is determined by its value on* v_1:

$$d(v_1) = 0 \implies d = 0;$$

(4) coordinate bijection: $\psi(d) := d(v_1)$ *defines a* Φ*-linear bijection*

$$\psi\colon \mathcal{D} \longrightarrow \mathrm{J}_1.$$

PROOF. By the Connection Involution Lemma 10.1.3 we know that bar is an involution of the Jordan algebra which interchanges e_2, e_0, hence their diagonal Peirce spaces. For bar reversal (1) we induct on n, $n = 0$ being vacuous, and for $n+1$ elements $a_i \in \mathrm{J}_2$ we have

$$\begin{aligned}
&\sigma(a_1)\sigma(a_2)\cdots\sigma(a_n)\sigma(a_{n+1})(v) \\
&\quad = \sigma(a_1)\sigma(a_2)\cdots\sigma(a_n)\sigma\big(\overline{a_{n+1}}\big)(v) && \text{[by Connection Fixed Point]} \\
&\quad = \sigma(\overline{a_{n+1}})\sigma(a_1)\sigma(a_2)\cdots\sigma(a_n)(v) && \text{[by Peirce Associativity, } \overline{a_{n+1}} \in \mathrm{J}_0\text{]} \\
&\quad = \sigma(\overline{a_{n+1}})\sigma(\overline{a_n})\cdots\sigma(\overline{a_2})\sigma(\overline{a_1})(v) && \text{[by the induction hypothesis].}
\end{aligned}$$

This establishes (1) by induction, and shows that $\mathcal{D}_2(v) = \mathcal{D}_0(v)$ as in bicyclicity (2). For faithfulness (3), $d(v) = 0 \implies d(\mathrm{J}_1) = d(\mathcal{D}_0(v))$ [by (2)] $=$ $\mathcal{D}_0(d(v))$ [by Peirce Associativity] $= 0$, hence $d = 0$ in $\mathcal{E}nd_\Phi(\mathrm{J}_1)$. For bijectivity (4), ψ is injective by (3) and surjective by (2) for $\mathcal{D} = \mathcal{D}_2$. $\square$

Reversal Involution Lemma 12.2.3 *When* $\{e_2, v_1, e_0\}$ *is a strong cyclic* 2*-frame, the coordinate algebra* $\mathcal{D}$ *carries a* **reversal involution** ρ (*written as a superscript*),

$$\big(\sigma(a_1)\sigma(a_2)\cdots\sigma(a_n)\big)^\rho := \sigma(a_n)\cdots\sigma(a_2)\sigma(a_1),$$

which interacts with the coordinate map ψ *by*:

(1) ψ intertwines ρ *and bar*: $d^\rho(v_1) = \overline{d(v_1)}$, $\psi(d^\rho) = \overline{\psi(d)}$;

(2) q–t reduction: $q_2(d(v_1), c(v_1)) = t_2(cd^\rho(v_1)) = t_2\big(\psi(cd^\rho)\big)$;

(3) trace-relation: $d + d^\rho = \sigma\big(t_2(d(v_1))\big) = \sigma\big(t_2(\psi(d))\big)$;

(4) norm relation: $dd^\rho = \sigma\big(q_2(d(v_1))\big) = \sigma\big(q_2(\psi(d))\big)$;

(5) hermitian relation: $\mathcal{H}(\mathcal{D}, \rho) = \sigma(\mathrm{J}_2)$.

PROOF. The invertibility of the coordinate map ψ in the Coordinate Map Lemma 12.2.2(4) allows us to pull back the connection involution on J_1 to an involutory linear transformation $\rho := \psi^{-1} \circ \overline{} \circ \psi$ on $\mathcal{D}$. Thus intertwining (1) holds by definition. We claim that ρ is just reversal on $\mathcal{D}$ since

$$\begin{aligned}
&\psi\Big(\big(\sigma(a_1)\sigma(a_2)\cdots\sigma(a_n)\big)^\rho\Big) \\
&\quad = \overline{\psi\big(\sigma(a_1)\sigma(a_2)\cdots\sigma(a_n)\big)} && \text{[by definition of } \rho\text{]} \\
&\quad = \overline{\sigma(a_1)\sigma(a_2)\cdots\sigma(a_n)(v)} && \text{[by definition of } \psi\text{]} \\
&\quad = \sigma\big(\overline{a_n}\big)\cdots\sigma\big(\overline{a_2}\big)\sigma\big(\overline{a_1}\big)(v) && \text{[by bar reversal 12.2.2(1)]} \\
&\quad = \sigma(a_n)\cdots\sigma(a_2)\sigma(a_1)(v) && \text{[bar is a Jordan algebra involution]} \\
&\quad = \psi\big(\sigma(a_n)\cdots\sigma(a_2)\sigma(a_1)\big) && \text{[by definition of } \psi\text{].}
\end{aligned}$$

Since the reversal mapping is always an algebra involution as soon as it is well-defined, we have a reversal involution ρ of the algebra $\mathcal{D}$.

The salient facts about the hermitian elements with respect to ρ now follow. The trace relation (3) follows easily: applying the invertible ψ, it suffices if the two sides agree on v, and $d(v)+d^\rho(v) = d(v)+\overline{d(v)}$ [by intertwining (1)] $= \sigma(t_2(d(v)))(v)$ [by Connection Fixed Point 10.1.3(3)]. The norm relation (4) follows for *monomials* $d = \sigma(a_{(1)})\sigma(a_{(2)})\cdots\sigma(a_{(n)})$ from

$$\begin{aligned}
&\sigma(q_2(d(v))) \\
&\quad = \sigma\big(q_2(\sigma(a_1)\sigma(a_2)\cdots\sigma(a_n)(v)\big) \\
&\quad = \sigma\big(U_{a_1}U_{a_2}\cdots U_{a_n}q_2(v)\big) && \text{[by repeated } q\text{-Composition 9.2.2(3)]} \\
&\quad = \sigma(a_1)\sigma(a_2)\cdots\sigma(a_n)1\sigma(a_n)\cdots\sigma(a_2)\sigma(a_1) \\
&\quad = dd^\rho && \text{[by repeated Peirce Specialization 9.1.1 and } \sigma(q_2(v)) = 1\text{].}
\end{aligned}$$

The quadratic norm relation (4) for *sums of monomials* will require the reduction (2) of q_2-values to t_2-values. It suffices to establish (2) for monomials $d = \sigma(a_{(1)})\sigma(a_{(2)})\cdots\sigma(a_{(n)})$, and here

$$\begin{aligned}
&q_2(d(v), c(v)) = q_2\big(\sigma(a_1)\sigma(a_2)\cdots\sigma(a_n)(v), c(v)\big) \\
&\quad = q_2\big(\sigma(\overline{a_n})\cdots\sigma(\overline{a_2})\sigma(\overline{a_1})(v), c(v)\big) && \text{[by bar reversal]} \\
&\quad = q_2\big(v, \sigma(\overline{a_1})\sigma(\overline{a_2})\cdots\sigma(\overline{a_n})c(v)\big) && \text{[by repeated } U1q \text{ 9.2.2(2)]} \\
&\quad = q_2\big(v, c\sigma(\overline{a_1})\sigma(\overline{a_2})\cdots\sigma(\overline{a_n})(v)\big) && \text{[by Peirce Associativity]} \\
&\quad = q_2\big(v, c\sigma(a_n)\cdots\sigma(a_2)\sigma(a_1)(v)\big) && \text{[by bar reversal]} \\
&\quad = q_2\big(v, cd^\rho(v)\big) && [\rho \text{ is reversal}] \\
&\quad = t_2\big(cd^\rho(v)\big) && \text{[by definition of the trace].}
\end{aligned}$$

Once we have the q–t relation (2) we can finish the norm relation (4): for mixed terms we have $\sigma(q_2(d(v), c(v))) = \sigma\big(t_2(cd^\rho(v))\big)$ [by (2)] $= cd^\rho + (cd^\rho)^\rho$ [by (3)] $= cd^\rho + dc^\rho$ [ρ is an involution]. Finally, for the hermitian relation (5) we have $d^\rho = d \Longrightarrow d = \frac{1}{2}[d + d^\rho] = \frac{1}{2}\sigma(t_2(d(v))) \in \sigma(\mathrm{J}_2)$ by the trace relation (3), and the converse is clear by the definition of reversal. □

12.3 Strong Hermitian Coordinatization

Now we are ready to establish the main result of this section, that Jordan algebras with strong hermitian frame can be coordinatized as hermitian algebras.

Strong 2×2 **Hermitian Coordinatization Theorem** 12.3.1 *A Jordan algebra* J *has a strong cyclic* 2*-frame* $\{e_2, v_1, e_0\}$,

$$v_1^2 = e_2 + e_0 = 1, \quad \mathrm{J}_1 = \mathcal{D}(v_1),$$

iff it is isomorphic to a hermitian 2×2 *matrix algebra with standard frame and symmetrically generated coordinate algebra. Indeed, if* J *has strong cyclic* 2*-frame then the map*

$$\varphi : a_2 \oplus d(v_1) \oplus b_0 \mapsto \sigma(a_2)[11] + d[12] + \sigma(\overline{b_0})[22]$$

is an isomorphism $\mathrm{J} \xrightarrow{\varphi} \mathcal{H}_2(\mathcal{D}, \rho)$ *sending the given cyclic* 2*-frame* $\{e_2, v_1, e_0\}$ *to the standard cyclic* 2*-frame* $\{1[11], 1[12], 1[22]\}$, *where the coordinate map* σ, *the coordinate involution* ρ, *and the coordinate algebra* $\mathcal{D}$ *are defined by*

$$\begin{aligned}\sigma(a_2) &:= V_{a_2}|_{\mathrm{J}_1},\\ d^\rho(v) &:= U_{v_1}(d(v_1^{-1})),\\ \mathcal{D} &:= \textit{the subalgebra of } \mathcal{E}nd_\Phi(\mathrm{J}_1) \textit{ generated by } \sigma(\mathrm{J}_2).\end{aligned}$$

PROOF. By coordinate bijectivity 12.2.2(4), every $x \in \mathrm{J}$ can be written *uniquely* as $x = a_2 \oplus d(v) \oplus b_0$. To show that our map $\varphi(x) = \sigma(a_2)[11] + d[12] + \sigma(\overline{b_0})[22] = \begin{pmatrix} \sigma(a_2) & d \\ d^\rho & \sigma(\overline{b_0}) \end{pmatrix}$ is a homomorphism, it suffices to prove that $\varphi(x^2) = \varphi(x)^2$. But by the Peirce Brace Rules 8.2.1, Peirce quadratic forms 9.2.1, Peirce associativity 9.1.3, and bar reversal 12.2.2(1) we have

$$\begin{aligned}x^2 &= [a_2^2 + q_2(d(v))] \oplus [2(a_2 + b_0) \bullet d(v)] \oplus [b_0^2 + q_0(d(v))]\\ &= [a_2^2 + q_2(d(v))] \oplus [\sigma(a_2)d(v) + d\sigma(b_0)(v)] \oplus [b_0^2 + q_0(d(v))]\\ &= [a_2^2 + q_2(d(v))] \oplus [\sigma(a_2)d + d\sigma(\overline{b_0})](v) \oplus [b_0^2 + q_0(d(v))],\end{aligned}$$

so

$$\begin{aligned}\varphi(x^2) = \sigma\big(a_2^2 + q_2(d(v))\big)[11] + \big(\sigma(a_2)d + d\sigma(\overline{b_0})\big)[12]\\ +\sigma\big(\overline{b_0^2 + q_0(d(v))}\big)[22].\end{aligned}$$

On the other hand, by the Basic Brace Products in the Hermitian Matrix Example 3.2.4, for $\mathcal{H}_2(\mathcal{D}, \rho)$ we have

$$\begin{aligned}\varphi(x)^2 &= \big(\sigma(a_2)[11] + d[12] + \sigma(\overline{b_0})[22]\big)^2\\ &= \big(\sigma(a_2)^2 + dd^\rho\big)[11] + \big(\sigma(a_2)d + d\sigma(\overline{b_0})\big)[12] + \big(\sigma(\overline{b_0})^2 + d^\rho d\big)[22],\end{aligned}$$

Comparing these two, $\varphi(x^2) = \varphi(x)^2$ follows from the diagonal facts that $\sigma(a_2^2) = \sigma(a_2)^2$ and $\sigma(\overline{b_0^2}) = \sigma((\overline{b_0})^2) = \sigma(\overline{b_0})^2$ [by the fact that bar is an automorphism and by Peirce Specialization Rules 9.1.1], and from the off-diagonal squaring facts that $\sigma(q_2(d(v))) = dd^\rho$ [by the norm relation (4) in the Reversal Involution Lemma 12.2.3] and $\sigma(\overline{q_0(d(v))}) = \sigma(q_2(\overline{d(v)}))$ [by Connection Action 10.1.3(2)] $= \sigma(q_2(d^\rho(v)))$ [by intertwining (1) of ρ and bar in Reversal Involution] $= d^\rho d$ [by the norm relation (4) again].

To show that φ is an isomorphism it remains only to prove that it is a linear bijection on each Peirce space. By Reversal Involution and Peirce Injectivity 9.1.2(1), φ is a bijection σ of J_2 on $\mathcal{H}_2(\mathcal{D},\rho)[11]$ and a bijection $\sigma\circ\overline{}$ of J_0 on $\mathcal{H}_2(\mathcal{D},\rho)[22]$, and by coordinate bijection in the Coordinate Map Lemma 12.2.2(4), φ is a bijection $\psi^{-1}:\mathrm{J}_1\longrightarrow\mathcal{D}\longrightarrow\mathcal{D}[12]$.

Finally, φ clearly sends $e_2\mapsto 1[11]$ [since $\sigma(e_2)=1$], $e_0\mapsto 1[22]$ [since $\sigma(\overline{e_0})=\sigma(e_2)=1$], and $v=1(v)\mapsto 1[12]$.

We have seen, conversely, by Twisted Hermitian Cyclicity Example 12.1.2 that every $\mathcal{H}_2(\mathrm{A})$ for symmetrically generated A has a strong cyclic 2-frame, so these are precisely all Jordan algebras with such a frame. □

12.4 Hermitian Coordinatization

Coordinatization is most clearly described for strong frames. When the frames are not strong, we again follow the trail blazed by Jacobson to a nearby strongly connected isotope, muttering the magic word "diagonal isotope" to convert the general case into a strong case.

2×2 **Hermitian Coordinatization** 12.4.1 *A Jordan algebra* J *has a cyclic* 2*-frame* $\{e_2,v_1,e_0\}$,

$$v_1^2=v_2+v_0 \text{ for } v_i \text{ invertible in } \mathrm{J}_i,\ \mathrm{J}_1=\mathcal{D}(v_1),$$

iff it is isomorphic to a twisted hermitian 2×2 *matrix algebra with standard frame and symmetrically generated coordinate algebra. Indeed, if* J *has a cyclic* 2*-frame then the map*

$$\varphi: a_2\oplus d(v_1)\oplus b_0\mapsto \sigma(a_2)[11]_\Gamma+d[12]_\Gamma+\sigma(\overline{b_0})[22]_\Gamma$$
$$(\Gamma:=\mathrm{diag}\{1,\mu\},\quad \mu:=\sigma(v_2))$$

is an isomorphism $\mathrm{J}\longrightarrow\mathcal{H}_2(\mathcal{D},\Gamma)$ *sending the given cyclic* 2*-frame* $\{e_2,v_1,e_0\}$ *to the standard cyclic frame* $\{1[11]_\Gamma,1[12]_\Gamma,\mu^{-1}[22]_\Gamma\}$*, where the coordinate map* σ*, coordinate involution* ρ*, and coordinate algebra* $\mathcal{D}$ *are defined by*

$$\sigma(a_2):=V_{a_2}|_{\mathrm{J}_1},$$
$$d^\rho(v):=U_{v_1}(d(v_1^{-1})),$$
$$\mathcal{D}:=\text{the subalgebra of } \mathcal{E}nd_\Phi(\mathrm{J}_1) \text{ generated by } \sigma(\mathrm{J}_2).$$

PROOF. The argument follows the same path as that of Spin Coordinatization 11.4.1. By the Diagonal Isotope Lemma 10.2.1, the diagonal isotope $\widetilde{\mathrm{J}}:=\mathrm{J}^{(u)}$ for $u=e_2+v_0^{-1}$ has the same Peirce decomposition as J and still satisfies the cyclic Peirce condition (since all diagonal isotopes do by the Cyclic Diagonal Isotopes Lemma 12.2.1), but is now *strongly* connected, so by the Strong Hermitian Coordinatization Theorem $\widetilde{\mathrm{J}}\cong\mathcal{H}_2(\widetilde{\mathcal{D}},\tilde{\rho})$. Then

$\mathrm{J} = \big(\widetilde{\mathrm{J}}\big)^{(u^{-2})}$ [by Isotope Symmetry (4) in the Jordan Homotope Proposition 7.2.1, for diagonal $u^{-2} = e_2 + v_0^2$] $\cong \mathcal{H}_2(\tilde{\mathcal{D}}, \tilde{\rho})^{(\Gamma)} \cong \mathcal{H}_2(\mathcal{D}, \Gamma)$ [by Twisted Matrix Example 7.5.3] is a twisted hermitian matrix algebra. This completes the proof that J is twisted hermitian.

If we want more detail about the form of the isomorphism, as in Spin Coordinatization we must argue at greater length. $\widetilde{\mathrm{J}} = \mathrm{J}^{(u)}$ has, by Creating Involutions 10.2.2(1), supplementary orthogonal idempotents $\tilde{e}_2 = e_2$, $\tilde{e}_0 = v_0$ strongly connected by the same old v_1, and with the same old Peirce decomposition; by the Strong Coordinatization Theorem 12.3.1 we have an explicit isomorphism $\widetilde{\mathrm{J}} \to \mathcal{H}_2(\tilde{\mathcal{D}}, \tilde{\rho})$ by $\tilde{\varphi}(a_2 \oplus \tilde{d}(v) \oplus b_0) = \tilde{\sigma}(a_2)[11] + \tilde{d}[12] + \tilde{\sigma}(\overline{b_0})[22]$. Here the coordinate map, coordinate ring, connection involution, and coordinate involution in $\widetilde{\mathrm{J}}$ can be expressed in J as

$$\tilde{\sigma} = \sigma, \qquad \tilde{\mathcal{D}} = \mathcal{D}, \qquad \overline{b_2} = U_v(b_2), \qquad \overline{b_0} = U_v^{-1}(b_0), \qquad \tilde{\rho} = \rho.$$

Indeed, by Strong Coordinatization the coordinate map is $\tilde{\sigma}(a_2) = \sigma_2(a_2)$ [by the Diagonal Isotope Lemma] $= \sigma(a_2)$ [by definition of σ], so by Strong Coordinatization the coordinate ring is $\tilde{\mathcal{D}} = \langle \tilde{\sigma}(\widetilde{\mathrm{J}}_2)\rangle = \langle \sigma(\mathrm{J}_2)\rangle = \mathcal{D}$ as above. By Creating Involutions (1)–(2) the connection involution has $\overline{b_2} = U_v(b_2)$, $\overline{b_0} = U_v^{-1}(b_0)$, $\overline{x_1} = 2U_v(v_0^{-1} \bullet x_1)$. Then by Strong Coordinatization $d^{\tilde{\rho}}(v) = \overline{d(v)} = 2U_v(v_0^{-1} \bullet d(v)) = 2U_v(d(v_0^{-1} \bullet v))$ [by Peirce Associativity 9.1.3] $= U_v(d(v^{-1})) = d^{\rho}(v)$ as above, because of the general relation

$$2q_0(v_1)^{-1} \bullet v_1 = v_1^{-1}$$

resulting from canceling U_v from the equation $2U_v(v_0^{-1} \bullet v) = U_v V_v(v_0^{-1})$ $= U_{v^2,v} v_0^{-1} = \{v_2 + v_0, v_0^{-1}, v\} = \{v_0, v_0^{-1}, v\}$ [by Peirce Orthogonality Rules 8.2.1] $= v$ [by Peirce Specialization Rules 9.1.1] $= U_v(v^{-1})$.

By Strong Coordinatization the map $\tilde{\varphi}$ reduces to

$$\tilde{\varphi}:\ a_2 \oplus d(v) \oplus b_0 \ \mapsto\ \sigma(a_2)[11] + d[12] + \sigma\big(U_v^{-1}(b_0)\big)[22].$$

Under the mapping $\tilde{\varphi}$ the Spin frame is sent as follows:

$$e_2 \mapsto 1[11],\ v \mapsto 1[12],\ e_0 \mapsto \mu^{-1}[22],\ u^{-2} \mapsto \mathrm{diag}\{1, \mu\} = \Gamma\ (\mu := \sigma(v_2))$$

since $\sigma(e_2) = 1$, and by Creating Involutions (3) we have $\sigma\big(U_v^{-1}(v_0^k)\big) = \mu^{k-1}$, so for $k = 0, 2$ we have $\sigma\big(U_v^{-1}(e_0)\big) = \mu^{-1}$, $\sigma\big(U_v^{-1}(v_0^2)\big) = \mu$, hence $\tilde{\varphi}(u^{-2}) = \tilde{\varphi}(e_2 + v_0^2) = 1[11] + \mu[22] = \mathrm{diag}\{1, \mu\}$.

The isomorphism $\tilde{\varphi}$ is at the same time an isomorphism

$$\tilde{\varphi} : \mathrm{J} = \big(\mathrm{J}^{(u)}\big)^{(u^{-2})} = \widetilde{\mathrm{J}}^{(u^{-2})} \longrightarrow \mathcal{H}_2(\tilde{\mathcal{D}}, \tilde{\rho})^{(\Gamma)} = \mathcal{H}_2(\mathcal{D}, \rho)^{(\Gamma)}$$

[using Isotope Symmetry (4) in Jordan Homotope] of J with a diagonal isotope of $\mathcal{H}_2(\mathcal{D}, \rho)$. By the Twisted Matrix Example with $\gamma_2 = \mu$, the map $L_\Gamma : a[11] + d[12] + b[22] \longrightarrow a[11]_\Gamma + d[12]_\Gamma + b[22]_\Gamma$ is an isomorphism

$\mathcal{H}_2(\mathcal{D},\rho)^{(\Gamma)} \longrightarrow \mathcal{H}_2(\mathcal{D},\Gamma)$ of Jordan algebras; combining this with the isomorphism $\tilde{\varphi}$ gives an isomorphism $\mathrm{J} \longrightarrow \mathcal{H}_2(\mathcal{D},\Gamma)$ given explicitly as in (2) by $a_2 \oplus d(v) \oplus b_0 \mapsto \sigma(a_2)[11] + d[12] + \sigma\big(U_v^{-1}b_0\big)[22] \mapsto \sigma(a_2)[11]_\Gamma + d[12]_\Gamma + \sigma(U_v^{-1}b_0)[22]_\Gamma$, sending e_2, v, e_0 to $1[11]_\Gamma, 1[12]_\Gamma, \mu^{-1}[22]_\Gamma$.

We have seen, conversely, by Twisted Hermitian Cyclicity 12.1.2 that every $\mathcal{H}_2(\mathrm{A},\Gamma)$ for which A is symmetrically generated has a cyclic 2-frame, so these are again precisely *all* Jordan algebras with such a frame. $\square$

Third Phase: Three's a Crowd

Ashes to ashes, dust to dust
Peirce decompositions are simply a must
–Old Jordan folksong, c. 40 B.Z.E.

In this phase we will investigate the structure of Jordan algebras having an orthogonal family of three or more connected supplementary idempotents. These five chapters are the heart of the classical approach, Peirce decompositions and coordinatization via hermitian matrix units. It is a very nuts-and-bolts approach: we reach deep inside the Jordan algebra, take out and examine the small Peirce parts of which it is composed, and then see how they fit together to create the living organism, the Jordan algebra.

In Chapter 13 we develop the necessary machinery for multiple Peirce decompositions with respect to a finite family of mutually orthogonal idempotents, featuring a larger cast of Peirce Multiplication Rules: Four Brace Product, Three U-Product, one Triple Product, and Three Orthogonality Rules, and a Peirce Identity Principle. We give an easy proof that the Albert algebra is exceptional, exhibiting a simple s-identity which it fails to satisfy.

In Chapter 14 we gather consequences of Peirce decompositions which follow easily from the Peirce Identity Principle, or from the two-idempotent case. Key technical tools are again Peirce specializations, Peirce quadratic forms, and connection involutions. Connection and strong connection of orthogonal idempotents are transitive, so if e_1 is connected to each e_j we obtain connectors for all pairs e_i, e_j. Moreover, a family of off-diagonal elements v_{1j} connecting e_1 to e_j becomes strongly connecting in some diagonal isotope.

Jordan algebras with Peirce frames of length $n \geq 3$ come in only one basic flavor: the hermitian algebras of $n \times n$ hermitian matrices with entries from a $*$-algebra. This uniform homogeneous structure is due to the rich group of hermitian symmetries $\mathcal{U}_\pi$ for all permutations π (automorphisms permuting idempotents and Peirce spaces in the same way that π permutes the n indices). This is the focus of Chapter 15. Strongly connecting v_{1j} can be completed to a family of hermitian matrix units $\{h_{ij}\}$, yielding hermitian involutions $\mathcal{U}_{(ij)}$ corresponding to the transpositions, which then generate all other hermitian symmetries $\mathcal{U}_\pi$.

In Chapter 16 we show that, once $n \geq 3$, the space J_{12} can be endowed with a $*$-algebra structure $(\mathrm{D}, -)$ which coordinatizes all the off-diagonal Peirce spaces, while $\mathcal{H}(\mathrm{D}, -)$ coordinatizes all the diagonal Peirce spaces.

In Chapter 17 we establish our main result in this phase, the Jacobson Coordinatization Theorem, which asserts that Jordan algebras with hermitian n-frames for $n \geq 3$ (but without any further nondegeneracy hypotheses) are automatically hermitian matrix algebras $\mathcal{H}_n(\mathrm{D}, \Gamma)$, where the Jordan Coordinates Theorem shows that D must be alternative with nuclear involution if $n = 3$ and associative if $n \geq 4$. As usual, we establish this for strongly connected Peirce frames, then deduce the result for general Peirce frames via the magic wand of isotopy.

13

Multiple Peirce Decompositions

The time has come, the walrus said, to talk of Peirce decompositions with respect to more than one idempotent. By now the reader has gained some familiarity with Peirce decompositions with respect to a single idempotent, and will not be surprised or overwhelmed when digesting the general case. This time we will immediately apply Peircers to obtain the basic facts of Peirce decompositions: as the World War I song says, “How are you going to keep them down on the farm, after they’ve seen Peircers?” Not only do Peircers show *that* Peirce decomposition works, they show *why* it works, as an instance of toral actions which play important roles in many areas of mathematics.[1]

13.1 Decomposition

Throughout this chapter we will fix an **orthogonal family** of idempotents $\mathcal{E} = \{e_1, \ldots, e_n\}$ in a Jordan algebra J, a set of **mutually orthogonal** idempotents,

$$e_i^2 = e_i, \quad e_i \bullet e_j = 0 \qquad (i \neq j).$$

These are often called *pairwise orthogonal* idempotents, since orthogonality takes place between idempotents two at a time. We will always supplement these to form a *supplementary* set of $n+1$ orthogonal idempotents in the unital hull $\widehat{\mathrm{J}}$ by adjoining

$$e_0 := \hat{1} - (\textstyle\sum_{i=1}^n e_i) \in \widehat{\mathrm{J}}.$$

Peirce Projections Definition 13.1.1 *The* **Peirce projections** $E_{ij} = E_{ji}$ *in* J *determined by an orthogonal family of idempotents* $\mathcal{E}$ *are the linear transformations on* $\mathrm{J} \lhd \widehat{\mathrm{J}}$ *given for* $i, j = 0, 1, \ldots, n$ *by*

$$E_{ii} := U_{e_i}, \qquad E_{ij} := U_{e_i, e_j} = E_{ji} \qquad (i \neq j).$$

[1] Multiple Peirce decompositions and the associated Multiplication Rules were mentioned in I.6.1.

Notice that for indices $i, j \neq 0$ the Peirce projection E_{ij} is intrinsically determined on J *by e_i and e_j themselves; it is only for index $i = 0$ that E_{0j} depends on the full collection $\mathcal{E}$ of idempotents in order to form $e_0 = \hat{1} - \sum_{\mathcal{E}} e_i$.*

Peircer Definition 13.1.2 *For any $(n+1)$-tuple of scalars $\mathbf{t} = (t_0, t_1, \dots, t_n) \in \Phi^{n+1}$ we define the* **Peircer** *$E(\mathbf{t})$ to be the linear transformation on $\mathrm{J} \triangleleft \widehat{\mathrm{J}}$ given by the U-operator of the element $e(\mathbf{t}) = \sum_{i=0}^{n} t_i e_i \in \widehat{\mathrm{J}}$,*

$$E(\mathbf{t}) := U_{e(\mathbf{t})}|_{\mathrm{J}} = \sum_{0 \leq i \leq j \leq n} t_i t_j E_{ij}.$$

It is very important that the Peircer is the U-operator determined by an element, since this guarantees (by the Fundamental Formula) that it is a structural transformation on J.

Peircer Torus Proposition 13.1.3 *The* **Peircer torus** *of operators on* J *determined by $\mathcal{E}$ is the strict homomorphism $\mathbf{t} \mapsto E(\mathbf{t})$ of multiplicative monoids $\Phi^{n+1} \to \mathcal{E}nd_{\Phi}(\mathrm{J})$: the* **Peircer Toral Property**

$$E(\mathbf{1}) = \sum_{0 \leq i \leq j \leq n} E_{ij} = \mathbb{1}_{\mathrm{J}}, \qquad E(\mathbf{st}) = E(\mathbf{s})E(\mathbf{t}),$$

holds strictly *in the sense that it continues to hold over all scalar extensions Ω, i.e., holds as operators on J_{Ω} for any $(n+1)$-tuples $\mathbf{s} = (s_0, s_1, \dots, s_n)$, $\mathbf{t} = (t_0, t_1, \dots, t_n) \in \Omega^{n+1}$.*

PROOF. While it was easy to establish $U_{e(\mathbf{s})} U_{e(\mathbf{t})} = U_{e(\mathbf{st})}$ for a single idempotent by power-associativity, for more than one idempotent we cannot apply power-associativity or Macdonald directly, and must digress through standard, but lengthy, arguments about scalar extensions and localizations in order to justify the Toral Property. (We sketch this *sotto voce* in Section 6.)

The most elementary route to show that $e(\mathbf{s}), e(\mathbf{t})$ lead an associative life together uses only properties of Peirce decompositions relative to a single idempotent. If the multiplication operators $E(\mathbf{t})$ are toral on the unital hull, their restrictions will be toral on the original algebra. Because of this, it suffices to work in the unital hull, so *we again assume from the start that $\mathrm{J} = \widehat{\mathrm{J}}$ is unital, and we can treat the idempotent e_0 and the index 0 like all the others.*

The E_{ij} are clearly supplementary *operators* since the e_i are supplementary *elements*: $\sum_i E_{ii} + \sum_{i<j} E_{ij} = \sum_i U_{e_i} + \sum_{i<j} U_{e_i, e_j} = U_{\sum_i e_i} = U_{e(\mathbf{1})} = U_1 = \mathbb{1}_{\mathrm{J}}$ by the basic property of the unit element. We have

$$E(\mathbf{s})E(\mathbf{t}) - E(\mathbf{s}\,\mathbf{t}) = \sum_{i \leq j\,;\, k \leq \ell} (s_i s_j)(t_k t_\ell)(E_{ij} E_{k\ell} - \delta_{ik}\delta_{j\ell} E_{ij}),$$

so the Toral Property holds strictly (equivalently, holds for all indeterminates in $\Phi[S, T]$) iff the Peirce projections E_{ij} satisfy $E_{ij}E_{k\ell} - \delta_{ik}\delta_{j\ell}E_{ij} = 0$, i.e., form a family of *mutually orthogonal projections*: $E_{ij}^2 = E_{ij}$, $E_{ij}E_{kl} = 0$ if $\{i, j\} \neq \{k, \ell\}$. We now establish this projection property.

If $\{i, j\}$ does not contain $\{k, \ell\}$, set $e = e_i + e_j$ (or $e = e_i$ if $i = j$), so $e_i, e_j \in \mathrm{J}_2(e)$ and one of $e_k, e_\ell \in \mathrm{J}_0(e)$ [since at least one of k, ℓ doesn't appear in $\{i,j\}$]; then orthogonality $E_{ij}E_{k\ell} = 0$ follows from $U_{J_2}U_{J_0,J}J \subseteq U_{J_2}(J_0 + J_1) = 0$ from Peirce Product Rules and Orthogonality 8.2.1.

Dually, if $\{i,j\}$ is not contained in $\{k,\ell\}$ then $e_k, e_\ell \in J_2(e)$ and one of $e_i, e_j \in J_0(e)$ for $e = e_k + e_\ell$ (or $e = e_k$ if $k = \ell$); then orthogonality $E_{ij}E_{k\ell} = 0$ follows from $U_{J_0,J}U_{J_2}J \subseteq U_{J_0,J}J_2 = 0$.

When $\{i,j\} = \{k,\ell\}$ we have $E_{ij}E_{k\ell} = E_{ij}E_{ij} = E_{ij}\sum_{r,s} E_{rs}$ [by the above orthogonality of distinct projections] $= E_{ij}\mathbb{1}_J = E_{ij}$. □

Recall that one virtue of the Peircers is that they clearly reveal *why* the Peirce projections are a supplementary family of mutually orthogonal projections; unfortunately, our elementary proof required us to *prove* this directly instead of *harvesting* it as a consequence.

Exercise 13.1.3A* In any associative algebra with $\frac{1}{2}$, show that $E^2 = E, EA + AE = 0$ imply $EA = AE = 0$. If $\frac{1}{2} \notin \Phi$, show that $E^2 = E, EA + AE = 0, EAE = 0$ imply $EA = AE = 0$. Use this to derive the orthogonality of the Peirce projections directly from the Jordan identity $U_{x^2} = U_x^2$ and its linearizations: if $e, f, g, h \perp$ are orthogonal idempotents show that (1) $U_e^2 = U_e$, (2) $U_{e,f}^2 = U_{e,f}$, (3) $U_eU_f = 0$, (4) $U_eU_{e,f} = U_{e,f}U_e = 0$, (5) $U_eU_{f,g} = U_{f,g}U_e = 0$, (6) $U_{e,f}U_{e,g} = 0$, (7) $U_{e,f}U_{g,h} = 0$. Give a different direct proof of (2) using the general identity $U_{x,y}^2 = U_{x^2,y^2} + V_{x,y}V_{y,x} - V_{U_xy^2}$.

Exercise 13.1.3B* A slightly less elegant method than the Peircer Toral Property for establishing the orthogonality of Peirce projections uses a non-controversial application of Macdonald's Principle, $U_xU_{x^3} = U_{x^4}$ for $x = e(\mathbf{t}) \in J[T]$, but to recover the individual E_{ij} without the help of the independent $s_i \in S$ requires a more delicate combinatorial identification of coefficients of various powers of the t_i. (1) Verify that no two distinct pairs $\{i,j\}, \{k,\ell\}$ can give rise to the same product $t_it_jt_k^3t_\ell^3$. (2) Use this to show that identifying coefficients of $t_i^4t_j^4$ in $(\sum_{i\le j} t_it_jE_{ij})(\sum_{k\le\ell} t_k^3t_\ell^3E_{k\ell}) = \sum_{p\le q} t_p^4t_q^4E_{pq}$ yields $E_{ij}^2 = E_{ij}$, and that identifying coefficients of $t_it_jt_k^3t_\ell^3$ yields $E_{ij}E_{k\ell} = 0$ if $\{i,j\} \neq \{k,\ell\}$.

Either way, elegant or not, we have our crucial decomposition of the identity operator and hence the module.

Peirce Decomposition Theorem 13.1.4 (1) *The Peirce projections with respect to an orthogonal family of idempotents form a supplementary family of projection operators on* J,

$$\mathbb{1}_J = \sum\nolimits_{0\le i\le j\le n} E_{ij}, \qquad E_{ij}E_{k\ell} = \delta_{i,k}\delta_{j,\ell}E_{ij},$$

and therefore the space J *breaks up as the direct sum of the ranges*: *we have the* **Peirce Decomposition** *of* J *into* **Peirce subspaces**,

$$J = \bigoplus\nolimits_{i\le j} J_{ij} \quad \textit{for} \quad J_{ij} = J_{ji} := E_{ij}(J).$$

(2) *Peirce decompositions are inherited by ideals or by subalgebras containing* $\mathcal{E}$: *we have* **Peirce Inheritance**,

$$K = \bigoplus\nolimits_{i\le j} K_{ij} \ \textit{for} \ K_{ij} = E_{ij}(K) = K \cap J_{ij} \quad (K \triangleleft J \ \textit{or} \ \mathcal{E} \subseteq K \le J).$$

PROOF. (1) We have already decomposed the identity operator into supplementary orthogonal projections, and this always leads to a decomposition of the underlying Φ-submodule into the direct sum of the Peirce subspaces $\mathrm{J}_{ij} = E_{ij}(\mathrm{J})$.

(2) For Peirce Inheritance, just as in 8.1.2(2) the Peirce projections E_{ij} are by definition multiplications by 1 and the e_i, so they map into itself any ideal or any subalgebra containing $\mathcal{E}$, therefore induce by restriction a decomposition $\mathrm{K} = \bigoplus_{i\le j} \mathrm{K}_{ij}$ for $\mathrm{K}_{ij} = E_{ij}(\mathrm{K})$; clearly this is contained in $\mathrm{K} \cap \mathrm{J}_{ij}$, and conversely if $x \in \mathrm{K} \cap \mathrm{J}_{ij}$ then $x = E_{ij}(x) \in E_{ij}(\mathrm{K})$. □

We will usually denote the Peirce projections by E_{ij}; if there were any danger of confusion (which there won't be), then we would use the more explicit notation $E_{ij}(\mathcal{E})$ to indicate which family of idempotents gives rise to the Peirce decomposition.

Peirce Eigenspace Laws 13.1.5 *The Peirce subspace* J_{ij} *is the intersection of* J *with the eigenspace with eigenvalue* $t_i t_j$ *of the indeterminate Peircer* $E(\mathbf{t})$ $(\mathbf{t} = (t_0, t_1, \ldots, t_n)$, *on* $\mathrm{J}[T]$ *for independent indeterminates* $t_i \in \Phi[T])$, *and also the common eigenspaces of the* V*-operators* V_{e_i}, V_{e_j} *on* J *with eigenvalue* 1 (*or* 2 *if* $i = j$): *we have the* **Peircer-** *and* V**-Eigenspace Laws**:

$$\begin{aligned}
\mathrm{J}_{ij} &= \{x \in \mathrm{J} \mid E(\mathbf{t})x = t_i t_j x\},\\
\mathrm{J}_{ii} &= \mathrm{J}_2(e_i) = \{x \in \mathrm{J} \mid V_{e_i}(x) = 2x\},\\
\mathrm{J}_{ij} &= \mathrm{J}_1(e_i) \cap \mathrm{J}_1(e_j) = \{x \in \mathrm{J} \mid V_{e_i}(x) = V_{e_j}(x) = x\} \quad (i \ne j),
\end{aligned}$$

and all other V_{e_k} $(k \ne i, j)$ *have eigenvalue* 0 *on* $\mathrm{J}_{ii}, \mathrm{J}_{ij}$:

$$\begin{aligned}
\mathrm{J}_{ii} &= \mathrm{J}_2(e_i) \cap \Big(\bigcap_{k\ne i} \mathrm{J}_0(e_k)\Big)\\
&= \{x \in \mathrm{J} \mid V_{e_i}(x) = 2x,\ V_{e_k}(x) = 0\ (k \ne i)\},\\
\mathrm{J}_{ij} &= \mathrm{J}_1(e_i) \cap \mathrm{J}_1(e_j) \cap \Big(\bigcap_{k\ne i,j} \mathrm{J}_0(e_k)\Big)\\
&= \{x \in \mathrm{J} \mid V_{e_i}(x) = V_{e_j}(x) = x,\ V_{e_k}(x) = 0 \text{ for } k \ne i, j\}.
\end{aligned}$$

Thus on a Peirce space either some idempotent's V*-operator has eigenvalue* 2 (*and all the rest have eigenvalue* 0), *or else two separate idempotents'* V*-operators have eigenvalue* 1 (*and all the rest have eigenvalue* 0). *We have exactly the same result for the* L*'s, scaled by a factor* $\frac{1}{2}$.

PROOF. As for the one-idempotent Peircer Eigenspace Law 8.1.4, the multiple Peircer Eigenspace Law follows from the definition of the Peircer $E(\mathbf{t}) = \sum t_i t_j E_{ij}$, since the "eigenvalues" $t_i t_j$ are by construction independent. Similarly, the V-Eigenspace Laws follow because $V_{e_i} = U_{e_i,1} = U_{e_i,\sum_j e_j} = U_{e_i,e_i} + \sum_{j\ne i} U_{e_i,e_j} = 2E_{ii} + \sum_{j\ne i} E_{ij}$ where the eigenvalues $2, 1, 0$ are distinct when $\frac{1}{2} \in \Phi$. □

13.2 Recovery

We will frequently need to focus on a single idempotent e_i out of the family $\mathcal{E}$, and it will be useful to recognize the Peirce decomposition relative to e_i inside the general Peirce decomposition.

Peirce Recovery Theorem 13.2.1 (1) *The Peirce decomposition of* J *relative to a single idempotent* e_i *in the family can be recovered as*

$$\mathrm{J}_2(e_i) = \mathrm{J}_{ii}, \qquad \mathrm{J}_1(e_i) = \bigoplus_{j \neq i} \mathrm{J}_{ij}, \qquad \mathrm{J}_0(e_i) = \bigoplus_{j,k \neq i} \mathrm{J}_{jk},$$

so the Peirce $2, 1,$ *or* 0 *space relative to* e_i *is the sum of the Peirce spaces where* $2, 1,$ *or* 0 *of the indices equals* i.

(2) *More generally, the Peirce decomposition relative to any sub-sum* $e_I = \sum_{i \in I} e_i$ *for a subset* $I \subseteq \{0, 1, \ldots, n\}$ *is given by*

$$\mathrm{J}_2(e_I) = \sum_{i \leq j \in I} \mathrm{J}_{ij}, \quad \mathrm{J}_1(e_I) = \sum_{i \in I, k \notin I} \mathrm{J}_{ik}, \quad \mathrm{J}_0(e_I) = \sum_{k \leq \ell \notin I} \mathrm{J}_{k\ell},$$

so again the Peirce $2, 1,$ *or* 0 *space relative to* e_I *is the sum of all Peirce spaces* J_{ij} *where* $2, 1,$ *or* 0 *of the indices fall in* I.

PROOF. It suffices to establish the general case (2), and here by the Peirce projection definitions 8.2, 13.1 we have

$$E_2(e_I) = U_{e_I} = U_{\sum_{i \in I} e_i} = \sum_{i \in I} U_{e_i} + \sum_{i < j \in I} U_{e_i, e_j} = \sum_{i \leq j \in I} E_{ij},$$

$$E_1(e_I) = U_{e_I, 1-e_I} = U_{\sum_{i \in I} e_i, \sum_{k \notin I} e_k} = \sum_{i \in I, k \notin I} U_{e_i, e_k} = \sum_{i \in I, k \notin I} E_{ik},$$

$$E_0(e_I) = U_{1-e_I} = U_{\sum_{k \notin I} e_k} = \sum_{k \leq l \notin I} E_{k\ell}. \qquad \square$$

13.3 Multiplication

The whole rationale for Peirce decomposition is that it breaks an indigestible algebra down into bite-sized Peirce pieces, which have simpler multiplication rules. As in the one-idempotent case, the underlying philosophy is that Peirce spaces behave like the Φ-submodules $\mathrm{D}E_{ij} \subseteq \mathcal{M}_n(\mathrm{D})$ and multiply like the matrix units E_{ij} themselves. We must again keep in mind that in the Jordan case the off-diagonal spaces $\mathrm{D}E_{ij}, \mathrm{D}E_{ji}$ cannot be separated, they are lumped into a single space $\mathrm{J}_{ij} = \mathrm{D}[ij] = \mathrm{J}_{ji}$. This symmetry in indices is important to keep in mind when considering "linked" or "connected" indices.

We have the following crucial rules for multiplying multiple Peirce spaces, generalizing Peirce Multiplication Rules for Peirce spaces relative to a single idempotent e (which is now subsumed under the general case by taking $e_1 = e, e_0 = 1-e$). Again the Peircer is effective because it satisfies the Fundamental Formula.

Peirce Multiplication Theorem 13.3.1 *The Peirce spaces multiply according to the following rules.*

(1) *For bilinear products we have* **Four Peirce Brace Rules** *for* distinct *indices* i, j, k:

$$\mathrm{J}_{ii}^2 \subseteq \mathrm{J}_{ii}, \quad \mathrm{J}_{ij}^2 \subseteq \mathrm{J}_{ii} + \mathrm{J}_{jj}, \quad \{\mathrm{J}_{ii}, \mathrm{J}_{ij}\} \subseteq \mathrm{J}_{ij}, \quad \{\mathrm{J}_{ij}, \mathrm{J}_{jk}\} \subseteq \mathrm{J}_{ik}.$$

(2) *For distinct indices* i, j *we have* **Three Peirce** U**-Product Rules**

$$U_{\mathrm{J}_{ii}}(\mathrm{J}_{ii}) \subseteq \mathrm{J}_{ii}, \quad U_{\mathrm{J}_{ij}}(\mathrm{J}_{ii}) \subseteq \mathrm{J}_{jj}, \quad U_{\mathrm{J}_{ij}}(\mathrm{J}_{ij}) \subseteq \mathrm{J}_{ij},$$

and for arbitrary i, j, k, ℓ *we have the* **Peirce Triple Product Rule**

$$\{\mathrm{J}_{ij}, \mathrm{J}_{jk}, \mathrm{J}_{k\ell}\} \subseteq \mathrm{J}_{i\ell}.$$

(3) *We have* **Peirce Brace**, U-, *and* **Triple Orthogonality Rules**

$$\{\mathrm{J}_{ij}, \mathrm{J}_{k\ell}\} = 0 \text{ if the indices don't link, } \{i, j\} \cap \{k, \ell\} = \emptyset,$$

$$U_{\mathrm{J}_{ij}}(\mathrm{J}_{k\ell}) = 0 \text{ if } \{k, l\} \not\subseteq \{i, j\},$$

$$\{\mathrm{J}_{ij}, \mathrm{J}_{k\ell}, \mathrm{J}_{rs}\} = 0 \text{ if the indices cannot be linked}$$

(*for possible linkages keep in mind that* $\mathrm{J}_{rs} = \mathrm{J}_{sr}$). *In particular, the diagonal spaces* J_{ii} *are inner ideals.*

PROOF. Just as in the single-idempotent case 8.2.1, for the multiplication rules it is crucial that we have scalars t_i^{-1} to work with, so we pass to the formal *Laurent polynomials* $\mathrm{J}[T, T^{-1}]$ in $T = \{t_0, t_1, \ldots, t_n\}$ (consisting of all *finite* sums $\sum_{-N}^{M} x_{k_0,k_1,\ldots,k_n} t_0^{k_0} t_1^{k_1} \cdots t_n^{k_n}$ with coefficients $x_{\mathbf{k}}$ from J), with the natural operations. This is an algebra over the *Laurent polynomials* $\Omega := \Phi[T, T^{-1}]$ (consisting of all finite sums $\sum_{-N}^{M} \alpha_{k_0,k_1,\ldots,k_n} t_0^{k_0} t_1^{k_1} \cdots t_n^{k_n}$ with coefficients $\alpha_{\vec{k}}$ from Φ), with the natural operations. In particular, Ω is a free Φ-module with basis of all monomials $t_0^{k_0} t_1^{k_1} \cdots t_n^{k_n}$ ($k_i \in \mathbb{Z}$) over Φ, so J is imbedded in $\mathrm{J}_\Omega \cong \mathrm{J}[T, T^{-1}]$.

Since by the Peircer Definition 13.1.2 the Peircer is a U-operator $E(\mathbf{t}) = U_{e(\mathbf{t})}$ restricted to J, and $E(\mathbf{t})$ now has an inverse $E(\mathbf{t}^{-1})$ since the Toral Property 13.1.3 continues [by strictness] to hold in J_Ω, we have for the U-rules $E(\mathbf{t})\big(U_{x_{ij}}(y_{k\ell})\big) = E(\mathbf{t})U_{x_{ij}}E(\mathbf{t})E(\mathbf{t}^{-1})(y_{k\ell})$ [by the Toral Property] $= U_{E(\mathbf{t})(x_{ij})}E(\mathbf{t}^{-1})(y_{k\ell})$ [by the Fundamental Formula] $= U_{t_i t_j x_{ij}} t_k^{-1} t_\ell^{-1}(y_{k\ell})$ [by the Peircer Eigenspace Law 13.1.5] $= t_i^2 t_j^2 t_k^{-1} t_\ell^{-1} U_{x_{ij}}(y_{k\ell})$, so the product is zero as in (3) unless $t_i^2 t_j^2 t_k^{-1} t_\ell^{-1} = t_r t_s$ for some r, s, i.e.,

$$\{k, \ell\} \subseteq \{i, j\}.$$

If k and ℓ coincide, then by symmetry we may assume that $k = \ell = i$, in which case $t_r t_s = t_j^2$ and $U_{x_{ij}} y_{ii}$ lies in the eigenspace J_{jj} as in the first (if $i = j$) or second (if $i \neq j$) part of (2). If k and ℓ are distinct, then so are i, j, and by symmetry we may assume that $k = i, \ell = j$, $t_r t_s = t_i t_j$, so $U_{x_{ij}} y_{ij}$ lies in the eigenspace J_{ij} as in the third part of (2).

Similarly for the trilinear products:

$$E(\mathbf{t}\,)\{x_{ij}, y_{k\ell}, z_{rs}\} = E(\mathbf{t}\,)U_{x_{ij},z_{rs}} E(\mathbf{t}\,)E(\mathbf{t}^{\,-1})(y_{k\ell})$$

$$= U_{E(\mathbf{t}\,)(x_{ij}),E(\mathbf{t}\,)(z_{rs})} E(\mathbf{t}^{\,-1})(y_{k\ell}), \qquad \text{[by Toral and Fundamental again]}$$

$$= \{t_i t_j x_{ij},\ t_k^{-1} t_\ell^{-1} y_{k\ell},\ t_r t_s z_{rs}\} \qquad \text{[by the Peircer Eigenspace Law again]}$$

$$= t_i t_j t_k^{-1} t_\ell^{-1} t_r t_s\ \{x_{ij}, y_{k\ell}, z_{rs}\}$$

is zero as in (3) unless the indices can be linked $j = k, \ell = r$ so $t_i t_j t_k^{-1} t_\ell^{-1} t_r t_s = t_i t_s$, in which case the product falls in the eigenspace J_{is} as in (2). [Note that the middle indices k, ℓ can't both be linked only to the same side, for example the left, since if $k = j, \ell = i$ but k, ℓ are not linked to the right ($k, \ell \neq r, s$) then $x_{ij}, y_{k\ell} = y_{ji} \in J_2(e_I)$ ($I = \{i, j\}$) but $z_{rs} \in J_0(e_I)$, so $\{x_{ij}, y_{k\ell}, z_{rs}\} \in \{J_2, J_2, J_0\} = 0$ by Peirce Orthogonality 8.2.1.]

The triple products come uniformly because the Peircer is a U-operator and satisfies the Fundamental Formula. There is no corresponding uniform derivation for the brace products (1), instead we derive them individually from the triple products: from (2)–(3) we see that $x_{ii}^2 = U_{x_{ii}} 1 = U_{x_{ii}}(\sum e_k) = U_{x_{ii}} e_i \in J_{ii}$; $x_{ij}^2 = U_{x_{ij}} 1 = U_{x_{ij}}(\sum e_k) = U_{x_{ij}}(e_i + e_j) \in J_{jj} + J_{ii}$; $\{x_{ij}, y_{k\ell}\} = \{x_{ij}, 1, y_{k\ell}\} = \sum \{x_{ij}, e_p, y_{k\ell}\}$ is 0 as in (3) unless $\{i, j\} \cap \{k, \ell\} \neq \emptyset$ contains a linking index p, and if (by symmetry) $p = j = k$ then $\{x_{ij}, y_{k\ell}\} = \{x_{ij}, e_j, y_{j\ell}\} \in \{J_{ij}, J_{jj}, J_{j\ell}\} \subseteq J_{i\ell}$ as in the third [if $i = j \neq \ell$ or $i \neq j = \ell$] or fourth [if i, j, ℓ are distinct] part of (1). □

As in so many instances in mathematics, the picture is clearest if we step back and take a broad view: to understand radii of convergence of real power series we need to step back to the complex domain, so to understand the Peirce decomposition in J it is best to step back to $J[T, T^{-1}]$, where the J_{ij} appear as eigenspaces of one particular operator relative to distinct eigenvalues $t_i t_j$, and their triple products are governed by the condition that the Peircer is a U-operator satisfying the Fundamental Formula.

13.4 The Matrix Archetype

We now give examples of idempotents and their Peirce decompositions. The guiding light for Peirce decompositions is the example set by the associative matrix algebras, the generalization of Associative, Full, and Hermitian Peirce Decomposition 8.3.1, 8.3.2, 8.3.3.

Full and Hermitian Peirce Decomposition 13.4.1 (1) *If* $\mathcal{E} = \{e_1, \dots, e_n\}$ *is a family of orthogonal idempotents in an associative algebra* A*, we have the* **associative Peirce decomposition**

$$\mathrm{A} = \bigoplus_{i,j=0}^{n} \mathrm{A}_{ij} \qquad (\mathrm{A}_{ij} := e_i \mathrm{A} e_j)$$

relative to $\mathcal{E}$*. These satisfy the* **Associative Peirce Multiplication Rules**

$$\mathrm{A}_{ij}\mathrm{A}_{k\ell} \quad \subseteq \quad \delta_{jk}\mathrm{A}_{i\ell}.$$

The associated Jordan algebra $\mathrm{J} = \mathrm{A}^+$ *has Jordan Peirce decomposition*

$$\mathrm{A}^+ = \bigoplus_{i \le j=0}^{n} \mathrm{J}_{ij} \quad \textit{for} \quad \mathrm{J}_{ii} := \mathrm{A}_{ii},\ \mathrm{J}_{ij} := \mathrm{A}_{ij} \oplus \mathrm{A}_{ji}\ (i \neq j).$$

If A *has an involution* $*$*, and* $e_i^* = e_i$ *are symmetric idempotents, then the associative Peirce spaces satisfy* $\mathrm{A}_{ij}^* = \mathrm{A}_{ji}$*, the Jordan algebra* $\mathcal{H}(\mathrm{A}, *)$ *contains the family* $\mathcal{E}$*, and the Peirce decomposition is precisely that induced from that of* A^+*:*

$$\mathcal{H}(\mathrm{A}, *) = \bigoplus_{i,j=0}^{n} \mathcal{H}_{ij} \quad \textit{for} \quad \mathcal{H}_{ii} = \mathcal{H}(\mathrm{A}_{ii}, *),$$
$$\mathcal{H}_{ij} = \mathcal{H}(\mathrm{A}_{ij} + \mathrm{A}_{ji}, *) = \{a_{ij} + a_{ij}^* \mid a_{ij} \in \mathrm{A}_{ij}\}.$$

(2) *If* $\mathrm{A} = \mathcal{M}_n(\mathrm{D})$ *is the algebra of* $n \times n$ *matrices over an associative algebra* D *and* $e_i = E_{ii}$ *are the diagonal matrix units, the Peirce spaces* A_{ij} *relative to* $\mathcal{E} = \{E_{11}, \dots, E_{nn}\}$ *are just the matrices having all entries* 0 *except for the* ij*-entry:* $\mathrm{A}_{ij} = \mathrm{D}E_{ij}$*. When* $\mathrm{J} = \mathrm{A}^+$ *we have the Jordan Peirce decomposition*

$$\mathrm{J}_{ii} = \mathrm{D}E_{ii}, \qquad \mathrm{J}_{ij} = \mathrm{D}E_{ij} + \mathrm{D}E_{ji}.$$

If D *carries an involution* $\overline{}$ *then* $\mathrm{A} = \mathcal{M}_n(\mathrm{D})$ *carries the conjugate-transpose involution* $X^* = \overline{X}^{tr}$*, and the Jordan algebra of hermitian elements* $\mathrm{J} = \mathcal{H}_n(\mathrm{D}, -) := \mathcal{H}(\mathcal{M}_n(\mathrm{D}), *)$ *has Peirce decomposition*

$$\mathrm{J}_{ii} = \mathcal{H}(\mathrm{D}, -)[ii] = \{aE_{ii} \mid a \in \mathcal{H}(\mathrm{D}, -)\},$$
$$\mathrm{J}_{ij} = \mathrm{D}[ij] = \{dE_{ij} + \bar{d}E_{ji} \mid d \in \mathrm{D}\}. \qquad \square$$

The Peirce Multiplication Rules 13.3.1 are precisely the rules for multiplying matrices, and one should always think of Peirce decompositions and rules as matrix decompositions and multiplications. As with the case of a single idempotent, we are led by the archetypal example of hermitian matrices to call the Peirce spaces $\mathrm{J}_{ii} = \mathrm{J}_2(e_i)$ the **diagonal** Peirce spaces, and the $\mathrm{J}_{ij} = \mathrm{J}_1(e_i) \cap \mathrm{J}_1(e_j)$ $(i \neq j)$ the **off-diagonal** Peirce spaces.

We cannot give an example of a multiple Peirce decomposition for a respectable quadratic factor extending the Reduced Spin Decomposition 8.3.5, since by law these algebras are limited to two orthogonal idempotents to a customer. The respectable cubic factors are allowed three orthogonal idempotents (but no more), so we can give an example extending 3×3 Cubic Peirce

Decomposition 8.3.7, analogous to that for any associative matrix algebra 13.4.1(2).

3×3 **Hermitian Peirce Decomposition** 13.4.2 *In a cubic factor* $\mathrm{J} = \mathcal{H}_3(\mathrm{D}, -)$ *for any alternative* $*$*-algebra* D *with nuclear involution in the Freudenthal Construction* 4.4.1*, the three diagonal idempotents* E_{11}, E_{22}, E_{33}*, form a supplementary orthogonal family, with Peirce decomposition*

$$\begin{aligned} \mathrm{J} &= \bigoplus\nolimits_{1 \le i \le j \le 3} \mathrm{J}_{ij}, \\ \mathrm{J}_{ii} &= \mathcal{H}(\mathrm{D})[ii] = \{aE_{ii} \mid a \in \mathcal{H}(\mathrm{D})\}, \\ \mathrm{J}_{ij} &= \mathrm{D}[ij] = \{dE_{ij} + \bar{d}E_{ji} \mid d \in \mathrm{D}\}. \end{aligned}$$

In the important case of an Albert algebra $\mathcal{H}_3(\mathrm{O})$*, we have* $\mathcal{H}(\mathrm{O}) = \Phi$*, so the three diagonal Peirce spaces* $\mathrm{J}_{ii} = \Phi E_{ii}$ *are one-dimensional, and the three off-diagonal* $\mathrm{J}_{ij} = \mathrm{O}[ij]$ *are eight-dimensional, leading to a* 27*-dimensional algebra.* □

13.5 The Peirce Principle

Macdonald's Principle provides us with a powerful method for verifying facts about multiple Peirce decompositions in Jordan algebras which extends the Peirce Principle 8.4.1 of a single Peirce decomposition.

Peirce Principle 13.5.1 *Any set of Peirce elements* $\{x_{ij}\}$*, from distinct Peirce spaces* J_{ij} *relative to a family* $\mathcal{E}$ *of orthogonal idempotents, lie in a special subalgebra* B *of* J *containing* $\mathcal{E}$*. Therefore, all Jordan behavior of distinct Peirce spaces in associative algebras persists in all Jordan algebras: if* f_α, f *are Jordan polynomials, a set of relations*

$$f_\alpha(e_1, \ldots, e_n, x_{i_1 j_1}, \ldots, x_{i_r j_r}) = 0$$

among Peirce elements x_{ij} *with distinct index pairs will imply a relation*

$$f(e_1, \ldots, e_n, x_{i_1 j_1}, \ldots, x_{i_r j_r}) = 0$$

in all *Jordan algebras if it does so in all* special *Jordan algebras, i.e., if the relations* $f_\alpha(e_1, \ldots, e_n, a_{i_1 j_1} + a_{j_1 i_1}, \ldots, a_{i_r j_r} + a_{j_r i_r}) = 0$ *imply* $f(e_1, \ldots, e_n, a_{i_1 j_1} + a_{j_1 i_1}, \ldots, a_{i_r j_r} + a_{j_r i_r}) = 0$ *for Peirce elements* $a_{ij} \in \mathrm{A}_{ij}$ *relative to idempotents* $\mathcal{E}$ *in all unital associative algebras* A*.*

In particular, we have the **Peirce Identity Principle**: *any Peirce identity* $f(e_1, \ldots, e_n, x_{i_1 j_1}, \ldots, x_{i_r j_r}) = 0$ *for a Jordan polynomial* f *will hold for Peirce elements* x_{ij} *with distinct index pairs in all Jordan algebras* J *if* $f(e_1, \ldots, e_n, a_{i_1 j_1} + a_{j_1 i_1}, \ldots, a_{i_r j_r} + a_{j_r i_r}) = 0$ *holds for all Peirce elements* $a_{ij} \in \mathrm{A}_{ij}$ *in all associative algebras* A*.*

PROOF. Everything takes place in the subalgebra

$$\begin{aligned} \mathrm{B} &:= \Phi[e_0, e_1, \ldots, e_n, x_{i_1 j_1}, \ldots, x_{i_r j_r}] \\ &= \Phi[e_0, e_1, \ldots, e_n, x] \subseteq \widehat{\mathrm{J}} \quad (x := \textstyle\sum_{i<j} x_{ij}), \end{aligned}$$

since *by distinctness* we can recover the individual components x_{ij} from the sum x by the Peirce projections, $x_{ij} = E_{ij}(x)$. Our goal is to show that this subalgebra B is special. A standard localization procedure (which we will explain in detail in Section 6) produces a *faithful cyclifying* scalar extension $\Omega = \Phi[T]_S \supseteq \Phi[T] := \Phi[t_0, t_1, \ldots, t_n]$ in the sense that

(1) Ω is faithful: $\widehat{\mathrm{J}}$ is faithfully imbedded in $\widehat{\mathrm{J}}_\Omega$;

(2) Ω cyclifies the e_i: $\Omega[e_0, e_1, \ldots, e_n] = \Omega[e(\mathbf{t})]$

with cyclic Ω-generator $e(\mathbf{t}) := \sum_{i=0}^{n} t_i e_i$.

Then B will be special since it imbeds in the special algebra B_Ω: $\mathrm{B} \subseteq \mathrm{B}_\Omega$ [because of faithfulness (1)] $= \Omega[e_0, e_1, \ldots, e_n, x]$ [by the above expression for B] $= \Omega[e(\mathbf{t}), x]$ [by cyclification (2)], where the latter is generated over Ω by two elements and hence is special by the Shirshov–Cohn Theorem 5.2.

The Peirce decomposition of the special algebra $\mathrm{B}_\Omega \subseteq \mathrm{A}^+$ is related to that of A by Full Example 13.4.1(1), so we can write $x_{ij} = a_{ij} + a_{ji}$ for $i \leq j$ (if $i = j$ we can artificially write $x_{ii} = a_{ii} = \frac{1}{2}a_{ii} + \frac{1}{2}a_{ii}$). Then $f(e_1, \ldots, e_n, a_{i_1 j_1} + a_{j_1 i_1}, \ldots, a_{i_r j_r} + a_{j_r i_r}) = 0$ in A implies that we have $f(e_1, \ldots, e_n, x_{i_1 j_1}, \ldots, x_{i_r j_r}) = 0$ in J. $\square$

Just as in the one-idempotent Peirce Principle, this Principle gives yet another method for establishing the Peirce Toral Property 13.1.3, Peirce Decomposition 13.1.4, Eigenspace Laws 13.1.5, and Peirce Multiplication Rules 13.3.1. The toral property, decomposition, and eigenvalues involve only the e_i and a general $x = \sum_{i,j} x_{ij}$. The associative Peirce decomposition 13.4.1 has $\mathrm{A}_{ij}\mathrm{A}_{k\ell} = \delta_{jk}\mathrm{A}_{i\ell}$ with orthogonal Peirce projections $C_{ij}(a) = e_i a e_j$, so the Jordan Peirce projections $E_{ii} = C_{ii}$, $E_{ij} = C_{ij} + C_{ji}$ are also supplementary orthogonal projections as in the Peirce Decomposition 13.1.4. We can recover all Peirce Brace Multiplication rules in 13.3.1: for squaring we have $x_{ii}^2 \in \mathrm{A}_{ii}\mathrm{A}_{ii} \subseteq \mathrm{A}_{ii}$ and $x_{ij}^2 \in (\mathrm{A}_{ij} + \mathrm{A}_{ji})^2 = \mathrm{A}_{ij}\mathrm{A}_{ji} + \mathrm{A}_{ji}\mathrm{A}_{ij} \subseteq \mathrm{A}_{ii} + \mathrm{A}_{jj}$, while for the brace we have $\{x_{ii}, x_{ij}\} \in \mathrm{A}_{ii}(\mathrm{A}_{ij} + \mathrm{A}_{ji}) + (\mathrm{A}_{ij} + \mathrm{A}_{ji})\mathrm{A}_{ii} \subseteq \mathrm{A}_{ij} + \mathrm{A}_{ji}$ and $\{x_{ij}, x_{k\ell}\} = 0$ since $(\mathrm{A}_{ij} + \mathrm{A}_{ji})(\mathrm{A}_{k\ell} + \mathrm{A}_{\ell k}) = (\mathrm{A}_{k\ell} + \mathrm{A}_{\ell k})(\mathrm{A}_{ij} + \mathrm{A}_{ji}) = 0$. Similarly, we obtain Triple Orthogonality Rules for distinct index pairs; we use Triple Switching to reduce repeated index pairs to $\{x_{ij}, y_{k\ell}, z_{ij}\} = 0$, which follows as a linearization of $U_{x_{ij}} y_{k\ell} = 0$. For the Basic U-Product Rules we get $U_{x_{ij}} y_{ii} \in (\mathrm{A}_{ij} + \mathrm{A}_{ji})\mathrm{A}_{ii}(\mathrm{A}_{ij} + \mathrm{A}_{ji}) \subseteq \mathrm{A}_{jj}$, *but we do not get* the U-product rule $U_{x_{ij}} y_{ij} \in \mathrm{J}_{ij}$ *directly* since it involves terms x_{ij}, y_{ij} from the same Peirce space [we must get it by linearizing $U_{x_{ij}} x_{ij} \in \mathrm{J}_{ij}$, or by building the U-operator out of bullets and using (1)]. We obtain the non-orthogonal triple products for distinct index pairs by $\{x_{ij}, y_{jk}, z_{k\ell}\} \in$

$\{A_{ij}+A_{ji}, A_{jk}+A_{kj}, A_{k\ell}+A_{\ell k}\} \subseteq A_{i\ell}+A_{\ell i}$, while we use Triple Switching to reduce repeated pairs to linearizations of U-products; or we could again build the triple products out of bullets and use (1).

Indistinct Indices Example 13.5.2 *The requirement of distinct index pairs in the Peirce Principle is crucial. We have a Jordan Peirce relation*

$$f(x_{12}, y_{12}, x_{13}, y_{13}, z_{23}) = [V_{x_{12},y_{12}}, V_{x_{13},y_{13}}](z_{23}) = 0$$

for Peirce elements (but with repeated index pairs) which holds in all associative algebras with at least three mutually orthogonal idempotents, but does not *hold in the Albert algebra $\mathcal{H}_3(\mathrm{O})$.*

Indeed, writing $x_{ij} = a_{ij} + a_{ji}$, $y_{ij} = b_{ij} + b_{ji}$, $z_{23} = c_{23} + c_{32}$ as in the 3×3 Hermitian Peirce Decomposition 13.4.2 we have $V_{x_{12},y_{12}}V_{x_{13},y_{13}}(z_{23}) = a_{21}b_{12}c_{23}b_{31}a_{13} + a_{31}b_{13}c_{32}b_{21}a_{12} = V_{x_{13},y_{13}}V_{x_{12},y_{12}}(z_{23})$. Yet this is *not* a Peirce relation for *all* Jordan algebras, since if in the Albert algebra $\mathcal{H}_3(\mathrm{O})$ we take $x_{12} = a[21]$, $y_{12} = 1[12]$, $x_{13} = c[13]$, $y_{13} = 1[31]$, $z_{23} = b[23]$ for arbitrary Cayley elements $a, b, c \in \mathrm{O}$, we have

$$\begin{aligned} V_{x_{12},y_{12}}V_{x_{13},y_{13}}(z_{23}) &= \{a[21], 1[12], \{b[23], 1[31], c[13]\}\} \\ &= a(1((b1)c)[23] = a(bc)[23], \\ V_{x_{13},y_{13}}V_{x_{12},y_{12}}(z_{23}) &= \{\{a[21], 1[12], b[23]\}, 1[31], c[13]\} \\ &= ((a(1b))1)c[23] = (ab)c[23], \end{aligned}$$

so $f(a[21], 1[12], b[23], 1[31], c[13]) = -[a, b, c][23]$, where the associator $[a, b, c]$ does *not* vanish on the nonassociative algebra O (recall that $a = i, b = j, c = \ell$ have $[a, b, c] = 2k\ell \neq 0$ in characteristic $\neq 2$). ☐

Such an identity distinguishing special algebras from all Jordan algebras is called a **Peirce s-identity**; this shows in a very painless way the exceptionality of reduced Albert algebras.

Albert's Exceptional Theorem 13.5.3 *A reduced Albert algebra $\mathcal{H}_3(\mathrm{O})$ is exceptional.* ☐

13.6 Modular Digression

The key step in the Peirce Principle is to show that there is a faithful cyclifying extension $\Omega = \Phi[T]_S \supseteq \Phi[T] := \Phi[t_0, t_1, \ldots, t_n]$ where all e_i lie in the "cyclic" subalgebra $\Omega[e(\mathbf{t})]$ generated by a single element (a Peircer, no less). [Note that this digression was unnecessary in the case of a single idempotent: $\Phi[e_0, e_1] = \Phi[1-e, e] = \Phi[e]$ is already cyclified in a unital algebra.] This is a purely module-theoretic result, having nothing to do with Jordan algebras. It is not hard to create the localization $\Phi[T]_S$ and see why it achieves *cyclification*, the only delicate point is to see that it is *faithful*. The argument that the imbedding is indeed faithful uses standard facts about 'localizing," forming rings and modules of "fractions." We will go through this in some detail in case the reader is rusty on this material, but its length should not obscure the fact that the Jordan proof has quickly been reduced to standard results about modules.

Cyclification

We have seen that $\mathrm{J} \cong \mathrm{J} \otimes 1$ is faithfully imbedded in $\mathrm{J}_{\Phi[T]} \cong \mathrm{J}[T]$ for the free Φ-module $\Phi[T]$ of polynomials. For cyclification, there is a standard interpolation argument allowing us to recover the individual idempotents from a distinct linear combination:

$$\text{if } (\alpha_i - \alpha_k)^{-1} \in \Omega \text{ for all } i \neq k, \text{ then the interpolating polynomials}$$

$$p_i(\lambda) := \prod_{k \neq i} \frac{\lambda - \alpha_k}{\alpha_i - \alpha_k} \in \Omega[\lambda] \quad \text{satisfy} \quad p_i(\alpha_i) = 1,\ p_i(\alpha_j) = 0\ (j \neq i),$$

$$\text{and hence recover } e_i = p_i(x) \in \Omega[x] \text{ from } x = \sum_{i=1}^n \alpha_i e_i.$$

Indeed, by power-associativity and definition of orthogonal idempotents, for *any* polynomial p we have $p(x) = p(\sum_k \alpha_k e_k) = \sum_k p(\alpha_k) e_k$. In particular, the interpolating polynomials produce $p_i(x) = \sum_k p_i(\alpha_k) e_k = e_i$. To create these interpolating polynomials we need only an Ω containing n scalars α_i with invertible differences. For example, if we were working over a field with at least n elements this would be automatic.

To do this "generically" for independent indeterminates t_i, we define

$$\Omega := \Phi[T][\Delta^{-1}] = \Phi[T]|_S \quad \text{(usually written just } \Phi[T]_S)$$

$$(\Delta := \prod_{i<j} (t_i - t_j), \quad S := \{\Delta^n \mid n \geq 0\})$$

obtained by localizing $\Phi[T]$ at the multiplicative submonoid S consisting of all non-negative powers of Δ. The resulting set of "fractions" $f(T)\Delta^{-n}$, with "numerator" $f(T) \in \Phi[T]$ forms a commutative $\Phi[T]$-algebra, and the fact that Δ is invertible guarantees that each of its factors $t_i - t_j$ is invertible there, $(t_i - t_j)^{-1} \in \Omega$ for all $i \neq j$ as required to construct the interpolating polynomials. Note that Φ, $\Phi[T]$, $\Omega = \Phi[T]|_S$ need not themselves be integral domains in order to make Δ invertible.

Faithfulness

In localizing non-integral domains, some of the original scalars are generally collapsed in the process of making Δ invertible. We need to know that in our particular case $\widetilde{\mathrm{J}} \subseteq \widetilde{\mathrm{J}}_\Omega$ remains faithfully imbedded in this larger algebra $\widetilde{\mathrm{J}}_\Omega := \widetilde{\mathrm{J}} \otimes_\Phi \Omega \cong \left(\widetilde{\mathrm{J}} \otimes_\Phi \Phi[T]\right) \otimes_{\Phi[T]} \Omega$ [by a basic transitivity property of tensor products] $\cong \widetilde{\mathrm{J}}[T] \otimes_{\Phi[T]} \Phi[T]|_S \cong \widetilde{\mathrm{J}}[T]|_S$ [by the basic property of localization of modules that for an R-module M, $M \otimes_R R|_S$ is isomorphic to the module localization $M|_S$ consisting of all "fractions" ms^{-1} with "numerator" $m \in M$ and "denominator" $s \in S$].

It is another standard fact about localizations that the kernel of the natural inclusion $\sigma : \widehat{\mathrm{J}} \to \widehat{\mathrm{J}}[T]|_S$ is precisely the set of elements of $\widehat{\mathrm{J}}[T]$ killed by some element of S (such elements must die in order for the s to become invertible, just as in a pride of lions some innocent bystanders are unavoidably killed when new rulers depose the old). We now prove that in our case the takeover is pacific, and σ injective, since all elements of S are injective (not zero-divisors) on $\widehat{\mathrm{J}}$.

In our case, we must show that the generator Δ of S acts injectively,

$$j(T) \cdot \Delta = 0 \implies j(T) = 0 \in \widehat{\mathrm{J}}[T].$$

For this we mimic the standard argument used to show that in the polynomial ring in one variable a polynomial $f(t)$ will be a non-zero-divisor in $R[t]$ if its top term a_n is a non-zero-divisor in R: its product with a nonzero polynomial $g(t)$ can't vanish since the top degree term of the product is the nonzero product $a_n b_m$ of the top degree terms of the two factors. In our several-variable case, to identify the "top" term we use *lexicographic ordering* on the monomials induced by the ordering $t_0 > t_1 > \cdots > t_n$ of the variables. In "vector-notation" we decree $\mathbf{t}^{\vec{e}} := t_0^{e_0} t_1^{e_1} \cdots t_n^{e_n}$ is bigger than $\mathbf{t}^{\vec{f}} := t_0^{f_0} t_1^{f_1} \cdots t_n^{f_n}$ if it has bigger *exponent*, $\vec{e} > \vec{f}$ in the sense that $e_i > f_i$ in the first place they differ. In this lexicographic ordering, $\Delta = \prod_{0<j\le n}[t_0 - t_j] \cdot \prod_{1<j\le n}[t_1 - t_j] \cdots \prod_{n-1<j\le n}[t_{n-1} - t_j]$ has *monic* lexicographically leading term $t_0^n t_1^{n-1} \cdots t_{n-1}^1 t_n^0$. From this fact we can see that Δ is nonsingular, since the lexicographically leading term of the product $j(T) \cdot \Delta$ is the product of the two lexicographically leading terms, which doesn't vanish in the present case (as long as $j(T) \neq 0$) because the leading term of Δ is monic.

Conclusions

The scalar extension $\Omega = \Phi[T]|_S$ achieves cyclicification due to invertibility of Δ, and the imbedding $\widehat{\mathrm{J}} \subseteq \widehat{\mathrm{J}}[T] \subseteq \widehat{\mathrm{J}}[T]|_S$ is faithful because Δ acts injectively on $\widehat{\mathrm{J}}[T]$. The existence of a faithful cyclifying extension establishes the Peirce Principle 13.5.1.

Notice that existence of Ω yields the multiplicative property in the Peircer Torus 13.1.3 directly: $E(\mathbf{st}) = U_{e(\mathbf{st})}$ [by definition 13.1.2] $= U_{e(\mathbf{s})\bullet e(\mathbf{t})}$ [by definition of *mutually orthogonal idempotents*] $= U_{e(\mathbf{s})} U_{e(\mathbf{t})}$ $\big[$by Macdonald's Principle 5.1.2 since everything takes place in $\Omega'[e(\mathbf{t})]$, $\Omega' = \Omega[s_0, \ldots, s_n]$ $\big]$ $= E(\mathbf{s})E(\mathbf{t})$. As usual, the Peirce Decomposition 13.1.4(1) is an immediate consequence of the this: identifying coefficients of $s_i s_j t_k t_\ell$ in $\sum_{p\le q} s_p s_q t_p t_q E_{pq} = \sum_{p\le q} (s_p t_p)(s_q t_q) E_{pq} = E(\mathbf{st}) = E(\mathbf{s})E(\mathbf{t}) = \Big(\sum_{i\le j} s_i s_j E_{ij}\Big)\Big(\sum_{k\le l} t_k t_\ell E_{k\ell}\Big) = \sum_{i\le j,\, k\le l} s_i s_j t_k t_\ell E_{ij} E_{k\ell}$ gives $E_{ij} E_{k\ell} = 0$ unless $\{i,j\} = \{k,\ell\} = \{p,q\}$, in which case $E_{pq} E_{pq} = E_{pq}$, so the E_{ij} are supplementary orthogonal idempotent operators.

Exercise 13.6.0* Prove that the above Δ is injective on $\widehat{\mathrm{J}}[T]$ by using induction on n.

13.7 Problems for Chapter 13

Problem 13.1* Another method of justifying the Toral Property 13.1.3 is to establish a general result that the U-operators permit composition (not just Jordan composition, as in the Fundamental Formula) for elements x, y which *operator-commute* (L_x, L_{x^2} commute with L_y, L_{y^2} on J), generalizing the case Power-Associativity 5.2.2(2) of elements p, q which are polynomials in a single element. (1) Prove that if elements x, y of a Jordan algebra J operator-commute, then they satisfy the strong version $U_{x\bullet y} = U_x U_y$ of the Fundamental Formula. (2) Conclude that if B is an operator-commutative subalgebra of J in the sense that L_b, L_c commute for all elements $b, c \in$ B, then B satisfies the *Commutative Fundamental Formula* $U_{b\bullet c} = U_b U_c$ on J for all $b, c \in$ B. (3) Show that any two polynomials $x = p(z), y = q(z)$ in a common element z operator-commute. (4) Show that any two *orthogonal idempotents* e, f operator-commute. (5) Prove the *Peirce Commutativity Theorem*: For any supplementary orthogonal family of idempotents $\mathcal{E}$, the subalgebra $\mathrm{B} := \Phi e_1 \boxplus \cdots \boxplus \Phi e_n$ is operator-commutative, and satisfies the Commutative Fundamental Formula $U_{p\bullet q} = U_p U_q$ for any $p, q \in$ B.

Problem 13.2* In general, just because x, y operator-commute on J does not guarantee that they continue to operator-commute on larger J′. (1) Show that any left, right multiplications in a linear algebra commute on the unit element, $[L_x, R_y]1 = 0$. Conclude that in any commutative algebra $[L_x, L_y]1 = 0$. (2) Show that if x, y operator-commute on J, then they continue to operator-commute on $\widehat{\mathrm{J}}$. (3) Show that in an associative algebra A two elements x, y commute on $z \in \mathrm{A}^+$, $[V_x, V_y]z = 0$, iff their commutator commutes with z, $[[x, y], z] = 0$. Conclude that x, y operator-commute in A^+ iff their commutator is central, $[x, y] \in \mathcal{C}ent(\mathrm{A})$. (4) Give an example of x, y which operator-commute on A^+ but not on a larger $(\mathrm{A}')^+$. Can you find an example for semiprime A, A′?

Problem 13.3 (1) If R is any commutative ring with unit and S a multiplicative submonoid of R, show that we can create a "ring $R|_S$ of S-fractions" r/s under the usual operations on fractions and the (slightly unusual) equivalence relation $r/s \sim r'/s' \iff$ there exists $t \in S$ such that $(rs' - r's)t = 0$. (2) Show that this ring has the properties (a) the images of the "denominators" $s \in S$ all become *invertible* in $R|_S$, (b) there is a universal homomorphism $R \xrightarrow{\sigma} R|_S$ of rings with the universal property that any homomorphism $R \xrightarrow{\tilde{f}} \widetilde{R}$ such that the image $\tilde{f}(S)$ is invertible in $\widetilde{R}$ can be written uniquely as a composite of σ and a homomorphism $R|_S \xrightarrow{f} \widetilde{R}$. (3) Show that the kernel of σ is precisely the set of elements of R killed by some element of S. Prove that σ injective iff all elements of S are injective on R. (4) Show that, similarly, for any right R-module M we can form a "module $M|_S$ of S-fractions" m/s with the properties (a) $M|_S$ is an $R|_S$-module, (b) there is a natural homomorphism $M \xrightarrow{\sigma} M|_S$ of R-modules whose kernel is the set of $m \in M$ killed by some element of S. (5) Show that the correspondences $M \to M|_S$, $\varphi \to \varphi|_S$ (defined by $\varphi|_S(m/s) = \varphi(m)/s$) give a functor from the category of R-modules to the category of $R|_S$-modules.

14

Multiple Peirce Consequences

Once more we stop to gather consequences from the multiple Peirce decomposition. First we use Peirce decomposition and orthogonality to establish the Jordan criterion for the hermitian matrix algebras. Then we extend the results on Peirce specializations, quadratic forms, and connection involutions to multiple Peirce decompositions.

14.1 Jordan Coordinate Conditions

Earlier we indicated, by ingenious substitutions into the linearized Jordan identities, why alternativity or associativity of the coordinates is necessary in order for the hermitian matrices to form a Jordan algebra. Now that we are more sophisticated, in particular know about Peirce decompositions in Jordan algebras, we can see the necessity of these conditions another way.[1]

Jordan Coordinates Theorem 14.1.1 *If the hermitian matrix algebra* $\mathcal{H}_n(\mathrm{D}, -)$ *for* $n \geq 3$ *satisfies the Jordan identities, then the coordinate* $*$*-algebra* D *must be alternative with nuclear involution,* $\mathcal{H}(\mathrm{D}, -) \subseteq \mathcal{N}uc(\mathrm{D})$*, and must even be associative if* $n \geq 4$.

PROOF. We show that

(1) $\quad [a, b, c] = 0 \qquad (a, b, c \in \mathrm{D},\ n \geq 4);$

(2) $\quad [\bar{a}, a, b] = 0 \qquad (a, b \in \mathrm{D});$

(3) $\quad [\alpha, b, c] = 0 \qquad (b, c \in \mathrm{D}, \alpha \in \mathcal{H}(\mathrm{D}, -)).$

These are enough to derive the theorem: (1) shows that D must be associative for $n \geq 4$, (3) and (2) show that left alternativity $[a, a, b] = [(a + \bar{a}), a, b] - [\bar{a}, a, b] = 0$ [since $a + \bar{a} = \alpha \in \mathcal{H}(\mathrm{D}, -)$], hence right alternativity by the involution, which shows D is alternative by Alternative Algebra Definition

[1] The Associative Coordinates Theorem for $n \geq 4$ was stated in I.2.6, and the Alternative Coordinates Theorem for $n = 3$ was stated in I.2.7; here we give full detailed proofs. Connected idempotents were defined in I.5.1 in discussing connected capacity.

2.1.1. Then (3) shows that $\mathcal{H}(\mathrm{D},-)$ lies in the nucleus by Nucleus Definition 2.2.1. We claim that for $a,b,c \in \mathrm{D}$ and $\alpha \in \mathcal{H}(\mathrm{D},-)$ we have

$$(1') \qquad 0 = 2\{b[13],a[21],c[34]\} = [a,b,c][24] \qquad (n \geq 4);$$

$$(2') \qquad 0 = 2U_{a[12]}(b[23]) = -[\bar{a},a,b][23];$$

$$(3') \qquad 0 = 2\{b[21],\alpha[22],c[13]\} = [\alpha,b,c][23].$$

On the one hand, all these disconnected triple products in $(1'),(2'),(3')$ *must vanish by Peirce Orthogonality* 13.3.1(3). On the other hand, using the brace definition of the U-operator, $2U_xy = \left(V_x^2 - V_{x^2}\right)y = \{x,\{x,y\}\} - \{x^2,y\}$ and its linearization, together with the Basic Products 3.2.4(2) for brace multiplication in hermitian matrix algebras, the right sides of $(1')$, $(2')$, $(3')$ reduce respectively to

$$\{b[13],\{a[21],c[34]\}\} + \{\{a[21],b[13]\},c[34]\} - \{a[21],\{b[13],c[34]\}\}$$
$$= 0 + (ab)c[24] - a(bc)[24] = [a,b,c][24];$$

$$\{\bar{a}[21],\{a[12],b[23]\}\} - \{a[12]^2,b[23]\}$$
$$= \bar{a}(ab)[23] - \{a\bar{a}[11] + \bar{a}a[22],b[23]\} = -[\bar{a},a,b][23];$$

$$\{b[21],\{\alpha[22],c[13]\}\} + \{\{\alpha[22],b[21]\},c[13]\} - \{\alpha[22],\{b[21],c[13]\}\}$$
$$= 0 + (\alpha b)c[23] - \alpha(bc)[23] = [\alpha,b,c][23].$$

This finishes the proof. □

Note how clearly the associators materialize out of the brace products.

The above conditions are in fact necessary and sufficient for $\mathcal{H}_n(\mathrm{D},-)$ to form a linear Jordan algebra. The converse when $n \geq 4$ is easy: whenever D is associative, $\mathcal{H}_n(\mathrm{D},-)$ is a special Jordan subalgebra of $\mathcal{M}_n(\mathrm{D})^+$. The converse when $n=3$ is considerably messier, so we leave it to Appendix C; in our work we will need only the easier case of a *scalar* involution.

Exercise 14.1.1A Carry out a direct frontal attack on the Jordan identity without the aid of Peirce decompositions. (1) Multiply the Jordan Definition (JAX2)$''$ in 1.8.1 by 8 to turn all associators into *brace associators* $[z,y,z]' := \{\{x,y\},z\} - \{x,\{y,z\}\}$. For $x = c[21]$, $z = b[32]$, $w = 1[11]$, $y = a[j3]$ ($j = 3$ or 4) show that $\{x,z\} = bc[31], \{z,w\} = 0, \{w,x\} = c[21]$, and that the brace version of (JAX2)$''$ reduces to $-[a,b,c][j1] = 0$. Conclude that 14.1.1(1) holds, hence D must be associative. (3) When $n=3$ take $j=3$, $y = \alpha[33] \in \mathcal{H}_n(\mathrm{D},-)$ for any $\alpha \in \mathcal{H}(\mathrm{D},-)$ to obtain 14.1.1(3). (4) Use a different substitution $x = a[12]$, $z = b[23]$, $y = 1[33]$ in the brace version of (JAX2)$'$ $[x^2,y,z]' + [\{x,z\},y,x]' = 0$ to get $-[\bar{a},a,b][23] = 0$ as in 14.1.1(2). Conclude as in 14.1.1 that in this case D is alternative with nuclear involution.

Exercise 14.1.1B Deduce the vanishing of 14.1.1($1'$-$3'$) from single-idempotent Peirce Orthogonality 8.2.1.

14.2 Peirce Specializations

The basic facts about Peirce specializations and associativity follow directly from the results for a single idempotent. Throughout the rest of this chapter we continue the conventions of the previous chapter: we fix an orthogonal family $\mathcal{E} = \{e_1, \ldots, e_n\}$ of idempotents in a Jordan algebra J with supplementation $\{e_0, e_1, \ldots, e_n\}$ in $\widehat{\mathrm{J}}$, and denote the Peirce subspaces by J_{ij}. With this understanding, we will not explicitly mention the family $\mathcal{E}$ again.

Peirce Specialization Proposition 14.2.1 *For* $k \neq i$ *the* **Peirce specializations** $\sigma_{ii}(a) := V_a \mid_{\mathrm{J}_{ik}}$ *is a Jordan homomorphism* $\mathrm{J}_{ii} \xrightarrow{\sigma_{ii}} \mathcal{E}nd_\Phi(\mathrm{J}_{ik})$:

$$\sigma_{ii}(e_i) = \mathbb{1}_{\mathrm{J}_{ik}} \qquad \sigma_{ii}(U_x y) = \sigma_{ii}(a)\sigma_{ii}(b)\sigma_{ii}(a),$$
$$\sigma_{ii}(a^2) = \sigma_{ii}(a)^2, \qquad \sigma_{ii}(a)\sigma_{ii}(b) = V_{x,y} \mid_{\mathrm{J}_{ik}} .$$

Similarly, for distinct i, j, k *the* **Peirce specializations** $\sigma_{ij}(x) := V_x \mid_{\mathrm{J}_{ik}+\mathrm{J}_{jk}}$ *is a Jordan homomorphism* $\mathrm{J}_{ii} + \mathrm{J}_{ij} + \mathrm{J}_{jj} \xrightarrow{\sigma_{ij}} \mathcal{E}nd_\Phi(\mathrm{J}_{ik} + \mathrm{J}_{jk})$:

$$\sigma_{ij}(x^2) = \sigma_{ij}(x)^2, \qquad \sigma_{ij}(U_x y) = \sigma_{ij}(x)\sigma_{ij}(y)\sigma_{ij}(x),$$
$$\sigma_{ij}(x)\sigma_{ij}(y) = V_{x,y} \mid_{\mathrm{J}_{ik}+\mathrm{J}_{jk}} .$$

PROOF. By Peirce Recovery 13.2.1 this follows directly from the single-idempotent case Peirce Specialization 9.1.1 (taking $e = e_i, e = e_i + e_j$ respectively), because by the Peirce Multiplication Rules 8.2.1 $\mathrm{J}_{ik} \subseteq \mathrm{J}_1(e)$ is invariant under $V_{\mathrm{J}_{ii}}$, is killed by $V_{\mathrm{J}_{jj}}$, and is mapped into J_{jk} by $V_{\mathrm{J}_{ij}}$. □

EXERCISE 14.2.1 Just as in 9.1.1, establish the above directly from the Peirce relations and Peirce Orthogonality in 13.3.1, using the Specialization Formulas (FFIII)′ and Triple Switch (FFIV) $\{x, \{z, y\}\} = \{x, z, y\} + \{x, y, z\}$.

Already we can see one benefit of Peirce analysis: we see that the diagonal Peirce subalgebras have a more associative nature, because they act associatively on the off-diagonal spaces by Peirce specialization. In respectable cases, where these specializations are faithful, this guarantees that the diagonal subalgebras are special Jordan algebras, so the only possible exceptionality resides in the off-diagonal part of the algebra.

Once more, the diagonal Peirce spaces Alphonse (J_{ii}) and Gaston (J_{jj}) courteously commute with each other as they take turns feeding on J_{ij}.

Peirce Associativity Proposition 14.2.2 *When* $\{i, j\} \cap \{k, l\} = \emptyset$ *the Peirce specializations of* J_{ij} *and* $\mathrm{J}_{k\ell}$ *on* J_{jk} *commute*: *we have the operator relations*

$$V_{x_{ij}} V_{z_{kl}}(y_{jk}) = \{x_{ij}, y_{jk}, z_{kl}\} = V_{z_{k\ell}} V_{x_{ij}}(y_{jk})$$

for elements $w_{rs} \in \mathrm{J}_{rs}$, *equivalently the elemental relations*

$$[x_{ij}, y_{jk}, z_{k\ell}] = 0, \qquad (x_{ij} \bullet y_{jk}) \bullet z_{k\ell} = x_{ij} \bullet (y_{jk} \bullet z_{k\ell}).$$

PROOF. This follows immediately from the Peirce Identity Principle 13.5.1 (by a somewhat messy computation), or from the single-idempotent case of Peirce Associativity 9.1.3 relative to $e = e_i$ (if $i = j$) or $e = e_i + e_j$ (if $i \neq j$) [since then by Peirce Recovery 13.2.1 $x \in \mathrm{J}_2(e)$, $y \in J_1(e)$, and $z \in \mathrm{J}_0(e)$], or best of all from Triple Switching [since $\{x_{ij}, \{y_{jk}, z_{k\ell}\}\} = \{x_{ij}, y_{jk}, z_{k\ell}\} + \{x_{ij}, z_{k\ell}, y_{jk}\} = \{x_{ij}, y_{jk}, z_{k\ell}\}$ by Peirce Orthogonality 13.3.1(3) when $\{i, j\}$, $\{k, \ell\}$ can't be linked]. □

14.3 Peirce Quadratic Forms

Again, the diagonal behavior of off-diagonal spaces J_{ij} is captured in the Peirce quadratic form q_{ii}.

q-Properties Proposition 14.3.1 *For off-diagonal elements $x_{ij} \in \mathrm{J}_{ij}$ $(i \neq j)$ we define Peirce quadratic forms $q_{ii} : \mathrm{J}_{ij} \to \mathrm{J}_{ii}$ by*

$$q_{ii}(x_{ij}) := E_{ii}(x_{ij}^2) = U_{x_{ij}} e_j.$$

(1) *We have* **Cube Recovery** *and recovery of fourth powers by*

$$x_{ij}^3 = \{q_{ii}(x_{ij}), x_{ij}\}, \quad E_{ii}(x_{ij}^4) = U_{x_{ij}} q_{jj}(x_{ij}) = q_{ii}(x_{ij})^2.$$

(2) *The operator $U_{x_{ij}}$ can be expressed in terms of q's: for $y_{ij} \in \mathrm{J}_{ij}$, $a_{ii} \in \mathrm{J}_{ii}$, $a_{jj} \in \mathrm{J}_{jj}$ for $j \neq i$ we have the $Uijq$* **Rules**

$$\begin{aligned} U_{x_{ij}} y_{ij} &= \{q_{ii}(x_{ij}, y_{ij}), x_{ij}\} - \{q_{jj}(x_{ij}), y_{ij}\}, \\ U_{x_{ij}} a_{jj} &= q_{ii}(x_{ij}, a_{jj} \bullet x_{ij}), \\ \{x_{ij}, a_{jj}, z_{ij}\} &= q_{ii}(\{x_{ij}, a_{jj}\}, z_{ij}) = q_{ii}(x_{ij}, \{a_{jj}, z_{ij}\}). \end{aligned}$$

(3) *The Peirce quadratic forms permit composition with braces: for distinct i, j, k we have the q-***Composition Rules**

$$\begin{aligned} q_{ii}(\{a_{ii}, x_{ij}\}) &= U_{a_{ii}} q_{ii}(x_{ij}), \\ q_{ii}(\{a_{ii}, x_{ij}\}, x_{ij}) &= V_{a_{ii}} q_{ii}(x_{ij}), \\ q_{ii}(\{a_{jj}, x_{ij}\}) &= U_{x_{ij}}(a_{jj}^2), \\ q_{ii}(\{x_{ij}, y_{jk}\}) &= U_{x_{ij}} q_{jj}(y_{jk}). \end{aligned}$$

(4) *We have a q-***Nondegeneracy Condition**: *If the algebra* J *is nondegenerate, then the Peirce quadratic forms q_{ii} are nondegenerate (in the sense that $\mathcal{R}ad(q_{ii}) = 0$).*

PROOF. Everything except (4) and the first and third parts of (2), (4) follows easily from the Peirce Identity Principle 13.5.1 (note that $q_{ii}(x_{ij}) = a_{ij}a_{ji}$ if $x_{ij} = a_{ij} + a_{ji}$ in A).

Alternately, everything but the fourth part of (3) follows from the q-Properties 9.2.2 and q-Nondegeneracy 9.2.3(2) for the single-idempotent case, since it takes place in the Peirce subalgebra $\mathrm{J}_2(e_i+e_j)$ of $\widehat{\mathrm{J}}$ where $e = e_i, \hat{1}-e = e_j$ are supplementary orthogonal idempotents with $x_{ij} \in \mathrm{J}_1(e_i)$ (note that by Diagonal Inheritance 10.1.1(1) this Peirce subalgebra inherits nondegeneracy from J). For the fourth part of (3), if we eschew the Peirce Identity Principle we are reduced to calculating: $2U_{x_{ij}} q_{jj}(y_{jk}) = 2q_{ii}(x_{ij}, q_{jj}(y_{jk}) \bullet x_{ij})$ [by the second part of (2)] $= 2q_{ii}(x_{ij}, y_{jk}^2 \bullet x_{ij})$ [since $q_{kk}(y_{jk})$ acts trivially by Peirce Orthogonality] $= q_{ii}(x_{ij}, \{y_{jk}^2, x_{ij}\}) = q_{ii}(x_{ij}, \{y_{jk}, \{y_{jk}, x_{ij}\}\})$ [by Peirce Specialization 14.2.1] $= q_{ii}(\{x_{ij}, y_{jk}\}, \{y_{jk}, x_{ij}\})$ [by symmetry in the third part of (2) with e_i, e_j replaced by $e_j + e_k, e_i$] $= 2q_{ii}(\{x_{ij}, y_{jk}\})$, and we scale by $\frac{1}{2}$. □

As with products in matrix algebras (cf. 3.2.4), most of these formulas are more natural for brace products than bullet products.

Exercise 14.3.1 Establish the fourth part of q-Composition (3) above without invoking $\frac{1}{2}$ by setting $x = x_{ij}, y = y_{jk}$ in the identity $U_x U_y + U_y U_x + V_x U_y V_x = U_{\{x,y\}} + U_{U_x(y),y}$ applied to the element 1.

14.4 Connected Idempotents

Orthogonal idempotents e_i, e_j which belong together in the same "part" of the algebra are "connected" by an off-diagonal element $u_{ij} \in \mathrm{J}_{ij}$ invertible in $\mathrm{J}_{ii} + \mathrm{J}_{ij} + \mathrm{J}_{jj}$. Invertibility of an *off-diagonal* element is equivalent to invertibility of its square, which is a *diagonal* element. We will frequently be concerned with diagonal elements in a Peirce decomposition, and it is important that the only way they can be invertible is for each diagonal entry to be invertible.

Diagonal Invertibility Lemma 14.4.1 *If $x = \sum_{i=0}^{n} x_{ii}$ for $x_{ii} \in \mathrm{J}_{ii}$ is a diagonal element in a unital Jordan algebra* J*, then x has an inverse x^{-1} in* J *iff each component x_{ii} has an inverse x_{ii}^{-1} in J_{ii}, in which case $x^{-1} = \sum_i x_{ii}^{-1}$ is also diagonal.*

PROOF. It will be easier for us to work with the quadratic conditions (QJInv1)–(QJInv2) of 6.1.1. If each x_{ii} has inverse x_{ii}^{-1} in J_{ii} then it is easy to see that $y = \sum x_{ii}^{-1}$ is the (unique) inverse of x: $U_x y^k = x^{2-k}$ for $k = 1, 2$. Conversely, if x has inverse $y = \sum_{i \le j} y_{ij}$ then U_x is invertible on J, and since U_x leaves each Peirce subspace invariant, each restriction $U_x|_{\mathrm{J}_{ii}} = U_{x_{ii}}$ and $U_x|_{\mathrm{J}_{ij}} = U_{x_{ii},x_{jj}}$ $(i \ne j)$ is invertible. From $x = U_x y$ we see that $x_{ii} = U_{x_{ii}} y_{ii}$ and $0 = U_{x_{ii},x_{jj}} y_{ij}$ for $i \ne j$, which by invertibility on J_{ij} forces $y_{ij} = 0, y = \sum y_{ii}$; then $1 = U_x y^2 = U_x(\sum y_{ii}^2)$ implies that each $e_i = U_{x_{ii}} y_{ii}^2$, so $y_{ii} = x_{ii}^{-1}$ and $y = \sum x_{ii}^{-1}$. □

Exercise 14.4.1* Prove Diagonal Invertibility using the Linear Jordan Inverse Conditions 6.1.7 (LJInv1)–(LJInv2) $x \bullet y = 1,\ x^2 \bullet y = x$. (1) If each x_{ii} has inverse y_{ii} in J_{ii}, show that $y = \sum_i y_{ii}$ acts as inverse in J. (2) If x has inverse $y = \sum_{i \leq j} y_{ij}$ show that each y_{ii} must be the inverse of x_{ii}; conclude by (1) that $y' = \sum_i y_{ii}$ is an inverse of x. (3) Conclude that $y = y'$.

Using the Peirce decomposition we can introduce an equivalence relation among orthogonal idempotents, which we will use in our structure theory to lump together idempotents which fall in the same "simple chunk."

Connection Definition 14.4.2 *Two orthogonal idempotents e_i, e_j are* **connected** *if they have a* **connecting element***, an element $v_{ij} \in \mathrm{J}_{ij}$ which is invertible in $\mathrm{J}_{ii} + \mathrm{J}_{ij} + \mathrm{J}_{jj} = \mathrm{J}_2(e_i + e_j)$. The idempotents are* **strongly connected** *if they have a* **strong connecting element***, one which is an involution in $\mathrm{J}_2(e_i + e_j)$. To indicate the connection together with the connecting element we use the notation*

$$e_i \overset{v_{ij}}{\sim} e_j.$$

Off-Diagonal Invertibility Lemma 14.4.3 (1) *We can detect off-diagonal invertibility of $v_{ij} \in \mathrm{J}_{ij}$ by invertibility of the Peirce quadratic forms in the diagonal spaces*:

$$\begin{aligned} v_{ij} \text{ is connecting } &\iff q_{ii}(v_{ij}), q_{jj}(v_{ij}) \text{ are invertible in } \mathrm{J}_{ii}, \mathrm{J}_{jj}; \\ v_{ij} \text{ is strongly connecting } &\iff q_{ii}(v_{ij}) = e_i, \quad q_{jj}(v_{ij}) = e_j. \end{aligned}$$

(2) *We can also detect off-diagonal invertibility of $v_{ij} \in \mathrm{J}_{ij}$ by invertibility of $U_{v_{ij}}$ on the diagonal spaces*:

$$\begin{aligned} v_{ij} \text{ is connecting } &\iff U_{v_{ij}}(\mathrm{J}_{ii}) = \mathrm{J}_{jj} \text{ and } U_{v_{ij}}(\mathrm{J}_{jj}) = \mathrm{J}_{ii} \\ &\iff e_j \in U_{v_{ij}}(\mathrm{J}_{ii}) \text{ and } e_i \in U_{v_{ij}}(\mathrm{J}_{jj}). \end{aligned}$$

proof. (1) For the first equivalence, by the Power Invertibility Criterion 6.1.8(2), v_{ij} is invertible in $\mathrm{B} := \mathrm{J}_{ii} + \mathrm{J}_{ij} + \mathrm{J}_{jj}$ iff $v_{ij}^2 = q_{ii}(v_{ij}) + q_{jj}(v_{ij})$ is, which is equivalent by Diagonal Invertibility 14.4.1 to each piece being invertible. For the second equivalence, clearly $v_{ij}^2 = e_i + e_j$ iff $q_{ii}(v_{ij}) = e_i,\ q_{jj}(v_{ij}) = e_j$.

(2) follows from the Invertibility Criterion 6.1.2: invertibility of v_{ij} implies surjectivity of $U_{v_{ij}}$ on B, so $U_{v_{ij}} \mathrm{J}_{ii} = \mathrm{J}_{jj}$ by the Peirce Multiplication Rules 13.3.1(2), surjectivity implies e_i, e_j are in the range, and their being in the range guarantees that the unit $e_i + e_j$ of B is in the range of $U_{v_{ij}}$, making v_{ij} invertible. ☐

Connection Equivalence Lemma 14.4.4 *Connectivity and strong connectivity are equivalence relations on orthogonal idempotents: if $\mathcal{E} = \{e_1, \ldots, e_n\}$ is an orthogonal family of idempotents in a Jordan algebra* J*, the* (*respectively strong*) **connection relation**

$$e_i \sim e_j \iff \begin{cases} e_i = e_j, & \text{if } i = j; \\ e_i, e_j \text{ (respectively strongly) connected}, & \text{if } i \neq j; \end{cases}$$

is an equivalence relation on the set $\mathcal{E}$:

$$e_i \overset{v_{ij}}{\sim} e_j,\ e_j \overset{v_{jk}}{\sim} e_k \Longrightarrow e_i \overset{v_{ik}}{\sim} e_k \quad (v_{ik} := \{v_{ij}, v_{jk}\},\ i, j, k \textit{ distinct})$$

(*and if the connecting elements* v_{ij}, v_{jk} *are strong, so is* v_{ik}).

PROOF. By definition $\sim$ is reflexive and symmetric, so we need only verify transitivity.

To establish transitivity in the weak connection case, by Off-Diagonal Invertibility 14.4.3(1) we must show that $q_{ii}(v_{ik})$ is invertible in J_{ii} [then dually for q_{kk}]. But $q_{ii}(v_{ik}) = q_{ii}(\{v_{ij}, v_{jk}\}) = U_{v_{ij}} q_{jj}(v_{jk})$ [by q-Composition 14.3.1(3)] is invertible by Diagonal Invertibility 14.4.1 in $\mathrm{B} := \mathrm{J}_{ii} + \mathrm{J}_{ij} + \mathrm{J}_{jj}$ as the ii-component of the invertible diagonal element $U_{v_{ij}}(q_{jj}(v_{jk}) + e_i)$ (which in turn is invertible by Invertible Products 6.1.8(1), since v_{ij} is invertible in B by hypothesis and the diagonal element $q_{jj}(v_{jk}) + e_i$ is invertible by Diagonal Invertibility again because each term is).

To show transitivity in the strong case, by Off-Diagonal Invertibility 14.4.3(1) we must show that $q_{ii}(v_{ik}) = e_i$. But again by q-Composition, $q_{ii}(v_{ik}) = U_{v_{ij}} q_{jj}(v_{jk}) = U_{v_{ij}} e_j$ [by strongness of v_{jk}] $= q_{ii}(v_{ij}) = e_i$ [by strongness of v_{ij}]. □

Once more, strong connection leads to involutions.

Connection Involution Proposition 14.4.5 (1) *If orthogonal idempotents* e_1, e_2 *in an arbitrary Jordan algebra* J *are strongly connected,* $v_{12}^2 = e_1 + e_2$ *for an element* v_{12} *in the Peirce space* J_{12}*, then the element*

$$u := 1 - e_1 - e_2 + v_{12}$$

is a connection involution in $\widehat{\mathrm{J}}$*, and* U_u *on* $\widehat{\mathrm{J}}$ *induces an involution* $\overline{}$ *on* J *which interchanges* e_1 *and* e_2: *we have* $\overline{e_i} = e_j$ *for* $i = 1, 2,\ j = 3 - i$, *and*

$$\overline{} \quad = \quad \begin{cases} U_{v_{12}} & \textit{on } \mathrm{J}_2(e_1 + e_2) = \mathrm{J}_{11} + \mathrm{J}_{12} + \mathrm{J}_{22}, \\ V_{v_{12}} & \textit{on } \mathrm{J}_1(e_1 + e_2) = \mathrm{J}_{10} + \mathrm{J}_{20}, \\ \mathbb{1}_{\mathrm{J}_{00}} & \textit{on } \mathrm{J}_0(e_1 + e_2) = \mathrm{J}_{00}. \end{cases}$$

(2) *On* $\mathrm{J}_2(e_1 + e_2)$ *we have a finer description of the action on the Peirce subspaces: if we define the* trace *by* $t_i(x) = q_i(v_{12}, x)$ $(i = 1, 2, j = 3 - i)$ *as in the single-variable case* 10.1.3(2), *then*

$$\overline{a_{ii}} = t_j(a_{ii} \bullet v_{12}) \qquad on \ \mathrm{J}_{ii},$$
$$\overline{x_{12}} = 2t_i(x_{12}) \bullet v_{12} - x_{12} \qquad on \ \mathrm{J}_{12},$$
$$\overline{q_i(x_{12})} = q_j(\overline{x_{12}}).$$

(3) *The fixed set of the connection involution* $\overline{}$ *on* J_{12} *is* $\{\mathrm{J}_{ii}, v_{12}\}$:

$$\overline{x_{ii} \bullet v_{12}} = \overline{x_{ii}} \bullet v_{12} = x_{ii} \bullet v_{12}, \qquad \overline{v_{12}} = v_{12}.$$

PROOF. (1) We have $u = e_0 + v$ for $e_0 = 1 - e_1 - e_2 \in \widehat{\mathrm{J}}_{00}$ an idempotent orthogonal to e_1, e_2, and hence by Peirce Orthogonality 13.3.1(1) $u^2 = e_0^2 + v^2 = e_0 + e_1 + e_2 = 1$. Therefore, by the Involution Lemma 6.1.10 the map U_u is an involutory automorphism on $\widehat{\mathrm{J}}$, and it (and its inverse!) leave the ideal J invariant, so it induces an involutory automorphism $\overline{}$ on J. Making heavy use of the Peirce Recovery formulas 13.2.1(2) for $e = e_1 + e_2$, we see that $\overline{} = U_u = U_{e_0+v}$ reduces by Peirce Orthogonality 13.3.1(3) to $U_{e_0} = \mathbb{1}_{\mathrm{J}_0}$ on $\mathrm{J}_0(e)$, to $U_{e_0,v} = U_{1,v} = V_v$ on $\mathrm{J}_1(e)$ [note that $U_{e,v}(\mathrm{J}_1(e)) \subseteq U_{\mathrm{J}_2}(\mathrm{J}_1) = 0$ by Peirce Orthogonality Rules 8.2.1], and to U_v on the Peirce subalgebra $\mathrm{J}_2(e)$ [where v is involutory, so $\overline{e_i} = e_j$ from Connection Action 10.1.3(2)]. Similarly, (2),(3) follow from Connection Involution 10.1.3(2),(3) applied to $\mathrm{J}_2(e)$. ☐

And once more we can strengthen connection by means of a diagonal isotope. We derive only the form we will need later, and leave the general case as an exercise.

Creating Involutions Theorem 14.4.6 (1) *We can strengthen the connectivity by passing to a diagonal isotope*: *if* $\mathcal{E} = \{e_1, \ldots, e_n\}$ *is a supplementary set of mutually orthogonal idempotents in a unital Jordan algebra* J *with* e_1 weakly *connected to* e_j *by connecting elements* $v_{1j} \in \mathrm{J}_{1j}$ $(j \neq 1)$, *then for* $u_{11} := e_1$, $u_{jj} := q_{jj}(v_{1j})^{-1}$ *the diagonal element* $u = \sum_{j=1}^{n} u_{jj}$ *is invertible, and the diagonal isotope* $\mathrm{J}^{(u)}$ *has a supplementary set* $\mathcal{E}^{(u)} = \{e_1^{(u)}, \ldots, e_n^{(u)}\}$ *of mutually orthogonal idempotents* $e_j^{(u)} := q_{jj}(v_{1j})$ strongly *connected to* $e_1^{(u)} := e_1$ *via the now-involutory* $v_{1j} \in \mathrm{J}_{ij}^{(u)}$,

$$v_{1j}^{(2,u)} = e_1^{(u)} + e_j^{(u)}.$$

(2) *This isotope has exactly the same Peirce decomposition as the original,*

$$\left(\mathrm{J}^{(u)}\right)_{ij}(\mathcal{E}^{(u)}) = \mathrm{J}_{ij}(\mathcal{E}),$$

and the new quadratic forms and Peirce specializations are given (*for distinct* $1, i, j$) *by*

$$q_{11}^{(u)}(x_{1j}) = q_{11}(x_{1j}), \qquad q_{ii}^{(u)}(x_{ij}) = q_{ii}(x_{ij}, q_{jj}(v_{1j})^{-1} \bullet x_{ij}),$$

$$\sigma^{(u)}(a_{11}) = \sigma(a_{11}), \qquad \sigma^{(u)}(a_{ii}) = \sigma(a_{ii})\sigma(q_{ii}(v_{1i}))^{-1}.$$

PROOF. We can derive this from the single-idempotent case Creating Involutions Proposition 10.2.2, noting that by Peirce Orthogonality the space $J_2(e_1+e_j)$ is a subalgebra of $\widetilde{J} := J^{(u)}$ coinciding with $J_2(e_1^{(u)}+e_j^{(u)})^{(e_1+u_{jj})}$. Alternately, we can derive it from our multiple Peirce information as follows: (1) We know that u is invertible with $u^{-1} = \sum_{i=1}^n u_{ii}^{-1}$ by Diagonal Invertibility 14.4.1, and in $\widetilde{J}$ has $\tilde{1} := 1^{(u)} = u^{-1} = \sum_{j=1}^n \tilde{e}_i$ for $\tilde{e}_i := e_i^{(u)} := u_{ii}^{-1}$ orthogonal idempotents since $\tilde{e}_i^{\tilde{2}} = U_{u_{ii}^{-1}} u = U_{u_{ii}^{-1}}(u_{ii}) = u_{ii}^{-1} = \tilde{e}_i$ and $\tilde{e}_i \tilde{\bullet} \tilde{e}_j = \frac{1}{2}\{\tilde{e}_i, u, \tilde{e}_j\} \in \{J_{ii}, u, J_{jj}\} = 0$ when $i \neq j$ by Peirce orthogonality and diagonality of u.

The new idempotents are strongly connected by the old v_{1j}: $v_{1j}^{(2,u)} = U_{v_{1j}}(u_{11}+u_{jj})$ by Peirce Orthogonality, where $U_{v_{1j}}(u_{11}) = U_{v_{1j}}(e_1) = q_{jj}(v_{1j}) = u_{jj}^{-1} = e_j^{(u)}$ and $U_{v_{1j}}(u_{jj}) = U_{v_{1j}} E_{jj}\big((v_{1j}^2)^{-1}\big) = E_{11}\big(U_{v_{1j}}(v_{1j}^{-2})\big) = E_{11}(e_1+e_j) = e_1 = e_1^{(u)}$ (as in Flipping 10.2.2(5)).

(2) The new Peirce spaces are $\widetilde{J}_{ij} = \tilde{U}_{\tilde{e}_i,\tilde{e}_j}\widetilde{J} \subseteq U_{J_{ii},J_{jj}}(J) \subseteq J_{ij}$ by the Peirce U and Triple Products 13.3.1(2), so $\widetilde{J} = \bigoplus \widetilde{J}_{ij} \subseteq \bigoplus J_{ij} = J$ implies that $\widetilde{J}_{ij} = J_{ij}$ for each i, j. The new Peirce Quadratic Forms 14.3.1(1) and Peirce Specializations 14.2.1 take the form $\tilde{E}_{ii}(x_{ij}^{\tilde{2}}) = E_{ii}(U_{x_{ij}}u)$ [using (1)] $= U_{x_{ij}}u_{jj} = q_{ii}(x_{ij}, u_{jj} \bullet x_{ij})$ [by $Uijq$ Rules 14.3.1(2)] and $\tilde{V}_{a_{ii}} = V_{a_{ii},u_{ii}}$ [by Jordan Homotope 7.2.1(2)] $= V_{a_{ii}}V_{u_{ii}}$ [by Peirce Specialization]. In particular, for $i = 1, u_{11} = e_1$ we have $\tilde{q}_{11}(x_{1j}) = q_{11}(x_{1j})$ and $\tilde{V}_{a_{ii}} = V_{a_{ii}}$, while for $i \neq 1$ we have $\tilde{q}_{ii}(x_{ij}) = q_{ii}(x_{ij}, q_{jj}(v_{1j})^{-1} \bullet x_{ij})$ and we have $\tilde{V}_{a_{ii}} = V_{a_{ii}}V_{q_{ii}(v_{1i})^{-1}} = V_{a_{ii}}V_{q_{ii}(v_{1i})}^{-1} = \sigma(a_{ii})\sigma(q_{ii}(v_{1i}))^{-1}$. ◻

EXERCISE 14.4.6 Let $\mathcal{E} = \{e_1, \ldots, e_n\}$ be a supplementary orthogonal family, and u_{jj} arbitrary invertible elements in J_{jj}. (1) Show that $u = \sum_{j=1}^n u_{jj}$ is invertible, and the diagonal isotope $J^{(u)}$ has a supplementary orthogonal family $\mathcal{E}^{(u)} = \{e_1^{(u)}, \ldots, e_n^{(u)}\}$ for $e_j^{(u)} := u_{jj}^{-1}$. (2) Show that the Peirce decompositions in $J^{(u)}$ and J coincide: $\big(J^{(u)}\big)_{ij}^{(\mathcal{E}^{(u)})} = J_{ij}^{(\mathcal{E})}$. (3) Show that the new quadratic forms and Peirce specializations are given by $q_{ii}^{(u)}(x_{ij}) = q_{ii}(x_{ij}, u_{jj} \bullet x_{ij})$, $\sigma^{(u)}(a_{ii}) = V_{a_{ii}}^{(u)} = V_{a_{ii}}V_{u_{ii}} = \sigma(a_{ii})\sigma(u_{ii})$ on J_{ij}.

15

Hermitian Symmetries

In the Hermitian and Spin Coordinatization Theorems, we needed two connected idempotents and a spin or hermitian condition. Once we get to three or more connected idempotents we don't need to impose additional behavioral conditions — the algebras all have a uniform pattern, and the key to having hermitian structure is the mere existence of hermitian frames.

15.1 Hermitian Frames

Hermitian frames consist of hermitian matrix units, the Jordan analogue of associative matrix units.

Hermitian n-Frame Definition 15.1.1 *For $n \geq 3$, a* **hermitian n-frame** *in a unital Jordan algebra* J *is a family of* **hermitian $n \times n$ matrix units** $\mathcal{H} = \{h_{ij} \mid 1 \leq i, j \leq n\}$, *consisting of orthogonal idempotents* $h_{11}, \dots, h_{nn}$ *together with strongly connecting elements* $h_{ij} = h_{ji}$ $(i \neq j)$ *satisfying*:

(1) *the* **Supplementary Rule** $\sum_{i=1}^{n} h_{ii} = 1$;

(2) *the* **Hermitian Product Rules** *for distinct indices* i, j, k

$$h_{ii}^2 = h_{ii}, \quad h_{ij}^2 = h_{ii} + h_{jj}, \quad \{h_{ii}, h_{ij}\} = h_{ij}, \quad \{h_{ij}, h_{jk}\} = h_{ik};$$

(3) *and the* **Hermitian Orthogonality Rule**

$$\{h_{ij}, h_{k\ell}\} = 0 \qquad \textit{if} \qquad \{i, j\} \cap \{k, \ell\} = \emptyset.$$

Note also that by definition *a hermitian n-frame is always strong.* In contrast to the case of 2-frames in Chapter 12, we will not need to impose any cyclicity conditions, hence our coordinate algebras will not have to be symmetrically generated.

Special Frame Example 15.1.2 (1) *The* **standard hermitian** n**-frame** *for* $\mathcal{H}_n(\mathrm{D}, -)$ *consists of the* **standard hermitian matrix units**

$$h_{ii} := 1[ii] = e_{ii}\,, \qquad h_{ij} := 1[ij] = e_{ij} + e_{ji} = h_{ji}.$$

(2) *In general, if* $\mathcal{E}_a = \{e_{ij} \mid 1 \le i, j \le n\}$ *is a family of* $n \times n$ **associative matrix units** *for an associative algebra* A*, in the usual sense that*

$$1 = \textstyle\sum_{i=1}^{n} e_{ii}, \qquad e_{ij}e_{k\ell} = \delta_{jk}e_{i\ell},$$

then the associated family of **symmetric matrix units**

$$\mathcal{H}(\mathcal{E}_a): \qquad h_{ii} := e_{ii}, \qquad h_{ij} := e_{ij} + e_{ji},$$

is a hermitian n-frame for the Jordan algebra A^+ (*or any special subalgebra* $\mathrm{J} \subseteq \mathrm{A}^+$ *containing* $\mathcal{H}$); *if an involution* $*$ *on* A *has* $e_{ij}^* = e_{ji}$ *for all* i, j*, then* $\mathcal{H}$ *is a hermitian n-frame for* $\mathcal{H}(\mathrm{A}, *)$.

(3) *Conversely, in any unital special Jordan algebra* $1 \in \mathrm{J} \subseteq \mathrm{A}^+$ *these are the only hermitian families*:

$$\mathcal{H} \text{ hermitian in J} \Longrightarrow \mathcal{H} = \mathcal{H}(\mathcal{E}_a) \text{ for } e_{ii} := h_{ii},\ e_{ij} := e_{ii}h_{ij}e_{jj}.$$

PROOF. (1), (2) are straightforward associative calculations; the condition $e_{ij}^* = e_{ji}$ in (2) guarantees that the h_{ij} are hermitian, $\mathcal{H}(\mathcal{E}_a) \subseteq \mathcal{H}(\mathrm{A}, *)$. To see the converse (3), we use the Peirce Decomposition 13.4.1 for A^+: by the hermitian matrix units condition 15.1.1 the $e_{ii} := h_{ii}$ are supplementary associative idempotents, and they are associatively orthogonal because (as we have seen before) $\{e, x\} = 0 \Rightarrow U_e x = \frac{1}{2}(\{e, \{e, x\}\} - \{e, x\}) = 0 \Rightarrow ex = e(ex + xe) - exe = 0$ and dually. Then the associative Peirce components $e_{ij} := e_{ii}h_{ij}e_{jj}$ satisfy $h_{ij} = e_{ij} + e_{ji}$, and by 15.1.1(2) $h_{ij}^2 = e_{ii} + e_{jj} \Rightarrow e_{ij}e_{ji} = e_{ii}$ for $i \ne j$, $\{h_{ij}, h_{jk}\} = h_{ik} \Rightarrow e_{ij}e_{jk} = e_{ik}$ for $i, j, k \ne$, $e_{ij} \in \mathrm{A}_{ij} \Rightarrow e_{ij}e_{k\ell} = 0$ if $j \ne k$, so $\mathcal{E}_a = \{e_{ij}\}_{1 \le i,j \le n}$ forms a family of associative matrix units. $\square$

Thus another way to say that $\mathcal{H}$ is a hermitian n-frame is that J *contains a copy of* $\mathcal{H}_n(\Phi) = (\bigoplus_{i=1}^{n} \Phi e_{ii}) \oplus (\bigoplus_{i<j} \Phi(e_{ij} + e_{ji}))$ *such that* $\mathcal{H}$ *is the copy of the standard symmetric matrix units.*

Having each pair h_{ii}, h_{jj} strongly connected by h_{ij} is not enough: the connecting elements themselves must interact according to $\{h_{ij}, h_{jk}\} = \{h_{ik}\}$. In case the h_{ij} are all tangled up, we can start all over again and replace them by an untangled family built up from only the connecting elements h_{1j}.

Hermitian Completion Lemma 15.1.3 *Any supplementary family* $\mathcal{E} = \{e_1, \ldots, e_n\}$ *of* n *orthogonal idempotents which are strongly connected can be imbedded in a hermitian n-frame*: *if* e_1 *is strongly connected to each* e_j ($j = 2, 3, \ldots, n$) *via* $v_{1j} \in \mathrm{J}_{1j}$*, then these can be completed to a family* $\mathcal{H} = \{h_{ij} \mid 1 \le i, j \le n\}$ *of hermitian matrix units by taking* (*for distinct indices* $1, i, j$)

$$h_{ii} := e_i, \quad h_{1j} := v_{1j} =: h_{j1}, \quad h_{ij} := \{v_{1i}, v_{1j}\} =: h_{ji}.$$

PROOF. Let J_{ij} denote the Peirce spaces with respect to the orthogonal idempotents $\{e_i\}$. By the Peirce Identity Principle 13.5.1 it suffices to prove this in special algebras. But the supplementary $e_{kk} := e_k$ are associative idempotents, and by the Special Example 15.1.2(3) in each $\mathrm{B}_{ij} = \mathrm{J}_{11} + \mathrm{J}_{1j} + \mathrm{J}_{jj}$ we have $v_{1j} := e_{1j} + e_{j1}$ for 2×2 associative matrix units $e_{11}, e_{1j}, e_{j1}, e_{jj}$, so we obtain a full family of $n \times n$ associative matrix units $\mathcal{E}_a = \{e_{ij} := e_{i1}e_{1j}\}$ because $e_{ij}e_{k\ell} = e_{i1}e_{1j}e_{k1}e_{1\ell} = \delta_{jk}e_{i1}e_{11}e_{1\ell} = \delta_{jk}e_{i1}e_{1\ell} = \delta_{jk}e_{i\ell}$. Hence the associated symmetric matrix units $\mathcal{H}(\mathcal{E}_a) = \{e_{ii}, e_{ij} + e_{ji}\}$ of the Special Example (2) are a hermitian family, given for $i = j$ by $h_{ii} = e_{ii} = e_i$, for $i = 1 \neq j$ by $h_{1j} = e_{1j} + e_{j1} = v_{1j}$, and for distinct $1, i, j \neq$ by $h_{ij} = e_{ij} + e_{ji} = e_{i1}e_{1j} + e_{j1}e_{1i} = \{e_{1i} + e_{i1}, e_{1j} + e_{j1}\} = \{v_{1i}, v_{1j}\}$. Thus our definitions above do lead to a hermitian completion. □

EXERCISE 15.1.3* Give a direct Jordan proof of Hermitian Completion by tedious calculation, showing that for $i, j \neq 1$ the new $h_{ij} := \{h_{i1}, h_{1j}\}$ continue to satisfy $h_{ij}^2 = h_{ii} + h_{jj}$, $\{h_{ii}, h_{ij}\} = h_{ij}$, $\{h_{ij}, h_{jk}\} = h_{ik}$, $\{h_{ij}, h_{k\ell}\} = 0$ if $\{i,j\} \cap \{k,\ell\} = \emptyset$. After all the calculations are over, step back and survey the carnage, then wistfully appreciate the power of the Peirce Identity Principle.

15.2 Hermitian Symmetries

The Coordinatization Theorem we are aiming for involves two aspects: finding the coordinate algebra D, and using it to coordinatize the entire algebra. The first involves Peirce relations just in the "Northwest" 3×3 chunk of the algebra, while the second involves constructing the connection symmetries that guarantee that all off-diagonal spaces look like D and all diagonal ones like $\mathcal{H}(\mathrm{D}, -)$. In this section our main goal is to construct Hermitian Symmetries corresponding to each permutation π of the indices, which flow directly out of Hermitian Involutions corresponding to the transpositions (ij).

Hermitian Involutions Lemma 15.2.1 *A hermitian n-frame $\mathcal{H} = \{h_{ij}\}$ gives rise to a family $\mathcal{U}(\mathcal{H})$ of* **hermitian involutions** *$\mathcal{U}_{ij}$ for $i \neq j$, involutory automorphisms of* J *given by*:

(1) $$\mathcal{U}_{ij} = \mathcal{U}_{ji} := U_{u_{ij}} \quad \textit{for} \quad u_{ij} := 1 - (h_{ii} + h_{jj}) + h_{ij}.$$

These permute the hermitian matrix units, involutions, and the Peirce spaces according to the transposition $\tau = (ij)$ on the index set as follows:

(2) *we have the* **Action Formula**

$$\mathcal{U}_{ij} = \begin{cases} \mathbb{1} & \textit{on} \quad \sum_{k,l \neq i,j} \mathrm{J}_{k\ell} \\ V_{h_{ij}} & \textit{on} \quad \sum_{k \neq i,j} (\mathrm{J}_{ki} + \mathrm{J}_{kj}) \\ U_{h_{ij}} & \textit{on} \quad \mathrm{J}_{ii} + \mathrm{J}_{ij} + \mathrm{J}_{jj}, \end{cases}$$

(3) *the* **Index Permutation Principle**

$$\mathcal{U}_{ij}(h_{kk}) = h_{\tau(k)\tau(k)}, \qquad \mathcal{U}_{ij}(h_{k\ell}) = h_{\tau(k)\tau(\ell)},$$
$$\mathcal{U}_{ij}(u_{k\ell}) = u_{\tau(k)\tau(\ell)}, \qquad \mathcal{U}_{ij}(\mathrm{J}_{k\ell}) = \mathrm{J}_{\tau(k)\tau(\ell)},$$

(4) *and the* **Fundamental Interaction Formula**

$$\mathcal{U}_{ij}\mathcal{U}_{k\ell}\mathcal{U}_{ij} = \mathcal{U}_{\tau(k)\tau(\ell)} \qquad (\tau := (ij)).$$

PROOF. (1)–(2) Note that the elements u_{ij} do satisfy $u_{ij} = u_{ji}$, $u_{ij}^2 = 1$, so by the Connection Involution Lemma 14.4.5(1) they determine involutory automorphisms $\mathcal{U}_{ij} = \mathcal{U}_{ji}$ on J, $\mathcal{U}_{ij}^2 = \mathbb{1}_{\mathrm{J}}$ as in (1), with action given in (2) using the Peirce Recovery Theorem 13.2.1(2) to identify $\mathrm{J}_0(h_{ii}+h_{jj}) = \sum_{k,\ell\neq i,j}\mathrm{J}_{k\ell}$, $\mathrm{J}_1(h_{ii}+h_{jj}) = \sum_{k\neq i,j}(\mathrm{J}_{ki}+\mathrm{J}_{kj})$, $\mathrm{J}_2(h_{ii}+h_{jj}) = \mathrm{J}_{ii}+\mathrm{J}_{ij}+\mathrm{J}_{jj}$.

(3) The first index permutation $\mathcal{U}_{ij}(h_{kk}) = h_{\tau(k)\tau(k)}$ of (3) follows for $k \neq i,j$ because then $\tau(k) = k$ and $\mathcal{U}$ fixes $h_{kk} \in \mathrm{J}_0$ by (2), and for $k = i$ follows by $\mathcal{U}_{ij}(h_{ii}) = h_{jj}$ from the Connection Involution Lemma again. The second permutation of (3) on matrix units $h_{k\ell}$ is trivial if k,ℓ are distinct from i,j [then τ fixes the indices k,ℓ and $\mathcal{U}$ fixes the element $h_{k\ell} \in \mathrm{J}_0$ by (2) again]; if $k,\ell = i,j$ agree entirely, then by symmetry we can assume that $(k,\ell) = (i,j)$, where $\mathcal{U}_{ij}(h_{ij}) = h_{ij}$ [by Connection Involution (3)] $= h_{ji} = h_{\tau(i)\tau(j)}$; while if k,ℓ agree just once with i,j we may assume that $k = i, \ell \neq i,j,k$, in which case $\mathcal{U}_{ij}(h_{i\ell}) = V_{h_{ji}}(h_{i\ell})$ [by (2)] $= h_{j\ell}$ [by the Hermitian Product Rule 15.1.1(2)] $= h_{\tau(i)\tau(\ell)}$. The third permutation of (3) on involutions $u_{k\ell}$ then follows because they are linear combinations of matrix units. The fourth permutation of (3) on spaces follows from the fact that an automorphism $\mathcal{U}$ takes Peirce spaces wherever their idempotents lead them, $\mathcal{U}(U_{e,f}\mathrm{J}) = U_{\mathcal{U}(e),\mathcal{U}(f)}(\mathrm{J})$.

(4) follows directly from these: $\mathcal{U}_{ij}\mathcal{U}_{k\ell}\mathcal{U}_{ij} = U_{u_{ij}}U_{u_{k\ell}}U_{u_{ij}}$ [by definition of $\mathcal{U}$] $= U_{U_{u_{ij}}(u_{k\ell})}$ [Fundamental Formula] $= U_{\mathcal{U}_{ij}(u_{k\ell})}$ [by definition again] $= U_{u_{\tau(k)\tau(\ell)}}$ [by (3)] $= \mathcal{U}_{\tau(k)\tau(\ell)}$ [by definition yet again]. □

EXERCISE 15.2.1A* Give an alternate proof using (as far you can) the Peirce Identity Principle 13.5.1 and calculations with associative matrix units of Special Example 15.1.2(3).

EXERCISE 15.2.1B* Establish the Lemma, discarding all your Principles, by direct calculation using Peirce Orthogonality on $u_{ij} = u_0 + h_{ij}$ for $u_0 \in \mathrm{J}_0(e_i + e_j)$.

Just as the transpositions generate all permutations, so the hermitian involutions generate all hermitian symmetries.

Hermitian Symmetries Theorem 15.2.2 *If $\mathcal{H} = \{h_{ij}\}$ is a hermitian n-frame, and $\mathcal{U}_{ij}$ are the associated hermitian involutions*

$$\mathcal{U}_{ij} = U_{u_{ij}} \in \mathcal{A}ut(\mathrm{J}), \qquad \mathcal{U}_{ij}^2 = \mathbb{1}_{\mathrm{J}},$$

then we have a monomorphism $\pi \mapsto \mathcal{U}_\pi$ of the symmetric group $S_n \to \mathcal{A}ut(\mathrm{J})$ extending the map $(ij) \mapsto \mathcal{U}_{(ij)} := \mathcal{U}_{ij}$,

$$(1) \qquad \mathcal{U}_1 = \mathbb{1}_{\mathrm{J}}, \qquad \mathcal{U}_{\sigma\circ\pi} = \mathcal{U}_\sigma \circ \mathcal{U}_\pi \qquad (\sigma, \pi \in S_n).$$

(2) *These* **hermitian symmetries** *naturally permute the hermitian matrix units and involutions and Peirce spaces by the* **Index Permutation Principle**

$$\mathcal{U}_\pi(h_{ii}) = h_{\pi(i)\pi(i)}, \qquad \mathcal{U}_\pi(h_{ij}) = h_{\pi(i)\pi(j)},$$
$$\mathcal{U}_\pi(u_{ij}) = u_{\pi(i)\pi(j)}, \qquad \mathcal{U}_\pi(\mathrm{J}_{ij}) = \mathrm{J}_{\pi(i)\pi(j)}.$$

(3) *Furthermore, we have the* **Agreement Principle**

$$\mathcal{U}_\pi = \mathbb{1} \quad \text{on } \mathrm{J}_{ij} \text{ if } \pi \text{ fixes both } i \text{ and } j\colon \pi(i) = i,\ \pi(j) = j,$$
$$\mathcal{U}_\pi = \mathcal{U}_\sigma \quad \text{on } \mathrm{J}_{ij} \text{ if } \pi(i) = \sigma(i),\ \pi(j) = \sigma(j).$$

PROOF. The hard part is showing (1), that the map $(ij) \mapsto \mathcal{U}_{(ij)}$ extends to a well-defined homomorphism $\pi \mapsto \mathcal{U}_\pi$ on all of S_n. Once we have established this, the rest will be an anticlimax. For (2) it suffices if Index Permutation holds for the generators $\mathcal{U}_{(ij)}$, which is just the import of Index Permutation Principle 15.2.1(3). For the First Agreement Principle in (3), if π fixes i, j it can be written as a product of transpositions $(k\ell)$ for $k, \ell \neq i, j$. Then [by (1) and well-definedness] $\mathcal{U}_\pi$ is a product of $\mathcal{U}_{(k\ell)}$, which by the Action Formula 15.2.1(2) are all the identity on J_{ij}. From this the Second Agreement Principle follows: $\pi(i) = \sigma(i),\ \pi(j) = \sigma(j) \Longrightarrow \sigma^{-1}\pi(i) = i, \sigma^{-1}\pi(j) = j \Longrightarrow \mathcal{U}_\sigma^{-1}\mathcal{U}_\pi = \mathcal{U}_{\sigma^{-1}\circ\pi}$ [by (2)] $= \mathbb{1}$ on J_{ij} [by First Agreement] $\Longrightarrow \mathcal{U}_\sigma = \mathcal{U}_\pi$ on J_{ij}.

The key to (1) is that S_n can be presented by generators $t_{ij} = t_{ji}$ for $i \neq j = 1, 2, \ldots, n$ and t-**relations**

$$t_{ij}^2 = 1, \qquad t_{ij}t_{k\ell}t_{ij} = t_{\tau(k)\tau(\ell)} \qquad (\tau = (ij)),$$

which are precisely the relations which we, with admirable foresight, have established in the Fundamental Interaction Formula 15.2.1(3). Let $\mathcal{F}$ be the free group on generators $t_{ij} = t_{ji}$ for $i \neq j = 1, 2, \ldots, n$, and $\mathcal{K}$ the normal subgroup generated by the t-relations, i.e., by all elements $t_{ij}^2,\ t_{ij}t_{k\ell}t_{ij}t_{\tau(k)\tau(\ell)}$. By the universal property of the free group, the set-theoretic map $t_{ij} \mapsto \mathcal{U}_{(ij)}$ extends to an epimorphism $\varphi_0 : \mathcal{F} \to \mathcal{U} \subseteq \mathcal{A}ut(\mathrm{J})$ for $\mathcal{U}$ the subgroup generated by the Hermitian Involutions Lemma 15.2.1. By (1), (3) of that Lemma the generators of $\mathcal{K}$ lie in the kernel of φ, so we have an induced epimorphism $\varphi : \mathcal{F}/\mathcal{K} \to \mathcal{U}$. We use $\tilde{t}$ to denote the coset of $t \in \mathcal{F}$ modulo $\mathcal{K}$. Furthermore, we have a restriction homomorphism $\rho : \mathcal{U} \to \ Symm(\mathcal{H}) = S_n$

for $\mathcal{H} = \{h_{11}, \ldots, h_{nn}\}$, since by the Index Permutation Principle (3) the $\mathcal{U}_{(ij)}$ (hence all of $\mathcal{U}$) stabilize $\mathcal{H}$; the image of ρ contains by Hermitian Involutions (1) the generators (ij) of S_n, hence is all of S_n, so ρ is an epimorphism too. The composite $\tilde{\rho} := \rho \circ \varphi : \mathcal{F}/\mathcal{K} \to \mathcal{U} \to \ Symm(\mathcal{H}) = S_n$ sends the coset $\tilde{t}_{ij} \mapsto \mathcal{U}_{(ij)} \mapsto (ij)$. To show that the epimorphisms φ, ρ are isomorphisms (so ρ^{-1} is an isomorphism $S_n \to \mathcal{U}$ as desired) it suffices if the composite is injective, i.e.,

$$\tilde{\rho}\colon \mathcal{F}/\mathcal{K} \to S_n \text{ via } \tilde{t}_{ij} \mapsto (ij) \text{ is a presentation of } S_n.$$

This is a moderately "well-known" group-theoretic result, which we proceed to prove *sotto voce.*

Suppose $\tilde{\rho}$ is *not* injective, and choose a product $p = \tilde{t}_{i_1 j_1} \cdots \tilde{t}_{i_N j_N} \neq 1$ of *shortest length* N in $\mathrm{Ker}(\tilde{\rho})$. We can rewrite p as $p^{(n)} p^{(n-1)} \ldots p^{(2)}$ without changing its length, where $p^{(k)} = \prod_{j<k} \tilde{t}_{kj}$ involves transpositions with indices $\leq k$. Indeed, we can move all $\tilde{t}_{ij}$ for $i, j < n$ to the right of all $\tilde{t}_{nk}$ (keeping the same overall length, but perhaps increasing the number of occurrences of the index n) via

$$\tilde{t}_{ij}\tilde{t}_{nk} = \begin{cases} \tilde{t}_{nk}\tilde{t}_{ij} & \text{if } i, j, k \text{ distinct } < n, \\ \tilde{t}_{nk}\tilde{t}_{ni} & \text{if } k = j \text{ (dually if } k = i), \end{cases}$$

in $\mathcal{F}/\mathcal{K}$, since for τ the transposition (nk) we have $\tilde{t}_{nk}^{-1}\tilde{t}_{ij}\tilde{t}_{nk} = \tilde{t}_{nk}\tilde{t}_{ij}\tilde{t}_{nk} = \tilde{t}_{\tau(1)\tau(j)}$ by the defining t-relations for $\mathcal{K}$, where $\tau(i) = i, \tau(j) = j$ if $i, j \neq n, k$, and $\tau(i) = i, \tau(j) = \tau(k) = n$ if $j = k$. Thus we get $p = p^{(n)} p'$ for p' a product of $\tilde{t}_{ij}$ for $i, j < n$; repeating this procedure, we get $p' = p^{(n-1)} \cdots p^{(2)}$ and $p = p^{(n)} p^{(n-1)} \cdots p^{(2)}$. Deleting any of these $p^{(k)}$ which reduce to 1 (empty products), we have the **p-product**

$$p = p^{(r)} p^{(s)} \cdots p^{(q)} \text{ for } n \geq r > s > \cdots > q \geq 2 \qquad (p^{(k)} \neq 1).$$

By minimality of N, the j's in $p^{(r)}$ *must be distinct*,

$$p^{(r)} = \tilde{t}_{r j_m} \cdots \tilde{t}_{r j_1} \Longrightarrow j_k \neq j_\ell,$$

since any repetition would lead to a sub-expression

$$\begin{aligned}
\tilde{t}_{rj}\left(\textstyle\prod_{r>k\neq j} \tilde{t}_{rk}\right)\tilde{t}_{rj} &= \tilde{t}_{rj}\left(\textstyle\prod_{r>k\neq j} \tilde{t}_{rk}\right)\tilde{t}_{rj}^{-1} && \text{[by the } t\text{-relations]} \\
&= \textstyle\prod_{r>k\neq j}\left(\tilde{t}_{rj}\tilde{t}_{rk}\tilde{t}_{rj}^{-1}\right) && \text{[conjugation by } \tilde{t}_{rj} \text{ is a group automorphism]} \\
&= \textstyle\prod_{r>k\neq j}\left(\tilde{t}_{rj}\tilde{t}_{rk}\tilde{t}_{rj}\right) && \text{[by the } t\text{-relations]} \\
&= \textstyle\prod_{r>k\neq j} \tilde{t}_{jk} && \text{[by the } t\text{-relations, since } k \neq r, j \text{ is fixed by } \tau].
\end{aligned}$$

But this would lead to an expression for p of length $N-2$, contrary to minimality of N. Applying $\tilde{\rho}$ to the p-product gives $1 = \tilde{\rho}(p) = \tilde{\rho}(p^{(r)})\tilde{\rho}(p^{(s)}) \cdots \tilde{\rho}(p^{(q)})$ in S_n; now $\tilde{\rho}(p^{(i)})$ for $i < r$ involves only transpositions (ik) for $r > i > k$, all of which fix the index r, so acting on r gives $r = 1(r) = \tilde{\rho}(p^{(r)})\tilde{\rho}(p^{(s)}) \cdots \tilde{\rho}(p^{(q)})(r)$ [by p-product and $\tilde{\rho}(p) = 1$] $= \tilde{\rho}(p^{(r)})(r) = \tau_{rj_m} \cdots \tau_{rj_1}(r)$ [by p-product] $= j_1$ [since *by distinctness* all other $r, j_k \neq j_1$ have τ_{rj_k} fixing j_1], contrary to $r > j_1$. Thus we have reached a contradiction, so no $p \neq 1$ in $\mathrm{Ker}(\tilde{\rho})$ exists, and the epimorphism $\tilde{\rho}$ is an isomorphism. Thus the t-relations are indeed a presentation of S_n by generators and relations. ☐

15.3 Problems for Chapter 15

Problem 15.1 (1) Show that the *scalar annihilators* $\mathcal{A}nn_\Phi(\mathrm{J}) := \{\alpha \in \Phi \mid \alpha \mathrm{J} = \mathbf{0}\}$ form an ideal in Φ, and that J is always a $\overline{\Phi}$-algebra for $\overline{\Phi} = \Phi/(\mathcal{A}nn_\Phi(\mathrm{J})$. We say that Φ acts *faithfully* on J if the scalar annihilators vanish, $\mathcal{A}nn_\Phi(\mathrm{J}) = 0$. (2) For a unital Jordan algebra show that the scalar annihilators coincide with the scalars that annihilate 1, $\{\alpha \in \Phi \mid \alpha 1 = 0\}$, so α acts faithfully on J iff it acts faithfully on the unit 1, $\alpha \mathrm{J} = \mathbf{0} \Leftrightarrow \alpha 1 = 0$. (3) Show that if $\mathcal{H}$ is a family of $n \times n$ hermitian matrix units in $\mathrm{J} \neq \mathbf{0}$, and Φ acts faithfully on J, then the units are linearly independent (in particular, nonzero) and J contains an isomorphic copy of $\mathcal{H}_n(\Phi)$. (4) Conclude that a Jordan algebra J contains a family of $n \times n$ hermitian matrix units iff it contains an isomorphic copy of $\mathcal{H}_n(\overline{\Phi})$ (in the category of unital algebras).

16

The Coordinate Algebra

In this chapter we take up the task of finding the coordinate algebra D. This is strictly a matter of the "upper 3×3 chunk" of the algebra. The coordinates themselves live in the upper 2×2 chunk. Recall from the 2×2 coordinatization in Chapters 11 and 12 that spin frames never did develop a coordinate algebra, and hermitian frames were only fitted unnatural coordinates from $\mathcal{E}nd_\Phi(\mathrm{J}_1(e))$ upon the condition that $\mathrm{J}_1(e)$ was cyclic as a $\mathrm{J}_2(e)$-module. But once a frame has room to expand into the Peirce spaces J_{13} and J_{23}, the coordinate algebra J_{12} automatically has a product of its very own.

16.1 The Coordinate Triple

In order for us to coordinatize hermitian matrices, we need a unital coordinate algebra with involution for the off-diagonal Peirce spaces, plus a designated coordinate subspace for the diagonal matrix entries. Luckily, in the presence of $\frac{1}{2}$ this diagonal coordinate subspace must be just the full space of all hermitian elements, but if we did not have $\frac{1}{2}$ our life would be more complicated and there could be many possibilities for the diagonal subspace.[1]

Hermitian Coordinate Algebra Definition 16.1.1 *If* $\mathcal{H} = \{h_{ij}\}$ *is a hermitian* n*-frame for* $n \geq 3$ *in a unital Jordan algebra* J*, then the* **hermitian coordinate $*$-algebra** D *for* J *determined by* $\mathcal{H}$ *is defined as follows. Recall that* $\mathcal{U}_{(12)} = U_{h_{12}}$ *on* J_{12}*, that* $\mathcal{U}_{(ij)} = V_{h_{ij}}$ *on* J_{ik} *for* $i, j, k \neq$ *by the Action Formula* 15.2.1(2)*, and that the fixed space of* $\mathcal{U}_{(12)}$ *on* J_{12} *is* $\{\mathrm{J}_{11}, h_{12}\}$ *by Connection Involution* 14.4.5(3)*.*

(1) *The* **coordinate algebra** *is the space* $\mathrm{D} := \mathrm{J}_{12}$ *with product*

[1] Over an imperfect field Φ of characteristic 2 (i.e., $\Phi^2 < \Phi$) with identity involution we would have to accept the simple algebra of symmetric $n \times n$ matrices with diagonal entries only from Φ^2, and over $\mathbb{Z}$ we would have to accept the prime ring of symmetric $n \times n$ matrices over $\mathbb{Z}[t]$ with diagonal entries only from $\mathbb{Z}[t]^2 = \mathbb{Z}[t^2] + 2\mathbb{Z}[t]$.

$$a \cdot b := \{\{a, h_{23}\}, \{h_{31}, b\}\} = \{\mathcal{U}_{(23)}(a), \mathcal{U}_{(13)}(b)\}$$
$$(a, b \in \mathrm{D} = \mathrm{J}_{12},\ \mathcal{U}_{(ij)} = U_{u_{ij}} \quad \textit{for} \quad u_{ij} = 1 - (e_i + e_j) + h_{ij}).$$

(2) *The* **coordinate involution** $\overline{}$ *on* D *is*

$$\bar{a} := U_{h_{12}}(a) = \mathcal{U}_{(12)}a.$$

(3) *The* **diagonal coordinate space** *is the image* $\mathrm{D}_0 = \mathcal{H}(\mathrm{D}, -) \subseteq \mathrm{D}$ *of the space* J_{11} *under the* **diagonalization map** $\delta_0 = V_{h_{12}}$:

$$\mathrm{D}_0 := \delta_0(\mathrm{J}_{11}) \qquad (\delta_0(a_{11}) := \{a_{11}, h_{12}\} = V_{h_{12}}(a_{11})).$$

In keeping with our long-standing terminology, the basic object of our functorial construction is the coordinate $*$-algebra $(\mathrm{D}, -)$, but by abuse of language we will refer to it as just plain D. To see that we have captured the correct notion, let us check our definition when the Jordan algebra is already a hermitian matrix algebra.

Hermitian Matrix Coordinates 16.1.2 *If* $\mathrm{J} = \mathcal{H}_n(\mathrm{D}, -)$ *as in Hermitian Matrix Example* 3.2.4*, and* $h_{ij} = 1[ij]$ *are the standard hermitian matrix units, then the coordinate algebra is the space* $\mathrm{J}_{12} = \mathrm{D}[12]$ *under the product* $a[12] \cdot b[12] = \{\{a[12], 1[23]\}, \{1[31], b[12]\}\} = \{a[13], b[32]\} = ab[12]$ *by Basic Brace Products* 3.7(2), *the diagonal coordinate space is* $\{\mathcal{H}(\mathrm{D}, -)[11], 1[12]\} = \mathcal{H}(\mathrm{D}, -)[12]$ *by Basic Brace Products, with coordinate involution* $\overline{a[12]} = U_{1[12]}(a[12]) = (1\bar{a}1)[12] = \bar{a}[12]$ *by Basic U Products* 3.7(3). *Thus the coordinate* $*$*-algebra is canonically isomorphic to* $(\mathrm{D}, -)$ *under the natural identifications, and the diagonal coordinate space* D_0 *is isomorphic to* $\mathcal{H}(\mathrm{D}, -)$.

More generally, if $1 \in \mathrm{J} \subseteq \mathrm{A}^+$ *is any unital special algebra then by the Special Example* 15.2(3) *any hermitian family in* J *has the form* $\mathcal{H} = \mathcal{H}(\mathcal{E}_a)$ *for a family* $\mathcal{E}_a = \{e_{ij}\}$ *of associative matrix units, and the product in the coordinate algebra* D*, the involution* $\overline{}$*, and the diagonalization map, are given by*

$$\begin{aligned} a \cdot b &= a_{12}e_{21}b_{12} + b_{21}e_{12}a_{21}, \\ \bar{a} &= e_{12}a_{21}e_{12} + e_{21}a_{12}e_{21}, \\ \delta_0(a_{11}) &= a_{11}e_{12} + e_{21}a_{11}. \end{aligned}$$

because the product is

$$\begin{aligned} &\{\{a_{12} + a_{21}, e_{23} + e_{32}\}, \{e_{31} + e_{13}, b_{12} + b_{21}\}\} \\ &\quad = \{a_{12}e_{23} + e_{32}a_{21}, e_{31}b_{12} + b_{21}e_{13}\} \\ &\quad = (a_{12}e_{23})(e_{31}b_{12}) + (b_{21}e_{13})(e_{32}a_{21}) \\ &\quad = a_{12}e_{21}b_{12} + b_{21}e_{12}a_{21}, \end{aligned}$$

the involution is $(e_{12}+e_{21})(a_{12}+a_{21})(e_{12}+e_{21}) = e_{12}b_{21}e_{12}+e_{21}b_{12}e_{21}$, *and the diagonal map is* $a_{11}(e_{12}+e_{21})+(e_{12}+e_{21})a_{11} = a_{11}e_{12}+e_{21}a_{11}$. □

The proof of the pudding is in the eating, and of the coordinate triple in the ability to coordinatize hermitian matrix algebras. We must now verify that the coordinate triple lives up to the high standards we have set for it. By Jordan Coordinates 14.1.1 we already know that it must be alternative with nuclear involution (even associative once $n \geq 4$).

Coordinate Algebra Theorem 16.1.3 *If* $(\mathrm{D}, -)$ *is the coordinate* $*$*-algebra of a unital Jordan algebra* J *with respect to a hermitian n-frame* $(n \geq 3)$ $\mathcal{H} = \{h_{ij}\}$, *then the algebra* D *is a unital linear algebra with unit* $1 = h_{12}$, *whose product mimics the brace product* $\{,\} : \mathrm{J}_{13} \times \mathrm{J}_{32} \to \mathrm{J}_{12}$, *and whose coordinate involution is a true involution on the coordinate algebra*:

$$(1) \qquad d \cdot h_{12} = h_{12} \cdot d = d;$$

$$(2) \qquad \{y_{13}, z_{32}\} = \mathcal{U}_{(23)}(y_{13}) \cdot \mathcal{U}_{(13)}(z_{32});$$

$$(3) \qquad \overline{d \cdot b} = \bar{b} \cdot \bar{d}, \qquad \bar{\bar{d}} = d, \qquad \overline{h_{12}} = h_{12}.$$

The diagonal coordinate space $\mathrm{D}_0 = \mathcal{H}(\mathrm{D}, -) = \{a \in \mathrm{D} \mid \bar{a} = a\}$ *is the space of symmetric elements under the coordinate involution, whose action on* D *mimics that of* J_{11} *on* J_{12} *and which is isomorphic as Jordan algebra to* J_{11} *under the diagonal coordinate map* $\delta_0 : \mathrm{J}_{11} \longrightarrow \mathrm{D}_0^+$, *where norms and traces given by the q- and t-forms*:

$$(4) \qquad \delta_0(a_{11}^2) = \delta_0(a_{11}) \cdot \delta_0(a_{11}) \qquad (a_{11} \in \mathrm{J}_{11});$$

$$(5) \qquad \delta_0(q_{11}(d)) = d \cdot \bar{d};$$

$$(5') \qquad \delta_0(t_{11}(d)) = d + \bar{d};$$

$$(6) \qquad \{a_{11}, d\} = \delta_0(a_{11}) \cdot d \qquad (a_{11} \in \mathrm{J}_{11},\ d \in \mathrm{J}_{12}).$$

PROOF. The Peirce relations show that the product gives a well-defined bilinear map $\mathrm{D} \times \mathrm{D} \longrightarrow \mathrm{D}$. Here the Hermitian Involutions Lemma 15.2.1 (1)-(3) shows that h_{12} is a left unit as in (1) since $h_{12} \cdot b = \{\mathcal{U}_{(23)}(h_{12}), \mathcal{U}_{(13)}(b)\}$ [by Definition 16.1.1(1)] $= \{h_{13}, \mathcal{U}_{(13)}(b)\}$ [by Index Permutation (3)] $= \mathcal{U}_{(13)}(\mathcal{U}_{(13)}(b))$ [by the Action Formula (2)] $= b$ [by the involutory nature (1) of the hermitian involutions $\mathcal{U}_{(ij)}$]. Dually h_{12} is a right unit (or we can use the result below that $\overline{}$ is an involution with $\overline{h_{12}} = h_{12}$).

(2) is just a reformulation of the Definition of the product: $\mathcal{U}_{(23)}(y_{13}) \cdot \mathcal{U}_{(13)}(z_{32}) = \{\mathcal{U}_{(23)}(\mathcal{U}_{(23)}(y_{13})), \mathcal{U}_{(13)}(\mathcal{U}_{(13)}(z_{32}))\} = \{y_{13}, z_{32}\}$ [since the $\mathcal{U}_{(ij)}$ are involutory].

For (3), the coordinate involution $\overline{}$ in the Definition 16.1.1(2) is certainly linear of period 2 since it is given by the hermitian involution $\mathcal{U}_{(12)}$, and it is an anti-homomorphism since $\overline{d \cdot b} = \mathcal{U}_{(12)}(\{\mathcal{U}_{(23)}(d), \mathcal{U}_{(13)}(b)\})$ [by the

Definitions (1),(3)] $= \{\mathcal{U}_{(12)}\mathcal{U}_{(23)}(d), \mathcal{U}_{(12)}\mathcal{U}_{(13)}(b)\}$ [since the hermitian involutions are algebra automorphisms] $= \{\mathcal{U}_{(13)}\mathcal{U}_{(12)}(d), \mathcal{U}_{(23)}\mathcal{U}_{(12)}(b)\}$ [by the Agreement Principle 15.2.2(3) since $(12)(23) = (13)(12)$, $(12)(13) = (23)(12)$] $= \{\mathcal{U}_{(23)}(\mathcal{U}_{(12)}(b)), \mathcal{U}_{(13)}(\mathcal{U}_{(12)}(d))\} = \bar{b} \cdot \bar{d}$ [by the Definitions (1), (3) again].

The involution must fix h_{12} since it is the unit element, but we can also see it directly: $\overline{h_{12}} = \mathcal{U}_{(12)}(h_{12}) = h_{12}$ by Index Permutation 15.2.1(3).

We already know that $\mathrm{D}_0 = \mathcal{H}(\mathrm{D}, -)$ from the Definition 16.1.1(3). The formulas (4)–(6) follow via the Peirce Identity Principle 13.5.2 from the Special Example 15.1.2(3), with the product, involution, and δ_0 given as in Hermitian Matrix Example 16.1.2. For (4), δ_0 is a Jordan homomorphism since $\delta_0(a_{11})\cdot\delta_0(a_{11}) = (a_{11}e_{12})e_{21}(a_{11}e_{12}) + (e_{21}a_{11})e_{12}(e_{21}a_{11}) = a_{11}^2 e_{12} + e_{21}a_{11}^2 = \delta_0(a_{11}^2)$. The map δ_0 is linear, and by definition a surjection; it is injective since $a_{11}e_{12} + e_{21}a_{11} = 0 \iff a_{11} = (a_{11}e_{12} + e_{21}a_{11})e_{21} = 0$. Thus δ_0 is an isomorphism.

To establish the relation (5) between the Peirce quadratic form and the norm, we compute: $d\bar{d} = a_{12}e_{21}(e_{12}a_{21}e_{12}) + (e_{21}a_{12}e_{21})e_{12}a_{21} = (a_{12}a_{21})e_{12} + e_{21}(a_{12}a_{21}) = \delta_0(a_{12}a_{21})$. Linearizing $d \mapsto d, h_{12}$ gives the trace relation (5′), since h_{12} is the unit by (1) above.

For the action (6) of D_0 on D we have $\delta_0(a_{11}) \cdot d = (a_{11}e_{12})e_{21}a_{12} + a_{21}e_{12}(e_{21}a_{11}) = a_{11}a_{12} + a_{21}a_{11} = \{a_{11}, a_{12} + a_{21}\} = \{a_{11}, d\}$. □

The results in (4)–(6) can also be established by direct calculation, without recourse to Peirce Principles.

Exercise 16.1.3A* Even without checking that h_{12} is symmetric in (1), show that if e is a left unit in a linear algebra D with involution then e *must* be a two-sided unit and *must* be symmetric.

Exercise 16.1.3B* Establish the results in (4)–(6) above by direct calculation. (1) Establish that δ_0 maps into hermitian elements. (2) Establish the norm relation: $d \cdot \bar{d} = \delta_0(q_{11}(d))$. (3) Establish the action of D_0. (4) Establish directly that δ_0 is a homomorphism. (5) Verify that δ_0 is injective.

17

Jacobson Coordinatization

We are now ready to establish the most powerful Jordan coordinatization theorem, the Jacobson Coordinatization Theorem[1] for algebras with three or more supplementary connected orthogonal idempotents. It is important to note that no "nondegeneracy" assumptions are made (which means that the theorem can be used to reduce the study of bimodules for such an algebra to the study of bimodules for its coordinate algebra, a topic we won't broach in this book).

17.1 Strong Coordinatization

As with the degree–2 coordinatization theorems (but with even more justification), the calculations will all be made for strongly connected algebras, and then the magic wand of isotopy will convert a merely-connected algebra to a strongly-connected one. So we begin with the conceptually simpler strongly connected case.

Jacobson Strong Coordinatization Theorem 17.1.1 *Any unital Jordan algebra* J *with a family* $\mathcal{H}$ *of hermitian* $n \times n$ *matrix units* ($n \geq 3$) *is an algebra of* $n \times n$ *hermitian matrices*: *there is an isomorphism* $\delta : \mathrm{J} \longrightarrow \mathcal{H}_n(\mathrm{D}, -)$ *taking the given family of matrix units to the standard family,*

$$\delta(h_{ii}) = 1[ii] = E_{ii}, \qquad \delta(h_{ij}) = 1[ij] = E_{ij} + E_{ji}.$$

Here D *is an alternative algebra with nuclear involution, which must be associative if* $n \geq 4$.

Indeed, if J *has merely a supplementary orthogonal family of idempotents* $\{e_i\}$ *with* e_1 *strongly connected to* e_j *by* $v_{1j} \in \mathrm{J}_{1j}$, *then there is an isomorphism* $\delta : \mathrm{J} \longrightarrow \mathcal{H}_n(\mathrm{D}, -)$ *with* $\delta(e_i) = 1[ii]$, $\delta(v_{1j}) = 1[1j]$.

[1] This was stated in I.5.2.

PROOF. By Hermitian Completion 15.1.3 any such family $\{e_i, v_{1j}\}$ can be completed to a Hermitian family $\{h_{ij}\}$, so we may assume from the start that we have a complete family $\mathcal{H}$. By the Coordinate Algebra Theorem 16.1.3 the hermitian 3×3 subfamily gives rise to a unital coordinate $*$-algebra D with unit $1 = h_{12}$; so far this is merely a unital linear algebra, but we can use it to build a unital linear matrix algebra $\mathcal{H}_n(\mathrm{D}, -)$ as in Hermitian Matrix Example 3.2.4, which is a direct sum of off-diagonal subspaces $\mathrm{D}[ij]$ $(i \neq j)$ and diagonal subspaces $\mathrm{D}_0[ii]$. (Once we know that $\mathcal{H}_n(\mathrm{D}, -)$ is Jordan, these will in fact be the Peirce spaces relative to the standard idempotents $1[ii]$).

Step 1: The Linear Coordinatization

We want to introduce coordinates into the Jordan algebra; all the off-diagonal Peirce spaces J_{ij} will be coordinatized by D (a copy of J_{12}), and all diagonal spaces J_{ii} will be coordinatized by D_0 (a copy of J_{11} in D under $\delta_0 = V_{h_{12}}$). To do this we use the Hermitian Symmetries $\mathcal{U}_\pi$ of 15.2.2(1), which allow us to move freely between diagonal (respectively off-diagonal) Peirce spaces. Since J_{12} and J_{11} are "self-coordinatized," it is natural to define the Peirce coordinatization maps

$$\begin{aligned} &\textit{for } i \neq j,\ \delta_{ij} : \mathrm{J}_{ij} \longrightarrow \mathrm{D} = \mathrm{J}_{12} \textit{ is given by } \delta_{ij} = \mathcal{U}_\pi \\ &\qquad\qquad \textit{for any } \pi \textit{ with } \pi(i) = 1,\ \pi(j) = 2; \\ &\textit{for } i = j,\ \delta_{ii} : \mathrm{J}_{ii} \longrightarrow \mathrm{D}_0 = \delta_0(\mathrm{J}_{11}) \textit{ is given by } \delta_{ii} = \delta_0 \circ \mathcal{U}_\pi \\ &\qquad\qquad \textit{for any } \pi \textit{ with } \pi(i) = 1. \end{aligned}$$

By the Agreement Principle 15.2.2(3) these are independent of the particular choice of π (any one will do equally well, giving the same map into D or D_0). We glue these pieces together to obtain a global coordinatization map

$$\delta\colon \mathrm{J} \longrightarrow \mathcal{H}_n(\mathrm{D}, -) \qquad \textit{via} \qquad \delta\Big(\sum_{i \leq j} x_{ij}\Big) = \sum_{i \leq j} \delta_{ij}(x)[ij],$$

which is automatically a linear bijection because J is the direct sum of its Peirce spaces J_{ij} [by Peirce Decomposition 13.1.4], $\mathcal{H}_n(\mathrm{D}, -)$ is the direct sum of the off-diagonal spaces $\mathrm{D}[ij]$ $(i \neq j)$ and the diagonal spaces $\mathrm{D}_0[ii]$, and the $\mathcal{U}_\pi$ and δ_0 are bijections.

Step 2: The Homomorphism Conditions

We must prove that δ is a homomorphism of algebras. If we can do this, everything else will follow: we will have the isomorphism δ of algebras, and it will take h_{ij} to $1[ij]$ as in the theorem because [by the Index Permutation Principle 15.2.2(2)] for $i \neq j$ we have $\delta_{ij}(h_{ij}) := \mathcal{U}_\pi(h_{ij}) = h_{\pi(1)\pi(j)} = h_{12}$ [by choice of π] $= 1$, while for $i = j$ we have $\delta_{ii}(h_{ii}) := \delta_0(\mathcal{U}_\pi(h_{ii})) = \delta_0(h_{\pi(1)\pi(1)}) = \delta_0(h_{11})$ [by choice of π] $= h_{12} = 1$, so in either case $\delta(h_{ij}) = \delta_{ij}(h_{ij})[ij] = 1[ij]$. Moreover, the alternativity or associativity of D will follow from the Jordan Coordinates Theorem 14.1.1 once we know that $\mathcal{H}_n(\mathrm{D}, -)$ is Jordan (being isomorphic to J).

While we have described δ in terms of Peirce spaces with indices $i \le j$, it is very inconvenient to restrict ourselves to such indices; it is important that δ preserves the natural symmetry $\mathrm{J}_{ji} = \mathrm{J}_{ij}$, $d[ji] = \bar{d}[ij]$:

$$\delta(x_{ij}) = \delta_{ij}(x_{ij})[ij] = \delta_{ji}(x_{ij})[ji], \ \ i.e., \ \overline{\delta_{ji}(x_{ij})} = \delta_{ij}(x_{ij}).$$

This is trivial if $i = j$, since δ_{ii} maps into $\mathrm{D}_0 = \mathcal{H}(\mathrm{D}, -)$. If $i \neq j$ and σ is any permutation with $\sigma(j) = 1, \sigma(i) = 2$, then $\pi = (12) \circ \sigma$ is a permutation with $\pi(j) = 2$, $\pi(i) = 1$, so by definition we have $\delta_{ji} = \mathcal{U}_\sigma, \delta_{ij} = \mathcal{U}_\pi$, hence $\overline{\delta_{ji}(x_{ij})} = \mathcal{U}_{(12)}(\delta_{ji}(x_{ij}))$ [by Definition of the involution] $= \mathcal{U}_{(12)}\mathcal{U}_\sigma(x_{ij}) = \mathcal{U}_\pi(x_{ij})$ [by homomorphicity of $\mathcal{U}$ from Hermitian Symmetries 15.2.2(1)] $= \delta_{ij}(x_{ij})$.

Step 3: Peirce Homomorphism Conditions

Thus it all boils down to homomorphicity $\delta(\{x, y\}) = \{\delta(x), \delta(y)\}$. It suffices to prove this for the spanning set of Peirce elements $x = x_{ij}, y = y_{kl}$. By the rules for multiplication in J and $\mathcal{H}_n(\mathrm{D}, -)$ given by the Peirce Multiplication Theorem 13.3.1 and the Hermitian Matrix Example 3.2.4, respectively, both J and $\mathcal{H}_n(\mathrm{D}, -)$ have (I) the same symmetry in the indices, $\delta_{ji}(x_{ij})[ji] = \overline{\delta_{ji}(x_{ij})}[ij]$ [by Step 2] and $d[ji] = \bar{d}[ij]$ [by Box Notation (1)], (II) the same four basic products $\mathrm{A}_{ii}^2 \subseteq \mathrm{A}_{ii}$, $\mathrm{A}_{ij}^2 \subseteq \mathrm{A}_{ii} + \mathrm{A}_{jj}$, $\{\mathrm{A}_{ii}, \mathrm{A}_{ij}\} \subseteq \mathrm{A}_{ij}$, $\{\mathrm{A}_{ij}, \mathrm{A}_{jk}\} \subseteq \mathrm{A}_{ik}$ for linked products (i, j, k distinct) [by the Peirce Brace Rules (1) and the Basic Brace Products (2)], and (III) the same basic orthogonality relations $\{\mathrm{A}_{ij}, \mathrm{A}_{k\ell}\} = \mathbf{0}$ if the indices cannot be linked [by Peirce Orthogonality Rules (3) and Basic Brace Orthogonality Rules (2)]. Thus homomorphicity reduces to four basic product conditions:

$$\delta_{ii}(a_{ii}^2) = \delta_{ii}(a_{ii}) \cdot \delta_{ii}(a_{ii}), \tag{3.1}$$

$$\delta_{ii}(q_{ii}(x_{ij})) = \delta_{ij}(x_{ij}) \cdot \delta_{ji}(x_{ij}), \tag{3.2}$$

$$\delta_{ij}(\{a_{ii}, x_{ij}\}) = \delta_{ii}(a_{ii}) \cdot \delta_{ij}(x_{ij}), \tag{3.4}$$

$$\delta_{ij}(\{y_{ik}, z_{kj}\}) = \delta_{ik}(y_{ik}) \cdot \delta_{kj}(z_{kj}), \tag{3.5}$$

for distinct indices i, j, k.

Step 4: As Easy as 1,2,3

Everything can be obtained from J_{11} or J_{12} by a hermitian symmetry, *so these reduce to the case of indices* $i = 1, j = 2, k = 3$:

$$\delta_{11}(a_{11}^2) = \delta_{11}(a_{11}) \cdot \delta_{11}(a_{11}), \tag{4.1}$$

$$\delta_{11}(q_{11}(x_{12})) = \delta_{12}(x_{12}) \cdot \delta_{21}(x_{12}), \tag{4.2}$$

$$\delta_{12}(\{a_{11}, x_{12}\}) = \delta_{11}(a_{11}) \cdot \delta_{12}(x_{12}), \tag{4.3}$$

$$\delta_{12}(\{y_{13}, z_{32}\}) = \delta_{13}(y_{13}) \cdot \delta_{32}(z_{32}). \tag{4.4}$$

But *we already know these* by the Coordinate Algebra Theorem 16.1.3 (4), (5), (6), (2) respectively, since by definition and symmetry

$$\delta_{11} = \delta_0, \quad \delta_{12} = \mathbb{1}_{J_{12}}, \quad \delta_{21} = \mathcal{U}_{(12)} = \overline{}, \quad \delta_{13} = \mathcal{U}_{(23)}, \quad \delta_{32} = \mathcal{U}_{(13)}.$$

To see the reduction from (3) to (4) in detail, choose any π such that $\pi(1) = i$, $\pi(2) = j$, $\pi(3) = k$; then by the Index Permutation Principle 15.5(2) we can find elements $a_{11}, x_{12}, y_{13}, z_{23}$ so that (*since the hermitian symmetries are algebra automorphisms*)

$$\begin{aligned}
a_{ii} &= \mathcal{U}_\pi(a_{11}), \qquad a_{ii}^2 = \mathcal{U}_\pi(a_{11}^2),\\
x_{ij} &= \mathcal{U}_\pi(x_{12}), \qquad q_{ii}(x_{ij}) = \mathcal{U}_\pi(q_{11}(x_{12})),\\
\{a_{ii}, x_{ij}\} &= \mathcal{U}_\pi(\{a_{11}, x_{12}\}),\\
y_{ik} &= \mathcal{U}_\pi(y_{13}), \qquad z_{kj} = \mathcal{U}_\pi(z_{32}), \quad \{y_{ik}, z_{kj}\} = \mathcal{U}_\pi(\{y_{13}, z_{32}\}).
\end{aligned}$$

By definition and the Agreement Principle

$$\delta_{ii}\mathcal{U}_\pi = \delta_{11}, \;\; \delta_{ij}\mathcal{U}_\pi = \delta_{12}, \;\; \delta_{ji}\mathcal{U}_\pi = \delta_{21}, \;\; \delta_{ik}\mathcal{U}_\pi = \delta_{13}, \;\; \delta_{kj}\mathcal{U}_\pi = \delta_{32}.$$

Thus

$$\begin{aligned}
&\delta_{ii}(a_{ii}) = \delta_{11}(a_{11}), && \delta_{ii}(a_{ii}^2) = \delta_{11}(a_{11}^2), && \delta_{ii}(q_{ii}(x_{ij})) = \delta_{11}(q_{11}(x_{12})),\\
&\delta_{ij}(x_{ij}) = \delta_{12}(x_{12}), && \delta_{ji}(x_{ij}) = \delta_{21}(x_{12}), && \delta_{ij}(\{a_{ii}, x_{ij}\}) = \delta_{12}(\{a_{11}, x_{12}\}),\\
&\delta_{ik}(y_{ik}) = \delta_{13}(y_{13}), && \delta_{kj}(z_{kj}) = \delta_{32}(z_{32}), && \delta_{ij}(\{y_{ik}, z_{kj}\}) = \delta_{12}(\{y_{13}, z_{32}\}),
\end{aligned}$$

and (3.1)–(3.4) reduce to (4.1)–(4.4). □

17.2 General Coordinatization

We obtain the general coordinatization by reduction to an isotope.

Jacobson Coordinatization Theorem 17.2.1 *Any unital Jordan algebra* J *with a supplementary orthogonal family of* $n \geq 3$ *connected idempotents* $\{e_i\}$ *(it suffices if* e_1 *is connected to each* e_j*) is isomorphic to an algebra of* $n \times n$ *twisted hermitian matrices: there is an isomorphism* $\varphi : \mathrm{J} \longrightarrow \mathcal{H}_n(\mathrm{D}, \Gamma)$ *taking the given family of idempotents to the standard family,*

$$\varphi(e_i) = 1[ii] = E_{ii}.$$

Here D *is an alternative algebra with nuclear involution, which must be associative if* $n \geq 4$.

PROOF. Suppose e_1 is connected to e_j by $v_{1j} \in \mathrm{J}_{1j}$; then by the Connection Definition 14.4.2 and the Off-Diagonal Invertibility Lemma 14.4.3(1), $q_{jj}(v_{1j})$ is invertible in J_{jj}. Set $u_{jj} = q_{jj}(v_{1j})^{-1}$, $u_{11} = e_1 = u_{11}^{-1}$, $u = \sum_j u_{jj}$. By the Creating Involutions Theorem 14.4.6, the isotope $\widetilde{\mathrm{J}} := \mathrm{J}^{(u)}$ has supplementary orthogonal idempotents $\tilde{e}_j = u_{jj}^{-1}$ yielding the same Peirce decomposition $\widetilde{\mathrm{J}}_{ij} = \mathrm{J}_{ij}$ as the original $\{e_j\}$, but with v_{1j} strongly connecting

$\tilde{e}_1$ to $\tilde{e}_j$. By the Strong Coordinatization Theorem we have an isomorphism $\tilde{\delta} : \widetilde{\mathrm{J}} \longrightarrow \mathcal{H}_n(\mathrm{D}, -)$ sending $\tilde{e}_j \mapsto 1[jj]$ and $v_{1j} \mapsto 1[1j]$. In particular, D is alternative with symmetric elements in the nucleus: $\mathrm{D}_0 \subseteq \mathcal{N}uc(\mathrm{D})$. Furthermore, by Jordan Isotope Symmetry 7.2.1(4) we can recover J as $\mathrm{J} = \widetilde{\mathrm{J}}^{(u^{-2})} \cong \mathcal{H}_n(\mathrm{D}, -)^{(\Gamma)} \cong \mathcal{H}_n(\mathrm{D}, \Gamma)$ via $\varphi = L_\Gamma \circ \tilde{\delta}$ since [by the Creating Involutions again] $u^{-2} = \sum_j u_{jj}^{-2} \in \sum_j \mathrm{J}_{jj}$ corresponds to a diagonal (hence, by the above, nuclear) element $\Gamma \in \sum_j \mathrm{D}_0[jj]$. Under this isomorphism e_j goes to an invertible idempotent in $\mathcal{H}_n(\mathrm{D}, \Gamma)_{jj}$, so it can only be $1[jj]$. We can also verify this directly: by the Jordan Homotope inverse recipe 7.2.1(3), e_j has inverse $U_{u_{jj}^{-1}}(e_j^{-1}) = u_{jj}^{-2}$ in $\mathrm{J}_{jj}^{(u)} = \mathrm{J}_{jj}^{(u_{jj})}$, so their images are inverses in $\mathcal{H}_n(\mathrm{D}, -)$; but $\tilde{\delta}(u_{jj}^{-2}) = \gamma_j[jj]$, $\tilde{\delta}(e_j) = \alpha_j[jj]$ implies that γ_j, α_j are inverses in D, so $\varphi(e_j) = L_\Gamma(\tilde{\delta}(e_j)) = \Gamma(\alpha_j[jj]) = \gamma_j\alpha_j[jj] = 1[jj] = E_{jj}$ as desired. $\square$

This completes our Peircian preliminaries, preparing us for the final push towards the structure of algebras with capacity.

Fourth Phase: Structure

In the final phase we will determine the structure of Jordan algebras having capacity. We begin in Chapter 18 with basic facts about regular elements and pairs. Structural pairs of transformations (interacting "fundamentally" with U-operators) are more useful than automorphisms for moving around among inner ideals. Structurally paired inner ideals have the same "inner shape".

Chapter 19 focuses on simple elements, those which are regular and whose principal inner ideal is simple (minimal and not trivial). The Minimal Inner Ideal Theorem shows that there are just three types of minimal inner ideals: principal inner ideals generated by trivial elements, simple (or division) idempotents, and simple nilpotents (which are structurally paired with simple idempotents). This guarantees that a nondegenerate algebra with d.c.c. on inner ideals always has a finite capacity (its unit is a sum of mutually orthogonal division idempotents). From this point on we concentrate on nondegenerate algebras with finite capacity.

Chapter 20 focuses on simple connectivity. Peirce arguments show that an algebra with capacity breaks into its connected components, and is connected iff it is simple, so an algebra with capacity is a direct sum of simple algebras with capacity.

Thus we narrow our focus to simple algebras. First we must describe their possible coordinate algebras. This requires a digression in Chapter 21 into alternative algebras, involving the Moufang identities, Artin's Theorem (that any two elements generate an associative subalgebra), together with a detailed study of the nucleus and center. The Herstein–Kleinfeld–Osborn Theorem shows that in a nondegenerate alternative algebra with nuclear involution whose nonzero hermitian elements are all invertible, the nucleus is either the whole algebra or the center, therefore the algebra is either associative (a division algebra, or the exchange algebra of a division algebra) or a composition algebra.

We divide the simple algebras up according to their capacity, capacity 1 being an instantaneous case (a division algebra). The case of capacity two in Chapter 22 is the most difficult: Osborn's Capacity Two Theorem shows that such an algebra satisfies either the Spin Peirce identity or the hermitian Peirce condition, so by Spin or Hermitian Coordinatization is a reduced spin factor or 2×2 hermitian algebra (which by H–K–O must be full or hermitian matrices over an associative division algebra).

Chapter 23 pulls it all together. In capacity three or more we can use Jacobson Coordinatization, augmented by the H-K-O to tell us what coordinates are allowable, to show that such an algebra is a hermitian matrix algebra over a division or composition algebra. Putting these pieces together gives the final Classical Classification Theorem, that simple Jordan algebras of finite capacity are either Jordan division algebras, reduced spin factors, hermitian matrix algebras (with respect to an exchange, orthogonal, or symplectic involution), or Albert algebras.

18

Von Neumann Regularity

In order to run at full capacity, we need a steady supply of idempotents. Before we can show in the next section that minimal inner ideals always lead to idempotents, we will have to gather some general observations about von Neumann regularity.

18.1 vNr Pairing

We begin with the concept of a regular element, showing that these naturally come in pairs; such pairs are precisely the idempotents in the theory of Jordan pairs.

vNr Definition 18.1.1 *An element $x \in \mathrm{J}$ is a* **von Neumann regular element**[1] *or* **vNr** *if*

$$x = U_x y \text{ for some } y \in \mathrm{J} \qquad (i.e., \ x \in U_x \mathrm{J}),$$

and x is a **double vNr** *if*

$$x = U_x U_x y \text{ for some } y \in \mathrm{J} \qquad (i.e., \ x \in U_x U_x \mathrm{J}).$$

A Jordan algebra is **von Neumann regular** (*or* **vNr**) *if all its elements are. Elements x, y are* **regularly paired**, *denoted $x \bowtie y$, if they are mutually paired with each other,*

$$x = U_x y, \qquad y = U_y x,$$

in which case we call (x, y) a **vNr pair**.

[1] In the literature this is usually just called *regular element*, but then, so are lots of other things. To avoid the overworked term "regular," we identify it by its originator, John von Neumann. But it becomes too much of a mouthful to call it by its full name, so once we get friendly with it we call it by its nickname "vNr," pronounced *vee-nurr* as in schnitzel; there will be no confusion between "Johnny" and "Norbert" in its parentage. We will happily use the notation vNr both as a noun, "is a vNr" (is a von Neumann regular element), and as an adjective, "is vNr" (is von Neumann regular).

EXERCISE 18.1.1 (1) Show that 0, all invertible elements, and all idempotents are always vNr, but that nonzero trivial elements never are: invertibles are paired with their inverses ($u \bowtie u^{-1}$), and idempotents are narcissistically paired with themselves ($e \bowtie e$). (2) Show that a vNr algebra is nondegenerate. (3) Show that if (x, y) is a vNr pair, then so is $(\alpha x, \alpha^{-1} y)$ for any invertible $\alpha \in \Phi$.

In associative algebras (especially matrix algebras) such a y paired with x is called a *generalized inverse* of x. While inverses are inherently monogamous, generalized inverses are bigamous by nature: a given x usually has several different generalized inverses. For example, while the matrix unit $x = E_{11}$ is happily paired with itself, it is also regularly paired with all the $y = E_{11} + \alpha E_{12}$ as α ranges over Φ. Any one of x's generalized inverses is equally useful in solving equations: when $x \in \mathrm{A}$ is invertible, the equation $x(u) = m$ for a fixed vector m in a left A-module M is always solvable uniquely for $u = x^{-1}(m) \in M$, but if x has only a generalized inverse y, then $x(u) = m$ is solvable iff $e(m) = m$ for the projection $e = xy$, in which case one solution (out of many) is $u = y(m)$.

vNr Pairing Lemma 18.1.2 (1) *Any vNr can be regularly paired*: *if* $x = U_x z$ *then* $x \bowtie y$ *for* $y = U_z x$. (2) *vNrs are independent of hats*: x *is a vNr in* $\widehat{\mathrm{J}}$ *iff it is a vNr in* J: *in the notation of the Principal Inner Proposition* 5.3.1,

$$x \textit{ is a vNr} \iff x \in (x) = U_x\mathrm{J} \iff x \in (x] = U_x\widehat{\mathrm{J}}.$$

PROOF. (1) We work entirely within the subalgebra $\Phi[x, z]$ of J generated by the two elements x and z, so by the Shirshov–Cohn Principle 5.1.4 we may assume that we are living in $\mathcal{H}(\mathrm{A}, *)$ for an associative $*$-algebra A where $xzx = x$. The element $y = zxz$ still satisfies $xyx = x(zxz)x = (xzx)zx = xzx = x$, and in addition satisfies $yxy = (zxz)x(zxz) = zxyxz = zxz = y$, so $x \bowtie y$. (2) if x is a vNr in the unital hull, $x = U_x\hat{z} \in U_x\widehat{\mathrm{J}} = (x]$ for some $\hat{z} \in \widehat{\mathrm{J}}$, then by (1) $x = U_x y$ is a vNr in the original algebra for $y = U_{\hat{z}} x \in \mathrm{J}$, so $x \in U_x\mathrm{J} = (x)$. The converse is clear, $x \in (x) \Longrightarrow x \in (x]$. □

EXERCISE 18.1.2* Prove the vNr Pairing Lemma using only the Fundamental Formula, with no reference to an associative algebra. Prove that x is regular iff all principal inner ideals $(x) = [x) = (x] = [x]$ agree.

A nice system (say, finite-dimensional semisimple) is always vNr, so every element can be regularly paired. In many ways a regular pair behaves like an idempotent. In the associative case, if $x = xyx$ then $e = xy, f = yx$ are true idempotents, but in a special Jordan algebra $\mathrm{J} \subseteq \mathrm{A}^+$ the elements e, f may no longer exist within J, only inside the associative envelope A.

If an element is a double vNr, it is invertible in its own little part of the algebra.

Double vNr Lemma 18.1.3 *If b is a double vNr, then its principal inner ideal $(b] = U_b\widehat{\mathrm{J}}$ is a unital Peirce subalgebra $\mathrm{J}_2(e)$ in which b is invertible: if $b = U_bU_ba$, then b is invertible in $(b] = \mathrm{J}_2(e)$ for the idempotent $e = U_{b^2}U_ab^2$.*

PROOF. First we work entirely within the subalgebra $\Phi[a,b]$, so by the Shirshov–Cohn Principle again we may assume that we are living in $\mathcal{H}(\mathrm{A},*)$. In A we have $b = b^2ab^2$ and $e = (b^2ab^2)ab^2 = bab^2$, hence $be = b(bab^2) = b^2ab^2 = b$ (dually, or via $*$, $eb = b$), so $b = ebe = U_eb$ and $e^2 = e(bab^2) = bab^2 = e$. Thus we deduce that in J we have the Jordan relation

$$b = U_eb \in \mathrm{J}_2(e) \quad \textit{for} \quad e^2 = e.$$

If we want results about principal inner ideals, we cannot work just within $\Phi[a,b]$, we must go back to J. By the nature of inner ideals, $e = U_{b^2}U_ab^2 \in U_{b^2}\mathrm{J}$, $b^2 = U_b\hat{1} \in U_b\widehat{\mathrm{J}}$, $b = U_eb \in U_e\widehat{\mathrm{J}}$ imply $(e] \subseteq (b^2] \subseteq (b] \subseteq (e]$, so

$$(b] = (b^2] = (e] = \mathrm{J}_2(e).$$

By the Invertibility Criterion 6.1.2(3), b is invertible in $(b]$ since its U-operator is surjective on the *unital subalgebra* $(b]$: $U_b(b] = U_b\left(U_b\widehat{\mathrm{J}}\right) = U_{b^2}\widehat{\mathrm{J}} = (b^2] = (b]$. ☐

Thus we see that little units e come, not from the stork, but from double vNrs.

EXERCISE 18.1.3A (1) As a lesson in humility, try to prove the vNr Pairing and Double vNr Lemmas strictly in Jordan terms, without using Shirshov–Cohn to work in an associative setting. You will come away with more respect for Shirshov and Cohn.

EXERCISE 18.1.3B (1) Show directly that for a vNr pair (x,y) the operators $E_2(x,y) := U_xU_y, E_1(x,y) := V_{x,y} - 2U_xU_y, E_0(x,y) := B_{x,y} = \mathbb{1}_{\mathrm{J}} - V_{x,y} + U_xU_y$ are supplementary orthogonal projection operators. (2) Alternately, show that the E_i are just the Peirce projections corresponding to the idempotent x in the homotope $\mathrm{J}^{(y)}$ (cf. Jordan Homotope Proposition 7.2.1). (3) If x,y are a vNr pair in a special algebra $\mathrm{J} \subseteq \mathrm{A}^+$, show that $e = xy, f = yx$ are idempotents in A, and in J there is a ghostly "Peirce decomposition," $\mathrm{J} = e\mathrm{J}f + ((1-e)\mathrm{J}f + e\mathrm{J}(1-f)) + (1-e)\mathrm{J}(1-f)$, and a similar one with e,f interchanged. Show that $ezf = E_2(x,y)z,\ (1-e)zf + ez(1-f) = E_1(x,y)z,\ (1-e)z(1-f) = E_0(x,y)z$. Thus in the Jordan algebra itself, the *pair* of elements x,y acts in some sense like an idempotent. This rich supply of "idempotents" was one of the sparks leading to Loos's creation of Jordan pairs (overcoming the severe idempotent-deficiency of Jordan triple systems); in the Jordan pair (J,J) obtained by doubling a Jordan algebra or triple system J, the "pair idempotents" are precisely the regular pairs (x,y), leading to pairs of Peirce decompositions $E_i(x,y), E_i(y,x)$, one on each copy of J.

The Invertibility Criterion assumes that the algebra is unital. Now we can show that not only does the existence of a surjective U-operator creates an inverse in the presence of a unit element, it even creates the unit element.

Surjective Unit Lemma 18.1.4 *If a Jordan algebra* J *contains an element with surjective U-operator,* $U_b\mathrm{J} = \mathrm{J}$*, then* J *has a unit element and* b *is invertible.*

PROOF. Such a b is a double vNr, $b \in \mathrm{J} = U_b\mathrm{J} = U_b(U_b\mathrm{J})$, so by the Double vNr Lemma 18.3 $\mathrm{J} = (b]$ is unital with b invertible. □

EXERCISE 18.1.4 Strengthen the Jordan Homotope Theorem 7.2.1(2) to show that if J is an arbitrary Jordan algebra and u an element such that the homotope $\mathrm{J}^{(u)}$ is unital, then J was necessarily unital to begin with, and u was an invertible element.

18.2 Structural Pairing

When x and y are regularly paired their principal inner ideals $(x]$ and $(y]$ are also paired up in a natural structure-preserving way, which is most clearly understood using the concept of structural transformation. This is a weaker notion than that of homomorphism because it includes "multiplications" such as U-operators, and is often convenient for dealing with inner ideals. Taking the Fundamental Formula as our guide, we consider operators which interact nicely with the U-operators. (We have crossed paths with these several times before.)

Structural Transformation Definition 18.2.1 (1) *A linear transformation* T *on* J *is* **weakly structural** *if it is* **structurally linked** *to some linear transformation* T^* *on* J *in the sense that*

$$U_{T(x)} = TU_xT^* \text{ on J for all } x \in \mathrm{J};$$

(2) T *is* **structural** *if it is weakly structural and remains so on the unital hull; more precisely,* T *is structurally linked to some* T^* *on* J *such that* T, T^* *have extensions to* $\widehat{\mathrm{J}}$ *(by abuse of notation still denoted by* T, T^**) which are structurally linked on* $\widehat{\mathrm{J}}$*,*

$$U_{T(\hat{x})} = TU_{\hat{x}}T^* \quad \text{on} \quad \widehat{\mathrm{J}} \quad \text{for all } \hat{x} \in \widehat{\mathrm{J}}.$$

(3) *A* **structural pair** (T, T^*) *consists of a pair of structural transformations linked to each other,*

$$U_{T(\hat{x})} = TU_{\hat{x}}T^*, \quad U_{T^*(\hat{x})} = T^*U_{\hat{x}}T \quad \text{on} \quad \widehat{\mathrm{J}} \quad \text{for all } \hat{x} \in \widehat{\mathrm{J}},$$

equivalently, T *is structurally linked to* T^* *and* T^* *is structurally linked to* $T^{**} = T$*.*

For unital algebras all weakly structural T are automatically structural (with extension $0 \boxplus T$ to $\widehat{\mathrm{J}} = \Phi e' \boxplus \mathrm{J}$). In non-unital Jordan algebras we usually demand that a transformation respect the square x^2 as well as the quadratic product $U_x y$, i.e., be structural on the unital hull, so we usually work with *structural* rather than *weakly structural* transformations.

For Jordan triples or pairs, or for non-invertible structural T in Jordan algebras, the "adjoint" T^* need not be uniquely determined. The accepted way to avoid this indeterminacy is to explicitly include the parameter T^* and deal always with the pair (T, T^*). Again, T^* need not be structural in general, but since it is for all the important structural T, the accepted way to avoid this nonstructurality is to impose it by fiat, and hence we consider structural *pairs* instead of mere structural *transformations.*

Clearly, any automorphism $T = \varphi$ determines a structural pair $(T, T^*) = (\varphi, \varphi^{-1})$ (i.e., $\varphi^* = \varphi^{-1}$). The Fundamental Formula says that all multiplications U_x determine a structural pair (U_x, U_x) (i.e., $(U_x)^* = U_x$), but the V_x and L_x are usually not structural. It can be shown[2] that the Bergmann operators $(B_{\alpha,x,y}, B_{\alpha,y,x})$ form a structural pair for any $\alpha \in \Phi, x \in \mathrm{J}, y \in \widehat{\mathrm{J}}$; in particular, the operator $U_{\alpha\hat{1}-x} = B_{\alpha,x,1}$ from the unital hull is structural on J. If $\mathrm{J} = \mathrm{A}^+$ for an associative algebra A, then each left and right associative multiplications L_x, R_x determine structural pairs, (L_x, R_x), (R_x, L_x) (i.e., $(L_x)^* = R_x$ and $(R_x)^* = L_x$), and so again $U_x = L_x R_x$ is structural with $U_x^* = U_x$, but $V_x = L_x + R_x$ is usually not structural.

Structural Innerness Lemma 18.2.2 (1) *If T is a structural transformation on* J, *then its range* $T(\mathrm{J})$ *is an inner ideal.* (2) *More generally, the image* $T(\mathrm{B})$ *of any inner ideal* B *of* J *is again an inner ideal.* (3) *If T, S are structural, then so is their composite $T \circ S$ with $(T \circ S)^* = S^* \circ T^*$.*

PROOF. The first assertion follows from the second because J itself is an inner ideal. For the second, if B is inner in J, $U_{\mathrm{B}}\widehat{\mathrm{J}} \subseteq \mathrm{B}$, then so is the Φ-submodule $T(\mathrm{B})$: $U_{T(\mathrm{B})}\widehat{\mathrm{J}} = TU_{\mathrm{B}}T^*(\widehat{\mathrm{J}})$ [by structurality] $\subseteq T(U_{\mathrm{B}}\widehat{\mathrm{J}}) \subseteq T(\mathrm{B})$. The third assertion is a direct calculation, $U_{T(S(x))} = TU_{S(x)}T^* = T\left(SU_xS^*\right)T^*$ by structurality of T and S. □

EXERCISE 18.2.2 (1) If T is weakly structural and B is a weak inner ideal, show that $T(\mathrm{B})$ is also weakly inner. Show that $(Tx) \subseteq T\big((x)\big)$ for weakly structural T, and $(Tx] \subseteq T\big((x]\big)$, $[Tx] \subseteq T\big([x]\big)$ for structural T. (2) If T, S are weakly structural show that $T \circ S$ is too. (3) Define a *weakly structural pair* (T, T^*), and show that if $(T, T^*), (S, S^*)$ are weakly structural pairs, then so is their product $(T \circ S, S^* \circ T^*)$. (4) If T is weakly structural and both T, T^* are invertible on J, show that T^{-1} is also weakly structural with $(T^{-1})^* = (T^*)^{-1}$. (5) Show that if (x, y) is a vNr pair then so is $(T(x), S(y))$ for any structural pairs $(T, T^*), (S, S^*)$ such that $T^*Sy = y, S^*Tx = x$ (in particular, if $S = (T^*)^{-1}$ where both T, T^* are invertible).

[2] cf. Exercise 5.3.1 and Bergmann Structurality IV.1.2.2.

Structural Pairing Definition 18.2.3 *Two inner ideals* B, D *in* J *are* **structurally paired** *if there exist structural transformations* T, S *on* J *which are inverse bijections between* B *and* D:

(SP1) $T(\mathrm{B}) \subseteq \mathrm{D}$, $S(\mathrm{D}) \subseteq \mathrm{B}$;
(SP2) $T \circ S = \mathbb{1}_{\mathrm{D}}$ *on* D, $S \circ T = \mathbb{1}_{\mathrm{B}}$ *on* B.

Structural pairing for inner ideals is a more general and more useful concept than conjugacy under a global isotopy of the algebra; it is a "local isotopy" condition, yet strong enough to preserve the lattice of inner ideals.

Structural Pairing Lemma 18.2.4 *If inner ideals* B, D *of* J *are structurally paired, then there is an isomorphism between the lattices of inner ideals of* J *contained in* B *and those contained in* D.

PROOF. If B, D are structurally paired by T, S, then $T|_{\mathrm{B}}, S|_{\mathrm{D}}$ are inverse bijections of modules which preserve inner ideals by Structural Innerness 18.2.2(2): if $\mathrm{B}' \subseteq \mathrm{B}$ is inner, then $\mathrm{D}' = T(\mathrm{B}')$ is again an inner ideal of J and is contained in $\mathrm{D} = T(\mathrm{B})$. Thus the pairing sets up inverse order-preserving bijections between the lattices of inner ideals in B and those in D, in short, a lattice isomorphism. □

Principal Pairing Lemma 18.2.5 *If elements* b, d *are regularly paired in a Jordan algebra, then their principal inner ideals* $(b], (d]$ *are structurally paired by* U_d, U_b: $(b] \xrightarrow{U_d} (d]$ *and* $(d] \xrightarrow{U_b} (b]$ *are inverse structural bijections preserving inner ideals.*

PROOF. U_b is certainly structural on $\widehat{\mathrm{J}}$ and maps all of $\widehat{\mathrm{J}}$ into $(b]$, dually for U_d, as in (SP1) of the above Structural Pairing Definition. For (SP2), on any $U_b\hat{a} \in U_b\widehat{\mathrm{J}} = (b]$ we have $U_bU_d(U_b\hat{a}) = U_{U_bd}\hat{a}$ [by the Fundamental Formula] $= U_b\hat{a}$ [by pairing], so $U_bU_d = \mathbb{1}_{(b]}$, and dually for U_dU_b. □

These Lemmas will be important in the next chapter, where we will be Desperately Seeking, not Susan, but Idempotents. The good minimal inner ideals are those governed by a division idempotent. At the other extreme are the minimal inner ideals governed by trivial elements, and these are so badly behaved that we will pass a law against them (the Nondegeneracy Law). The remaining minimal inner ideals B are nilpotent, so there is no hope of squeezing idempotents out of them. Structural Pairing shows that the image of a minimal inner ideal under a structural transformation T is again minimal, and luckily the nilpotent B's run around with respectable T(B)'s having idempotents. In this way the minimal inner ideals will provide us enough fuel (division idempotents) to run our structure theory.

18.3 Problems for Chapter 18

PROBLEM 18.1 (1) (cf. Problem 7.2(1)) If T is structural and *invertible* on a unital algebra, show that the adjoint T^* is uniquely determined as $T^* = T^{-1}U_{T(1)} = U_{T^{-1}(1)}T^{-1}$. (2) If $\mathrm{J} = \mathrm{A}^+$ for A the unital associative algebra of upper triangular 2×2 matrices over Φ, show that $T = U_{E_{11}}$ is structural for *any* $T^* = T + S$ as long as $S(\mathrm{A}) \subseteq \mathrm{A}E_{22}$, so T^* is far from unique. [In upper triangular matrices $E_{22}\mathrm{A}E_{11} = \mathbf{0}$.]

PROBLEM 18.2 Let J be a non-unital Jordan algebra (cf. Problem 7.2(2) for the case of a unital algebra). (1) If T is structurally linked to T^* and both T, T^* are invertible on J, show that T^{-1} is structurally linked to $(T^*)^{-1}$. (2) If (T, T^*) is an *invertible structural pair* (a structural pair with both T, T^* invertible), show that the *inverse structural pair* $(T^{-1}, (T^*)^{-1})$ is also structural. (3) Show that the set of invertible structural pairs forms a subgroup $\mathcal{S}trg(\mathrm{J}) \subset \mathcal{E}nd(J)^\times \times (\mathcal{E}nd(J)^\times)^{op}$, called the *structure group* of J. Show that in a natural way this contains a copy of the automorphism group $\mathcal{A}ut(\mathrm{J})$ as well as all invertible Bergmann operators $B_{\alpha,x,y}$. [There are usually many invertible Bergmann operators, corresponding to quasi-invertible pairs x, y, but there are invertible operators U_v only if J is unital, the situation of Problem 7.2.]

QUESTION 18.1* (1) If T is invertible and structurally linked to T^*, is T^* necessarily invertible? (2) If (T, T^*) is structural and T is invertible, is T^* necessarily invertible too? Equivalently, in a unital algebra must $T(1)$ or $T^{-1}(1)$ be invertible?

QUESTION 18.2* In order for a weakly structural transformation T linked to T^* to become strong, T, T^* must extend to $(\widehat{T}, \widehat{T}^*)$ on the unital hull, i.e., they must decide what to do to the element 1: $\widehat{T}(1) = \hat{t} = \tau 1 \oplus t, \widehat{T}^*(1) = \hat{t}^* = \tau^* 1 \oplus t^*$. (1) If τ is cancelable (e.g., if Φ is a field and $\tau \neq 0$) show that $\tau^* = \tau$, so it is natural to impose this as a general condition. (2) Assuming $\tau^* = \tau$, find conditions on τ, t, t^* which are necessary and sufficient for T structurally linked to T^* to extend to $\widehat{T}$ structurally linked to $\widehat{T^*}$ on $\widehat{\mathrm{J}}$. (3) If T, T^* are structurally linked to each other, find necessary and sufficient conditions on the $\tau = \tau^*, t, t^*$ of (1) for $(\widehat{T}, \widehat{T}^*)$ to be structurally linked to each other (i.e., (T, T^*) is a structural pair).

QUESTION 18.3 vNr-ity is defined as $x \in (x)$. Investigate what happens if we choose the other principal inner ideals. Is x vNr iff $x \in (x]$? Iff $x \in [x]$? What happens if we can generate the square: what is the connection between x being vNr and x^2 being in (x)? Or in $(x]$? Or in $[x]$? Either prove your assertions or give counterexamples.

QUESTION 18.4 Can you think of any conditions on the element x in $\mathrm{J} = \mathrm{A}^+$ for an associative algebra A in order for $V_x = L_x + R_x$ to be a structural transformation?

19

Inner Simplicity

We obtain an explicit description of all simple inner ideals, which shows that they all are closely related to simple idempotents. The main purpose of simple inner ideals is to provide us with simple idempotents, the fuel of classical structure theory.[1]

19.1 Simple Inner Ideals

We begin by defining simple elements and giving archetypal examples thereof in the basic examples of Jordan algebras.

Simple Inner Ideal Definition 19.1.1 *A nonzero inner ideal* B *of* J *is* **minimal** *if there is no inner ideal* $\mathbf{0} < \mathrm{C} < \mathrm{B}$ *of* J *properly contained inside it. An inner ideal is* **simple** *if it is both minimal and nontrivial,* $U_{\mathrm{B}}\widehat{\mathrm{J}} \neq \mathbf{0}$*. An element* b *of* J *is* **simple** *if its principal inner ideal* $(b]$ *is a simple inner ideal of* J *containing* b*; since* $b \in (b]$ *iff* $b \in (b)$ *by vNr Pairing* 18.1.2(2)*, this is the same as saying that* b *is a vNr and generates a simple inner ideal* $(b]$*. An idempotent* e *is a* **division idempotent** *if the Peirce subalgebra* $(e] = \mathrm{J}_2(e)$ *is a division algebra.*[2]

Simple Pairing Lemma 19.1.2 (1) *If inner ideals* B, D *in* J *are structurally paired, then* B *is minimal (respectively simple) iff* D *is minimal (respectively simple).* (2) *If elements* b, d *are regularly paired, then* b *is simple iff* d *is simple.*

PROOF. (1) Preservation of minimality follows from the lattice isomorphism in the Structural Pairing Lemma 18.2.4. Simplicity will be preserved

[1] The Minimal Inner Ideal Theorem and examples of simple inner ideals were given in I.6.4.

[2] In the literature such an idempotent is often called *completely primitive*, but the terminology is not particularly evocative. In general, one calls e a (*something-or-other*) *idempotent* if the subalgebra $U_e\mathrm{J}$ it governs is a (*something-or-other*) *algebra.* A good example is the notion of an *abelian* idempotent used in C^*-algebras.

if triviality is, and B trivial $\Longrightarrow U_{\mathrm{B}}\widehat{\mathrm{J}} = \mathbf{0} \Longrightarrow U_{T(\mathrm{B})}\widehat{\mathrm{J}} = TU_{\mathrm{B}}T^*(\widehat{\mathrm{J}}) \subseteq T\left(U_{\mathrm{B}}(\widehat{\mathrm{J}})\right) = \mathbf{0} \Longrightarrow T(\mathrm{B})$ trivial. (2) If b, d are regularly paired, then by the Principal Pairing Lemma 18.2.5 the inner ideals $(b], (d]$ are structurally paired, so by (1) $(b]$ is minimal $\Longleftrightarrow (d]$ is minimal, and since b, d are both automatically vNr if they are regularly paired, we see that b is simple $\Longleftrightarrow d$ is simple. □

Now we give examples of simple elements and pairings.

Full and Hermitian Idempotent Example 19.1.3 *In the Jordan matrix algebra* $\mathrm{J} = \mathcal{M}_n(\Delta)^+$ *or* $\mathcal{H}_n(\Delta, -)$ *for any associative division algebra* Δ *with involution, the diagonal matrix unit* $e = E_{ii}$ *is a simple idempotent and its principal inner ideal* $(e] = (e) = \Delta E_{ii}$ *or* $\mathcal{H}(\Delta)E_{ii}$ *is a simple inner ideal.*

The same holds in any diagonal isotope $\mathcal{H}_n(\Delta, \Gamma)$, *since the isotope* $\mathrm{J}_{ii}^{(u_{ii})} = \mathcal{H}(\Delta)^{(\gamma_i)}E_{ii}$ *is a division algebra iff* $\mathcal{H}(\Delta)E_{ii}$ *is by Jordan Homotope Proposition* 7.2.1(3).

In the Jordan matrix algebra $\mathcal{H}_n(\mathrm{C}, \Gamma)$ *for any composition algebra* C (*whether division or split*) *over a field* Φ, *the diagonal matrix unit* $b = E_{ii}$ *is a simple idempotent and its principal inner ideal* $\mathrm{J}_{ii} = \mathcal{H}(\mathrm{C})^{(\gamma_i)}E_{ii} \cong \Phi E_{ii}$ *is a simple inner ideal.* □

Full and Hermitian Nilpotent Example 19.1.4 *In* $\mathcal{M}_n(\Delta)^+$ *for an associative division algebra* Δ, *the off-diagonal matrix unit* $b = E_{ij}$ *is a simple nilpotent element, whose simple principal inner ideal* $\mathrm{B} = U_b(\mathrm{J}) = \Delta E_{ij}$ *is regularly paired* $b \bowtie d, b \bowtie e$ *with the nilpotent* $d = E_{ji}$ *and the simple idempotents* $e = E_{ii} + E_{ji}$ *or* $e = E_{jj} + E_{ji}$, *and is structurally paired with the simple idempotents* $U_{\frac{1}{2}+d}b = \frac{1}{2}E_{ii} + \frac{1}{4}E_{ij} + E_{ji} + \frac{1}{2}E_{jj}$ *and* $\frac{1}{2}U_{1+d}b = \frac{1}{2}\big(E_{ii} + E_{ij} + E_{ji} + E_{jj}\big)$.

In general, a hermitian algebra $\mathcal{H}_n(\Delta, -)$ *need not contain nilpotent elements* (*e.g., it may be formally real*). *If* Δ *contains an element* γ *of norm* $\gamma\gamma^* = -1$, *then the element* $b = E_{ii} + \gamma E_{ij} + \gamma^* E_{ji} - E_{jj}$ *is a simple nilpotent in* $\mathcal{H}_n(\Delta, -)$ *paired with the simple idempotent* $e = E_{ii}$. □

Triangular Trivial Example 19.1.5 *If* $\mathrm{J} = \mathrm{A}^+$ *for* $\mathrm{A} = \mathcal{T}_n(\Delta)$ *the upper-triangular* $n \times n$ *matrices over a division algebra* Δ, *then the off-diagonal matrix units* E_{ij} $(i < j)$ *are trivial elements, and the inner ideal* ΦE_{ij} *is a trivial inner ideal which is minimal if* Φ *is a field.* □

Division Example 19.1.6 *If* J *is a division algebra, then every nonzero element* b *is simple,* $(b] = (b) = \mathrm{J}$. □

Reduced Spin Idempotent Example 19.1.7 *The spin factor* $\mathcal{R}ed\mathcal{S}pin(q)$ *of Reduced Spin Example* 3.4.1 *has* $\mathrm{J}_{ii} = \Phi e_i \cong \Phi^+$ *for* $i = 1, 2$ *by construction. Thus the* e_i *are simple idempotents* (J_{ii} *is a division subalgebra and a simple inner ideal*) *iff* Φ *is a field.* □

19.2 Minimal Inner Ideals

We show that all minimal inner ideals (simple or not) are of the three types encountered in the examples: idempotent, nilpotent, and trivial.

Minimal Inner Ideal Theorem 19.2.1 (1) *The minimal inner ideals* B *in a Jordan algebra* J *over* Φ *are of the following Types*:

TRIVIAL: $\mathrm{B} = \Phi z$ *for a trivial* z $\quad (U_z\widehat{\mathrm{J}} = \mathbf{0})$;

IDEMPOTENT: $\mathrm{B} = (e] = U_e\mathrm{J}$ *for a simple idempotent* e
(*in which case* B *is a division subalgebra*);

NILPOTENT: $\mathrm{B} = (b] = U_b\mathrm{J}$ *for a simple nilpotent* b
(*then* B *is a trivial subalgebra,* $\mathrm{B}^2 = U_{\mathrm{B}}\mathrm{B} = \mathbf{0}$).

Any $(b]$ *of Nilpotent Type is structurally paired with a simple inner ideal* $(e]$ *for a simple idempotent* e.

(2) *An idempotent* e *is simple iff it is a division idempotent.*

(3) *An inner ideal is simple iff it is of Idempotent Type for a division idempotent, or of Nilpotent Type for a simple nilpotent structurally paired with a division idempotent.*

PROOF. Let B be a minimal inner ideal of J. We will break the proof into several small steps.

Step 1: The case where B contains a trivial element

If B contains a *single* nonzero trivial element z, then Φz is an inner ideal of J contained in B, so by minimality of B we must have $\mathrm{B} = \Phi z$ and B of Trivial Type is *entirely* trivial. FROM NOW ON WE ASSUME THAT B CONTAINS NO TRIVIAL ELEMENTS, in particular, $U_{\mathrm{B}}\widehat{\mathrm{J}} \neq 0$ implies that B is simple.

Step 2: Two properties

The absence of trivial elements guarantees that $b \neq 0$ in B implies that $(b] = U_b\widehat{\mathrm{J}} \neq \mathbf{0}$ is an inner ideal of J contained in B, so again by minimality we have

(2.1) $\quad \mathrm{B} = (b]$ for any $b \neq 0$ in B,

(2.2) $\quad$ all $b \neq 0$ in B are simple.

Step 3: The case $b^2 \neq 0$ for all $b \neq 0$ in B

If $b \neq 0$ in B, then by hypothesis $b^2 \neq 0$ lies in B [because inner ideals are also subalgebras], so $b \in \mathrm{B} = U_{b^2}\widehat{\mathrm{J}}$ [using (2.1) for b^2] $= U_b\left(U_b\widehat{\mathrm{J}}\right) = U_b\mathrm{B}$ [by (2.1)] implies that all U_b are surjective on B, so by the Surjective Unit Lemma 18.1.4 $\mathrm{B} = (e]$ is a unital subalgebra and all $b \neq 0$ are invertible. Then B is a division algebra with e a simple idempotent, and B is of Idempotent Type.

Step 4: The case some $b^2 = 0$ for $b \neq 0$ in B

Here $B^2 = U_B B = \mathbf{0}$, since $B^2 = (U_b \widehat{J})^2$ [by (2.1)] $= U_b U_{\widehat{J}} b^2$ [by the Fundamental Formula acting on 1] $= \mathbf{0}$, and once *all* $b^2 = 0$ we have all $U_b B = \mathbf{0}$ [if $U_b B \neq \mathbf{0}$ then, since it is again an inner ideal of J contained in B by Structural Innerness 18.2.2(2), by minimality it must be all of B, $B = U_b B = U_b(U_b B) = U_{b^2} B = \mathbf{0}$, a contradiction], and B is of Nilpotent Type.

Step 5: The Nilpotent Type in more detail

By (2.2), all $b \neq 0$ in B are simple, hence regular, so by vNr Pairing 18.1.2 we have b regularly paired with some d (which by Simple Pairing 19.1.2(2) is itself simple, so $(d]$ is again simple). If $d^2 \neq 0$ then by the above $(d] = (e]$ for a simple idempotent e, and we have established the final assertion of part (1) of the Theorem.

So assume that $b^2 = d^2 = 0$. Then $(b]$ can divorce $(d]$ and get structurally paired with $(e]$, where $e := U_{\frac{1}{2}\hat{1}+d}(b) \bowtie b$ is a simple (but honest) idempotent in J regularly paired with b. Indeed, by the Shirshov–Cohn Principle 5.1.3 we can work inside $\Phi[b,d] \cong \mathcal{H}(A,*) \subseteq \widehat{A}^+$, where the element $u := \frac{1}{2}\hat{1} + d \in \widehat{A}$ has

(5.1) $\quad bub = b,$ (5.2) $\quad bu^2b = b,$

since $bub = \frac{1}{2}b^2 + bdb = 0 + b$ and $bu^2b = \frac{1}{4}b^2 + bdb + bd^2b = bdb = b$ when $b^2 = d^2 = 0$. But then $e^2 = (ubu)(ubu) = u(bu^2b)u = ubu$ [using (5.2)] $= e$ is idempotent, and it is regularly paired with b because $beb = bubub = bub = b$ [using (5.1) twice] and $ebe = (ubu)b(ubu) = u(bubub)u = u(b)u$ [above] $= e$.

This establishes the tripartite division of minimal inner ideals into types as in part (1). For (2), the unital subalgebra e is a division algebra iff it contains no proper inner ideals by the Division Algebra Criterion 6.1.4, i.e., is simple. For (3), by (1) a simple inner ideal has one of these two types; conversely, by (2) the Idempotent Types are simple, and by Simple Pairing 19.1.2 *any* inner ideal (of nilpotent type or not) which is structurally paired with a simple $(e]$ is itself simple. □

Note that we have not claimed that *all* Φz of Trivial Type are minimal, nor have we said intrinsically which nilpotent b are simple.

Exercise 19.2.1A (1) Show that when Φ is a field, every trivial element z determines a trivial minimal inner ideal $B = \Phi z = [z]$. (2) Show that for a general ring of scalars Φ, $B = \Phi z$ for a trivial z is minimal iff $z^\perp = \{\alpha \in \Phi \mid \alpha z = 0\}$ is a maximal ideal $M \triangleleft \Phi$, so $B = \Phi' z$ is 1-dimensional over the field $\Phi' = \Phi/M$. (3) Show that a trivial element z is never simple, $(z) = (z] = 0$, and an inner ideal $B = \Phi z$ is never simple.

Exercise 19.2.1B* (1) In Step 5 where $b^2 = d^2 = 0$, show that b is regularly paired with $c := U_{1+d} b \bowtie b$ satisfying $c^2 = 2c$, and $e := \frac{1}{2}c$ is a simple idempotent with $(e] = (c]$ structurally paired with $(b]$. (2) Give a proof of the Nilpotent case $b^2 = d^2 = 0$ strictly in terms of the Fundamental Formula, with no reference to associative algebras.

19.3 Problems for Chapter 19

PROBLEM 19.1 Let T be a structural transformation on a nondegenerate algebra J. Show that for any simple inner B, $T(\mathrm{B})$ is either simple or zero. If b is simple, conclude that $T\,(\,(b]\,)$ is either simple or zero. (2) Define the *socle* of a nondegenerate Jordan algebra to be the sum of all its simple inner ideals. Show that this is invariant under all structural transformations, hence is an ideal. A deep analysis of the socle by Loos showed that it encompasses the "finite-capacity" parts of a nondegenerate algebra, and provides a useful tool to streamline some of Zel'manov's structural arguments.

20

Capacity

We are now ready to obtain a capacity from the d.c.c and break it up into connected capacities which represent the simple direct summands of the algebra.

20.1 Capacity Existence

In this section we will show that nondegenerate Jordan algebras with d.c.c. on inner ideals necessarily have finite capacity. This subsumes the classification of algebras with d.c.c. under the (slightly) more general classification of algebras with finite capacity, the ultimate achievement of the Classical Theory. Recall the notion of capacity that we are concerned with in this Phase.[1]

Capacity Definition 20.1.1 *A Jordan algebra has* **capacity** n *if it has a unit element which is a sum of* n *mutually orthogonal simple idempotents*:

$$1 = e_1 + \cdots + e_n \qquad (e_i \textit{ simple orthogonal}).$$

J *has* **connected** *or* **strongly connected capacity** n *if it has such a decomposition where* e_i, e_j *are* (*respectively strongly*) *connected for each pair* $i \neq j$. *An algebra* **has (finite) capacity** *if it has capacity* n *for some* n (*a priori there is no reason that an algebra couldn't have two different capacities at the same time*).

Straight from the definition and the Minimal Inner Ideal fact 19.2.1(2) that e is simple iff it is a division idempotent, we have the following theorem.

Capacity 1 Theorem 20.1.2 *A Jordan algebra has capacity* 1 *iff it is a division algebra.* □

[1] Capacity was defined in I.5.1, and capacity theorems were stated in I.6.5–6.

Thus Jordan division algebras have capacity 1 (and are nondegenerate). We saw in the Simple Reduced Spin Idempotent Example 19.1.6 that a reduced spin factor $\mathcal{R}ed\mathcal{S}pin(q)$ of a quadratic form q over a field Φ (which is nondegenerate iff q is nondegenerate, by Factor Triviality Example 5.3.6) has connected capacity 2 unless $q = 0$, in which case by nondegeneracy $\mathrm{J}_{12} = M = \mathbf{0}$ and it collapses to $\Phi e_1 \boxplus \Phi e_2$ of disconnected capacity 2. We also saw in Simple Matrix Idempotent Example 19.1.3, 19.1.7 the Jordan matrix algebras $\mathcal{M}_n(\Delta)^+$ and $\mathcal{H}_n(\Delta, \Gamma)$ for an associative division algebra Δ, or the $\mathcal{H}_n(\mathrm{C}, \Gamma)$ for a composition algebra C over a field, have connected capacity n (and are nondegenerate). The goal of this Phase is the converse: that any nondegenerate Jordan algebra with connected capacity is one of these types.

Let's show that algebras with d.c.c. have capacity.

Capacity Existence Theorem 20.1.3 *If* J *is nondegenerate with minimum condition on inner ideals, then* J *has a finite capacity.*

PROOF. We tacitly assume that $\mathrm{J} \neq \mathbf{0}$. Since J is nondegenerate, there are no simple inner ideals of trivial type, so the Simple Inner Ideal Theorem 19.3 guarantees that there exist simple idempotents in J. Among all idempotents $e = e_1 + \cdots + e_n$ for e_i simple orthogonal idempotents, we want to choose a maximal one and prove that $e = 1$; we don't have a maximum condition on inner ideals, so we can't choose e with *maximal* $\mathrm{J}_2(e)$, so instead we use the minimum condition to choose e with *minimal* Peirce inner ideal $\mathrm{J}_0(e)$. Here $\mathrm{J}_0(e)$ inherits nondegeneracy and minimum condition from J by Diagonal Inheritance 10.1.1, so *if* $\mathrm{J}_0(e) \neq \mathbf{0}$ it too has a simple idempotent e_{n+1} (which by Diagonal Inheritance is simple in J as well). Now $e_{n+1} \in \mathrm{J}_0(e)$ is orthogonal to all $e_i \in \mathrm{J}_2(e)$ by Peirce Orthogonality Rules 8.2.1, so $\tilde{e} = e + e_{n+1} = e_1 + \cdots + e_n + e_{n+1}$ is a bigger sum of simple orthogonal idempotents, with smaller Peirce space $\mathrm{J}_0(\tilde{e}) = \mathrm{J}_{00} < \mathrm{J}_{00} \oplus \mathrm{J}_{0,n+1} \oplus \mathrm{J}_{n+1,n+1} = \mathrm{J}_0(e)$ [applying Peirce Recovery 13.2.1(2) to $\widetilde{\mathcal{E}} = \{e_1, \ldots, e_n, e_{n+1}\}$] since $e_{n+1} \in \mathrm{J}_{n+1,n+1}$. But this contradicts minimality of $\mathrm{J}_0(e)$, so we *must have* $\mathrm{J}_0(e) = \mathbf{0}$. But then $e = 1$ by the Idempotent Unit Theorem 10.1.2. □

20.2 Connected Capacity

Once J has a capacity, we can forget about the minimum condition: we discard it as soon as we have sucked out its capacity. To analyze algebras with capacity, we first we break them up into connected components. To insure that the components are mutually orthogonal we need to know what connection and non-connection amount to for simple idempotents, and here the key is what invertibility and non-invertibility amount to.

Off-Diagonal Non-Invertibility Criterion 20.2.1 *Let e_1, e_2 be orthogonal simple idempotents in* J*. Then the following six conditions on an element x_{12} of the off-diagonal Peirce space* J_{12} *are equivalent*:

(1) x_{12} *is not invertible in* $\mathrm{J}_2(e_1+e_2) = \mathrm{J}_{11} + \mathrm{J}_{12} + \mathrm{J}_{22}$;

(2a) $U_{x_{12}}\mathrm{J}_{11} = \mathbf{0}$;

(2b) $U_{x_{12}}\mathrm{J}_{22} = \mathbf{0}$;

(3a) $q_{22}(x_{12}) = 0$;

(3b) $q_{11}(x_{12}) = 0$;

(4) $x_{12}^2 = 0$.

PROOF. The reader will have to draw a diagram for this play — watch our moves! We first show that (4) $\iff$ (3*a*) AND (3*b*), and (1) $\iff$ (3*a*) OR (3*b*). Then we establish a "left hook" (3*b*) $\Longrightarrow$ (2*a*) $\Longrightarrow$ (3*a*), so dually we have a "right hook" (3*a*) $\Longrightarrow$ (2*b*) $\Longrightarrow$ (3*b*), and putting the two together gives a "cycle" (2*a*) $\iff$ (2*b*) $\iff$ (3*a*) $\iff$ (3*b*) showing that all are equivalent [knocking out the distinction between "and" and "or" for (3a),(3b)], hence also equivalent to (1) and to (4).

From $x_{12}^2 = q_{22}(x_{12}) + q_{11}(x_{12})$ we immediately see "and," and to see "or" note that x_{12} not invertible $\iff x_{12}^2 = q_{22}(x_{12}) + q_{11}(x_{12})$ not invertible [by Power Invertibility Criterion 6.1.8(2)] $\iff q_{22}(x_{12})$ or $q_{11}(x_{12})$ not invertible [by the Diagonal Invertibility Lemma 14.4.1] $\iff q_{22}(x_{12}) = 0$ or $q_{11}(x_{12}) = 0$ [since J_{ii} are division algebras].

For the left hook, (3b) passes to (2a) by $(U_{x_{12}}\mathrm{J}_{11})^2 = U_{x_{12}}U_{\mathrm{J}_{11}}\, x_{12}^2$ [by the Fundamental Formula] $= U_{x_{12}}U_{\mathrm{J}_{11}}\, q_{11}(x_{12})$ [by Peirce Orthogonality] $= 0$ [by (3b)] $\Longrightarrow U_{x_{12}}\mathrm{J}_{11} = \mathbf{0}$ [by the fact that the division algebra J_{22} has no nilpotent elements], and (2a) hands off to (3a) by the definition of q in the q-Proposition 14.3(1). Is that footwork deft, or what! □

The crucial fact about connectivity between simple idempotents is that it is an all-or-nothing affair.

Simple Connection Lemma 20.2.2 *If e_1, e_2 are orthogonal simple idempotents in a nondegenerate Jordan algebra* J*, then either e_1, e_2 are connected or else* $\mathrm{J}_{12} = U_{e_1,e_2}\mathrm{J} = \mathbf{0}$.

PROOF. e_1, e_2 not connected $\iff$ no $x_{12} \in \mathrm{J}_{12}$ is invertible $\iff q_{ii}(\mathrm{J}_{12}) = U_{\mathrm{J}_{12}}\mathrm{J}_{ii} = \mathbf{0}$ for $i = 1, 2$ [by the previous Non-invertibility Criterion 20.2.1(2ab), (3ab)] $\iff U_{\mathrm{J}_{12}}(\mathrm{J}_{11} + \mathrm{J}_{12} + \mathrm{J}_{22}) = \mathbf{0}$ [by the $Uijq$-Rules 14.3.1(2)] $\iff U_{\mathrm{J}_{12}}\mathrm{J} = \mathbf{0}$ [by Peirce Orthogonality 13.3.1(3)] $\iff \mathrm{J}_{12} = \mathbf{0}$ [by nondegeneracy]. □

From this the decomposition into a direct sum of connected components falls right into our laps.

Connected Capacity Theorem 20.2.3 *A nondegenerate Jordan algebra with capacity splits into the direct sum* $\mathrm{J} = \mathrm{J}_1 \boxplus \cdots \boxplus \mathrm{J}_n$ *of a finite number of nondegenerate ideals* J_k *having connected capacity.*

PROOF. Let $1 = \sum_{i \in I} e_i$ for simple e_i as in the Capacity Definition 20.1.1. By Connection Equivalence 14.4.4, connectivity of the e_i defines an equivalence relation on the index set $I : \ i \sim j$ iff $i = j$ or e_i, e_j are connected. If we break I into connectivity classes K and let the $f_K = \sum\{e_i | i \in K\}$ be the class sums, then for $K \neq L$, $U_{f_K, f_L}\mathrm{J} = \sum_{k \in K, \ell \in L} U_{e_k, e_\ell}\mathrm{J} = \mathbf{0}$ by Simple Connection 20.2.2, so that $\mathrm{J} = U_1\mathrm{J} = U_{\sum_K f_K}\mathrm{J} = \sum_K U_{f_K}\mathrm{J} = \boxplus_K \mathrm{J}_K$ is [by Peirce Orthogonality 8.2.1] an algebra-direct sum of Peirce subalgebras $\mathrm{J}_K = U_{f_K}\mathrm{J}$ (which are then automatically ideals) having unit f_K with connected capacity. □

An easy argument using Peirce decompositions shows the following.

Simple Capacity Theorem 20.2.4 *A nondegenerate algebra with capacity is simple iff its capacity is connected.*

PROOF. If J is simple there can be only one summand in the Connected Capacity Theorem 20.2.3, so the capacity must be connected. To see that connection implies simplicity, suppose K were a nonzero ideal in J where any two e_i, e_j are connected by some v_{ij}. Then by Peirce Inheritance 13.1.4(2) we would have the Peirce decomposition $\mathrm{K} = \bigoplus_{i \leq j} \mathrm{K}_{ij}$ where some $\mathrm{K}_{ij} \neq \mathbf{0}$ is nonzero. We can assume that some diagonal $\mathrm{K}_{ii} \neq \mathbf{0}$, since if an off-diagonal $\mathrm{K}_{ij} \neq \mathbf{0}$ then also $\mathrm{K}_{ii} = \mathrm{K} \cap \mathrm{J}_{ii} \supseteq q_{ii}(\mathrm{K}_{ij}, \mathrm{J}_{ij}) \neq 0$ by q-Nondegeneracy 14.3.1(4). But $\mathrm{K}_{ii} = \mathrm{K} \cap \mathrm{J}_{ii}$ is an ideal in the division algebra J_{ii}, so $\mathrm{K}_{ii} = \mathrm{J}_{ii}$ and for all $j \neq i$ $\mathrm{K} \supseteq U_{v_{ij}}\mathrm{K}_{ii} = U_{v_{ij}}\mathrm{J}_{ii} = \mathrm{J}_{jj}$ [by connectivity] as well, so $\mathrm{K} \supseteq \sum \mathrm{J}_{kk} \supseteq \sum e_k = 1$ and $\mathrm{K} = \mathrm{J}$. □

20.3 Problems for Chapter 20

In the above proofs we freely availed ourselves of multiple Peirce decompositions. Your assignment, should you choose to accept it, is to exercise mathematical frugality (avoiding the more cumbersome notation and details of the multiple case) and derive everything from properties of single Peirce decompositions.

PROBLEM 20.1* (Capacity Existence) Show, in the notation of the proof of 20.1.3, that (1) $\mathrm{J}_0(e+e_{n+1}) \subseteq \mathrm{J}_0(e)$, (2) the inclusion is proper.

PROBLEM 20.2* (Simple Connection) Show that $U_{e_1,e_2}(\mathrm{J}) = \mathbf{0}$ for disconnected e_1, e_2 as in 20.2.2. (1) Show that it suffices to *assume from the start that* J *is unital with* $1 = e_1 + e_2$, where $e = e_1$ is an idempotent with $1 - e = e_2$, $\mathrm{J}_1(e) = U_{e_1,e_2}(\mathrm{J})$. (2) Show that $\mathrm{J}_2(e), \mathrm{J}_0(e)$ are division algebras, in particular have no nilpotent elements. (3) Show that $q_2(x_1) = 0 \Longleftrightarrow q_0(x_1) = 0$. (4) If the quadratic form $q_2(\mathrm{J}_1)$ vanishes identically, *show that* J_1 *vanishes.* (5) On the other hand, if $q_2(\mathrm{J}_1) \neq 0$, *show that there exists an invertible* v.

PROBLEM 20.3* (Connected Capacity) Show that connection is transitive as in 20.2.3. Assume v_{ij} connects e_i, e_j and v_{jk} connects e_j, e_k. (1) Show that $v = \{v_{ij}, v_{jk}\} \in \mathrm{J}'_1(e)$ for $\mathrm{J}' = \mathrm{J}_2(e_i + e_k)$, $e = e_i$, $1' - e = e_j$. (2) Show that both $\mathrm{J}'_2(e), \mathrm{J}'_0(e)$ are division algebras, so v will be invertible if $v^2 = q_2(v) + q_0(v)$ is *invertible*, hence if $q_2(v)$ (dually $q_0(v)$) is *nonzero.* (3) Show that $q_2(v) = E_2\left(U_{v_{ij}}(a_j)\right)$ where $(v_{jk})^2 = a_j + a_k$ for *nonzero* $a_r \in \mathrm{J}_2(e_r)$. (4) Conclude that $q_2(v)$ is nonzero because $U_{v_{ij}}$ is invertible on the nonzero element $a_j \in \mathrm{J}_2(e_i + e_j)$.

PROBLEM 20.4* (Simple Capacity) To see that connection implies simplicity in 20.2.4, suppose K were a proper ideal. (1) Show that $U_{e_i}(\mathrm{K}) = \mathbf{0}$ for each i. (2) Then show that $U_{e_i,e_j}(\mathrm{K}) = \mathbf{0}$ for each pair $i \neq j$. (3) Use (1) and (2) to show that $\mathrm{K} = U_1(\mathrm{K}) = 0$.

21

Herstein–Kleinfeld–Osborn Theorem

In classifying Jordan algebras with nondegenerate capacity we will need information about the algebras that arise as coordinates. Recall that the Hermitian and Jacobson Coordinatization Theorems have, as their punch line, that the Jordan algebra in question looks like $\mathcal{H}_n(\mathrm{D}, \Gamma)$ for an alternative algebra D with involution whose hermitian elements are all nuclear. Here D coordinatizes the off-diagonal Peirce spaces and $\mathcal{H}(\mathrm{D}, -)$ coordinatizes the diagonal spaces. The condition that $\mathcal{H}_n(\mathrm{D}, \Gamma)$ be *nondegenerate with capacity* n relative to the standard E_{ii} reduces by Jordan Matrix Nondegeneracy 5.3.5 to the condition that the coordinates be nondegenerate and the diagonal coordinates be a division algebra. In this chapter we will give a precise characterization of these algebras.

Along the way we need to become better acquainted with alternative algebras, gathering a few additional facts about their nuclei and centers, and about additional identities they satisfy.

21.1 Alternative Algebras Revisited

Alternative algebras get their name from the fact that the associator is an alternating function of its arguments. Alternative algebras satisfy the Moufang Laws, which play as important a role in the theory as do the alternative laws themselves.

Moufang Lemma 21.1.1 *An alternative algebra is automatically* **flexible***,*

$$[x, y, x] = 0, \qquad (1)$$

so alternativity is the condition that the associator $[x, y, z]$ *be an alternating multilinear function of its arguments, in the sense that it vanishes if any two of its variables are equal (equivalently, in the presence of* $\frac{1}{2}$*, the condition that the associator is a skew-symmetric function of its arguments).*

Alternative algebras automatically satisfy the **Left**, **Middle**, *and* **Right Moufang Laws**

$$(2)\quad x(y(xz)) = (xyx)x, \quad (xy)(zx) = x(yz)x, \quad ((zx)y)x = z(xyx),$$

as well as the **Left Bumping Formula**

$$(3)\qquad [x, y, zx] = x[y, z, x].$$

Any linear algebra satisfies the **Teichmüller Identity**

$$(4)\qquad [xy, z, w] - [x, yz, w] + [x, y, zw] = [x, y, z]w + x[y, z, w].$$

PROOF. (1) As we have seen before, our old friend flexibility comes from linearizing alternativity, $[x, y, x] = -[y, x, x] = 0$. Thus we are entitled to write xyx without ambiguity in place of $(xy)x$ and $x(yx)$.

Left Moufang is equivalent to *Left Bumping* because

$$\begin{aligned} x(z(xy)) - (xzx)y &= x\big((zx)y - [z, x, y]\big) - \big([x, zx, y] + x((zx)y)\big) \\ &= -x[y, z, x] + [x, y, zx], \qquad \text{[by alternativity]}, \end{aligned}$$

which in turn is equivalent to *Middle Moufang* because

$$\begin{aligned} -x[y, z, x] + [x, y, zx] &= \big(-x(yz)x + x(y(zx))\big) + \big((xy)(zx) - x(y(zx))\big) \\ &= -x(yz)x + (xy)(zx). \end{aligned}$$

Dually for *Right Moufang,* so all the Moufang Laws (2) are equivalent to Left Bumping (3). To see that *Left Bumping* holds, note that $[x, y, x^2] = 0$ [L_x commutes with R_x by flexibility, hence also with $R_{x^2} = R_x^2$ by right alternativity], so linearizing $x \mapsto x, z$ in this shows that

$$\begin{aligned} [x, y, zx] &= -[x, y, xz] - [z, y, x^2] \\ &= +[x, xz, y] - [x^2, z, y] && \text{[by alternativity]} \\ &= (x^2z)y - x((xz)y) - (x^2z)y + x^2(zy) \\ &= -x((xz)y) + x(x(zy) && \text{[by left alternativity]} \\ &= -x[x, z, y] = x[y, z, x] && \text{[by alternativity].} \end{aligned}$$

Teichmüller (4) can be verified by direct calculation to hold in all nonassociative algebras:

$$\begin{aligned} &[xy, z, w] - [x, yz, w] + [x, y, zw] \\ &\quad = ((xy)z)w - (xy)(zw) - (x(yz))w + x((yz)w) + (xy)(zw) - x(y(zw)) \\ &\quad = ((xy)z)w - (x(yz))w + x((yz)w) - x(y(zw)) \\ &\quad = [x, y, z]w + x[y, z, w]. \qquad \square \end{aligned}$$

Exercise 21.1.1* Alternative implies Moufang, but in characteristic 2 situations the left alternative law is not strong enough to guarantee left Moufang, resulting in some pathological [= not-alternative] simple left alternative algebras; the proper notion is that of a *left Moufang* algebra, satisfying both the left alternative and left Moufang laws, and the satisfying theorem states that every simple left Moufang algebra is actually alternative. (1) Show that the left alternativity $[x, x, y] = 0$ plus flexibility $[x, y, x] = 0$ implies alternativity. (2) Show that the left alternativity implies the Left Moufang Law $(x(yx))z = x(y(xz))$ if $\frac{1}{2} \in \Phi$ [you must be careful not to assume flexibility, so carefully distinguish $x(yx)$ from $(xy)x$!]

Emil Artin gave a beautiful characterization of alternative algebras: they are precisely the nonassociative algebras in which every subalgebra generated by *two* elements is associative (just as the power-associative algebras are precisely the nonassociative algebras in which every subalgebra generated by *one* element is associative). This gives us an analogue of Macdonald's Theorem: A polynomial in two variables will vanish in all alternative algebras if it vanishes in all associative algebras.

Artin's Theorem 21.1.2 *A linear algebra* A *is alternative iff every subalgebra generated by two elements is associative. In particular, alternative algebras satisfy every identity in two variables that is satisfied by all associative algebras.*

Proof. The proof requires only the alternative laws, Middle Moufang, and Teichmüller. If the subalgebra $\Phi[x, y] \subseteq$ A is associative, then we certainly have $[x, x, y] = [y, x, x] = 0$ for all $x, y \in$ A, and therefore A is alternative by Alternative Definition 2.1.1.

The nontrivial part is the converse: if A is alternative then every $\Phi[x, y]$ is associative, equivalently, $[p(x, y), q(x, y), r(x, y)] = 0$ for all nonassociative polynomials p, q, r in two variables. By linearity it suffices to prove this for *monomials* p, q, r, and we may induct on the total degree $n = \partial p + \partial q + \partial r$ (using the obvious notion of degree for a nonassociative monomial). The result is vacuous for $n = 0, 1, 2$ (if we are willing to work in the category of unital algebras where $\Phi[x, y]$ is understood to contain the "trivial" monomial 1 of degree 0, the result is no longer vacuous, but it is still trivial, since if the three degrees sum to at most 2, then at least one factor must be 1, and any associator with term 1 vanishes). The result is also trivial if $n = 3$: if none of the factors is 1, all must be degree 1 monomials x or y, and two of the three must agree, so the associator vanishes by the alternative and flexible laws.

Assume now we have proven the result for monomials of total degree $< n$, and consider p, q, r of total degree $= n$. The induction hypothesis that lower associators vanish shows, by the usual argument (Generalized Associative Law), that *we can rearrange parentheses at will inside any monomial of degree* $< n$. We may *assume that* p, q, r *are all of degree* ≥ 1 (the associator vanishes trivially if any term has degree 0, i.e., is 1); then each of p, q, r has degree $< n$, and therefore *we can rearrange parentheses at will inside* p, q, r.

We have a general *principle*: *the associator* $[p, q, r]$ *vanishes if one term begins with a variable* x *or* y*, and another term ends in that same variable.* To see this, by rearranging (using alternation of the associator, and symmetry in x and y) we may suppose that p begins with x and r ends with x: then $[p, q, r] = [xp', q, r'x]$ [for $p', q' \in \widehat{\mathrm{A}}$ monomials of degree ≥ 0, using the induction associative hypothesis to rearrange parentheses in $p, r] = \big((xp')q\big)(r'x) - (xp')\big(q(r'x)\big) = \big(x(p'q)\big)(r'x) - (xp')\big((qr')x\big)$ [again using the induction associativity hypothesis] $= x\big((p'q)r'\big)x) - x\big(p'(qr')\big)x$ [by the Middle Moufang Law] $= x[p', q, r']x = 0$ [by the induction hypothesis again].

Suppose $[p, q, r] \neq 0$, where by symmetry we may assume that $p = xp'$ begins with x. Then by the *Principle* neither q nor r can end in x, they must both end in y. But then by the *Principle* q (respectively r) ending in y forces r (respectively q) to begin with x. But then $[p, q, r] = [xp', q, r] = [x, p'q, r] - [x, p', qr] + [x, p', q]r + x[p', q, r]$ [by Teichmüller], where the last two associators vanish because they are of lower degree $< n$, and the first two vanish by the above *principle,* since in both of them x certainly *ends* with x, while in the first r, and in the second qr, *begins* with x. Thus $[p, q, r] \neq 0$ leads to a contradiction, and we have established the induction step for degree n. □

EXERCISE 21.1.2 Thinking of Macdonald's Theorem, can you find an identity in three variables, of degree one in one of the variables, that holds in all associative algebras but *not* in all alternative algebras?

21.2 A Brief Tour of the Alternative Nucleus

Alternative algebras come in only two basic flavors, associative (where the nucleus is the whole algebra) and octonion (where the nucleus reduces to the center). This incipient dichotomy is already indicated by basic properties of the nucleus,

Alternative Nucleus Lemma 21.2.1 *If* D *is alternative with nucleus* $\mathcal{N} = \mathcal{N}uc(\mathrm{D})$*, then for any elements* $n \in \mathcal{N}, x, y, z \in \mathrm{D}$ *we have the* **Nuclear Slipping Formula**:

(1) $$n[x, y, z] = [nx, y, z] = [xn, y, z].$$

The nucleus is **commutator-closed**:

(2) $$[\mathcal{N}, \mathrm{D}] \subseteq \mathcal{N}.$$

We have nuclear product relations

(3) $$[\mathcal{N}, x]x \subseteq \mathcal{N}, \quad [\mathcal{N}, x][x, y, z] = 0 \quad \textit{for any } x, y, z \in \mathrm{D}.$$

Nuclear commutators absorb D *and kill associators,*

$$(4) \qquad [\mathcal{N},\mathcal{N}]\mathrm{D} \subseteq \mathcal{N}, \qquad [\mathcal{N},\mathcal{N}]\,[\mathrm{D},\mathrm{D},\mathrm{D}] = 0.$$

Indeed, any nuclear subalgebra closed under commutators automatically satisfies these last two inclusions, and if it contains an invertible commutator then the nuclear subalgebra must be the whole algebra:

(5) *if* B *satisfies* $[\mathrm{B},\mathrm{D}] \subseteq \mathrm{B} \subseteq \mathcal{N}$, *then it also has*:

(5*a*) $[\mathrm{B},\mathrm{B}]\mathrm{D} + 2[\mathrm{B},x]x \subseteq \mathrm{B}$ *for all* $x \in \mathrm{B}$,

(5*b*) *if* $b \in [\mathrm{B},\mathrm{B}]$ *has* $b^{-1} \in \mathrm{D}$, *then* $\mathrm{D} = \mathrm{B}$ *is associative.*

PROOF. Notice that all of these properties are true in associative algebras (where $[x,y,z] = 0$ and $\mathcal{N} = \mathrm{D}$) and in octonion algebras (where $\mathcal{N} = \mathcal{C}ent(\mathrm{D})$, $[\mathcal{N},\mathrm{D}] = 0$).

A nuclear element *slips* in and out of parentheses, so $n[x,y,z] = [nx,y,z]$ and $[xn,y,z] = [x,ny,z]$ in any linear algebra; what is different about alternative algebras is that nuclear elements can *hop* because of the alternating nature of the associator: $n[x,y,z] = -n[y,x,z] = -[ny,x,z] = [x,ny,z] = [xn,y,z]$ as in (1). Subtracting the last two terms in (1) gives $0 = [[n,x],y,z]$, so $[[n,x],\mathrm{D},\mathrm{D}] = \mathbf{0}$, which implies that $[n,x] \in \mathcal{N}$ in an alternative algebra, establishing (2). Since $[n,x]x \in \mathcal{N} \Leftrightarrow [[n,x]x,\mathrm{D},\mathrm{D}] = 0 \Leftrightarrow [n,x][x,y,z]$ by slipping (1) for the nuclear $[n,x]$ by (2), we see that the two parts of (3) are equivalent; the second holds because $[n,x][x,y,z] = nx[y,z,x] - xn[y,z,x]$ [by nuclearity of n and alternativity] $= n[x,y,zx] - x[ny,z,x]$ [bump and slip] $= [x,ny,zx] - [x,ny,zx]$ [hop and bump] $= 0$. Linearizing $x \mapsto x,m$ for $m \in \mathcal{N}$ in (3) gives (4): $[n,m]x \in -[n,x]m + \mathcal{N} \subseteq -\mathcal{N}m + \mathcal{N}$ [using (2)] $\subseteq \mathcal{N}$, and $[n,m][x,y,z] = -[n,x][m,y,z] = 0$ [for nuclear m].

The first part of (4) is also a special case of the first part of (5a), which follows easily from $[b,c]x = [b,cx] - c[b,x]$ [by nuclearity of b,c] $\subseteq \mathrm{B} - \mathrm{BB}$ [by hypothesis (5)] $\subseteq \mathrm{B}$. Then the implication (5b) holds for such commutators b since $\mathrm{B} \supseteq b\mathrm{D} \supseteq b(b^{-2}(b\mathrm{D})) = (bb^{-2}b)(\mathrm{D})$ [by Left Moufang] $= 1\mathrm{D} = \mathrm{D}$, and $\mathcal{N} \supseteq \mathrm{B} \supseteq \mathrm{D} \supseteq \mathcal{N}$ implies that $\mathcal{N} = \mathrm{B} = \mathrm{D}$ is associative. Finally, for the second part of (5a) (with the annoying factor 2) we have $2[b,x]x = ([b,x]x + x[b,x]) + ([b,x]x - x[b,x]) = [b,x^2] + [[b,x],x] \in \mathrm{B}$ [using alternativity and the hypothesis (5) thrice]. □

EXERCISE 21.2.1A Enlarge on the Alternative Nucleus Lemma 21.2.1. (1) Show that in alternative algebras we have unrestricted nuclear slipping $n[x,y,z] = [nx,y,z] = [xn,y,z] = [x,ny,z] = [x,yn,z] = [x,y,nz] = [x,y,zn] = [x,y,z]n$; in particular, the nucleus commutes with associators. (2) Show that nuclear commutators annihilate associators, $[n,m][x,y,z] = 0$.

EXERCISE 21.2.1B* Enlarging on 21.2.1(5b), if all nonzero elements of $\mathcal{H}$ are invertible, show that *any* skew element $b = -b^* \in \mathrm{B} = \Phi\langle\mathcal{H}\rangle$ (the subalgebra generated by hermitian elements) which is invertible in D is actually invertible in B itself.

Next we look inside the nucleus at the center. We can always replace the current ring of scalars by the center, since multiplication is still bilinear over the center. A $*$-algebra becomes an algebra over the $*$-center $\mathcal{H}(\mathcal{C}ent(\mathrm{D}), *)$ (in order that the involution remain linear).

Central Involution Theorem 21.2.2 *Every unital alternative algebras with central involution has* norms $n(x) := x\bar{x}$ *and* traces $t(x) := x + \bar{x}$ *in the* $*$*-center satisfying*

(1)	$t(\bar{x}) = t(x), \quad n(\bar{x}) = n(x)$	(**Bar Invariance**),
(2)	$t(xy) = n(\bar{x}, y) = t(yx)$	(**Trace Commutativity**),
(3)	$t((xy)z) = t(x(yz))$	(**Trace Associativity**),
(4)	$n(xy) = n(x)n(y)$	(**Norm Composition**),
(5)	$\bar{x}(xy) = n(x)y = (yx)\bar{x}$	(**Kirmse Identity**),
(6)	$x^2 - t(x)x + n(x)1 = 0$	(**Degree–2 Identity**).

Every alternative algebra with central involution and no nil ideals of index 2 *(i.e., ideals where every element z satisfies $z^2 = 0$) is a composition algebra with standard involution $\bar{x} = t(x)1 - x$ over its $*$-center.*

PROOF. Note that since $x + \bar{x} \in \mathcal{C}ent(\mathrm{D}) \subseteq \mathcal{N}uc(\mathrm{D})$, we can *remove a bar anywhere in an associator for the price of a minus sign*; e.g., $[a, b, \bar{x}] = -[a, b, x]$. For Bar Invariance (1), the involution trivially leaves the trace invariant, $t(\bar{x}) = t(x)$, and it also leaves the norm invariant: $n(\bar{x}) = \bar{x}x = (t(x) - x)x = t(x)x - x^2 = x(t(x) - x) = x\bar{x} = n(x)$. Along the way we established the Degree–2 Identity (6). The linearization of the norm is a twist of the trace bilinear form: $n(x, \bar{y}) = x\bar{\bar{y}} + \bar{y}\bar{x} = xy + \overline{xy}$ [since bar is an involution] $= t(xy)$ [by definition]. Trace Commutativity (2) then follows from symmetry, $n(x, \bar{y}) = n(\bar{x}, \bar{\bar{y}})$ [by bar-invariance (1) of the norm] $= n(\bar{x}, y)$.

For Trace Associativity (3), associators $[x, y, z]$ are skew because $\overline{[x, y, z]} = -[\bar{z}, \bar{y}, \bar{x}]$ [by the involution] $= +[z, y, x]$ [removing three bars] $= -[x, y, z]$ [by alternativity], so $0 = t([x, y, z]) = t((xy)z) - t(x(yz))$.

From Artin's Theorem 21.2 we know that $x, y, \bar{x}, \bar{y}$ all lie in a *unital associative* subalgebra B generated by two elements x, y over the $*$-center. Inside the associative subalgebra B, the calculations for norm composition (4) and Kirmse (5) are trivial (compare the blood, sweat, and tears of Exercise 21.2.2 below using only alternativity!): $n(xy)1 = (xy)\overline{(xy)} = (xy)(\bar{y}\,\bar{x}) = x(y\bar{y})\bar{x} = x(n(y)1)\bar{x} = n(y)x\bar{x} = n(y)n(x)1 = n(x)n(y)1$ and $n(x)y = (\bar{x}x)y = \bar{x}(xy)$, dually on the right.

Finally, we check that absence of ideals nil of index 2 guarantees that n is nondegenerate. The radical of n is an ideal: it is clearly a Φ-submodule by definition of quadratic form, and it is a left (dually right) ideal because linearizing $x \mapsto x, 1,\ y \mapsto y, z$ in $n(xy) = n(x)n(y)$ gives $n(xy, z) + n(xz, y) = t(x)n(y, z)$, hence for radical z we see that $n(xz, y) = 0$ and $xz \in \mathcal{R}ad(n)$. Its elements z all have $t(z) = n(z, 1) = 0, n(z) = \frac{1}{2}n(z, z) = 0$ and so square to 0 by the Degree–2 Identity (6). If there are no nil ideals of index 2 then radical $\mathcal{R}ad(n)$ vanishes, and n is nondegenerate permitting composition, so we have a composition algebra. □

Exercise 21.2.2* Establish Norm Composition and the Kirmse Identity in the Central Involution Theorem (4), (5) without invoking Artin's Theorem, expanding $\big(n(xy) - n(x)n(y)\big)1$ for norm composition and $n(x)y - \bar{x}(xy)$ for Kirmse.

In studying the octonions and the eight-square problem, J. Kirmse in 1924 considered linear $*$-algebras satisfying $x^*(xy) = n(x)y = (yx)x^*$ for a quadratic form n. In 1930 Artin and his student Max Zorn (he of the famous lemma) dropped the involution condition and invented the category of alternative algebras.

21.3 Herstein–Kleinfeld–Osborn Theorem

Now we have enough information about alternative algebras to embark on a proof of our main result.

Herstein–Kleinfeld–Osborn Theorem 21.3.1 *A nondegenerate alternative $*$-algebra has all its hermitian elements invertible and nuclear iff it is isomorphic to one of the following*:

Noncommutative Exchange Type: *the exchange algebra $\mathcal{E}x(\Delta)$ of a noncommutative associative division algebra Δ*;

Division Type: *an associative division $*$-algebra Δ with non-central involution*;

Composition Type: *a composition $*$-algebra of dimension $1, 2, 4,$ or 8 over a field Ω (with central standard involution) : the ground field (unarion), a quadratic extension (binarion), a quaternion algebra, or an octonion algebra.*

In particular, the algebra is automatically $*$-simple, and is associative unless it is an octonion algebra. We can list the possibilities in another way: the algebra is one of

Exchange Type′: *the direct sum $\Delta \boxplus \Delta^{op}$ of an associative division algebra Δ and its opposite, under the exchange involution*;

Division Type′: *an associative division algebra Δ with involution*:

Split Quaternion Type′: *a split quaternion algebra of dimension 4 over its center Ω with standard involution; equivalently, 2×2 matrices $\mathcal{M}_2(\Omega)$ under the symplectic involution $x^{sp} := sx^{tr}s^{-1}$ for symplectic $s := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$*;

Octonion Type′: *an octonion algebra O of dimension 8 over its center Ω with standard involution.*

proof. We will break the proof into bite-sized steps, each one of some intrinsic interest.

Step 1: Reduction to the $*$-simple case

We claim that D is AUTOMATICALLY $*$-simple:[1] a proper $*$-ideal $\mathrm{I} \triangleleft \mathrm{D}$ would consist entirely of trivial elements, so by nondegeneracy no such ideal exists. Indeed, $\mathrm{I} \neq \mathrm{D}$ contains no invertible elements, hence no nonzero hermitian elements; yet for $z \in \mathrm{I}$, $x \in \mathrm{D}$ the elements $z + \bar{z}$, $z\bar{z}$, $xz + \bar{z}\bar{x}$ are hermitian and still in I, so they vanish: I is skew $[\bar{z} = -z]$, nil $[z^2 = -z\bar{z} = 0]$, and $*$-commutes with D $[xz = z\bar{x}]$. But then all $z \in \mathrm{I}$ are trivial: $z(xz) = z(z\bar{x}) = z^2\bar{x}$ [using alternativity] $= 0$. FROM NOW ON WE ASSUME THAT D IS $*$-SIMPLE.

Step 2: Reduction to the simple case

We can easily take care of the case where D is $*$-simple but NOT simple: this is exactly Exchange Type by the general $*$-Simple Theorem 1.5.4 for $*$-simple linear algebras (alternative or not). We may assume that Δ is noncommutative in Exchange Type, since if $\Delta = \Omega$ is a field then $\Delta \boxplus \Delta^{op} = \Omega \boxplus \Omega^{op}$ is merely a split 2-dimensional binarion algebra over its $*$-center $\Omega(1 \boxplus 1)$, and can be included under Composition Type. FROM NOW ON WE ASSUME THAT D IS SIMPLE.

Step 3: Reduction to the case of non-central $\mathcal{H}$

Next we take care of the case of a central involution where the hermitian elements all lie in the center (hence in the $*$-center): this is exactly Composition Type. Indeed, by simplicity D contains no ideals nil of index 2, so by the Central Involution Theorem 21.2.2,

(3.1) If D has central involution, it is a composition algebra over its $*$-center with standard involution.

By Hurwitz's Theorem 2.6.2 we know that the composition algebras are of dimension $1, 2, 4$, or 8. FROM NOW ON WE ASSUME THAT THE HERMITIAN ELEMENTS ARE NOT CENTRAL.

Step 4: Reduction to the associative, hermitian-generated case

Since hermitian elements are nuclear by hypothesis, they are central as soon as they commute with D, so by our non-centrality assumption we must have $[\mathcal{H}, \mathrm{D}] \neq \mathbf{0}$. We will go part way towards establishing Division Type by showing that D is associative and hermitian-generated:

(4.1) if D is simple with $[\mathcal{H}, \mathrm{D}] \neq 0$, then $\mathrm{D} = \langle \mathcal{H} \rangle$ is associative and hermitian-generated.

Let $\mathrm{B} \subseteq \mathcal{N}$ denote the subalgebra generated by $\mathcal{H}$. Note that if $\bar{x} = \epsilon x$, $\bar{y} = \delta y$, then $\overline{[x, y]} = [\bar{y}, \bar{x}] = -[\bar{x}, \bar{y}] = -\epsilon\delta[x, y]$, so if we denote the skew elements

[1] We want to classify only simple algebras with capacity, so we could have assumed from the start that D is $*$-simple. Instead, we have shown that nondegeneracy implies $*$-simplicity in coordinates where hermitians are invertible, analogous to the Simple Capacity Theorem 20.2.4 that connection implies simplicity in Jordan algebras with capacity.

under the involution by $\mathcal{S}$, we have $[\mathcal{H},\mathcal{H}] \subseteq \mathcal{S}$, $[\mathcal{H},\mathcal{S}] \subseteq \mathcal{H}$. We automatically have $[\mathcal{H},\mathcal{H}] \subseteq \mathrm{B}$ since B is a subalgebra, and $[\mathcal{H},\mathcal{S}] \subseteq \mathcal{H} \subseteq \mathrm{B}$, so $[\mathcal{H},\mathrm{D}] \subseteq \mathrm{B}$; thus commutation by D maps $\mathcal{H}$ into B, hence maps the nuclear subalgebra B generated by $\mathcal{H}$ back into B, and B itself is invariant under commutation:

$$(4.2) \qquad [\mathrm{B},\mathrm{D}] \subseteq \mathrm{B}.$$

Thus B satisfies the hypotheses (5) of Alternative Nucleus 21.2.1, so by (5b) we will have $\mathrm{D} = \mathrm{B} = \mathcal{N}$ associative if there is an invertible B-commutator $b \in [\mathrm{B},\mathrm{B}]$. We now exhibit such a b.

By non-centrality $\mathbf{0} \neq [\mathcal{H},\mathrm{D}] = [\mathcal{H},\mathcal{S}+\mathcal{H}]$, so either (Case 1) $[\mathcal{H},\mathcal{S}] \neq \mathbf{0}$, or (Case 2) $[\mathcal{H},\mathcal{S}] = \mathbf{0}$ but $[\mathcal{H},\mathcal{H}] \neq \mathbf{0}$. In **Case 1**, some $b = [h,s]$ is a nonzero hermitian element (hence by hypothesis invertible), which is also a B-commutator because we have $bs = [h,s]s \in [\mathrm{B},s]s \subseteq \mathrm{B}$ *using* $\frac{1}{2}$ *with Alternative Nucleus* (5a), so $s = b^{-1}(bs)$ [by nuclearity of $b \in \mathcal{H}$] $\in b^{-1}\mathrm{B} \subseteq \mathrm{B}$ [since $b^{-1} \in \mathcal{H} \subseteq \mathrm{B}$] and $b = [h,s] \in [\mathrm{B},\mathrm{B}]$.

In **Case 2**, $\mathbf{0} \neq [\mathcal{H},\mathcal{H}]$ contains a nonzero B-commutator $b \in [\mathcal{H},\mathcal{H}] \subseteq [\mathrm{B},\mathrm{B}] \cap \mathcal{S}$, which we claim is invertible in D because it lies in the center of D, which is a field by simplicity of D. To see centrality, $[b,\mathcal{H}] \subseteq [\mathcal{S},\mathcal{H}] = \mathbf{0}$ by hypothesis in Case 2, and $[b,\mathcal{S}] \subseteq [[\mathcal{H},\mathcal{H}],\mathcal{S}] = [[\mathcal{H},\mathcal{S}],\mathcal{H}] + [\mathcal{H},[\mathcal{H},\mathcal{S}]]$ [by Jacobi's Identity and nuclearity of $\mathcal{H}$] $= \mathbf{0}$ by our hypothesis again, so b commutes with $\mathcal{S}+\mathcal{H} = \mathrm{D}$ and therefore is central.

Thus we have exhibited our invertible B-commutator b, so by Alternative Nucleus $\mathrm{D} = \mathrm{B} = \mathcal{N}$ is associative and hermitian-generated as claimed in (4.1). FROM NOW ON ASSUME THAT D IS SIMPLE, ASSOCIATIVE, HERMITIAN-GENERATED, WITH NON-CENTRAL INVOLUTION.

Step 5: Reduction to the division algebra case

Let Z be the set of non-invertible elements. We want these rowdy elements to vanish, so that everybody but 0 is invertible and we have an associative division algebra with non-central involution as in Division Type. We know that Z *misses* the hermitian elements, $\mathcal{H} \cap Z = \mathbf{0}$, since nonzero hermitian elements are invertible; we want to show that it *kills* hermitian elements in the sense that

$$(5.1) \qquad z \in Z \Longrightarrow z\mathcal{H}\bar{z} = \bar{z}\mathcal{H}z = \mathbf{0}.$$

To see this, remember that in $\mathcal{H}$ each element must be invertible or die, and if $0 \neq z \in Z$ then $z\bar{z}, \bar{z}z \in \mathcal{H}$ can't both be invertible [$z\bar{z}a = 1 = b\bar{z}z \Rightarrow z$ has right, left inverses $\Rightarrow z$ is invertible, contrary to $z \in Z$], and as soon as one dies the other does too [if $z\bar{z} = 0$ then $\bar{z}z$ kills $z \neq 0$, $z(\bar{z}z) = (z\bar{z})z = 0z = 0$, is thereby barred from invertibility, and so is condemned to die]. Once $\bar{z}z = z\bar{z} = 0$ the hermitian elements $zh\bar{z}, \bar{z}hz \in \mathcal{H}$ both kill $z \neq 0$ [by$(zh\bar{z})z = z(\bar{z}hz) = 0$] and likewise join their comrades in death, establishing (5.1).

We claim that once z has tasted the blood of *hermitian elements* it will go on to kill *all* elements,

$$(5.2) \qquad z \in Z \Longrightarrow z\mathrm{D}\bar{z} = \mathbf{0}.$$

For $d \in \mathrm{D}$ the element $w := zd\bar{z}$ is skew, $\bar{w} = z\bar{d}\bar{z} = z(d+\bar{d})\bar{z} - zd\bar{z} = 0 - w$ by (5.1), and for $h \in \mathcal{H}$ the commutator $h' = [h, w] \in [\mathcal{H}, \mathcal{S}] \subseteq \mathcal{H}$ is not invertible because it kills z, $h'z = [h(zd\bar{z}) - (zd\bar{z})h]z = hzd(\bar{z}z) - zd(\bar{z}hz) = 0$ by (5.1). Therefore h' is condemned to death, and $h' = 0$ means that w commutes with $\mathcal{H}$. BECAUSE D IS HERMITIAN-GENERATED this implies that w lies in the *center*, yet is nilpotent since $w^2 = zd(\bar{z}z)d\bar{z} = 0$ by (5.1). The center is a field, so like $\mathcal{H}$ its elements must invert or die; therefore, each $w = zd\bar{z}$ must die, establishing (5.2). But in any simple (even prime) associative algebra $x\mathrm{D}y = \mathbf{0}$ implies that $x = 0$ or $y = 0$ [if $x, y \neq 0$ then the three nonzero ideals $\mathrm{I}_x := \widehat{\mathrm{D}}x\widehat{\mathrm{D}}, \mathrm{D}, \mathrm{I}_y := \widehat{\mathrm{D}}y\widehat{\mathrm{D}}$ have zero product $\mathrm{I}_x\mathrm{D}\mathrm{I}_y = 0$], so we see that $z = 0$. Thus $Z = \mathbf{0}$, and we return to a tranquil life in a division algebra with non-central involution.

Step 6: Converse

For the converse, clearly Exchange, Division, and Composition Types are nondegenerate, and they have invertible hermitian elements in the nucleus: the hermitians in Exchange Type are all $\delta \oplus \delta$ which are thus *isomorphic to* Δ^+, in Division Type they are *in* Δ^+, and in Composition Type they are *equal to* Ω, so in all cases are invertible and nuclear.

Step 7: Alternate List

Finally, we check the alternate list, where we divvy up the algebras of Composition Type and parcel them among the other Types. The composition algebras of dimension 1 and the non-split ones of dimension 2 and 4 are associative division algebras and so go into Division Type′ (where of course we now drop the assumption of non-central involution). A split composition algebra of dimension 2 is just $\Omega \oplus \Omega^{op}$, so it can be included in Exchange Type′ (where of course we must drop the assumption of noncommutativity). A split composition algebra of dimension 4 is a split quaternion algebra, isomorphic to $\mathcal{M}_2(\Omega)$, and the standard involution corresponds to the involution on matrix units given by $E_{ii}^* = E_{jj}$, $E_{ij}^* = -E_{ij}$ $(i = 1, 2, j = 3 - i)$, which is just the symplectic involution $x^{sp} = sx^{tr}s^{-1}$. This particular situation gets its own Split Quaternion Type′. The lone nonassociative algebra, the octonion algebra of dimension 8 (split or not), gets its own Octonion Type′. □

In the Hermitian Coordinatization of Chapter 12 a large role was played by the fact that the coordinate algebra was hermitian-generated (generated as an associative algebra by its hermitian elements). The above proof can be modified (cf. Problem 21.1 below) to show that every division algebra with non-central involution is automatically hermitian-generated, so that the crucial issue is whether the involution is central or not.

21.4 Problems for Chapter 21

PROBLEM 21.1* Establish the *Hermitian Generation Theorem*: In the Herstein–Kleinfeld–Osborn Theorem, every $*$-algebra D of Exchange or Division Type is hermitian-generated, and an algebra D of Composition Type is hermitian-generated iff D has dimension 1.

(I) Show that $\mathrm{D} = \mathcal{E}x(\Delta)$ for an associative division algebra Δ is hermitian-generated iff the involution is non-central iff Δ is noncommutative. (1) Show that the center of *any* $\mathrm{D} = \mathcal{E}x(\mathrm{B})$ is $\Omega \boxplus \Omega$ for Ω the center of B, and the exchange involution is *central*, i.e., $\mathcal{H} = \{(b,b) \mid b \in \mathrm{B}\}$ lies in the center, iff $\mathrm{B} = \Omega$ is *commutative*. Conclude that the involution is *noncentral* iff B is *noncommutative*. (2) If B is *commutative* show that $\mathcal{H} = \{(b,b)\}$ does not generate all of D, only itself. (3) Show that if $\mathrm{B} = \Delta$ is *noncommutative*, then $\mathrm{D} = \mathcal{E}x(\Delta)$ is hermitian-generated. (4) Conclude that $\mathrm{D} = \mathcal{E}x(\Delta)$ is hermitian-generated iff Δ is noncommutative.

(II) Show that if $\mathrm{D} = \Delta$ is an associative division algebra with involution, then D is hermitian-generated iff the involution is either trivial (so Δ is commutative) or non-central (so Δ is noncommutative).

(III) Show that a simple D with central involution is hermitian-generated iff $\mathrm{D} = \Omega$ has trivial involution (hence is a composition algebra of dimension 1).

Our proof of the Herstein–Kleinfeld–Osborn Theorem 21.5 is "slick," but a more leisurely proof may put it in a broader perspective.

PROBLEM 21.2* Show that (3.1) in Step 3 of HKO works for $*$-simple D (even for $*$-semiprime D). (1) If D is alternative with central involution and norm Q ($Q(x)1 = x\bar{x}$), show that for $z \in \mathcal{R}ad(Q)$, $x \in \mathrm{D}$ we have $z^2 = 0$, $\bar{z} = -z$, $zx = \bar{x}z$, $zxz = 0$. (2) For $z \in \mathcal{R}ad(Q)$ show that $\mathrm{Z} := z\mathrm{D} = \mathrm{D}z$ is a trivial $*$-ideal $\mathrm{ZZ} = \mathbf{0}$. (3) Conclude that if D has no trivial $*$-ideals then $\mathcal{R}ad(Q) = \mathbf{0}$, and if D is $*$-simple then Q is nondegenerate over a field Ω.

PROBLEM 21.3* Enlarge on (4.1) in Step 4 of HKO. Let A be any linear algebra. (1) Use the Teichmüller Identity $[xy,z,w] - [x,yz,w] + [x,y,zw] = x[y,z,w] + [x,y,z]w$ to show that $[\mathrm{A},\mathrm{A},\mathrm{A}]\widehat{\mathrm{A}} = \widehat{\mathrm{A}}[\mathrm{A},\mathrm{A},\mathrm{A}] =: \mathcal{A}(\mathrm{A})$ forms an ideal (the *associator ideal* generated by all associators $[x,y,z]$). (2) Show that if A is simple, then either $\mathcal{A}(\mathrm{A}) = \mathrm{A}$ is "totally nonassociative," or $\mathcal{A}(\mathrm{A}) = \mathbf{0}$ and A is associative. (3) Show that if A is totally nonassociative simple, then no nuclear element $n \neq 0$ can kill all associators $n[\mathrm{A},\mathrm{A},\mathrm{A}] = [\mathrm{A},\mathrm{A},\mathrm{A}]n = \mathbf{0}$. (4) Conclude that if A is simple alternative but not associative, then $[\mathcal{N},\mathcal{N}] = \mathbf{0}$. (5) If A is simple alternative with nuclear involution ($\mathcal{H} \subseteq \mathcal{N}$), show that either (*i*) A is associative, (*ii*) the involution is central ($[\mathcal{H},\mathrm{D}] = \mathbf{0}$), or (*iii*) $[\mathcal{H},\mathcal{S}] \neq \mathbf{0}$. Conclude that a simple alternative algebra with nuclear involution either has central involution or is associative.

PROBLEM 21.4* In (5.1) of Step 5 of HKO, let D be a unital associative algebra. (1) Prove that you can't be invertible if you kill someone (nonzero, i.e., who is not already dead, as pointed out by Hercule Poirot in *Murder on the Orient Express*). Even more, if $xy = 0$, then not only x, but also yx can't be invertible. (2) Prove again that if an element has a right and a left inverse, these coincide and are the unique two-sided inverse. Conclude that the set Z of non-invertible elements is the union $Z = Z_r \cup Z_\ell$ for Z_r the elements with no right inverse and Z_ℓ the elements with no left inverse. (3) Show that if an element has a right multiple which is right invertible, then the element itself is right invertible (and dually for left). Conclude that $Z_r D \subseteq Z_r$, $DZ_\ell \subseteq Z_\ell$. (4) Conclude that if an element has an invertible right multiple and an invertible left multiple, then it is itself invertible. (5) When $\mathcal{H} = \mathcal{H}(\mathrm{D}, -)$ is a division algebra, show that $z \in Z \Rightarrow z\bar{z} = \bar{z}z = 0$. Use this to show that $z\mathcal{H}\bar{z} = \mathbf{0}$. (6) Show that in any *prime* algebra (one with no orthogonal ideals), $x\mathrm{D}y = \mathbf{0} \Rightarrow x = 0$ *or* $y = 0$.

Problem 21.5 In (5.1) of HKO, reveal the Peirce decomposition lurking behind Step 5. Assume that the set of non-invertible elements is $\mathrm{Z} \neq 0$. (1) Show that $\mathrm{Z} = \{z \mid z\bar{z} = 0\} = \{z \mid \bar{z}z = 0\}$ has $\mathbf{0} < \mathrm{Z} < \mathrm{D}$. (2) Show that Z is invariant under multiplications from D (in particular, from Φ), but Z cannot by simplicity be an ideal, so it must fail the only other ideal criterion: $\mathrm{Z} + \mathrm{Z} \not\subseteq \mathrm{Z}$. Conclude that there exist non-invertible $z_i \in \mathrm{Z}$ with $z_1 + z_2 = u$ invertible. (3) Show that $e_i = z_i u^{-1}$ are conjugate supplementary orthogonal idempotents. (4) Show that the off-diagonal Peirce spaces $\mathrm{D}_{ij} = e_i\mathrm{D}e_j = e_i\mathrm{D}\overline{e_i}$ are skew, and $\mathcal{H}(\mathrm{D}, -) = \{a_{ii} + \overline{a_{ii}} \mid a_{ii} \in \mathrm{D}_{ii}\}$ commutes with $\mathrm{I} = \mathrm{D}_{12}\mathrm{D}_{21} + \mathrm{D}_{12} + \mathrm{D}_{21} + \mathrm{D}_{21}\mathrm{D}_{12}$. (5) Show that $\mathrm{I} < \mathrm{D}$ is the ideal generated by the off-diagonal Peirce spaces, so $\mathrm{I} = 0$ by simplicity; but then $\mathrm{D} = \mathrm{D}_{11} \oplus \mathrm{D}_{22}$ contradicts simplicity too, a contradiction.

Problem 21.6* Show that unital alternative algebras have a well-defined notion of inverse. (1) If the element x has y with $xy = yx = 1$, show that $y := x^{-1}$ is unique, and that $L_{x^{-1}} = (L_x)^{-1}$, $R_{x^{-1}} = (R_x)^{-1}$. (2) Conclude (cf. 21.2.1(5b)) that $b\mathrm{D} \subseteq \mathrm{B} \implies \mathrm{D} \subseteq b^{-1}\mathrm{B} \subseteq \mathrm{B}$ if b^{-1} belongs to the subalgebra B. The Moufang identities for the U-operator $U_x := L_x R_x$ allow us to show that alternative inverses behave much like Jordan inverses. (3) Show that x has an inverse $\iff 1 \in \mathrm{Im}(L_x) \cap \mathrm{Im}(R_x) \iff 1 \in \mathrm{Im}(U_x) \iff U_x$ invertible, in which case $U_{x^{-1}} = U_x^{-1}$. (4) Show that x, y invertible $\implies xy$ invertible $\implies x$ has a right inverse, y has a left inverse. (5) Give an associative example where xy is invertible but neither x nor y is. (6) Show that xy, zx invertible $\implies x$ invertible. (7) Prove the *Fundamental Formulas* $U_{xy} = L_x U_y R_x = R_y U_x L_y$, $U_{xyx} = U_x U_y U_x$ in alternative algebras, and use them to re-establish (6) [use the inverse criterion (3)].

Problem 21.7 (1) Show that for any invertible elements u, v in a unital alternative algebra the product $x \cdot_{u,v} y = (xu)(vy)$ gives a new alternative algebra $\mathrm{A}^{(u,v)}$, the *elemental isotope*, with unit $v^{-1}u^{-1}$. (2) Show that in a composition algebra with norm Q, the isotope determined by invertible elements u, v is again a composition algebra with norm $Q^{(u,v)} = Q(u)Q(v)Q$.

(3) If A is associative, show that $\mathrm{A}^{(u,v)} = \mathrm{A}^{(uv)}$ is the usual associative isotope $\mathrm{A}^{(w)} : x \cdot_w y = xwy$.

Problem 21.8 (1) Show $[x, yz] = [x, y]z + y[x, z] - [x, y, z] + [y, x, z] - [y, z, x]$ in any linear algebra A; conclude that $\mathrm{ad}_x : y \mapsto [x, y]$ is a derivation of A for any nuclear element x. (2) In an alternative algebra show ad_x is a derivation iff $3x \in \mathcal{N}uc(\mathrm{A})$ is nuclear, in particular ad_x is *always* a derivation in alternative algebras of characteristic 3, and in algebras without 3-torsion ad_x is *only* a derivation for nuclear x. (3) Show ad_x is a derivation of Jordan structure A^+ for any alternative algebra: $[x, y^2] = \{y, [x, y]\}$, $[x, yzy] = y[x, z]y + \{y, z, [x, y]\}$ (where $\{x, y, z\} := x(yz) + z(yx) = (xy)z + (zy)x$ is the linearization of xyx).

Problem 21.9 In coordinatizing projective planes, an important role is played by isotopy. An *isotopy* of a linear algebras $\mathrm{A} \to \mathrm{A}'$ is a triple (T_1, T_2, T_3) of invertible linear transformations $\mathrm{A} \to \mathrm{A}'$ with $T_1(x \cdot y) = T_2(x) \cdot' T_3(y)$ for all $x, y \in \mathrm{A}$. (1) Show that the composition of two isotopies $\mathrm{A} \to \mathrm{A}' \to \mathrm{A}''$ is again an isotopy, as is the inverse of any isotopy, and the identity isotopy $(\mathbb{1}_\mathrm{A}, \mathbb{1}_\mathrm{A}, \mathbb{1}_\mathrm{A})$ is always an *autotopy* (= self-isotopy) of A. (2) Show that if A is unital then necessarily $T_2 = R^{-1}_{T_3(1)} T_1$, $T_3 = L^{-1}_{T_2(1)} T_1$, so the isotopy takes the form $T_1(xy) = R^{-1}_{T_3(1)}(T_1(x)) \cdot' L^{-1}_{T_2(1)}(T_1(y))$. (3) In a unital alternative algebra (satisfying the left and right inverse properties $M_x^{-1} = M_{x^{-1}}$ for $M = L, R$), show that isotopies are precisely the maps T satisfying $T(xy) = \big(T(x)u\big)\big(vT(y)\big)$ for some invertible u, v and all x, y. (4) Show that the isotopies (T_1, T_2, T_3) of unital alternative algebras are in 1-to-1 correspondence with the isomorphisms $T : \mathrm{A} \to (\mathrm{A}')^{(u',v')}$ onto elemental isotopes. (5) Use the Moufang identities to show that for an invertible element u of an alternative algebra the triples $(U_u, L_u, R_u), (L_u, U_u, L_u^{-1}), (R_u, R_u^{-1}, U_u)$ are autotopies. (6) Establish the Principle of Triality: if (T_1, T_2, T_3) is an autotopy of a unital alternative algebra A, then so are $(T_2, R_{u_2}^{-1} T_2, U_{u_2} T_3)$ and $(T_3, U_{u_3} T_2, L_{u_3}^{-1} T_3)$ for $u_2 := T_3(1)^{-1}, u_3 := T_2(1)^{-1}$.

Question 21.1 (1) Does the Fundamental Formula $U_{U_x y} = U_x U_y U_x$ hold for the operator $U_x := L_x R_x$ in all alternative algebras? In all left alternative algebras? (2) Does an alternative algebra A become a Jordan algebra A^+ via $x \bullet y := \frac{1}{2}(xy + yx)$? Does this hold for left or right alternative algebras? (3) Does a unital alternative algebra become a Jordan algebra A^+ via $U_x y := x(yx)$? Does this work for left or right alternative algebras? Are the resulting algebras ever special?

22

Osborn's Capacity 2 Theorem

The hardest case in the classification is the case of capacity two; once we get to capacity three or more we have a uniform answer, due to the Jacobson Coordinatization Theorem. For capacity two we have two coordinatization theorems, and the hard part will be proving that at least one of them is applicable.[1]

22.1 Commutators

Zel'manov has shown us that commutators $[x, y]$, despite the fact that they don't exist, are crucial ingredients of Jordan theory. While they don't exist in a Jordan algebra itself, they lead an ethereal existence lurking at the fringe of the Jordan algebra, just waiting to manifest themselves. They do leave footprints: *double* commutators, *squares* of commutators, and *U-operators* of commutators do exist within the Jordan algebra. In associative algebras we have

$$\begin{aligned}
[[x,y],z] &= (xy-yx)z - z(xy-yx) = (xyz+zyx) - (yxz+zxy) \\
[x,y]^2 &= (xy-yx)^2 = (xy+yx)^2 - 2(xyyx+yxxy) \\
[x,y]z[x,y] &= (xy+yx)z(xy+yx) - 2(xyzyx+yxzxy), \\
[[x,y]^3,z] &= \Big[[x,y],\Big[[x,y],[[x,y],z]\Big]\Big] + 3\Big[[x,y],[x,y]z[x,y]\Big].
\end{aligned}$$

Since each of these right-hand expressions can be formulated in Jordan terms which make sense in any Jordan algebra, we are motivated to make the following definition. (Note that the expression $[x, y]$ does not make Jordan sense by itself, so these "commutator products" cannot be regarded as true products.)

[1] cf. I.6.6.

Commutator Definition 22.1.1 *In any Jordan algebra we introduce the following mnemonic shorthands*:

$$[[x,y],z] := (V_{x,y} - V_{y,x})\,z =: D_{x,y}(z), \qquad (\textbf{Double Commutator}),$$

$$[x,y]^2 := \{x,y\}^2 - 2\big(U_x(y^2) + U_y(x^2)\big) \quad (\textbf{Commutator Square}),$$

$$U_{[x,y]}z := \big(U_{\{x,y\}} - 2\{U_x, U_y\}\big)\,z \qquad (U\ \textbf{Commutator}),$$

$$[[x,y]^3, z] := \big(D_{x,y}^3 + 3D_{x,y}U_{[x,y]}\big)\,z \qquad (\textbf{Commutator Cube}).$$

Note that the usual inner derivation $D_{x,y} := V_{x,y} - V_{y,x}$ determined by x and y agrees with $[V_x, V_y]$ by Macdonald's Principle; it is a manifestation of $Ad_{[x,y]}$ (*where the ghostly $[x,y]$ itself becomes visible only in associative envelopes of special Jordan algebras*)*.*

By our calculation above, in special Jordan algebras these fictitious Jordan commutators reduce to products of actual associative commutators.

Exercise 22.1.1A (1) Show that we have an alternate expression $[x,y]^2 = \{x, U_y x\} - U_x y^2 - U_y x^2$ for Square Commutator. (2) Show that we have operator identities $W_{x,y} := U_{x,y}^2 - U_{x^2,y^2} = V_{x,y}V_{y,x} - V_{U_x y^2} = V_x U_y V_x - U_{U_x y, y}$ in any Jordan algebra; show that in special algebras this operator acting on z reduces to the "pentad" $\{x,y,z,x,y\}$. (3) Show that we have the alternate expression $U_{[x,y]} = W_{x,y} - U_x U_y - U_y U_x$ for U Commutator. (4) Show that in any Jordan algebra $U_{[x,y]}1 = [x,y]^2$.

Exercise 22.1.1B* (1) If e is an idempotent in a Jordan algebra and $a_2, b_2 \in \mathrm{J}_2$, $x_1 \in \mathrm{J}_1$, show that we have $q_0([[a_2,b_2],x_1],y_1) = -q_0(x_1,[[a_2,b_2],y_1])$. (2) Conclude that we have $q_0([[a_2,b_2],x_1]) = -U_{x_1}[a_2,b_2]^2$. (3) Show that $q_2([[a_2,b_2],x_1],x_1) = [[a_2,b_2],q_2(x_1)]$ two different ways: firstly, use q-Composition Rules 9.2.2(3) twice and subtract, and secondly, note that $D = D_{a_2,b_2}$ is a derivation with $D(e_0) = 0$, so apply it to $q_2(x_1)$. (4) Show that $q_2([[a_2,b_2],x_1]) = U_{[a_2,b_2]}q_2(x_1)$.

Throughout the rest of the chapter we will be working with a connection involution $\bar{x}$ determined by a strong connecting element as in the Connection Involution Lemma 10.1.3(1). Our proof of Osborn's Theorem will frequently depend on whether the elements at issue are skew or symmetric, and it will be convenient to introduce some notation for these.

Skewtrace Definition 22.1.2 (1) *Let e_2 be an idempotent in a Jordan algebra strongly connected to e_0 via $v_1 \in \mathrm{J}_1$, with connection involution $x \mapsto \bar{x} := U_v(x)$. We define the* **trace** *and* **skewtrace** (*pronounced* tur *and* skewtur) *of an element x by*

$$\mathrm{Tr}(x) := x + \bar{x}, \quad \mathrm{Sktr}(x) := x - \bar{x}$$

We depart from our standard notation $t(x) = x + x^*$ for traces of an arbitrary involution (*cf. Morphism Definition* 1.5.1), *using capital letters to carefully* *distinguish between traces of the "global" involution* $\overline{}$ *on* J *and those of the "local" involution* ρ *on the coordinate algebra* $\mathcal{D}$ *of* J *which will be induced by Hermitian Coordinatization.*

Because we have the luxury of a scalar $\frac{1}{2}$*, our module decomposes into symmetric and skew submodules with respect to the involution: every symmetric element is a trace, every skew element is a skewtrace, and every element* $x = \frac{1}{2}\big((x+\bar{x})+(x-\bar{x})\big) = \frac{1}{2}\mathrm{Tr}(x)+\frac{1}{2}\mathrm{Sktr}(x)$ *can be represented as the average of its symmetric and skew parts.*

(2) *We will also save ourselves a lot of V's and subscripts by using the notation of the Peirce specializations* 9.1.1*,* $\sigma(a_i) = V_{a_i}|_{\mathrm{J}_1}$ *of* $\mathrm{J}_2 + \mathrm{J}_0$ *on* J_1 *for an idempotent e, and define corresponding action of skewtraces*

$$\sigma\kappa(a_i) := \sigma(a_i - \overline{a_i}) = V_{\mathrm{Sktr}(a_i)}|_{\mathrm{J}_1}.$$

Peirce Commutator Lemma 22.1.3 *If* e_2 *is an idempotent in a Jordan algebra strongly connected to* e_0 *via* $v_1 \in \mathrm{J}_1$*, with connection involution* $x \mapsto \bar{x}$*, then*

$$\sigma\kappa(a_2)(v_1) = 0, \qquad \sigma\kappa(a_2)(\sigma(b_2)(v_1)) = [[a_2, b_2], v_1].$$

PROOF. The first holds because $\sigma(\overline{a_2})(v) = \sigma(a_2)(v)$ by the Connection Fixed Points 10.1.3(3). Then the second results from $\sigma\kappa(a_2)\sigma(b_2)(v) = [\sigma\kappa(a_2), \sigma(b_2)](v)$ [by the first part] $= [\sigma(a_2), \sigma(b_2)](v)$ [since $\sigma(\overline{a_2}) \in \sigma(\mathrm{J}_0)$ commutes with $\sigma(b_2)$ by Peirce Associativity 9.1.3] $= [[a_2, b_2], v]$ [by the Peirce Specialization Rules 9.1.1]. □

One suspects that commutators will help future generations understand the classical structure theory in a new light. For example, the Spin Peirce Relation 11.1.2 can be formulated as $[[e, y_1], [e, x_1]^2] = 0$, and we have $U_{[[a_2,b_2],x_1]} = U_{[a_2,b_2]}U_{x_1}$ on J_0, $q_2\left([[a_2, b_2], x_1]\right)\big) = U_{[a_2,b_2]}q_2(x_1)$. If u is an involution in J, the involution $\mathcal{U} = U_u$ on J has the form $\mathcal{U}(x) = x - \frac{1}{2}[[x, u], u]$ (so in some sense it equals $\mathbb{1} - \frac{1}{2}Ad_u^2$).

As another example, commutators provide clear examples of s-identities. I once blithely took it for granted that $U_{[x,y]}$, like any self-respecting U-operator, belonged to the structure semigroup: $U_{U_{[x,y]}z} - U_{[x,y]}U_zU_{[x,y]}$. Armin Thedy, coming from right alternative algebras, showed that this is *not* an identity for all Jordan algebras, indeed $T_{11}(x, y, z) := U_{U_{[x,y]}z} - U_{[x,y]}U_zU_{[x,y]}$ is an operator s-identity separating special from exceptional Jordan algebras. As this book went to press, Ivan Shestakov gave a lovely commutator-version of Glennie's identities G_8 and G_9, namely that $[[x, y]^3, z]$ is a derivation. The derivation rule $[[x, y]^3, z^2] = \{z, [[x, y]^3, z]\}$ for squares is just G_8, and the derivation rule for cubes $[[x, y]^3, z^3] = \{z^2, [[x, y]^3, z]\} + U_z[[x, y]^3, z]$ is just G_9. This makes it crystal clear that G_8 implies G_9: from the square we get the bullet product, and hence $j(x, y, \cdot)$ is a derivation on *any* auxiliary product, for example we have another s-identity $[[x, y]^3, U_zw] = \{z, w, [[x, y]^3, z]\} + U_z[[x, y]^3, w]$.

Exercise 22.1.3A Justify the above remark about commutators. (1) Show the Spin Peirce Relation 11.1.2 can be formulated as $[[e,y],[e,x]^2] = 0$ for all $x, y \in \mathrm{J}_1(e)$, and (2) $U_{[[a_2,b_2],x_1]} = U_{[a_2,b_2]}U_{x_1}$ on J_0, $q_2([[a_2,b_2],x_1])) = U_{[a_2,b_2]}(q_2(x_1))$. (3) An alternate proof of (2) uses the fact (easily seen from Macdonald's Principle) that $D = D_{a_2,b_2}$ is a derivation, so $D(q_2(x_1)) = D(U_{x_1}e_0) = \big(U_{D(x_1),x_1} + U_{x_1}D\big)e_0 = q_2(D(x_1), x_1)$ since $D(\mathrm{J}_1) \subseteq \mathrm{J}_1$ and $D(\mathrm{J}_0) = \mathbf{0}$. (4) If u is an involution in J, show that the involution $\mathcal{U} = U_u$ on J has the form $\mathcal{U}(x) = x - \frac{1}{2}[[x,u],u]$, so that in some sense $\mathcal{U} = \mathbb{1} - \frac{1}{2}Ad(u)^2$. (5) If e_2 is strongly connected to e_0 by v_1 with connection involution $\overline{}$, mimic the proof in 22.1.3 to show that $V_{\mathrm{Sktr}(a_2)}([[b_2,c_2],v_1]) = 2[a_2,[b_2,c_2]] \bullet v_1$.

Exercise 22.1.3B (1) Show that $T_{10}(x,y,z) = U_{U_{[x,y]}z} - U_{[x,y]}U_zU_{[x,y]} = 2K_{x,y,z}$ collapses entirely in characteristic 2 ($K = \frac{1}{2}T_{10}(x,y,z)$ survives as an s-identity, but an unlovely one). (2) Show that, for any derivation D, $D^3(ab) - D^3(a)b - aD^3(b) = 3f(a,b,D(a),D(b))$; conclude that $\text{Ш}_8(x,y,z) = [[x,y]^3, z^2] - \{z,[[x,y]^3,z]\} = 3F(x,y,z)$ collapses entirely in characteristic 3 ($F = \frac{1}{3}\text{Ш}_8(x,y,z)$ survives as an s-identity, but again an unlovely one).

22.2 Capacity Two

Recall that an algebra has *capacity* two if it has unit the sum of two supplementary *division* idempotents (i.e., the diagonal Peirce spaces J_i are division algebras), and is *nondegenerate* if it has no trivial elements $U_z = 0$.

Osborn's Capacity Two Theorem 22.2.1 *A Jordan algebra is simple nondegenerate of capacity* 2 *iff it is isomorphic to one of*:

Full 2 × 2 Type: $\mathcal{M}_2(\Delta)^+ \cong \mathcal{H}_2(\mathcal{E}x(\Delta), ex)$ *for a noncommutative associative division algebra* Δ;

Hermitian 2 × 2 Type: $\mathcal{H}_2(\Delta, \Gamma)$ *for an associative division algebra* Δ *with non-central involution*;

Reduced Spin Type: $\mathcal{R}ed\mathcal{S}pin(q)$ *for a nondegenerate quadratic form* q *over a field* Ω.

proof. This will be another long proof, divided into a series of short steps.

Step 1: Reduction to the strong case

By Creating Involutions Proposition 10.2.2 some diagonal isotope $\widetilde{\mathrm{J}} = \mathrm{J}^{(u)}$ has *strong* capacity 2 (the diagonal Peirce spaces $\widetilde{\mathrm{J}}_i = \mathrm{J}_i^{(u_i)}$ are still division algebras), and $\mathrm{J} \cong \widetilde{\mathrm{J}}^{(\tilde{u})}$ is a diagonal isotope of $\widetilde{\mathrm{J}}$. If we can prove that $\widetilde{\mathrm{J}}$ is of one of the three Types (with $\Gamma = 1$ in the second Type), then so is $\mathrm{J} = \widetilde{\mathrm{J}}^{(\tilde{u})}$: any diagonal isotope of $\mathcal{R}ed\mathcal{S}pin(q)$ is reduced spin by Quadratic Factor Isotopes Example 7.3.1(2); any isotope $(A^+)^{(u)} = (A_u)^+ \cong A^+$ of a full $\mathcal{M}_2(\mathrm{D})^+ = A^+$ is isomorphic to $\mathcal{M}_2(\mathrm{D})^+$; and any diagonal isotope of $\mathcal{H}_2(\mathrm{D}, -)$ is $\mathcal{H}_2(\mathrm{D}, \Gamma)$ by Twisted Matrix Example 7.5.3(3).

FROM NOW ON WE ASSUME THAT J HAS STRONG CAPACITY 2: let $1 = e_2 + e_0$ for J_i division idempotents strongly connected by $v = v_1 \in J_1$, let $\bar{x} = U_v(x)$ be the connection involution, and let $\mathcal{D} \subseteq \mathcal{E}nd(J_1)$ be the subalgebra generated by $\sigma(J_2)$. We claim that the following five properties hold:

$$(1.1) \qquad J_1 = \mathcal{H}_1 \oplus \mathcal{S}_1 = \mathrm{Tr}(J_1) \oplus \mathrm{Sktr}(J_1) \text{ for } \mathcal{H}_1 := \{x \in J_1 \mid \bar{x} = x\},$$
$$\mathcal{S}_1 := \{x \in J_1 \mid \bar{x} = -x\};$$

$$(1.2) \qquad x + \bar{x} = \sigma(t_i(x))v \;\; (x \in J_1,\ t_i(x) := q_i(x, v),\ i = 2, 0);$$

$$(1.3) \qquad \mathcal{H}_1 = \sigma(J_2)v : \;\; \overline{\sigma(a_2)v} = \sigma(\overline{a_2})v = \sigma(a_2)v;$$

$$(1.4) \qquad \mathcal{S}_1 = \{x \in J_1 \mid t_2(x) = 0\} = \{x \in J_1 \mid t_0(x) = 0\};$$

(1.5) $\qquad \mathcal{H}_1$ is a "division triple": all $h_1 \neq 0$ are invertible in J.

Indeed, the decomposition (1.1) follows from the Skewtrace Definition 22.1.2. (1.2)–(1.3) follow from the Connection Involution Lemma 10.1.3(2)–(3). For (1.4), $\mathcal{S}_1$ consists by (1.2) of elements with $\sigma(t_i(x)) = 0$, which is equivalent to $t_i(x) = 0$ by Peirce Injectivity 9.1.2(1) [from injectivity of U_{v_1}]. For (1.5), if $h_1 = \sigma(a_2)v$ weren't invertible then by the Off-Diagonal Non-Invertibility Criterion 20.2.1 we would have $0 = q_2(h_1) = q_2(\{a_2, v\}) = U_{a_2}q_2(v)$ [by the q-Composition Rules 9.2.2(3)] $= a_2^2$, which forces a_2 to vanish in the division algebra J_2, hence $h_1 = \sigma(a_2)v = 0$ too.

Step 2: The case where diagonal skewtraces kill J_1

We come to the First Dichotomy, the first branching in our family tree, where the spin factors diverge from the main line. The decision whether the algebra is a spin factor or not hinges on whether diagonal skewtraces vanish identically on the off-diagonal space. We will show that the diagonal skewtraces kill J_1 precisely when J is of Reduced Spin Type.

The Spin Peirce Relation 11.1.2 for norms is equivalent to the Spin Bar Relation 11.2.3(1), which can be reformulated as $(a_2 - \overline{a_2}) \bullet J_1 = \mathbf{0}$, or in terms of diagonal skewtraces $\sigma\kappa(J_2)J_1 = \mathbf{0}$. When this vanishes identically, then by the Strong Spin Coordinatization Theorem 11.3.1 $J \cong \mathcal{R}ed\mathcal{S}pin(q)$ over $\Omega = \Omega^+ \cong J_2^+$. Then Ω is a field because it is commutative and J_2 is a Jordan division algebra. Here $\mathcal{R}ed\mathcal{S}pin(q)$ is nondegenerate iff q is a nondegenerate quadratic form by Factor Nondegeneracy 5.3.6(3), and we have Reduced Spin Type.

FROM NOW ON WE ASSUME THAT DIAGONAL SKEWTRACES DO NOT ALL KILL J_1, so in the notation of Skewtrace Definition 22.1.2(2)

$$(2.1) \qquad \sigma\kappa(J_2)J_1 \neq \mathbf{0}.$$

Step 3: The case where diagonal skewtraces kill $\mathcal{S}_1$ but not J_1

By (1.1) and assumption (2.1), $\sigma\kappa(J_2)\mathcal{H}_1 \oplus \sigma\kappa(J_2)\mathcal{S}_1 = \sigma\kappa(J_2)J_1 \neq \mathbf{0}$, so one of the two pieces must fail to vanish. In principle there is a branching into the case where skewtraces kill $\mathcal{S}_1$ but not $\mathcal{H}_1$, and the case where they do not kill $\mathcal{S}_1$. We will show that the first branch withers away and dies.

We begin by showing that $\sigma\kappa(\mathrm{J}_2)\mathcal{S}_1 = \mathbf{0}$ leads to $\mathcal{D}$-*invariance* and q_0-*orthogonality* of $\mathcal{S}_1, \mathcal{D}v$:

$$(3.1) \qquad \sigma\kappa(\mathrm{J}_2)\mathcal{S}_1 = \mathbf{0} \implies \mathcal{D}(\mathcal{S}_1) \subseteq \mathcal{S}_1, \quad q_0(\mathcal{S}_1, \mathcal{D}v) = \mathbf{0}.$$

Invariance holds because the generators $\sigma(a_2)$ of $\mathcal{D}$ leave $\mathcal{S}_1$ invariant,

$$\overline{\sigma(a_2)s_1} = \sigma(\overline{a_2})\overline{s_1} = -\sigma(\overline{a_2})s_1 = -\sigma(a_2)s_1$$

by the skewtrace assumption. Orthogonality holds because $q_0(\mathcal{S}_1, \mathcal{D}v) = q_0(\mathcal{D}\mathcal{S}_1, v)$ [by repeated use of $U1q$ Rules 9.2.2(2)] $= t_0(\mathcal{D}\mathcal{S}_1) \subseteq t_0(\mathcal{S}_1)$ [by invariance] $= 0$ [by (1.3)]. Thus (3.1) holds.

From this we can show that if skewtraces die on $\mathcal{S}_1$, then on $\mathcal{H}_1$ they shrink into the radical:

$$(3.2) \qquad \sigma\kappa(\mathrm{J}_2)\mathcal{S}_1 = \mathbf{0} \implies \sigma\kappa(\mathrm{J}_2)\mathcal{H}_1 \subseteq \mathcal{R}ad(q_0).$$

Indeed, any such value $z_1 = \sigma\kappa(a_2)h_1 = \sigma(a_2 - \overline{a_2})\sigma(b_2)v$ [by (1.3)] $= [[a_2, b_2], v] \in \mathcal{D}v$ [by Peirce Commutator 22.1.3] is skew because skewtraces are skew and h is symmetric: $\overline{z_1} = \sigma(\overline{a_2} - a_2)h_1 = -z_1$. Thus $z_1 \in \mathcal{S}_1 \cap \mathcal{D}v$ is q_0-orthogonal by (3.1) to both to $\mathcal{S}_1$ and to $\mathcal{H}_1 \subseteq \mathcal{D}v$ [by (1.3)], hence to all of J_1, and we have q_0-*radicality* $q_0(z_1, \mathrm{J}_1) = \mathbf{0}$. This establishes (3.2).

But J is nondegenerate, so q_0 is too by q-Nondegeneracy 9.2.3(2), therefore (3.2) implies that $\sigma\kappa(\mathrm{J}_2)\mathcal{H}_1 = \mathbf{0}$ and skewtraces kill $\mathcal{H}_1$ too. Thus if skewtraces killed the skew part, they would be forced to kill the symmetric part as well, hence all of $\mathrm{J}_1 = \mathcal{H}_1 + \mathcal{S}_1$ [by (1.1)], which is forbidden by (2.1) above. This allows us to ASSUME FROM NOW ON THAT DIAGONAL SKEWTRACES DO NOT KILL $\mathcal{S}_1$,

$$(3.3) \qquad \sigma\kappa(\mathrm{J}_2)\mathcal{S}_1 \neq \mathbf{0}.$$

Step 4: The case where diagonal skewtraces do not all kill $\mathcal{S}_1$

The final branching in the family tree, the Second Dichotomy, is where the hermitian algebras of Full and Hermitian Types diverge. We claim that if diagonal skewtraces fail to kill $\mathcal{S}_1$, $\sigma\kappa(\mathrm{J}_2)\mathcal{S}_1 \neq \mathbf{0}$, then J is of hermitian type. The crux of the matter is that each off-diagonal skew element $s_1 \in \mathcal{S}_1$ faces a *Zariski dichotomy*: it must either live invertibly in $\mathcal{D}(v)$ or be killed by all diagonal skewtraces,

$$(4.1) \qquad \sigma\kappa(\mathrm{J}_2)s_1 \neq \mathbf{0} \implies s_1 \in \mathcal{D}(v) \text{ is invertible in J}.$$

Indeed, by hypothesis (3.3) some $h := \sigma\kappa(a_2)s \neq 0$. Now this h is symmetric because s and $\mathrm{Sktr}(a_2)$ are both skew, so by (1.5) it is invertible. We can use this to show that s itself is invertible: if s were *not* invertible, then for the diagonal $b = \mathrm{Sktr}(a_2) \in \mathrm{J}_2 + \mathrm{J}_0$ we would have $U_s(b), U_s(b^2), s^2 \in$

$U_s(\mathrm{J}_2 + \mathrm{J}_0) = \mathbf{0}$ by the Off-Diagonal Non-Invertibility Criterion again, so $h^2 = \{b, s\}^2 = \{b, U_s(b)\} + U_s(b^2) + U_b(s^2)$ [by Macdonald's Principle] $= 0$, contradicting the invertibility of h.

Thus s and h are invertible in J. We will have to work harder to show that s belongs to $\mathcal{D}(v)$ as in (4.1). Since s is invertible, we may set

$$d_2 := U_{s^{-1}}(\overline{a_2}),\ s_2 := q_2(s), \qquad c_2 := a_2^2 - \{a_2, s_2, d_2\} + U_{d_2}(s_2^2) \in \mathrm{J}_2,$$

$$d := \sigma(a_2) - \sigma(s_2)\sigma(d_2), \qquad d^\rho := \sigma(a_2) - \sigma(d_2)\sigma(s_2) \in \mathcal{D}.$$

In these terms we claim that

$$h = d(s), \quad d^\rho d = \sigma(c_2). \tag{4.2}$$

The first holds because

$$\begin{aligned}
h &= \sigma(a_2 - \overline{a_2})s = \sigma(a_2)s - V_s(\overline{a_2}) && \text{[by definition of } h\text{]}\\
&= \sigma(a_2)(s) - V_sU_s(d_2) && \text{[by definition of } d_2\text{]}\\
&= \sigma(a_2)(s) - U_{s^2,s}(d_2) && \text{[by Commuting (FFII)]}\\
&= \sigma(a_2)(s) - \{q_2(s), d_2, s\} && \text{[by Peirce Orthogonality 8.2.1]}\\
&= \big(\sigma(a_2) - \sigma(s_2)\sigma(d_2)\big)(s) && \text{[by Peirce Specialization 9.1.1]}\\
&= d(s) && \text{[by definition of } d, s_2\text{]},
\end{aligned}$$

while for the second, as operators on J_1 we have

$$\begin{aligned}
d^\rho d &= \big(\sigma(a_2) - \sigma(d_2)\sigma(s_2)\big)\big(\sigma(a_2) - \sigma(s_2)\sigma(d_2)\big) && \text{[by definition of } d\text{]}\\
&= \sigma(a_2)\sigma(a_2) - \big(\sigma(a_2)\sigma(s_2)\sigma(d_2) + \sigma(d_2)\sigma(s_2)\sigma(a_2)\big)\\
&\quad + \sigma(d_2)\sigma(s_2)\sigma(s_2)\sigma(d_2)\\
&= \sigma(a_2^2 - \{a_2, s_2, d_2\} + U_{d_2}(s_2^2)) && \text{[by Peirce Specialization Rules]}\\
&= \sigma(c_2) && \text{[by definition of } c_2\text{]}.
\end{aligned}$$

By invertibility of h and the Non-Invertibility Criterion again, we have $0 \neq 2q_0(h) = q_0(d(s), d(s)) = q_0(d^\rho d(s), s)$ [by the $U1q$ Rules twice] $= q_0(\sigma(c_2)s, s)$ [by (4.2)], so the element c_2 must be nonzero, therefore invertible in the division algebra J_2, hence from the formula $\sigma(c_2)(s) = d^\rho d(s) = d^\rho(h)$ [by (4.2)] we get an explicit expression

$$s = \sigma(c_2^{-1})d^\rho(h) \in \mathcal{D}(v),$$

since $h \in \mathcal{H}_1 \subseteq \mathcal{D}(v)$ [by (1.3)]. Thus s lies in $\mathcal{D}(v)$, completing the argument for (4.1).

Applying a Zariski-density argument to the Zariski dichotomy (4.1) shows that all skew elements make a common decision: if *each* s_1 must submit to being killed by all skewtraces or else live in $\mathcal{D}(v)$, then *all* $\mathcal{S}_1$ must submit to being killed by skewtraces or else *all* must live together in $\mathcal{D}(v)$. Since by hypothesis they aren't all killed by skewtraces, they must all live in $\mathcal{D}(v)$:

$$(4.3) \qquad \mathcal{S}_1 \subseteq \mathcal{D}(v).$$

Indeed, by (3.3) there exists at least one resistant $s \in \mathcal{S}_1$ which refuses to be killed by skewtraces, and he drags all submissive t's into resistant $s+t$'s $[\sigma\kappa(\mathrm{J}_2)t = \mathbf{0} \implies \sigma\kappa(\mathrm{J}_2)(s+t) \neq \mathbf{0}]$; but *all* resisters live in $\mathcal{D}(v)$ by (4.1), so even the submissive t's must live there too: $s, s+t \in \mathcal{D}(v) \implies t = (s+t)-s \in \mathcal{D}(v)$. Thus $\mathcal{S}_1 \subseteq \mathcal{D}(v)$.

By (4.3) all of $\mathcal{S}_1$ moves to $\mathcal{D}(v)$ to join $\mathcal{H}_1$ (which already lives there by (1.3)), so we have all $\mathrm{J}_1 \subseteq \mathcal{D}(v)$. Once the Hermitian Peirce Condition $\mathrm{J}_1 = \mathcal{D}(v)$ holds, the Strong 2×2 Hermitian Coordinatization Theorem 12.3.1 shows that $\mathrm{J} \cong \mathcal{H}_2(\mathcal{D})$ with $v \cong 1[12]$ and $\mathcal{D}$ hermitian-generated. (2.1) together with the Hermitian Matrix formulas 3.2.4 then imply the involution is not central. Also, $\mathcal{H}_2(\mathcal{D}) \cong \mathrm{J}$ nondegenerate implies $\mathcal{D}$ nondegenerate by Jordan Matrix Nondegeneracy 5.3.5, and $\mathcal{H}(\mathcal{D}) \cong \mathrm{J}_2$ a Jordan division algebra by capacity 2 implies that the hermitian $\mathcal{H}(\mathcal{D})$ are invertible in $\mathcal{D}$. By the Herstein–Kleinfeld–Osborn Theorem 21.3.1, either $\mathcal{D} = \mathcal{E}x(\Delta)$ of Full Type, or $\mathcal{D} = \Delta$ of Hermitian Type, or else $\mathcal{D}$ is a composition algebra over its $*$-center Ω, which is excluded because its standard involution is central, violating (2.1).

Step 5: The converse

We have shown the hard part, that every simple J has one of the given types. We now check conversely that these types are always simple and nondegenerate. Since they all have connected capacity 2, by Simple Capacity 20.2.4 we know that they will be simple as soon as they are nondegenerate. But all types are nondegenerate: $\mathcal{R}ed\mathcal{S}pin(q)$ is nondegenerate iff q is by Factor Nondegeneracy (3) again, $\mathcal{H}_2(\mathrm{D},\Gamma) \cong \mathcal{H}_2(\mathrm{D},-)^{(\Gamma)}$ is nondegenerate since isotopes inherit nondegeneracy (by Jordan Homotope 7.2.1(3)) and $\mathcal{H}_2(\mathrm{D},-)$ is nondegenerate by Jordan Matrix Nondegeneracy again.

Thus we have the simples, the whole simples, and nothing but the simples. □

One way to measure the depth of a result is to count the number of previous results that it depends on. I count 23 different numbered results from 11 chapters used in the above proof!

Exercise 22.2.1A Assume that e_2 is strongly connected to e_0 by v_1. (1) Show that $V_{[[a_2,b_2],c_2]} = [[V_{\mathrm{Sktr}(a_2)}, V_{b_2}], V_{\mathrm{Sktr}(c_2)}] \in \left(\mathcal{D}V_{\mathrm{Sktr}(\mathrm{J}_2)}\mathcal{D}\right)^2$ on J_1. (2) If $\sigma\kappa(\mathrm{J}_2)\mathcal{S}_1 = \mathbf{0}$ (as in Step 3 of the above proof), show that $[[\mathrm{J}_2,\mathrm{J}_2],\mathrm{J}_1] \subseteq \mathcal{S}_1$ and $[[\mathrm{J}_2,\mathrm{J}_2],\mathrm{J}_2] = \mathbf{0}$.

Exercise 22.2.1B In Step 4 we called on Macdonald to certify the identity $\{x,y\}^2 = \{x, U_y x\} + U_x y^2 + U_y x^2$. (1) Verify that this does indeed hold in all special algebras, hence in all Jordan algebras. (2) Less loftily, derive it by setting $y \mapsto 1, z \mapsto y$ in (FFI′) acting on 1, using $\{x, U_y x\} = \{y, U_x y\}$ from setting $z \mapsto 1$ in (FFIIIe).

23

Classical Classification

The denouement, the final classification theorem, comes as an anticlimax, since all the hard work has already been done. If we wish, we may treat this section as an easy victory lap in celebration of our achievement.

23.1 Capacity $n \geq 3$

There is a uniform pattern to all simple nondegenerate Jordan algebras of capacity at least three: they are all matrix algebras.

Capacity ≥ 3 **Theorem** 23.1.1 *A Jordan algebra is simple nondegenerate of capacity $n \geq 3$ iff it is a Jordan matrix algebra isomorphic to one of the following*:

> $\mathcal{H}_n(\mathcal{E}x(\Delta), ex) \cong \mathcal{M}_n(\Delta)^+$ *for Δ a noncommutative associative division algebra*;
>
> $\mathcal{H}_n(\Delta, \Gamma)$ *for an associative division algebra Δ with non-central involution*;
>
> $\mathcal{H}_n(\mathrm{C}, \Gamma)$ *for a composition algebra* C *of dimension* $1, 2, 4,$ *or* 8 *over its center* Ω *with standard involution* [*dimension* 8 *only for* $n = 3$].

In particular, it is automatically special unless it is a reduced 27*-dimensional Albert algebra. We can list the possibilities in another way*: *the algebra is isomorphic to one of*

> $\mathcal{H}_n(\mathcal{E}x(\Delta), ex) \cong \mathcal{M}_n(\Delta)^+$ *for Δ an associative division algebra*;
>
> $\mathcal{H}_n(\Delta, \Gamma)$ *for an associative division $*$-algebra Δ*;
>
> $\mathcal{H}_n(\mathcal{Q}(\Omega), \Gamma) = \mathcal{H}_{2n}(\Omega, sp)$ *for $\mathcal{Q}(\Omega)$ the split quaternion algebra over a field Ω, equivalently, the hermitian $2n \times 2n$ matrices over Ω under the symplectic involution $X^{sp} = SX^{tr}S^{-1}$, for S the symplectic $2n \times 2n$ matrix* $\mathrm{diag}\{s, s, \ldots, s\}$ *with* $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$;
>
> $\mathcal{H}_3(\mathbf{O}, \Gamma)$ *for an octonion algebra* $\mathbf{O}$ *of dimension* 8 *over its center* Ω [$n = 3$ *only*].

PROOF. If J is simple nondegenerate of capacity n, then by the Simple Capacity Theorem 20.2.4 it has n connected idempotents with diagonal Peirce spaces J_{ii} division algebras. By Jacobson's Coordinatization Theorem 17.2.1, when $n \geq 3$ the algebra J is isomorphic to $\mathcal{H}_n(\mathrm{D}, \Gamma)$ for D alternative with nuclear involution, and J_{ii} is isomorphic to $\mathcal{H}(\mathrm{D}, -)$, so the latter is a division algebra, and (as we noted before Osborn's Theorem) D is nondegenerate.

By the Herstein–Kleinfeld–Osborn Theorem 21.3.1, D is one of the above types. The only case that needs some explaining is the split quaternion case. Here $\mathrm{D} = \mathcal{M}_2(\Omega)$ under $\bar{x} = sx^{tr}s^{-1}$. $\mathcal{M}_n(\mathcal{M}_2(\Omega)) \cong \mathcal{M}_{2n}(\Omega)$ by regarding an $n \times n$ matrix $X = (x_{ij})$ of 2×2 matrices $x_{ij} \in \mathcal{M}_2(\Omega)$ as a $2n \times 2n$ matrix decomposed into 2×2 blocks x_{ij}. The involution $(\overline{X})^{tr}$ of $\mathcal{M}_n(\mathcal{M}_2(\Omega))$ yields a matrix whose ij-block is $\overline{x_{ji}} = sx_{ji}^{tr}s^{-1}$. Any time that $Y \in \mathcal{M}_{2n}(\Omega)$ is decomposed into 2×2 blocks $Y = (y_{ij})$, the conjugate $SYS^{-1} = (sI_n)Y(sI_n)^{-1}$ has as its ij-block $sy_{ij}s^{-1}$, and Y^{tr} has as its ij block y_{ji}^{tr}, so $SX^{tr}S^{-1}$ has as its ij-block $sx_{ji}^{tr}s^{-1}$, showing that $\overline{X}^{tr} = SX^{tr}S^{-1}$ corresponds to the symplectic involution.

We have thus shown that every simple J has one of the given types, and it remains to check conversely that these types are always simple and nondegenerate. Just as in Osborn's Capacity 2 Theorem 22.2.1, since they all have connected capacity n it suffices by Simple Capacity 20.2.4 to verify nondegeneracy. But $\mathcal{H}(\mathrm{D}, -)$ is a division algebra and certainly semiprime, and each algebra $\mathrm{D} = \Delta \oplus \Delta^{op}$ or Δ or C is semiprime, so $\mathcal{H}_n(\mathrm{D}, -)$ is nondegenerate by Jordan Matrix Nondegeneracy 5.3.5. Then its isotope $\mathcal{H}_n(\mathrm{D}, \Gamma)$ is also nondegenerate [by Twisted Matrix Isotopy 7.5.3(3) and Jordan Homotope 7.2.1(3)]. Once more we have captured precisely the simple algebras. □

We can sum up the cases of Capacity 1 (20.1.2), Capacity 2 (22.2.1), and Capacity ≥ 3 (23.1.1) to construct our final edifice.

Classical Structure Theorem 23.1.2 *A Jordan algebra is nondegenerate with finite capacity iff it is a direct sum of a finite number of simple ideals of the following Division, Spin, or Matrix Types:*

J_0	$(n = 1)$	*a Jordan division algebra;*
$\mathcal{R}ed\mathcal{S}pin(q)$	$(n = 2)$	*for a nondegenerate quadratic form q over a field Ω;*
$\mathcal{M}_n(\Delta)^+$	$(n \geq 2)$	*for a noncommutative associative division algebra Δ;*
$\mathcal{H}_n(\Delta, \Gamma)$	$(n \geq 2)$	*for an associative division algebra Δ with noncentral involution;*
$\mathcal{H}_n(\mathrm{C}, \Gamma)$	$(n \geq 3)$	*for a composition algebra C of dimension 1, 2, 4, or 8 over its center Ω (dimension 8 only for $n = 3$).*

This classification[1] *is focused on the capacity n of the algebra. If we organize the algebras according to structure (as Zel'manov has taught us to understand*

[1] See the Classical Structure Theorem I.5.2.

it), and allow some overlapping of Types, we can describe the algebras as Division, Quadratic, Albert, and Hermitian Types:

Division Type: *a Jordan division algebra*;

Quadratic Type: *a quadratic factor* $\mathcal{J}ord(Q,c)$ *determined by a nondegenerate quadratic form* Q *with basepoint* c *over a field* Ω (*not split of dimension* 2);

Albert Type: *a cubic factor* $\mathcal{J}ord(N,c)$ *determined by a Jordan cubic form* N *with basepoint* c (*an exceptional Albert algebra*) *of dimension* 27 *over a field* Ω;

Hermitian Type: $\mathcal{H}(\mathrm{A},*)$ *for a* $*$*-simple artinian associative algebra* A.

In more detail, the algebras of Hermitian Type are twisted Jordan matrix algebras:

Exchange Type: $\mathcal{M}_n(\Delta)^+$ *for an associative division ring* Δ (A *is* $*$*-simple but not simple,* $\mathrm{A} = \mathcal{E}x(\mathrm{B})$ *with exchange involution for a simple artinian algebra* $\mathrm{B} = \mathcal{M}_n(\Delta)$);

Orthogonal Type: $\mathcal{H}_n(\Delta,\Gamma)$ *for an associative division ring* Δ *with involution* ($\mathrm{A} = \mathcal{M}_n(\Delta)$ *simple artinian with involution*);

Symplectic Type: $\mathcal{H}_n(\mathrm{Q},\Gamma)$ *for a quaternion algebra* Q *over a field* Ω *with standard involution* ($\mathrm{A} = \mathcal{M}_n(\mathrm{Q})$ *simple artinian with involution of symplectic type*). □

Strictly speaking, if Δ is a composition algebra with its usual central involution over Ω, the involution in Hermitian Orthogonal Type is not truly of orthogonal type (a form of the transpose on some $\mathrm{A}_\Omega \cong \mathcal{M}_{nm}(\Omega)$); if $\Delta = \mathrm{Q}$ is quaternion the involution on A is of symplectic type; if $\Delta = \mathrm{B}$ is binarion the involution on A is of exchange type; only the composition algebra $\Delta = \Omega$ yields an involution of orthogonal type.

Notice that there is no twisting by Γ in the Hermitian Exchange Type: the Full Isotope Example 7.5.1 shows that $(\mathrm{A}^+)^{(u)} \cong \mathrm{A}^+$ for *any* invertible $u \in \mathrm{A}$. There is also no diagonal twisting needed in Hermitian Symplectic Type when $\mathrm{Q} = \mathcal{Q}(\Omega)$ is *split* quaternion (see the following exercise).

Exercise 23.1.2* Let $\mathrm{J} := \mathcal{H}(\mathrm{A},*)$ for an associative $*$-algebra A. (1) If $a \in \mathrm{A}$, $\alpha \in \Phi$ are invertible, show that $T(x) := \alpha a x a^*$ is a structural transformation on J with $T(1) = \alpha a a^*$, hence $\mathrm{J} \cong \mathrm{J}^{(u)}$ if $u^{-1} = \alpha a a^*$. (2) If all invertible $v \in \mathrm{J}$ are norms $v = bb^*$ for invertible $b \in \mathrm{A}$, show all isotopes $\mathrm{J}^{(u)}$ are isomorphic to J. (3) If $\mathrm{A} = \mathcal{M}_n(\mathrm{D})$ for D with involution $\overline{}$, and $x^* = \bar{x}^{tr}$ is the standard involution, and $\Gamma = \mathrm{diag}(\gamma_1, \dots, \gamma_n)$ for invertible $\gamma_i = \overline{\gamma_i} \in \mathrm{D}$, show that if each $\gamma_i = d_i \overline{d_i}$ is a norm of an invertible $d_i \in \mathrm{D}$, then Γ is an invertible norm in $\mathcal{M}_n(\mathrm{D},-)$ and thus $\mathcal{H}_n(\mathrm{D},\Gamma) \cong \mathcal{H}_n(\mathrm{D},-)$. (4) Use this to prove that all diagonal isotopes of J are isomorphic to J when (i) $\mathrm{D} = \mathcal{E}x(\mathrm{B})$ under the exchange involution, (ii) $\mathrm{D} = \mathcal{Q}(\Omega) = \mathcal{M}_2(\Omega)$ is split quaternion under the symplectic involution.

And thus we come to the end of a Theory, an Era, and the Classical Part of our story.

Part III

Zel'manov's Exceptional Theorem

Introduction

In Part III we will give a fairly self-contained treatment of Zel'manov's celebrated theorem that the only prime i-exceptional Jordan algebras are forms of Albert algebras, filling in the details sketched in Chapter 8 of the Historical Survey (Part I). Throughout this part we again work with linear Jordan algebras over a fixed (unital, commutative, associative) ring of scalars Φ containing $\frac{1}{2}$, but we conduct a quadratic argument as long as it is convenient and illuminating. We will make use of a few results from Part II on Peirce decompositions and invertibility, and of course will invoke Macdonald's Principle at the drop of a hat, but otherwise require very little from Part II. The actual classification of simple exceptional algebras shows in essence that they must have capacity, and then falls back on the classification in Part II to conclude that they are Albert algebras. It does not provide an independent route to Albert algebras, but rather shows that finiteness is unavoidable in the exceptional setting.

First Phase: Begetting Capacity

In this phase we describe general situations that lead to the classical algebras of finite capacity considered in Part II. An algebra over a large field will have a finite capacity if its elements have small spectra. The crucial result is that a non-vanishing f always puts a bound on the f-spectra of elements (which is close to putting a bound on ordinary spectra); this means that i-exceptionality, the non-vanishing of some s-identity f, is an incipient finiteness condition.

Chapter 1 reviews standard features of invertibility and extends them to quasi-invertibility and proper quasi-invertibility: for an element x, invertibility is reflected in invertibility of U_x (as operator), q.i. is reflected in invertibility of $U_{1-x} = B_{x,1}$, and p.q.i. in invertibility of all Bergmann operators $B_{x,y}$. Structural transformations which are congruent to $\mathbb{1}$ are surjective or invertible on $\widehat{\mathrm{J}}$ iff they are on J, allowing us to pass to the unital hull whenever the mood strikes us. The Jacobson radical has the elemental characterization as all p.q.i. elements. It reduces to the nil radical for algebraic or I-genic algebras, and algebras over a big field; it reduces to the degenerate radical for algebras with d.c.c. on inner ideals (a nondegenerate algebra avoids the radical of any larger algebra with d.c.c.)

Chapter 2 tells how to beget and bound idempotents. Begetting is rampant in I-genic algebras (e.g., algebraic algebras), where every non-nilpotent element generates a nonzero idempotent in its principal inner ideal. An algebra is I-finite if it has no infinite orthogonal family of idempotents; this is equivalent to the a.c.c. on idempotents (the d.c.c. on idempotents is equivalent to having no infinite orthogonal family of bounded idempotents). These two conditions together beget capacity: a semiprimitive I-finite I-genic algebra always has finite capacity.

Chapter 3 brings in the concepts of spectra and bigness for Jordan algebras over a field. The usual spectrum of an element x is the set of scalars for which $\lambda 1 - x$ is not invertible ($U_{\lambda 1-x}(\mathrm{J})$ is not all of J). Slightly smaller than the spectrum is the f-spectrum for a nonvanishing polynomial f (the scalars for which f vanishes on $U_{\lambda 1-x}(\mathrm{J})$). The ingenious f-Spectral Bound Theorem shows that the f-spectrum is bounded in size by $2N$ if f has degree N. In a semiprimitive algebra, a global bound on the spectra of elements yields I-gene and I-finiteness, hence a capacity. Thus capacity will follow whenever we can translate the f-spectral bound into an ordinary spectral bound.

The resolvent is the complement of the spectrum. An infinite set of scalars from a field is big with respect to an algebra if its cardinality is greater than the dimension of the algebra. Amitsur's Big Resolvent Theorem says that an element with a big resolvent must be algebraic, and the amazing Division Evaporation Theorem guarantees that the only division algebra over a big algebraically closed field is the field itself.

1

The Radical

We begin with some basic results about the semiprimitive radical; the connection between nondegeneracy and primitivity depends on a detailed analysis of the radical. Although our final structure theorem concerns nondegenerate algebras, our proof proceeds by classifying the more restrictive primitive algebras. At the very end we imbed a prime nondegenerate algebra in a semiprimitive algebra, wave a magic ultrafilter wand, and presto — all nondegenerate prime exceptional algebras are turned into Albert forms.

The Jacobson radical $\mathcal{R}ad(\mathrm{A})$ of an associative algebra A can be defined in three ways. Its origin and its primary importance come from its role as the obstacle to irreducible representation: the part of the algebra that refuses to act on irreducible modules (the intersection of the kernels of all *irreducible representations*). Secondly, it is the maximal *quasi-invertible* ideal (ideal consisting entirely of quasi-invertible elements, elements z such that $\hat{1} - z$ is formally invertible in the unital hull). Thirdly, and most useful in practice, it can be precisely described elementally as the set of all elements which are *properly quasi-invertible* (where in general the adverb "properly" means "all multiples remain such").

In Jordan theory there are no irreducible representations or irreducible modules $\mathrm{M} = \mathrm{A}/\mathrm{B}$ for maximal modular left ideals B, though there are ghosts of such modules (namely, the maximal modular *inner* ideals we will meet in Chapter 5). Jordan algebras do have an exact analogue for the associative theory of inverses and quasi-inverses, and so we will define the radical to be the maximal quasi-invertible ideal. But we will spend the rest of this chapter making sure that we again have the user-friendly *elemental* characterization in terms of properly quasi-invertible elements. This is especially important in Jordan algebras, where *ideals* are hard to generate.

1.1 Invertibility

We begin by establishing the basic facts about quasi-invertibility. Since quasi-invertibility of z means invertibility of $\hat{1}-z$, many results are mere translations of results about inverses, which we now "recall."[1]

Basic Inverse Theorem 1.1.1 *We have the following basic facts about inverses in Jordan algebras.*

(1) **Existence:** *An element u in a unital Jordan algebra* J *is defined to be* **invertible** *if there exists an element v satisfying the* **Quadratic Jordan Inverse Conditions**

$$\text{(QJInv1)} \qquad U_u v = u, \qquad \text{(QJInv2)} \qquad U_u v^2 = 1.$$

The element v is called the **inverse** u^{-1} *of u.*

(2) **Extension:** *If u is invertible in* J, *then it remains invertible (with the same inverse) in any algebra $\widetilde{\mathrm{J}} \supseteq \mathrm{J}$ having the same unit as* J.

(3) **Criterion:** *The following are equivalent for an element u of a unital Jordan algebra* J:

(*i*) *u is an invertible element of* J;
(*ii*) *U_u is an invertible operator on* J;
(*iii*) *U_u is surjective on* J: $U_u(\mathrm{J}) = \mathrm{J}$;
(*iv*) *the image of U_u contains the unit*: $1 \in U_u(\mathrm{J})$;
(*v*) *$U_u(\mathrm{J})$ contains some invertible element.*

(4) **Consequences:** *If u is invertible in* J *with inverse v, then*:

(*i*) *U_u, U_v are inverse operators*;
(*ii*) *the inverse is uniquely determined as $v = U_u^{-1}u$*;
(*iii*) *invertibility is symmetric: v is invertible with inverse u*;
(*iv*) $\{u, v\} = 2$.

PROOF. (1) is just the definition of inverse as given in II.6.1.1. (2) holds because invertibility is strictly a relation between u, v, and the unit element 1. (3) We repeat the arguments of the Invertibility Criterion II.6.1.2 (with the wrinkle (v) thrown in). $(i) \Rightarrow (ii)$: Applying the Fundamental Formula to the relations (QJInv1)–(QJInv2) shows that $U_uU_vU_u = U_u, U_uU_vU_vU_u = \mathbb{1}_{\mathrm{J}}$; the latter shows that the operator U_u is surjective and injective, hence invertible, and then canceling U_u from the former gives $U_uU_v = U_vU_u = \mathbb{1}_{\mathrm{J}}$, so U_u and U_v are inverses. [This also proves (4)(i).] Clearly $(ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v)$. On the other hand, $(v) \Rightarrow (ii)$ follows from the Fundamental Formula: if U_ua is invertible, so is the operator $U_{U_ua} = U_uU_aU_u$ [by $(i) \Rightarrow (ii)$], which again implies that U_u is invertible. $(ii) \Rightarrow (i)$ because $v := U_u^{-1}u$ satisfies $U_uv = u$ [by definition], $U_uv^2 = U_uU_v1 = U_uU_vU_uU_u^{-1}1 = U_{U_uv}U_u^{-1}1$ [by the Fundamental Formula] $= U_uU_u^{-1}1 = 1$. [This also establishes (4)(ii).]

[1] cf. II Section 6.1.

(4) We saw (i) and (ii) above. (iii) follows from (ii): the conditions (QJInv1)–(QJInv2) are satisfied with the roles of u and v switched, since $U_v u = U_v(U_u(v)) = \mathbb{1}_{\mathrm{J}}(v) = v$, $U_v u^2 = U_v(U_u(1)) = \mathbb{1}_{\mathrm{J}}(1) = 1$. (iv) results by canceling U_u from $U_u(\{u,v\}) = U_u V_u(v) = V_u U_u(v)$ [by Commuting (FFII)] $= V_u(u) = 2u^2 = U_u(2)$. □

1.2 Structurality

Naturally we are going to derive the basic facts about *quasi-inverses* from facts about *ordinary inverses* making use of the unital hull $\widehat{\mathrm{J}} = \Phi\hat{1} \oplus \mathrm{J}$. First we need to recall the results on structural transformations[2] about operators on $\widehat{\mathrm{J}}$ and their restrictions to J.

Structural Transformation Definition 1.2.1 (1) *A* **structural transformation** *T on a Jordan algebra* J *is a linear transformation for which there exists a linear T^* on* J *such that both T, T^* extend to $\widehat{\mathrm{J}}$ and satisfy*

$$U_{T(\hat{x})} = T U_{\hat{x}} T^* \quad \text{on } \widehat{\mathrm{J}} \text{ for all } \hat{x} \in \widehat{\mathrm{J}}.$$

If T^* is also structural with $T^{**} = T$, we say that (T, T^*) is a **structural pair** *on* J.

(2) *We say that a linear transformation S on $\widehat{\mathrm{J}}$ is* **congruent to** $\widehat{\mathbb{1}} := \mathbb{1}_{\hat{\mathrm{J}}}$ **mod J** *if*

$$(\widehat{\mathbb{1}} - S)(\widehat{\mathrm{J}}) \subseteq \mathrm{J}, \ \textit{i.e.,}\ S(\mathrm{J}) \subseteq \mathrm{J} \ \textit{and}\ S(\hat{1}) = \hat{1} - c$$

for some c in J, *more explicitly,*

$$S(\alpha\hat{1} \oplus x) = \alpha\hat{1} \oplus (-\alpha c + S(x)).$$

Important examples of structural T congruent to $\widehat{\mathbb{1}}$ are the **quasi-U operators** $B_{x,\hat{1}} = \mathbb{1}_{\mathrm{J}} - V_x + U_x = U_{\hat{1}-x}$ for $x \in \mathrm{J}$, more generally the **Bergmann operators** $B_{x,\hat{y}} := \mathbb{1}_{\mathrm{J}} - V_{x,\hat{y}} + U_x U_{\hat{y}}$ for $x \in \mathrm{J}, \hat{y} \in \widehat{\mathrm{J}}$. We have mentioned several times that these Bergmann operators are structural, and it is now time to put our proof where our mouth is. This is one important identity that will *not* succumb to Macdonald's blandishments, so the fact that it is easy to verify in associative algebras won't save us from the calculations that follow.

Bergmann Structurality Proposition 1.2.2 *The generalized Bergmann operators*

$$B_{\alpha,x,y} := \alpha^2 \mathbb{1}_{\mathrm{J}} - \alpha V_{x,y} + U_x U_y \qquad (x, y \in \mathrm{J}, \alpha \in \Phi)$$

form a structural pair $(B_{\alpha,x,y}, B_{\alpha,y,x})$,

$$U_{B_{\alpha,x,y}(z)} = B_{\alpha,x,y} U_z B_{\alpha,y,x}.$$

[2] cf. II.18.2.1.

PROOF. Expanding this out in powers of α, for $B := B_{\alpha,x,y}$ we have $U_{B(z)} = U_{\alpha^2 z - \alpha\{x,y,z\} + U_xU_y(z)} = \alpha^4 U_z - \alpha^3 U_{z,\{x,y,z\}} + \alpha^2\big(U_{\{x,y,z\}} + U_{U_xU_y(z),z}\big) - \alpha U_{U_xU_y(z),\{x,y,z\}} + U_{U_xU_y(z)}$, while $BU_zB^* = \alpha^4 U_z - \alpha^3\big(V_{x,y}U_z + U_zV_{y,x}\big) + \alpha^2\big(U_xU_yU_z + U_zU_yU_x + V_{x,y}U_zV_{y,x}\big) - \alpha\big(U_xU_yU_zV_{y,x} + V_{x,y}U_zU_yU_x\big) + U_xU_yU_zU_yU_x$. To show that these coincide for all x, y, α in all situations, we must prove that the coefficients of like powers of α coincide. This is trivial for α^4, is the Fundamental Lie Formula (FFV) for α^3, is Alternate Fundamental Formula (FFI)′ for α^2, and is the Fundamental Formula (FFI) for α^0, but for α^1 we have a totally unfamiliar bulky formula:

$$(\alpha^1) \qquad U_{U_xU_y(z),\{x,y,z\}} = U_xU_yU_zV_{y,x} + V_{x,y}U_zU_yU_x.$$

For this we need the following Formulas from II Section 5.2: Commuting (FFII), Triple Shift (FFIIIe), Fundamental Lie (FFV), and a linearized Fundamental (FFI) $U_{U_x(y),U_{x,z}(y)} = U_xU_yU_{x,z} + U_{x,z}U_yU_x$ [replace $x \mapsto x + \lambda z$ in (FFI) and equate coefficients of λ]. Using this linearized (FFI) we compute

$$\begin{aligned} U_{U_xU_y(z),\{x,y,z\}} &= U_{U_x(U_y(z)),U_{x,z}(y)} \\ &= -U_{U_x(y),U_{x,z}(U_y(z))} + U_xU_{U_y(z),y}U_{x,z} + U_{x,z}U_{y,U_y(z)}U_x \\ &= -U_{U_x(y),U_{x,U_z(y)}(y)} + U_x\big(U_yV_{z,y}\big)U_{x,z} + U_{x,z}\big(V_{y,z}U_y\big)U_x \end{aligned}$$

[by Triple Shift on the first term, and Commuting (FFII) on the second and third terms]. In the first term we replace $y \mapsto U_zy$ in linearized (FFI) to get

$$-U_{U_x(y),U_{x,U_z(y)}(y)} = -\big(U_xU_yU_{x,U_z(y)} + U_{x,U_z(y)}U_yU_x\big).$$

On the second term above we apply $V_{z,y}U_{x,z} = U_{x,U_z(y)} + U_zV_{y,x}$ [noting that $\{z, y, \{x, a, z\}\} = U_z\{a, x, y\} + \{x, a, U_zy\}$ by Fundamental Lie (FFVe)] to get

$$U_xU_y\big(V_{z,y}U_{x,z}\big) = U_xU_y\big(U_{x,U_z(y)} + U_zV_{y,x}\big).$$

Finally, on the third term above we apply $U_{x,z}V_{y,z} = U_{x,U_zy} + V_{x,y}U_z$ [noting that $\{x, \{y, z, b\}, z\} = \{x, b, U_zy\} + \{x, y, U_zb\}$ by linearized Triple Shift (FFIIIe)] to get

$$\big(U_{x,z}V_{y,z}\big)U_yU_x = \big(U_{x,U_zy} + V_{x,y}U_z\big)U_yU_x.$$

When we add these three expressions, four terms cancel, and the expression of α^1 becomes

$$U_{U_xU_y(z),\{x,y,z\}} = U_xU_yU_zV_{y,x} + V_{x,y}U_zU_yU_x$$

as claimed. This finishes α^1, and hence the structurality of the Bergmann operators. □

The pain of the above proof could have been lessened by an injection of Koecher's Principle (see Problems 1.6, 1.7 below).

Congruent to $\widehat{\mathbb{1}}$ Lemma 1.2.3 (1) *If a structural transformation T on a unital Jordan algebra* J *is invertible, then its adjoint is T^* is invertible too.*

(2) *If T is structural and has the property that it is invertible as soon as it is surjective (for example, a U-operator), then we have* **contagious invertibility**: *if T ever takes on an invertible value, then it and its adjoint are invertible operators,*

$$T(x) \text{ is invertible iff } T,\ T^*,\ x \text{ are invertible.}$$

(3) *If a structural T is congruent to $\widehat{\mathbb{1}}$ mod* J *on a general* J*, then T is injective (respectively surjective) on $\widehat{\mathrm{J}}$ iff it is injective (respectively surjective) on* J*. In particular, a Bergmann operator $B_{x,y}$ is invertible or surjective on* J *iff it is on $\widehat{\mathrm{J}}$.*

PROOF. (1) If T is invertible, then $T(x) = 1$ for some x, so $\mathbb{1}_{\mathrm{J}} = U_{T(x)} = TU_xT^*$ [by Structurality 1.2.1(1)] implies that $U_xT^* = T^{-1}$ invertible; then U_x is surjective, hence invertible [by Inverse Criterion 1.1(3)$(ii) = (iii)$], so $T^* = U_x^{-1}T^{-1}$ is invertible too.

(2) A similar argument works whenever $T(x)$ is invertible *if we know that T, like U_x, becomes invertible once it is surjective*: $U_{T(x)} = TU_xT^*$ invertible [by the Inverse Criterion $(i) = (ii)$ and Structurality] $\Longrightarrow T$ is surjective $\Longrightarrow T$ is invertible [by the *hypothesis* on T] $\Longrightarrow U_xT^* = T^{-1}U_{T(x)}$ is invertible $\Longrightarrow U_x$ surjective $\Longrightarrow U_x$ invertible $\Longrightarrow x$ invertible [by the Inverse Criterion] $\Longrightarrow T^* = U_x^{-1}T^{-1}$ invertible.

(3) In $\widehat{\mathrm{J}} = \Phi\hat{1} \oplus \mathrm{J}$, the submodule $\Phi\hat{1}$ is a faithful copy of Φ, $\alpha\hat{1} = 0 \Longrightarrow \alpha = 0$. Applying Congruence to $\widehat{\mathbb{1}}$ 1.2.1(2) with S replaced by T, and using directness, the kernel and cokernel of T live in J: any killing takes place in J because $T(\alpha\hat{1} \oplus x) = \alpha\hat{1} \oplus (-\alpha c + T(x) = 0 \Longleftrightarrow \alpha = 0, T(x) = 0$; similarly, any non-imaging takes place in J because $\alpha\hat{1} \oplus x \notin T(\widehat{\mathrm{J}}) \Longleftrightarrow \alpha\hat{1} \oplus x - T(\alpha\hat{1}) \notin T(\widehat{\mathrm{J}}) \Longleftrightarrow x + \alpha c \notin T(\mathrm{J})$. □

1.3 Quasi-Invertibility

Quasi-invertibility is determined by Bergmann operators, which are congruent to $\widehat{\mathbb{1}}$.

Radical Definition 1.3.1 (1) *An element x of an arbitrary Jordan algebra* J *is* **quasi-invertible** (**q.i.** *for short*) *if $\hat{1} - z$ is invertible in the unital hull $\widehat{\mathrm{J}}$; if $(\hat{1} - z)^{-1} = \hat{1} - w$ then w is called the* **quasi-inverse** *of z, denoted by $qi(z)$.*[3]

[3] The notation $q(x)$ has already been used for the Peirce quadratic forms; to avoid confusion, we add the i to make $qi(\cdot)$ (pronounced "kyoo-eye") to make the reference to quasi-inverse clearer, but we save space by staying aperiodic and not going the whole way to write $q.i.(\cdot)$. Some authors use the negation of our definition, $(\hat{1} - z)^{-1} = \hat{1} + w$, so that the quasi-inverse is the geometric series $w = +\sum z^i$. We prefer to keep the symmetry in z and w, so that $(\hat{1} - w)^{-1} = \hat{1} - z$ just as $\left(u^{-1}\right)^{-1} = u$ for ordinary inverses.

The set of quasi-invertible elements of J *is denoted by* $\mathcal{QI}(\mathrm{J})$. *An algebra or ideal is called quasi-invertible (q.i.) if all its elements are q.i.*

(2) *The* **Jacobson radical** $\mathcal{R}ad(\mathrm{J})$ *is the maximal q.i. ideal, i.e., the maximal ideal contained inside the set* $\mathcal{QI}(\mathrm{J})$.[4] *An algebra is called* **semiprimitive**[5] *if it has no q.i. ideals, equivalently its Jacobson radical vanishes,* $\mathcal{R}ad(\mathrm{J}) = \mathbf{0}$.

(3) *An element is* **nilpotent** *if some power vanishes,* $x^n = 0$; *then all higher powers* $x^m = x^n \bullet x^{m-n} = 0$ *for* $m \geq n$ *also vanish. Nilpotent elements are always q.i.:*

$$z^n = 0 \Longrightarrow \begin{cases} (\hat{1} - z)^{-1} & = \hat{1} + z + z^2 + \cdots + z^{n-1}, \\ -qi(z) & = z + z^2 + \cdots + z^{n-1} \end{cases}$$

by the usual telescoping sum, since by power-associativity this all takes place in the special subalgebra $\Phi[z]$.

(4) *The nilpotent elements are the most important single source of q.i. elements. An algebra or ideal is called* **nil** *if all its elements are nilpotent. The* **nil radical** $\mathcal{N}il(\mathrm{J})$ *is defined as the maximal* **nil ideal**.[6] *Since nil implies q.i. by (3), we have*

$$\mathcal{N}il(\mathrm{J}) \subseteq \mathcal{R}ad(\mathrm{J}).$$

It is always useful to think of $(\hat{1} - z)^{-1}$ and the *negative* $-qi(z)$ as given by the **geometric series**

$$(\hat{1} - z)^{-1} \approx \sum_{n=0}^{\infty} z^n, \qquad -qi(z) \approx \sum_{n=1}^{\infty} z^n.$$

This is strictly true whenever the infinite series converges respectably, such as for an element of norm < 1 in a Banach algebra (see Problem 1.1), or if z is nilpotent so the series is finite as in (3). Amitsur's Tricks (see Problem 1.2 and Theorem 3.2.2) show that the nil elements are the only die-hard q.i. elements, the only ones that remain q.i. over a big field or when multiplied by a scalar indeterminate. In some sense the nil radical is the "stable" form of the Jacobson radical: in Chapter 5 we will see that if we perturb the algebra enough, the radical will shrink down into the nil radical. But the Jacobson radical has proven to be the most useful radical, both structurally and practically. We will soon obtain a handy elemental characterization of the Jacobson radical, but there is no such characterization known for the nil radical: even in associative algebras, the Köthe Conjecture (that the nil radical consists precisely of the properly nilpotent elements) remains unsettled.

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[4] Defining is not creating: we have to show that there *is* just one such maximal ideal, which we will do in 1.5.1.

[5] Philosophically, an algebra is *semi-blah* iff it is a subdirect product of *blah* algebras. Later, in Chapter 5, we will see what a vanishing Jacobson radical has to do with primitivity. For the present, just think of semiprimitivity as some form of "semisimplicity."

[6] This is to be carefully distinguished from a *nilpotent* ideal, which means a power of the ideal (not just of its individual elements) vanishes. The nil radical is sometimes called the *Köthe* radical.

Basic Quasi-Inverse Theorem 1.3.2 *We have the following basic facts about quasi-inverses in Jordan algebras.*

(1) **Existence:** *An element z is quasi-invertible in* J *with quasi-inverse w iff it satisfies the* **Quasi-Inverse Conditions**

$$\text{(QInv1)} \quad U_{\hat{1}-z}w = z^2 - z, \qquad \text{(QInv2)} \quad U_{\hat{1}-z}w^2 = z^2.$$

(2) **Extension:** *If z is quasi-invertible in* J, *then it remains quasi-invertible* (*with the same quasi-inverse*) *in any extension algebra* $\widetilde{\mathrm{J}} \supseteq \mathrm{J}$.

(3) **Criterion:** *The following are equivalent for an element z of a Jordan algebra* J:

(*i*) *z is a quasi-invertible element of* J;
(*ii*) *the quasi-U-operator $U_{\hat{1}-z}$ is an invertible operator on* J
(*equivalently, invertible on* $\widehat{\mathrm{J}}$);
(*iii*) *$U_{\hat{1}-z}$ is surjective on* J, $U_{\hat{1}-z}(\mathrm{J}) = \mathrm{J}$
(*equivalently, surjective on* $\widehat{\mathrm{J}}$, $U_{\hat{1}-z}(\widehat{\mathrm{J}}) = \widehat{\mathrm{J}}$);
(*iv*) *the image of $U_{\hat{1}-z}$ contains* $2z - z^2 \in U_{\hat{1}-z}(\mathrm{J})$;
(*v*) *$U_{\hat{1}-z}(\mathrm{J})$ contains $2z - z^2 - u$ for some q.i. u.*

(4) **Consequences:** *If z is quasi-invertible in* J *with quasi-inverse $w = qi(z)$, then*:

(*i*) *the quasi-U-operators $U_{\hat{1}-z}, U_{\hat{1}-w}$ are inverse operators*;
(*ii*) *w is uniquely determined as $w = U_{\hat{1}-z}^{-1}(z^2 - z)$*;
(*iii*) *w is quasi-invertible with quasi-inverse $z = qi(w)$*;
(*iv*) $\{z, w\} = 2(z + w)$, $U_z w = w + z + z^2$,

showing that the quasi-inverse w necessarily lies in the original J.

PROOF. We apply the Basic Inverse Theorem 1.1.1 to $u = \hat{1} - z$ in the algebra $\widehat{\mathrm{J}}$. (1) The invertibility conditions (QJInv1)–(QJInv2) of the Definition 1.1.1(1) for $u = \hat{1} - z, v = \hat{1} - w$ become $\hat{1} - z = U_{\hat{1}-z}(\hat{1} - w) = (\hat{1} - 2z + z^2) - U_{\hat{1}-z}(w) \iff U_{\hat{1}-z}(w) = -z + z^2$ as in (QInv1) and $\hat{1} = U_{\hat{1}-z}(\hat{1} - w)^2 = U_{\hat{1}-z}(\hat{1} - 2w + w^2) = U_{\hat{1}-z}(2(\hat{1} - w) - \hat{1} + w^2) = 2(\hat{1} - z) - (\hat{1} - 2z + z^2) + U_{\hat{1}-z}(w^2)$ [by the above] $= \hat{1} - z^2 + U_{\hat{1}-z}(w^2) \iff U_{\hat{1}-z}w^2 = z^2$ as in (QInv2).

(2) is clear, since quasi-invertibility is strictly between z and w.

(3) Since z is q.i. iff $\hat{1} - z$ is invertible, Invertible Criterion 1.1.1(3)(*i*)–(*v*) implies the equivalence of (*i*)–(*v*). For (*ii*) and (*iii*) use the Congruent to $\widehat{\mathbb{1}}$ Lemma 1.2.3(3) for $T = U_{\hat{1}-z}$ to see that invertibility or surjectivity on J is equivalent to invertibility or surjectivity on $\widehat{\mathrm{J}}$. For (*iv*), since $U_{\hat{1}-z}(\hat{1}) = \hat{1} - 2z + z^2$, the element $\hat{1}$ lies in the range of $U_{\hat{1}-z}$ iff $2z - z^2$ does (and its preimage must be in J by directness); (*iv*) is the special case $u = 0$ of (*v*), where $U_{\hat{1}-z}(a) = 2z - z^2 - u$ for q.i. u iff $U_{\hat{1}-z}(\hat{1} + a) = \hat{1} - u$ is an invertible value of $U_{\hat{1}-z}$.

(4) Since z is q.i. with quasi-inverse w iff $\hat{1} - z$ is invertible with inverse $\hat{1} - w$ in $\widehat{\mathrm{J}}$, (*i*)–(*iii*) follow from Inverse Consequences 1.1.1(4) and (1). For (*iv*), note that $\hat{2} = \{(\hat{1} - z), (\hat{1} - w)\} = \hat{2} - 2w - 2z + \{z, w\} \iff \{z, w\} = 2(z + w)$,

so $-z+z^2 = U_{\hat{1}-z}w$ [by the above] $= w - \{z,w\} + U_z w = w - 2w - 2z + U_z w$. Since $U_z(w) \in \mathrm{J}$, this shows that w lies in J, not just $\widehat{\mathrm{J}}$. □

Exercise 1.3.2* Show that z^n q.i. implies z q.i. ; show that the converse is false.

1.4 Proper Quasi-Invertibility

The Jacobson radical consists precisely of the *properly* quasi-invertible elements. In general, in Jordan algebras the adverb "properly" means "in all homotopes"; in Jordan theory we don't have a notion of "left multiple" yz, but the notion "z in the y-homotope $\mathrm{J}^{(y)}$" provides a workable substitute. We will write the unital hull of the homotope as

$$\widehat{\mathrm{J}^{(y)}} := \Phi 1^{(y)} \oplus \mathrm{J}^{(y)},$$

so that the fictitious unit $1^{(y)}$ wears a (y) instead of a hat $\hat{}$.

Proper Quasi-Invertible Definition 1.4.1 (1) *A pair (z,y) of elements in a Jordan algebra* J *is called a* **quasi-invertible (q.i.) pair** *if z is quasi-invertible in the homotope $\mathrm{J}^{(y)}$, in which case its quasi-inverse is denoted by $z^y = qi(z,y)$ and called the* **quasi-inverse of the pair**. *The quasi-U operators which determine quasi-invertibility become, in the y-homotope, the Bergmann operators* 1.2.2:

$$U^{(y)}_{1^{(y)}-z} = \mathbb{1}_{\mathrm{J}} - V^{(y)}_z + U^{(y)}_z = \mathbb{1}_{\mathrm{J}} - V_{z,y} + U_z U_y =: B_{z,y}.$$

Thus (z,y) is q.i. iff $B_{z,y}$ is an invertible operator on J.

(2) *An element z is called* **properly quasi-invertible** (**p.q.i.** *for short*) *if it remains quasi-invertible in* all *homotopes $\mathrm{J}^{(y)}$ of* J, *i.e.,* all *pairs (z,y) are quasi-invertible and all $B_{z,y}$ are invertible on* J. *The set of properly quasi-invertible elements of* J *is denoted by $\mathcal{PQI}(\mathrm{J})$. Proper quasi-invertibility is a stronger condition than mere quasi-invertibility*:

$$z \ p.q.i. \implies (z,z) \ q.i. \iff z^2 \ \ q.i. \iff z, \, -z \ \ q.i.$$

since [*by Macdonald or Section* II.5.2] $B_{z,z} = \mathbb{1}_{\mathrm{J}} - V_{z,z} + U_z U_z = \mathbb{1}_{\mathrm{J}} - V_{z^2} + U_{z^2} = U_{\hat{1}-z^2} = U_{\hat{1}-z} U_{\hat{1}+z} = U_{\hat{1}+z} U_{\hat{1}-z}$ *invertible on* J [*equivalently, on* $\widehat{\mathrm{J}}$ *by the Congruent to* $\widehat{\mathbb{1}}$ *Lemma* 1.2.3(3)] $\iff U_{\hat{1}-z}, U_{\hat{1}+z}$ *invertible on* $\widehat{\mathrm{J}}$ $\iff \hat{1}-z, \hat{1}+z$ *invertible in* $\widehat{\mathrm{J}} \iff z, -z$ *q.i. in* J.

(3) *An element z is called* **properly nilpotent** (**p.n.** *for short*) *if it remains nilpotent in* all *homotopes $\mathrm{J}^{(y)}$ of* J, *i.e.,* all *pairs $U^{(y)}_z = U_z U_y$ are nilpotent operators. The set of properly nilpotent elements of* J *is denoted*

by $\mathcal{P}nil(\mathrm{J})$. Just as nilness is more restrictive than quasi-invertibility, proper nilpotence is more restrictive than proper quasi-invertibility:[7]

$$z \ p.n. \implies z \ p.q.i.$$

We should think associatively that $B_{z,y}(x) \approx (1 - zy)x(1 - yz)$. Again, a good mnemonic device is to think of the *negative* of the quasi-inverse of the pair as a geometric series related to the associative quasi-inverses of zy and yz; since the homotope is obtained by sticking a factor y in the middle of all products, the geometric approach to quasi-inverses leads to

$$\begin{aligned} -qi(z,y) &\approx \ z + zyz + zyzyz + \cdots \ = \ \textstyle\sum_{n=1}^{\infty} z^{(n,y)} \\ &\approx \ z(\hat{1} + yz + yzyz + \cdots) \ = \ z(\hat{1} - yz)^{-1} \approx (\hat{1} - zy)^{-1} z \\ &\approx \ z + z(y + yzy + \cdots)z \ = \ z + U_z(-qi(y,z)). \end{aligned}$$

This helps us understand some of the facts about q.i. pairs.

Basic Q.I. Pair Theorem 1.4.2 *We have the following basic facts about quasi-invertible pairs in Jordan algebras.*

(1) **Existence:** (z,y) *is quasi-invertible in* J *with quasi-inverse* $w = qi(z,y)$ *iff it satisfies the* **Quasi-Inverse PairConditions**

$$\text{(QInvP1)} \quad B_{z,y}w = U_z y - z, \qquad \text{(QInvP2)} \quad B_{z,y}U_w y = U_z y.$$

(2) **Extension:** *If* (z,y) *is quasi-invertible in* J, *then it remains quasi-invertible* (*with the same quasi-inverse* $qi(z,y)$) *in any extension algebra* $\widetilde{\mathrm{J}} \supseteq \mathrm{J}$.

(3) **Criterion:** *The following are equivalent for a pair of elements* (z,y):

(*i*) (z,y) *is quasi-invertible in* J;
(*ii*) *the Bergmann operator* $B_{z,y}$ *is an invertible operator on* J (*equivalently, on* $\widehat{\mathrm{J}}$);
(*iii*) $B_{z,y}$ *is surjective on* J, $B_{z,y}(\mathrm{J}) = \mathrm{J}$ (*equivalently, on* $\widehat{\mathrm{J}}$, $B_{z,y}(\widehat{\mathrm{J}}) = \widehat{\mathrm{J}}$);
(*iv*) $2z - U_z y \in B_{z,y}(\mathrm{J})$;
(*v*) $B_{z,y}(\mathrm{J})$ *contains* $2z - U_z y - u$ *for some q.i. pair* (u,y);
(*vi*) $\{z,y\} - U_z y^2 \in B_{z,y}(\mathrm{J})$;
(*vii*) $\{z,y\} - U_z y^2$ *is q.i. in* J;
(*i*)* (y,z) *is quasi-invertible in* J;
(*ii*)* *the Bergmann operator* $B_{y,z}$ *is an invertible operator on* J (*equivalently, on* $\widehat{\mathrm{J}}$).

[7] The Köthe conjecture is that, as with the radical, the nil radical consists of the entire set $\mathcal{P}nil(\mathrm{J})$. Most ring theorists believe that the conjecture is false, but no one has been able to come up with a counter-example, even in associative algebras.

(4) **Consequences:** *If* (z, y) *is quasi-invertible in* J *with quasi-inverse* $w = qi(z, y)$*, then:*

(*i*) *the Bergmann operators* $B_{z,y}, B_{w,y}$ *are inverse operators*;
(*ii*) w *is uniquely determined as* $w = B_{z,y}^{-1}(U_z y - z)$;
(*iii*) (w, y) *is quasi-invertible with quasi-inverse* $z = qi(w, y)$;
(*iv*) $\{z, y, w\} = 2(z + w), \quad U_z U_y w = w + z + U_z y$;
$(i)^*$ *the Bergmann operators* $B_{y,z}, B_{y,w}$ *are inverse operators,*

so again necessarily $w \in$ J.

PROOF. Before we begin we must be clear about a subtle point: though the homotope $\mathrm{J}^{(y)}$ has the same underlying space as the algebra J, the unital hull $\widehat{\mathrm{J}^{(y)}} := \Phi 1^{(y)} \oplus \mathrm{J}$ of the homotope does *not* have the same underlying space as the unital hull $\widehat{\mathrm{J}} := \Phi\hat{1} \oplus \mathrm{J}$ of J (and the *hull of the homotope* $\widehat{\mathrm{J}^{(y)}}$ is definitely *not* the *homotope of the hull* $\big(\widehat{\mathrm{J}}\big)^{(y)}$, since the latter is *never* unital). By the Structural Transformation Definition 1.2.1 and Bergmann Structurality 1.2.2, as an operator on the y-homotope the Bergmann operator $B_{z,y}$ is a symmetric structural transformation (a U-operator $B_{z,y} = U^{(y)}_{1^{(y)}-z} = B^*_{z,y}$) which is congruent to $\mathbb{1}^{(y)}$ since $B_{z,y}(1^{(y)}) = 1^{(y)} - 2z + U_z(y)$. At the same time it is a non-symmetric structural transformation ($B^*_{z,y} = B_{y,z}$) on J itself which is congruent to $\widehat{\mathbb{1}}$ since $B_{z,y}(\hat{1}) = \hat{1} - \{z, y\} + U_z(y^2)$. Thus by the Congruent to $\widehat{\mathbb{1}}$ Lemma 1.2.1(3) we have

$$
\begin{aligned}
&B_{z,y} \textit{ is invertible on } \widehat{\mathrm{J}^{(y)}} \iff \textit{invertible on } \mathrm{J}^{(y)} \iff \textit{surjective on } \mathrm{J}^{(y)} \\
&\quad \iff \textit{invertible on } \widehat{\mathrm{J}} \quad \iff \textit{invertible on } \mathrm{J} \quad \iff \textit{surjective on } \mathrm{J}.
\end{aligned}
$$

Thus when we talk about the Bergmann operator being invertible or surjective "on the unital hull," it doesn't matter which hull we are talking about. In consequence, we will entirely neglect the hull of the homotope, with its funny $1^{(y)}$, and deal entirely with our usual hull.

We apply the Basic Q.I. Theorem 1.3.2 in the homotope $\mathrm{J}^{(y)}$. Since in $\mathrm{J}^{(y)}$ the operator $U_{\hat{1}-z}$ becomes the Bergmann operator $B_{z,y}$, and the squares and braces are $x^{(2,y)} = U_x y$ and $\{x, z\}^{(y)} = \{x, y, z\}$, Q.I. Existence (1) gives (1); Q.I. Extension (2) gives (2) [for $y \in$ J we still have inclusion $\widetilde{\mathrm{J}}^{(y)} \supseteq \mathrm{J}^{(y)}$]; Q.I. Criterion (3)(*i*)–(*v*) gives the equivalence of (3)(*i*)–(*v*); and Q.I. Consequences (4)(*i*)–(*iv*) gives the consequences (4)(*i*)–(*iv*). An immediate consequence of Congruent to $\widehat{\mathbb{1}}$ 1.2.3(1) applied to $T = B_{z,y}$ is that *q.i. pairs are symmetric:* in (3), $(ii) \iff (ii)^*$, and hence $(i) \iff (i)^*$.

In any structural pair (T, T^*) as defined in 1.2.1(1), $T = \widehat{\mathbb{1}} \Longrightarrow T^* = TU_{\hat{1}}T^* = U_{T(\hat{1})} = U_{\hat{1}} = \widehat{\mathbb{1}}$ too. Applying this to $T := B_{z,y}B_{w,y}$ and $T^* := B_{y,w}B_{y,z}$ (which form a structural pair as the "product" of the structural pairs $(B_{z,y}, B_{y,z})$ and $(B_{w,y}, B_{y,w})$ from Bergmann Structurality 1.2.2), and dually to the pair (S, S^*) for $S := B_{w,y}B_{z,y}$ and $S^* := B_{y,z}B_{y,w}$, we have $(4)(i) \iff T = S = \widehat{\mathbb{1}} \iff T^* = S^* = \widehat{\mathbb{1}} \iff (4)(i)^*$. Thus $(4)(i)^*$ is a consequence.

To get the remaining equivalences with $(vi), (vii)$ in (3) *we work in the unital hull* $\widehat{\mathrm{J}}$ *instead of the hull of the homotope* $\widehat{\mathrm{J}^{(y)}}$. Set

$$B := B_{z,y},\ B^* := B_{y,z},\ x := \{z,y\} - U_z y^2,\quad \text{so } B(\hat{1}) = \hat{1} - x.$$

We noted at the start that B is structural on the unital algebra $\widehat{\mathrm{J}}$, and is invertible iff it is surjective [on the module $\widehat{\mathrm{J}}$ or on the module J], so Contagious Inversion 1.2.3(2) tells us that

$$\text{(C.I)}\qquad B(\hat{a}) \text{ invertible in } \widehat{\mathrm{J}} \Longrightarrow B, B^*, \hat{a} \text{ invertible.}$$

Thus $(ii) \iff (ii) + (ii)^*$ [from $(3)(ii) = (ii)^*$] $\iff B, B^*$ invertible $\iff U_{\hat{1}-x} = U_{B(\hat{1})} = BU_{\hat{1}}B^* = BB^*$ invertible [$\Rightarrow$ is clear, $\Leftarrow$ follows from (C.I.) with $\hat{a} = \hat{1}$] $\iff x$ q.i. $\iff (vii)$. This establishes the equivalence of (vii). For the equivalence of (vi), clearly $(iii) \Longrightarrow (vi)$, while conversely $(vi) \iff x = B(a)$ [for some $a \in \mathrm{J}$] $\iff \hat{1} = x + (\hat{1} - x) = B(a) + B(\hat{1}) = B(\hat{1} + a) \in B(\widehat{\mathrm{J}}) \Longrightarrow B$ invertible [by (C.I.)] $\Longrightarrow (iii)$. Thus all conditions in (3) are equivalent. □

Three Basic Q.I. Pair Principles 1.4.3 (1) **Symmetry Principle**: *Quasi-invertibility is symmetric*: (z, y) *is quasi-invertible in* J *iff* (y, z) *is, in which case*

$$qi(y, z) = U_y qi(z, y) - y.$$

(2) **Structural Shifting Principle**: *If* T *is a structural transformation on* J, *then it is a homomorphism* $\mathrm{J}^{(T^*(y))} \to \mathrm{J}^{(y)}$ *of Jordan algebras for any* $y \in \mathrm{J}$, *and it can be shifted in quasi-invertible pairs: if* $(z, T^*(y))$ *is quasi-invertible, so is* $(T(z), y)$, *in which case*

$$qi(T(z), y) = T(qi(z, T^*(y))).$$

If (T, T^*) *is a structural pair, then* $(T(z), y)$ *is quasi-invertible iff* $(z, T^*(y))$ *is quasi-invertible.*

(3) **Addition Principle**: *Quasi-invertibility is additive: if* x, y, z *are elements of* J *with* (z, y) *quasi-invertible, then* $(x + z, y)$ *is quasi-invertible iff* $(x, -qi(y, z))$ *is quasi-invertible,*

$$B_{x+z,y} = B_{x,-qi(y,z)} B_{z,y} \qquad (\text{if } (z, y) \text{ q.i.}).$$

PROOF. (1) We noticed in constructing Principal Inner Ideals II.5.3.1 the general formula

$$(4)\qquad U_{y - U_y z} = U_y B_{z,y} = B_{y,z} U_y,$$

which follows from Macdonald, since it involves only two elements, or directly from $U_y - U_{y,U_y w} + U_{U_y w} = U_y(\mathbb{1}_{\mathrm{J}} - V_{w,y} + U_w U_y)$ by the Commuting Formula (FFII) and the Fundamental Formula (FFI). Basic Q.I.

Pair Facts 1.4.2(3)(i) = (i)* shows symmetry (z,y) q.i. $\iff$ (y,z) q.i., and then the formula for $qi(y,z)$ follows from Q.I. Pair Consequences 1.4.2(4)(ii): $qi(y,z) = B_{y,z}^{-1}(U_y z - y) = B_{y,z}^{-1}\big(U_y[U_z y - z] - [y - \{y,z,y\} + U_y U_z y]\big) = B_{y,z}^{-1}\big(U_y[B_{z,y} qi(z,y)] - [B_{y,z} y]\big)$ [by Q.I. Pair Consequences (4)]] $= B_{y,z}^{-1}\big(B_{y,z}[U_y qi(z,y)] - B_{y,z}[y]\big)$ [by (4) above] $= U_y qi(z,y) - y$.

(2) T is a homomorphism because it preserves squares, $T(x^{(2,T^*(y))}) = T(U_x T^*(y)) = U_{T(x)} y$ [by Structurality 1.2.1(1)] $= T(x)^{(2,y)}$, therefore T takes q.i. elements $z \in \mathrm{J}^{(T^*(y))}$ to q.i. elements $T(z) \in \mathrm{J}^{(y)}$. Then takes quasi-inverses $qi(z,T^*(y))$ to quasi-inverses $qi(T(z),y)$ as in (2). Conversely, if $T^{**} = T$ then $(T(z),y)\, q.i. \Longrightarrow (y,T(z)) = (y,T^{**}(z))\, q.i.$ [by Symmetry (1) above] $\Longrightarrow (T^*(y),z)\, q.i.$ [by the above shift applied to T^*] $\Longrightarrow (z,T^*(y)) q.i.$ [by Symmetry again].

(3) We have another general formula

$$(5) \qquad B_{x,y-U_y w} = B^{(y)}_{x,1^{(y)}-w} \ on\ \widehat{\mathrm{J}^{(y)}} = \Phi 1^{(y)} \oplus \mathrm{J}^{(y)},$$

which follows from $B_{x,y-U_y w} = \widehat{\mathbb{1}} - (V_{x,y} - V_{x,U_y w}) + U_x U_{y-U_y w} = \widehat{\mathbb{1}} - (V_x^{(y)} - V_{x,w}^{(y)}) + U_x U_y B_{w,y}$ [using (4)] $= \widehat{\mathbb{1}} - V^{(y)}_{x,1^{(y)}-w} + U_x^{(y)} U^{(y)}_{1^{(y)}-w}$ [by P.Q.I. Definition 1.4.1(1)] $= B^{(y)}_{x,1^{(y)}-w}$.

For invertible u we have formulas

$$(6)\ V_{U_u w,u^{-1}} = V_{u,w}, \quad (7)\ B_{U_u w,u^{-1}} = B_{u,w}, \quad (8)\ B_{x,u^{-1}} U_u = U_{u-x}.$$

To see (6), on $\mathrm{J} = U_u \mathrm{J}$ we have $\{U_u w, u^{-1}, U_u a\} = U_u U_{w,a} U_u u^{-1}$ [Fundamental Formula (FFI)] $= U_u\{w,u,a\} = \{u,w,U_u a\}$ [by Commuting (FFII)]. For (7), $\mathbb{1}_{\mathrm{J}} - V_{U_u w,u^{-1}} + U_{U_u w} U_{u^{-1}} = \mathbb{1}_{\mathrm{J}} - V_{u,w} + U_u U_w$ [by (6), the Fundamental Formula, and Inverse Consequences 1.1.1(2)]. For (8), $U_{u-x} = U_{u-U_u w}$ [for $w = U_u^{-1} x$] $= B_{u,w} U_u$ [by (4)] $= B_{U_u w,u^{-1}} U_u$ [by (7)] $= B_{x,u^{-1}} U_u$.

Applying (8) to $u = \widehat{1^{(y)}} - z$, $u^{-1} = \widehat{1^{(y)}} - qi(z,y)$ in $\widehat{\mathrm{J}^{(y)}}$ gives $B^{(y)}_{x,\widehat{1^{(y)}}-qi(z,y)} U^{(y)}_{\widehat{1^{(y)}}-z} = U^{(y)}_{(\widehat{1^{(y)}}-z)-x} = U^{(y)}_{\widehat{1^{(y)}}-(z+x)}$. By (5) and Proper Q.I. Definition 1.4.1(1) this converts to $B_{x,y-U_y qi(z,y)} B_{z,y} = B_{x+z,y}$, where $y - U_y qi(z,y) = -qi(y,z)$ by Symmetry (1), establishing the formula in part (3). Thus $(x+z,y)$ q.i. $\iff$ $B_{x+z,y}$ invertible [by the Basic Q.I. Criterion 1.3.2(3)(i) = (ii))] $\iff$ $B_{x,-qi(y,z)}$ invertible [from the formula in part (3), since $B_{y,z}$ is already assumed invertible] $\iff$ $(x,-qi(y,z))$ q.i. □

Exercise 1.4.3* (1) Recall that T is *weakly structural* if there is a T^* such that $U_{T(x)} = TU_x T^*$ just on J. Prove that $T : \mathrm{J}^{(T^*(y))} \to \mathrm{J}^{(y)}$ is a homomorphism of quadratic Jordan algebras with $\frac{1}{2}$ held behind your back [i.e., prove $T(x^{(2,T^*(y))}) = T(x)^{(2,y)}$, $T(U_x^{(T^*(y))} z) = U^{(y)}_{T(x)} T(z)$ directly from structurality]; prove $TB_{x,T^*(y)} = B_{T(x),y} T$ for all x,y. (2) Use this to verify that $T(qi(z,T^*(y))) = qi(T(z),y)$. (3) If (T,T^*) is a weakly structural pair, show that $(z,T^*(y))$ q.i. $\iff$ $(T(z),y)$ q.i.

1.5 Elemental Characterization

Now we are ready to give our elemental characterization of the radical as the properly quasi-invertible elements. In the next section we will harvest some of the consequences of this characterization.

Elemental Characterization Theorem 1.5.1 (1) *If* K *is a q.i. ideal of* J, *then*

$$\mathrm{K} \subseteq \mathcal{PQI}(\mathrm{J}) \subseteq \mathcal{QI}(\mathrm{J}).$$

(2) *The Jacobson radical coincides with the set of all properly quasi-invertible elements:*

$$\mathcal{R}ad(\mathrm{J}) = \mathcal{PQI}(\mathrm{J}),$$

which therefore forms a structurally invariant ideal, the unique maximal q.i. ideal (*containing all others*).

(3) *Thus an algebra is semiprimitive iff it contains no nonzero properly quasi-invertible elements.*

PROOF. (1) The second inclusion holds by the P.Q.I. Definition 1.4.1(2). The first inclusion holds because the elements of any q.i. ideal must actually be *properly* q.i.: z in a q.i. ideal K $\Longrightarrow$ all $\{z, y\} - U_z y^2 \in \mathrm{K} \Longrightarrow$ all $\{z, y\} - U_z y^2$ q.i. $\Longrightarrow$ all (z, y) are q.i. [by Q.I. Pair Criterion 1.4.2(4)(vi)] $\Longrightarrow$ z is p.q.i. Thus the radical, and all q.i. ideals, are contained in $\mathcal{PQI}(\mathrm{J})$.

(2)–(3) We claim that $\mathcal{PQI}(\mathrm{J})$ itself is a *structurally invariant ideal*, and therefore from the above is the maximal q.i. ideal $\mathcal{R}ad(\mathrm{J})$ as in (2), so that an algebra is semiprimitive iff $\mathcal{PQI}(\mathrm{J}) = \mathbf{0}$ as in (3). $\mathcal{PQI}(\mathrm{J})$ is closed under *addition*: if x, z are p.q.i., then for any y the pairs (z, y) and $(x, -qi(z, y))$ are quasi-invertible, implying that $(x + z, y)$ is quasi-invertible by the Addition Principle, so $x + z$ is p.q.i. It is closed under *structural transformations* T: z p.q.i. $\Longrightarrow$ $(z, T^*(y))$ quasi-invertible for all $y \Longrightarrow (T(z), y)$ is quasi-invertible for all y by Structural Shifting $\Longrightarrow T(z)$ is p.q.i. In particular, $\mathcal{PQI}(\mathrm{J})$ is invariant under *scalar multiplications*, so it is a *linear subspace*, and it is invariant under all *outer multiplications* $T = U_a$ $(a \in \widehat{\mathrm{J}})$, hence also all $L_a = \frac{1}{2}V_a = \frac{1}{2}(U_{a+\hat{1}} - U_a - \mathbb{1}_{\mathrm{J}})$, so it is an *ideal*. □

EXERCISE 1.5.1* (1) Use the elemental characterization to show that $\mathcal{R}ad(U_e\mathrm{J}) = U_e\mathrm{J} \cap \mathcal{R}ad(\mathrm{J})$ for any idempotent e. (2) Show that $\mathcal{R}ad(\mathrm{J})$ is invariant under multiplication by all elements γ of the centroid (all linear transformations with $\gamma(x \bullet y) = \gamma(x) \bullet y = x \bullet \gamma(y)$, cf. II Section 1.6.2).

1.6 Radical Inheritance

This elemental characterization of the radical has several immediate consequences. It allows us to relate semiprimitivity to nondegeneracy, and to prove that the radical is *hereditary*, in the sense that the property of *radicality* ($\mathcal{R}ad(\mathrm{J}) = \mathrm{J}$) is inherited by ideals.[8] The useful version of the hereditary property is that the radical of an ideal (or Peirce inner ideal) is just the part inherited from the global radical.

Hereditary Radical Theorem 1.6.1 (1) *Quasi-invertibility takes place in inner ideals: if an element of an inner ideal* B *is q.i. in* J, *then its quasi-inverse lies back in* B, *and similarly for p.q.i.*:

$$qi(\mathrm{B}) \subseteq \mathrm{B}, \qquad \mathrm{B} \cap \mathcal{QI}(\mathrm{J}) \subseteq \mathcal{QI}(\mathrm{B}),$$
$$pqi(\mathrm{B}, \mathrm{J}) \subseteq \mathrm{B}, \qquad \mathrm{B} \cap \mathcal{PQI}(\mathrm{J}) \subseteq \mathcal{PQI}(\mathrm{B}).$$

(2) *We have equality for ideals or Peirce inner ideals*:

$$\mathcal{PQI}(\mathrm{K}) = \mathrm{K} \cap \mathcal{PQI}(\mathrm{J}) \ \textit{if} \ \mathrm{K} \triangleleft \mathrm{J},$$
$$\mathcal{PQI}(\mathrm{J}_k(e)) = \mathrm{J}_k(e) \cap \mathcal{PQI}(\mathrm{J}) \quad (k = 2, 0, \ e \ \textit{idempotent}).$$

(3) *In particular, the radical remains radical in the unital hull*:

$$\mathcal{PQI}(\mathrm{J}) = \mathrm{J} \cap \mathcal{PQI}(\widehat{\mathrm{J}}).$$

(4) *These guarantee that ideals and Peirce inner ideals inherit semiprimitivity*: *we have the* **Ideal** *and* **Peirce Inheritance Principles**

$$\mathcal{R}ad(\mathrm{J}) = \mathbf{0} \Longrightarrow \mathcal{R}ad(\mathrm{K}) = \mathcal{R}ad(\mathrm{J}_k(e)) = \mathbf{0}.$$

(5) *Semiprimitivity implies nondegeneracy*:

$$\mathcal{R}ad(\mathrm{J}) = \mathbf{0} \Longrightarrow \mathrm{J} \ \textit{nondegenerate.}$$

PROOF. (1) By Q.I. Existence 1.3.2(1), the quasi-inverse $qi(z) = -z - z^2 + U_z qi(z)$ lies in the inner ideal B if z does (thanks to our strongness requirement that all inner ideals be subalgebras). The same holds for all $qi(z, y)$ by applying this in the y-homotope where B remains inner (cf. Q.I. Pair Existence 1.4.2(1)).

(2) There are lots of *inner ideals* where the p.q.i. inclusion is strict (an upstanding algebra with no radical can have nilpotent inner ideals which are entirely radical), but for *ideals* K or *Peirce inner ideals* $\mathrm{J}_2(e), \mathrm{J}_0(e)$ we always have $\mathcal{PQI}(\mathrm{B}) \subseteq \mathcal{PQI}(\mathrm{J})$, since if $z \in \mathcal{PQI}(\mathrm{B})$, $y \in \mathrm{J}$, then in the ideal case the square $y^{(2,z)} = U_y z \in \mathrm{K}$ is q.i. in $\mathrm{K}^{(z)}$, therefore in $\mathrm{J}^{(z)}$, so by P.Q.I. Definition 1.4.1(2) y itself is q.i. in $\mathrm{J}^{(z)}$ too and [by Symmetry 1.4.3(1)] z is

[8] It is a fact from general radical theory that *radical-freedom* $\mathcal{R}ad(\mathrm{J}) = \mathbf{0}$ is *always* inherited by ideals!

p.q.i. in J. In the Peirce case, similarly, for the Peirce projection $T = T^* = U_e$ or $U_{\hat{1}-e}$, the element $T(y) \in \mathrm{B}$ is q.i. in $\mathrm{B}^{(z)}$, therefore in $\mathrm{J}^{(z)}$, so by Structural Shifting 1.4.3(2) y itself is q.i. in $\mathrm{J}^{(T^*(z))} = \mathrm{J}^{(z)}$, too [since $z = T(z) \in \mathrm{J}_k(e)$]. Thus [by Symmetry again] z is p.q.i. in J.

(3) is an immediate consequence of (2), since J is an ideal in $\widehat{\mathrm{J}}$. (4) follows immediately from (2). For (5), any trivial z would be p.q.i. (even properly nilpotent, $z^{(2,y)} = U_z y = 0$ for all y), so $z \in \mathcal{PQI}(\mathrm{J}) = \mathcal{R}ad(\mathrm{J}) = \mathbf{0}$. □

1.7 Radical Surgery

For a "bad" property $\mathcal{P}$ of algebras (such as trivial, q.i., nil, degenerate, or nilpotent), the $\mathcal{P}$-radical $\mathcal{P}(\mathrm{J})$ is designed to isolate the bad part of the algebra so that its surgical removal (forming the quotient algebra $\mathrm{J}/\mathcal{P}(\mathrm{J})$, referred to as **radical surgery**) creates an algebra without badness: the radical is the smallest ideal whose quotient is "free of $\mathcal{P}$" (has no $\mathcal{P}$-ideals). On the other hand, for a "nice" property $\mathcal{P}$ of algebras (such as primitive or simple or prime), the $\mathcal{P}$-radical $\mathcal{P}(\mathrm{J})$ is designed to isolate the obstacle to niceness so that its surgical removal creates "semi-niceness": the radical is the smallest ideal whose quotient is **semi-$\mathcal{P}$** [= **subdirect product** of $\mathcal{P}$-algebras]. Then an algebra is semi-$\mathcal{P}$ iff $\mathcal{P}(\mathrm{J})$ vanishes. In the case of the Jacobson radical, the bad property is quasi-invertibility and the nice property is primitivity. Let us convince ourselves that removing the radical does indeed stamp out quasi-invertibility and create semiprimitivity.

Radical Surgery Theorem 1.7.1 *The quotient of a Jordan algebra by its radical is semiprimitive:*

$$\mathcal{R}ad\big(\mathrm{J}/\mathcal{R}ad(\mathrm{J})\big) = \overline{\mathbf{0}}.$$

PROOF. There is no q.i. ideal $\overline{\mathrm{I}}$ left in $\overline{\mathrm{J}} = \mathrm{J}/\mathcal{R}ad(\mathrm{J})$ because its preimage I would be q.i. in J [hence contained in the radical and $\overline{\mathrm{I}} = \overline{\mathbf{0}}$]: quasi-invertibility is a "recoverable" property in the sense that if $\mathrm{K} \subseteq \mathrm{I} \subseteq \widehat{\mathrm{J}}$ where both I/K and K have the property, so did I to begin with. Indeed, note first that hat interacts smoothly with quotient bars: $\overline{\widehat{\mathrm{J}}} = \big(\Phi\hat{1} \oplus \mathrm{J}\big)/\mathrm{K} = \Phi\bar{\hat{1}} \oplus \big(\mathrm{J}/\mathrm{K}\big) = \widehat{\overline{\mathrm{J}}}$. Since any $z \in \mathrm{I}$ becomes q.i. in $\overline{\mathrm{I}}$, in $\widehat{\overline{\mathrm{J}}} = \overline{\widehat{\mathrm{J}}}$ we have $\bar{\hat{1}} = \overline{U}_{\bar{\hat{1}}-\bar{z}}\bar{\hat{x}}$ for some $\hat{x} \in \widehat{\mathrm{J}}$, $\hat{1} = U_{\hat{1}-z}\hat{x} + k$ in $\widehat{\mathrm{J}}$ for some k in the q.i. K. Then $\hat{1} - k = U_{\hat{1}-z}\hat{x}$ is invertible, hence $\hat{1} - z$ is too [by Invertibility Criterion 1.1.1(3)(v)], and z is quasi-invertible. □

So far we have an idea of semi-niceness, but we will have to wait till Chapter 6 to meet the really nice primitive algebras. Now we derive some relations between radicals that will be needed for the last step of our summit assault, the Prime Dichotomy Theorem 9.2.1. In keeping with the above philosophy, we define the **degenerate radical** $\mathcal{D}eg(\mathrm{J})$ to be the smallest ideal whose quotient is nondegenerate. Such a smallest ideal exists: it is just the intersection $\mathrm{K} = \bigcap_\alpha \mathrm{K}_\alpha$ of *all* such nonzero ideals. Indeed, this ideal is clearly smallest, and it still has nondegenerate quotient: $U_{\bar z}\overline{\mathrm{J}} = \overline{\mathbf{0}} \Rightarrow U_z\mathrm{J} \subseteq \mathrm{K} \subseteq \mathrm{K}_\alpha \Rightarrow z \in \mathrm{K}_\alpha$ [for all α, since $\mathrm{J}/\mathrm{K}_\alpha$ is nondegenerate] $\Rightarrow z \in \bigcap_\alpha \mathrm{K}_\alpha = \mathrm{K}$.

Jacobson radical surgery creates nondegeneracy, rather brutally for most algebras, but for algebras with d.c.c. the surgery is non-invasive: it removes nothing but the degenerate radical.[9]

Radical Equality Theorem 1.7.2 (1) $\mathcal{R}ad(\mathrm{J})$ *contains all weakly trivial elements* z ($U_z\mathrm{J} = \mathbf{0}$), *but contains no nonzero vNrs* (*von Neumann regular elements* $x \in U_x\widehat{\mathrm{J}}$). *In particular, the radical contains no nonzero idempotents.*

(2) *The degenerate radical is always contained in the semiprimitive radical,* $\mathcal{D}eg(\mathrm{J}) \subseteq \mathcal{R}ad(\mathrm{J})$.

(3) *A nondegenerate algebra with d.c.c. on inner ideals is semiprimitive.*

(4) *The degenerate and semiprimitive radical coincide,* $\mathcal{D}eg(\mathrm{J}) = \mathcal{R}ad(\mathrm{J})$, *for algebras* J *which have the d.c.c. on inner ideals.*

PROOF. Set $\mathrm{R} := \mathcal{R}ad(\mathrm{J}), \mathrm{Z} := \mathcal{D}eg(\mathrm{J})$ for brevity. (1): If z is weakly trivial, then $z^{(2,y)} = U_z y = 0$ for all y, so z is properly nilpotent, hence properly quasi-invertible, and therefore lies in R. If $x \in (x]$ then by the vNr Pairing Lemma II.18.1.2 it is regularly paired $x \bowtie y$ with some $y \in \mathrm{J}$, $x = U_x y$, $y = U_y x$. But then $B_{x,y}(x) = x - \{x, y, x\} + U_x U_y x = x - 2U_x y + U_x y = x - 2x + x = 0$. If x is nonzero then $B_{x,y}$ kills x and is not invertible, so (x, y) is not q.i. [by Basic Q.I. Theorem 1.4.2(ii)], therefore x is not p.q.i., and so does not lie in the radical R [by Elemental Characterization 1.5.1]. All idempotents are clearly vNr's, $e = U_e e$ [or: no $e \neq 0$ is q.i., no $\hat{1} - e$ is ever invertible, because it kills its brother, $U_{\hat{1}-e}e = 0$], so no nonzero idempotents lie in the radical.

(2) By Radical Surgery 1.7.1 above, $\overline{\mathrm{J}} := \mathrm{J}/\mathrm{R}$ is a semiprimitive algebra, $\mathcal{R}ad(\overline{\mathrm{J}}) = \overline{\mathbf{0}}$, and so by (1) $\overline{\mathrm{J}}$ has no trivial elements. But then R creates nondegeneracy by surgery, hence contains the smallest ideal $\mathcal{D}eg(\mathrm{J})$ with that property.

(3) will follow as the particular case $\mathrm{J}' = \mathrm{J}$ of the following general result (which is what we really need for Dichotomy 9.2.1).

Radical Avoidance Lemma 1.7.3 *A nondegenerate Jordan* Φ*-algebra* J *avoids the radical of any larger Jordan* Φ'*-algebra* $\mathrm{J}' \supseteq \mathrm{J}$ *over* $\Phi' \supseteq \Phi$ *which has d.c.c. on principal inner* Φ'*-ideals* (*e.g., if* J' *is finite-dimensional over a field* Φ'): $\mathrm{J} \cap \mathcal{R}ad(\mathrm{J}') = \mathbf{0}$.

PROOF. If the intersection is *nonzero,* choose a principal inner Φ'-ideal $(x]' = U_x\widehat{\mathrm{J}}'$ of J' minimal among all those determined by elements $0 \neq x \in \mathrm{J} \cap \mathcal{R}ad(\mathrm{J}')$ (which we can do, thanks to the d.c.c. in J'). By *nondegeneracy* of J, x is not trivial in J, $0 \neq U_x\widehat{\mathrm{J}} \subseteq U_x\widehat{\mathrm{J}}' = (x]'$, so there exists $0 \neq$

[9] cf. the A–W–J Structure Theorem in I.4.11.

$y \in U_x\widehat{\mathrm{J}} \subseteq (x]' \cap \mathrm{J} \cap \mathcal{R}ad(\mathrm{J}')$ [since the ideal $\mathcal{R}ad(\mathrm{J}')$ is closed under inner multiplication] of the same type as x. But then $(y]' \subseteq (x]'$ by innerness [or directly, $(y]' := U_y\widehat{\mathrm{J}'} \subseteq U_x U_{\widehat{\mathrm{J}}} U_x \widehat{\mathrm{J}'} \subseteq U_x\widehat{\mathrm{J}'} =: (x]'$], so $(y]' = (x]'$ by minimality of $(x]'$, and therefore $y \in (x]' = (y]'$ is a nonzero vNr contained in $\mathcal{R}ad(\mathrm{J}')$, contradicting (1) above. □

Returning to the proof of (3), taking $\mathrm{J}' = \mathrm{J}, \Phi' = \Phi$ we see that a nondegenerate algebra with d.c.c. avoids its own radical, $\mathrm{J} \cap \mathcal{R}ad(\mathrm{J}) = \mathbf{0}$, i.e., $\mathcal{R}ad(\mathrm{J}) = \mathbf{0}$ and J is semiprimitive.

(4) follows from (3) by radical surgery: the reverse inclusion $\mathrm{Z} \supseteq \mathrm{R}$ to (2) follows because the q.i. ideal R/Z vanishes in the nondegenerate algebra J/Z by (3) [note that any quotient $\overline{\mathrm{J}} := \mathrm{J}/\mathrm{I}$ inherits the d.c.c. from J, since the inner ideals of $\overline{\mathrm{J}}$ are in 1–to–1 order–preserving correspondence with the inner ideals of J containing I]. □

Radical Avoidance is a poor man's way of avoiding having to prove that the radical is actually *nilpotent* in the presence of the d.c.c.

Exercise 1.7.3* Show that the Radical Avoidance Lemma goes through if J' has d.c.c. on those *open* principal inner ideals $(x)' := U_x\mathrm{J}'$ which are determined by elements $x \in \mathrm{J}$.

1.8 Problems for Chapter 1

Problem 1.1* Let A be an associative (real or complex) unital *Banach algebra* (a complete normed vector space with norm $\|x\|$ satisfying $\|\alpha x\| = \|\alpha\|\|x\|$, $\|x+y\| \leq \|x\| + \|y\|$, $\|xy\| \leq \|x\|\,\|y\|$). (1) Show that every element of norm $\|x\| < 1$ is quasi-invertible, with $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$. (2) Show that in a Banach algebra α lies in the resolvent of x (i.e., $\alpha 1 - x$ is invertible) for all α with $\alpha > \|x\|$. (3) Show more generally that the set of invertible elements is an open subset: if u is invertible, then so are all v with $\|u-v\| < \|u^{-1}\|^{-1}$.

Problem 1.2* (Amitsur's Polynomial Trick) If $x \in \mathrm{J}$ is such that the element tx is quasi-invertible in the unital Jordan algebra $\mathrm{J}[t] = \Phi[t] \otimes_\Phi \mathrm{J}$ of polynomials in the scalar indeterminate t, show that x must be nilpotent in J: the quasi-inverse must be the geometric series $(\hat{1} - tx)^{-1} = \sum_{n=0} t^n x^n$, and since this exists as a finite polynomial, we must eventually have $x^n = 0$.

Problem 1.3 (1) If u, v are inverses in a unital Jordan algebra, show that the operators U_u, U_v, V_u, V_v all commute, and hence all operators $U_{p(u)}, V_{q(u)}$ commute with all $U_{r(v)}, V_{s(v)}$ for polynomials p, q, r, s. Use $\frac{1}{2} \in \Phi$ to show that the subalgebra $\Phi[u, v]$ is a commutative associative linear algebra. (2) If z, w are quasi-inverses in a Jordan algebra, show that the operators U_z, U_w, V_z, V_w all commute, and therefore all operators $U_{p(z)}, V_{q(z)}$ commute with all $U_{r(w)}, V_{s(w)}$ for polynomials p, q, r, s; and the subalgebra $\Phi[z, w]$ is a commutative associative linear algebra.

Problem 1.4* (1) For quadratic Jordan algebras the argument of Elemental Characterization 1.5.1 establishes that $\mathcal{PQI}(\mathrm{J})$ is an outer ideal, but it requires more effort to prove that it is also inner. One needs a Fourth Principle for q.i. pairs: the *Homotope Shifting Principle*. Show that $(x, U_z y)$ is quasi-invertible in J iff (x, y) is quasi-invertible in $\mathrm{J}^{(z)}$ iff (w, z) is quasi-invertible in J for $w := \{x, z, y\} - U_x U_z U_y z$. (2) Use this Principle to show that $\mathcal{R}ad(\mathrm{J}^{(z)}) = \{x \in \mathrm{J} \mid U_z x \in \mathcal{R}ad(\mathrm{J})\}$. (3) Show that J is radical iff all $\mathrm{J}^{(z)}$ are radical.

Problem 1.5* (1) If u is invertible in a unital J, show that $u - x$ is invertible $\Longleftrightarrow x$ is q.i. in $\mathrm{J}^{(u^{-1})}$; conclude that x p.q.i. in a unital J implies that $u - x$ is invertible for all invertible u. (2) Show that x p.q.i. , y q.i. in J $\Longrightarrow x + y$ q.i. in J. (3) Conclude that x, y p.q.i. in J $\Longrightarrow x + y$ p.q.i. in J.

Problem 1.6 Show that J is radical ($\mathcal{R}ad(\mathrm{J}) = \mathrm{J}$) iff all $\mathrm{J}^{(z)}$ are radical.

Problem 1.7* A useful sidekick of Macdonald and Shirshov–Cohn is *Koecher's Principle*, which says that if a homogeneous Jordan polynomial $f(x_1, \ldots, x_n)$ vanishes whenever the x_i are *invertible* elements of *unital* Jordan algebras, then f vanishes for *all* elements x_i in *all* Jordan algebras. (1) Establish Koecher's Principle. (2) Show that Koecher's Principle for finite-dimensional algebras over an infinite field Φ follows from Zariski-density.

Problem 1.8* Bergmann Structurality 1.2.2 has an easy proof using Koecher's Principle. (1) Show as in the P.Q.I. Definition 1.4.1 that $B_{\alpha,x,y} = U^{(y)}_{\alpha\widehat{1^{(y)}}-x}$ for $\widehat{1^{(y)}}$ the formal unit for $\widehat{J^{(y)}} = \Phi\widehat{1^{(y)}} \oplus \mathrm{J}^{(y)}$, and as in the proof of 1.4.3(4) that $B_{\alpha,x,y}U_x = U_{\alpha x - U_x(y)} = U_x B_{\alpha,y,x}$. (2) Show that $[U_{B_{\alpha,x,y}(z)} - B_{\alpha,x,y}U_z B_{\alpha,y,x}]U_y = 0$ by the Fundamental Formula in $\widehat{J^{(y)}}$. (3) Show that the Bergmann structural identity holds for all $z \in U_x\mathrm{J}$. (4) Conclude that the Bergmann structural identity holds whenever x or y is invertible. Use Koecher's Principle to show that the identity holds for all elements of all Jordan algebras.

Problem 1.9* (1) Examine the proof of the Minimal Inner Ideal Theorem II.19.2.1 to show that every minimal inner ideal in a nondegenerate Jordan algebra contains an element which is regularly (not just structurally) paired with a nonzero idempotent. Conclude that the radical of a nondegenerate Jordan algebra with d.c.c. on inner ideals must vanish. (2) Argue directly from the d.c.c. that if $(c]$ (respectively, (c)) is minimal among nonzero principal (respectively, open principal) inner ideals in a nondegenerate Jordan algebra, then *every* one of its nonzero elements b is a vNr. (3) Conclude that in a nondegenerate Jordan algebra with d.c.c. on principal inner ideals $(x]$ (respectively, open principal inner ideals (x)), *every* nonzero inner ideal $\mathrm{I} \neq \mathbf{0}$ contains a von Neumann regular element (vNr) $b \neq 0$. (4) Deduce from this a slight improvement on Radical Equality Theorem 1.7.2(3): a nondegenerate Jordan algebra J with d.c.c. on principal inner ideals $(x]$ (respectively, open principal inner ideals (x)) is semiprimitive, $\mathcal{R}ad(\mathrm{J}) = \mathbf{0}$.

Question 1.1 Is Koecher's Principle valid for inhomogeneous f (so that *any* f which vanishes on invertible elements vanishes on all elements)?

Question 1.2 The Zariski topology (where the closed sets are the zero-sets of polynomial functions, so nonempty open sets are always dense) is usually defined only for finite-dimensional vector spaces over an infinite field. Can the restriction of finite-dimensionality be removed?

Question 1.3* Our entire treatment of quasi-invertibility has taken place in the formal unital hull $\widehat{\mathrm{J}} = \Phi\hat{1} \oplus \mathrm{J}$. What if J already has a perfectly good unit element? Can we replace $\hat{1}$ by 1 in all definitions and results, and obtain an analogous theory entirely within the category of unital Jordan algebras?

Question 1.4* It was easy to see that J/I inherits the d.c.c. from J. Does it always inherit the principal d.c.c. (with respect to either open (x), principal $(x]$, or closed $[x]$ principal inner ideals)?

2

Begetting and Bounding Idempotents

In this chapter we give some natural conditions on an algebra which guarantee that it has a finite capacity.[1] These are mild enough that they will automatically be satisfied by primitive i-exceptional algebras over big fields, and so will serve to cast such algebras back to the classical theory, where we know that they must be Albert algebras. First we formulate *axiomatically* the crucial property of having a rich supply of idempotents, and show that this property can be derived from *algebraicness*[2] rather than a *finiteness* condition. Once we are guaranteed a rich supply of idempotents, our second step is to insure that we don't have *too many*: all things in moderation, even idempotents. The reasonable restriction is that there be no infinite family of orthgonal idempotents. Thirdly, when we mix in semiprimitivity, these three conditions are sufficient to produce a capacity.

2.1 I-gene

An associative algebra is called *I-genic*, or an *I-algebra*, if every non-nilpotent element generates an idempotent as a left multiple, equivalently, if every non-nil left ideal contains an idempotent. In Jordan algebras we don't have left or right, but we do have inner and outer.

I-Genic Definition 2.1.1 (1) *An algebra is* **I-genic** (*idempotent-generating*), *if it has the* **I-gene** (*idempotent-generating*) *property that every non-nilpotent element b has an idempotent inner multiple, i.e., generates a nonzero idempotent in its principal inner ideal*:

[1] cf. I.18.1, where the concepts of this chapter (I-gene, I-finiteness) were introduced, and proofs sketched for the basic results (Algebraic-I Proposition, I-Finite Proposition, I-Finite Capacity Theorem).

[2] Most authors use the term *algebraicity* for the property of being algebraic, but I prefer a plebeian *ness* to an elevated *icity*: al–juh–bray–iss–it–tee is too much for my common tongue.

$$b \textit{ not nilpotent } \Longrightarrow (b] = U_b\widehat{\mathrm{J}} \textit{ contains } e \neq 0.$$

(2) *I-gene is equivalent to the condition that every non-nil inner ideal contains a nonzero idempotent,*

$$\mathrm{B} \textit{ not nil } \Longrightarrow \mathrm{B} \textit{ contains } e \neq 0.$$

Indeed, the second condition implies the first because if b is not nilpotent then $\mathrm{B} = (b]$ is not nil [it contains the non-nilpotent b^2], so contains an idempotent. Conversely, the first implies the second because any non-nil inner ideal B contains a non-nilpotent b, and $e \in (b]$ implies $e \in B$.

Exercise 2.1.1A* An associative algebra A is called a *left I-algebra* if every non-nilpotent element $x \in \mathrm{A}$ has a nonzero idempotent left multiple $e = ax$; dually for *right I-algebra.* (1) Show that A is a left I-algebra iff every non-nil left ideal contains a nonzero idempotent. (2) Show that A is a left I-algebra iff it is a right I-algebra; show that this happens if A^+ is I-genic. (3) Does left-I imply inner-I?

Exercise 2.1.1B* The Principal Inner Proposition II.5.3.1 shows that $(b) \subseteq (b] \subseteq [b],\ (b) \subseteq [b) \subseteq [b],\ U_{[b]}\mathrm{J} \subseteq (b)$ for the principal inner ideals $(b) := U_b(\mathrm{J}),\ (b] = U_b(\widehat{\mathrm{J}}),\ [b] = \Phi b + U_b(\widehat{\mathrm{J}})$ and the weak inner ideal $[b) := \Phi b + U_b(\mathrm{J})$. Show that we could have defined I-gene using weak inner ideals instead of all inner ones, by showing that the following are equivalent: there is a nonzero idempotent e in (1) every non-nil weak inner ideal; (2) every non-nil inner ideal; every principal inner ideal (3) (b); (4) $(b]$; (5) $[b)$; or (6) $[b]$ for non-nilpotent b.

I-gene need not be inherited by all *subalgebras*: if $\Phi[b]$ inherited I-gene it would imply that b generated an idempotent which is a *polynomial* in b, which is too strong a condition – too close to algebraicness. But I-gene is inherited by *inner ideals.*

I-Gene Inheritance Lemma 2.1.2 *In a Jordan algebra* J*, any* principal *inner ideal of an* inner *ideal is again* inner:

$$b \in \mathrm{B} \textit{ inner in } \mathrm{J} \Longrightarrow U_b(\widehat{\mathrm{B}}) \textit{ is inner in } \mathrm{J}.$$

I-gene is inherited by all inner ideals:

$$\mathrm{B} \textit{ inner in I-genic } \mathrm{J} \Longrightarrow \mathrm{B} \textit{ is I-genic.}$$

proof. The first follows from the invariance of inner ideals under structural transformations (Structural Innerness II.18.2.2), or directly from the Fundamental Formula: $U_{U_b(\widehat{\mathrm{B}})}(\widehat{\mathrm{J}}) = U_b U_{\widehat{\mathrm{B}}} U_b(\widehat{\mathrm{J}}) \subseteq U_b U_{\widehat{\mathrm{B}}}(\mathrm{B})$ [by innerness of B] $\subseteq U_b(\widehat{\mathrm{B}})$. [An *arbitrary* inner ideal of B need not remain inner in J; for example, $\mathrm{B}' = \Phi E_{12}$ in $\mathrm{B} = \Omega E_{12}$ in $\mathrm{J} = \mathcal{M}_2(\Omega)$.]

For the second part, if $b \in \mathrm{B}$ is non-nilpotent then the B-principal inner ideal $(b)_{\mathrm{B}} = U_b(\widehat{\mathrm{B}}) \subseteq \mathrm{B}$ is not nil (it contains b^2) and is inner in J by the first part, so by the I-Gene property 2.1.1(2) in J, it contains an idempotent. □

2.2 Algebraic Implies I-Genic

I-genic algebras are precisely those that are rich in idempotents, the very stuff of the classical approach. A crucial fact for Zel'manov's approach is that algebraicness, a condition of "local" finiteness at each individual element (every subalgebra $\Phi[x]$ generated by one element is finite-dimensional over Φ), suffices to produce the required idempotents.

Algebraic Definition 2.2.1 *An element x of a Jordan Φ-algebra is* **algebraic** *if it satisfies a* monic *polynomial with zero constant term,*

$$p(x) = 0 \text{ for some } p(t) = t^n + \alpha_{n-1}t^{n-1} + \cdots + \alpha_1 t^1 \in t\Phi[t].$$

For unital algebras the condition is equivalent to the condition that it satisfy a monic polynomial

$$q(x) = 0 \text{ for some } q(t) = t^n + \alpha_{n-1}t^{n-1} + \cdots + \alpha_0 1 \in \Phi[t],$$

since if x satisfies a general $q(t)$, it also satisfies $p(t) = tp(t)$ with zero constant term. Algebraicness just means that some power x^n can be expressed as a Φ-linear combination of lower powers. Notice in particular, that nilpotent *elements are always algebraic, as are* idempotents. *Over a field it suffices if x satisfies a* nonzero *polynomial, since any nonzero polynomial over a field can be made monic.*

An algebra is **algebraic** *if all of its elements are algebraic.*

Algebraic I Proposition 2.2.2 *Every algebraic algebra over a field is I-genic.*

PROOF. If b is non-nilpotent, then there is a nonzero idempotent in the subalgebra $C = \Phi[c]_0 \subseteq (b]$ of polynomials with zero constant term in $c = b^2 = U_b\hat{1} \in (b]$ [which is non-nilpotent since b is]: C is a finite-dimensional commutative associative algebra which is not nil, since it contains c, so by standard associative results it contains an idempotent. [Or directly: by finite-dimensionality the decreasing chain of subalgebras $C_n = \widehat{C}c^n$ eventually stabilizes, and when $C_n = C_{2n}$ we have $c^n = ac^{2n} = ac^nc^n$ for some $a \in \widehat{C}$, $e := ac^n \in C$ has $c^ne = ec^n = c^n \neq 0$, so $e \neq 0$ and $e^2 = ac^ne = ac^n = e$ is idempotent]. □

EXERCISE 2.2.2* An element y is *von Neumann regular* (a vNr, as in II.18.1.1) if $y \in (y)$, and *doubly vNr* if $y = U_{y^2}a \in (y^2)$, in which case by the Double vNr Lemma II.18.1.3 $e = U_{y^2}U_ay^2$ is a nonzero idempotent in (y). (1) If $(x^m) = (x^{4m+2})$ show that $y = x^{2m+1}$ is doubly vNr. (3) Prove *Morgan's Theorem* that if J has d.c.c. on the open principal inner ideals $\{(x^m)\}$, then J is I-genic. In particular, any Jordan algebra with d.c.c. on inner ideals is I-genic.

2.3 I-genic Nilness

In I-genic algebras, proper quasi-invertibility shrinks to nilness.

Radical Nilness Proposition 2.3.1 (1) *The Jacobson radical of an I-genic algebra reduces to the nil radical,*

$$\mathcal{R}ad(\mathrm{J}) = \mathcal{N}il(\mathrm{J}).$$

(2) *In particular, this holds for algebraic algebras over fields. More generally, as soon as an individual element of the Jacobson radical is algebraic, it must be nilpotent:*

$$x \in \mathcal{R}ad(\mathrm{J}) \textit{ algebraic over a field } \implies x \textit{ is nilpotent.}$$

(3) *Indeed, if we call an element* co-algebraic *if it satisfies a* co-monic *polynomial* (*one whose* lowest *term has coefficient* 1), *then for an arbitrary ring of scalars we have*

$$x \in \mathcal{R}ad(\mathrm{J}) \text{ co-algebraic over any } \Phi \implies x \text{ is nilpotent.}$$

PROOF. (1) If $x \in \mathcal{R}ad(\mathrm{J})$ were *not* nilpotent, then by I-gene it would generate a nonzero idempotent $e \in (x] \subseteq \mathcal{R}ad(\mathrm{J})$, contrary to the Radical Equality Theorem 1.7.2(1).

(3) Whenever we have a co-algebraic relation $x^n + \alpha_1 x^{n+1} + \cdots + \alpha_m x^{n+m} = 0$ we have $0 = x^n \bullet (\hat{1} + \alpha_1 x^1 + \cdots + \alpha_m x^m) = x^n \bullet (\hat{1} - z)$ for z a polynomial in x with zero constant term, therefore if x is radical so is z, and hence $\hat{1} - z$ is invertible. But then [by power-associativity in the commutative associative subalgebra $\Phi[x]$ with $\frac{1}{2} \in \Phi$] we also have $U_{\hat{1}-z} x^n = (\hat{1} - z) \bullet x^n \bullet (\hat{1} - z) = 0$, and we can cancel $U_{\hat{1}-z}$ to get nilpotence $x^n = 0$.

(2) follows from (3) since over a field algebraic is the same as co-algebraic [we can divide by the lowest nonzero coefficient in any monic polynomial to make it co-monic, and vice versa]. □

2.4 I-Finiteness

The crucial finiteness condition turns out to be the rather mild one that there not be an *infinite* family of *orthogonal* idempotents.[3] We denote *orthgonality* $e \bullet f = 0$ of idempotents by $e \perp f$.

I-Finite Proposition 2.4.1 (1) *An algebra* J *is defined to be idempotent-finite* (**I-finite**) *if it has no infinite orthogonal family* $e_1, e_2, \ldots$ *of nonzero idempotents.*

[3] The I-Finiteness Proposition and I-Finite Capacity Theorem were sketched in I.8.1.

(2) *We order idempotents according to whether one dominates the other under multiplication,*

$$g \geq f \iff g \bullet f = f \iff U_g f = f \iff f \in \mathrm{J}_2(g),$$
$$g > f \iff g \geq f,\ g \neq f,$$

equivalently, has larger Peirce 2*-space* (*hence smaller* 0*-space*),

$$g \geq f \iff \mathrm{J}_2(g) \supseteq \mathrm{J}_2(f) \implies \mathrm{J}_0(g) \subseteq \mathrm{J}_0(f),$$
$$g > f \iff \mathrm{J}_2(g) > \mathrm{J}_2(f) \implies \mathrm{J}_0(g) < \mathrm{J}_0(f).$$

An idempotent g is bigger than f iff it is built by adjoining to f an orthogonal idempotent e :

$$g \geq f \iff g = f + e \ \textit{ for } e \perp f,$$

equivalently, if

$$e := g - f \in \mathrm{J}_2(g) \cap \mathrm{J}_0(f).$$

(3) *I-finiteness is equivalent to the* **a.c.c. on idempotents**, *since strictly increasing families of idempotents in* J *are equivalent to nonzero orthogonal families:*

$$f_1 < f_2 < \cdots \ \textit{is a strictly increasing family}\ (f_{i+1} = f_i + e_i)$$
$$\iff e_1, e_2, \ldots \ \textit{is a nonzero orthogonal family}$$
$$0 \neq e_i = f_{i+1} - f_i \in \mathrm{J}_2(f_{i+1}) \cap \mathrm{J}_0(f_i).$$

(4) *I-finiteness implies the* **d.c.c. on idempotents** *too, since decreasing families are equivalent to bounded orthogonal families:*

$$g_1 > g_2 > \cdots \ \textit{is a strictly decreasing family}\ (g_i = g_{i+1} + e_i)$$
$$\iff e_1, e_2, \ldots \ \textit{are nonzero orthogonal, bounded above by}\ g_1$$
$$0 \neq e_i = g_i - g_{i+1} \in \mathrm{J}_2(g_1) \cap \mathrm{J}_2(g_i) \cap \mathrm{J}_0(g_{i+1}).$$

(5) *Any algebra with d.c.c. on inner ideals is I-finite.*

PROOF. The first equivalences in (2) follow from the characterizations of the Peirce space $\mathrm{J}_2(g)$ in the Eigenspace Laws II.8.1.4, or from the Shirshov–Cohn Principle: in an associative algebra, $2f = gf + fg \implies 4f = 2(gf + fg) = (g^2 f + gfg) + (gfg + fg^2) = gf + 2gfg + fg = 2f + 2gfg \implies f = fg = gf = gfg$, and conversely $f = gfg \implies gf = fg = gfg = f \implies gf + fg = 2f$. For the second equivalence, $f \in \mathrm{J}_2(g) \iff \mathrm{J}_2(f) = U_f \mathrm{J} \subseteq \mathrm{J}_2(g)$ [inner ideal] and $\hat{1} - g \leq \hat{1} - f$ in $\widehat{\mathrm{J}} \implies \mathrm{J}_0(f) = \widehat{\mathrm{J}}_2(\hat{1} - f) \cap \mathrm{J} \supseteq \widehat{\mathrm{J}}_2(\hat{1} - g) \cap \mathrm{J} = \mathrm{J}_0(g)$. For the third equivalence, if $g = f + e$ for $e \perp f$ then $g \bullet f = f \bullet f + e \bullet f = f + 0 = f$. Conversely, if $g \geq f$ then $e := g - f \in \mathrm{J}_2(g)$ is orthogonal to f (lies in $\mathrm{J}_0(f)$) since $e \bullet f = g \bullet f - f \bullet f = f - f = 0$; it is idempotent since

$e^2 = (g-f) \bullet e = e - 0 = e$. For the strict inequalities, $g > f \implies 0 \neq e \in \mathrm{J}_2(g) \cap \mathrm{J}_0(f) \implies \mathrm{J}_2(g) > \mathrm{J}_2(f)$ and $\mathrm{J}_0(g) < \mathrm{J}_0(f)$.

For (3), by (2) above an ascending chain of f_k leads to an orthogonal family $0 \neq e_k = f_{k+1} - f_k \in \mathrm{J}_2(f_{k+1}) \cap \mathrm{J}_0(f_k)$. Indeed, e_i, e_j are orthogonal if $i < j$ (i.e., $i+1 \leq j$) because $e_i \in \mathrm{J}_2(f_{i+1}) \subseteq \mathrm{J}_2(f_j)$ but $e_j \in \mathrm{J}_0(f_j)$, where $\mathrm{J}_2(f_j), \mathrm{J}_0(f_j)$ are orthogonal by the Peirce Orthogonality relation II.8.2.1, or by the Shirshov–Cohn Principle: in an associative algebra $e_i f_j = e_i,\ f_j e_j = 0$, so $e_i e_j = e_i f_j e_j = 0$, and dually $e_j e_i = 0$. Conversely, any orthogonal family of nonzero idempotents $\{e_i\}$ leads to an ascending chain $0 = f_1 < f_2 < \cdots$ for $f_{i+1} = e_1 + e_2 + \cdots + e_i$.

For (4), by (2) again a descending chain of g_k leads to an orthogonal family $0 \neq e_k = g_k - g_{k+1} \in \mathrm{J}_2(g_k) \cap \mathrm{J}_0(g_{k+1})$. Indeed, e_i, e_j are orthogonal if $i < j$ (i.e., $i+1 \leq j$) because $e_j \in \mathrm{J}_2(g_j) \subseteq \mathrm{J}_2(g_{i+1})$ but $e_i \in \mathrm{J}_0(g_{i+1})$, so are again orthogonal. Conversely, any othogonal family of nonzero idempotents $\{e_i\}$ bounded above by g leads to a descending chain $g = g_1 > g_2 > \cdots$ for $g_{i+1} = g - (e_1 + e_2 + \cdots + e_i)$.

For (5), I-finiteness is by (3) equivalent to the a.c.c. on idempotents $\{f_i\}$, which follows by (2) from the d.c.c. on inner ideals $\mathrm{J}_0(f_i)$. □

Notice that I-finiteness is an "elemental" condition, so it is automatically inherited by *all* subalgebras (whereas I-gene was inherited only by inner ideals).

Exercise 2.4.1* Show that in the presence of a unit the a.c.c. and d.c.c. on idempotents are equivalent, since $\{f_i\}$ is an increasing (respectively decreasing) family iff $\{1 - f_i\}$ is a decreasing (respectively increasing) family.

The next step shows that the classical finiteness condition, having a capacity, is a consequence of our generating property (I-gene) and our bounding property (I-finiteness).

I-Finite Capacity Theorem 2.4.2 *A semiprimitive Jordan algebra which is I-genic and I-finite necessarily has a capacity.*

proof. Semiprimitivity implies that the nil radical vanishes by Radical Definition 1.3.1(4). In particular, J can't be entirely nil, so by I-gene nonzero idempotents exist. From the d.c.c. on idempotents we get minimal idempotents e; we claim that these are necessarily division idempotents. Any non-nilpotent element b of $\mathrm{J}_2(e)$ generates a nonzero idempotent, which by minimality must be e; from $b = U_e b$ we get (without explicitly invoking any Peirce facts from Part II) $U_b = U_e U_b U_e, U_b U_e = U_b$, so $e \in (b] = U_b(\widehat{\mathrm{J}}) = U_b U_e(\widehat{\mathrm{J}}) = U_b(\mathrm{J}_2(e))$ and by Invertibility Criterion 1.1.1(4)(*iii*) b is invertible in $\mathrm{J}_2(e)$. Thus every element of $\mathrm{J}_2(e)$ is invertible or nilpotent.

We claim that by semiprimitivity there can't be any nilpotent elements at all in $\mathrm{J}_2(e)$. If z is nilpotent, x, y arbitrary in $\mathrm{J}_2(e)$, then $U_x U_z y$ can't be invertible [recall that $U_a b$ is invertible iff both a, b are invertible], so it must be nilpotent. Then the operator $U_{U_xU_zy} = (U_xU_zU_y)(U_zU_x)$ is nilpotent too; now in general ST is nilpotent iff TS is $[(ST)^n = 0 \Rightarrow (TS)^{n+1} = T(ST)^nS = 0]$, so $(U_zU_x)(U_xU_zU_y) = U_{U_z(x^2)}U_y = U_{z'}^{(y)}$ is nilpotent also. Then the element $z' := U_z(x^2)$ is nilpotent in the y-homotope for every y, i.e., is properly nilpotent and hence properly quasi-invertible, so by the Elemental Characterization Theorem 1.5.1(2) it lies in the radical. But $\mathrm{J}_2(e)$ inherits semiprimitivity by Hereditary Radical Theorem 1.6.1(4), so $z' = 0$. Linearizing $x \mapsto y, \frac{1}{2}e$ in $0 = U_z(x^2)$ gives $0 = U_z(e \bullet y) = U_zy$ for all $y \in \mathrm{J}_2(e)$. Since $\mathrm{J}_2(e)$ also inherits nondegeneracy by Hereditary Radical (4)–(5), we get $z = 0$ again.

Thus there are no nilpotent elements after all: every element of $\mathrm{J}_2(e)$ is invertible, $\mathrm{J}_2(e)$ is a division algebra, and e is a division idempotent.

Once we get division idempotents, we build an orthogonal family of them reaching up to 1: if $e = e_1 + \cdots + e_n$ is *maximal* among all idempotents which are *finite sums of division idempotents* e_i (such exists by the a.c.c.), we claim that $\mathrm{J}_0(e) = \mathbf{0}$. Indeed, $\mathrm{J}_0(e)$ inherits the three properties *semiprimitivity* [by Hereditary Radical (4) again], *I-gene* [by I-Gene Inheritance 2.1.2 since $\mathrm{J}_0(e)$ is an inner ideal], and *I-finiteness* [since $\mathrm{J}_0(e)$ is a subalgebra]. Thus if it were nonzero we could as above find a nonzero division idempotent $e_{n+1} \in \mathrm{J}_0(e)$ orthogonal to all the others, and $e + e_{n+1} > e$ would contradict the maximality of e. Therefore we must have $\mathrm{J}_0(e) = \mathbf{0}$, in which case e is the unit by the Idempotent Unit Theorem II.10.1.2 [stating that iff J is nondegenerate and e is an idempotent with nothing orthogonal to it, $\mathrm{J}_0(e) = \mathbf{0}$, then $e = 1$ is the unit for J].

Therefore $1 = e_1 + \cdots + e_n$ for division idempotents e_i, and we have reached our capacity. □

2.5 Problems for Chapter 2

Problem 2.1 Show that nilpotence is symmetric in any associative algebra A: for elements x, y the product xy is nilpotent iff the product yx is. Show that x is nilpotent in the homotope A_y (where $a \cdot_y b := ayb$) iff xy is nilpotent, so an element is properly nilpotent (nilpotent in all homotopes) iff all multiples xy (equivalently, all yx) are nilpotent. Conclude that any element of a nil one-sided ideal is properly nilpotent. Show that a nil ideal I is always properly nil: $\mathrm{I} \subseteq \mathcal{P}nil(\mathrm{A})$ (just as a q.i. ideal is always properly q.i.).

Problem 2.2* If $\mathrm{J} = \boxplus_{i=1}^{\infty} \Phi e_i$ for a field Φ, show that J has d.c.c. on all *principal* inner ideals, but not on *all* inner ideals, and is *not I-finite*, so we cannot weaken the condition in the I-Finite Proposition 2.4.1(5) to d.c.c. on principal inner ideals only.

3

Bounded Spectra Beget Capacity

The next stage in the argument shows that having a capacity is a consequence, for algebras over a suitably big field, of having a finite bound on the spectra of all the elements: the local finiteness condition of algebraicness appears as soon as we have a finiteness condition on spectra.[1]

In this chapter we will adopt an ambivalent attitude to unital hulls: our basic unital hull will be denoted by $\widetilde{\mathrm{J}}$ and its unit by $\tilde{1}$. These are defined to be the original algebra $\widetilde{\mathrm{J}} := \mathrm{J}$, $\tilde{1} := 1$ if J is already unital, and the formal unital hull $\widetilde{\mathrm{J}} := \widehat{\mathrm{J}}$, $\tilde{1} := \hat{1}$ if J is not unital. This distinction matters only in deciding whether or not 0 belongs to a spectrum.

3.1 Spectra

Let us recall the definitions, familiar to us from the theory of operators on Hilbert space. Shimshon Amitsur was once teaching an analysis course on differential equations when he realized that the notion of spectrum and resolvent made perfectly good sense in algebra, and was related to quasi-invertibility.

Φ–Spectrum Proposition 3.1.1 (1) *If* J *is a Jordan algebra over a field* Φ*, the* Φ–**spectrum** $\mathcal{S}pec_{\Phi,\mathrm{J}}(z)$ *of an element* $z \in \mathrm{J}$ *is defined to be the set of scalars* $\lambda \in \Phi$ *such that* $\lambda\tilde{1} - z$ *is not invertible in the unital hull* $\widetilde{\mathrm{J}}$*, or equivalently, the principal inner ideal* $U_{\lambda\tilde{1}-z}(\widetilde{\mathrm{J}})$ *can be distinguished from* $\widetilde{\mathrm{J}}$*,*

$$\begin{aligned}\mathcal{S}pec_{\Phi,\mathrm{J}}(z) &:= \{\lambda \in \Phi \mid \lambda\tilde{1} - z \textit{ not invertible in } \widetilde{\mathrm{J}}\}\\ &= \{\lambda \in \Phi \mid U_{\lambda\tilde{1}-z}(\widetilde{\mathrm{J}}) < \widetilde{\mathrm{J}}\}.\end{aligned}$$

The spectrum can also be defined by the action of $\lambda\tilde{1} - z$ *on* J *itself,*

[1] cf. I.8.6–8.8, where bigness and the Φ-,f-, and absorber-spectra were introduced, and proofs of the basic results (Spectral Relations Proposition, f-Spectral Bound Theorem, Big Resolvent Trick, Division Evaporation Theorem) were sketched.

$$\mathcal{S}pec_{\Phi,\mathrm{J}}(z) = \{\lambda \in \Phi \mid U_{\lambda\tilde{1}-z}(\mathrm{J}) < \mathrm{J}\}.$$

In the non-unital case $\widetilde{\mathrm{J}} = \widehat{\mathrm{J}}$ *and no* $z \in \mathrm{J}$ *is ever invertible, so always* $0 \in \mathcal{S}pec_{\Phi,\mathrm{J}}(z)$.

The **Φ-resolvent** $\mathcal{R}es_{\Phi,\mathrm{J}}(z)$ *of* z *is defined to be the complement of the spectrum, the set of* $\lambda \in \Phi$ *such that* $\lambda\tilde{1} - z$ is *invertible in the unital hull.*

(2) *Spectrum and resolvent are intimately related to quasi-invertibility*:

$$\text{For } \lambda = 0\text{:} \begin{cases} 0 \in \mathcal{R}es_{\Phi,\mathrm{J}}(z) & \Longleftrightarrow z \text{ is invertible in J},\\ 0 \in \mathcal{S}pec_{\Phi,\mathrm{J}}(z) & \Longleftrightarrow z \text{ is not invertible in J};\end{cases}$$
$$\text{For } \lambda \neq 0\text{:} \begin{cases} \lambda \in \mathcal{R}es_{\Phi,\mathrm{J}}(z) & \Longleftrightarrow \lambda^{-1}z \text{ is q.i.},\\ \lambda \in \mathcal{S}pec_{\Phi,\mathrm{J}}(z) & \Longleftrightarrow \lambda^{-1}z \text{ is not q.i.}\end{cases}$$

(3) *The nonzero part of the spectrum and resolvent are* independent of ideal extensions: *If* z *is an element of an ideal* I *of* J *then*

$$\mathcal{S}pec_{\Phi,\mathrm{I}}(z) \setminus \{0\} = \mathcal{S}pec_{\Phi,\mathrm{J}}(z) \setminus \{0\} \text{ and } \mathcal{R}es_{\Phi,\mathrm{I}}(z) \setminus \{0\} = \mathcal{R}es_{\Phi,\mathrm{J}}(z) \setminus \{0\}.$$

PROOF. (1) To see the equivalence between non-surjectivity of $U_{\lambda\tilde{1}-z}$ on $\widetilde{\mathrm{J}}$ and on J, for $\lambda \neq 0$ we can apply the Congruent to $\widehat{1}$ Lemma 1.2.3(3): for $\lambda \neq 0$ in the field Φ we can divide by λ and replace $\lambda\tilde{1} - z$ by $\tilde{1} - \lambda^{-1}z$, where surjectivity on $\widetilde{\mathrm{J}}$ and J coincide by the Lemma. As usual, it is only $\lambda = 0$ that causes us any trouble; here the equivalence is trivial in the unital case $\widetilde{\mathrm{J}} = \mathrm{J}$, and in the nonunital case $\widetilde{\mathrm{J}} = \widehat{\mathrm{J}}$ equivalence is easy because $\lambda = 0$ *always* belongs to the spectrum under either criterion: $U_{0-z}(\widehat{\mathrm{J}}) \subseteq \mathrm{J}$ is *never* $\widehat{\mathrm{J}}$ and $U_{0-z}(\mathrm{J})$ is *never* J [by the Surjective Unit II.18.1.4, equality would force J to be unital].

(2) For $\lambda \neq 0$ in a field Φ, $\lambda\tilde{1} - z$ is invertible iff $\tilde{1} - \lambda^{-1}z$ is invertible, i.e., iff $\lambda^{-1}z$ is quasi-invertible. Quasi-invertibility doesn't depend on the hull, or even on what ideal z belongs to: if $z \in \mathrm{I} \triangleleft \widetilde{\mathrm{J}}$ (e.g., I = J) has a quasi-inverse $w \in \widetilde{\mathrm{J}}$, then by Basic Q.I. Theorem 1.3.2(4)(iv) $w = U_z w - z^2 - z$ must belong to the ideal I too (c.f. Hereditary Radical Theorem 1.6.1(1)), and z is q.i. in I. This already establishes (3), which finesses the question about $\lambda = 0$.

For $\lambda = 0$, we have already seen in (1) that in the non-unital case 0 always belongs to the spectrum and never to the resolvent, and z is never invertible in J. In the unital case $\widetilde{\mathrm{J}} = \mathrm{J}$ we have by definition that $0 \in \mathcal{S}pec_{\Phi,\mathrm{J}}(z) \Longleftrightarrow 0 - z$ is not invertible in J $\Longleftrightarrow$ z is not invertible in J. □

EXERCISE 3.1.1A* Let J be a Jordan algebra over a field Φ. (1) [$\lambda = 0$] Show that $z \in \mathrm{J}$ is invertible in $\widetilde{\mathrm{J}}$ iff z is invertible in $\widetilde{\mathrm{J}} = \mathrm{J}$. (2) [$\lambda \neq 0$] Show that $\lambda\tilde{1} - z$ is invertible iff there is $w_\lambda \in \mathrm{J}$ with $U_{\lambda\tilde{1}-z}w_\lambda = \lambda z - z^2$, $U_{\lambda\tilde{1}-z}(w_\lambda)^2 = z^2$. (3) Show that $\mathcal{S}pec_{\mathrm{B}}(z) \supseteq \mathcal{S}pec_{\mathrm{J}}(z)$, $\mathcal{R}es_{\mathrm{B}}(z) \subseteq \mathcal{R}es_{\mathrm{J}}(z)$ for all elements z of all *unital* subalgebras B of J. (4) Give a counter-example to equality $\mathcal{S}pec_{\mathrm{I}}(z) = \mathcal{S}pec_{\mathrm{J}}(z)$, $\mathcal{R}es_{\mathrm{I}}(z) = \mathcal{R}es_{\mathrm{J}}(z)$ for an element z of an ideal I of J (naturally it will be $\lambda = 0$ that misbehaves).

EXERCISE 3.1.1B Let J be a unital Jordan algebra over a field Φ. (1) If $z \in \mathrm{J}$ is nilpotent, show that $1 - z$ is invertible with inverse given by the (short) geometric series; show that $\lambda\tilde{1} - z$ is invertible iff $\lambda \neq 0$. Conclude that $\mathcal{S}pec_{\Phi,\mathrm{J}}(z) = \{0\}$. (2) If $1 = e_1 + \cdots + e_n$ for orthogonal idempotents e_i, show that $x = \sum_i x_i$ $(x_i \in \mathrm{J}_2(e_i))$ is invertible in J iff each x_i is invertible in $\mathrm{J}_2(e_i)$; show that the spectrum of any $x = \sum_i \lambda_i e_i + z_i$ $(z_i \in \mathrm{J}_2(e_i)$ nilpotent) is $\mathcal{S}pec_{\Phi,\mathrm{J}}(x) = \{\lambda_1, \ldots, \lambda_n\}$.

f-Spectrum Definition 3.1.2 *If f is a Jordan polynomial which does not vanish strictly on* J*, the* **f-spectrum** *f-$\mathcal{S}pec_{\Phi,\mathrm{J}}(z)$ of z is defined to be the set of scalars such that the inner ideal $U_{\lambda\tilde{1}-z}(\mathrm{J})$ can be distinguished from* J *by the strict vanishing of f:*

$$f\text{-}\mathcal{S}pec_{\Phi,\mathrm{J}}(z) := \{\lambda \in \Phi \mid U_{\lambda\tilde{1}-z}(\mathrm{J}) \text{ satisfies } f \text{ strictly}\} \subseteq \mathcal{S}pec_{\Phi,\mathrm{J}}(z).$$

Here we say that a Jordan polynomial $f(x_1, \ldots, x_n)$ vanishes **strictly** *on* J *if f and all its linearizations f' vanish on* J*, $f'(a_1, \ldots, a_n) = 0$ for all $a_i \in \mathrm{J}$.*

We often use these concepts when J is a Jordan algebra over a ring of scalars Ω, and Φ is some subfield of it. When Φ is understood we leave it out of the notation, writing $\mathcal{S}pec_{\mathrm{J}}(z)$, etc.

3.2 Bigness

To a finite-dimensional algebra the real numbers look really big; Amitsur was the first to discover that amazing things happen whenever you work over a really big field.

Big Definition 3.2.1 *If* J *is a Jordan algebra over a field Φ, a set of scalars $\Phi_0 \subseteq \Phi$ will be called* **big** *(with respect to Φ and* J*) if its* cardinality *is infinite and greater than the* dimension *of the algebra*

$$|\Phi_0| \text{ is infinite, and } |\Phi_0| > \dim_\Phi(\mathrm{J}).$$

Because of this infiniteness, we still have

$$|\Phi_0 \setminus \{0\}| \geq |\Phi_0| - 1 > \dim_\Phi(\widehat{\mathrm{J}}).$$

When Φ and J *are understood, we say simply that Φ_0 is big. We will be particularly interested in the case where* J *is a* **Jordan algebra over a big field** *in the sense that $\Phi_0 = \Phi$ is itself big relative to* J.

Because we demand that a big Φ_0 be infinite, it won't be fazed by the fact that $\dim_\Phi(\widehat{\mathrm{J}})$ may be 1 larger than $\dim_\Phi(\mathrm{J})$, even if we take 1 away: the number of *nonzero* elements in Φ_0 is (at least) $|\Phi_0| - 1$, and this is strictly bigger than the dimension of the unital hull (which is where the resolvent inverses $(\lambda\tilde{1} - z)^{-1}$ live).

Amitsur's Big Resolvent Trick 3.2.2 (1) *If* J *is a Jordan algebra over a field* Φ*, any element* $z \in \mathrm{J}$ *with big resolvent is algebraic*:

$$|\mathcal{R}es_{\Phi,\mathrm{J}}(z)| \text{ big} \implies z \text{ algebraic over } \Phi,$$

and an element with small (for example, finite) spectrum over a big field has big resolvent and is algebraic:

$$|\mathcal{S}pec_{\Phi,\mathrm{J}}(z)| < |\Phi| \text{ big} \implies z \text{ algebraic.}$$

(2) *In particular, the radical of a Jordan algebra over a big field is always nil and coincides with the set of properly nilpotent elements*:

$$\mathcal{R}ad(\mathrm{J}) = \mathcal{N}il(\mathrm{J}) = \mathcal{P}nil(\mathrm{J}) = \mathcal{PQI}(\mathrm{J}) \qquad (\Phi \text{ big field}).$$

PROOF. (1) The reason that a big resolvent forces algebraicness is beautifully simple: bigness 3.2.1 implies that $|\mathcal{R}es_{\Phi,\mathrm{J}}(z))| > \dim_\Phi(\widehat{\mathrm{J}})$, and *there are too many inverses* $x_\lambda := (\lambda\tilde{1} - z)^{-1}$ in $\widehat{\mathrm{J}}$ ($\lambda \in \mathcal{R}es_{\Phi,\mathrm{J}}(z)$) *for all of them to be linearly independent* in $\widehat{\mathrm{J}}$ over Φ. Thus there must exist a *nontrivial linear dependence relation* over Φ among distinct x_{λ_i}. By clearing denominators this gives a *nontrivial algebraic dependence relation* for z : $\sum_i \alpha_i x_{\lambda_i} = 0 \implies 0 = \big(\prod_k L_{\lambda_k 1 - z}\big)\big(\sum_i \alpha_i(\lambda_i 1 - z)^{-1}\big) = \sum_i \alpha_i \prod_{k\neq i}(\lambda_k 1 - z) = p(z)$. The reason that this relation is nontrivial is that if $p(t)$ were the zero polynomial, then for each j we would have $0 = p(\lambda_j) = \sum_i \alpha_i \prod_{k\neq i}(\lambda_k - \lambda_j) = \alpha_j \prod_{k\neq j}(\lambda_k - \lambda_j)$ and thus α_j would be zero by the distinctness of the λ's, contradicting the nontriviality of the original linear dependence relation.

If z has small spectrum over a big field, then its resolvent must be big because it has the same size as $|\Phi|$: $|\mathcal{R}es_{\Phi,\mathrm{J}}(z)| = |\Phi| - |\mathcal{S}pec_{\Phi,\mathrm{J}}(z)| = |\Phi|$ [because $|\mathcal{S}pec_{\mathrm{J}}(z)| < |\Phi|$ and $|\Phi|$ is infinite].

(2) If $z \in \mathcal{R}ad(\mathrm{J})$, then it is p.q.i. by the Elemental Characterization 1.5.1(2); then all scalar multiples $\lambda^{-1}z$ are q.i., so by Φ–Spectrum Proposition 3.1.1(2)–(3) all $\lambda \neq 0$ fall in the resolvent, and $|\mathcal{R}es(z)| \geq |\Phi\backslash\{0\}| = |\Phi|$ is big. By (1) this forces z to be algebraic, and then by Radical Nilness 2.3.1(2) to be nilpotent. In view of Radical Definition 1.3.1(4), this gives the equality of the Jacobson and nil radicals. For the last equality, we always have $\mathcal{P}nil(\mathrm{J}) \subseteq \mathcal{PQI}(\mathrm{J})$ because in any homotope nil implies q.i.; conversely, if z is p.q.i. it remains p.q.i. in each homotope $\mathrm{J}^{(x)}$ [it is q.i. in $\mathrm{J}^{(U_x y)} = \big(\mathrm{J}^{(x)}\big)^{(y)}$ for all y], which is still an algebra over a big field, so by the first equality $z \in \mathcal{R}ad(\mathrm{J}^{(x)}) = \mathcal{N}il(\mathrm{J}^{(x)})$ is nilpotent in each $\mathrm{J}^{(x)}$, i.e., $z \in \mathcal{P}nil(\mathrm{J})$ is properly nilpotent. □

EXERCISE 3.2.2 (1) For quadratic Jordan algebras we cannot boost a relation among inverses into an algebraic relation by multiplying by L (and $V = 2L$ is bad in characteristic 2), so we must do our boosting by a U. (1) If $\{\lambda_1, \ldots, \lambda_n\}$ are distinct scalars and $\{\alpha_1, \ldots, \alpha_n\}$ are not all zero in the field Φ, show that $q(t) = \big(\prod_{i=1}^n U_{t-\lambda_i}\big)\big(\sum \alpha_i(t-\lambda_i)^{-2}\big)$ is a nonzero polynomial in $\Phi[t]$ with $q(\lambda_i) = \alpha_i \prod_{j\neq i}(\lambda_i - \lambda_j)^2$ for each i. (2) Use this to show that if there are too many inverses $(\lambda_i 1 - z)^{-2}$, then z is algebraic over arbitrary Φ (not necessarily containing $\frac{1}{2}$).

3.3 Evaporating Division Algebras

Although Zel'manov gave a surprising classification of all division algebras (which, you will remember, was an open question in the classical theory), in his final classification of prime algebras he was able to finesse the problem of division algebras entirely: if you pass to a suitable scalar extension, they disappear!

Division Evaporation Theorem 3.3.1 (1) *If* J *is a Jordan division algebra over* Ω *containing a big algebraically closed field* Φ, $|\Phi| > \dim_\Phi(\mathrm{J})$, *then* $\mathrm{J} = \Phi 1$.

(2) *Any division algebra of characteristic* $\neq 2$ *can be split by a scalar extension: if* $\mathrm{J} > \Gamma 1$ *is a division algebra over its centroid* Γ, *and* Φ *a big algebraically closed extension field of* Γ, $|\Phi| > \dim_\Gamma(\mathrm{J})$, *then the scalar extension* $\mathrm{J}_\Phi = \Phi \otimes_\Gamma \mathrm{J}$ *is central-simple over* Φ *but* not *a division algebra.*

PROOF. (1) If $x \notin \Phi 1$, then $x - \lambda 1 \neq 0$ for all λ in Φ and hence [since J is a division algebra] all $U_{x-\lambda 1}$ are invertible and all λ lie in the resolvent of x; by bigness of Φ with respect to Φ and J, the Big Resolvent Trick 3.2.2 guarantees that x is algebraic over Φ, so $p(x) = 0$ for some monic polynomial $p(t)$, which must factor as $p(t) = \prod_i (t - \lambda_i)$ for some $\lambda_i \in \Phi$ by the *algebraic closure* of Φ; but then $0 = U_{p(x)} 1 = \prod_i U_{x-\lambda_i 1} 1$ contradicts the fact that all $U_{x-\lambda_i 1}$ are invertible [cf. the Invertible Products Proposition II.6.1.8(3)]. Thus $\mathrm{J} = \Phi 1$.

(2) If J is a proper division algebra over Γ, and Φ an algebraically closed (hence infinite!) extension field with $|\Phi| > \dim_\Gamma(\mathrm{J})$, the scalar extension $\mathrm{J}_\Phi = \Phi \otimes_\Gamma \mathrm{J}$ remains central-simple over Φ by the Strict Simplicity Theorem II.1.7.1. But the dimension never grows under scalar extension, $\dim_\Phi(\mathrm{J}_\Phi) = \dim_\Gamma(\mathrm{J})$ [any Γ-basis for J remains a Φ-basis for J_Φ], so Φ will still be big for J_Φ. Furthermore, $\mathrm{J} > \Gamma 1$ implies that $\mathrm{J}_\Phi > \Phi 1$, so by the first part J_Φ cannot still be a division algebra. □

At first glance it seems a paradox that $\mathrm{J} = \Phi 1$ instead of $\mathrm{J} = \Omega 1$, but a little reflection or exercise will allay your suspicions; we must already have $\Omega = \Phi = \Gamma$: $\Phi 1$ is as far as J can go and remain a division algebra with center Γ!

3.4 Spectral Bounds and Capacity

Zel'manov showed that getting a bound on the spectra of elements produces a capacity, and non-vanishing of an s-identity f put a bound at least on the f-spectra. In retrospect, these two steps doomed all i-exceptional algebras to a finite-dimensional life.

Bounded Spectrum Theorem 3.4.1 (1) *If* $x = \sum_{i=1}^{n} \lambda_i e_i$ *for a family of nonzero orthogonal idempotents* e_i*, then* $\mathcal{S}pec_{\Phi,\mathrm{J}}(x) \supseteq \{\lambda_1, \ldots, \lambda_n\}$.

(2) *If there is a universal bound* $|\mathcal{S}pec_{\Phi,\mathrm{J}}(x)| \leq N$ *on spectra, where* Φ *contains at least* $N+1$ *distinct elements, then there is a universal bound* N *on the size of any family of orthogonal idempotents, in particular,* J *is I-finite.*

(3) *If* J *is a semiprimitive Jordan algebra over a big field with a uniform bound on all spectra,* $|\mathcal{S}pec_{\mathrm{J}}(z)| \leq N$ *for some finite* N *and all* $z \in \mathrm{J}$*, then* J *is algebraic with capacity.*

PROOF. (1) Making use of Peirce decompositions of II.1.3.4, we can write $\hat{1} = e_0 + \sum_{i=1}^{n} e_i$, so $\lambda\tilde{1} - x = \lambda e_0 + \sum_{i=1}^{n}(\lambda - \lambda_i)e_i$ is not invertible if $\lambda = \lambda_j$ because it kills the nonzero element e_j, $U_{\lambda_j \tilde{1} - x}(e_j) = 0$ [using the Peirce multiplication formulas in the commutative associative subalgebra $\Phi[e_1, \ldots, e_n] = \Phi e_1 \boxplus \cdots \boxplus \Phi e_n$].

(2) If there are $N+1$ distinct scalars λ_i, there can't be $N+1$ orthogonal idempotents e_i; otherwise, by (1) $x = \sum_{i=1}^{N+1} \lambda_i e_i$ would have at least $N+1$ elements in its spectrum, contrary to the bound N, so the longest possible orthogonal family has length at most N.

(3) Assume now that Φ is big. In addition to I-finiteness (2), by Amitsur's Big Resolvent Trick 3.2.2 the finite spectral bound N implies algebraicness, hence the Algebraic I Proposition 2.2.2 implies that J is I-genic. If we now throw semiprimitivity into the mix, the I-Finite Capacity Theorem 2.4.2 yields capacity. □

Zel'manov gave an ingenious combinatorial argument to show that a non-vanishing polynomial f puts a bound on that part of the spectrum where f vanishes strictly, the f-spectrum. This turns out to be the crucial finiteness condition: the finite degree of the polynomial puts a finite bound on this spectrum.

***f*-Spectral Bound Theorem** 3.4.2 (1) *If a polynomial* f *of degree* N *does not vanish strictly on a Jordan algebra* J*, then* J *can contain at most* $2N$ *inner ideals* B_k *where* f *does vanish strictly and which are* **relatively prime** *in the sense that*

$$\mathrm{J} = \sum_{i,j} \mathrm{C}_{ij} \qquad \textit{for} \qquad \mathrm{C}_{ij} = \bigcap_{k \neq i,j} \mathrm{B}_k.$$

(2) *In particular, in a Jordan algebra over a field a non-vanishing* f *provides a uniform bound* $2N$ *on the size of* f*-spectra,*

$$|f\text{-}\mathcal{S}pec_{\Phi,\mathrm{J}}(z)| \leq 2N.$$

PROOF. Replacing f by one of its homogeneous components, we can assume that $f(x_1, \ldots, x_n)$ is homogeneous in each variable and has total degree N, so its linearizations $f'(x_1, \ldots, x_{n'})$ are still homogeneous and have total degree N. By relative primeness we have

$$f'(\mathrm{J}, \ldots, \mathrm{J}) = f'\left(\sum \mathrm{C}_{ij}, \ldots, \sum \mathrm{C}_{ij}\right) = \sum f''(\mathrm{C}_{i_1 j_1}, \ldots, \mathrm{C}_{i_{n''} j_{n''}})$$

summed over further linearizations f'' of f'. For any collection $\{\mathrm{C}_{i_1j_1},\ldots,\mathrm{C}_{i_{n''}j_{n''}}\}$ ($n''\le N$), each C_{ij} avoids at most two indices i,j, hence lies in B_k for all others, so n'' of the C_{ij}'s avoid at most $2n''\le 2N$ indices, so *when there are more than* $2N$ *of the* B_k*'s then at least one index* k *is unavoided,* and $\mathrm{C}_{i_1j_1},\ldots,\mathrm{C}_{i_{n''}j_{n''}}$ all lie in this B_k (which depends on the collection of C's), and

$$f''(\mathrm{C}_{i_1j_1},\ldots,\mathrm{C}_{i_{n''}j_{n''}})\subseteq f''(\mathrm{B}_k,\ldots,\mathrm{B}_k)=\mathbf{0}$$

by the hypothesis that f *vanishes strictly on each individual* B_k. Since this happens for each f'' and each collection $\{\mathrm{C}_{i_1j_1},\ldots,\mathrm{C}_{i_{n''}j_{n''}}\}$, we see that $f'(\mathrm{J},\ldots,\mathrm{J})=\mathbf{0}$ for each linearization f', contrary to the hypothesis that f does *not* vanish strictly on J.

(2) In particular, there are at most $2N$ distinct λ's for which the inner ideals $\mathrm{B}_\lambda := U_{\lambda\tilde{1}-z}(\mathrm{J})$ satisfy f strictly, since the family $\{\mathrm{B}_{\lambda_k}\}_{k=1}^n$ is automatically relatively prime: the scalar interpolating polynomials

$$p_i(t)=\prod_{k\neq i}\frac{\lambda_k-t}{\lambda_k-\lambda_i}$$

of degree $n-1$ have $p_i(\lambda_j)=\delta_{ij}$, so $p(t)=\sum_i p_i(t)$ has $p(\lambda_k)=1$ for all k, $p(t)-1$ of degree $<n$ has n distinct roots λ_k, and so vanishes identically: $p(t)=1$. Substituting z for t yields $\mathrm{J}=U_{\tilde{1}}(\mathrm{J})=U_{\sum p_i(z)}(\mathrm{J})=\sum_{i,j}\mathrm{J}_{ij}$, where $\mathrm{J}_{ii}:=U_{p_i(z)}(\mathrm{J})\subseteq\bigcap_{k\neq i}U_{\lambda_k\tilde{1}-z}(\mathrm{J})=\bigcap_{k\neq i}\mathrm{B}_{\lambda_k}=\mathrm{C}_{ii}$, and $\mathrm{J}_{ij}=U_{p_i(z),p_j(z)}(\mathrm{J})\subseteq\bigcap_{k\neq i,j}U_{\lambda_k\tilde{1}-z}(\mathrm{J})=\bigcap_{k\neq i,j}\mathrm{B}_{\lambda_k}=\mathrm{C}_{ij}$. Thus $\mathrm{J}=\sum_{i,j}\mathrm{J}_{ij}\subseteq\sum_{i,j}\mathrm{C}_{ij}$. □

A good example of such relatively prime inner ideals are the Peirce inner ideals $\mathrm{B}_k:=\mathrm{J}_2(\sum_{i\neq k}e_i)$ for a supplementary orthogonal family of idempotents e_i in J. Here $\mathrm{C}_{ii}=\mathrm{J}_2(e_i)=\mathrm{J}_{ii},\mathrm{C}_{ij}=\mathrm{J}_2(e_i+e_j)=\mathrm{J}_{ii}+\mathrm{J}_{ij}+\mathrm{J}_{jj}$ (in the notation of the Multiple Peirce Decomposition Theorem II.13.1.4), so $\mathrm{J}=\sum_{i,j}\mathrm{C}_{ij}$ is the Peirce decomposition of J with respect to the family. In nice cases the B_k have faithful Peirce specializations on $\mathrm{J}_1(\sum_{i\neq k}e_i)$, and therefore are special and strictly satisfy Glennie's Identity G_8. In order for J to fail globally to satisfy $f=G_8$ of degree $N=8$ strictly, the number of relatively prime B_k would have to be bounded by $2N=16$: as soon as a nice algebra contains 17 mutually orthogonal idempotents, it is forced to satisfy Glennie's identity. *Any* finite bound from *any* non-vanishing s-identity will already be enough to force a nice i-exceptional algebras back to the classical finite-capacity case, where we know the answer: the only nice i-exceptional algebras are Albert algebras, so they in fact have at most three orthogonal idempotents.

We will have to relate the f-spectra to the ordinary spectra. This will take place only in the heart of the algebra, and will involve yet another kind of spectrum coming from the absorbers of inner ideals discussed in the next chapter.

3.5 Problems for Chapter 3

PROBLEM 3.1 (1) If J is an algebra over a big field, show that $|\mathrm{J}| = |\Phi|$. (2) Show in general that if J has dimension d over a field Φ of cardinality f, then $|\mathrm{J}| \leq \sum_{n=0}^{\infty}(df)^n$; if d or f is infinite, show that $|\mathrm{J}| \leq \max(d, f)$, while if d and f are both finite, then $|\mathrm{J}| = f^d$. (3) Conclude that always $|\mathrm{J}| \leq \max(d, f, \aleph_0)$.

PROBLEM 3.2 Let A be a unital algebra over an algebraically closed field Φ. (1) Show that if $a \in \mathrm{A}$ has $\lambda 1 - a$ invertible for *all* $\lambda \in \Phi$, then a is *not* algebraic over Φ, and the $(\lambda 1 - a)^{-1}$ are linearly *independent* over Φ, so $\dim_\Phi(\mathrm{A}) \geq |\Phi|$. (2) Conclude that there cannot be a proper field extension $\Omega \supset \Phi$ such that Φ is big with respect to the algebra Ω, i.e., $|\Phi| > \dim_\Phi(\Omega)$.

PROBLEM 3.3* Suppose V is a vector space over a field Γ, and let $\Phi \subseteq \Omega \subseteq \Gamma$ be subfields. (1) Show that we have $|\Gamma| \geq |\Omega| \geq |\Phi|$ and (denoting $\dim_\Phi(V)$ by $[V : \Phi]$, etc.) we have $[V : \Phi] = [V : \Omega][\Omega : \Phi] \geq [V : \Omega] = [V : \Gamma][\Gamma : \Omega] \geq [V : \Gamma]$. (2) Show that if J is an algebra whose centroid Γ is a field, and some subfield is big with respect to J, then any larger subfield is even bigger: if $\Phi \subseteq \Omega \subseteq \Gamma$ and $|\Phi| > [\mathrm{J} : \Phi]$, then $|\Omega| > [\mathrm{J} : \Omega]$ too for any intermediate subfield $\Phi \subseteq \Omega \subseteq \Gamma$.

PROBLEM 3.4* Prove the associative analogue of the f-Spectral Bound Theorem: (1) If a polynomial f of degree N does not vanish strictly on an associative algebra A, then A can contain at most N relatively prime left ideals B_k ($\mathrm{A} = \sum_i \mathrm{C}_i$ for $\mathrm{C}_i = \bigcap_{k \neq i} \mathrm{B}_k$) on which f vanishes strictly. (2) In particular, in an associative algebra over a field a non-vanishing f provides a uniform bound $|f\text{-}\mathcal{S}pec_{\Phi,\mathrm{A}}(z)| \leq N$ on the size of f-spectra $f\text{-}\mathcal{S}pec_{\Phi,\mathrm{A}}(z) := \{\lambda \in \Phi \mid \mathrm{A}(\lambda\tilde{1} - z) \textit{ satisfies } f \textit{ strictly}\}$. (3) Give an elementary proof of this spectral bound in the special case of the polynomial ring $\mathrm{A} = \Phi[s]$.

Second Phase: The Inner Life of Jordan Algebras

The key to Zel'manov's Exceptional Theorem for prime and simple algebras is the case of *primitive* algebras over *big algebraically closed fields*. Once we establish this, the final phase will be a mere logical mopping-up operation. In this Second Phase we examine the Jordan analogues of three associative concepts: absorber, primitivity, and heart. The absorber of an inner ideal plays the role of core, the primitizer is an inner ideal which render an algebra primitive, and the heart is the nonzero minimal ideal.

Chapter 4 focuses on the hero of this phase, the quadratic absorber. The linear absorber of an inner ideal absorbs linear multiplications by the ambient algebra, and by Zel'manov's specialization contains the i-specializing ideal consisting of all refugees from special algebras (values of s-identities). The quadratic absorber absorbs quadratic multiplications; it is the double linear absorber, and is not much smaller than the linear absorber itself (in a nondegenerate algebra, if the quadratic absorber vanishes, so must the linear absorber). It is not quite an ideal, but the ideal it generates is not much bigger (it is nil modulo the absorber by the Absorber Nilness Theorem); it contains the cube of the i-specializer (and in nondegenerate algebras, the ideal itself). As a result, absorberless inner ideals are i-special. The absorber also provides a new spectrum for an element x, the scalars for which the principal inner ideal $(\lambda 1 - x]$ is so small as to be absorberless.

Chapter 5 develops the basic facts of primitivity. The Semiprimitive Imbedding Theorem says that every nondegenerate Jordan algebra imbeds in a subdirect product of primitive algebras satisfying the same strict identities over a big algebraically closed field. A primitizer is a modular inner ideal which supplements all nonzero ideals; a modulus is an element c which acts as "outer unit," in the sense that $1 - c$ maps the whole algebra into the inner ideal. A proper inner ideal cannot contain its modulus (or any power or translate thereof), so the Absorber Nilness Theorem forces a primitizer to be absorberless.

Chapter 6 brings us to the heart of the matter. Every nonzero ideal in a primitive algebra contains the i-specializer, so an i-exceptional algebra has the i-specializer as its heart. A nonvanishing s-identity f puts a global bound on on f-spectra, absorber spectra, and ordinary spectra *of hearty elements*; over a big algebraically closed field, this forces the heart to have a simple capacity, whence the whole algebra is heart and falls under the control of the Clasical Structure Theory. Once the division algebras are removed from contention by the Division Evaporation Theorem over a big field, the only possibility that remains is an Albert algebra.

4

Absorbers of Inner Ideals

A key new concept is that of *outer* absorbers of *inner* ideals, analogous to *right* absorbers $r(L) = \{z \in L \mid z\mathrm{A} \subseteq L\}$ of *left* ideals L in associative algebras A. In the associative case these absorbers are ideals, precisely the cores of the one-sided ideals. In the Jordan case the quadratic absorber is only an *inner* ideal in J, but on the one hand, it is large enough to govern the obstacles to speciality, and on the other, it is close enough to being an ideal that the ideal in J it generates is nil modulo the original inner ideal.[1]

4.1 Linear Absorbers

The quadratic absorber is the double linear absorber, and the linear absorber has its origins in a specialization. The Peirce specialization of the inner ideals $\mathrm{J}_2, \mathrm{J}_0$ played an important role in the classical theory. Zel'manov discovered a beautiful generalization for *arbitrary* inner ideals.

Z-Specialization Definition 4.1.1 *The* **Z-specialization** *of an inner ideal* B *of* J *is the map from* B *into the associative algebra* $\mathcal{E}nd_\Phi(M)$ *of endomorphisms of the quotient* Φ*-module* $M := J/B$ *defined by*

$$Z(b) := \overline{V_b} \qquad (\overline{V_b}(\overline{x}) := \overline{V_b(x)} = \overline{\{b, x\}}).$$

Note that because B is a subalgebra we have $V_b(\mathrm{B}) \subseteq \mathrm{B}$; thus each V_b stabilizes the submodule B, and so induces a well-defined linear transformation $\overline{V_b}$ on the quotient module $M = \overline{J}$. (Recall our convention that J, B denote spaces with algebraic structure, but J, B are merely linear spaces.)

[1] cf. I.8.2 (also 8.6–8.7), where the concepts (i-specializer, absorber, absorber spectrum) of this chapter were introduced, and proofs of the results (Quadratic Absorber Theorem, Absorber Nilness Theorem, Spectral Relations Proposition) were sketched.

Z-Specialization Lemma 4.1.2 *Let* B *be an inner ideal of a Jordan algebra* J*, and $M = J/B$ the quotient Φ-module.*

(1) *The Z-specialization is a true specialization, a Jordan homomorphism of* B *into the special Jordan algebra $\mathcal{E}nd_\Phi(M)^+$:*

$$\overline{V_{b^2}} = \overline{V_b}^2, \qquad \overline{V_{U_b c}} = \overline{V_b}\,\overline{V_c}\,\overline{V_b}.$$

(2) *The kernel of this homomorphism is the* **linear absorber**

$$\mathrm{Ker}(Z) = \ell a(\mathrm{B}) := \{z \in \mathrm{B} \mid V_{\mathrm{J}} z \subseteq \mathrm{B}\},$$

which is an ideal of B *and an inner ideal of* J*,*

$$\ell a(\mathrm{B}) \lhd \mathrm{B}, \qquad \ell a(\mathrm{B}) \textit{ inner in } \mathrm{J}.$$

PROOF. (1) It suffices to prove that Z preserves squares, and this follows from the general formula $V_{b^2} = V_b^2 - 2U_b$ together with the fact that U_b disappears in the quotient, since it maps J down into B by definition of inner ideal. For linear Jordan algebras this implies that Z preserves U-products, but we can also see this directly for quadratic Jordan algebras from the Specialization Formula (FFIII)$'$, $V_{U_b c} = V_b V_c V_b - U_{b,c} V_b - U_b V_c$, where for $b, c \in \mathrm{B}$ both $U_{b,c}, U_b$ map J into B.

(2) Clearly, the kernel consists of those $z \in \mathrm{B}$ which act trivially on M because their V-operator shoves J down into B, $V_z \mathrm{J} = V_{\mathrm{J}} z \subseteq \mathrm{B}$ as in (2).

As the kernel of a homomorphism, $\ell a(\mathrm{B})$ is automatically an ideal of B. But it is also an inner ideal in all of J: if $b \in \ell a(\mathrm{B})$ and $x \in \mathrm{J}$ then $U_b x \in \ell a(\mathrm{B})$. Indeed, $V_{U_b x} = V_{b,x} V_b - U_b V_x$ [using Triple Switch (FFIV) in the Specialization Formula (FFIII)$'$] shoves J down into B because U_b does by innerness of B, and $V_{b,x} V_b$ does too, since first V_b shoves J down into B [don't forget that $b \in \ell a(\mathrm{B})$!], and then $V_{b,x}$ won't let it escape [$V_{b,x}(\mathrm{B}) = U_{b,\mathrm{B}} x \subseteq \mathrm{B}$ by innerness]. □

Note that M is just a module, with no further algebraic structure. When $\mathrm{B} = \mathrm{J}_2(e)$ is the Peirce inner ideal determined by an idempotent, the module M can be identified with the Peirce complement $\mathrm{J}_1(e) \oplus \mathrm{J}_0(e)$, and the Z-specialization is essentially the Peirce specialization on $\mathrm{J}_1(e)$ (with a trivial action on $\mathrm{J}_0(e)$ glued on). The linear absorber is the set of $b \in \mathrm{J}_2$ which kill J_1, which in this case is an ideal in all of J and is precisely the core (cf. Exercise 4.1.3B).

The linear absorber is overshadowed by its big brother, the quadratic absorber, which is a user-friendly substitute for the core of an inner ideal: it is the *square* of the linear absorber, and is *close* to being an ideal, since, as we will see later, the ideal it generates is not too far removed from B.

Absorbers Definition 4.1.3 (1) *The linear absorber* $\ell a(\mathrm{B})$ *of an inner ideal* B *in a Jordan algebra* J *absorbs linear multiplication by* J *into* B:

$$\ell a(\mathrm{B}) = \{z \in \mathrm{B} \mid L_{\mathrm{J}} z \subseteq \mathrm{B}\} = \mathrm{Ker}(Z).$$

The **double absorber** *absorbs two linear multiplications by* J *into* B:

$$\ell a^2(\mathrm{B}) := \ell a(\ell a(\mathrm{B})) = \{z \in \mathrm{B} \mid L_{\mathrm{J}} L_{\widehat{\mathrm{J}}} z \subseteq \mathrm{B}.\}$$

(2) *The* **quadratic absorber**[2] $qa(\mathrm{B})$ *absorbs quadratic multiplications by* J *into* B:

$$qa(\mathrm{B}) = \{z \in \mathrm{B} \mid V_{\mathrm{J},\widehat{\mathrm{J}}} z + U_{\mathrm{J}} z \subseteq \mathrm{B}\}.$$

(3) *The* **higher linear** *and* **quadratic absorbers** *are defined inductively by*

$$\ell a^n(\mathrm{B}) := \ell a(\ell a^{n-1}(\mathrm{B})), \qquad qa^n(\mathrm{B}) := qa(qa^{n-1}(\mathrm{B}));$$

these absorb strings of n *linear or quadratic products respectively. The absorbers of* B *are the same whether we take them in* J *or its unital hull* $\widehat{\mathrm{J}}$, *so there is no loss of generality in working always with unital* J.

Exercise 4.1.3A Let L be a left ideal and R a right ideal in an associative algebra A. Show that for the inner ideal $\mathrm{B} = \mathrm{L} \cap \mathrm{R}$ of A^+, the linear and quadratic absorbers coincide with the *core* of B (the largest associative ideal of A contained in B): $\ell a(\mathrm{B}) = qa(\mathrm{B}) = \{z \in \mathrm{B} \mid z\mathrm{A} \subseteq \mathrm{L},\ \mathrm{A}z \subseteq \mathrm{R}\}$.

Exercise 4.1.3B Show that for a Peirce inner ideal $\mathrm{B} = U_e \mathrm{J} = \mathrm{J}_2(e)$ determined by an idempotent in a Jordan algebra, the core of B (the largest Jordan ideal of J contained in B) is $\ell a(\mathrm{B}) = qa(\mathrm{B}) = \{z \in \mathrm{B} \mid z \bullet \mathrm{J}_1(e) = \mathbf{0}\}$, which is just the kernel of the *Peirce specialization* $x \mapsto V_x$ of B in $\mathcal{E}nd_\Phi(\mathrm{J}_1(e))^+$, so $\mathrm{B}/\ell a(\mathrm{B}) \cong V_{\mathrm{J}_2(e)}|\mathrm{J}_1(e)$.

The absorbers were born of specialization, and in turn give birth to speciality: the linear absorber contains all obstacles to speciality and i-speciality.

Specializer Definition 4.1.4 *The* **specializer** $\mathcal{S}pecializer(\mathrm{J})$ *of any Jordan algebra* J *is the smallest ideal of* J *whose quotient is special*; J *is special iff* $\mathcal{S}pecializer(\mathrm{J}) = \mathbf{0}$. *The* **i-specializer** $i\text{-}\mathcal{S}pecializer(\mathrm{J})$ *is the smallest ideal of* J *whose quotient is* **identity-special** (**i-special**); J *is i-special iff* $i\text{-}\mathcal{S}pecializer(\mathrm{J}) = \mathbf{0}$. $i\text{-}\mathcal{S}pecializer(\mathrm{J})$ *consists precisely of all values* $f(\mathrm{J}, \dots, \mathrm{J})$ *on* J *of all s-identities* $f(x_1, \dots, x_n)$ (*all those Jordan polynomials that should have vanished if* J *were special, constituting the ideal* $i\text{-}\mathcal{S}pecializer(X)$ *of all s-identities in the free Jordan algebra* $\mathcal{FJ}[X]$ *on a countable number of indeterminates* X). *Since it is easier to be i-special than to be special, we don't need to remove as much of the algebra to create i-speciality, and we always have* $i\text{-}\mathcal{S}pecializer(\mathrm{J}) \subseteq \mathcal{S}pecializer(\mathrm{J})$.

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[2] Like *quasi-inverse qi*, *quadratic absorber* is written aperiodically as qa and pronounced "kyoo-ay." In the same way, *linear absorber* is written ℓa and pronounced "ell-ay."

Exercise 4.1.4* An ideal I is *co-special* or *co-i-special* if its quotient J/I is special or i-special. (1) Show that a smallest co-special ideal always exists: an arbitrary intersection of co-special ideals is co-special, so that $\mathcal{S}pecializer(\mathrm{J}) = \bigcap\{\mathrm{I} \mid \mathrm{I}\ \textit{is co-special}\}$. (2) Show that a smallest co-i-special ideal always exists: an arbitrary intersection of co-i-special ideals is co-i-special, so that $i\text{-}\mathcal{S}pecializer(\mathrm{J}) = \bigcap\{\mathrm{I} \mid \mathrm{I}\ \text{is co-i-special}\}$.

4.2 Quadratic Absorbers

We can't obtain the properties of the quadratic absorber quite so easily and naturally.

Quadratic Absorber Theorem 4.2.1 (1) *The linear and quadratic absorbers of an inner ideal* B *in a Jordan algebra* J *are ideals in* B*, and they together with all higher absorbers are again inner ideals in* J*. The linear absorber also absorbs obstacles to speciality: we have* **Specializer Absorption**

$$i\text{-}\mathcal{S}pecializer(\mathrm{B}) \subseteq \mathcal{S}pecializer(\mathrm{B}) \subseteq \ell a(\mathrm{B}) \quad (\mathrm{B}/\ell a(\mathrm{B})\ \textit{is special}).$$

(2) *The double absorber coincides with the quadratic absorber,*

$$qa(\mathrm{B}) = \ell a^2(\mathrm{B}).$$

(3) *The linear absorber already absorbs* $V_{\mathrm{J},\widehat{\mathrm{B}}}$,

$$V_{\mathrm{J},\widehat{\mathrm{B}}}(\ell a(\mathrm{B})) = U_{\mathrm{J},\ell a(\mathrm{B})}(\widehat{\mathrm{B}}) \subseteq \mathrm{B},$$

and we have an **Absorber Boosting Principle**: *internal multiplications boost the absorbent power,*

$$i\text{-}\mathcal{S}pecializer(\mathrm{B})^3 \subseteq i\text{-}\mathcal{S}pecializer(\mathrm{B})^2 \subseteq U_{\ell a(\mathrm{B})}(\widehat{\mathrm{B}}) \subseteq qa(\mathrm{B}),$$
$$U_{qa(\mathrm{B})}(\widehat{\mathrm{B}}) \subseteq qa^2(\mathrm{B}).$$

(4) *If* J *is nondegenerate and the quadratic absorber of an inner ideal vanishes, then its i-specializer and linear absorber vanish too, and its* Z*-specialization is injective*:

$$qa(\mathrm{B}) = \mathbf{0} \Rightarrow \ell a(\mathrm{B}) = i\text{-}\mathcal{S}pecializer(\mathrm{B}) = \mathbf{0}.$$

Proof. (2) Since $\frac{1}{2} \in \Phi$ the double and quadratic absorbers coincide: all $L_x,\ L_x L_{\hat{y}}$ map z into B $\iff$ all $V_x = 2L_x,\ V_x V_y = 4L_x L_y$ do $\iff$ all $V_x,\ 2U_x = V_x^2 - V_{x^2},\ V_{x,y} = V_x V_y - U_{x,y}$ do $\iff$ all $U_x,\ V_{x,\hat{y}}$ do.

(1) The Jordan algebra $\mathrm{B}/\ell a(\mathrm{B})$ is manifestly special, since by Z-Specialization 4.1.2(2) Z imbeds $\mathrm{B}/\ell a(\mathrm{B})$ in the special Jordan algebra $\mathcal{E}nd_\Phi(J/B)^+$, and therefore $\ell a(\mathrm{B})$ swallows up the obstacle $\mathcal{S}pecializer(\mathrm{B})$ and hence $i\text{-}\mathcal{S}pecializer(\mathrm{B})$ as well.

We saw that the linear absorber is an ideal in B and an inner ideal in J in Z-Specialization (2). Thus all higher absorbers $\ell a^n(\mathrm{B})$ are also inner [but only ideals in $\ell a^{n-1}(\mathrm{B})$, not all of B]. This shows that the quadratic absorber too, as somebody's linear absorber by (2), is an inner ideal in J. Thus again the higher absorbers are inner ideals.

Though it doesn't arise as a kernel, the quadratic absorber is nevertheless an ideal in B: if $z \in qa(\mathrm{B})$ and $b \in \mathrm{B}$ then $V_b z \in qa(\mathrm{B})$ because it absorbs V's by $V_{x,\hat{y}}(V_b z) = (V_b V_{x,\hat{y}} + V_{V_{x,\hat{y}}(b)} - V_{b,\{\hat{y},x\}})z$ [from the 5-Linear Identity (FFV)$'$ with $z, w \mapsto b, 1$] $\subseteq V_b(\mathrm{B}) + \mathrm{B} - \mathrm{B}$ [since z absorbs $V_{\mathrm{J},\widehat{\mathrm{J}}}$] $\subseteq \mathrm{B}$, and it absorbs U's by $U_x(V_b z) = (U_{V_b(x),x} - V_b U_x)z$ [from Fundamental Lie (FFV)] $\subseteq \mathrm{B} - V_b\mathrm{B} \subseteq \mathrm{B}$ [since z absorbs U_J]. [In contrast to linear absorbers, the higher quadratic absorbers *are* ideals in B, see Problem 4.1.]

(3) For the first part, for $z \in \ell a(\mathrm{B})$ we have $\{\mathrm{J}, \hat{1}, z\} = \{\mathrm{J}, z\} \subseteq \mathrm{B}$ by absorption and $\{\mathrm{J}, \mathrm{B}, z\} = \{\{\mathrm{J}, \mathrm{B}\}, z\} - \{\mathrm{B}, \mathrm{J}, z\}$ [by Triple Switch (FFIVe)] $\subseteq \{\mathrm{J}, z\} - \{\mathrm{B}, \mathrm{J}, \mathrm{B}\} \subseteq \mathrm{B}$ by absorption and innerness, so together $\{\mathrm{J}, \widehat{\mathrm{B}}, z\} \subseteq \mathrm{B}$.

For the second part, to see that products boost the absorptive power, let $z \in \ell a(\mathrm{B}), \hat{b} \in \widehat{\mathrm{B}}$. Then $U_z\hat{b}$ absorbs $V_{x,\hat{y}}$'s because $V_{x,\hat{y}}(U_z\hat{b}) = (U_{V_{x,\hat{y}}(z),z} - U_z V_{\hat{y},x})\hat{b}$ [by Fundamental Lie again] $\subseteq U_{\mathrm{J},z}(\widehat{\mathrm{B}}) - U_\mathrm{B}(\mathrm{J}) \subseteq \mathrm{B}$ by the first part and by innerness of B. Moreover, $U_z\hat{b}$ also absorbs U_x's because $U_x(U_z\hat{b}) = (U_{\{x,z\}} + U_{x^2,z^2} - U_z U_x - U_{x,z}^2)\hat{b}$ [from Macdonald or linearized Fundamental $U_{x^2} = U_x^2$] $\subseteq U_\mathrm{B} J + U_{\mathrm{J},\ell a(\mathrm{B})}\hat{b} - U_\mathrm{B}\mathrm{J} - U^2_{\mathrm{J},\ell a(\mathrm{B})}\hat{b} \subseteq \mathrm{B}$ by innerness and the absorption of the first part. Thus $U_z\hat{b}$ absorbs its way into $qa(\mathrm{B})$ as in the third inclusion. The first two inclusions then follow from $i\text{-}\mathcal{S}pecializer(\mathrm{B}) \subseteq \ell a(\mathrm{B})$, since then $i\text{-}\mathcal{S}pecializer(\mathrm{B})^3 \subseteq i\text{-}\mathcal{S}pecializer(\mathrm{B})^2 \subseteq U_{\ell a(\mathrm{B})}\widehat{\mathrm{B}}$.

For the third and final inclusion, that $w := U_z\hat{b} \in qa^2(\mathrm{B}) = qa(\mathrm{Q})$ is even more absorptive when $z \in \mathrm{Q} := qa(\mathrm{B})$, note that w absorbs $V_{x,\hat{y}}$ as always from Fundamental Lie by $V_{x,\hat{y}}(U_z\hat{b}) = (U_{V_{x,\hat{y}}(z),z} - U_z V_{\hat{y},x})\hat{b} \subseteq U_{\mathrm{B},\mathrm{Q}}(\widehat{\mathrm{B}}) - U_\mathrm{Q}(\mathrm{J}) \subseteq \mathrm{Q}$ by *idealness and innerness* of Q established in (2). But to show that w absorbs U_x we cannot use our usual argument, because we do not know that $U_{\{x,z\}}\hat{b} \in \mathrm{Q}$. So we go back to the definition: $U_x w \in \mathrm{Q} = qa(\mathrm{B})$ because it absorbs all $V_{r,\hat{s}}, U_r$ for $r \in \mathrm{J}$, $\hat{s} \in \widehat{\mathrm{J}}$. Indeed, it absorbs V's because $V_{r,\hat{s}}(U_x w) = V_{r,\hat{s}}(U_x U_z\hat{b}) \subseteq (U_{x',x}U_z - U_x U_{z',z} + U_x U_z V_{r,\hat{s}})\hat{b}$ [using Fundamental Lie twice] $\subseteq U_\mathrm{J} U_z(\mathrm{J}) - U_\mathrm{J} U_{\mathrm{B},z}(\widehat{\mathrm{B}}) + U_\mathrm{J} U_z(\mathrm{J}) \subseteq U_\mathrm{J}\mathrm{Q}$ [by *idealness and innerness* of Q again] $\subseteq \mathrm{B}$ [since Q absorbs U_J by definition]. To see that $U_x w$ also absorbs U's, note that $U_r(U_x w) = U_r(U_x U_z\hat{b}) \subseteq \big(U_{\{r,x,z\}} + U_{U_r U_x z, z} - U_z U_x U_r - (U_{V_{r,x}(z),z} - U_z V_{x,r})V_{x,r}\big)\hat{b}$ [by Alternate Fundamental (FFI)$'$ and Fundamental Lie] $\subseteq U_\mathrm{B}(\widehat{\mathrm{J}}) + U_{\mathrm{J},qa(\mathrm{B})}(\widehat{\mathrm{B}})$ [since z absorbs $V_{r,x}$] $\subseteq \mathrm{B}$ [by innerness and the first part of (3)]. Thus we can double our absorption, double our fun, by U-ing.

(4) If J is nondegenerate and $qa(\mathrm{B}) = \mathbf{0}$, we must show that all $z \in \ell a(\mathrm{B})$ vanish. By nondegeneracy it suffices if all $w = U_z x$ vanish for $x \in \mathrm{J}$, or in turn

if all $U_w = U_z U_x U_z$ vanish. But here $U_z U_x U_z = U_z(U_{\{x,z\}} + U_{x^2,z^2} - U_z U_x - U^2_{x,z})$ [by linearized $U_{x^2} = U_x U_x$ again] $= U_z U_{\{x,z\}} + U_z U_{x^2,z^2} - U_{z^2} U_x - (V_{z^2} V_{z,x} - V_{z^3,x}) U_{x,z}$ [using Macdonald to expand $U_z U_{z,x}$]; this vanishes because $z^2, z^3 \in U_z \widehat{\mathrm{B}} \subseteq qa(\mathrm{B}) = \mathbf{0}$ by (3), and $U_z U_{\{x,z\}}(\mathrm{J}) \subseteq U_z U_{\mathrm{B}}(\mathrm{J}) \subseteq U_{\ell a(\mathrm{B})}(\mathrm{B})$ [by innerness] $\subseteq qa(\mathrm{B})$ [by (3) again] $= \mathbf{0}$ too. Thus nondegeneracy allows us to go from *i-Specializer*$(\mathrm{B})^2 = \mathbf{0}$ in (3) to *i-Specializer*$(\mathrm{B}) = \mathbf{0}$ in (4). □

We emphasize cubes rather than squares of ideals, since the square is not an ideal but the cube always is. In general, the product $\mathrm{B} \bullet \mathrm{C}$ of two ideals is not again an ideal, since there is no identity of degree 3 in Jordan algebras to re-express $x \bullet (b \bullet c)$, but the Jordan identity of degree 4 is enough to show that $U_{\mathrm{B}}(\mathrm{C})$ is again an ideal, since by Fundamental Lie $x \bullet U_b c = U_{x \bullet b, b} c - U_b(x \bullet c) \in U_{\mathrm{B}}(\mathrm{C})$.

Exercise 4.2.1A* Ignore the linear absorber and show directly that the quadratic absorber is an inner ideal in J: if $z \in qa(\mathrm{B}), r \in \mathrm{J}, \hat{x}, \hat{s} \in \widehat{\mathrm{J}}$, then $U_z \hat{x}$ absorbs double $V_{r,s}$ by Fundamental Lie (FFV), and U_r by $U_{\{r,z\}} + U_{U_r z, z} = U_r U_z + +U_z U_r + (U_{\{r,z\},z} - U_z V_r) V_r$.

Exercise 4.2.1B* Show that $V_{\mathrm{J},\widehat{\mathrm{J}}} V_{\mathrm{B}} \subseteq V_{\widehat{\mathrm{B}}} V_{\mathrm{J},\widehat{\mathrm{J}}}$ and $U_{\mathrm{J}} V_{\mathrm{B}} \subseteq V_{\widehat{\mathrm{B}}} U_{\mathrm{J}}$.

The new notion of absorber brings with it a new notion of spectrum, the *absorber spectrum* consisting of those λ for which $U_{\lambda \tilde{1} - z}(\mathrm{J})$ is not merely different from J (as in the ordinary spectrum), or distinguishable from J by means of some f (as in the f-spectrum), but so far from J that its absorber is zero. For convenience, we denote the i-specializer of the free Jordan algebra $\mathcal{FJ}[X]$ by *i-Specializer*(X); it consists of all s-identities in the variables X, all Jordan polynomials which vanish on all special algebras (equivalently, on all associative algebras).

Absorber Spectrum Definition 4.2.2 *$\mathcal{A}bs\mathcal{S}pec_{\Phi,\mathrm{J}}(z)$, the* **$\Phi$-absorber spectrum** *of an element z, is the set of scalars giving rise to an absorberless inner ideal:*

$$\mathcal{A}bs\mathcal{S}pec_{\Phi,\mathrm{J}}(z) := \{\lambda \in \Phi \mid qa(U_{\lambda \tilde{1} - z}(\mathrm{J})) = \mathbf{0}\}.$$

Note that since $U_{\lambda \hat{1} - z} = U_{\lambda \tilde{1} - z}$ on J (and equals $U_{\lambda 1 - z}$ if J is unital), the absorber spectrum is indifferent to our choice of unit $1, \hat{1}$, or $\tilde{1}$.

Spectral Relations Proposition 4.2.3 *If f is an s-identity in the cube i-Specializer$(X)^3$ (if J is nondegenerate we can even take any f from the ideal i-Specializer(X) itself) which does not vanish strictly on an i-exceptional algebra J, then we have the following* **Spectral Relations** *for any element $z \in \mathrm{J}$:*

$$\mathcal{A}bs\mathcal{S}pec_{\Phi,\mathrm{J}}(z) \subseteq f\text{-}\mathcal{S}pec_{\Phi,\mathrm{J}}(z) \subseteq \mathcal{S}pec_{\Phi,\mathrm{J}}(z).$$

PROOF. By Quadratic Absorber 4.2.1(3)–(4), an absorberless inner ideal strictly satisfies *all* $f \in i\text{-}\mathcal{S}pecializer(X)^3$ (even all $f \in i\text{-}\mathcal{S}pecializer(X)$ if J is nondegenerate), so for λ in the absorber spectrum of z the inner ideal $U_{\lambda\tilde{1}-z}(\mathrm{J})$ is absorberless, hence it satisfies all such f strictly, so $\lambda \in f\text{-}\mathcal{S}pec_{\Phi,\mathrm{J}}(z)$ by the f-Spectrum Definition 3.1.2. We already know the last inclusion by the f-Spectrum Theorem. □

EXERCISE 4.2.3 Show directly from the definitions that $\mathcal{A}bs\,\mathcal{S}pec_{\Phi,\mathrm{J}}(z) \subseteq \mathcal{S}pec_{\Phi,\mathrm{J}}(z)$. (2) Where does the space $\mathcal{E}ig_{\Phi,\mathrm{J}}(z)$ sit in the above chain of spectra?

4.3 Absorber Nilness

We now embark on a long, delicate argument to establish the crucial property of the quadratic absorber $\mathrm{Q} = qa(\mathrm{B})$: that the ideal $\mathcal{I}_{\mathrm{J}}(\mathrm{Q})$ it generates is **nil modulo** Q in the sense that for every element z of the ideal some power $z^n \in \mathrm{Q}$. Notice that if the quadratic absorber is equal to the core (as in the associative case, and for Peirce inner ideals), then it already *is* an ideal, $\mathrm{Q} = \mathcal{I}_{\mathrm{J}}(\mathrm{Q})$, and the result is vacuous. The import of the result is that the absorber $\mathrm{Q} \subseteq \mathrm{I}$ is sufficiently *close* to being an ideal $\mathcal{I}_{\mathrm{J}}(\mathrm{Q})$ that it behaves like an ideal for many purposes.

Absorber Nilness Theorem 4.3.1 *The ideal* $\mathcal{I}_{\mathrm{J}}(qa(\mathrm{B}))$ *in* J *generated by the quadratic absorber* $qa(\mathrm{B})$ *of an inner ideal* B *is nil mod* $qa(\mathrm{B})$, *hence nil mod* B.

PROOF. Throughout the proof we let $\mathrm{Q} := qa(\mathrm{B})$ and $\mathrm{I} := \mathcal{I}_{\mathrm{J}}(\mathrm{Q})$, and $\mathcal{T}$ be the set of all monomial operators $T = \alpha U_{x_1} \cdots U_{x_r}$ for $\alpha \in \Phi$, $x_i \in \widehat{\mathrm{J}}$ of *length* $r \geq 0$. We will break the proof into a series of steps.

Step 1: I is spanned by $\mathcal{T}(\mathrm{Q})$

Indeed, the sum of all monomials $w = T(z)$ for $z \in \mathrm{Q}$, $T \in \mathcal{T}$ is a linear subspace which contains Q, and is closed under arbitrary algebra multiples since $2L_x = V_x = U_{x,1} = U_{x+1} - U_x - U_1$ is a sum of U-operators. Because they are structural transformations, products of U's are much easier to work with than products of L's.

Step 2: Boosting nilpotence

We claim that if $x \in \mathrm{J}$ is properly nilpotent mod $qa(\mathrm{B})$, then it is also properly nilpotent modulo any higher absorber:

$$(2.1) \qquad x^{(m,y)} \in \mathrm{Q} \Longrightarrow x^{(3^n m,y)} \in qa^n(\mathrm{Q}) \qquad (y \in \widehat{\mathrm{J}}).$$

This follows by applying to $z = x^{(m,y)}$ the following general result:

$$(2.2) \qquad z \in \mathrm{Q} \Longrightarrow z^{(3^n,y)} \in qa^n(\mathrm{Q}) \text{ for all } y \in \widehat{\mathrm{J}}.$$

For this we induct on n, $n = 0$ being trivial, and for the induction step we have $w = z^{(3^n,y)} \in qa^n(\mathrm{Q}) =: qa(\mathrm{C}) \Longrightarrow z^{(3^{n+1},y)} = w^{(3,y)} = U_w U_y w \in U_{qa(\mathrm{C})}\mathrm{C}$ [since $w \in qa(\mathrm{C})$ absorbs U_y's into C] $\subseteq qa^2(\mathrm{C})$ [by Absorber Boosting 4.2.1(3) applied to C] $= qa^{n+1}(\mathrm{Q})$.

Step 3: Bounding proper nilpotence

Recall that the *length* of T is the number of factors U_{x_i}, so by definition it is absorbed by a suitably high absorber:

$$(3.1) \qquad T\big(qa^r(\mathrm{Q})\big) \subseteq \mathrm{Q} \quad (r = \text{length}(T)).$$

We claim that each monomial $w = T(z)$ is properly nilpotent mod Q with index r which depends only on the length of T:

$$(3.2) \qquad w^{(3^r,y)} \in \mathrm{Q} \qquad (r = \text{length}(T)).$$

From Structural Shifting 1.9.3(2) we have that $w^{(3^r,y)} = T(z)^{(3^r,y)} = T(z^{(3^r,T^*(y))}) \in T(qa^r(\mathrm{Q}))$ [by (2.2) above] $\subset$ Q [from (3.1)], which establishes (3.2).

Step 4: Modular Interlude.

That wasn't painful, was it? Now it's one thing to prove that *monomials* are nilpotent mod Q, but quite another to prove that *finite sums* of such monomials remain nilpotent: as we raise the sum to higher and higher powers the number of terms in the expansion proliferates hopelessly. If you want to get dispirited, just sit down and try to show from scratch that the sum of *two* monomial elements is nilpotent. But Zel'manov saw a way out of this quagmire, finessing the difficulty by showing that the finite sums remain *quasi-invertible* mod Q in all extensions, and then using Amitsur's Polynomial Trick to deduce *nilness* mod Q.

Invertible Modulo an Inner Ideal Definition 4.3.2 *If* C *is an inner ideal in* J, *we say that two elements* x, y *are* **equivalent mod** C, *written* $x \equiv_{\mathrm{C}} y$, *if* $x - y \in \mathrm{C}$. *We say that an element* $u \in \widehat{\mathrm{J}}$ *is* **invertible mod** C *if there is an element* $v \in \widehat{\mathrm{J}}$ *such that*

$$U_u v \equiv_{\mathrm{C}} \hat{1}.$$

We say that z *is* **quasi-invertible mod** C *if* $\hat{1} - z$ *is invertible mod* C *in* $\widehat{\mathrm{J}}$ (*where* C *remains inner*), *in which case the element* v *necessarily has the form* $\hat{1} - w$ *for some* $w \in \mathrm{J}$:

$$U_{\hat{1}-z}(\hat{1} - w) \equiv_{\mathrm{C}} \hat{1}.$$

Invertible Modulo an Absorber Lemma 4.3.3 *Let $qa(\mathrm{B})$ be the quadratic absorber of an inner ideal* B *in* J. *Then we have the following general principles.*

- **Multiplication of Equivalences Principle:** *we can multiply an equivalence by U, at the price of one degree of absorption*:

$$(1) \qquad x \equiv_{qa(\mathrm{B})} y \implies U_x a \equiv_{\mathrm{B}} U_y a,\ U_a x \equiv_{\mathrm{B}} U_a y \quad (a \in \widehat{\mathrm{J}}).$$

- **Higher Invertibility Principle:** *invertibility mod an absorber passes to all higher absorbers,*

$$(2) \quad u \textit{ invertible mod } qa(\mathrm{B}) \implies u \textit{ invertible mod } qa^n(\mathrm{B}) \textit{ for any } n.$$

- **Cancellation Principle:** *we can cancel an invertible factor U at the cost of two degrees of absorbency,*

$$(3) \qquad u \textit{ invertible mod } qa(\mathrm{B}),\ U_u x \equiv_{qa^2(\mathrm{B})} U_u y \implies x \equiv_{\mathrm{B}} y.$$

- **Invertibility of Factors Principle:** *as with ordinary inverses, if a product is invertible so are its factors,*

$$(4) \qquad U_x y \textit{ invertible mod } qa(\mathrm{B}) \implies x, y \textit{ invertible mod } qa(\mathrm{B}).$$

- **Inverse Equivalence Principle:** *invertibility passes to sufficiently equivalent elements,*

$$(5)\ u \textit{ invertible mod } qa(\mathrm{B}),\ u' \equiv_{qa^2(\mathrm{B})} u \implies u' \textit{ invertible mod } qa(\mathrm{B}).$$

- **Nilpotence Implies Quasi-Invertibility Principle:** *nilpotence modulo an absorber implies quasi-invertibility,*

$$(6) \qquad z \textit{ nilpotent mod } qa(\mathrm{B}) \implies z \textit{ q.i. mod } qa(\mathrm{B}).$$

PROOF. For Multiplication of Equivalences (1), if $x-y \in qa(\mathrm{B})$ then $U_x a - U_y a = (U_{(x-y)+y} - U_y)a = (U_{x-y} + U_{x-y,y})a \in U_{qa(\mathrm{B})}(\mathrm{J}) + \{qa(\mathrm{B}), \mathrm{J}, \mathrm{J}\} \subseteq \mathrm{B}$, and $U_a x - U_a y \in U_{\mathrm{J}} qa(\mathrm{B}) \subseteq \mathrm{B}$.

For Higher Invertibility (2), it suffices to show that if u is invertible mod $qa(\mathrm{C}) = qa^n(\mathrm{B})$ for some $n \geq 1$, then it is also invertible mod the next-higher absorber $qa^2(\mathrm{C}) = qa^{n+1}(\mathrm{B})$. By invertibility mod $qa(\mathrm{C})$ we have v such that $U_u v = \hat{1} - c$ for $c \in qa(\mathrm{C})$, so $U_u(U_v U_u U_{\hat{1}+c}\hat{1}) = U_{U_u v} U_{\hat{1}+c}\hat{1} = U_{\hat{1}-c} U_{\hat{1}+c}\hat{1} = U_{\hat{1}-c^2}\hat{1}$ [by Macdonald] $= \hat{1} - c'$ for $c' = 2c^2 - c^4 = U_c(\hat{2} - c^2) \in U_{qa(\mathrm{C})}(\widehat{\mathrm{C}}) \subseteq qa^2(\mathrm{C})$ by Absorber Boosting 4.2.1(3).

For Cancellation (3), set $d := x - y$; then $U_u d \in qa^2(\mathrm{B})$ and $U_u v \equiv_{qa(\mathrm{B})} \hat{1}$ together imply that $x - y = U_{\hat{1}} d \equiv_{\mathrm{B}} U_{U_u v} d$ [using Multiplication of Equivalences (1)] $= U_u U_v (U_u d) \in U_u U_v (qa^2(\mathrm{B}))$ [by hypothesis] $\subseteq \mathrm{B}$ [since $qa^2(\mathrm{B})$ absorbs $U_{\mathrm{J}} U_{\mathrm{J}}$ by Higher Absorber Definition 4.1.3(3)].

For Invertibility of Factors (4), if $U_x y$ is invertible mod $qa(\mathrm{B})$ then by Higher Invertibility (2) it is also invertible mod $qa^4(\mathrm{B})$, so $U_{U_x y}v \equiv_{qa^4(\mathrm{B})} \hat{1}$ for some $v \in \widehat{\mathrm{J}}$. Then $\hat{1} \equiv_{qa^4(\mathrm{B})} U_x(U_y U_x v) = U_x(U_y w)$ for $w := U_x v$ shows that x is invertible mod $qa^4(\mathrm{B})$ too, whence $U_x\hat{1} = U_{\hat{1}}(U_x\hat{1}) \equiv_{qa^3(\mathrm{B})} U_{U_x U_y w}(U_x\hat{1})$ [by Multiplication of Equivalences (1)] $= U_x U_y\ (U_w U_y U_x U_x\hat{1})$ [by the Fundamental Formula]. This implies that $\hat{1} \equiv_{qa(\mathrm{B})} U_y(U_w U_y U_x U_x\hat{1})$ by Cancellation (3) of U_x, and y is invertible too mod $qa(\mathrm{B})$.

For Inverse Equivalence (5), if $U_u v \equiv_{qa(\mathrm{B})} \hat{1}$ then $U_{u'}v \equiv_{qa(\mathrm{B})} U_u v$ [by Multiplication of Equivalences (1)] $\equiv_{qa(\mathrm{B})} \hat{1}$, so u' is invertible mod $qa(\mathrm{B})$ too.

For Nilpotence Implies Quasi-Invertibility (6), if $z^n \in qa(\mathrm{B})$ then $U_{\hat{1}-z}\big(\hat{1}+z+\cdots+z^{n-1}\big)^2 = \big(\hat{1}-z^n\big)^2 \equiv_{qa(\mathrm{B})} \hat{1}$, so z is q.i. mod $qa(\mathrm{B})$. □

This Lemma will be so useful in the rest of the proof that instead of calling it the Invertibility Modulo an Absorber Lemma, we will just call it IMAL. We have now gathered enough tools concerning modular quasi-invertibility to resume the proof of the Theorem.

Step 5: Congruence to $\hat{1}$

We claim that for every monomial w we can transform $\hat{1}-w$ to be congruent to $\hat{1}$: we will find a polynomial $p(t)$ with constant term 1 such that

$$U_{p(w)}(\hat{1}-w) \equiv_{qa^2(\mathrm{Q})} \hat{1}. \tag{5.1}$$

Indeed, from Step 3 w is properly nilpotent mod Q, hence mod any higher absorber by Step 2, so there is an m such that $w^m \in qa^2(\mathrm{Q})$. Then for all $k \geq 2m+1$ we have $w^k \in U_{w^m}(\mathrm{J}) \subseteq U_{qa^2(\mathrm{Q})}(\mathrm{J}) \subseteq qa^2(\mathrm{Q})$ [by innerness]. Now *because we have a scalar* $\frac{1}{2}$, *we can extract square roots*: the binomial series $(1-t)^{-\frac{1}{2}} = \sum_{i=0}^{\infty} \alpha_i t^i$ has coefficients $\alpha_0 = 1,\ \alpha_i = (-1)^i \binom{-\frac{1}{2}}{i}$ [by the Binomial Theorem] $= \binom{2i-1}{i}(\frac{1}{2})^{2i-1}$ [by virtuoso fiddling, heard in Exercise 4.3.3 below] $\in \mathbb{Z}[\frac{1}{2}]1 \subseteq \Phi$. The partial sum $p(t) = \sum_{i=0}^{2m} \alpha_i t^i$ is a polynomial of degree $2m$ with constant term 1, satisfying

$$p(t)^2 = 1 + t + \cdots + t^{2m} + \beta_{2m+1}t^{2m+1} + \cdots + \beta_{4m}t^{4m}$$

because $p(t)^2$ is a polynomial of degree $4m$ which as a formal series is congruent mod t^{2m+1} to $(\sum_{i=0}^{\infty} \alpha_i t^i)^2 = (1-t)^{-1} = \sum_{i=0}^{\infty} t^i$, so it must coincide with that power series up through degree $2m$. Thus if we evaluate this on w in the hull $\widehat{\mathrm{J}}$, we get $U_{p(w)}(\hat{1}-w) = p(w)^2 \bullet (\hat{1}-w)$ [by Macdonald] $= (\hat{1} + w + \cdots + w^{2m} + \beta_{2m+1}w^{2m+1} + \cdots + \beta_{4m}w^{4m}) \bullet (\hat{1}-w) = (\hat{1} - w^{2m+1}) + \sum_{k=2m+1}^{4m} \beta_k(w^k - w^{k+1}) \equiv_{qa^2(\mathrm{Q})} \hat{1}$ [since we noted that $w^k \in qa^2(\mathrm{Q})$ for all $k \geq 2m+1$], establishing (5.1).

Step 6: Quasi-invertibility Mod Q

We claim that the elements of I are all q.i. modulo Q. In view of Step 1 this means that every finite sum $v_n = w_1 + \cdots w_n$ of monomials is q.i. mod Q. We of course prove this by induction on n, the case $n = 1$ being settled by Step 3 and the Nilpotent-Implies-Q.I. Principle IMAL(6). Assume the result for sums of n monomials, and consider a sum $v_{n+1} = w_1 + \cdots w_n + w_{n+1} = v_n + w$ of $n+1$ monomials. We must show that $\hat{1} - v_{n+1}$ is invertible mod Q. We squash it down to a shorter sum (which is therefore invertible by induction), using the multiplier p of Step 5 to "eliminate" the last term: if $p = p(w)$ as in (5.1) then

$$(6.1) \qquad U_p(\hat{1} - v_{n+1}) \equiv_{qa^2(\mathrm{Q})} \hat{1} - v'_n$$

since $U_p(\hat{1} - v_{n+1}) = U_p((\hat{1} - w) - v_n) \equiv_{qa^2(\mathrm{Q})} \hat{1} - v'_n$ [by (5.1)], where $v'_n := U_p v_n$ again a sum of n monomials $w'_i := U_p w_i$. Now *by induction* v'_n is q.i. and $\hat{1} - v'_n$ is invertible mod Q, so by the Inverse Equivalence Principle IMAL(5) [here's where we need congruence mod $qa^2(\mathrm{Q})$] $U_p(\hat{1} - v_{n+1})$ is invertible mod Q too, so by the Invertibility of Factors Principle IMAL(4) $\hat{1} - v_{n+1}$ is too, completing our induction.

Now we come to Amitsur's magic wand, which turns quasi-invertibility into nilpotence.

Step 7: Modular Amitsur Polynomial Trick

We claim that if $x \in \mathrm{J}$ has tx q.i. mod $\mathrm{Q}[t]$ in $\mathrm{J}[t]$, then x is nilpotent mod $\mathrm{Q} = qa(\mathrm{B})$. Indeed, tx q.i. means that $\hat{1} - tx$ is invertible mod $\mathrm{Q}[t]$, so by the Higher Invertibility Principle IMAL(2) we can find $\hat{v}$ with $U_{\hat{1}-tx}\hat{v} \equiv_{qa^3(\mathrm{B})[t]} \hat{1}$. We claim that we can find $\hat{y} \in \widehat{\mathrm{J}}[t]$ with

$$(7.1) \qquad U_{\hat{1}-tx}(\hat{y}) \equiv_{qa^2(\mathrm{B})[t]} \hat{1} - tx.$$

By Multiplication of Equivalences IMAL(1) we have $\hat{1} - tx = U_{\hat{1}}(\hat{1} - tx) \equiv_{qa^2(\mathrm{B})[t]} U_{U_{\hat{1}-tx}\hat{v}}(\hat{1} - tx) = U_{\hat{1}-tx}U_{\hat{v}}U_{\hat{1}-tx}(\hat{1} - tx) =: U_{\hat{1}-tx}\hat{y}$. Writing this $\hat{y}$ as $\hat{y} = \sum_{i=0}^{N} t^i y_i$ for coefficients $y_i \in \widehat{\mathrm{J}}$, we will show that this $\hat{y}$ is congruent to the geometric series:

$$(7.2) \qquad y_i \equiv_{\mathrm{Q}} x^i.$$

Since the polynomial $\hat{y}$ must eventually have $y_i = 0$ for all $i > N$ for some N, this will imply $0 \equiv_{\mathrm{Q}} x^i$ for all $i > N$, and x will be nilpotent mod Q, and the Trick will have succeeded.

It will be sufficient if

$$(7.3) \qquad c'_i := y_i - x^i \in \mathcal{M}_x qa^2(\mathrm{B})$$

for $\mathcal{M}_x$ the algebra of multiplication operators involving only the element x.

At first glance the arbitrarily long strings of multiplications by x in $\mathcal{M}_x$ look frightening, since we are down to our last qa and our absorption is reduced by 1 each time we multiply by Multiplication of Equivalences IMAL(1). Luckily, the crucial fact is that no matter how long a string of multiplications by x we have, it can always be shortened to sums of monomials $U_{p(x)} \in U_{\Phi[x]}$ of length 1:

$$\mathcal{M}_x = U_{\Phi[x]}. \tag{7.4}$$

The reason is that this span of U-operators absorbs any further multiplications by x, in view of the relation $L_{x^k}U_{p(x)} = U_{x^k \bullet p(x), p(x)}$ by Macdonald's Principle [cf. the Operator Power-Associativity Rules II.5.2.2(2)], where $U_{r(x),p(x)} = U_{r(x)+p(x)} - U_{r(x)} - U_{p(x)} \in U_{\Phi[x]}$.

Once we have cut $\mathcal{M}_x$ down to a reasonable length 1 by (7.4), we can show why (7.3) is sufficient to establish (7.2): $y_i - x^i = c'_i \in \mathcal{M}_x qa^2(\mathrm{B})$ [by (7.3)] $= U_{\Phi[x]} qa^2(\mathrm{B})$ [by (7.4)] $\subseteq U_{\mathrm{J}} qa^2(\mathrm{B}) \subseteq qa(\mathrm{B}) = \mathrm{Q}$ [by definition of absorption] as in (7.2).

Thus we are finally reduced to showing that $c'_i \in \mathcal{M}_x qa(\mathrm{Q})$ as in (7.3). For this, we start from the definition of the y_i as coefficients of $\hat{y}$, where $\hat{1} - tx$ is equivalent mod $qa^2(\mathrm{B})[t] = qa(\mathrm{Q})[t]$ to $U_{\hat{1}-tx}\hat{y} = (\mathbb{1}_{\mathrm{J}[t]} - tV_x + t^2U_x)(\sum_{i=0}^{N} t^i y_i) = \sum_{j=0}^{N+2} t^j (y_j - V_x y_{j-1} + U_x y_{j-2})$. Identifying coefficients of like powers of t, we obtain elements $c_k \in qa(\mathrm{Q})$ such that

$$\begin{aligned} \hat{1} + c_0 &= y_0, \\ -x + c_1 &= y_1 - V_x y_0, \\ 0 + c_j &= y_j - V_x y_{j-1} + U_x y_{j-2} \text{ for all } j \geq 2. \end{aligned}$$

Solving this recursively, we get $y_0 = \hat{1} + c'_0$ for $c'_0 := c_0 \in qa(\mathrm{Q}) \subseteq \mathcal{M}_x(qa(\mathrm{Q}))$, then $y_1 = V_x(\hat{1} + c'_0) - x + c_1 = x + c'_1$ for $c'_1 := V_x c'_0 + c_1 \in \mathcal{M}_x(qa(\mathrm{Q}))$, and if the assertion is true for consecutive $j, j+1$, then $y_{j+2} = V_x(y_{j+1}) - U_x(y_j) + c_{j+2} = V_x(x^{j+1} + c'_{j+1}) - U_x(x^j + c'_j) + c_{j+2} = (2x^{j+2} + V_x(c'_{j+1})) - (x^{j+2} + U_x(c'_j)) + c_{j+2} = x^{j+2} + c'_{j+2}$, where the error term $c'_{j+2} := V_x(c'_{j+1}) - U_x(c'_j) + c_{j+2} \in \mathcal{M}_x(\mathcal{M}_x(qa(\mathrm{Q})) \subseteq \mathcal{M}_x(qa(\mathrm{Q}))$. This completes the recursive construction of the $c'_i \in \mathcal{M}_x qa(\mathrm{Q})$ as in (7.3), establishing the Amitsur Trick.

Step 8: Q.I. Implies Nil

We claim that if $\mathcal{I}_{\mathrm{J}}(\mathrm{Q})$ is quasi-invertible mod Q for all $\mathrm{Q} = q(\mathrm{B})$ for all inner ideals B in all Jordan algebras J, then $\mathcal{I}_{\mathrm{J}}(\mathrm{Q})$ is nil mod Q for all B and J. Indeed, $\widetilde{\mathrm{B}} = \mathrm{B}[t]$ remains an inner ideal in the Jordan algebra $\widetilde{\mathrm{J}} = \mathrm{J}[t] = \mathrm{J} \otimes_{\Phi} \Phi[t]$ of polynomials in the scalar indeterminate t with coefficients in J, and $\widetilde{\mathrm{Q}} := qa(\widetilde{\mathrm{B}}) = qa(\mathrm{B})[t] = \mathrm{Q}[t]$. [This holds because a polynomial $\sum_i t^i z_i$ belongs to $qa(\widetilde{\mathrm{B}})$ iff it absorbs $V_{\mathrm{J},\mathrm{J}}, U_{\mathrm{J}}$ into $\mathrm{B}[t]$ since $\Phi[t]$ is automatically absorbed into $\mathrm{B}[t]$, and by equating coefficients of t^i this happens iff each coefficient z_i absorbs $V_{\mathrm{J},\mathrm{J}}, U_{\mathrm{J}}$ into B, i.e., belongs to $\mathrm{Q} = qa(\mathrm{B})$.]

Thus $\mathcal{I}_{\widetilde{\mathrm{J}}}(\widetilde{\mathrm{Q}}) = \mathcal{I}_{\mathrm{J}}(\mathrm{Q})[t]$, so if $x \in \mathcal{I}_{\mathrm{J}}(\mathrm{Q})$ then $\tilde{x} := tx \in \mathcal{I}_{\widetilde{\mathrm{J}}}(\widetilde{\mathrm{Q}})$ must remain q.i. modulo $\widetilde{\mathrm{Q}}$ by Step 6. But this implies that x is actually *nilpotent* modulo Q by the Modular Amitsur Polynomial Trick Step 7. Thus every $x \in \mathrm{I} = \mathcal{I}_{\mathrm{J}}(\mathrm{Q})$ is nilpotent mod Q, and Absorber Nilness is established. □

This is the crucial result which will make the absorber vanish and create an exceptional heart in primitive algebras.

EXERCISE 4.3.3* Fiddle with binomial coefficients to show that we have a binomial indentity $(-1)^i \binom{-\frac{1}{2}}{i} = \binom{2i-1}{i}(\frac{1}{2})^{2i-1}$.

4.4 Problems for Chapter 4

PROBLEM 4.1 Let B be an inner ideal in J. (1) Show that $\mathrm{J} \bullet \ell a^n(\mathrm{B}) \subseteq \ell a^{n-1}(\mathrm{B})$ for all n. (2) If $\ell a^{n-1}(\mathrm{B})$ is an ideal in B, show that $\ell a^n(\mathrm{B})$ is an ideal in B iff $\{\mathrm{B}, \mathrm{J}, \ell a^n(\mathrm{B})\} \subseteq \ell a^{n-1}(\mathrm{B})$ iff $\{\mathrm{J}, \mathrm{B}, \ell a^n(\mathrm{B})\} \subseteq \ell a^{n-1}(\mathrm{B})$ iff $\{\mathrm{B}, \ell a^n(\mathrm{B}), \mathrm{J}\} \subseteq \ell a^{n-1}(\mathrm{B})$. (3) Show that if C is an ideal of B and an inner ideal of J, then $qa(\mathrm{C})$ is again an ideal of B and an inner ideal of J. (4) Conclude that $qa^n(\mathrm{B})$ is always an ideal of B.

PROBLEM 4.2 (1) We say that z is *properly quasi-invertible mod* C if $1^{(y)} - z$ is invertible mod C in $\widehat{\mathrm{J}^{(y)}}$ (where C remains inner) for each $y \in \mathrm{J}$: $U^{(y)}_{1^{(y)}-z}(1^{(y)} - w^{(y)}) \equiv_{\mathrm{C}} 1^{(y)}$ for each y and some $w^{(y)} \in \mathrm{J}$. Show that z p.n. mod $qa(\mathrm{B}) \Longrightarrow z$ p.q.i. mod $qa(\mathrm{B})$. (2) Show that $x \equiv_{qa(\mathrm{B})} y \Longrightarrow B_{x,a}(b) \equiv_{\mathrm{B}} B_{y,a}(b)$.

PROBLEM 4.3 Repeat the argument of the Absorber Nilness Theorem to show that the ideal $\mathcal{I}_{\mathrm{J}}(qa(\mathrm{B}))$ is *properly nil* mod $qa(\mathrm{B})$. (Step 1) Modify the definition of monomial $T(z)$ for $T = T_1 \cdots T_s$ to include Bergman operators B_{x_i,y_i} $(x_i, y_i \in \widehat{\mathrm{J}})$ among the T_i; instead of *length* of T we work with *width* of T, defined to be the sum of the widths of its constituent T_i, where U_{x_i} has width 1 and B_{x_i,y_i} width 2. (Step 3) Show that equation (3.1) in the proof holds for these more general T if we replace length by width. (Step 4) Show that for every y we can transform every $1^{(y)} - w$ for monomial w to be congruent to $1^{(y)}$: we can find a polynomial $p(t)$ with constant term 1 such that $U^{(y)}_{p^{(y)}(w)}(1^{(y)} - w) \equiv_{qa^2(\mathrm{Q})} 1^{(y)}$. (Step 5) Show that $U^{(y)}_p(1^{(y)} - v_{n+1}) \equiv_{qa^2(\mathrm{Q})} 1^{(y)} - v'_n$ where v'_n is a sum of n new monomials $w'_i = U^{(y)}_p w_i = B_{s,y} w_i$ *by the inclusion of Bergmann B's in our monomials* (w'_i has width 2 greater than the original w_i) [recall that p had constant term 1, so we can write $p = 1^{(y)} - s$ for $s = s(w, y) \in \mathrm{J}$]. (Step 5) Show that from Amitsur's Polynomial Trick that if tx remains p.q.i. modulo $qa(\mathrm{B})[t]$ in $\mathrm{J}[t]$, then x is p.n. modulo $qa(\mathrm{B})$.

5

Primitivity

The next new concepts are those of *modular inner ideal* and *primitivity,* which again have an illustrious associative pedigree but require a careful formulation to prosper in the Jordan setting.[1]

5.1 Modularity

In his work with the radical, Zel'manov had already introduced the notion of modular inner ideal and primitive Jordan algebra. In the associative case a *left* ideal L is *modular* if it has a *modulus* c, an element which acts like a *right unit* ("modulus" in the older literature) for A modulo L: $ac \equiv a$ modulo L for each $a \in$ A, or globally $\mathrm{A}(\hat{1} - c) \subseteq$ L. If A is unital, then *all* left ideals are modular with modulus $c = 1$. Such c remains a modulus for any larger $\mathrm{L}' \supseteq$ L, any translate $c + b$ by $b \in$ L is another modulus, and as soon as L contains one of its moduli, then it must be all of A.

The concept of modularity was invented for the Jacobson radical in non-unital algebras: in the unital case $\mathcal{R}ad(\mathrm{A})$ is the intersection of all maximal left ideals, in the non-unital case it is the intersection of all maximal *modular* left ideals. To stress that the moduli are meant for nonunital situations, we consistently use $\hat{1}$ to indicate the external nature of the unit. (If an algebra already has a unit, we denote it simply by 1.)

In Jordan algebras we don't have *left* or *right*, we have *inner* and *outer*. The analogue of a *left* ideal L is an *inner* ideal B, and the analogue of a *right* unit is an *outer* unit for J modulo B, $U_{\hat{1}-c}(\mathrm{J}) \subseteq$ B. This turns out not to be quite enough to get a satisfactory theory.

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[1] cf. I.8.3–8.4 for the concepts of this chapter (modulus, modular exclusion, primitivity), and sketches of the results (Absorberless Primitivity Proposition, Semiprimitivity Theorem).

Modularity Definition 5.1.1 *An inner ideal* B *in a Jordan algebra* J *is* **modular** *with* **modulus** *c if it satisfies the* **Modulus Conditions**

$$\text{(Mod 1)}\ \ U_{\hat{1}-c}(\mathrm{J}) \subseteq \mathrm{B}, \quad \text{(Mod 2)}\ c - c^2 \in \mathrm{B}, \quad \text{(Mod 3)}\ \ \{\hat{1} - c, \widehat{\mathrm{J}}, \mathrm{B}\} \subseteq \mathrm{B}.$$

The last condition can be written in terms of J *as*

$$\text{(Mod 3a)}\quad \{\hat{1} - c, \mathrm{J}, \mathrm{B}\} \subseteq \mathrm{B}, \qquad \text{(Mod 3b)}\quad \{c, \mathrm{B}\} \subseteq \mathrm{B}.$$

Exercise 5.1.1* (1) Check that if J is unital, then indeed all inner ideals are modular with modulus 1. (2) Show that all inner B in J remain inner in $\widehat{\mathrm{J}}$ with modulus $\hat{1}$, but as an inner ideal in $\widehat{\mathrm{J}}$ it *never* has a modulus $c \in \mathrm{J}$. (3) Show that an inner ideal B in J has modulus $c \in \mathrm{J}$ iff its "c-hull" $\widehat{\mathrm{B}}^c := \Phi(\hat{1} - c) + \mathrm{B}$ is inner in $\widehat{\mathrm{J}}$ with modulus $\hat{1}$ *and* modulus c, in which case $\widehat{\mathrm{B}}^c \cap \mathrm{J} = \mathrm{B}$. (4) Show that if B' is an inner ideal in $\widehat{\mathrm{J}}$ with a modulus $c \in \mathrm{J}$, then its "contraction" $\mathrm{B} := \mathrm{J} \cap \mathrm{B}'$ is an inner ideal in J with modulus c, and $\widehat{\mathrm{B}}^c = \mathrm{B}'$. (5) Conclude that hull and contraction are inverse bijections between c-modular inner ideals in J and $\widehat{\mathrm{J}}$, in particular, $\mathrm{B} = \mathrm{J} \iff \widehat{\mathrm{B}}^c = \widehat{\mathrm{J}} \iff c \in \mathrm{B}$. (6) Conclude that the modular ideals are precisely the traces (intersections) on J of those inner ideals of the immortal hull which have a mortal modulus $c \in \mathrm{J}$ in addition to their immortal modulus $\hat{1}$.

It is important that we can adjust the modulus to adapt to circumstances.

Modulus Shifting Lemma 5.1.2 (1) *If c is a modulus for an inner ideal* B *in a Jordan algebra* J*, so is any* **translate** *or* **power** *of c*:

$$c + b,\ c^n \text{ are moduli for any } b \in \mathrm{B},\ n \geq 1.$$

Indeed, powers are merely translates:

$$c - c^n \in \mathrm{B} \text{ for all } n \geq 1.$$

(2) *A modulus c remains a modulus for a larger inner ideal* $\mathrm{B}' \supseteq \mathrm{B}$ *only if* $\{\hat{1} - c, \widehat{\mathrm{J}}, \mathrm{B}'\} \subseteq \mathrm{B}'$, *which may not happen for all enlargements* B', *but does happen for ideals. We have the* **Ideal Enlargement Property**:

$$c \text{ remains a modulus for any } \mathrm{B}' = \mathrm{B} + \mathrm{I} \text{ for } \mathrm{I} \lhd \mathrm{J}.$$

(3) *The* **Modulus Exclusion Property** *says that a proper inner ideal cannot contain its modulus*:

$$\text{If } \mathrm{B} \text{ has a modulus } c \in \mathrm{B}, \text{ then } \mathrm{B} = \mathrm{J}.$$

(4) *The* **Strong Modulus Exclusion Property** *says that the ideal generated by the quadratic absorber of a proper inner ideal cannot contain a modulus either*:

$$\text{If } \mathrm{B} \text{ has a modulus } c \in \mathcal{I}_{\mathrm{J}}(qa(B)), \text{ then } \mathrm{B} = \mathrm{J}.$$

proof. (1) For translation shifting from c to $c' = c + b$, we must use the Modularity conditions (Mod 1)–(Mod 3) of 5.1.1 for c to verify the corresponding conditions (Mod 1)′–(Mod 3)′ for c'. (Mod 1)′ holds because $U_{\hat 1-c'} = U_{\hat 1-c-b} = U_{\hat 1-c} - U_{\hat 1-c,b} + U_b$, where each piece maps J into B by (Mod 1), (Mod 3), and innerness of B. (Mod 2)′ holds because $c' - (c')^2 = (c - c^2) + (b - \{c, b\} - b^2)$, where the first term is in B by (Mod 2) and the second term by (Mod 3b) and the fact that inner ideals are subalgebras. (Mod 3)′ holds because $U_{\hat 1-c',\mathrm{B}} = U_{\hat 1-c,\mathrm{B}} - U_{b,\mathrm{B}}$ maps $\widehat{\mathrm{J}}$ into B, since the first term does by (Mod 3) and the second term by innerness of B.

For power shifting from c to c^n, we use recursion on n, $n = 1$ being trivial, $n = 2$ being (Mod 2), and for the recursion step to $n+2$ we note that $(c^{n+2} - c) - 2(c^{n+1} - c) + (c^n - c) = c^{n+2} - 2c^{n+1} + c^n = U_{\hat 1-c}c^n \in \mathrm{B}$ by (Mod 1); since $c^{n+1} - c$ and $c^n - c$ lie in B by the recursion hypothesis, we see that $c^{n+2} - c$ does too, completing the recursion step.

(2) Indeed, note that (Mod 1)–(Mod 2) certainly hold in *any* enlargement, and (Mod 3) holds in any *ideal* enlargement since $\{\hat 1 - c, \widehat{\mathrm{J}}, \mathrm{B}\} \subseteq \mathrm{B}$ by (Mod 3) for B and $\{\hat 1 - c, \widehat{\mathrm{J}}, \mathrm{I}\} \subseteq \mathrm{I}$ by definition of ideal.

(3) follows because if $c \in \mathrm{B}$, then by (1) the translate $c' = c - c = 0$ would be a modulus, which by (Mod 1) clearly implies that $\mathrm{J} = \mathrm{B}$. □

EXERCISE 5.1.2 Verify that for a left ideal L of an associative algebra A with associative modulus c (1) any translate or power of c is again a modulus; (2) the inner ideal $\mathrm{B} = \mathrm{L}$ in $\mathrm{J} = \mathrm{A}^+$ is modular with Jordan modulus c.

An important source of modular inner ideals is structural transformations.

Structural Inner Ideal Example 5.1.3 *If T is a structural transformation on* J *with T^* (as in the Structural Transformation Definition* 1.2.1) *which are both congruent to $\hat{\mathbb{1}}$ mod* J (*as in the Congruent to $\hat{\mathbb{1}}$ Lemma* 1.2.3),

$$(\hat{\mathbb{1}} - T)(\widehat{\mathrm{J}}) + (\hat{\mathbb{1}} - T^*)(\widehat{\mathrm{J}}) \subseteq \mathrm{J}.$$

then the structural inner ideal $T(\mathrm{J})$ is a modular inner ideal:

(1) $T(\mathrm{J})$ *is inner with modulus* $c = \hat 1 - T(\hat 1) \in \mathrm{J}$.

In particular,

(2) $B_{x,y}(\mathrm{J})$ *is inner with modulus* $c = \{x, y\} - U_x y^2$,
$U_{\hat 1-x}(\mathrm{J})$ *is inner with modulus* $c = 2x - x^2$.

PROOF. (1) Acting separately on $\hat 1$, J, the congruence condition in (1) reduces to

(1′) $T(\hat 1) = \hat 1 - c, \quad T^*(\hat 1) = \hat 1 - c^*, \quad T(\mathrm{J}) + T^*(\mathrm{J}) \subseteq \mathrm{J}.$

for some $c, c^* \in \mathrm{J}$. The linear subspace $\mathrm{B} := T(\mathrm{J})$ is an *inner ideal* [cf. Structural Innerness II.18.2.2] since $U_{Tx}(\widehat{\mathrm{J}}) = TU_xT^*(\widehat{\mathrm{J}})$ [by structurality] $\subseteq T(\mathrm{J}) = \mathrm{B}$. To check that c is a *modulus* as in the Definition 5.1.1, for (Mod

1) we have $U_{\hat{1}-c}(\mathrm{J}) = U_{T(\hat{1})}(\mathrm{J}) = TT^*(\mathrm{J})$ [by structurality] $\subseteq T(\mathrm{J}) = \mathrm{B}$; for (Mod 2) we have $c^2 - c = (\hat{1} - c) - (\hat{1} - c)^2 = T(\hat{1}) - T(\hat{1})^2$ [by (1′)] $= T(\hat{1}) - U_{T(\hat{1})}(\hat{1}) = T(\hat{1} - T^*(\hat{1}))$ [by structurality] $= T(c^*) \in T(\mathrm{J})$ [by (1′)] $= \mathrm{B}$; and for (Mod 3) we have $\{\hat{1} - c, \widehat{\mathrm{J}}, \mathrm{B}\} = \{T(\hat{1}), \widehat{\mathrm{J}}, T(\mathrm{J})\}$ [by (1′)] $= TU_{\hat{1},\mathrm{J}}(T^*(\widehat{\mathrm{J}}))$ [by structurality] $\subseteq T(\mathrm{J}) = \mathrm{B}$. Applying (1) to $T = B_{x,y}$, $c = 1 - B_{x,y}(1) = \{x, y\} - U_y U_y 1$ and $T = U_{\hat{1}-x} = B_{x,\hat{1}}$, $c = 1 - U_{\hat{1}-x}(1) = 2x - U_x 1$ gives (2). □

Exercise 5.1.3 Show that $T = \widehat{\mathbb{1}} - S$ is structural with $T^* = \widehat{\mathbb{1}} - S^*$ on $\widehat{\mathrm{J}}$ iff S, S^* satisfy $U_{x,Sx} - (SU_x + U_x S^*) = U_{S(x)} - SU_x S^*$, and show that $T(\mathrm{J}) \subseteq \mathrm{J}$ iff $S(\mathrm{J}) \subseteq \mathrm{J}$, and dually for T^*, S^*.

5.2 Primitivity

An associative algebra A is *primitive* if it has a faithful irreducible representation, or in more concrete terms if there exists a left ideal such that A/L is a faithful irreducible left A-module. *Irreducibility* means that L is *maximal modular*, while *faithfulness* means that L has zero *core* (the maximal ideal of A contained in L, which is just its right absorber $\{z \in \mathrm{A} \mid z\mathrm{A} \subseteq \mathrm{L}\}$); this core condition (that no nonzero ideal I is contained in L) means that $\mathrm{I} + \mathrm{L} > \mathrm{L}$, hence in the presence of maximality means that $\mathrm{I} + \mathrm{L} = \mathrm{A}$, and L *supplements all nonzero ideals.* Once a modular L_0 has this property, it can always be enlarged to a maximal modular left ideal L which is even more supplementary.

In the Jordan case A/L is not going to provide a representation anyway, so there is no need to work hard to get the maximal L (the "irreducible" representation), any supplementary L_0 will do. In the Jordan case we may not have representations on modules, but we do still have maximal modular inner ideals and cores. The **core** $\mathcal{C}ore(S)$ of any subset S is the maximal two-sided ideal contained in B (the sum of all such ideals).

Primitivity Definition 5.2.1 *A Jordan algebra is* **primitive** *if it has a* **primitizer** P, *a proper modular inner ideal* $\mathrm{P} \neq \mathrm{J}$ *which has the* **Supplementation Property:**

$$\mathrm{I} + \mathrm{P} = \mathrm{J} \textit{ for all nonzero ideals } \mathrm{I} \textit{ of } \mathrm{J}.$$

Another way to express this is the **Modulus Containment Property**:

every nonzero ideal I *contains a modulus for* P.

Indeed, if $\mathrm{I} + \mathrm{P} = \mathrm{J}$ we can write the modulus for P as $c = i + p$, in which case the translate $i = c - p$ remains a modulus and lies in I. Conversely, if I contains a modulus c for P, then by the Ideal Enlargement Property 5.1.2(2) it remains a modulus for the inner ideal $\mathrm{I} + \mathrm{P}$, which then by the Modulus Exclusion Property 5.1.2(3) must be all of J.

The terminology "semiprimitive" for Jacobson-radical-less algebras means "subdirect product of primitives," so we certainly want to be reassured that totally-primitive algebras are always semi-primitive!

Primitive Proposition 5.2.2 (1) *Any primitive Jordan algebra is semiprimitive, hence nondegenerate*:

$$\mathrm{J}\ \textit{primitive} \implies \mathcal{R}ad(\mathrm{J}) = \mathbf{0} \implies \mathrm{J}\ \textit{nondegenerate.}$$

(2) *In particular, primitive algebras have no nil ideals; indeed, they have no ideals nil modulo* P,

$$\mathrm{I} \lhd \mathrm{J}, \mathrm{I}\ \textit{nil mod}\ \mathrm{P} \implies \mathrm{I} = \mathbf{0}.$$

(3) *Although quadratic absorber and core do not coincide for general inner ideals, the core, absorber, and s-identities all vanish on the primitizer. We have the* **Absorberless Primitizer Property**:

$$\mathcal{C}ore(\mathrm{P}) = qa(\mathrm{P}) = i\text{-}\mathcal{S}pecializer(\mathrm{P}) = \mathbf{0}.$$

PROOF. (1) If $\mathrm{I} = \mathcal{R}ad(\mathrm{J}) \neq \mathbf{0}$, then I contains a modulus c for P by the Modulus Containment Property 5.2.1; then $U_{\hat{1}-c}(\mathrm{J}) \subseteq \mathrm{P} < \mathrm{J}$ by (Mod 1), yet $\hat{1} - c$ is invertible in $\widehat{\mathrm{J}}$ because the radical c is quasi-invertible, forcing $U_{\hat{1}-c}(\mathrm{J}) = \mathrm{J}$, a contradiction. Once J is semiprimitive, $\mathcal{R}ad(\mathrm{J}) = \mathbf{0}$, it is nondegenerate by Hereditary Radical Theorem 1.6.1(5).

(2) If I were nonzero, it would contain a modulus c for P by Modulus Containment, hence by nilness modulo P some power c^n would fall in P, yet remain a modulus for P by Modulus Shifting 5.1.2(1), contrary to Modulus Exclusion 5.1.2(3).

(3) We introduced Modulus Containment as a surrogate for corelessness; note that if $\mathrm{I} = \mathcal{C}ore(\mathrm{P}) \subseteq \mathrm{P}$ were nonzero, then the Supplementation Property in Definition 5.2.1 would imply $\mathrm{P} = \mathrm{P} + \mathrm{I} = \mathrm{J}$, contrary to its assumed properness, so I must vanish. The vanishing of $qa(\mathrm{P})$ is much deeper: the ideal $\mathrm{I} = \mathcal{I}_{\mathrm{J}}(qa(\mathrm{P}))$ is usually not contained in P, but it is nil mod P by the Absorber Nilness Theorem 4.3.1, so from (2) we see that $\mathrm{I} = \mathbf{0}$ and hence the generator of I vanishes, $qa(\mathrm{P}) = \mathbf{0}$ too. Once the absorber vanishes, Quadratic Absorber 4.2.1(4) says that $i\text{-}\mathcal{S}pecializer(\mathrm{P}) = \mathbf{0}$ by nondegeneracy (1). □

Recall that if M is a maximal modular left ideal in an associative algebra A, then removing the core $\mathrm{K} = \mathcal{C}ore(\mathrm{M})$ produces a primitive algebra A/K with faithful irreducible representation on the left module A/M. Primitizers always have zero core, and once again removing the core from a maximal modular inner ideal is enough to create primitivity.

Coring Proposition 5.2.3 *If* B *is maximal among all proper c-modular inner ideals, then removing its core creates a primitive algebra:* $J/\mathcal{C}ore(B)$ *is primitive with primitizer* $B/\mathcal{C}ore(B)$.

PROOF. We have $K = \mathcal{C}ore(B) \subseteq B < J$, so the image $\overline{B} := B/K < J/K =: \overline{J}$ remains a proper inner ideal in $\overline{J}$ with modulus $\bar{c}$ by taking images of (Mod 1)–(Mod 3), but now it also has the Supplementation Property 5.2.1 for a primitizer. Indeed, if $\overline{I}$ is nonzero in $\overline{J}$, then its pre-image is an ideal $I > K$ in J which is not contained in the core, hence not entirely contained in B; then the ideal enlargement $I + B > B$ is still a c-modular inner ideal by Ideal Enlargement 5.1.1(2), so by *maximality* of B it must not be proper, so $I + B = J$ and therefore $\overline{I} + \overline{B} = \overline{J}$ as required for the Supplementation Property. □

5.3 Semiprimitivity

We are finally ready to connect the Jacobson radical to primitivity, showing that the radical vanishes iff the algebra is a subdirect product of primitive algebras, thereby justifying our use of the term "semiprimitive" for such algebras. Recall that the Jacobson radical of an associative algebra A is the intersection of all maximal modular left ideals M, and also the intersection of their cores $K = \mathcal{C}ore(M)$. In the Jordan case we have a similar core characterization.

Semiprimitivity Theorem 5.3.1 (1) *The Jacobson radical is the intersection of the cores of all maximal modular inner ideals:* $\mathcal{R}ad(J) = \bigcap\{\mathcal{C}ore(B)\}$, *where the intersection is taken over all inner ideals* B *which are maximal among all c-modular inner ideals for some* $c = c(B)$ *depending on* B.

(2) J *is semiprimitive iff it is a subdirect product* $J = \prod_\alpha J_\alpha$ *of primitive Jordan algebras* $J_\alpha = J/K_\alpha$, *i.e., iff the co-primitive ideals* $K_\alpha \triangleleft J$ *separate points,* $\bigcap K_\alpha = \mathbf{0}$.

PROOF. (1) $\mathcal{R}ad(J) \subseteq \bigcap K_\alpha$ since for each $K = \mathcal{C}ore(B)$ we have $\overline{J} = J/K$ primitive by the Coring Proposition 5.2.3, and therefore semiprimitive by the Primitive Proposition 5.2.2(1). Thus the image of $\mathcal{R}ad(J)$, as a q.i. ideal in the semiprimitive J/K, must vanish, and $\mathcal{R}ad(J) \subseteq K$.

The converse $\mathcal{R}ad(J) \supseteq \bigcap K_\alpha$ is more delicate: if $z \notin \mathcal{R}ad(J)$, we must construct a maximal modular B (depending on z) with $z \notin \mathcal{C}ore(B)$. Here the Elemental Characterization Theorem 1.5.1(2) of the radical as the p.q.i. elements comes to our rescue: $z \notin \mathcal{R}ad(J) \Longrightarrow z$ not p.q.i. $\Longrightarrow$ some (z, y) not q.i., $B_0 = B_{z,y}(J) < J$ [by non-surjectivity Criterion 1.4.2(4*iii*)] with modulus $c = \{z, y\} - U_z y^2$ [by Bergmann Modularity 5.1.3(2)]. Properness $B < J$ of a c-modular inner ideal is equivalent [by the Modulus Exclusion Property 5.1.2(3)] to $c \notin \mathrm{B}$, so we can apply Zorn's Lemma to find a maximal c-modular (proper) inner ideal B containing B_0, and we claim that $z \notin \mathcal{C}ore(\mathrm{B})$: if z belonged to the ideal $\mathcal{C}ore(\mathrm{B}) \subseteq \mathrm{B}$, so would $c = \{z, y\} - U_z y^2$, a contradiction.

(2) J semiprimitive $\iff \bigcap \mathrm{K}_\alpha = \mathcal{R}ad(\mathrm{J}) = \mathbf{0}$ [by definition and (1)] $\iff$ the co-primitive ideals K_α [by the Coring Proposition again] separate points, and in this case J is a subdirect product of the primitive $\mathrm{J}_\alpha = \mathrm{J}/\mathrm{K}_\alpha$. If $\mathrm{J} = \prod_\alpha \mathrm{J}_\alpha$ is the subdirect product of *some* primitive algebras $\mathrm{J}_\alpha = \mathrm{J}/\mathrm{K}_\alpha$ (perhaps unrelated to maximal modular inner ideals), then [by the Primitive Proposition (1) again] the radical image $\pi_\alpha(\mathcal{R}ad(\mathrm{J}))$ in each primitive J_α is zero, $\mathcal{R}ad(\mathrm{J}) \subseteq \bigcap \mathrm{Ker}(\pi_\alpha) = \bigcap \mathrm{K}_\alpha = \mathbf{0}$ by definition of semi-direct product, and J is semiprimitive. □

We remark that also in the Jordan case the radical is the intersection of all maximal modular inner ideals, not just their cores (see Problem 2).

5.4 Imbedding Nondegenerates in Semiprimitives

There are standard methods, familiar from associative theory, for shrinking the Jacobson radical into the degenerate radical. We will show that we can imbed any nondegenerate algebra into a semiprimitive algebra, a subdirect product of primitive algebras, by a multi-step process passing through various scalar extensions and direct products. In the end we want to apply this result to prime algebras, and assuming primeness from the start saves a few steps, so we will only imbed prime nondegenerate algebras (the general case is left to Problem 6). Each imbedding step creates a larger algebra where more of the radical vanishes, starting from an algebra where the degenerate radical vanishes and eventually reaching an algebra where the entire Jacobson radical vanishes. Furthermore, at each step in the process the strict identities are preserved, so that the final semiprimitive algebra satisfies *exactly the same strict identities* as the original, and each primitive factor of the subdirect product inherits these identities (but might satisfy additional ones).

Recall that a polynomial is said to vanish *strictly* on J if it vanishes on *all* scalar extensions $\Omega\mathrm{J}$. This happens iff it vanishes on the "generic" extension $\Phi[T] \otimes \mathrm{J} = \mathrm{J}[T]$ of polynomials (with coefficients from J) in a countable set of indeterminates T. A polynomial which vanishes on J but not all extensions vanishes only "fortuitously" on J, due to a lack of sufficient scalar power; only the polynomials which vanish strictly are "really" satisfied by J. For example, the identity $x^2 = x$ defining Boolean algebras does not hold strictly; indeed, the only scalars that Boolean algebras can tolerate are those in $\mathbb{Z}_2$; Boolean algebras have a fleeting existence over the scalars $\mathbb{Z}_2$, then disappear.

Semiprimitive Imbedding Theorem 5.4.1 *Every prime nondegenerate Jordan algebra* J *can be imbedded in a semiprimitive Jordan algebra* $\widetilde{\mathrm{J}}$ *over a big algebraically closed field* Ω, $|\Omega| > \dim_\Omega \widetilde{\mathrm{J}}$, *in such a way that* J *and* $\widetilde{\mathrm{J}}$ *satisfy exactly the same strict identities. In particular,* J *is i-exceptional iff* $\widetilde{\mathrm{J}}$ *is.*

Here $\widetilde{\mathrm{J}} \lessapprox \prod_\alpha \widetilde{\mathrm{J}}_\alpha$ *is a subdirect product of primitive algebras* $\widetilde{\mathrm{J}}_\alpha$*, where the* $\widetilde{\mathrm{J}}_\alpha$ *are also algebras over the big algebraically closed field* Ω, $|\Omega| > \dim_\Omega \widetilde{\mathrm{J}_\alpha}$. *We may also imbed* J *in a larger* $\vec{\mathrm{J}} := \prod_\alpha \widetilde{\mathrm{J}}_\alpha$ *which is the full direct product of the* $\widetilde{\mathrm{J}}_\alpha$*, still satisfying exactly the same strict identities as* J (*through now* Ω *may no longer be big for* $\vec{\mathrm{J}}$).

PROOF. As usual, we break this long proof into small steps.[2]

Step 1: Avoiding Degeneracy

We first imbed J in an algebra J_1 over a field Ω_0 where no element of J is *trivial* in J_1. By primeness of J its centroid Γ is an integral domain acting faithfully on J [by the Centroid Theorem II.1.6.3]. The usual construction of the algebra of fractions $\Gamma^{-1}\mathrm{J}$ leads (just as in the "scalar case" constructing the field of fractions of an integral domain) to a Jordan algebra over the field of fractions $\Omega_0 = \Gamma^{-1}\Gamma$. The fact that the action of Γ is faithful guarantees that J is imbedded in $\mathrm{J}_1 := \Gamma^{-1}\mathrm{J}$, and we claim that

(1.1) there is no nonzero $z \in \mathrm{J}$ which is trivial in J_1.

Certainly, by nondegeneracy of J none of its elements can be trivial in this larger algebra, $U_z\mathrm{J}_1 \supseteq U_z\mathrm{J} \neq \mathbf{0}$. Thus we can pass from a prime nondegenerate algebra to an algebra over a *field* where J avoids any possible degeneracy in J_1. Primeness is used only to get to a field quickly. In fact, J_1 remains prime and nondegenerate (see Problem 5.7 at the end of the chapter), but from this point on we discard primeness and use only nondegeneracy.

Step 2: Avoid Proper Nilness of Bounded Index

We next imbed J_1 in an Ω_0-algebra J_2 where no element of J is *properly nilpotent of bounded index* (p.n.b.i.) in J_2. Let $\mathrm{J}_2 := \mathrm{J}_1[T] = \mathrm{J} \otimes_{\Omega_0} \Omega_0[T] = \widetilde{\mathrm{J}}$ be the algebra of polynomials in an infinite set of scalar indeterminates T. In fact, in a pinch you can get by with just one indeterminate (see Problem 5.5 below). We claim that

(2.1) there is no $0 \neq z \in \mathrm{J}$ such that $z^{(n,\widetilde{\mathrm{J}})} = \mathbf{0}$ for some $n = n(z)$.

Indeed, if *some* nth power vanished for all $\tilde{y} \in \widetilde{\mathrm{J}}$, then *all* mth powers for $m \geq n$ would vanish because $z^{(n+k,\tilde{y})} = z^{(n,\tilde{y})} \bullet_{\tilde{y}} z^{(k,\tilde{y})} = \frac{1}{2}\{z^{(n,\tilde{y})}, \tilde{y}, z^{(k,\tilde{y})}\} = 0$. The *p.n.b. index* of z is the smallest n such that $z^{(m,\widetilde{\mathrm{J}})} = \mathbf{0}$ for all $m \geq n$.

We claim that no nonzero z can have proper index $n > 1$. Suppose that $z^{(m,\widetilde{\mathrm{J}})} = \mathbf{0}$ for all $m \geq n > 1$, but that some $z^{(n-1,\tilde{y})} \neq 0$. Because T is infinite

[2] cf. the proof sketch in I.8.9.

we can choose a $t \in T$ which does not appear in the polynomial $\tilde{y}$. For any $x \in \mathrm{J}$ set $\tilde{w} := \tilde{y} + tx$. Since $n > 1$, we have $m := 2n-2 = n+(n-2) \geq n$, and therefore by definition of the p.n.b. index we have $z^{(2n-2,\tilde{w})} = z^{(m,\tilde{w})} = 0$. Then the coefficients of all powers of t in the expansion of $z^{(2n-2,\tilde{w})} = 0$ must vanish, in particular, that of t itself. The coefficient of t^1 in $z^{(2n-2,\tilde{w})} = \big(U_z^{(\tilde{w})}\big)^{n-2} z^{(2,\tilde{w})} = (U_z U_{\tilde{w}})^{n-2} U_z \tilde{w}$ consists of all terms with a single factor tx and the rest of the factors z and $\tilde{y}$, namely,

$$\begin{aligned} 0 = &\ \big(U_z U_{\tilde{y}}\big)^{n-2} U_z x \ + \ \textstyle\sum_{k=1}^{n-2} \big(U_z U_{\tilde{y}}\big)^{n-2-k} \big(U_z U_{\tilde{y},x}\big) \big(U_z U_{\tilde{y}}\big)^{k-1} U_z \tilde{y} \\ = &\ U_{z^{(n-1,\tilde{y})}}\, x \ + \ \textstyle\sum_{k=1}^{n-2} \{ z^{(n+k-1,\tilde{y})}, x, z^{(n-k-1,\tilde{y})} \} \\ = &\ U_{z^{(n-1,\tilde{y})}} x, \end{aligned}$$

since $z^{(n+k-1,\tilde{y})} = 0$ for $k \geq 1$ by hypothesis that n is a proper nilpotence bound.

Here we have used Macdonald to write the sum in terms of $\tilde{y}$-homotope powers, since the kth term can be written in associative algebras as

$$\begin{aligned} &\{\overbrace{z, \tilde{y}, \dots, z, \tilde{y}, z}^{n-1-k}, \tilde{y}, \overbrace{z, \tilde{y}, \dots, \tilde{y}, z}^{2k}, x, \overbrace{z, \tilde{y}, z, \dots, \tilde{y}, z}^{n-1-k}\} \\ &\qquad = \{\overbrace{z, \tilde{y}, \dots, \tilde{y}, z}^{n+k-1}, x, \overbrace{z, \tilde{y}, \dots, \tilde{y}, z}^{n-k-1}\}) \end{aligned}$$

(where the label on the brace tells the number of factors z in the alternating list).

If we order the variables in T, and the lexicographically leading term of $0 \neq z^{(n-1,\tilde{y})} \in \widetilde{\mathrm{J}}$ is $0 \neq z' \in \mathrm{J}$, then the lexicographically leading term of $U_{z^{(n-1,\tilde{y})}} x$ is $U_{z'} x$. Thus we have $U_{z'} x = 0$ for all $x \in \mathrm{J}$, so $0 \neq z'$ is trivial in J, contradicting *nondegeneracy* of J. Thus $n > 1$ is impossible, as we claimed.

Thus the only possible index is $n = 1$; but then $z = z^{(1,\tilde{y})} = 0$ anyway. This establishes (2). Thus we can pass to an algebra over a field where J avoids any proper nilpotence of bounded index in J_2.

Step 3: Avoid Proper Nilpotence Entirely

We imbed J_2 in an Ω_0-algebra J_3 such that no element of J is *properly nilpotent* in J_3. An algebra J always imbeds as *constant sequences* $z \mapsto (z, z, \dots)$ in the *sequence algebra* $\mathcal{S}eq(\mathrm{J}) = \prod_1^\infty \mathrm{J}$. Let $\mathrm{J}_3 := \mathcal{S}eq(\mathrm{J}_2)$ consist of all sequences from J_2, where J_2 is identified with the constant sequences. We claim that

(3.1) there is no nonzero $z \in \mathrm{J}$ which is properly nil in J_3.

Indeed, by Step 2 there is no global bound on the index of nilpotence of z in the various $\mathrm{J}^{(y)}$'s for $y \in \mathrm{J}_2$: for each n there is a $y_n \in \mathrm{J}_2$ with $z^{(n,y_n)} \neq 0$; then the element $\vec{y} = (y_1, y_2, y_3, \dots)$ and the copy $\vec{z} = (z, z, z, \dots)$ live in the sequence algebra J_3, but for any n we have $\vec{z}^{(n,\vec{y})} = (z^{(n,y_1)}, z^{(n,y_2)}, \dots, z^{(n,y_n)}, \dots) \neq (0, 0, \dots, 0, \dots)$ because it differs from zero in the nth place, so z is not nil in the homotope $J_3^{(\vec{y})}$, i.e., z is not properly nilpotent. Thus we can pass to an algebra over a field where J avoids all proper nilpotence whatsoever in J_3.

Step 4: Avoid Proper Quasi-Invertibility

We imbed J_3 in an algebra J_4 over a big algebraically closed field such that no element of J is *properly quasi-invertible* in J_4. If $J_4 := \Omega \otimes_{\Omega_0} J_3$ for Ω algebraically closed with $|\Omega| > \dim_{\Omega_0} J_3 = \dim_\Omega J_4$, then J_4 is an algebra over a big algebraically closed field. [Recall that, as in our discussion of bigness, dimension stays the same under scalar extension: if the x_i form a basis for J_3 over Ω_0, then the $1 \otimes x_i$ form a basis for $J_4 = \Omega \otimes_{\Omega_0} J_3$ over Ω, so $\dim_{\Omega_0} J_3 = \dim_\Omega J_4$.] We claim that

(4.1) no nonzero $z \in J$ is properly quasi-invertible in J_4

because any such z would be properly nilpotent in J_4 (hence even more properly in J_3) since by Amitsur's Big Resolvent Trick 3.2.2(2) the Jacobson radical of the Jordan algebra J_4 over the big field Ω is properly nil, and by Step 3 there aren't any such elements in J. Thus we can pass to an algebra over a big algebraically closed field where J avoids all proper quasi-invertibility in J_4.

Step 5: Obtain Semiprimitivity with the Same Identities

Finally, we imbed J_4 in a semiprimitive algebra J_5 which satisfies exactly the same strict identities as J. Once we have passed to an algebra J_4 where J avoids all proper quasi-invertibility, we can surgically remove the proper quasi-invertibility to create a semiprimitive algebra, yet without disturbing the original algebra J: (4) and the Elemental Characterization 1.5.1(2) guarantee that

(5.1) $$J \cap \mathcal{R}ad(J_4) = \mathbf{0},$$

so J remains imbedded in the semiprimitive Ω-algebra $J_5 := J_4/\mathcal{R}ad(J_4)$ [by the Radical Surgery Theorem 1.7.1]. Moreover, the scalar extension J_1 inherits all *strict* identities from J; the scalar extension J_2 inherits all *strict* identities from J_1; the direct product J_3 inherits *all* identities from J_2; the scalar extension J_4 inherits all *strict* identities from J_3; and the quotient J_5 inherits *all* identities from J_4. Thus J_5 inherits *all strict* identities from J. Conversely, the subalgebra $J \subseteq J_5$ inherits *all* identities from J_5, so they have exactly the same strict identities.

At last we have shrunk the Jacobson radical away to zero, and the Ω-algebra $\widetilde{J} := J_5$ now fulfills all the requirements of our theorem: it is semiprimitive and has the same strict identities as J. By Semiprimitivity 5.3.1(2), semiprimitive Ω-algebras are subdirect products $\widetilde{J} \mathrel{\widetilde{\approx}} \prod_\alpha \widetilde{J}_\alpha$ for primitive Ω-algebras $\widetilde{J}_\alpha = \widetilde{J}/\widetilde{K}_\alpha$ for Ω-ideals $\widetilde{K}_\alpha$ in $\widetilde{J}$ with $\bigcap_\alpha \widetilde{K}_\alpha = \mathbf{0}$. Here the primitive $\widetilde{J}_\alpha$ inherit all strict identities from J_5, hence from the original J, and the algebraically closed field Ω remains big for $\widetilde{\mathrm{J}}$ and even bigger for each $\widetilde{\mathrm{J}}_\alpha$: $|\Omega| > \dim_\Omega \mathrm{J}_4 \geq \dim_\Omega \mathrm{J}_5 = \dim_\Omega \widetilde{\mathrm{J}} \geq \dim_\Omega \widetilde{\mathrm{J}}_\alpha$. J is also imbedded in the full direct product $\vec{\mathrm{J}} := \prod_\alpha \widetilde{\mathrm{J}}_\alpha$, which inherits all strict identities of J from the individual $\widetilde{\mathrm{J}}_\alpha$ (but we no longer know whether $|\Omega| > \dim_\Phi \vec{\mathrm{J}}$). $\square$

EXERCISE 5.4.1* In quadratic Jordan algebras $z^{(n,\tilde{y})} = 0$ does *not* imply $z^{(m,\tilde{y})} = 0$ for all $m \geq n$. Show instead (in a pacific manner, with no bullets, only U's) that $z^{(n,\tilde{y})} = 0$ implies that $z^{(k,\tilde{y})} = 0$ for all $m \geq 2n$.

5.5 Problems for Chapter 5

PROBLEM 5.1 (1) Show that if J is unital with unit e, so $\widehat{\mathrm{J}} = \Phi e' \boxplus \mathrm{J}$, then all structural transformations T on J extend to $\widehat{T} := \mathbb{1}' \boxplus T$, $\widehat{T^*} := \mathbb{1}' \boxplus T^*$, which are congruent to $\widehat{\mathbb{1}}$ mod J via $c = e - T(e)$. [Since all inner ideals of J are modular with modulus e, by the Modulus Shifting Lemma 5.1.2 it is not surprising that $c = e - T(e)$ is a modulus for $T(\mathrm{J})$!] (2) Show that a structural transformation T is congruent to $\widehat{\mathbb{1}}$ mod J iff $S := \widehat{\mathbb{1}} - T, S^* := \widehat{\mathbb{1}} - T^*$ satisfy $S(\mathrm{J}) + S^*(\mathrm{J}) \subseteq \mathrm{J},\ S(\hat{1}) := c, S^*(\hat{1}) := c^* \in \mathrm{J}$. (3) Show that the set $\mathcal{S}$ of structural transformations congruent to $\widehat{\mathbb{1}}$ mod J forms a monoid: $\mathbb{1}_\mathrm{J} \in \mathcal{S}$ with $c = c^* = 0$, and if $T_1, T_2 \in \mathcal{S}$ with $S_i(\hat{1}) = c_i$, then $T_1T_2 \in \mathcal{S}$ with $S_{12}(\hat{1}) = c_1 + T_1(c_2) = c_1 + c_2 - S_1(c_2)$. (3) Conclude that if $T_i \in \mathcal{S}$ with $S_i(\hat{1}) = c_i$, then $T_1 \cdots T_n \in \mathcal{S}$ with $S_{1\ldots n}(\hat{1}) = c_1 + T_1(c_2) + T_1T_2(c_3) + \cdots + T_1 \cdots T_{n-1}(c_n) = \sum_i c_i - \sum_{i<j} S_i(c_j) + \sum_{i<j<k} S_iS_j(c_k) + \cdots \pm S_1 \cdots S_{n-1}(c_n)$.

PROBLEM 5.2* Show that in fact, as in the associative case, $\mathcal{R}ad(\mathrm{J}) = \bigcap \mathrm{B}$ is the intersection of all maximal-modular inner ideals (not merely their cores). The inclusion $\mathcal{R}ad(\mathrm{J}) \subseteq \bigcap \mathrm{B}$ is clear from the core result; for the reverse inclusion $\mathcal{R}ad(\mathrm{J}) \supseteq \bigcap \mathrm{B}$, we must show that for each $z \notin \mathcal{R}ad(\mathrm{J})$ there exists a maximal modular B (depending on z) with $z \notin \mathrm{B}$. This involves a *tricky* dichotomy, considering separately the two cases (1) $z^2 \in \mathcal{R}ad(\mathrm{J})$ and (2) $z^2 \notin \mathcal{R}ad(\mathrm{J})$. In case (1) choose B containing $\mathrm{B}_0 = B_{z,y}(\mathrm{J}) < \mathrm{J}$ maximal with modulus $c = \{z, y\} - U_z y^2$ as in the above proof, and show that $z \in \mathrm{B}$ would lead to a contradiction $c = B_{z,y}(c) - U_{\hat{1}-c,z}y + \{z, B_{z,y}(y)\} + U_z(2y^2 - V_z y^3 - U_y c) + \{z^2, y^2\} \in B$. In case (2) show that there is y which is not q.i. in $\mathrm{J}^{(U_z\hat{1})}$, hence $U_z y$ which is not q.i. in J. Therefore $\mathrm{B}_0 = U_{\hat{1}-U_z y}\mathrm{J} < \mathrm{J}$ with modulus $c = 2U_z y - (U_z y)^2 \in [z]$ imbeds in a maximal c-modular B; show that $z \in \mathrm{B}$ would lead to a contradiction $c \in [z] \subseteq \mathrm{B}$.

PROBLEM 5.3* Establish the beautiful *Amitsur Polynomial Trick* for associative algebras A: If $z \in \mathrm{A}$ and the element tz is q.i. in $\mathrm{A}[t]$, then z must be nilpotent in A.

PROBLEM 5.4* Establish the equally beautiful *Zel'manov Polynomial Trick* for Jordan algebras J: If $z \in \mathrm{J}$ is properly nil of bounded index in $\widetilde{\mathrm{J}} := \mathrm{J}[T]$

for an infinite set of indeterminates, then z must be degenerate in J (in the sense of belonging to the degenerate radical $\mathcal{D}eg(\mathrm{J})$, the smallest ideal I such that J/I is nondegenerate). (1) Use *radical surgery* to reduce the problem to the case of showing that $z = 0$ when J is nondegenerate. (2) Show that if J is nondegenerate, so is any polynomial extension $\mathrm{J}[T]$. (3) When J is nondegenerate, show that no z can have bounded proper nilpotence index $n > 1$ ($z^{(m,\widetilde{\mathrm{J}})} = \mathbf{0}$ for all $m \geq n$ but some $z^{(n-1,\tilde{y})} \neq 0$). (4) Conclude that if z is properly nilpotent of bounded degree, then its index must be 1, and $z = 0$.

Problem 5.5 Tie all but one indeterminate behind your back and prove the Zel'manov Polynomial Trick for a single indeterminate: if $z \in \mathrm{J}$ is properly nil of bounded index in $\widetilde{\mathrm{J}} := \mathrm{J}[t]$, then z must be degenerate in J. Even extend the result to showing that if J is nondegenerate, then there are no elements $\tilde{z} \in \widetilde{\mathrm{J}}$ (not just in J) which are properly nilpotent of bounded index in $\widetilde{\mathrm{J}}$. Here you will have to get downright combinatorial to keep the powers of t separate. (1) Let $x \in \mathrm{J}$ and let d be the maximum of 1 and the degrees in t of $\tilde{z}, \tilde{y} \in \widetilde{\mathrm{J}}$. Show that if $s \in \Phi[t]$ is a scalar, then $\tilde{z}^{(2n,\tilde{y}+sx)} = \sum_{k=0}^{2n-1} s^k p_k(\tilde{z};\tilde{y};x)$, where each $p_k(\tilde{z};\tilde{y};x)$ is homogeneous of degree $2n, 2n-1-k, k$, respectively, in $\tilde{z}, \tilde{y}, x$. Conclude that p_k is a polynomial in t of degree $\leq (2n)d+(2n-1-k)d+(k)0 \leq (4n-1)d < 4nd$. (2) If $\tilde{z}$ is p.n.b. of index $n > 1$, $\tilde{z}^{(m,\widetilde{\mathrm{J}})} = \mathbf{0}$ but $\tilde{z}^{(n-1,\tilde{y})} \neq 0$, and set $s = t^f, f = 4nd$ and show that $0 = \tilde{z}^{(2n,\tilde{y}+sx)} = \sum_{k=0}^{2n-1} s^k p_k(\tilde{z};\tilde{y};x)$, where each $s^k p_k(\tilde{z};\tilde{y};x)$ is a sum of powers t^j whose exponents lie in the range $fk \leq j < f(k+1)$. Since the ranges are non-overlapping, identify coefficients in the range $f \leq j < 2f$ to conclude that $U_{\tilde{z}^{(n-1,\tilde{y})}} x = 0$. (3) Show that nondegeneracy of $\widetilde{\mathrm{J}}$ leads to a contradiction $\tilde{z}^{(n-1,\tilde{y})} = 0$, so again the only possibility is $n = 1, \tilde{z} = 0$.

Problem 5.6 Strengthen the Semiprimitive Imbedding Theorem 5.4.1 to show that *any* nondegenerate Jordan algebra J (prime or not) is imbedded in a semiprimitive algebra which satisfies exactly the same strict identities as J does. (1) If $\mathrm{J}_2 := \mathrm{J}[T]$ for a countable set of indeterminates T, and $\mathcal{PNBI}(\mathrm{J})$ denotes the set of all p.n.b.i. elements of J, show that $\mathrm{J} \cap \mathcal{PNBI}(\mathrm{J}_2) \subseteq \mathcal{D}eg(\mathrm{J})$. (2) If $\mathrm{J}_3 := \mathcal{S}eq(\mathrm{J}_2)$, show that $\mathrm{J}_2 \cap \mathcal{P}nil(\mathrm{J}_3) \subseteq \mathcal{PNBI}(\mathrm{J}_2)$. (3) If $\mathrm{J}_4 := \mathrm{J}_3[t]$, show that $\mathrm{J}_3 \cap \mathcal{PQI}(\mathrm{J}_4) \subseteq \mathcal{P}nil(\mathrm{J}_2)$. (4) Conclude that J is imbedded in $\widetilde{\mathrm{J}} = \mathrm{J}_4/\mathcal{R}ad(\mathrm{J}_4)$ having exactly the same strict identities as J.

Problem 5.7 Show that the algebra of fractions $\mathrm{J}_1 = \Gamma^{-1}\mathrm{J}$ of a prime nondegenerate algebra J by its centroid Γ (appearing in Step 1 of the proof of Semiprimitive Imbedding 5.4.1) is again prime nondegenerate.

6

The Primitive Heart

The final concept is that of the *heart*. It is difficult to have a heart: for nice algebras, there must be a simple ideal at the algebra's core, which makes the whole algebra but a heartbeat away from simplicity. Zel'manov made the amazing anatomical discovery that i-exceptional algebras (like Tin Woodsmen) always have hearts, consisting of the values taken on by all special identities. This is the key to the classification of primitive exceptional algebras over big fields (after which it is just a mopping-up operation to classify prime exceptional algebras in general).[1]

6.1 Hearts and Spectra

Just as in the associative case, the heart is the minimal nonzero ideal.

Heart Definition 6.1.1 *The* **heart** *of a Jordan algebra is its smallest nonzero ideal, if such exists:* $\mathbf{0} \neq \heartsuit(\mathrm{J}) = \bigcap\{\mathrm{I} \lhd \mathrm{J} \mid \mathrm{I} \neq \mathbf{0}\}$.

Of course, most algebras are heartless, and many hearts are trivial (for $\mathrm{A} = \Phi E_{11} + \Phi E_{12} \subseteq \mathcal{M}_2(\Phi)$ over a field, the heart $\heartsuit = \Phi E_{12}$ is trivial). But if there happens to be a semiprimitive heart, bounding the spectra of its elements forces simplicity of the heart.

Heart Principles 6.1.2 *Hearts have the following influences on the ambient algebra.*

(1) **Heart Indecomposability Principle**: *If* J *has a heart* $\heartsuit(\mathrm{J})$*, then* J *is indecomposable,* $\mathrm{J} \neq \mathrm{J}_1 \boxplus \mathrm{J}_2$.

(2) **Unital Heart Principle**: *If* $\heartsuit(\mathrm{J})$ *has a unit element, then* J *is all heart,* $\heartsuit(\mathrm{J}) = \mathrm{J}$.

[1] cf. I.8.5, 8.7, and 8.8, where hearts were discussed and sketches were given of the Primitive Exceptional Heart Theorem, Hearty Spectral Relations Theorem, and Big Primitive Exceptional Theorem.

(3) **Nontrivial Heart Principle**: J *is all nontrivial heart* (J $= \heartsuit$(J) *nontrivial*) *iff* J *is simple.*

(4) **Capacious Heart Principle**: *If* $\heartsuit$(J) *has a capacity* (*in particular, if* J *is semiprimitive and the elements of* $\heartsuit$(J) *have bounded spectra over a big field*), *then* J *is simple with capacity.*

PROOF. (1) If $J = J_1 \boxplus J_2$ for $J_i \neq \mathbf{0}$, we would have $\heartsuit \subseteq J_1 \cap J_2 = \mathbf{0}$, resulting in heartlessness.

(2) If $\heartsuit$ has a unit e, then $\heartsuit \subseteq J_2(e) \subseteq J_2(e) + J_1(e) = U_e(J) + U_{e,\hat{1}-e}(J) \subseteq \heartsuit$ because $\heartsuit$ is an ideal, which implies that $\heartsuit = J_2(e)$, $J_1(e) = \mathbf{0}$, $J = J_2(e) \boxplus J_0(e)$, so by (1), $J_0(e) = \mathbf{0}$ and $J = J_2(e)$ has unit e.

(3) Here $\Longleftarrow$ is clear, and $\Longrightarrow$ holds because any ideal $I \neq \mathbf{0}$ has $I \supseteq \heartsuit = J$, so J has no proper ideals and by hypothesis is nontrivial, i.e., is simple.

(4) If $\heartsuit$ has capacity, then by the Capacity Definition II.20.1.1 it has a unit, so by (2), $\heartsuit = J$ is all heart with a nontrivial idempotent, hence by (3) is simple.

To see the "in particular" statement, if J is semiprimitive over a big field the ideal $\heartsuit$ inherits semiprimitivity by Ideal Inheritance 1.6.1(4), and the field is still just as big for $\heartsuit$, so by Bounded Spectrum 3.4.1(3) the ideal $\heartsuit$ has a capacity. $\square$

For elements of the heart, the absorber spectrum and ordinary spectrum are practically the same.

Hearty Spectral Relations Proposition 6.1.3 *If z is an element of the heart* $\heartsuit$(J) *of a Jordan algebra* J *over a field* Φ, *then its absorber spectrum almost coincides with its spectrum,*

$$\mathcal{S}pec_{\Phi,J}(z) \subseteq \mathcal{A}bs\,\mathcal{S}pec_{\Phi,J}(z) \cup \{0\} \subseteq \mathcal{S}pec_{\Phi,J}(z) \cup \{0\}.$$

PROOF. We already recalled that $\mathcal{A}bs\,\mathcal{S}pec(z) \subseteq \mathcal{S}pec(z)$ in the Spectral Relations Proposition 4.2.3, so the last inclusion is clear. For the first, consider any $0 \neq \lambda$ in $\mathcal{S}pec(z)$; then $J > B = U_{\lambda\hat{1}-z}(J) = U_{\lambda(\hat{1}-w)}(J) = U_{\hat{1}-w}(J)$ for $w = \lambda^{-1}z$ [it is important here that Φ is a field]. B is an inner ideal with modulus $c = 2w - w^2 \in \heartsuit(J)$ [since $z, w \in \heartsuit(J)$] by Structural Inner Example 5.1.3(2). To show that $\lambda \in \mathcal{A}bs\,\mathcal{S}pec(z)$ we must show that $qa(B) = \mathbf{0}$. But $J > B \implies c \notin \mathcal{I}_J(qa(B))$ [by the Strong Modulus Exclusion Property 5.1.2(4)] $\implies \heartsuit(J) \not\subseteq \mathcal{I}_J(qa(B))$ [since $c \in \heartsuit(J)$] $\implies \mathcal{I}_J(qa(B)) = \mathbf{0}$ [by definition of heart] $\implies qa(B) = \mathbf{0}$. $\square$

6.2 Primitive Hearts

Zel'manov opened up a primitive i-exceptional algebra and found a very natural heart.

Primitive Exceptional Heart Theorem 6.2.1 *A primitive i-exceptional Jordan algebra has heart* $\heartsuit(\mathrm{J}) = i\text{-}\mathcal{S}pecializer(\mathrm{J})$ *consisting of all values on* J *of all s-identities.*

PROOF. $i\text{-}\mathcal{S}pecializer(\mathrm{J})$ is an ideal, and it is a *nonzero* ideal since J is *i-exceptional, i.e., not i-special,* and therefore does not satisfy all s-identities. We need to show that each nonzero ideal I contains $i\text{-}\mathcal{S}pecializer(\mathrm{J})$. Now containment $\mathrm{I} \supseteq i\text{-}\mathcal{S}pecializer(\mathrm{J})$ is equivalent to $\mathbf{0} = i\text{-}\mathcal{S}pecializer(\mathrm{J})/\mathrm{I} = i\text{-}\mathcal{S}pecializer(\mathrm{J}/\mathrm{I})$ in J/I. But from the Supplementation Property 5.2.1(1) of the primitizer P of J we have $\mathrm{J}/\mathrm{I} = (\mathrm{I}+\mathrm{P})/\mathrm{I} \cong \mathrm{P}/\mathrm{P}\cap\mathrm{I}$ where we know that $i\text{-}\mathcal{S}pecializer(\mathrm{P}/\mathrm{P}\cap\mathrm{I}) = i\text{-}\mathcal{S}pecializer(\mathrm{P})/\mathrm{P}\cap\mathrm{I} = \mathbf{0}$ from $i\text{-}\mathcal{S}pecializer(\mathrm{P}) = \mathbf{0}$ by the Primitive Proposition 5.2.2(3). □

The key to the entire classification of i-exceptional algebras turns out to be the case of primitive algebras over big fields.

Big Primitive Exceptional Theorem 6.2.2 *A primitive i-exceptional Jordan algebra over a big algebraically closed field* Φ *is a simple split Albert algebra* $\mathcal{A}lb(\Phi)$. *A primitive Jordan algebra over a big algebraically closed field is either i-special or a split Albert algebra.*

PROOF. Since every algebra is either i-special or i-exceptional, it suffices to prove the first assertion. We break the proof for a primitive i-exceptional algebra J into a few dainty steps.

Step 1: The heart $\heartsuit = i\text{-}\mathcal{S}pecializer(\mathrm{J}) \neq \mathbf{0}$ exists by the Primitive Exceptional Heart Theorem 6.2.1, so there *exists* an $f \in i\text{-}\mathcal{S}pecializer(X)$ of some finite degree N which does not vanish strictly on J. By f-Spectral Bound 3.4.2(2) there is a uniform bound $2N$ on *f–spectra* of elements of J.

Step 2: Since the absorber spectrum is contained in the f-spectrum for all nonvanishing $f \in i\text{-}\mathcal{S}pecializer(X)$ by the Spectral Relations Proposition 4.2.3 [since J primitive implies nondegenerate by Primitive Proposition 5.2.2(1)], there is a bound $2N$ on the size of *absorber spectra* of elements of J.

Step 3: Since the spectrum and absorber spectrum for a hearty element differ in size by at most 1 by the Hearty Spectral Relations Proposition 6.1.3, there is a bound $2N+1$ on *ordinary spectra* of *hearty* elements.

Step 4: Once the heart has a global bound on spectra over a *big field*, by the Capacious Heart Principle 6.1.2(4) $\heartsuit$ has capacity and $\mathrm{J} = \heartsuit$ is simple with capacity.

Step 5: Once J has simple capacity, by the Classical Structure Theorem II.23.1.2, the only *i-exceptional* simple algebra it can be is an algebra $\mathcal{H}_3(\mathrm{O}, \Gamma)$

of Albert Type: it is not of Division Type by Division Evaporation 3.3.1 over the *big field* Φ. But over an *algebraically closed field* O must be split, hence J is a split Albert algebra $\mathcal{A}lb(\Omega)$ over its center $\Omega \supseteq \Phi$. But Division Evaporation forces $\Omega = \Phi$ by bigness $\Phi > \dim_\Phi(\mathrm{J})$. □

It is not surprising that students of Jordan algebras can often be found humming to themselves the song "You've got to have heart, All you really need is heart ... ."

6.3 Problems for Chapter 6

QUESTION 6.1* Let $\mathrm{A} = \mathcal{M}_\infty(\Delta)$ be the primitive unital associative algebra of all *row-and-column-finite* $\infty \times \infty$ matrices (the matrices having only a finite number of nonzero entries from Δ in each row and column) over an associative division algebra Δ. Does A have an associative heart, or A^+ a Jordan heart?

QUESTION 6.2* (1) Let $\mathrm{A} = \mathcal{E}nd(V_\Delta)$ be the primitive unital associative algebra of all linear transformations on an infinite-dimensional left vector space V over a division ring Δ. Does A have an associative heart, or A^+ a Jordan heart? (2) Answer the same question for A the *column-finite* $\infty \times \infty$ matrices (the matrices having only a finite number of nonzero entries from Δ in each column, but no restriction on the row entries).

QUESTION 6.3* Let $\mathrm{A} = \mathcal{B}(H)$ be the unital associative algebra of all bounded linear operators on a (real or complex) Hilbert space H. Does A have a *closed heart*? In the topological category, only *closed* ideals are considered; although there are several different important topologies on bounded operators, we mean here the *norm* topology, with metric $d(T,S) := \|T-S\|$ for $\|T\| := \sup_{\|x\|\leq 1} \|T(x)\|$. The closed heart would be the nonzero intersection of all nonzero closed ideals.

QUESTION 6.4* If an associative algebra A has a simple ideal H, under what conditions is H the heart of A?

Third Phase: Logical Conclusions

In this final phase we bring in ultrafilters to show that, from purely logical considerations, our classification of i-exceptional primitive algebras over big fields is sufficient to classify i-exceptional prime nondegenerate algebras over arbitrary scalars. A nondegenerate algebra imbeds in a subdirect product of primitive algebras over big algebraically closed fields, and by our classification we know that these factors are i-special or split Albert algebras. Prime algebras face a Dichotomy: they must either imbed entirely in an ultraproduct of i-special algebras (which is itself i-special, as is any subalgebra), or they imbed entirely in an ultraproduct of split Albert algebras (which is itself a split Albert algebra over a field). The prime i-exceptional algebras opt for Albert, and we show that they actually imbed as *forms* of split Albert algebras.

We begin in Chapter 7 with the basic facts about filters, which we can think of as the collection of neighborhoods of a point. The filters we will be interested in are filters on the index set of a direct product of algebras. In general, any filter can be restricted to a subset, and any downward-directed collection of nonempty subsets can be enlarged to a filter; an important example of such a collection is the support sets of nonzero elements of a prime subalgebra of a direct product. Ultrafilters are maximal filters, satisfying the immeasurably powerful condition that for any subset, either it or its complement belongs to the filter.

In Chapter 8 we use ultrafilters to construct filtered products, quotient algebras of the direct product where two elements are equivalent if they agree on a set in the filter ("agree in a neighborhood of the point"); this is the usual quotient by the filter ideal (consisting of all elements which vanish on a set of the filter). An ultraproduct is just the filtered product by an ultrafilter. It inherits all the "elementary" algebraic properties satisfied by all the factors. Thus ultraproducts of division algebras, fields, algebraically closed fields, quadratic forms, quadratic or cubic factors, or split Albert algebras are again such.

In Chapter 9 we the examine the consequences for prime Jordan algebras. The Finite Dichotomy Principle says that an ultraproduct of algebras of a finite number of flavors must itself have one of those flavors. In particular, the Prime Dichotomy Theorem says that a prime nondegenerate Jordan algebra must either be i-special or a form of an Albert algebra, and a simple algebra must either be i-special or itself a 27-dimensional Albert algebra. This establishes Zel'manov's Exceptional Theorem: the only simple (or prime) nondegenerate i-exceptional Jordan algebras are Albert algebras (or forms thereof).

7

Filters and Ultrafilters

We have actually done all the structural work for classifying simple and prime exceptional algebras. The rest of the book will show how the (seemingly restrictive) classification over big algebraically closed fields can be spruced up to apply in complete generality.

7.1 Filters in General

We begin with the concept of filter.[1] Ultimately we will be concerned only with ultrafilters on the set of indices for a direct product, but we will start from basics with the general concepts. For any set X we denote by $\mathcal{P}(X)$ the *power set* of X, the set of all subsets of X. For any such subset $Y \subseteq X$ we denote by $Y' := X \setminus Y$ the set-theoretic complement of Y in the ambient set X.

Filter Definition 7.1.1 *A nonempty collection $\mathcal{F} \subseteq \mathcal{P}(X)$ of subsets of a given set X is called a* **filter** *on X if it has the following three properties:*

(Filt1) *it is closed under* intersections: $Y_1, Y_2 \in \mathcal{F} \implies Y_1 \cap Y_2 \in \mathcal{F}$;

(Filt2) *it is closed under* enlargement: $Z \supseteq Y \in \mathcal{F} \implies Z \in \mathcal{F}$;

(Filt3) *it is* proper: $\emptyset \notin \mathcal{F}$.

(Filt1) *is equivalent to closure under* finite intersections,

(Filt1)′ $Y_1, \ldots, Y_n \in \mathcal{F} \implies Y_1 \cap \cdots \cap Y_n \in \mathcal{F}$.

(Filt2) *is equivalent to closure under* unions,

(Filt2)′ $Y \in \mathcal{F},\ Z \subseteq X \implies Y \cup Z \in \mathcal{F}$,

and in particular, implies

(Filt2)″ $X \in \mathcal{F}$.

[1] Filters, ultrafilters, and support filters were introduced in I.8.10–8.11.

(Filt3) *is equivalent, in view of* (Filt2), *to*

(Filt3)′ $\mathcal{F} \neq \mathcal{P}(X)$.

Filters are "dual ideals" in the lattice $\mathcal{P}(X)$; just as an ideal $\mathrm{I} \triangleleft \mathrm{A}$ has $y_1, y_2 \in \mathrm{I} \Longrightarrow y_1 + y_2 \in \mathrm{I}$ and $y \in \mathrm{I}, z \in \mathrm{A} \Longrightarrow y \cdot z \in \mathrm{I}$, $\mathcal{F}$ satisfies the corresponding properties (Filt1), (Filt2) with $+, \cdot$ replaced by $\cap, \cup$.

Principal Example 7.1.2 *The* **principal filter** $\mathcal{F}_{X_0}$ *on* X *determined by a subset* $X_0 \subseteq X$ *consists of all subsets* $Y \supseteq X_0$ *containing* X_0. □

Restriction and Intersection Proposition 7.1.3 *The* **restriction filter** $\mathcal{F}|_Y$ *on* Y *determined by a subset* $Y \in \mathcal{F}$ *consists of all subsets of* $\mathcal{F}$ *contained in* Y,

$$\mathcal{F}|_Y = \{Z \in \mathcal{F} \mid Z \subseteq Y\} = \mathcal{F} \cap \mathcal{P}(Y).$$

This coincides with the **intersection filter** $\mathcal{F} \cap Y$ *consisting of all intersections of elements of* $\mathcal{F}$ *with* Y,

$$\mathcal{F} \cap Y = \{Z \cap Y | Z \in \mathcal{F}\}.$$

PROOF. $\mathcal{F}|_Y$ effortlessly inherits (Filt1), (Filt2), (Filt3) from $\mathcal{F}$, hence is a filter on Y; it is equal to $\mathcal{F} \cap Y$ because $Z \in \mathcal{F}|_Y \Longrightarrow Z = Z \cap Y \in \mathcal{F} \cap Y$, and $Z \cap Y \in \mathcal{F} \cap Y$ for $Z \in \mathcal{F} \Longrightarrow Z \cap Y \in \mathcal{F}$ [by (Filt1) for $Z, Y \in \mathcal{F}$] $\Longrightarrow Z \cap Y \in \mathcal{F}|_Y$. □

EXERCISE 7.1.3 Let $X_0 \subseteq X$ be a subset, and $\mathcal{F}$ a filter on X. (1) Show that $\mathcal{F}|_{X_0}$ automatically satisfies (Filt1), (Filt2), (Filt3), but is nonempty iff $X_0 \in \mathcal{F}$. (2) Show that $\mathcal{F} \cap X_0$ automatically satisfies (Filt1), (Filt2), and is nonempty, but satisfies (Filt3) iff $X \setminus X_0 \notin \mathcal{F}$.

Enlargement Proposition 7.1.4 *Every nonempty collection* $\mathcal{F}_0$ *of nonempty subsets of* X *which is* directed downwards ($Y_1, Y_2 \in \mathcal{F}_0 \Longrightarrow Y_1 \cap Y_2 \supseteq Y_3$ *for some* $Y_3 \in \mathcal{F}_0$) *generates on* X *an* **enlargement filter**

$$\overline{\mathcal{F}_0} = \{Z \mid Z \supseteq Y \textit{ for some } Y \in \mathcal{F}_0\}.$$

PROOF. (Filt2) is trivial [anything larger than something larger than $Y \in \mathcal{F}_0$ is even larger than that same Y], (Filt3) is easy [if $\emptyset$ wasn't in the collection before, you won't get it by taking enlargements: $\emptyset \notin \mathcal{F}_0 \Longrightarrow \emptyset \notin \overline{\mathcal{F}}_0$], while (Filt1) uses *downward directedness* [$Z_i \supseteq Y_i \in \mathcal{F}_0 \Longrightarrow Z_1 \cap Z_2 \supseteq Y_1 \cap Y_2 \supseteq Y_3 \in \mathcal{F}_0$]. □

7.2 Filters from Primes

In algebra, the most important way to construct downwardly directed collections $\mathcal{F}_0$ (hence filters $\overline{\mathcal{F}}_0$) is through prime subalgebras of direct products.

Recall from the Direct Sum Definition II.1.2.6 the *direct product* $\prod_{x\in X} \mathrm{A}_x$ of algebraic systems A_x consists of all "X-tuples" $a = \prod_x a_x$ of elements $a_x \in \mathrm{A}_x$, or more usefully all *functions* on X whose value $a(x) := a_x$ at any x lies in A_x, under the *pointwise operations* on functions.

Support Set Definition 7.2.1 (1) *If* $\mathrm{A} = \prod_{x\in X} \mathrm{A}_x$ *is a direct product of linear algebraic systems (additive abelian groups with additional structure), for* $a \in \mathrm{A}$ *let the* **zero set** *and the* **support set** *of* a *be*

$$\mathcal{Z}ero(a) = \{x \in X \mid a(x) = 0\}, \quad \mathcal{S}upp(a) = \{x \in X \mid a(x) \neq 0\},$$

so

$$\mathcal{Z}ero(a) = X \iff \mathcal{S}upp(a) = \emptyset \iff a = 0.$$

(2) *If* $\mathcal{I}_{\mathrm{A}}(a)$ *denotes the ideal in* A *generated by* a, *we have*

$$b \in \mathcal{I}_{\mathrm{A}}(a) \Longrightarrow \mathcal{Z}ero(b) \supseteq \mathcal{Z}ero(a),\ \mathcal{S}upp(b) \subseteq \mathcal{S}upp(a),$$

since if $\pi_x(a) = a(x)$ *denotes the projection of* A *onto the* x*th coordinate* A_x, $x \in \mathcal{Z}ero(a) \iff \pi_x(a) = 0 \iff a \in \mathrm{Ker}(\pi_x) \Longrightarrow b \in \mathcal{I}_{\mathrm{A}}(a) \subseteq \mathrm{Ker}(\pi_x) \Longrightarrow \pi_x(b) = 0 \Longrightarrow x \in \mathcal{Z}ero(b)$.

Prime algebras are precisely the algebraic creatures who benefit most from filtration.

Prime Example 7.2.2 *An algebraic system is* **prime** *if the* product *of two nonzero ideals is again nonzero. For linear algebras the product is taken to be the usual bilinear product* $\mathrm{Prod}(I_1, I_2) := I_1 I_2$, *but for Jordan algebras the correct product is the quadratic product* $\mathrm{Prod}(I_1, I_2) := U_{I_1}(I_2)$. *In either case, two nonzero ideals have nonzero* intersection $\mathrm{I}_1 \cap \mathrm{I}_2 \supseteq \mathrm{Prod}(\mathrm{I}_1, \mathrm{I}_2) \neq \mathbf{0}$. *If* $\mathrm{A}_0 \neq \mathbf{0}$ *is a prime nonzero subsystem of the direct product* $\mathrm{A} = \prod_{x\in X} \mathrm{A}_x$, *then*

$$\mathcal{S}upp(\mathrm{A}_0) = \{\mathcal{S}upp(a_0) \mid 0 \neq a_0 \in \mathrm{A}_0\}$$

is a nonempty collection [since $\mathrm{A}_0 \neq \mathbf{0}$*] of nonempty subsets [by the Support Set Definition* 7.2.1(2), *since* $a_0 \neq 0$*] which is directed downwards: if* $a_1, a_2 \neq 0$ *in* A_0, *then* by primeness $\mathcal{I}_{\mathrm{A}}(a_1) \cap \mathcal{I}_{\mathrm{A}}(a_2) \neq 0$ *contains some* $a_3 \neq 0$, *hence* $\mathcal{S}upp(a_3) \subseteq \mathcal{S}upp(a_1) \cap \mathcal{S}upp(a_2)$ *by Support Set* (2) *again. Therefore by the Enlargement Proposition* 7.1.4 *we always have the* **support filter** $\mathcal{F}(\mathrm{A}_0)$ *of* A_0, *the enlargement filter*

$$\mathcal{F}(\mathrm{A}_0) := \overline{\mathcal{S}upp(\mathrm{A}_0)} = \{Z \mid Z \supseteq \mathcal{S}upp(a_0) \text{ for some } 0 \neq a_0 \in \mathrm{A}_0\}.$$

Recall that we are trying to analyze prime Jordan algebras A_0. We will eventually imbed these in such a direct product $\mathrm{A} = \prod_{x\in X} \mathrm{A}_x$ (which is very far from being prime, since any two of its factors $\mathrm{A}_x, \mathrm{A}_y$ are orthgonal), and will use the support sets of A_0 to generate a filter as above, and then return to primeness via an ultraproduct. We now turn to these mysterious beasts.

7.3 Ultimate Filters

An ultrafilter is just a filter which is as big as it can be (and still remain a filter). We will use filters to chop down a direct product; to get the result as tight as possible, we must make the filter as large as possible. Ultrafilters are so large that their power is almost magical.

Ultrafilter Definition 7.3.1 *An* **ultrafilter** *is a maximal filter.*

Principal Ultrafilter Example 7.3.2 *Any element $x_0 \in X$ determines a* **principal ultrafilter** $\mathcal{F}_{x_0}$ *consisting of all subsets containing the point x_0.*

It is useful to think of an ultrafilter as the collection $\mathcal{F}_{x_\infty}$ of all neighborhoods of an ideal point x_∞ of some logical "closure" or "completion" $\overline{X}$ of X. The case of a principal ultrafilter is precisely the case where this ideal point actually exists inside X, $x_\infty = x_0$. We will see that these are the uninteresting ultrafilters.

Just as every proper ideal in a unital algebra can be imbedded in a maximal ideal, we have an analogous result for filters.

Ultra Imbedding Theorem 7.3.3 *Every filter is contained in an ultrafilter.*

PROOF. We apply Zorn's Lemma to the collection of filters containing a given filter $\mathcal{F}_0$. Notice that the union $\mathcal{F}$ of a chain (or even upwardly directed set) of filters $\{\mathcal{F}_i\}$ is again a filter. Indeed, (Filt1) holds by directedness: if $Y_1, Y_2 \in \mathcal{F}$ then $Y_1 \in \mathcal{F}_i, Y_2 \in \mathcal{F}_j$ for some $i, j \Longrightarrow Y_1, Y_2 \in \mathcal{F}_k$ for some k by directedness, so $Y_1 \cap Y_2 \in \mathcal{F}_k$ [by (Filt1) for $\mathcal{F}_k$], hence $Y_1 \cap Y_2 \in \mathcal{F}$. (Filt2) is obvious. (Filt3) holds because properness can be phrased in terms of avoiding a specific element $\emptyset$. Thus the hypotheses of Zorn's Lemma are met, and it guarantees the existence of a maximal filter, i.e., an ultrafilter, containing $\mathcal{F}_0$. □

Outside of the rather trivial principal ultrafilters, *all* ultrafilters arise by imbedding via Zorn's Lemma, and so have no concrete description at all! Indeed, one can show that to require that a countable set of natural numbers $\mathbb{N}$ *have* a non-principal ultrafilter on it is a set-theoretic assumption stronger than the Countable Axiom of Choice!

Prime ideals in a commutative associative ring are characterized by the property $y_1 y_2 \in \mathrm{I} \Longrightarrow y_1 \in \mathrm{I}$ or $y_2 \in \mathrm{I}$. Ultrafilters have an analogous characterization.

Ultrafilter Characterization Theorem 7.3.4 *The following conditions on a filter $\mathcal{F}$ are equivalent:*

(UFilt1) $\mathcal{F}$ *is an ultrafilter*;

(UFilt2) $Y_1 \cup Y_2 \in \mathcal{F} \implies Y_1 \in \mathcal{F}$ *or* $Y_2 \in \mathcal{F}$;

(UFilt3) $Y_1 \cup \cdots \cup Y_n \in \mathcal{F} \implies$ *some* $Y_i \in \mathcal{F}$;

(UFilt4) *for all* $Y \subseteq X$, *either* $Y \in \mathcal{F}$ *or* $Y' \in \mathcal{F}$.

PROOF. (UFilt1) $\implies$ (UFilt2): if $Y_1 \cup Y_2 \in \mathcal{F}$ but $Y_1, Y_2 \notin \mathcal{F}$, we claim that $\mathcal{G} = \{Z | Z \supseteq W \cap Y_1 \textit{ for some } W \in \mathcal{F}\} > \mathcal{F}$ is a strictly larger filter (which will contradict the maximality (UFilt1)). Certainly, $\mathcal{G} \supseteq \mathcal{F}$: any $W \in \mathcal{F}$ contains $W \cap Y_1$, and so falls in $\mathcal{G}$. The containment is strict because $Y_1 \notin \mathcal{F}$ by hypothesis, but $Y_1 \in \mathcal{G}$ because $Y_1 = (Y_1 \cup Y_2) \cap Y_1 = W \cap Y_1$ for $W = Y_1 \cup Y_2 \in \mathcal{F}$ by hypothesis. Thus $\mathcal{G}$ is strictly larger.

Now we check that $\mathcal{G}$ is a filter: (Filt2) is trivial; (Filt1) holds because $Z_i \supseteq W_i \cap Y_1$ for $W_i \in \mathcal{F} \implies Z_1 \cap Z_2 \supseteq W_1 \cap W_2 \cap Y_1$ for $W_1 \cap W_2 \in \mathcal{F}$ [by (Filt1) for $\mathcal{F}$]; and (Filt3) must hold or else for some $W \in \mathcal{F}$ we would have $W \cap Y_1 = \emptyset \implies W \subseteq Y_1' \implies Y_1' \in \mathcal{F}$ [by (Filt2) for $\mathcal{F}$] $\implies Y_1' \cap Y_2 = Y_1' \cap (Y_1 \cup Y_2) \in \mathcal{F}$ [by (Filt1) for $\mathcal{F}$ because $Y_1 \cup Y_2 \in \mathcal{F}$ by hypothesis] $\implies Y_2 \in \mathcal{F}$ [by (Filt2) for $\mathcal{F}$], contrary to hypothesis.

(UFilt2) $\iff$ (UFilt3): $\impliedby$ is clear, and $\implies$ follows by an easy induction.

(UFilt2) $\implies$ (UFilt4): apply (UFilt2) with $Y_1 = Y, Y_2 = Y'$, since $Y \cup Y' = X$ is always in $\mathcal{F}$ by (F2$''$).

(UFilt4) $\implies$ (UFilt1): if $\mathcal{F}$ were properly contained in a filter $\mathcal{G}$ there would exist $Z \in \mathcal{G}$ with $Z \notin \mathcal{F}$; then $Z' \in \mathcal{F} \subseteq \mathcal{G}$ by (UFilt4), and so $\emptyset = Z \cap Z' \in \mathcal{G}$ by (Filt1) for $\mathcal{G}$, contradicting (Filt3) for $\mathcal{G}$. □

EXERCISE 7.3.4 Show that a collection of subsets of X is an ultrafilter on X iff it satisfies (Filt1), (UFilt4), (Filt3) (i.e., (Filt2) is superceded by (UFilt4)).

Ultrafilter Restriction Example 7.3.5 *The restriction $\mathcal{F}|_Y$ of an ultrafilter $\mathcal{F}$ on a set X to a set $Y \in \mathcal{F}$ is an ultrafilter on Y.*

PROOF. $\mathcal{F}|_Y$ is a filter on Y by the Restriction and Intersection Proposition 7.1.3; we verify that $\mathcal{F}|_Y$ has the complementation property (UFilt4) of an ultrafilter: $Z \subseteq Y$, $Z \notin \mathcal{F}|_Y \implies Z \notin \mathcal{F} \implies Z' \in \mathcal{F}$ [by (UFilt4) for $\mathcal{F}$] $\implies Y \setminus Z = Y \cap Z' \in \mathcal{F} \implies Y \setminus Z \in \mathcal{F}|_Y$. □

7.4 Problems for Chapter 7

PROBLEM 7.1* (1) Show that the collection of all subsets of $\mathbb{N}$ with finite complement, or of $\mathbb{R}^+$ with bounded complement, forms a filter $\mathcal{F}_0$. Show that $\mathcal{F}_0$ can be described as the enlargement filter (as in Enlargement Proposition 7.1.4) of the collection of intervals (N, ∞). If $\mathcal{F}$ is an ultrafilter containing $\mathcal{F}_0$, show that $\mathcal{F}$ cannot be principal. (2) Let X be the set $\mathbb{Z}$ of all integers or the set $\mathbb{R}$ of all real numbers. Show similarly that the collection of all subsets of X with *finite* complement forms a filter $\mathcal{F}_0$; if $\mathcal{F}$ is an ultrafilter which contains $\mathcal{F}_0$, show that $\mathcal{F}$ is not principal. Show that $\mathcal{F}_0$ is contained in the filter $\mathcal{F}_1$ of all subsets with *bounded* complement.

PROBLEM 7.2* Show that if an ultrafilter contains a finite subset Y, it must be a principal ultrafilter.

PROBLEM 7.3* Let $\mathcal{F}$ be the set of neighborhoods of a point x_0 of a topological space X (the subsets $Y \supseteq U \ni x_0$ containing an open set U around x_0); show that $\mathcal{F}$ is a filter.

8

Ultraproducts

The chief use made of ultrafilters in algebra is in the construction of ultraproducts, which are quotients of direct products. We will show that every prime subalgebra of a direct product imbeds in an ultraproduct. Ultraproducts are logical "models" of the original algebra, retaining all its elementary algebraic properties.[1]

8.1 Ultraproducts

We begin with the basic facts about filtered products in general.

Filtered Product Definition 8.1.1 (1) *Let* $\mathrm{A} = \prod \mathrm{A}_x$ *be a direct product of algebraic systems indexed by a set* X*, and let* $\mathcal{F}$ *be a filter on* X*. The* **congruence** $\equiv_{\mathcal{F}}$ *determined by* $\mathcal{F}$ *is*

$$a \equiv_{\mathcal{F}} b \textit{ iff } a \textit{ agrees with } b \textit{ on some } Y \in \mathcal{F} \ (a(x) = b(x) \textit{ for all } x \in Y);$$

equivalently, (*in view of the enlargement property* (Filt2) *of filters*), *their* **agreement set** *belongs to* $\mathcal{F}$:

$$\mathcal{A}gree(a, b) := \{x \in X \mid a(x) = b(x)\} \in \mathcal{F}.$$

The **filtered product** $(\prod \mathrm{A}_x)/\mathcal{F}$ *is the quotient* $\mathrm{A}/\equiv_{\mathcal{F}}$*, under the induced operations. Intuitively, this consists of "germs" of functions* (*as in the theory of varieties or manifolds*), *where we identify two functions if they agree "locally" on some "neighborhood"* Y*.*

(2) *When the* A_x *are* linear *algebraic systems with underlying additive abelian groups, then the agreement set can be replaced by the zero set of the difference,* $\mathcal{A}gree(a, b) = \mathcal{Z}ero(a - b)$*, and the congruence can be replaced by the* **filter ideal**

[1] The definitions and basic facts about ultraproducts were sketched in I.8.10.

$$\mathrm{I}(\mathcal{F}) = \{a \in \mathrm{A} \mid a \equiv_{\mathcal{F}} 0\} = \{a \in \mathrm{A} \mid \mathcal{Z}ero(a) \in \mathcal{F}\}$$

(*where* $a \equiv_{\mathcal{F}} b$ *iff* $a-b \equiv_{\mathcal{F}} 0$ *iff* $a-b \in \mathrm{I}(\mathcal{F})$). *Then the filtered product modulo the congruence* $\equiv_{\mathcal{F}}$ *can be replaced by the quotient modulo the filter ideal*:

$$(\prod \mathrm{A}_x)/\mathcal{F} = (\prod \mathrm{A}_x)/\mathrm{I}(\mathcal{F}).$$

Ultraproduct Definition 8.1.2 *An* **ultraproduct** *is just a product filtered by an ultrafilter.*

In the ultraproduct we are identifying two functions if they agree "on a neighborhood of the ideal point x_∞," and the resulting quotient can be thought of as an "ideal factor A_{x_∞}." Note that if $\mathcal{F}$ is the principal ultrafilter $\mathcal{F}_{x_0}$, then the ultraproduct $(\prod \mathrm{A}_x)/\mathcal{F} \cong \mathrm{A}_{x_0}$ is precisely the x_0th factor. From this point of view, the principal ultrafilters are worthless, producing no new ultraproducts.

Filter Restriction Theorem 8.1.3 *Let* $\mathrm{A} = \prod_{x \in X} \mathrm{A}_x$ *be a direct product of linear algebraic systems.*

(1) *The bigger the filter on* X, *the smaller the filtered product*: *if* $\mathcal{F} \subseteq \mathcal{G}$ *are filters on* X, *then* $\mathrm{I}(\mathcal{F}) \subseteq \mathrm{I}(\mathcal{G})$ *induces a canonical projection*

$$\mathrm{A}/\mathcal{F} \longrightarrow \mathrm{A}/\mathcal{G}.$$

(2) *If* $X_0 \in \mathcal{F}$, *then for the direct product* $\mathrm{A}_0 = \prod_{x \in X_0} \mathrm{A}_x$ *and the restriction filter* $\mathcal{F}_0 = \mathcal{F}|_{X_0}$ *on* X_0, *we have a natural isomorphism*

$$\mathrm{A}/\mathcal{F} \cong \mathrm{A}_0/\mathcal{F}_0$$

so that for any $X_0 \in \mathcal{F}$ *we can discard all factors* A_x *for* $x \notin X_0$.

(3) *If* A_0 *is a prime subalgebra of* A, *then* A_0 *remains imbedded in the filtered product* $\mathrm{A}/\mathcal{F}$ *for any filter* $\mathcal{F} \supseteq \mathcal{F}(\mathrm{A}_0)$ *containing the support filter of* A_0.

(4) *In particular, any prime subalgebra* A_0 *of* A *imbeds in an ultraproduct* $\mathrm{A}/\mathcal{F}$ *for* $\mathcal{F} \supseteq \mathcal{F}(\mathrm{A}_0)$.

PROOF. (1) is clear from the Filtered Product Definition 8.1.1(1), since $a \equiv_{\mathcal{F}} b \Longrightarrow a \equiv_{\mathcal{G}} b$.

(2) We have a canonical inclusion in: $\mathrm{A}_0 \hookrightarrow \mathrm{A}$ by $in(a_0)(x) = a_0(x)$ if $x \in X_0$, $in(a_0)(x) = 0$ if $x \notin X_0$. This induces an epimorphism $f = \pi \circ in$: $\mathrm{A}_0 \longrightarrow \mathrm{A}/\mathcal{F}$, since if $a \in \mathrm{A}$ then its restriction $a_0 \in \mathrm{A}_0$ to X_0 has $in(a_0) = a$ on $X_0 \in \mathcal{F}$, $in(a_0) \equiv_{\mathcal{F}} a$, $f(a_0) = \pi(in(a_0)) = \pi(a)$. The kernel of f consists of all a_0 with $in(a_0) = 0$ on some $Y \in \mathcal{F}$, i.e., $a_0 = 0$ on $Y \cap X_0 \in \mathcal{F} \cap X_0 = \mathcal{F}|_{X_0}$ [by the Restriction Filter Proposition 7.1.3] $= \mathcal{F}_0$, i.e., $a_0 \in \mathrm{I}(\mathcal{F}_0)$, so f induces an isomorphism $\mathrm{A}_0/\mathcal{F}_0 = \mathrm{A}_0/\mathrm{I}(\mathcal{F}_0) \longrightarrow \mathrm{A}/\mathcal{F}$.

(3) No nonzero a_0 is killed by the imbedding, because if $0 \neq a_0 \in \mathrm{A}_0$ has $a_0 \equiv_{\mathcal{F}} 0$, then $\mathcal{Z}ero(a_0) \in \mathcal{F}$ by the Filtered Product Definition of the

congruence, yet $\mathcal{S}upp(a_0) \in \mathcal{F}_0 \subseteq \mathcal{F}$ by hypothesis on $\mathcal{F}$, so $\emptyset = \mathcal{Z}ero(a_0) \cap \mathcal{S}upp(a_0) \in \mathcal{F}$ by (Filt2) for $\mathcal{F}$, which would contradict (Filt3).

(4) follows because every filter $\mathcal{F}_0$ imbeds in an ultrafilter $\mathcal{F}$, and we apply (3). □

Though we will not take the long detour necessary to *prove* it, we want to at least *state* the result which guarantees that ultraproducts are tight models.

Basic Ultraproduct Fact 8.1.4 *Any elementary property true of all factors* A_x *is inherited by any ultraproduct* $(\prod \mathrm{A}_x)/\mathcal{F}$. □

Elementary is here a technical term from mathematical logic. Roughly, it refers to a property describable (using universal quantifiers) in terms of a finite number of elements of the system. For example, algebraic closure of a field requires that each nonconstant polynomial have a root in the field, and this is elementary [for each *fixed* $n > 1$ and *fixed* $a_0, \ldots, \alpha_n$ in Φ there *exists* a $\lambda \in \Phi$ with $\sum_{i=0}^n \alpha_i \lambda^i = 0$]. However, simplicity of an algebra makes a requirement on *sets* of elements (ideals), or on existence of a finite number n of elements without any bound on n [for each fixed $a \neq 0$ and b in A there *exists* an n and a set $c_1, \ldots, c_n; d_1, \ldots, d_n$ of $2n$ elements with $b = \sum_{i=1}^n c_i a d_i$]. The trouble with such a condition is that as x ranges over X the numbers $n(x)$ may tend to infinity, so that there is no *finite* set of elements $c_i(x), d_i(x)$ in the direct product with $b(x) = \sum_{i=1}^n c_i(x)a(x)d_i(x)$ for *all* x.

8.2 Examples

Rather than give a precise definition of "elementary," we will go through some examples in detail, including the few that we need.

Identities Example 8.2.1 *The property of having a multiplicative unit or of satisfying some identical relation (such as the commutative law, anti-commutative law, associative law, Jacobi identity, Jordan identity, left or right alternative law) is inherited by direct products and homomorphic images, so certainly is inherited by the ultraproduct* $(\prod \mathrm{A}_x)/\mathcal{F}$. □

Division Algebra Example 8.2.2 *Any ultraproduct of division algebras is a division algebra. In particular, any ultraproduct of fields is a field. Moreover, any ultraproduct of algebraically closed fields is again an algebraically closed field.*

PROOF. The division algebra condition is that every element $a \neq 0$ has a multiplicative inverse b ($ab = ba = 1$ in associative or alternative algebras, $a \bullet b = 1$ and $a^2 \bullet b = a$ in Jordan algebras, or $U_a b = a, U_a b^2 = 1$ in quadratic Jordan algebras). Here the direct product $\prod \mathrm{A}_x$ most definitely does *not* inherit this condition – there are lots of nonzero functions having many zero values, whereas an invertible element of the direct product must have *all* its values invertible (hence nonzero). However, if a function $a \in \mathrm{A}$ is nonzero in the ultraproduct $\mathrm{A}/\mathrm{I}(\mathcal{F})$, then *most* of its values are nonzero: $a \notin \mathrm{I}(\mathcal{F}) \Longrightarrow \mathcal{Z}ero(a) \notin \mathcal{F}$ by Filtered Product Definition 8.1.1(2). Then the complement $Y = \mathcal{S}upp(a)$ *must be in* $\mathcal{F}$ *by ultrafilter property* (UFilt4). Since a is nonzero on Y, for each $y \in Y$ the element $a(y) \neq 0$ has an inverse $b(y)$ in the division algebra A_y, and we can *define* an element b of the direct product by $b(x) = b(y)$ if $x = y \in Y$ and $b(x) = 0$ if $x \notin Y$; then the requisite relations ($ab = 1$ or whatever) hold at each $y \in Y$, therefore hold globally in the quotient $\mathrm{A}/\mathcal{F}$ ($ab \equiv_{\mathcal{F}} 1$ or whatever), and b is the desired inverse of a.

Now consider algebraic closure. Recall that a field Ω is *algebraically closed* if every monic polynomial of degree ≥ 1 has a root in Ω (this guarantees that all non-constant polynomials split entirely into linear factors over Ω). If $\Omega = \left(\prod_{x \in X} \Omega_x\right)/\mathcal{F}$ is an ultraproduct of algebraically closed fields Ω_x, we know that Ω is itself a field; to show that it is algebraically closed, we must show that any monic polynomial $\pi(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots + t^n \in \Omega[t]$ of degree $n \geq 1$ over Ω has a root $\beta \in \Omega$. This is easy to do, since it is already true at the level of the direct product (the filter $\mathcal{F}$ is superfluous). Choose pre-images $a_i \in \prod \Omega_x$ of the $\alpha_i \in \Omega$, and consider the polynomial $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + t^n \in (\prod \Omega_x)[t]$ over the direct product. For each fixed x the x-coordinate or value $p(t)(x) = a_0(x) + a_1(x)t + a_2(x)t^2 + \cdots + t^n \in \Omega_x[t]$ is a monic polynomial of degree $n \geq 1$, so *by the hypothesis that* Ω_x *is algebraically closed* it has a root $b(x) \in \Omega_x$. These $b(x)$ *define* an element b of the direct product which is a root of $p(t)$: $p(b) = a_0 + a_1 b + a_2 b^2 + \cdots + b^n = 0$ in $\prod \Omega_x$ because $p(b(x)) = a_0(x) + a_1(x)b(x) + a_2(x)b(x)^2 + \cdots + b(x)^n = 0$ in Ω_x for each x. Then the image β of b in Ω is the desired root of $\pi(t)$.[2] $\square$

Quadratic Form Example 8.2.3 *Any ultraproduct of quadratic forms over fields is again a quadratic form over a field. If the individual quadratic forms are nondegenerate, so is the ultraproduct.*

PROOF. If Q_x are quadratic forms on vector spaces V_x over fields Φ_x, set $V := \prod V_x$, $\Phi := \prod \Phi_x$, $Q := \prod Q_x$; then it is easy to check that Q is a quadratic form on V over the ring Φ of scalars. If $\mathcal{F}$ is an ultrafilter on X, set $V' := V/\mathcal{F}$, $\Phi' := \Phi/\mathcal{F}$; by the previous Division Algebra Example 8.2.2 Φ' is a field, and it is routine to check that V' remains a vector space over Φ'. We claim that Q induces a quadratic form $Q' = Q/\mathcal{F}$ on V' over Φ' via $Q'(a') := Q(a)'$ (where $'$ denotes coset mod $\mathrm{I}(\mathcal{F})$ as in the Filtered Product Definition 8.1.1). It suffices if this is a well-defined map, since it automatically inherits the identities which characterize a quadratic form. But if $a \equiv_{\mathcal{F}} b$ in V, then for some $Y \in \mathcal{F}$ and all $y \in Y$ we have $a(y) = b(y)$ in V_y, $Q(a)(y) = Q_y(a(y)) = Q_y(b(y)) = Q(b)(y)$ in Φ_y, therefore $Q(a) \equiv_{\mathcal{F}} Q(b)$ in $\Phi/\mathcal{F}$, and $Q'(a') = Q'(b')$ in $\Phi' = \Phi/\mathcal{F}$.

[2] Note that at each x there are n choices for $b(x)$ in Ω_x, so there are usually *infinitely many* roots b in the direct product. But since the ultrafilter creates a field, magically it must reduce this plethora of roots to exactly n, counting multiplicities!

It remains only to check that Q' is nondegenerate as a quadratic form if each Q_x is. If $z' = z_{\mathcal{F}} \neq 0'$ in V', then $\mathcal{Z}ero(z) \notin \mathcal{F} \Longrightarrow Y = \mathcal{S}upp(z) \in \mathcal{F}$ by (UFilt4) as always; since each Q_x is nondegenerate, once $z_y \neq 0$ there exists a_y with $Q_y(z_y, a_y) \neq 0$ [here we are using the fact that $\frac{1}{2} \in \Phi$; for general Φ the argument is a bit longer]. The element $a \in V$ *defined* by $a(x) = a_y$ if $x = y \in Y$, and $a(x) = 0$ if $x \notin Y$, has $Q(z,a)(y) \neq 0$ for all $y \in Y \in \mathcal{F}$, so $Q'(z', a') \neq 0'$ in $\Phi' = \Phi/\mathcal{F}$, and Q' is nondegenerate. □

EXERCISE 8.2.3* (1) Argue a bit longer in the nondegeneracy argument to show that over *arbitrary* fields (allowing characteristic 2) an ultraproduct $Q' = Q/\mathcal{F}$ on V' over Φ' of nondegenerate quadratic forms remains nondegenerate. As above, show that if $z' \neq 0'$ in V', then $\mathcal{S}upp(z) = Y = Y_1 \cup Y_2 \in \mathcal{F}$ for $Y_1 := \{y \in Y \mid \text{there exists } a(y) \text{ with } Q_y(z(y), a(y)) \neq 0\}$, and $Y_2 := \{y \in Y \mid Q_y(z(y)) \neq 0\}$, and use the properties of ultraproducts to show that either $Q'(z') \neq 0'$ or $Q'(z', a') \neq 0'$ for some a', hence $z' \notin \mathcal{R}ad(Q')$. (2) Show more generally that $\mathcal{R}ad(Q') \cong \big(\prod \mathcal{R}ad(Q_x)\big)/\mathcal{F}$.

A similar result holds for cubic factors constructed by the Freudenthal–Springer–Tits Constructions in Chapter II.4 from Jordan cubic forms N with adjoints $\#$ and basepoints c.

Split Albert Example 8.2.4 *Any ultraproduct of split Albert algebras over fields is a split Albert algebra over a field. If the factors are split over algebraically closed fields, so is the ultraproduct.*

PROOF. Recall the *Split Albert Algebra* $\mathcal{A}lb(\Phi)$ over any scalar ring Φ in the Reduced Albert Algebra Theorem II.4.4.2. If we set $\mathrm{A}_x := \mathcal{A}lb(\Phi_x)$, $\mathrm{A} := \prod \mathrm{A}_x$, $\Phi := \prod \Phi_x$ then $\mathrm{A} := \prod \mathcal{A}lb(\Phi_x) \cong \mathcal{A}lb(\prod \Phi_x) = \mathcal{A}lb(\Phi)$, and for any ultrafilter on X we have $\mathrm{A}/\mathcal{F} \cong \mathcal{A}lb(\Phi/\mathcal{F})$ [the ideal $\mathrm{I}(\mathcal{F})$ in the direct product A as in Filtered Product Definition 8.1.1(2) consists of all hermitian 3×3 matrices with entries in the corresponding ideal $\mathcal{O}\big(\mathrm{I}_\Phi(\mathcal{F})\big)$ of split octonion elements of the direct product $\mathcal{O}\big(\Phi\big)$, which by abuse of language we may write as $\mathrm{I}(\mathcal{F}) = \mathcal{A}lb(\mathrm{I}_\Phi(\mathcal{F}))$, and we have $\mathcal{A}lb(\Phi)/\mathcal{A}lb(\mathrm{I}_\Phi(\mathcal{F})) = \mathcal{A}lb(\Phi/\mathrm{I}_\Phi(\mathcal{F}))$]. By the Division Algebra Example 8.2.2, $\Phi/\mathrm{I}_\Phi(\mathcal{F})$ is a field of the required type. □

We remark that the same argument will work for any functor from rings of scalars to any category, as long as the functor commutes with direct products and quotients.

Quadratic Factor Example 8.2.5 *Any ultraproduct of Jordan quadratic factors over fields is a quadratic factor over a field. If the factors are all nondegenerate, so is the ultraproduct.*

PROOF. If $\mathrm{A}_x = \mathcal{J}ord(Q_x, c_x)$ is a Jordan algebra determined by a quadratic form Q_x with basepoint c_x over a field Φ_x, the Jordan structure is determined by having c_x as unit and $U_a b = Q_x(a, \bar{b})a - Q_x(a)\bar{b}$ for

$\bar{b} = Q_x(b, c_x)c_x - b$. Set $\mathrm{A} = \prod \mathrm{A}_x,\ \Phi = \prod \Phi_x,\ Q = \prod Q_x,\ c = \prod c_x$; then we noted in the Quadratic Form Example 8.2.3 that Q is a quadratic form on A over the ring Φ of scalars, and it is easy to check that c is a basepoint which determines the pointwise Jordan structure of A: $\mathrm{A} = \mathcal{J}ord(Q, c)$.

If $\mathcal{F}$ is an ultrafilter on X, then by the Quadratic Form Example $Q' := Q/\mathcal{F}$ is a (respectively nondegenerate) quadratic form on $\mathrm{A}' := \mathrm{A}/\mathcal{F}$ over the field $\Phi' := \Phi/\mathcal{F}$. It is easy to check that $c' := c_{\mathcal{F}}$ is a basepoint for Q' determining the quotient Jordan structure of A′: $\mathrm{A}' = \mathcal{J}ord(Q', c')$. Thus the ultraproduct $\mathrm{A}/\mathcal{F}$ is just the quadratic factor $\mathcal{J}ord(Q', c')$. □

We noted in the case of simplicity that not all algebraic properties carry over to ultraproducts: properties of the form "for each element a there is an $n = n(a)$ such that ... holds" usually fail in the direct product and the ultraproduct because in an infinitely long string $(a_1, a_2, \dots)$ there may be no upper bound to the $n(a_i)$. Another easy example is provided by a countably infinite **ultrapower** $\mathrm{A}^X/\mathcal{F} = (\prod_{x\in X} \mathrm{A})/\mathcal{F}$ (ultraproduct based on the direct *power*, where all the factors are the same, instead of the direct *product* of different factors).

Nil Algebra Non-Example 8.2.6 *A countable ultrapower* $\mathrm{A}^{\mathbb{N}}/\mathcal{F}$ *of a nil algebra* A *is a nil algebra if and only if either* (1) A *is nil of bounded index, or* (2) *the ultrafilter* $\mathcal{F}$ *is principal.*

PROOF. If the algebra is nil of bounded index as in (1), it satisfies a polynomial identity $a^n = 0$ for some fixed n independent of a, hence the direct product and its quotient ultrapower do too. If the ultrafilter is principal $\mathcal{F} = \mathcal{F}(x_0)$ as in (2), then the ultraproduct is just the x_0th factor $\mathrm{A} \cong \mathrm{A}_{x_0}$, which is certainly nil. All of this works for any ultrapower $\mathrm{A}^X/\mathcal{F}$.

The harder part is the converse; here we require $X = \mathbb{N}$ to be countably infinite. Assume that (1) fails, but $\mathrm{A}^{\mathbb{N}}/\mathcal{F}$ is nil; we will show that the ultrafilter is principal, equivalently (cf. Problem 7.2) that $\mathcal{F}$ contains some finite set Y. Since A does not have bounded index, there are elements $a_k \in \mathrm{A}$ with $a_k^k \neq 0$. Let $a = (a_1, a_2, \dots) \in \mathrm{A}^{\mathbb{N}}$ result from stringing these a_k's together. If the image $\pi(a)$ of a is nilpotent in $\mathrm{A}^{\mathbb{N}}/\mathcal{F}$, $\pi(a)^n = 0$ for some fixed n, then back in $\mathrm{A}^{\mathbb{N}}$ we have $a^n(k) = 0$ on some subset $Y \in \mathcal{F}$, i.e., $a_k^n = 0$, and therefore $n > k$ for all $k \in Y$. But then $Y \subseteq \{1, 2, \dots, n-1\}$ is the desired finite set in the filter $\mathcal{F}$. □

8.3 Problems for Chapter 8

PROBLEM 8.1* (1) Show that if a prime algebra A_0 is imbedded in a direct sum $\mathrm{A} = \mathrm{A}_1 \boxplus \mathrm{A}_2$, then it is already imbedded in one of the summands A_1 or A_2. Generalize this to $\mathrm{A}_0 \hookrightarrow \mathrm{A}_i$ for some i in any *finite* direct sum $\mathrm{A} := \mathrm{A}_1 \boxplus \cdots \boxplus \mathrm{A}_n$. (2) Show that if a prime algebra A_0 is imbedded in a direct product $\prod_{x\in X} \mathrm{A}_x$ and $X = X_1 \cup X_2$ is a disjoint union of two subsets, then A_0 is imbedded in $\prod_{x_i\in X_i} \mathrm{A}_{x_i}$ for $i = 1$ or $i = 2$. Extend this to the case of a finite disjoint union $X = X_1 \cup \cdots \cup X_n$.

QUESTION 8.1* How is the dimension of the ultraproduct related to the dimensions of the individual factors A_x? Is an ultraproduct of composition algebras again a composition algebra? What about split composition algebras? Quaternion algebras? Octonion algebras? Split octonion algebras?

9

The Final Argument

We are finally ready to analyze the structure of prime algebras. The game plan is (1) to go from *prime nondegenerate* algebras to *semiprimitive* algebras over big fields by an imbedding process, (2) pass via subdirect products from semiprimitive to *primitive* algebras over big fields where the true classification takes place, then (3) form an *ultraproduct* to get back down to a "model" of the original prime algebra. Note that we go directly from primitive to prime without passing simple, so that the structure of simple algebras follows from the more general structure of prime algebras. Even if we started with a simple algebra, the simplicity would be destroyed in the passage from nondegenerate to semiprimitive (even in the associative theory there are simple radical rings).

We have completed the first two steps of our game plan in the Semiprimitive Imbedding Theorem 5.4.1. What remains is the ultraproduct step.[1]

9.1 Dichotomy

Because ultraproducts provide tight algebraic models of the factors, if the factors all come in only a *finite* number of algebraic flavors, then the ultraproduct too must have *exactly one* of those flavors.

Finite Dichotomy Principle 9.1.1 *If each factor* A_x *in an ultrafilter belongs to one of a finite number of types* $\{T_1, \ldots, T_n\}$*, then an ultraproduct is isomorphic to a* homogeneous *ultraproduct of a single type* T_i*: if* $X_i = \{x \in X \mid \mathrm{A}_x$ *has type* $T_i\}$*, for some* $i = 1, 2, \ldots, n$ *we have*

$$\mathrm{A} = \left(\prod_{x \in X} \mathrm{A}_x\right)/\mathcal{F} \quad \cong \quad \mathrm{A}_i := \left(\prod_{x \in X_i} \mathrm{A}_x\right)/\left(\mathcal{F}|_{X_i}\right).$$

PROOF. By hypothesis $X = X_1 \cup \cdots \cup X_n$ is a finite union [for any x the factor A_x has some type T_i, so $x \in X_i$], so by property (UFilt3) of Ultrafilter Characterization Theorem 7.3.4, some $X_i \in \mathcal{F}$. Then by the Filter

[1] Prime Dichotomy was described in I.8.11.

Restriction Theorem 8.1.3(2) $\big(\prod_{x\in X} \mathrm{A}_x\big)/\mathcal{F} \cong \big(\prod_{x\in X_i} \mathrm{A}_x\big)/\big(\mathcal{F}|_{X_i}\big)$, where by Ultra Restriction Example 7.3.5 $\mathcal{F}|_{X_i}$ is an ultrafilter on the homogeneous direct product $\prod_{x\in X_i} \mathrm{A}_x$. □

9.2 The Prime Dichotomy

There are only two options for a primitive Jordan algebra over a big field: satisfy all s-identities and be an i-special algebra, or be an Albert algebra. Applying this dichotomy to a prime nondegenerate Jordan algebra yields the main theorem of Part III.

Prime Dichotomy Theorem 9.2.1 (1) *Every prime nondegenerate Jordan Φ-algebra with $\frac{1}{2} \in \Phi$ is either i-special or a form of a split Albert algebra.* (2) *Every simple nondegenerate Jordan algebra of characteristic $\neq 2$ is either i-special or a 27-dimensional Albert algebra $\mathcal{J}ord(N, c)$ over its center.*

PROOF. (1) We have already noted that the first two steps in this argument, going from prime to semiprimitive to primitive, have been taken: by the Semiprimitive Imbedding Theorem 5.4.1, a *prime nondegenerate* J imbeds in a direct product $\vec{\mathrm{J}} = \prod_\alpha \widetilde{\mathrm{J}}_\alpha$ for *primitive* i-exceptional algebras $\widetilde{\mathrm{J}}_\alpha$ over one big algebraically closed field Ω, $|\Omega| > \dim_\Omega \widetilde{\mathrm{J}}_\alpha$. Further, we clearly understand the resulting factors: by the Big Primitive Exceptional Theorem 6.2.2, these factors are either i-special or split Albert algebras $\mathcal{A}lb(\Omega)$ over their center Ω.

So far we have an enlargement of J with a precise structure. Now we must recapture the original J without losing our grip on the structure. To tighten, we need an ultrafilter, and this is where for the first time primeness is truly essential. By the Prime Example 7.2.2, any prime subalgebra of a direct product has a support filter $\mathcal{F}(\mathrm{J})$ generated by the supports $\mathcal{S}upp(a) = \{\alpha \mid a(\alpha) \neq 0\}$ of its nonzero elements, and by the Filter Restriction Theorem 8.1.3(4) J remains imbedded in some ultraproduct $\vec{\mathrm{J}}/\mathcal{F}$ determined by an ultrafilter $\mathcal{F} \supseteq \mathcal{F}(\mathrm{J})$. By the Finite Dichotomy Theorem 9.1.1 we can replace this ultraproduct by a *homogeneous* ultraproduct where *all* the factors are i-special or *all* the factors are split Albert algebras.

In the first case the algebra J inherits i-speciality as a *subalgebra* of a *quotient* $\vec{\mathrm{J}}/\mathcal{F}$ of a *direct product* $\vec{\mathrm{J}} = \prod_\alpha \widetilde{\mathrm{J}}_\alpha$ of i-special factors.

In the second case the algebra J is a subalgebra of an ultraproduct of split Albert factors, which by the Split Albert Example 8.2.4 is itself an 27-dimensional Albert algebra $\overline{\mathrm{J}} = \mathcal{A}lb(\overline{\Omega})$ over a big algebraically closed field $\overline{\Omega}$. Here $\overline{\Omega} = \big(\prod_\alpha \Omega\big)/\mathcal{F}$ is an ultrapower of Ω; it remains an algebraically closed field over Ω by the Division Algebra Example 8.2.2, so it also remains big since $|\overline{\Omega}| \geq |\Omega|$.

We claim that in fact $\mathrm{J}' = \overline{\Omega}\mathrm{J}$ is *all* of the 27-dimensional algebra $\overline{\mathrm{J}}$, so J is indeed a form of a split Albert algebra as claimed in the theorem. Otherwise, J' would have dimension < 27 over $\overline{\Omega}$, as would the semisimple algebra $J'' := J'/\mathcal{Rad}(J')$ and all of its simple summands. But by the finite-dimensional (or finite-capacity) theory, these simple summands of dimensions < 27 over their centers must all be *special*, so their direct sum J'' would be special too. But J remains imbedded in $J'' = J'/\mathcal{Rad}(J')$ since $J \cap \mathcal{Rad}(J') = \mathbf{0}$ by the Radical Avoidance Lemma 1.7.3 [J is nondegenerate *by hypothesis*, and the finite-dimensional J' certainly has the d.c.c. on all inner ideals (indeed, all subspaces!)],[2] so J would be special too, contrary to the assumed i-exceptionality of J.

(2) This establishes the theorem for prime algebras. If J is *simple* nondegenerate i-exceptional, we will show that it is already 27*-dimensional over its centroid*, which is a field Φ by the Centroid Theorem 1.6.3, and therefore again by the finite-dimensional theory it will follow that $J = \mathcal{J}ord(N, c)$. Now J is also prime and nondegenerate, so applying the prime case gives $\overline{\Omega}J = \mathcal{A}lb(\overline{\Omega})$. We have a natural epimorphism $J_{\overline{\Omega}} := \overline{\Omega} \otimes_\Phi J \twoheadrightarrow \overline{\Omega}J = \mathcal{A}lb(\overline{\Omega})$. *In characteristic* $\neq 2$ the scalar extension $J_{\overline{\Omega}}$ of the central-simple *linear Jordan* algebra J remains simple over $\overline{\Omega}$ by the Strict Simplicity Theorem II.1.7.1, so this epimorphism must be an isomorphism, and $\dim_\Phi(J) = \dim_{\overline{\Omega}}(J_{\overline{\Omega}}) = \dim_{\overline{\Omega}}(\mathcal{A}lb(\overline{\Omega})) = 27$. □

This completes our proof of this powerful theorem. We can reformulate it as proving the nonexistence of i-exceptional Jordan algebras which might provide a home for a non-Copenhagen quantum mechanics.

Zel'manov's Exceptional Theorem 9.2.2 (1) *The only i-exceptional prime nondegenerate Jordan* Φ*-algebras with* $\frac{1}{2} \in \Phi$ *are forms of split Albert algebras.*
(2) *The only i-exceptional simple nondegenerate Jordan algebras of characteristic* $\neq 2$ *are* 27*-dimensional Albert algebras over their centers.* □

We remark that a highly nontrivial proof (which we have chosen not to include) shows that simple Jordan algebras are *automatically nondegenerate.* This is not true for prime algebras: there exist prime degenerate Jordan algebras, known as Pchelintsev Monsters.

This theorem leaves open the slight hope that there exist prime nondegenerate Jordan algebras which are i-special but not special, and therefore would qualify as (perhaps substandard) exceptional housing for non-Copenhagen quantum mechanics. Perhaps someday there will be a direct proof that *all* nondegenerate i-special algebras are in fact special, but at the present time one has to work hard to show that *prime* nondegenerate i-special algebras are special. The only way Zel'manov could do this (and finally dash all hopes for an exceptional home for quantum mechanics) was to first classify *all prime nondegenerate algebras whatsoever,* and then notice that they turn out to be either *special* or Albert. But that general structure theory, with epic battles against the tetrad eaters, is another story, and it's time to close this Part III and this taste of Jordan structure theory.

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[2] This avoidance would have been even easier if we had stopped to prove that $\mathcal{Rad}(J')$ is nilpotent.

9.3 Problems for Chapter 9

Problem 9.1* From Dichotomy 9.1.1 we know that an ultraproduct of fields Φ_x of characteristics $0, p_1, \ldots, p_n$ must be a field of characteristic 0 or $p = p_i$ for some $i = 1, \ldots, n$. Explain what happens in an ultraproduct of fields Φ_x of infinitely many distinct characteristics — by Division Algebras Example 8.2.2 this must be a field of some characteristic, but where does the characteristic come from? Consider a very specific case, $\big(\prod_p \mathbb{Z}_p\big)/\mathcal{F}$ for an ultrafilter $\mathcal{F}$ on the set of prime numbers.

Problem 9.2 (1) If $\Omega \supseteq \Phi$ is a small extension of a field Φ ($\dim_\Phi \Omega < |\Phi|$, i.e., Φ is big for Ω), show that Ω is algebraic over Φ. Conclude that if Φ is algebraically closed as well, then $\Omega = \Phi$: algebraically closed fields admit no small extensions. (2) Show that bigness passes to the algebraic closure: if Φ is big for J, then its algebraic closure $\overline{\Phi}$ remains big for $\overline{\mathrm{J}} := \overline{\Phi} \otimes_\Phi \mathrm{J}$. (3) If J is an algebra over a field Ω and Φ is a subfield of Ω, use transitivity of degree $[\mathrm{J} : \Phi] = [\mathrm{J} : \Omega]\,[\Omega : \Phi]$ to show that if Φ is big for J then Φ is also big for Ω: $|\Phi| > [\mathrm{J} : \Phi] \Longrightarrow |\Phi| > [\Omega : \Phi]$, and Ω is a small extension of Φ. (4) Show that a Jordan algebra over a big algebraically closed field Φ whose centroid is a field (e.g., a simple algebra) is already central over Φ: $\Gamma(\mathrm{J}) = \Phi$.

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Part IV

Appendices

Introduction

These appendices serve as eddies, where we can leisurely establish important results whose technical proofs would disrupt the narrative flow of the main body of the text. We have made free use of some of these results, especially Macdonald's Theorem, in our treatment, but their proofs are long, combinatorial, or computational, and do not contribute ideas and methods of proof which are important for the mainstream of our story. We emphasize that these are digressions from the main path, and should be consulted only after the reader has gained a global picture. A hypertext version of this book would have links at this point to the appendices which could be opened only after the main body of text had been perused at least once.

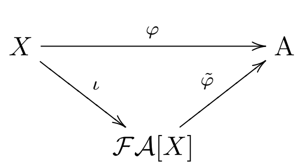
A

Cohn's Special Theorems

In this chapter we obtain some purely associative results due to P.M. Cohn in 1954, relating the symmetric elements and the Jordan elements in a free associative algebra, providing a criterion for the homomorphic image of a special algebra to be special. For convenience, *we will work entirely within the category of unital algebras.* The corresponding non-unital theory will be left as a worthwhile project for the reader at the end of the appendix.

A.1 Free Gadgets

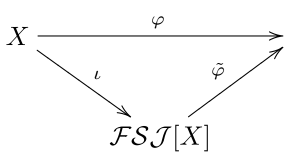
Let $\mathcal{FA}[X]$ denote the **free unital associative algebra** on the set X, the free Φ-module spanned by all *monomials* $x_1 \cdots x_n$ for all $n \geq 0$ (the *empty product* for $n = 0$ serving as unit) and all $x_i \in X$ (not necessarily distinct), with product determined by linearity and "concatenation" $(x_1 \cdots x_n)(x_{n+1} \cdots x_{n+m}) = x_1 \cdots x_n x_{n+1} \cdots x_{n+m}$. We think of the elements of the free algebra as *free* or *generic associative polynomials* $p(x_1, \ldots, x_n)$ in the variables $x \in X$. Let ι be the canonical set-theoretic imbedding of X in $\mathcal{FA}[X]$. The free algebra has the **universal property** that any set-theoretic mapping φ of X into a unital associative algebra A factors uniquely through the canonical imbedding $\iota\colon X \to \mathcal{FA}[X]$ via a homomorphism $\widetilde{\varphi} : \mathcal{FA}[X] \to$ A of unital associative algebras; we say that φ *extends uniquely to* a unital associative algebra homomorphism.



This universal property completely characterizes the free algebra. Indeed, rather than single out one construction as *the* free algebra, most authors define *a* free algebra to be anything satisfying the universal property, then show that any two such are naturally isomorphic. If we wish to stress the functoriality of the free algebra, we denote the extension $\widetilde{\varphi}$ by $\mathcal{FA}[\varphi]$. You should think of this as an *evaluation map* $p(x_1, \ldots, x_n) \mapsto p(a_1, \ldots, a_n)$, obtained by *substituting* the $a_i = \varphi(x_i) \in$ A for the variables x_i. *Free associative polynomials were born to be evaluated* on associative Φ-algebras, in the same way that ordinary commutative polynomials in $\Phi[X]$ were born to be evaluated on commutative Φ-algebras.

The free algebra $\mathcal{FA}[X]$ also has a unique **reversal involution** ρ fixing the generators $x \in X$: $\rho(x_1 x_2 \cdots x_n) = x_n \cdots x_2 x_1$. The $*$-algebra $(\mathcal{FA}[X], \rho)$ has the universal property that any set-theoretic mapping of X into the hermitian part $\mathcal{H}(\mathrm{A}, *)$ of a unital associative $*$-algebra $(\mathrm{A}, *)$ extends uniquely to a $*$-homomorphism $(\mathcal{FA}[X], \rho) \to (\mathrm{A}, *)$ of unital $*$-algebras factoring through the canonical $*$-imbedding $X \to \mathcal{H}(\mathcal{FA}[X], \rho)$.

The *reversible* elements $\rho(p) = p$ in $\mathcal{FA}[X]$ form a unital Jordan algebra $\mathcal{H}(\mathcal{FA}[X], \rho)$ containing X; the Jordan subalgebra generated by X is called the **free special unital Jordan algebra** $\mathcal{FSJ}[X]$. We will call an element of $\mathcal{FSJ}[X]$ a *free special* or *generic special Jordan polynomial* on X. $\mathcal{FSJ}[X]$ has the **universal property** that any set-theoretic mapping φ of X into a special unital Jordan algebra J extends *uniquely* to a homomorphism $\widetilde{\varphi} : \mathcal{FSJ}[X] \to$ J of unital Jordan algebras factoring through the canonical imbedding $\iota: X \to \mathcal{FSJ}[X]$.



The map $\widetilde{\varphi}$ (denoted by $\mathcal{FSJ}[\varphi]$ if we wish to stress the functoriality of the free gadget) is again evaluation $p(x_1, \ldots, x_n) \mapsto p(a_1, \ldots, a_n)$ at $a_i = \varphi(x_i)$ for all Jordan polynomials p in the x's. Again, *special Jordan polynomials were born to be evaluated* (though only on special Jordan algebras).

We obtain a **free associative** or **free special Jordan functor** $\mathcal{F}$ ($\mathcal{F} = \mathcal{FA}$ or $\mathcal{FSJ}$) from the category of sets to the category of unital associative or special Jordan algebras, sending a set X to the algebra $\mathcal{F}[X]$, and a map $f: X \to Y$ to the homomorphism $\mathcal{F}[f]: \mathcal{F}[X] \to \mathcal{F}[Y]$ induced by $X \to Y \to \mathcal{F}[Y]$. In particular, for $X \subseteq Y$ the natural set inclusion $X \hookrightarrow Y$ induces a canonical homomorphism $\mathcal{F}[X] \to \mathcal{F}[Y]$, and it is clear that this is a monomorphism, so we will always identify $\mathcal{FA}[X]$ with the subalgebra of $\mathcal{FA}[Y]$ generated by X. Under this associative homomorphism, Jordan products go into Jordan products, inducing an identification of $\mathcal{FSJ}[X]$ with the Jordan subalgebra of $\mathcal{FSJ}[Y]$ generated by X.

A.2 Cohn Symmetry

Clearly, $\mathcal{FSJ}[X] \subseteq \mathcal{H}(\mathcal{FA}[X], \rho)$, and the question is how close these are. As long as Φ contains a scalar $\frac{1}{2}$, this is easy to answer: on the one hand, the **tetrads** $\{x_1, x_2, x_3, x_4\} := x_1x_2x_3x_4 + x_4x_3x_2x_1$ are reversible elements *but not Jordan products* in $\mathcal{FSJ}[X]$. Indeed, in the exterior algebra $\Lambda[X]$ the subspace ΦX (which, by convention, means all Φ-linear combinations of generators, consisting of wedges of length 1) forms a special Jordan subalgebra with trivial products $x \wedge x = x \wedge y \wedge x = x_1 \wedge x_2 + x_2 \wedge x_1 = x_1 \wedge x_2 \wedge x_3 + x_3 \wedge x_2 \wedge x_1 = 0$, but not containing tetrads because $x_1 \wedge x_2 \wedge x_3 \wedge x_4 + x_4 \wedge x_3 \wedge x_2 \wedge x_1 = 2x_1 \wedge x_2 \wedge x_3 \wedge x_4 \notin \Phi X$. On the other hand, as soon as we adjoin these tetrads to the generators X we can generate all reversible elements.

Cohn Reversible Theorem A.2.1 *The Jordan algebra $\mathcal{H}(\mathcal{FA}[X], \rho)$ of reversible elements of the free associative algebra over $\Phi \ni \frac{1}{2}$ is precisely the Jordan subalgebra generated by $\mathcal{FSJ}[X]$ together with all increasing tetrads $\{x_1, x_2, x_3, x_4\}$ for distinct $x_1 < x_2 < x_3 < x_4$ (in some ordering of X).*

PROOF. Let B denote the subalgebra generated by X and the increasing tetrads. Since $\frac{1}{2} \in \Phi$, the reversible elements are all "traces" $\mathrm{tr}(a) := a + \rho(a)$, hence are spanned by the traces of the monomials, which are just the n-**tads** $\{x_1, x_2, \ldots, x_n\} := \mathrm{tr}(x_1x_2 \cdots x_n) = x_1x_2 \cdots x_n + x_n \cdots x_2x_1$ for $x_i \in X$. [WARNING: for $n = 0, 1$ we fudge and decree that the 0-tad is 1 and the 1-tad is $\{x_1\} = x_1$ (not $x_1 + \rho(x_1) = 2x_1$!!)] We will prove that all n-tads are $\equiv 0$ modulo B by induction on n. The cases $n = 0, 1, 2, 3$ are trivial (they are Jordan products of x's).

For $n = 4$ the tetrads can all be replaced by $\pm$ tetrads with distinct x's arranged in increasing order, since $\{x_1, x_2, x_3, x_4\}$ is an alternating function of its arguments mod B (note that a tetrad with two adjacent elements equal reduces to a Jordan triad in $\mathcal{FSJ}[X] \subseteq$ B, e.g., $\{x_1, x, x, x_4\} = \{x_1, x^2, x_4\}$).

Now assume that $n > 4$ and that the result has been proven for all $m < n$, and consider an n-tad $x_I := \{x_1, x_2, \ldots, x_n\}$ determined by the n-tuple $I = (1, 2, \ldots, n)$. By induction we have mod B that $0 \equiv \{x_1, \{x_2, x_3, \ldots, x_n\}\} = \{x_1, x_2, \ldots, x_n\} + \{x_2, \ldots, x_n, x_1\} = x_I + x_{\sigma(I)}$, where σ is the n-cycle $(12 \ldots n)$. Thus

(1) $$x_{\sigma(I)} \equiv -x_I \mod \mathrm{B}.$$

Applying this repeatedly gives $x_I = x_{\sigma^n(I)} \equiv (-1)^n x_I$, so that when n is *odd* we have $2x_I \equiv 0$, hence $x_I \equiv 0$ mod B in the presence of $\frac{1}{2}$.

From now on assume that n is *even,* so the n-cycle σ is an odd permutation, so (1) becomes

(2) $$x_{\sigma(I)} \equiv (-1)^\sigma x_I \mod \mathrm{B}$$

(where $(-1)^\pi$ denotes the signature of the permutation π). We have the same for the transposition $\tau = (12)$: by induction $0 \equiv \{x_1, x_2, \{x_3, x_4, \ldots, x_n\}\} =$

$\{x_1, x_2, x_3, \ldots, x_n\} + \{x_3, x_4, \ldots, x_n, x_2, x_1\} = x_I + x_{\sigma^2\tau(I)} \equiv x_I + (-1)^2 x_{\tau(I)}$ and

$$(3) \qquad x_{\tau(I)} \equiv (-1)^\tau x_I \mod \mathrm{B}.$$

Putting (2), (3) together gives

$$(4) \qquad x_{\pi(I)} \equiv (-1)^\pi x_I \mod \mathrm{B}$$

for all permutations π in S_n, since the transposition τ and the n-cycle σ generate S_n.

Now we bring in the tetrads. Since B contains the tetrads and is closed under Jordan products, we have by induction

$$\begin{aligned} 0 &\equiv \{\{x_1, x_2, x_3, x_4\}, \{x_5, \ldots, x_n\}\} \\ &= \{x_1, x_2, x_3, x_4, x_5, \ldots, x_n\} + \{x_4, x_3, x_2, x_1, x_5, \ldots, x_n\} \\ &\quad + \{x_5, \ldots, x_n, x_1, x_2, x_3, x_4\} + \{x_5, \ldots, x_n, x_4, x_3, x_2, x_1\} \\ &= x_I + x_{\tau_{(14)}\tau_{(23)}(I)} + x_{\sigma^4(I)} + x_{\tau_{(14)}\tau_{(23)}\sigma^4(I)} \\ &\equiv x_I + (-1)^2 x_I + (-1)^4 x_I + (-1)^2(-1)^4 x_I \\ &= 4x_I, \end{aligned}$$

[using (4)], so again in the presence of $\frac{1}{2}$ we can conclude that $x_I \equiv 0$. This completes the induction that all n-tads fall in B. □

When $| X | \leq 3$, there are no tetrads with distinct variables, so we have no need of tetrads. Here we will break our long-standing policy of using the word *hermitian* in place of *symmetric*, and will speak colloquially of *symmetric elements* or *symmetric expressions* in the variables x, y, z.

Cohn Symmetry Theorem A.2.2 (1) *We have equality* $\mathcal{H}(\mathcal{FA}[X], \rho) = \mathcal{FSJ}[X]$ *when* $|X| \leq 3$*: any symmetric associative expression in at most three variables is a Jordan product. In particular, we have*:

(2) **Cohn 2-Symmetry**: $\mathcal{FSJ}[x, y] = \mathcal{H}(\mathcal{FA}[x, y], \rho)$;

(3) **Cohn 3-Symmetry**: $\mathcal{FSJ}[x, y, z] = \mathcal{H}(\mathcal{FA}[x, y, z], \rho)$.

A.3 Cohn Speciality

Now we turn to the question of which images of the free special Jordan algebra remain special. Surprisingly, speciality is not inherited by all images: the class of special algebras does not form a variety defined by identities, only the larger class of homomorphic images of special algebras (the *identity-special* or *i-special* algebras) can be defined by identities (namely, the s-identities, those that vanish on all special algebras).

Cohn Speciality Criterion A.3.1 *Any homomorphic image of a special algebra is isomorphic to $\mathcal{FSJ}[X]/\mathrm{K}$ for some set of generators X and some ideal $\mathrm{K} \triangleleft \mathcal{FSJ}[X]$. Such an image will be special iff the kernel K is "closed," in the sense that the associative ideal $\overline{\mathrm{K}}$ it generates still intersects $\mathcal{FSJ}[X]$ precisely in the original kernel*:

$$(1) \qquad \overline{\mathrm{K}} \cap \mathcal{FSJ}[X] = \mathrm{K} \qquad (\overline{\mathrm{K}} := \mathcal{I}_{\mathcal{FA}[X]}(\mathrm{K})).$$

Put in seemingly more general terms,

$$(2) \quad \mathcal{FSJ}[X]/\mathrm{K} \ \textit{special iff} \ \ \mathrm{K} = \mathrm{I} \cap \mathcal{FSJ}[X] \ \ \textit{for some} \ \ \mathrm{I} \triangleleft \mathcal{FA}[X].$$

PROOF. Every special Jordan algebra J is a homomorphic image of some $\mathcal{FSJ}[X]$: by the universal property of the free special Jordan algebra there is a homomorphism of $\mathcal{FSJ}[X]$ to J for any set X of generators of J as Φ-algebra [at a pinch, $X = \mathrm{J}$ will do], which is an epimorphism because the image contains the entire generating set X. Thus every image of a special algebra is also an image of some $\mathcal{FSJ}[X]$ and thus is isomorphic to $\mathcal{FSJ}[X]/\mathrm{K}$ for K the kernel of the epimorphism.

We now investigate speciality of such an $\mathcal{FSJ}[X]/\mathrm{K}$. The seeming generality of (2) is illusory: the trace conditions in (1) and (2) are equivalent, since as soon as $\mathrm{K} = \mathrm{I} \cap \mathcal{FSJ}[X]$ is the trace on $\mathcal{FSJ}[X]$ of *some* ideal it is immediately the trace of its *closure*: $\mathrm{K} \subseteq \mathrm{I} \triangleleft \mathcal{FA}[X] \Longrightarrow \mathrm{K} \subseteq \overline{\mathrm{K}} \subseteq \mathrm{I} \Longrightarrow \mathrm{K} \subseteq \mathcal{FSJ}[X] \cap \overline{\mathrm{K}} \subseteq \mathcal{FSJ}[X] \cap \mathrm{I} = \mathrm{K}$ forces $\mathrm{K} = \mathcal{FSJ}[X] \cap \overline{\mathrm{K}}$.

Thus it suffices to prove (2). For the direction $\Longleftarrow$, as soon as K is the trace of *some* associative ideal as in (2), the quotient is immediately seen to be special by the Third Fundamental Homomorphism Theorem:

$$\mathcal{FSJ}[X]/\mathrm{K} = \mathcal{FSJ}[X]/(\mathrm{I} \cap \mathcal{FSJ}[X]) \cong \big(\mathcal{FSJ}[X] + \mathrm{I}\big)/\mathrm{I} \subseteq \big(\mathcal{FA}[X]/\mathrm{I}\big)^+$$

is imbedded in the associative algebra $\mathcal{FA}[X]/\mathrm{I}$.

For the converse direction $\Longrightarrow$, we suppose that the quotient is special, so there exists a faithful specialization $\mathcal{FSJ}[X]/\mathrm{K} \xrightarrow{\phi} \mathrm{A}^+$. The map σ: $X \xhookrightarrow{\iota} \mathcal{FSJ}[X] \overset{\pi}{\twoheadrightarrow} \mathcal{FSJ}[X]/\mathrm{K} \xrightarrow{\phi} \mathrm{A}^+$ is a map of sets, hence induces a unital associative homomorphism $\sigma_A \colon \mathcal{FA}[X] \to \mathrm{A}$ by the associative universal property. Moreover, since $\sigma_A|_{\mathcal{FSJ}[X]}$ and $\phi \circ \pi$ are both *Jordan* homomorphisms on $\mathcal{FSJ}[X]$ which agree with σ on the generators X, they must agree everywhere by the uniqueness in the Jordan universal property: $\sigma_A|_{\mathcal{FSJ}[X]} = \phi \circ \pi$. Thus $\ker(\sigma_A) \cap \mathcal{FSJ}[X] = \ker(\sigma_A|_{\mathcal{FSJ}[X]}) = \ker(\phi \circ \pi) = \mathrm{K}$ exhibits K as the trace of an associative ideal. □

When X consists of at most two variables, *all* homomorphic images are special.

Cohn 2-Speciality Theorem A.3.2 *If $|X| \leq 2$, then every homomorphic image of $\mathcal{FSJ}[X]$ is special: every i-special algebra on at most two generators is actually special, indeed, is isomorphic to $\mathcal{H}(\mathrm{A}, *)$ for an associative algebra* A *with involution* $*$.

PROOF. For convenience, we work only with the case of two generators $X = \{x, y\}$ (for one generator, just replace all y's by x's). We may represent our two-generator unital algebra as

$$\text{(1)} \qquad \mathrm{J} \cong \mathcal{FSJ}[x,y]/\mathrm{K}$$

for a Jordan ideal $\mathrm{K} \triangleleft \mathcal{FSJ}[x,y] \cong \mathcal{H}(\mathcal{FA}[x,y],\rho)$ (using Cohn 2–Symmetry). The associative closure $\overline{\mathrm{K}}$ is spanned by all pkq for associative monomials p, q in x, y and $k \in \mathrm{K}$, and is thus automatically a $*$-ideal invariant under the reversal involution ρ (remember that $k \in \mathcal{FSJ}[x,y]$ is reversible, and it is a general fact that if the generating set S is closed under an involution, $S^* \subseteq S$, then the ideal $\mathcal{I}_{\mathrm{A}}(S)$ generated by S in A is automatically a $*$-ideal invariant under the involution).

By the Cohn Speciality Criterion A.3.1, the homomorphic image (1) is special iff

$$\text{(2)} \qquad \overline{\mathrm{K}} \cap \mathcal{FSJ}[x,y] = \mathrm{K}.$$

By Cohn 2–Symmetry the trace $\overline{\mathrm{K}} \cap \mathcal{FSJ}[x,y] = \overline{\mathrm{K}} \cap \mathcal{H}(\mathcal{FA}[x,y],\rho)$ consists precisely of all reversible elements of $\overline{\mathrm{K}}$, and due to the presence of $\frac{1}{2}$ the reversible elements are all traces $\mathrm{tr}(u) = u + \rho(u)$ of elements of $\overline{\mathrm{K}}$, which are spanned by all $m(k) := pkq + \rho(q)k\rho(p)$. We claim that *each individual* $m(k)$ lies in K. In the free associative algebra $\mathcal{FA}[x,y,z]$ on three generators the element $m(z) := pzq + \rho(q)z\rho(p)$ is reversible, so by Cohn 3–Symmetry it is a Jordan product which is homogeneous of degree 1 in z, thus of the form $m(z) = M_{x,y}(z)$ for some Jordan multiplication operator in x, y. Under the associative homomorphism $\mathcal{FA}[x,y,z] \to \mathcal{FA}[x,y]$ induced by $x, y, z \mapsto x, y, k$, this Jordan polynomial is sent to $m(k) = M_{x,y}(k)$, so the latter is a Jordan multiplication acting on $k \in \mathrm{K}$ and hence falls back in the ideal K, establishing (2) and speciality. Recall from Cohn Speciality A.3.1 the explicit imbedding $\mathrm{J} \cong \mathcal{FSJ}[x,y]/\mathrm{K}$ [by (1)] $= \mathcal{FSJ}[x,y]/(\mathcal{FSJ}[x,y] \cap \overline{\mathrm{K}})$ [by (2)] $\cong (\mathcal{FSJ}[x,y] + \overline{\mathrm{K}})/\overline{\mathrm{K}} \subseteq \mathcal{FA}[x,y]/\overline{\mathrm{K}} =: \mathrm{A}$.

But we are not content with mere speciality. Since $\overline{\mathrm{K}}$ is invariant under ρ, the associative algebra A inherits a reversal involution $*$ from ρ on $\mathcal{FA}[x,y]$, whose symmetric elements are (again thanks to the presence of $\frac{1}{2}$) precisely all traces $a + a^* = \pi(u) + \pi(\rho(u))$ (π the canonical projection of $\mathcal{FA}[x,y]$ on A) $= \pi(u + \rho(u)) = \pi(h)$ for $h \in \mathcal{H}(\mathcal{FA}[x,y],\rho) = \mathcal{FSJ}[x,y]$ (by Cohn 2–Symmetry), so $\mathcal{H}(\mathrm{A},*) = \pi(\mathcal{FSJ}[x,y]) = (\mathcal{FSJ}[x,y] + \overline{\mathrm{K}})/\overline{\mathrm{K}} \cong \mathrm{J}$. Thus J arises as the full set of symmetric elements of an associative algebra with involution. □

This result fails as soon as we reach three variables. We can give our first example of a homomorphic image of a special algebra which is not special, merely i-special.

i-Special-but-Not-Special Example A.3.3 *If* K *is the Jordan ideal generated by* $k = x^2 - y^2$*, the homomorphic image* $\mathcal{FSJ}[x, y, z]/\mathrm{K}$ *is not special.*

PROOF. By Cohn's Criterion we must show that $\overline{\mathrm{K}} \cap \mathcal{FSJ}[X] > \mathrm{K}$. The reversible tetrad $\bar{k} := \{k, x, y, z\} = \{x^2, x, y, z\} - \{y^2, x, y, z\}$ falls in $\mathcal{FSJ}[x, y, z]$ by Cohn 3–Symmetry, and it certainly lies in the associative ideal $\overline{\mathrm{K}}$ generated by K, but we claim that it does *not* lie in K itself. Indeed, the elements of K are precisely all images $M(k)$ of $M(t)$ for Jordan multiplications M by x, y, z acting on $\mathcal{FSJ}[x, y, z, t]$. Since $M(t)$ of degree i, j, ℓ in x, y, z maps under $t \mapsto x^2 - y^2$ to an element with terms of degrees $i+2, j, \ell$ and $i, j+2, \ell$ in x, y, z, it contributes to the element $\bar{k}$ with terms of degrees $3, 1, 1$ and $1, 3, 1$ only when $i = j = \ell = 1$. Thus we may assume that $M = M_{1,1,1}$ is homogeneous of degree 1 in each of x, y, z.

As elements of $\mathcal{H}[x, y, z, t]$ (which, as we've noted, in four variables is slightly larger than $\mathcal{FSJ}[x, y, z, t]$), we can certainly write such an $M(t)$ as a linear combination of the 12 possible tetrads of degree 1 in each of x, y, z, t:

$$\begin{aligned} M(t) = {} & \alpha_1\{txyz\} + \alpha_2\{tyxz\} + \alpha_3\{xtyz\} + \alpha_4\{xytz\} \\ & + \alpha_5\{yxtz\} + \alpha_6\{ytxz\} + \alpha_7\{txzy\} + \alpha_8\{xtzy\} \\ & + \alpha_9\{tyzx\} + \alpha_{10}\{ytzx\} + \alpha_{11}\{xyzt\} + \alpha_{12}\{yxzt\}. \end{aligned}$$

By assumption this maps to

$$\begin{aligned} \{x^3yz\} - {} & \{y^2xyz\} = \bar{k} = M(k) \\ = {} & \alpha_1\big(\{x^3yz\} - \{y^2xyz\}\big) + \alpha_2\big(\{x^2yxz\} - \{y^3xz\}\big) \\ & + \alpha_3\big(\{x^3yz\} - \{xy^3z\}\big) + \alpha_4\big(\{xyx^2z\} - \{xy^3z\}\big) \\ & + \alpha_5\big(\{yx^3z\} - \{yxy^2z\}\big) + \alpha_6\big(\{yx^3z\} - \{y^3xz\}\big) \\ & + \alpha_7\big(\{x^3zy\} - \{y^2xzy\}\big) + \alpha_8\big(\{x^3zy\} - \{xy^2zy\}\big) \\ & + \alpha_9\big(\{x^2yzx\} - \{y^3zx\}\big) + \alpha_{10}\big(\{yx^2zx\} - \{y^3zx\}\big) \\ & + \alpha_{11}\big(\{xyzx^2\} - \{xyzy^2\}\big) + \alpha_{12}\big(\{yxzx^2\} - \{yxzy^2\}\big) \\ = {} & \big(\alpha_1 + \alpha_3\big)\{x^3yz\} - \alpha_1\{y^2xyz\} - \big(\alpha_2 + \alpha_6\big)\{y^3xz\} \\ & + \big(\alpha_5 + \alpha_6\big)\{yx^3z\} - \big(\alpha_3 + \alpha_4\big)\{xy^3z\} + \big(\alpha_8 + \alpha_7\big)\{x^3zy\} \\ & - \big(\alpha_{10} + \alpha_9\big)\{y^3zx\} + \alpha_2\{x^2yxz\} + \alpha_9\{x^2yzx\} - \alpha_7\{y^2xzy\} \\ & + \alpha_4\{xyx^2z\} - \alpha_5\{yxy^2z\} - \alpha_8\{xy^2zy\} + \alpha_{10}\{yx^2zx\} \\ & + \alpha_{11}\{xyzx^2\} + \alpha_{12}\{yxzx^2\} - \alpha_{11}\{xyzy^2\} - \alpha_{12}\{yxzy^2\}. \end{aligned}$$

Identifying coefficients of the 18 independent 5-tads on both sides of the equation gives $\alpha_1 + \alpha_3 = \alpha_1 = 1, \alpha_2 + \alpha_6 = \alpha_3 + \alpha_4 = 0$, all nine single-coefficient α_i for $i = 2, 9, 7, 4, 5, 8, 10, 11, 12$ vanish, which in turn implies from the double-coefficient terms that the two remaining α_i for $i = 3, 6$ vanish, leaving only $\alpha_1 = 1$ nonzero. But then $M(t) = \{t, x, y, z\}$ would be a tetrad, contradicting the fact that $M(t)$ is a Jordan product. □

We remark that this argument fails if we take $k = x^2$ instead of $k = x^2 - y^2$: then $\bar{k} := \{x, y, z, k\} = \{xyzx^2\} = \frac{1}{2}\big(V_{\{x,y,z\}} - U_{x,z}V_y + V_{x,y}V_z\big)(x^2)$ is a Jordan product back in K. Of course, this by itself doesn't prove that $\mathcal{FSJ}[x, y, z]/\mathcal{I}(x^2)$ is special — I don't know the answer to that one.

A.4 Problems for Appendix A

Problem A.1 As good practice in understanding the statements and proofs of the basic results in this appendix, go back and develop a completely analogous *non-unital* theory, starring the free (unit-less) associative algebra $\mathcal{FA}_0[X]$ and the free special Jordan algebra $\mathcal{FSJ}_0[X]$, dressed in their universal properties. The plot should involve finding a Cohn Reversible Theorem, Cohn Symmetry Theorem, Cohn Speciality Criterion, and Cohn 2-Speciality Theorem for algebras without units.

Question A.1 Is there a way to quickly derive the unit-less theory from the unital theory, and vice versa? For these unit-less free gadgets $\mathcal{F}_0$ ($= \mathcal{FA}_0, \mathcal{FSJ}_0$), to what extent is it true that $\mathcal{F}[X] = \widehat{\mathcal{F}_0[X]} := \Phi 1 \oplus \mathcal{F}_0[X]$? Is it true that $\mathcal{F}_0[X]$ can be identified with the ideal in $\mathcal{F}[X]$ spanned by all monomials of codegree ≥ 1 (equivalently, those with zero "constant term", i.e., those that vanish under the specialization $\mathcal{F}[X] \to \mathcal{F}[X]$ sending all $x \mapsto 0$)?

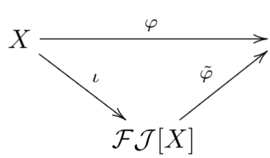
B

Macdonald's Theorem

In this appendix we will descend into the combinatorics of the free Jordan algebra on three generators in order to obtain Macdonald's Principle. Once more, *all algebras in this chapter will be unital.* Because we have so many formulas in this appendix, we will omit the chapter designation: the notation 1.2.356 will mean Section 1, Statement 2, Formula 356.[1]

B.1 The Free Jordan Algebra

We have heretofore been silent about the **free (unital) Jordan Φ-algebra** on a set of generators X, consisting of a unital Jordan Φ-algebra[2] $\mathcal{FJ}[X]$ with a given mapping ι: $X \to \mathcal{FJ}[X]$ satisfying the following **universal property**: any mapping φ: $X \to$ J of the set X into a unital Jordan Φ-algebra J extends uniquely to a homomorphism (or factors uniquely through ι via) $\widetilde{\varphi} : \mathcal{FJ}[X] \to$ J of unital Jordan Φ-algebras.



The condition that the extension be unique just means that the free algebra is generated (as unital Φ-algebra) by the set X; if we wish to stress the functoriality of the free gadget, we denote $\widetilde{\varphi}$ by $\mathcal{FJ}[\varphi]$. The explicit construction indicated below shows that ι is injective (we can also see this by noting that

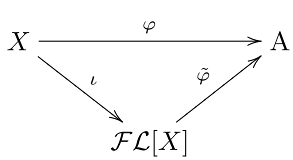
[1] The concepts of speciality and exceptionality were introduced in II.3.1.2, the Macdonald, Shirshov, and Shirshov–Cohn Theorems in II Section 5.1.

[2] Note that we don't bother to indicate the base scalars in the notation; a more precise notation would be $\mathcal{FJ}_\Phi[X]$, but we will never be this picky, relying on the reader's by now finely-honed common sense.

the map $X \to \boxplus_{x \in X} \Phi 1_x$ (each $\Phi 1_x$ a faithful copy of Φ^+) via $x \mapsto 1_x$ is injective, and if it factors through ι, then ι must also be injective). *We will always assume that X is contained in $\mathcal{FJ}[X]$.*

Like all free gadgets, the free algebra is determined up to isomorphism by its universal property: an algebra that *acts* like a free algebra with respect to X *is* a free algebra. Although by this approach we cannot speak of *the* free algebra, there is a canonical construction which we always keep in the back of our minds as "the" free algebra: $\mathcal{FJ}[X] := \mathcal{FL}[X]/\mathrm{K}$, where $\mathcal{FL}[X] = \Phi[\mathcal{M}[X]]$ is the *free unital linear Φ-algebra* generated by X, and K is the ideal of "Jordan relations." Here the **free monad** $\mathcal{M}[X]$ on the set X is a *nonassociative monoid* (set with unit and closed under a binary product); it has the universal property that any set-theoretic mapping $\varphi : X \to M$ into a monad M extends uniquely to a homomorphism $\widetilde{\varphi} : \mathcal{M}[X] \to M$ of monads. This is a graded set which can be constructed recursively as follows: the only monomial of degree 0 is the unit $\mathbf{1}$; the monomials of degree 1 are precisely the elements $\mathbf{x}$ of X; and if the monomials of degree $< n$ have been constructed, the monomials of degree n are precisely all $\mathbf{m} = \mathbf{(pq)}$ (an object consisting of an ordered pair of monomials $\mathbf{p}$, $\mathbf{q}$ surrounded by parentheses) for $\mathbf{p}$, $\mathbf{q}$ monomials of degrees $p, q > 0$ with $p + q = n$. For example, the elements of degree 2, 3 are all $\mathbf{(xy)}$, $\mathbf{((xy)z)}$, $\mathbf{(x(yz))}$. The mapping $(\mathbf{p}, \mathbf{q}) \mapsto (\mathbf{pq})$ for $\mathbf{p}, \mathbf{q} \neq \mathbf{1}$, $(\mathbf{1}, \mathbf{p}) \mapsto \mathbf{p}$, $(\mathbf{p}, \mathbf{1}) \mapsto \mathbf{p}$ gives a "totally nonassociative" product on $\mathcal{M}[X]$ with unit $\mathbf{1}$.

The **free (unital) linear algebra** $\mathcal{FL}[X] = \Phi[\mathcal{M}[X]]$ is the *monad algebra* generated by the free monad $\mathcal{M}[X]$, i.e., the free Φ-module spanned by all *monomials* $\mathbf{m}$, consisting of all $\mathbf{a} = \sum_{\mathbf{m}} \alpha_{\mathbf{m}} \mathbf{m}$ with bilinear multiplication $\mathbf{a} \cdot \mathbf{b} = \big(\sum_{\mathbf{m}} \alpha_{\mathbf{m}} \mathbf{m}\big)\big(\sum_{\mathbf{n}} \beta_{\mathbf{n}} \mathbf{n}\big) := \sum_{\mathbf{m},\mathbf{n}} \alpha_{\mathbf{m}} \beta_{\mathbf{n}} (\mathbf{mn})$, with canonical inclusion ι: $x \mapsto \mathbf{x}$ of $X \hookrightarrow \mathcal{M}[X] \hookrightarrow \Phi[\mathcal{M}[X]]$. This has the universal property that every set-theoretic map φ: $X \to \mathrm{A}$ into an arbitrary unital linear Φ-algebra extends uniquely (factors through ι) to a homomorphism $\widetilde{\varphi}$: $\mathcal{FL}[X] \to \mathrm{A}$ of unital Φ-algebras.



Namely, $\widetilde{\varphi}$ is the unique linear map whose values on monomials are defined recursively on degree 0 by $\widetilde{\varphi}(\mathbf{1}) := 1_{\mathrm{A}}$, on degree 1 by $\widetilde{\varphi}(\mathbf{x}) := \varphi(x)$ for the generators $x \in X$, and if defined up to degree n, then $\widetilde{\varphi}((\mathbf{pq})) := \widetilde{\varphi}(\mathbf{p})\widetilde{\varphi}(\mathbf{q})$.

The ideal K of relations which must be divided out is the ideal generated by all $\mathbf{(ab)}-\mathbf{(ba)}$, $\mathbf{(((aa)b)a)}-\mathbf{((aa)(ba))}$ for polynomials $\mathbf{a}$, $\mathbf{b} \in \mathcal{FL}[X]$, precisely the elements that must vanish in the quotient in order for the commutative law and Jordan identity II.1.8.1(JAX1)–(JAX2) to hold. Thus we have a representation of the free Jordan algebra in the form $\mathcal{FJ}[X] = \mathcal{FL}[X]/\mathrm{K}$.

EXERCISE B.1.0A Show that the ideal K is generated in terms of *monomials* **m**, **n**, **p**, **q** $\in \mathcal{M}[X]$ by all (1) **(mn)** - **(nm)**, (2) **(((mm)n)m)** - **((mm)(nm))**, (3) **(((mm)n)p)** + 2**(((mp)n)m)** - **((mm)(np))** - 2**((mp)(nm))**, (4) 2**(((mq)n)p)** + 2**(((mp)n)q)** + 2**(((qp)n)m)** - 2**((mq)(np))** - 2**((qp)(nm))** - 2**((mp)(nq))**.

EXERCISE B.1.0B* Show that the free linear algebra $\mathcal{FL}[X]$ carries a reversal involution ρ (involutory isomorphism from $\mathcal{FL}[X]$ to its opposite $\mathcal{FL}[X]^{op}$) uniquely determined by $\rho(x) = x$ for all $x \in X$. Show that the Jordan kernel K is invariant under the involution, $\rho(\mathrm{K}) \subseteq \mathrm{K}$, so that $\mathcal{FJ}[X]$ inherits a reversal involution. Why is this involution never mentioned in the literature?

Irrespective of how we represent the free algebra, we have a standard result that if $X \subset Y$ then $\mathcal{FJ}[X]$ can be canonically identified with the Jordan subalgebra of $\mathcal{FJ}[Y]$ generated by Y: the canonical epimorphism of $\mathcal{FJ}[X]$ onto this subalgebra, induced by $X \to Y \to \mathcal{FJ}[Y]$, is an isomorphism (it is injective because it has as left inverse the homomorphism $\mathcal{FJ}[Y] \to \mathcal{FJ}[X]$ induced by $Y \to X \cup \{0\}$ via $x \mapsto x$ $(x \in X), y \mapsto 0$ $(y \in Y \setminus X)$). In particular, we will always identify $\mathcal{FJ}[x,y]$ with the elements of $\mathcal{FJ}[x,y,z]$ of degree 0 in z.

A **Jordan polynomial** or **Jordan product** $f(x,y,z)$ in three variables is just an element of the free Jordan algebra $\mathcal{FJ}[x,y,z]$. Like any polynomial, it determines a mapping $(a,b,c) \mapsto f(a,b,c)$ of $\mathrm{J}^3 \to \mathrm{J}$ for any Jordan algebra J by *evaluating* the variables x,y,z to specific elements a,b,c. This, of course, is just the universal property: the map $\varphi : (x,y,z) \mapsto (a,b,c)$ induces a homomorphism $\widetilde{\varphi} : \mathcal{FJ}[x,y,z] \to \mathrm{J}$, and $f(a,b,c)$ is just $\widetilde{\varphi}(f(x,y,z))$. Such a polynomial **vanishes on** J iff $f(a,b,c) = 0$ for all elements $a,b,c \in \mathrm{J}$, i.e., iff all evaluations in J produce the value 0.[3]

We have a canonical specialization $\mathcal{FJ}[x,y,z] \to \mathcal{FSJ}[x,y,z]$ (in the Jordan sense of homomorphism into a special algebra) onto the free special Jordan algebra (inside the free associative $\mathcal{FA}[x,y,z]$) fixing x,y,z; the kernel of this homomorphism is the ideal of *s-identities,* those Jordan polynomials in three variables that vanish on (x,y,z) in $\mathcal{FSJ}[X]$ and therefore, by the universal property of the free special algebra, on any (a,b,c) in any *special* Jordan algebra $\mathrm{J} \subseteq \mathrm{A}^+$.

[3] This standard process of evaluating polynomials is called *specializing* the indeterminates x,y,z to the values a,b,c, going from general or "generic" values to particular or "special" values. But here the Jordan polynomial can be "specialized" to values in *any* Jordan algebra, not just *special Jordan algebras* in the technical sense, so to avoid confusion with "special" in the Jordan sense we will speak of *evaluating* f at a,b,c.

B.2 Identities

Remember that we have used The Macdonald[4] for simply *everything* we have done so far, so to prove The Macdonald itself we have to go back to the very beginning, to the **Basic Identities**

Basic Identities Lemma B.2.1 *From the Jordan identity we obtain the following operator identities*:

$$
\begin{aligned}
&(2.1.1)\quad [L_x, L_{x^2}] = [V_x, U_x] = 0,\\
&(2.1.2)\quad L_{x^2\bullet y} = -2L_xL_yL_x + L_{x^2}L_y + 2L_{x\bullet y}L_x\\
&\qquad\qquad = -2L_xL_yL_x + L_yL_{x^2} + 2L_xL_{x\bullet y},\\
&(2.1.3)\quad L_{(x\bullet z)\bullet y} = -L_xL_yL_z - L_zL_yL_x + L_{z\bullet x}L_y + L_{x\bullet y}L_z + L_{z\bullet y}L_x\\
&\qquad\qquad = -L_xL_yL_z - L_zL_yL_x + L_yL_{z\bullet x} + L_zL_{x\bullet y} + L_xL_{z\bullet y},\\
&(2.1.4)\quad U_x = L_x^2 - L_{x^2}, \qquad U_{x,y} = 2\big(L_xL_y + L_yL_x - L_{x\bullet y}\big),\\
&(2.1.5)\quad \{x,y,y\} = \{x,y^2\}, \qquad V_{x,y} = V_xV_y - U_{x,y}, \quad V_x := 2L_x.
\end{aligned}
$$

PROOF. (1) The Jordan identity (JAX2) $[x^2, y, x] = 0$ in operator form says L_x commutes with L_{x^2}, hence also with $U_x = 2L_x^2 - L_{x^2}$. (2) The linearization (c.f. the Linearization Proposition II.1.8.5) (JAX2)$'$ $[x^2, y, z] + 2[x\bullet z, y, x] = 0$ acting on z becomes $\big(L_{x^2\bullet y} - L_{x^2}L_y\big) + 2\big(L_xL_y - L_{x\bullet y}\big)L_x = 0$, while the linearization (JAX2)$''$ $x \mapsto x, \frac{1}{2}z$ of (2) yields (3). The second equality in (2), (3) follows from linearizing $L_xL_{x^2} = L_{x^2}L_x$. (4) is just the definition of the U-operators. For the first part of (5), note by (4) that $\{x,y,y\} = U_{x,y}y = 2(x\bullet y^2 + y\bullet(x\bullet y) - (x\bullet y)\bullet y) = 2x\bullet y^2 = \{x,y^2\}$. Linearizing $y \mapsto y, z$ gives $\{x,y,z\} + \{x,z,y\} = \{x,\{y,z\}\}$; interpreting this as an operator on z yields $V_{x,y} + U_{x,y} = V_xV_y$ as required for the second part of (5). □

The reader will notice that we have abandoned our practice of giving mnemonic mnicknames to results and formulas; this appendix in particular is filled with technical results only a lemma could love, and we will keep our acquaintance with them as brief as possible and fasten our attention on the final goal.

From Basic (2.1.3) we can immediately establish operator-commutativity and associativity of powers.

[4] As Ivanna Trump would have said. In the old Scottish clan of MacDonalds, the current clan chief or principal would be referred to simply as "The MacDonald." And so it is in the Jordan clan, where we refer to our chief principle as *The Macdonald.*

Power Associativity Lemma B.2.2 (1) *If the powers of an element x in a (unital) Jordan algebra are defined recursively by $x^0 := 1, x^1 := x$, and $x^{n+1} := x \bullet x^n$, then all the multiplication operators L_{x^n} are polynomials in the commutating operators L_x, L_{x^2}, and therefore they all commute with each other.*

(2) *The element x generates a commutative associative subalgebra $\Phi[x]$: we have power-associativity*

$$x^n \bullet x^m = x^{n+m}.$$

PROOF. We prove (1) by recursion on n, the cases $n = 0, 1, 2$ being trivial. Assuming it for powers $< n+2$ we have for the $(n+2)$nd power (setting $y = x, z = x^n$ in Basic (2.1.3) the result that $L_{x^{n+2}} = L_{(x \bullet x^n) \bullet x} = L_{x^{n+1}} L_x + \big(L_{x^{n+1}} - L_{x^n} L_x\big) L_x + \big(L_{x^2} - L_x L_x\big) L_{x^n}$, where by recursion $L_{x^n}, L_{x^{n+1}}$ are polynomials in L_x, L_{x^2} and hence $L_{x^{n+2}}$ is too.

(2) then follows easily by recursion on $n+m$; the result is trivial for $n+m = 0, 1, 2$, and assuming it for degrees $< n+m$, with $m \geq 2$, we obtain $x^n \bullet x^m = L_{x^n}(L_x x^{m-1}) = L_x(L_{x^n} x^{m-1})$ [by commutativity (1)] $= L_x(x^{n+m-1})$ [by the recursion hypothesis on (2)] $= x^{n+m}$ [by definition of the power]. □

In fact, B.2.2(1) is just the special case $X = \{x\}$ of the following result about generation of multiplication operators, which again flows out of Basic (2.1.3).

Generation Theorem B.2.3 *If X generates a unital subalgebra* B *of unital Jordan algebra* J, *then the multiplication algebra*[5] $\mathcal{M}_{\mathrm{B}}(\mathrm{J})$ *of* B *on* J *(generated by all multiplications L_b for elements $b \in \mathrm{B}$) is generated by all the operators $L_x, L_{x^2}, L_{x \bullet y}$ (or, equivalently, by all the $V_x, U_x, U_{x,y}$) for $x, y \in X$.*

PROOF. In view of Basic (2.1.4), $U_x, U_{x,y}$ are equivalent mod V_x, V_y to L_{x^2}, $L_{x \bullet y}$, so it suffices to prove the L-version. For this, it suffices to generate L_b for all monomials b of degree ≥ 3 in the elements from X. [Note that by convention the unit 1 is generated as the empty monomial on X, so the operator $\mathbb{1}_{\mathrm{J}} = L_1$ is considered to be generated from X.] Such b may be written as $b = (p \bullet q) \bullet r$ for monomials p, q, r, which by Basic (2.1.3) can be broken down into operators L_s of lower degree, so by repeating we can break all L_b down into a sum of products of L_c's for c of degree 1 or 2, each of which is x, x^2, or $x \bullet y$ for generators $x, y \in X$. □

We need the following general identities and identities for operator-commuting elements.

[5] The multiplication algebra of B on J is sometimes denoted by $\mathcal{M}_{\mathrm{J}}(\mathrm{B})$ or $\mathcal{M}(\mathrm{B})|_{\mathrm{J}}$ or just by $\mathcal{M}(\mathrm{B})$ (if J is understood), but we will use a subscript for B, since that is where the multipliers live in the operators $L_b, V_b, U_b, U_{b,c}$.

Operator Commuting Lemma B.2.4 (1) *Let x, y be arbitrary elements of a Jordan algebra. Then we have* **Commuting Formulas**

$$\begin{aligned} &(2.4.1) \qquad U_{x^2,x} = V_x U_x = U_x V_x, \\ &(2.4.2) \qquad U_{x^2,y} = V_x U_{x,y} - U_x V_y = U_{x,y} V_x - V_y U_x, \\ &(2.4.3) \qquad 2U_{x\bullet y,x} = V_y U_x + U_x V_y. \end{aligned}$$

(2) *More generally, if x' is an element which operator-commutes with x (all the operators $L_x, L_{x'}, L_{x^2}, L_{x'^2}, L_{x\bullet x'}$ commute), then we have*

$$\begin{aligned} &(2.4.4) \qquad U_{x'\bullet x,x} = V_{x'} U_x = U_x V_{x'}, \\ &(2.4.5) \qquad U_{x\bullet x'} = U_x U_{x'}, \\ &(2.4.6) \qquad U_{x^2\bullet x',y} = V_x U_{x\bullet x',y} - U_x U_{x',y} = U_{x\bullet x',y} V_x - U_{x',y} U_x, \\ &(2.4.7) \qquad U_{x^2\bullet x',y} = V_{x\bullet x'} U_{x,y} - U_x V_{x',y}. \end{aligned}$$

PROOF. (2.4.1) follows from (2.4.2) with $y = x$, in view of Basic (2.1.1). The first equality in (2.4.2) holds because $U_{x^2,y} - V_x U_{x,y} + U_x V_y = 2\big(L_{x^2} L_y + L_y L_{x^2} - L_{x^2\bullet y}\big) - \big(2L_x\big)2\big(L_x L_y + L_y L_x - L_{x\bullet y}\big) + \big(2L_x^2 - L_{x^2}\big)2L_y$ [using Basic (2.1.4)] $= 2\big(L_y L_{x^2} - L_{x^2\bullet y} - 2L_x L_y L_x + 2L_x L_{x\bullet y}\big) = 0$ [by Basic (2.1.2)]. The second equality in (2.4.2) holds by linearizing $x \mapsto x, y$ in Basic (2.1.1), $V_x U_x = U_x V_x$. (2.4.3) is equivalent to (2.4.2), since their sum is just the linearization $x \mapsto x, x, y$ in (2.4.1) $U_{x^2,x} = V_x U_x$. Notice that (2.4.1)–(2.4.3) are symmetric under reversal of products.

The first equality in (2.4.4) follows by linearizing $x \mapsto x, x'$ in Commuting (2.4.2) and setting $y = x$ to get $2U_{x\bullet x',x} = V_x U_{x',x} + V_{x'} U_{x,x} - U_{x,x'} V_x = V_{x'}(2U_x)$ by the assumed commutativity of L_x with $L_{x'}, L_{x\bullet x'}$ (hence $U_{x,x'}$, in view of Basic (2.1.4)), then canceling 2's. The second equality holds similarly by commutativity of $L_{x'}$ with L_x, L_{x^2} (hence U_x, in view of Basic (2.1.4)).

Applying this thrice yields (2.4.5): $2U_x U_{x'} = \big(V_x^2 - V_{x^2}\big)U_{x'}$ [by Basic (2.1.4)] $= V_x U_{x\bullet x',x'} - U_{x^2\bullet x',x'}$ [by (4) twice, noting that our hypotheses apply to x^2, x' in place of x', x since all multiplications by $b, c \in \mathrm{B} = \Phi[x, x']$ commute by the hypotheses and the Generation Theorem B.2.3] $= U_{x\bullet(x\bullet x'),x'} + U_{x\bullet x',x\bullet x'} - U_{x^2\bullet x',x'}$ [using linearized (2.4.4)] $= U_{x\bullet x',x\bullet x'}$ [since $x \bullet (x \bullet x') = x^2 \bullet x'$ by commutativity of $L_x, L_{x'}$ acting on x] $= 2U_{x\bullet x'}$, and we again divide by 2.

For the first equality in (2.4.6), we calculate $2\big(V_x U_{x\bullet x',y} - U_{x^2\bullet x',y} - U_x U_{x',y}\big) = V_x\big(V_x U_{x',y} + V_{x'} U_{x,y} - U_{x,x'} V_y\big) - \big(V_{x^2} U_{x',y} + V_{x'} U_{x^2,y} - U_{x^2,x'} V_y\big) - \big(V_x^2 - V_{x^2}\big)U_{x',y}$ [by linearizing $x \mapsto x, x'$ and also $x \mapsto x^2, x'$ in (2.4.2), and using Basic (2.1.4)] $= \big(U_{x^2,x'} - V_x U_{x,x'}\big)V_y + V_{x'}\big(-U_{x^2,y} + V_x U_{x,y}\big)$ [since $V_x \leftrightarrow V_{x'}$] $= \big(-U_x V_{x'}\big)V_y + V_{x'}\big(U_x V_y\big)$ [by (2.4.2) with $y \mapsto x'$, and by (2.4.2) itself] $= 0$ [by $V_{x'} \leftrightarrow U_x$]. The second equality in (2.4.6) follows by the dual argument (reversing all products, noting symmetry in (2.4.2)).

(2.4.7) is equivalent to the first equality in (2.4.6), since the sum of their right sides minus their left sides is $V_{x\bullet x'} U_{x,y} + V_x U_{x\bullet x',y} - U_{x\bullet x',x} V_y -$

$2U_{x\bullet(x\bullet x'),y}$ [by Basic (2.1.5) and (2.4.4), and noting that $x \bullet (x \bullet x') = x^2 \bullet x'$ by the assumed commutativity of $L_x, L_{x'}$ acting on x] $= 0$ [as the linearization $x \mapsto x, x \bullet x'$ of (2.4.2)]. □

EXERCISE B.2.4A Derive the first equality in (2.4.4) directly from (2.1.3).

Macdonald Tools Lemma B.2.5 *For any elements x, y in a unital Jordan algebra and $k, i \geq 1$ with $m = \min\{k, i\}$, we have the following operator identities*:

$$U_{x^k}U_{x^i} = U_{x^{k+i}}, \tag{2.5.1}$$

$$U_{x^k,y}U_{x^i} = U_{x^{k+i},y}V_{x^i} - U_{x^{k+2i},y}, \tag{2.5.2}$$

$$V_{x^k}U_{x^i,y} = U_{x^{k+i},y} + U_{x^m}\Xi_{k,i} \tag{2.5.3}$$

$$\text{for } \Xi_{k,i} := \begin{cases} U_{x^{i-k},y} \text{ if } i \geq k,\ m = k, \\ V_{x^{k-i},y} \text{ if } k \geq i,\ m = i, \end{cases}$$

$$V_{x^k}V_{x^i} = V_{x^{k+i}} + U_{x^m}U_{x^{k-m},x^{i-m}} = V_{x^{k+i}} + U_{x^k,x^i}. \tag{2.5.4}$$

PROOF. By the Power-Associativity Lemma B.2.2(1) any x^n, x^m operator-commute with $x^n \bullet x^m = x^{n+m}$, and we apply the results of Commuting B.2.4. (2.5.1) is a special case of Commuting (2.4.5) [replacing $x \mapsto x^k, x' \mapsto x^i, x \bullet x' \mapsto x^{k+i}$]; (2.5.2) is a special case of the second part of Commuting (2.4.6) [with $x \mapsto x^i$, $x' \mapsto x^k$, $x \bullet x' \mapsto x^{k+i}$, $x^2 \bullet x' \mapsto x^{k+2i}$]; while (2.5.3) follows from the first part of Commuting (2.4.6) [with $x \mapsto x^k$, $x' \mapsto x^{i-k}$, $x \bullet x' \mapsto x^i$, $x^2 \bullet x' \mapsto x^{k+i}$] when $i \geq k$, and from Commuting (2.4.7) [with $x \mapsto x^i, x' \mapsto x^{k-i}, x \bullet x' \mapsto x^k, x^2 \bullet x' \mapsto x^{k+i}$] when $k \geq i$. Since $U_{x^n,1} = U_{1,x^n} = V_{x^n,1} = V_{x^n}$, (2.5.4) is just the special case $y = 1$ of (2.5.3) [using linearized Commuting (2.4.5) for the last equality]. □

B.3 Normal Form for Multiplications

For convenience, we will denote the reversal involution $\rho(p)$ on $\mathcal{FA}[x, y]$ by the generic involution symbol p^*. In order to show that the free and free special multiplications in two variables are isomorphic, we put them in a standard form of operators $M_{p,q}$ which produce, acting on the element $z \in \mathcal{FSJ}[x, y, z]$, the basic reversible elements $m(p; z; q) := pzq^* + qzp^*$ homogeneous of degree 1 in z (p, q monomials in $\mathcal{FA}[x, y]$).

We define the operators and verify their action recursively, where the recursion is on the *weight* $\omega(p)$ of monomials p, defined as the number of alternating powers x^i, y^j ($i, j > 0$). If X, Y denote all monomials beginning with a positive power of x, y respectively, then $\omega(1) = 0,\ \omega(x^i) = \omega(y^j) = 1$, and $\omega(p) = 1 + \omega(p')$ if $p = x^i p'$ for $p' \in Y$ (respectively, $p = y^j p'$ for $p' \in X$). We define the weight of a *pair* of monomials to be the sum of the individual weights, $\omega(p, q) := \omega(p) + \omega(q)$.

$M_{p,q}$ **Definition** B.3.1 *In* $\mathcal{M}ult_{\mathcal{FJ}[x,y]}(\mathcal{FJ}[x, y, z])$, *we recursively define multiplication operators* $M_{p,q} = M_{q,p}$ *parameterized by associative monomials* $p, q \in \mathcal{FA}[x, y]$ *in terms of their weight* $\omega(p, q)$ *as follows* (*where* $i, j > 0$) :

$$\text{(3.1.1)} \qquad M_{1,1} = 2\,\mathbb{1}_{\mathcal{FJ}},\ M_{x^i,1} := V_{x^i},\ M_{y^j,1} := V_{y^j},\ M_{x^i,y^j} := U_{x^i,y^j};$$

$$\text{(3.1.2}_x\text{)} \quad M_{x^i p', x^j q'} := U_{x^i} M_{p', x^{j-i} q'} \qquad (\text{if } j \geq i,\ p', q' \in Y \cup \{1\}),$$

$$\text{(3.1.2}_y\text{)} \quad M_{y^i p', y^j q'} := U_{y^i} M_{p', y^{j-i} q'} \qquad (\text{if } j \geq i,\ p', q' \in X \cup \{1\}),$$

$$\text{(3.1.3)} \quad M_{x^i p', y^j q'} := U_{x^i, y^j} M_{p', q'} - M_{y^j p', x^i q'}$$
$$(p' \in Y \cup \{1\}, q' \in X \cup \{1\},\ \text{not } p' = q' = 1),$$

$$\text{(3.1.4}_x\text{)} \qquad M_{x^i p', 1} := V_{x^i} M_{p', 1} - M_{p', x^i}\ (p' \in Y),$$

$$\text{(3.1.4}_y\text{)} \qquad M_{y^i p', 1} := V_{y^i} M_{p', 1} - M_{p', y^i}\ (p' \in X). \qquad \square$$

Note that the $M_{p,q}$ are operators on the *free Jordan algebra* which are parameterized by elements of the *free associative algebra*. We now verify that they produce the reversible elements $m(p; z; q)$ in the free associative algebra. (Recall by Cohn 3-Symmetry A.2.2(3) that for $* := \rho$, $\mathcal{FSJ}[x, y, z] = \mathcal{H}(\mathcal{FA}[x, y, z], *)$ consists precisely of all reversible elements of the free associative algebra.)

$M_{p,q}$ **Action Lemma** B.3.2 *The multiplication operators* $M_{p,q}$ *act on the element* z *in the free special algebra* $\mathcal{FSJ}[x, y, z]$ *by*

$$M_{p,q}(z) = m(p; z; q) = pzq^* + qzp^*.$$

PROOF. We will prove this by recursion on the weight $\omega(p, q)$. For weights 0, 1, 2 as in (3.1.1), we have $M_{1,1}(z) = 2\,\mathbb{1}_{\mathcal{FJ}[x,y,z]}(z) = 2z = 1z1^* + 1z1^*$, $M_{x^i,1}(z) = V_{x^i}(z) = x^i z + z x^i = x^i z 1^* + 1z(x^i)^*$ [analogously for $M_{y^j,1}$], and $M_{x^i,y^j}(z) = U_{x^i,y^j}(z) = x^i z y^j + y^j z x^i = x^i z (y^j)^* + y^j z (x^i)^*$. For (3.1.2$_x$) [analogously (3.1.2$_y$)], if $j \geq i$, $p', q' \in Y \cup \{1\}$ we have $M_{x^i p', x^j q'}(z) = U_{x^i} M_{p', x^{j-i} q'}(z) = x^i \big(p' z (q')^* x^{j-i} + x^{j-i} q' z (p')^*\big) x^i$ [by recursion, since $\omega(p', x^{j-i} q') \leq \omega(p') + \omega(q') + 1 < \omega(p') + \omega(q') + 2 = \omega(p, q)$], which is just $pzq^* + qzp^*$.

For (3.1.3), if $p' \in Y \cup \{1\}, q' \in X \cup \{1\}$, not $p' = q' = 1$, then $M_{x^i p', y^j q'}(z) = U_{x^i, y^j} M_{p', q'}(z) - M_{y^j p', x^i q'}(z) = U_{x^i, y^j}\big(p' z (q')^* + q' z (p')^*\big) - \big(y^j p' z (q')^* x^i + x^i q' z (p')^* y^j\big)$ [by recursion, since (p', q') and $(y^j p', x^i q')$ have lower weight than (p, q) as long as one of p', q' is not 1 (e.g., if $1 \neq p' \in Y$ then $\omega(y^j p') = \omega(p')$)], which reduces to $x^i p' z (q')^* y^j + y^j q' z (p')^* x^i = pzq^* + qzp^*$.

Finally, for $(3.1.4_x)$ [analogously $(3.1.4_y)$] if $p' \in Y$ then $M_{x^ip',1}(z) = V_{x^i}M_{p',1}(z) - M_{p',x^i}(z) = V_{x^i}\big(p'z1^* + 1z(p')^*\big) - \big(p'zx^i + x^iz(p')^*\big)$ [by recursion, since $(p',1)$ has lower weight than $(p,1)$, and $(p',x^i) \in (Y,X)$ has the same weight as $(x^ip',1)$ but falls under case (3.1.3) handled above], which becomes $x^ip'z1 + 1z(p')^*x^i = pz1^* + 1zp^*$. □

The crux of The Macdonald is the following verification that these operators *span* all free multiplications by x and y; from here it will be easy to see that these map onto a *basis* for the free special multiplications by x and y, and therefore they must actually form a *basis* for the free multiplications.

$M_{p,q}$ **Closure Lemma** B.3.3 *The span $\mathcal{M}$ of all the Macdonald operators $M_{p,q} \in \mathcal{M}ult_{\mathcal{FJ}[x,y]}(\mathcal{FJ}[x,y,z])$ (p,q monomials in $\mathcal{FA}[x,y]$) is closed under left multiplication by all multiplication operators in x,y : for all monomials p,q in $\mathcal{FA}[x,y]$ and all $k,\ell \geq 0$ we have*[6]

$$(3.3.1)\qquad U_{x^k}M_{p,q} = M_{x^kp,x^kq}, \qquad U_{y^k}M_{p,q} = M_{y^kp,y^kq},$$

$$(3.3.2)\qquad U_{x^k,y^\ell}M_{p,q} = M_{x^kp,y^\ell q} + M_{y^\ell p,x^kq},$$

$$(3.3.3)\qquad V_{x^k}M_{p,q} = M_{x^kp,q} + M_{p,x^kq}, \quad V_{y^k}M_{p,q} = M_{y^kp,q} + M_{p,y^kq}.$$

Hence $\mathcal{M}$ is the entire multiplication algebra $\mathcal{M}_{\Phi[x,y]}(\mathcal{FJ}[x,y,z])$.

PROOF. (1) By symmetry in x,y we need only prove the x-version of (1), and we give a direct proof of this. If p or q lives in $Y \cup \{1\}$, then x^k is all the x you can extract simultaneously from both sides of (x^kp, x^kq), so the result follows directly from Definition $(3.1.2_x)$. If both p,q live in X, $p = x^ip', q = x^jq'$ for $j \geq i > 0$, $p',q' \in Y \cup \{1\}$, then $U_{x^k}M_{p,q} - M_{x^kp,x^kq} = U_{x^k}M_{x^ip',x^jq'} - M_{x^{k+i}p',x^{k+j}q'} = U_{x^k}U_{x^i}M_{p',x^{j-i}q'} - U_{x^{k+i}}M_{p',x^{j-i}q'}$ [by Definition $(3.1.2_x)$] $= 0$ [by Tool (2.5.1)].

(2) and (3) are considerably more delicate; we prove them by recursion on the total weight $\omega(p,q)$, treating three separate cases. The first case, where $p = q = 1$, is easy, since $M_{1,1} = 2\,\mathbb{1}_{\mathcal{FJ}[x,y,z]}$ and $M_{x^k,y^\ell} + M_{y^\ell,x^k} = 2U_{x^k,y^\ell}$, $M_{x^k,1} + M_{1,x^k} = 2V_{x^k}$ by Definition (3.1.1). Assume that we have proven (2), (3) for all lesser weights. Note that no limits are placed on k,ℓ; this will allow us to carry out a sub-recursion by moving factors outside to increase k,ℓ and reduce the (p',q') left inside.

For the second case, when p,q both lie in X (dually Y), we can handle both (2) and (3) together by allowing k,ℓ to be zero: $p = x^ip', q = x^jq'$, for $i,j \geq 1$, $p',q' \in Y \cup \{1\}$, where by symmetry we may assume $j \geq i$. Then the difference at issue is

[6] Note that by the Generation Theorem we need only prove closure in (1)–(3) for $k = \ell = 1$, but our inductive proof of (2)–(3) requires k,ℓ to be able to grow. The proof for (1) is just as easy for general k as it is for $k = 1$, so we may as well do the general case for all three parts.

$$\begin{aligned}
\Delta_{k,\ell}(p,q) &:= U_{x^k,y^\ell} M_{x^i p', x^j q'} - M_{x^{k+i}p', y^\ell x^j q'} - M_{y^\ell x^i p', x^{k+j} q'} \\
&= U_{x^k,y^\ell}\big(U_{x^i} M_{p', x^{j-i}q'}\big) - \big(U_{x^{k+i},y^\ell} M_{p', x^j q'} - M_{y^\ell p', x^{k+i+j}q'}\big) \\
&\quad - M_{y^\ell x^i p', x^{k+j}q'} \\
&= \big(U_{x^{k+i},y^\ell} V_{x^i} - U_{x^{k+2i},y^\ell}\big) M_{p', x^{j-i}q'} - U_{x^{k+i},y^\ell} M_{p', x^j q'} \\
&\quad + M_{y^\ell p', x^{k+i+j}q'} - M_{y^\ell x^i p', x^{k+j}q'} \\
&= U_{x^{k+i},y^\ell}\big(V_{x^i} M_{p', x^{j-i}q'} - M_{p', x^j q'}\big) \\
&\quad - \big(U_{x^{k+2i},y^\ell} M_{p', x^{j-i}q'} - M_{y^\ell p', x^{k+i+j}q'}\big) - M_{y^\ell x^i p', x^{k+j}q'} \\
&= U_{x^{k+i},y^\ell}\big(M_{x^i p', x^{j-i}q'}\big) - \big(M_{x^{k+2i}p', y^\ell x^{j-i}q'}\big) - M_{y^\ell x^i p', x^{k+j}q'} \\
&= \Delta_{k+i,\ell}(x^i p', x^{j-i}q'),
\end{aligned}$$

using, for each $=$ in succession: the definition of Δ; Definitions $(3.1.2_x, 3)$; Tool (2.5.2); rearrangement; recursion on (3) and (2) [since $(p', x^{j-i}q')$ has lesser weight than (p,q)]; the definition of Δ.

If $j = i$ then the pair $(x^i p', x^{j-i}q')$ has lesser weight than (p,q); otherwise, if $j > i$ it has the same weight but lower total x-degree $i + (j-i) < i + j$ in the initial x-factors of p' and q', so by recursion on this degree we can reduce to the case where at least one of the initial factors vanishes and we have lower weight. Thus in the second case, both subcases lead by recursion on total weight to $\Delta_{k,\ell}(p,q) = 0$.

For the remaining third case, where p, q lie in different spaces, by symmetry we can assume that (p,q) is contained in either $(1,Y)$ or (X,Y) or $(X,1)$. The result is easy when $k, \ell > 0$ (i.e., for (2)): $\Delta_{k,\ell}(p,q) := U_{x^k,y^\ell} M_{p,q} - M_{x^k p, y^\ell q} - M_{y^\ell p, x^k q} = 0$ because in all three subcases $(y^\ell p, x^k q)$ begin with exactly ℓ y's and k x's, so the result follows directly from the definition (3.1.3) of $M_{y^\ell p, x^k q}$. This finishes all cases of (2).

For (3) in this third case, it suffices by symmetry to consider only the x-version. We must consider the three subcases separately. The first subcase $(p,q) \in (1,Y)$ is easiest: here $p = 1, q = y^i q'$, and $\Delta_{k,0}(p,q) := V_{x^k} M_{1, y^i q'} - M_{x^k, y^i q'} - M_{1, x^k y^i q'} = 0$ by Definition $(3.1.4_x)$ of $M_{1, x^k y^i q'} = M_{x^k y^i q', 1}$.

The second subcase $(p,q) \in (X,Y)$ is messiest, since it depends on which of i or k is larger. Here $p = x^i p', q = y^j q'$ $(p' \in Y \cup \{1\}, q' \in X \cup \{1\})$, $m := \min\{i,k\}$, and the relevant difference becomes

$$\begin{aligned}
\Delta_{k,0}(p,q) &= V_{x^k} M_{x^i p', y^j q'} - M_{x^{k+i}p', y^j q'} - M_{x^i p', x^k y^j q'} \\
&= V_{x^k}\big(U_{x^i,y^j} M_{p',q'} - M_{y^j p', x^i q'}\big) - \big(U_{x^{i+k},y^j} M_{p',q'} - M_{y^j p', x^{k+i}q'}\big) \\
&\quad - U_{x^m} M_{x^{i-m}p', x^{k-m}y^j q'} \\
&= \big(V_{x^k} U_{x^i,y^j} - U_{x^{i+k},y^j}\big) M_{p',q'} - \big(V_{x^k} M_{y^j p', x^i q'} - M_{y^j p', x^{k+i}q'}\big) \\
&\quad - U_{x^m} M_{x^{i-m}p', x^{k-m}y^j q'} \\
&= U_{x^m}\big(\Xi_{k,i} M_{p',q'} - \Xi' - M_{x^{i-m}p', x^{k-m}y^j q'}\big)
\end{aligned}$$

where

$$\Xi' := \begin{cases} M_{x^{k-m}y^j p', x^{i-m}q'} \text{ if } (p',q') \neq (1,1), \\ \Xi_{k,i} \text{ if } (p',q') = (1,1), \end{cases} \qquad .$$

using, for each = in succession: the definition of Δ; Definitions (3.1.3, 3, 2_x); rearrangement; and, finally, using Tool (2.5.3) on the first term in parenthesis, and on the second term either Tool (2.5.3) [if $(p',q') = (1,1)$, in view of Definition (3.1.1)] or recursion [if $(p',q') \neq (1,1)$, so that $(y^j p', x^i q')$ has lower weight, and $M_{x^k y^j p', x^i q'} = U_{x^m} M_{x^{k-m} y^j p', x^{i-m} q'}$ by Definition (3.1.2_x)].

First consider the case where i is larger, $m = k \leq i$, $\Xi_{k,i} = U_{x^{i-k}, y^j}$ by Tool (2.5.3). If $(p',q') \neq (1,1)$ the inner term above becomes $U_{x^{i-k},y^j} M_{p',q'} - M_{y^j p', x^{i-k} q'} - M_{x^{i-k} p', y^j q'} = \Delta_{i-k,j}(p',q') = 0$ [by (2) for (p',q') of lesser weight], while if $(p',q') = (1,1)$ it becomes $U_{x^{i-k},y^j} M_{1,1} - U_{x^{i-k},y^j} - U_{x^{i-k},y^j} = 0$ [$\mathcal{M}_{1,1} = 2\mathbb{1}_{\mathcal{FJ}[x,y,z]}$ by Definition (3.1.1)]. In either case, $\Delta_{k,0}(p,q) = 0$.

Next consider the case where k is larger, $m = i \leq k$, $\Xi_{k,i} = V_{x^{k-i}, y^j}$ by Tool (2.5.3). If $(p',q') = (1,1)$ the inner term above becomes $V_{x^{k-i},y^j} M_{1,1} - V_{x^{k-i},y^j} - M_{1, x^{k-i} y^j} = \big(V_{x^{k-i}} V_{y^j} - U_{x^{k-i},y^j}\big) - \big(V_{x^{k-i}} M_{1,y^j} - M_{x^{k-i},y^j}\big)$ [by Basic (2.1.5), Definition (3.1.4_x)] $= 0$ [by Definition (3.1.1)]. The complicated case is $(p',q') \neq (1,1)$, where the inner term becomes

$$
\begin{aligned}
&V_{x^{k-i},y^j} M_{p',q'} - M_{x^{k-i} y^j p', q'} - M_{p', x^{k-i} y^j q'} \\
&\quad = \big(V_{x^{k-i}} V_{y^j} - U_{x^{k-i},y^j}\big) M_{p',q'} - \big(V_{x^{k-i}} M_{y^j p', q'} - M_{y^j p', x^{k-i} q'}\big) \\
&\qquad - \big(V_{x^{k-i}} M_{p', y^j q'} - M_{x^{k-i} p', y^j q'}\big) \\
&\quad = V_{x^{k-i}} \big(V_{y^j} M_{p',q'} - M_{y^j p', q'} - M_{p', y^j q'}\big) \\
&\qquad - \big(U_{x^{k-i},y^j} M_{p',q'} - M_{y^j p', x^{k-i} q'} - M_{x^{k-i} p', y^j q'}\big) \\
&\quad = V_{x^{k-i}} \Delta_{0,j}(p',q') - \Delta_{k-i,j}(p',q') = 0,
\end{aligned}
$$

using, for each = in succession: Basic (2.1.5) [since $\Delta_{k-i,0}$ vanishes by (3) on lower-weight terms $\omega(y^j p', q')$, $\omega(p', y^j q') < \omega(x^i p', y^j q') = \omega(p,q)$]; rearrangement; definition of Δ. Here the final $\Delta(p',q')$ vanish by the recursion hypotheses (3), (2) on lesser-weight term (p',q'). In either case, $\Delta_{k,0}(p,q) = 0$, finishing the second subcase $(p,q) \in (X,Y)$.

The third and final subcase $(p,q) \in (X,1)$ depends on the previous cases. Here $p = x^i p', p' \in Y \cup 1, q = 1$. If $p' = 1$ then $\Delta_{k,0}(p,q) := V_{x^k} M_{x^i,1} - M_{x^{k+i},1} - M_{x^i,x^k} = V_{x^k} V_{x^i} - V_{x^{k+i}} - U_{x^m} U_{x^{k-m},x^{i-m}}$ [by Definitions (3.1.1), (3.1.2_x)] $= 0$ [by Tool (2.5.4)]. Henceforth we assume $1 \neq p' \in Y$ and again set $m := \min(i,k)$. The difference is

$$
\begin{aligned}
&\Delta_{k,0}(x^i p', 1) \\
&\quad = V_{x^k} M_{x^i p', 1} - M_{x^{k+i} p', 1} - M_{x^i p', x^k} \\
&\quad = V_{x^k} \big(V_{x^i} M_{p',1} - M_{p',x^i}\big) - \big(V_{x^{k+i}} M_{p',1} - M_{p',x^{k+i}}\big) - U_{x^m} M_{x^{i-m} p', x^{k-m}} \\
&\quad = \big(V_{x^k} V_{x^i} - V_{x^{k+i}}\big) M_{p',1} - \big(V_{x^k} M_{p',x^i} - M_{p',x^{k+i}}\big) - U_{x^m} M_{x^{i-m} p', x^{k-m}} \\
&\quad = \big(U_{x^m} U_{x^{k-m},x^{i-m}}\big) M_{p',1} - \big(M_{x^k p', x^i}\big) - U_{x^m} M_{x^{i-m} p', x^{k-m}} \\
&\quad = U_{x^m} \big(U_{x^{k-m},x^{i-m}} M_{p',1} - M_{x^{k-m} p', x^{i-m}} - M_{x^{i-m} p', x^{k-m}}\big) = 0,
\end{aligned}
$$

using, for each = in succession: the definition of Δ; Definitions (3.1.4_x, 4_x, 2_x); rearrangement; Tool (2.5.4) and the second subcase above of (3) [on $M_{p',x^i} = M_{x^i,p'} \in M_{X,Y}$, since we are assuming $p' \in Y$]; Definition (3.1.2_x). The final term in parentheses vanishes, since if $m = i < k$ the term is $\Delta_{k-i,0}(p',1)$, and if $m = k < i$ it is $\Delta_{i-k,0}(p',1)$, both of which vanish by recursion on weight [or straight from Definition $(3.1.4_x)$], while if $m = k = i$ the result is trivial because $U_{1,1} = 2\, \mathbb{1}_{\mathcal{FJ}[x,y,z]}$. This completes the recursion for (3).

$\mathcal{M}$ contains all of $\mathcal{M}_{\Phi[x,y]}\,(\mathcal{FJ}[x,y,z])$ because it contains $1_{\mathcal{FJ}}$ and by (3.3.1)–(3.3.3) above is closed under left multiplication by the generators $V_x, V_y,\ U_x, U_y, U_{x,y}$ [by the Generation Theorem B.2.3] for x, y the generators of $\mathrm{B} := \Phi[x,y] \subseteq \mathrm{J} := \mathcal{FJ}[x,y,z]$. □

B.4 The Macdonald Principles

We are now ready to establish the fundamental Macdonald Theorem of 1958 in all its protean forms.

Macdonald Principles B.4.1 *We have the following equivalent versions of the Macdonald*:

(1) *There are no s-identities in three variables x, y, z of degree ≤ 1 in z.*

(2) *The canonical homomorphism of the free Jordan algebra $\mathcal{FJ}[x,y,z]$ onto the free special Jordan algebra $\mathcal{FSJ}[x,y,z]$ in three variables is injective on elements of degree ≤ 1 in z.*

(3) *Any Jordan polynomial $f(x,y,z)$ which is of degree ≤ 1 in z and vanishes in all associative algebras A^+ (equivalently, all special Jordan algebras $\mathrm{J} \subseteq \mathrm{A}^+$) vanishes in all Jordan algebras J.*

(4) *Any multiplication operator in two variables which vanishes on all special algebras will vanish on all Jordan algebras.*

(5) *The canonical mapping $\mathcal{M}_{\Phi[x,y]}(\mathcal{FJ}[x,y,z]) \xrightarrow{\sigma} \mathcal{M}_{\Phi[x,y]}(\mathcal{FSJ}[x,y,z])$ of the free multiplication subalgebra to the special multiplication subalgebra is injective.*

PROOF. Let us first convince ourselves that all five assertions are equivalent. (1) $\Longleftrightarrow$ (2) because the s-identities are precisely the elements of the kernel of the canonical homomorphism $\mathcal{FJ} \to \mathcal{FSJ}$. (2) $\Longleftrightarrow$ (3) because f vanishes on all (respectively all special) Jordan algebras iff it vanishes in $\mathcal{FJ}$ (respectively $\mathcal{FSJ}$) by the universal property for the free algebras: $f(x,y,z) = 0$ in $\mathcal{FJ}[x,y,z]$ (respectively $\mathcal{FSJ}[x,y,z]$) iff all specializations $f(a,b,c) = 0$ for a, b, c in any (respectively any special) J.

The tricky part is the equivalence of (3) and (4). The homogeneous polynomials of degree 1 in z are precisely all $f(x,y,z) = M_{x,y}(z)$ given by a multiplication operator in x, y, and [by definition of the zero operator] $f(x,y,z)$ vanishes on an algebra J iff the operator $M_{x,y}$ does. Thus (4) is equivalent to (3) for polynomials which are homogeneous of degree 1 in z. In particular, (3) $\Longrightarrow$ (4), but to prove (4) $\Longrightarrow$ (3) we must show that the addition of a constant term doesn't affect the result. We will do this by separating f into its homogeneous components, and reducing each to a multiplication operator. Any polynomial of degree ≤ 1 in z can be written as

$f(x,y,z) = f_0(x,y,z) + f_1(x,y,z) = f_0(x,y) + f_1(x,y,z)$ in terms of its homogeneous components of degree $0, 1$ in z. Now, it is a general fact that f vanishes on *all* algebras iff each of its homogeneous components does (see Problem B.2 below), but in the present case we can easily see the stronger result that our f vanishes on any *particular* algebra iff its components f_0, f_1 do: the homogeneous $f_1(x,y,z)$ automatically vanishes at $z = 0$ [$f_1(x,y,0) = f_1(x,y,0 \cdot 0) = 0^1 f_1(x,y,0) = 0$ by homogeneity of degree 1 in z], so all $f(a,b,0) = f_0(a,b) + f_1(a,b,0) = f_0(a,b)$, $f(a,b,c) = f_1(a,b,c) + f_0(a,b)$ vanish iff all $f_0(a,b)$ and $f_1(a,b,c)$ vanish. Thus we have reduced the problem for f to a problem about its separate components:

$$(6) \qquad f \text{ vanishes on J} \iff f_0, f_1 \text{ vanish on J}.$$

Now we reduce this problem about polynomials to a problem about operators. We have noticed that $f_1(x,y,z) = M_{x,y}(z)$ vanishes as a polynomial iff the multiplication operator $M_{x,y}$ vanishes as an operator, and surprisingly the same holds for $f_0(x,y)$: an operator $N = L_a$ vanishes on a unital algebra iff its value on 1 is zero, so the operator $N_{x,y} := L_{f_0(x,y)}$ vanishes iff $f_0(x,y) = N_{x,y}(1)$ vanishes as a polynomial.

$$(7) \qquad f_0(x,y),\ f_1(x,y,z) \text{ vanish on J} \iff N_{x,y},\ M_{x,y} \text{ do}.$$

From this we can show that (4) $\Longrightarrow$ (3): by (6), (7) f vanishes in all special algebras iff the operators N, M do; by (4) this happens iff $N_{x,y}, M_{x,y}$ vanish on the free algebra $\mathcal{FJ}$, in which case they vanish at $1, z \in \mathcal{FJ}$ and $f(x,y,z) = N_{x,y}(1) + M_{x,y}(z) = 0$ in $\mathcal{FJ}$.

Finally, (4) $\iff$ (5) because a multiplication operator $M_{x,y}$ is zero as an element of $\mathcal{M}_{\Phi[x,y]}(\mathcal{FJ}[x,y,z])$ (respectively, in $\mathcal{M}_{\Phi[x,y]}(\mathcal{FSJ}[x,y,z])$) iff it is zero on $\mathcal{FJ}[x,y,z]$ (respectively, on $\mathcal{FSJ}[x,y,z]$) [by definition of the zero operator] iff it is zero on all Jordan algebras (respectively, on all special Jordan algebras) J [evaluating x,y,z at any $a,b,c \in$ J].

So all five forms are equivalent, but are any of them true?

We will establish version (5). The canonical map σ will be injective (as asserted by (5)) if its composition $\tau := \varepsilon_z \circ \sigma$ with evaluation at $z \in \mathcal{FSJ}[x,y,z]$ is an injective map $\tau : \mathcal{M}_{\Phi[x,y]}(\mathcal{FJ}[x,y,z]) \to \mathcal{FSJ}[x,y,z]$. By the $M_{p,q}$ Closure Lemma B.3.3, the algebra $\mathcal{M}_{\Phi[x,y]}(\mathcal{FJ}[x,y,z])$ is spanned by the elements $M_{p,q} = M_{q,p}$ for associative monomials $p, q \in \mathcal{FA}[x,y]$, and we have a general result about linear transformations:

(8) A linear map $T : M \to N$ of Φ-modules will be injective
if it takes a spanning set $\{m_i\}$ into an independent set in N
(in which case the original spanning set was already a basis for M).

Indeed, $T(m) = T(\sum \alpha_i m_i)$ [since the m_i span M] $= \sum_i \alpha_i T(m_i) = 0 \Longrightarrow$ all $\alpha_i = 0$ [by independence of the $T(m_i)$ in N] $\Longrightarrow m = 0$.

Thus we need only verify that the $\tau(M_{p,q}) = M_{p,q}(z) = m(p;z;q)$ [by the $M_{p,q}$ Action Lemma B.3.2] are independent. But the distinct monomials pzr in the free algebra $\mathcal{FA}[x,y,z]$ are linearly independent, and two reversible

$pzq^* + qzp^*$, $p'z(q')^* + q'z(p')^*$ can share a common monomial only if either $pzq^* = p'z(q')^*$ [in which case, by the uniqueness of expression in the free algebra we have $p = p', q = q', M_{p,q} = M_{p',q'}$] or else $pzq^* = q'z(p')^*$ [in which case, $p = q', q = p', M_{p,q} = M_{q',p'} = M_{p',q'}$ by symmetry of the M's]. Thus in both cases distinct $M_{p,q}$'s contribute distinct monomials, and so are independent. In view of (8), this completes the proof of (5), and hence of the entire theorem. □

EXERCISE B.4.1 (1) Show that a Jordan polynomial $f(x_1, \ldots, x_n)$ in n variables in $\mathcal{FJ}[X]$ (respectively, in $\mathcal{FSJ}[X]$) can be uniquely decomposed into its homogeneous components of degree e_i in each variable x_i, $f = \sum_{e_1,\ldots,e_n} f_{e_1,\ldots,e_n}$. (2) Show (using indeterminate scalar extensions $\Omega = \Phi[t_1, \ldots, t_n]$) that f vanishes on all algebras (respectively, all special algebras) iff each of its homogeneous components $f_{e_1,\ldots,e_n}$ vanishes on all algebras (respectively, all special algebras). In particular, conclude that $f = 0$ in $\mathcal{FJ}[X]$ iff each $f_{e_1,\ldots,e_n} = 0$ in $\mathcal{FJ}[X]$.

Our version of Macdonald's Principles has subsumed Shirshov's earlier 1956 theorem. Actually, Macdonald's original 1958 theorem concerned only polynomials homogeneous of degree 1 in z, and thus amounted to the operator versions (4), (5). The assertion about polynomials homogeneous of degree 0 in z is precisely Shirshov's Theorem that $\mathcal{FJ}[x, y] \cong \mathcal{FSJ}[x, y]$. The reason (4) ⇒ (3) is tricky is that it is nothing but Macdonald swallowing up Shirshov, and any herpetologist can tell you that swallowing a major theorem requires a major distension of the jaws.

Shirshov's Theorem B.4.2 *The free Jordan algebra on two generators is special: the canonical homomorphism σ_2: $\mathcal{FJ}[x, y] \to \mathcal{FSJ}[x, y]$ is an isomorphism.*

PROOF. The canonical specialization σ_2 (determined by $\sigma_2(x) = x, \sigma_2 = y$) is an epimorphism because its image contains the generators x, y. We claim that it is also a monomorphism. We have noted before that we have a canonical identification of $\mathcal{FJ}[x, y], \mathcal{FSJ}[x, y]$ with the subalgebras $\mathrm{B} \subseteq \mathcal{FJ}[x, y, z]$, $\mathrm{B}_s \subseteq \mathcal{FSJ}[x, y, z]$ generated by x, y, under which the canonical projection σ_2 corresponds to the restriction of the canonical projection $\sigma_3 : \mathcal{FJ}[x, y, z] \twoheadrightarrow \mathcal{FSJ}[x, y, z]$ to $\mathrm{B} \twoheadrightarrow \mathrm{B}_s$. By Macdonald's Principle (2), σ_3 is injective on B, i.e., σ_2 is injective on $\mathcal{FJ}[x, y]$. This completes the proof that σ_2 is an isomorphism. □

In 1959 Cohn combined Shirshov's Theorem with his own Speciality Theorem A.3.2 to obtain the definitive result about Jordan algebras with two generators.

Shirshov–Cohn Theorem B.4.3 *Any Jordan algebra (unital or not) generated by two elements x, y is special, indeed is isomorphic to an algebra $\mathcal{H}(\mathrm{A}, *)$ for an associative algebra* A *with involution* $*$.

PROOF. We may restrict ourselves to unital algebras, since J is isomorphic to $\mathcal{H}(\mathrm{A},*)$ iff the formal unital hull $\widehat{\mathrm{J}}$ is isomorphic to $\mathcal{H}(\widehat{\mathrm{A}},*)$. In the unital case any J generated by two elements is a homomorphic image of $\mathcal{FJ}[x,y] \cong \mathcal{FSJ}[x,y]$ [by Shirshov's Theorem], and by Cohn's 2-Speciality Theorem any image of $\mathcal{FSJ}[x,y]$ is special of the form $\mathcal{H}(\mathrm{A},*)$. □

This establishes, once and for all, the validity of the basic principles we have used so frequently in our previous work.

B.5 Albert i-Exceptionality

We have just seen that any Jordan polynomial in three variables linear in one of them will vanish on all Jordan algebras if it vanishes on all associative algebras. This fails as soon as the polynomial has degree at least 2 in all variables: we can exhibit s-identities of degree 8, 9, 11 which hold in all special Jordan algebras but not in the Albert algebra. This reproves the exceptional nature of the Albert algebra; more, it shows that the Albert algebra cannot even be a *homomorphic image* of a special algebra, since it does not satisfy all their identities.

i-Special Definition B.5.1 *An* s-**identity** *is a Jordan polynomial* (*element of the free Jordan algebra*) *which is satisfied by all special algebras, but not by all algebras. Equivalently, the polynomial vanishes on all special algebras but is not zero in the free algebra. A Jordan algebra is* i-**special** *if it satisfies all the s-identities that the special algebras do, otherwise it is* i-**exceptional**.

Thus being i-special is easier than being special: the algebra need only obey externally the laws of speciality (*the s-identities*), *without being special in its heart* (*living an associative life*). *Correspondingly, being i-exceptional is harder than being exceptional: to be i-exceptional an algebra can't even* look *special as regards its identities — without any interior probing, it reveals externally its exceptional nature by refusing to obey one of the speciality laws.*

The first s-identities were discovered by Charles Glennie, later a more understandable one was discovered by Armin Thedy, and recently Glennie's original identities have been recast by Ivan Shestakov.

s-Identities Definition B.5.2 **Glennie's Identities** *are the Jordan polynomials* $G_n := H_n(x,y,z) - H_n(y,x,z)$ *of degrees* $n = 8, 9$ (*degree* 3 *in* x, y *and degrees* $2, 3$ *in* z) *expressing the symmetry in* x, y *of the products*

$$H_8(x,y,z) := \{U_xU_yz, z, \{x,y\}\} - U_xU_yU_z(\{x,y\}),$$

$$H_9(x,y,z) := \{U_xz, U_{y,x}U_zy^2\} - U_xU_zU_{x,y}U_yz.$$

We may also write G_8 *as*

$$\{[U_x,U_y]z, z, \{x,y\}\} - [U_x,U_y]U_z\{x,y\}.$$

Both G_8 and G_9 may be written in terms of commutators as **Shestakov's Identities**

$$Ш_8 := [[x,y]^3, z^2] - \{z, [[x,y]^3, z]\},$$
$$Ш_9 := [[x,y]^3, z^3] - \{z^2, [[x,y]^3, z]\} - U_z[[x,y]^3, z],$$

where the pseudo-derivation $\mathrm{ad}[[x,y]^3, \cdot]$ *is defined as* $D_{x,y}^3 + 3U_{[x,y]}D_{x,y}$ *in terms of the Jordan derivation* $\mathrm{ad}([x,y]) = [[x,y],\cdot] = D_{x,y} := V_{x,y} - V_{y,x} = [V_x, V_y]$ *and the pseudo-structural transformation* $U_{[x,y]} := U_{\{x,y\}} - 2\{U_x, U_y\}$.

Thedy's Identity T_{10} *is the operator Jordan polynomial*

$$T_{10}(x,y,z) := U_{U_{[x,y]}(z)} - U_{[x,y]}U_zU_{[x,y]}$$

of degree 10 (*degree* 4 *in* x, y *and degree* 2 *in* z) *expressing the structurality of* $U_{[x,y]}$; *acting on an element* w, *this produces an element polynomial*

$$T_{11}(x,y,z,w) := T_{10}(x,y,z)(w) = U_{U_{[x,y]}(z)}w - U_{[x,y]}U_zU_{[x,y]}w$$

of degree 11 (*degree* 4 *in* x, y, *degree* 2 *in* z, *and degree* 1 *in* w).

s-Identities Theorem B.5.3 *Glennie's and Thedy's Identities are s-identities: they vanish in all special algebras, but not in all Jordan algebras, since they do not vanish on the Albert algebra* $\mathcal{H}_3(\mathrm{O})$.

PROOF. The easy part is showing that these vanish in associative algebras: H_8 reduces to the symmetric 8-tad

$$\begin{aligned}&(xyzyx)z(xy+yx) + (xy+yx)z(xyzyx) - xyz(xy+yx)zyx\\ &\quad= \{x,y,z,y,x,z,x,y\} + \{x,y,z,y,x,z,y,x\} - \{x,y,z,y,x,z,y,x\}\\ &\quad= \{x,y,z,y,x,z,x,y\},\end{aligned}$$

and H_9 reduces to the symmetric 9-tad

$$\begin{aligned}&(xzx)\big(y(zy^2z)x + x(zy^2z)y\big) + \big(y(zy^2z)x + x(zy^2z)y\big)(xzx)\\ &\qquad -xz\big(x(yzy)y + y(yzy)x\big)zx\\ &\quad= \{x,z,x,y,z,y,y,z,x\} + \{x,z,x,x,z,y,y,z,y\} - \{x,z,x,y,z,y,y,z,x\}\\ &\quad= \{x,z,x,x,z,y,y,z,y\}.\end{aligned}$$

The operator $U_{[x,y]}$ on z in a special algebra reduces to $[x,y]z[x,y]$, involving honest commutators $[x,y]$ (which make sense in the associative algebra, but not the special Jordan algebra), so acting on w we have $T_{10}(w) = ([x,y]z[x,y])w([x,y]z[x,y]) - [x,y]\big(z([x,y]w[x,y])z\big)[x,y] = 0$.

The hard part is choosing manageable substitutions which don't vanish in the reduced Albert algebra $\mathcal{H}_3(\mathrm{O})$ for an octonion algebra O (with scalar involution, i.e., all traces and norms $t(d), n(d)$ lie in $\Phi 1$).

B.5.1 Nonvanishing of G_9

Throughout we use the Jacobson box notation and Hermitian Multiplication Rules II.3.2.4 for $\mathcal{H}_3(\mathrm{D},-)$ without further comment. For G_9 the calculations are not overly painful: take

$$(1)\qquad x := 1[12], \qquad y := 1[23], \qquad z := a[21] + b[13] + c[32].$$

We will not compute the entire value of $G_9(x,y,z)$, we will apply the Peirce projection E_{13} and show that already its component in the Peirce space J_{13} is nonzero. We claim that

$$(2)\qquad \begin{aligned} E_{13}U_x &= 0, \qquad E_{13}U_y = 0,\\ U_x z &= a[12], \qquad U_y z = c[23],\\ U_z x^2 &= n(a)(e_1+e_2) + n(b)e_3 + n(c)e_3 + ab[23] + ca[31],\\ U_z y^2 &= n(c)(e_3+e_2) + n(a)e_1 + n(b)e_1 + bc[12] + ca[31]. \end{aligned}$$

The Peirce Triple Product Rules II.13.3.1(2) show that $U_x\mathrm{J} \subseteq \mathrm{J}_{11}+\mathrm{J}_{12}+\mathrm{J}_{22}$, so $E_{13}U_x\mathrm{J} = 0$, analogously $E_{13}U_y = 0$, which lightens our burden by killing off two of the four terms. This establishes the first line of (2). The second line follows since $U_x z = U_x a[21]$ [by Peirce U-Orthogonality] $= 1a1[12] = a[12]$ [by Hermitian Multiplication], and analogously for $U_y z$. The third line follows from $\big(U_{a[21]} + U_{b[13]} + U_{c[32]} + U_{a[21],b[13]} + U_{b[13],c[32]} + U_{c[32],a[21]}\big)(e_1+e_2) = n(a)(e_1+e_2)+n(b)e_3+n(c)e_3+ab[23]+0+ca[31]$, and the fourth line follows analogously, due to symmetry in the indices 1, 3. From the Peirce relations we have

$$(3)\qquad \begin{aligned} E_{12}U_{x,y} &= U_{x,y}E_{23}, \qquad E_{13}V_{a[21]} = V_{a[21]}E_{23},\\ E_{23}U_{x,y} &= U_{x,y}E_{12}, \qquad E_{13}V_{c[32]} = V_{c[32]}E_{12}. \end{aligned}$$

(For example, $\{a[21],\mathrm{J}_{ij}\} = 0$ unless i or j links up, and $E_{13}\{a[21],\mathrm{J}_{2j}\} = 0$ unless $j = 3$, $E_{13}\{a[21],\mathrm{J}_{1j}\} = 0$, and dually for $c[32]$.) Using (2), (3), (3), (2) in succession, we compute

$$\begin{aligned} &E_{13}\big(\{U_x z, U_{y,x}U_z y^2\}\big) && E_{13}\big(\{U_y z, U_{x,y}U_z x^2\}\big)\\ &= E_{13}\big(V_{a[12]}U_{x,y}U_z y^2\big) && E_{13}\big(V_{c[23]}U_{x,y}U_z x^2\big)\\ &= V_{a[12]}E_{23}\big(U_{x,y}U_z y^2\big) && = V_{c[23]}E_{12}\big(U_{x,y}U_z x^2\big)\\ &= V_{a[12]}U_{x,y}E_{12}\big(U_z y^2\big) && = V_{c[23]}U_{x,y}E_{23}\big(U_z x^2\big)\\ &= V_{a[12]}U_{1[12],1[23]}(bc[12]) && = V_{c[23]}U_{1[12],1[23]}(ab[23])\\ &= V_{a[12]}(bc[23]) = a(bc)[13]. && = V_{c[23]}(ab[12]) = (ab)c[13]. \end{aligned}$$

Subtracting the second E_{13} component from the first gives

$$E_{13}\big(G_9(x,y,z)\big) = (a(bc) - (ab)c)[13] = -[a,b,c][13].$$

This vanishes on $\mathcal{H}_3(\mathrm{D})$ iff $[a,b,c]=0$ for all $a,b,c\in \mathrm{D}$, i.e., iff D is associative. Thus Glennie's Identity of degree 9 does not vanish for a non-associative octonion algebra O. □

B.5.2 Nonvanishing of G_8

For G_8 the calculations are considerably more painful. We set

$$(1') \quad x := e_1 - e_3, \quad y := a[12] + b[13] + c[23], \quad z := 1[12] + 1[23].$$

Since $\{e_i, a[ij]\} = a[ij],\ U_{1[i2]}a[i2] = a[2i] = \bar{a}[i2],\ \{1[j2], a[i2], 1[i2]\} = \bar{a}[j2]$ we have

$$(2') \quad \{x,y\} = a[12] - c[23], \quad U_z(\{x,y\}) = (a-\bar{c})[21] + (\bar{a}-c)[32].$$

Thus $z, U_z(\{x,y\})$ fall in the Peirce spaces $\mathrm{J}_{12} + \mathrm{J}_{23}$, and if we take $E := E_{12} + E_{23}$, then from $U_x = E_{11} - E_{13} + E_{33}$ we have

$$\begin{aligned}
U_xE = U_yU_xE = 0, \quad U_xU_yE &= (E_{11} - E_{13} + E_{33})\big(U_{a[12]} + U_{b[13]} \\
&\quad + U_{c[23]} + U_{a[12],b[13]} + U_{b[13],c[23]} + U_{c[23],a[12]}\big)(E_{12} + E_{23}) \\
&= (E_{11}U_{a[12],b[13]}E_{32} - E_{13}U_{a[12],b[13]}E_{12}) \\
&\quad + (E_{33}U_{b[13],c[23]}E_{12} - E_{13}U_{b[13],c[23]}E_{23}),
\end{aligned}$$

$$(3') \qquad E_{13}[U_x, U_y]E = -E_{13}U_{a[12],b[13]}E_{21} - E_{13}U_{b[13],c[23]}E_{32}.$$

We now examine only the 13-components of G_8. On the one hand, by $(2')$, $(3')$,

$$\begin{aligned}
&E_{13}\big([U_x,U_y]U_z\{x,y\}\big) = \big(E_{13}[U_x,U_y]E\big)\big((a-\bar{c})[21] + (\bar{a}-c)[32]\big) \\
&= -E_{13}\big(U_{a[12],b[13]}E_{12}\big((a-\bar{c})[21]\big)\big) - E_{13}\big(U_{b[13],c[23]}E_{32}\big((\bar{a}-c)[32]\big)\big) \\
&= -\{a[12], (a-\bar{c})[21], b[13]\} - \{b[13], (\bar{a}-c)[32], c[23]\} \\
&= \big(-a((a-\bar{c})b) - (b(\bar{a}-c))c\big)[13] \\
&= \big(-a^2b + \big(at(\bar{c}b) - a(\bar{b}c)\big) - \big(t(b\bar{a})c - (a\bar{b})c\big) + bc^2\big)[13] \qquad \text{[by alternativity]} \\
&= \big(-a^2b + at(\bar{c}b) - t(b\bar{a})c + [a,\bar{b},c] + bc^2\big)[13] \\
&= \big(bc^2 - a^2b + at(\bar{c}b) - t(b\bar{a})c - [a,b,c]\big)[13] \qquad \text{[by scalar involution].}
\end{aligned}$$

On the other hand, by $(2')$ we have

$$\begin{aligned}
&E_{13}\{[U_x,U_y]z, z, \{x,y\}\} = E_{13}\{[U_x,U_y]Ez, 1[12]+1[23], a[12]-c[23]\} \\
&= E_{13}\{[U_x,U_y]Ez, 1[12], a[12]\} - E_{13}\{[U_x,U_y]Ez, 1[23], c[23]\} \\
&\ - E_{13}\{[U_x,U_y]Ez, 1[12], c[23]\} + E_{13}\{[U_x,U_y]Ez, 1[23], a[12]\} \\
&= \{a[12], 1[21], E_{13}[U_x,U_y]Ez\} - \{E_{13}[U_x,U_y]Ez, 1[32], c[23]\} \\
&\ - \{E_{11}[U_x,U_y]Ez, 1[12], c[23]\} + \{a[12], 1[23], E_{33}[U_x,U_y]Ez\} \\
&= -\{a[12], 1[21], U_{a[12],b[13]}1[21] + U_{b[13],c[23]}1[32]\} \qquad \text{[using (3')]}
\end{aligned}$$

$$+\ \{U_{a[12],b[13]}1[21] + U_{b[13],c[23]}1[32], 1[32], c[23]\}$$
$$-\ \{U_{b[13],\bar{a}[21]}1[32], 1[12], c[23]\} + \{a[12], 1[23], U_{\bar{c}[32],b[13]}1[21]\}$$
$$= -\{a[12], 1[21], (ab+bc)[13]\} + \{(ab+bc)[13], 1[32], c[23]\}$$
$$-\ \{t(b\bar{a})[11], 1[12], c[23]\} + \{a[12], 1[23], t(\bar{c}b)[33]\}$$
$$= \big(-a(ab+bc) + (ab+bc)c - t(b\bar{a})c + at(\bar{c}b)\big)[13]$$
$$= \big(-a^2b - a(bc) + (ab)c + bc^2 - t(b\bar{a})c + at(\bar{c}b)\big)[13]$$
$$= \big(bc^2 - a^2b + [a,b,c] - t(b\bar{a})c + at(\bar{c}b)\big)[13].$$

Subtracting gives $E_{13}\big(G_8(x,y,z)\big) = 2[a,b,c][13] \neq 0$, so again $\mathcal{H}_3(\mathrm{D})$ satisfies G_8 iff D is associative, and the Albert algebra does not satisfy G_8. □

B.5.3 Nonvanishing of T_{11}

Now we turn to Thedy's Identity. For arbitrary $a, b, c \in \mathrm{O}$ we set

$$(1'') \quad x := e_1 - e_2, \quad y := 1[12] + a[23] + b[13], \quad z := 1[13] + c[12], \quad w = e_2.$$

We compute directly

$$(2'') \quad U_x = E_{11} - E_{12} + E_{22}, \qquad \{x,y\} = b[13] - a[23],$$
$$U_{\{x,y\}} = U_{b[13]} + U_{a[23]} - U_{b[13],a[23]},$$
$$U_{\{x,y\}}(w) = n(a)e_3,$$
$$U_{\{x,y\}}(z) = b^2[13] - ab[23] + t(\bar{b}ca)e_3,$$
$$(3'') \quad U_y = U_{1[12]} + U_{a[23]} + U_{b[13]} + U_{1[12],a[23]} + U_{1[12],b[13]} + U_{a[23],b[13]},$$
$$(4'') \quad U_y w = e_1 + n(a)e_3 + a[13],$$
$$U_y 1[13] = b^2[13] + t(a)e_2 + b[12] + ab[23],$$
$$U_y c[12] = \bar{c}[12] + ca[23] + \bar{c}b[13] + t(\bar{b}ca)e_3,$$
$$(5'') \quad U_{[x,y]}w = -4e_1 - n(a)e_3 - 2a[13],$$
$$U_{[x,y]}z = -\big(2t(a)\big)e_2 + 3t(\bar{b}ca)e_3 + \big(2b + 4\bar{c}\big)[12]$$
$$+\ \big(b^2 + 2\bar{c}b\big)[13] + \big(2ca - ab\big)[23],$$

where for these last two formulas we have used (2″), (4″) to compute

$$U_{[x,y]}e_2 = U_{\{x,y\}}e_2 - 2U_xU_ye_2 - 2U_yU_xe_2$$
$$= \big(n(a)e_3\big) - 2\big(e_1\big) - 2\big(e_1 + n(a)e_3 + a[13]\big)$$
$$= -n(a)e_3 - 4e_1 - 2a[13] \qquad \text{[for the first]}$$

$U_{[x,y]}z = U_{\{x,y\}}z - 2U_xU_yz - 2U_yU_xz$

$= U_{\{x,y\}}z - 2U_xU_y(1[13] + c[12]) + 2U_yc[12]$

$= U_{\{x,y\}}z - 2(E_{11} - E_{12} + E_{22})U_y1[13] + 2(E_{33} + E_{13} + E_{23} + 2E_{12})U_yc[12]$

$= \big(b^2[13] - ab[23] + t(\bar{b}ca)e_3\big) - 2\big(t(a)e_2 - b[12]\big)$

$\quad +2\big(t(\bar{b}ca)e_3 + \bar{c}b[13] + ca[23] + 2\bar{c}[12]\big)$

$= \big(b^2 + 2\bar{c}b\big)[13] + \big(- ab + 2ca\big)[23] + (1+2)t(\bar{b}ca)e_3$

$\quad +\big(- 2t(a)\big)e_2 + \big(+ 2b + 4\bar{c}\big)[12]$ [for the second].

Again we will examine only the 13-components of $T_{11}(x, y, z, w)$. Now in general we have $E_{13}\big(U_pe_2\big) = \{E_{12}(p), E_{23}(p)\}$, so for the left side of Thedy we get

$$(6'') \qquad E_{13}\big(U_{U_{[x,y]}(z)}w\big) = \big(4b(ca) + 8n(c)a - 2bab - 4\bar{c}(ab)\big)[13],$$

since by $(5'')$ $\{E_{12}(U_{[x,y]}(z)), E_{23}(U_{[x,y]}(z))\} = \{(2b + 4\bar{c})[12], (2ca - ab)[23]\} = \big(4b(ca) - 2b(ab) + 8\bar{c}(ca) - 4\bar{c}(ab)\big)[13]$, and we use the Kirmse Identity to simplify $\bar{c}(ca) = n(c)a$.

Attacking Thedy's right side, we start from $(5'')$ and compute

$$(7'') \qquad U_zU_{[x,y]}w = \big(U_{1[13]} + U_{c[12]} + U_{1[13],c[12]}\big)\big(- 4e_1 - n(a)e_3 - 2a[13]\big)$$

$$= \big(- 4e_3 - n(a)e_1 - 2\bar{a}[13]\big) + \big(- 4n(c)e_2\big) + \big(- 4c[32] - 2\bar{a}c[12]\big)$$

$$= - n(a)e_1 - 4n(c)e_2 - 4e_3 - 2\bar{a}c[12] - 2\bar{a}[13] - 4c[32].$$

$$(8'') \qquad E_{13}U_{[x,y]} = E_{13}\big(U_{\{x,y\}} - 2\{U_x, U_y\}\big)$$

$$= E_{13}\big(U_{b[13]} + U_{a[23]} - U_{b[13],a[23]}\big) - 2E_{13}\{(E_{11} - E_{12} + E_{22}), U_y\}$$

$$= E_{13}U_{b[13]}E_{13} + 0 - U_{b[13],a[23]}E_{23} - 0 - 2E_{13}U_y(E_{11} - E_{12} + E_{22})\big)$$

$$= E_{13}U_{b[13]}E_{13} - U_{b[13],a[23]}E_{23} - 2E_{13}\big(U_{1[12],a[23]}E_{22} - U_{1[12],b[13]}E_{12}\big)$$

$$= U_{b[13]}E_{13} - U_{b[13],a[23]}E_{23} - 2U_{1[12],a[23]}E_{22} + 2U_{1[12],b[13]}E_{12}.$$

using $(2'')$, $(3'')$. Applying this to $(7'')$ gives

$$(9'') \qquad E_{13}U_{[x,y]}\big(U_zU_{[x,y]}w\big) = U_{b[13]}(-2\bar{a})[13] - U_{b[13],a[23]}(-4c)[32]$$

$$\qquad - 2U_{1[12],a[23]}(-4n(c)e_2) + 2U_{1[12],b[13]}(-2\bar{a}c)[12]$$

$$= \big(- 2bab + 4(bc)a + 8n(c)a - 4(\bar{c}a)b\big)[13].$$

Subtracting $(9'')$ from $(6'')$ gives $E_{13}\big(T_{11}(x, y, z, w)\big) = \big(4b(ca) - 4\bar{c}(ab) - 4(bc)a + 4(\bar{c}a)b\big)[13] = \big(-4[b, c, a] + 4[\bar{c}, a, b]\big)[13] = \big(-4[a, b, c] + 4[a, b, \bar{c}]\big)[13] = -8[a, b, c][13]$. Once again, since O is nonassociative, we can find a, b, c for which this is nonzero, and therefore the Albert algebra does not satisfy T_{11}. $\square$

B.6 Problems for Appendix B

Problem B.1 Suppose that for each set nonempty set X we have a free gadget $\mathcal{FC}[X]$ in some category $\mathcal{C}$ of pointed objects (spaces with a distinguished element 0 or 1), together with a given map $\iota\colon X \to \mathcal{FC}[X]$ of sets, with the universal property that any set-theoretic mapping of X into a $\mathcal{C}$-object C extends uniquely (factors through ι) to a $\mathcal{C}$-morphism $\mathcal{FC}[X] \to C$. (1) Show that we have a functor from the category of sets to $\mathcal{C}$. (2) Show that if $X \hookrightarrow Y$ is an imbedding, then the induced $\mathcal{FC}[X] \to \mathcal{FC}[Y]$ is a monomorphism, allowing us to regard $\mathcal{FC}[X]$ as a subobject of $\mathcal{FC}[Y]$.

Problem B.2 (1) Verify that the free and free special functors are indeed functors from the category of sets to the category of (respectively, special) unital Jordan algebras. (2) Modify the constructions to provide (non-unital) free algebras $\mathcal{F}_0[X]$ (free Jordan $\mathcal{FJ}_0[X]$ or free special Jordan $\mathcal{FSJ}_0[X]$ respectively) with appropriate universal properties, and show that these provide functors from the category of sets to the category of (respectively special) non-unital Jordan algebras. (3) Show that the canonical homomorphism from the free non-unital gadget $\mathcal{F}_0[X]$ to the free unital gadget $\mathcal{F}[X]$ induced by the map $X \to \mathcal{F}[X]$ is a monomorphism, with $\mathcal{F}[X] = \Phi 1 \oplus \mathcal{F}_0[X] = \widehat{\mathcal{F}_0[X]}$ just the unital hull (cf. Question A.1). Conclude that $\mathcal{F}_0[X]$ is precisely the ideal of Jordan polynomials in $\mathcal{F}[X]$ with zero constant term.

Problem B.3 Show that a multiplication operator $M_{x_1,\dots,x_n}$ for $n < |X|$ vanishes on $\mathcal{FJ}[X]$ (respectively, on $\mathcal{FSJ}[X]$) iff it vanishes at a "generic element" $z \in X \setminus \{x_1, \dots, x_n\}$.

Problem B.4 (1) Show that instead of using a heavy dose of parentheses, the free monad $\mathcal{M}[X]$ (and hence the free linear algebra $\mathcal{F}[X]$) on a set X can be built recursively within the free associative monoid $\mathcal{FM}[X \cup \{p\}]$ (where the distinct element p will function as a product symbol): the monomial of degree 0 is the unit 1, the monomials of degree 1 are the $x \in X$, and the monomials of degree $n \geq 2$ are all $p\mathbf{pq}$ for monomials $\mathbf{p}$, $\mathbf{q}$ of degrees $i, j \geq 1$ with $i + j = n$. Rather surprisingly, the free associative gadget contains a recipe for the most highly nonassociative algebra imaginable. (2) Show that the free associative monoid $\mathcal{FM}[x, y]$ on two generators already contains inside it the free associative monoid $\mathcal{FM}[x_1, x_2, \dots]$ on countably many generators via $x_i \mapsto yx^iy$; extend this to show that $\mathcal{FA}[x_1, x_2, \dots] \hookrightarrow \mathcal{FA}[x, y]$.

C

Jordan Algebras of Degree 3

In II Chapter 4 we left the messy calculations to an appendix, and here we are. We will verify in detail that the Jordan identity holds for the Hermitian matrix algebras $\mathcal{H}_3(\mathrm{D}, -)$ for alternative D with nuclear involution, and for the algebras $\mathcal{J}ord(N, \#, c)$ of the general Cubic Construction, and that the Freudenthal and Tits norms are sharped.

C.1 Jordan Matrix Algebras

Here we will grind out the case of a 3×3 Jordan matrix algebra II.3.2.4 whose coordinate algebra is alternative with *nuclear* involution. Later we will consider the case of *scalar* involution in the Freudenthal Construction II.4.4.1, and show that it is a sharped cubic construction as in II.4.2.2.

Our argument requires a few additional facts about associators in alternative algebras and general matrix algebras.

Alternative Associator Facts Lemma C.1.1 *If* A *is an arbitrary linear algebra, then the algebra* A^+ *with brace product* $\{x, y\} := xy + yx$ *has associator*

(1.1.1) $$[x, y, z]^+ = ([x, y, z] - [z, y, x]) + ([y, x, z] - [z, x, y])$$
$$([x, z, y] - [y, z, x]) + [y, [x, z]].$$

Any alternative algebra D *satisfies the identity*

(1.1.2) $$[w, [x, y, z]] = [w, x, yz] + [w, y, zx] + [w, z, xy].$$

PROOF. (1) We compute

$$[x,y,z]^+ = (\underbrace{xy}_{1} + \underbrace{yx}_{2})z + z(\underbrace{xy}_{3} + \underbrace{yx}_{4}) - x(\underbrace{yz}_{1} + \underbrace{zy}_{5}) - (\underbrace{yz}_{6} + \underbrace{zy}_{4})x$$

$$= \overbrace{\underbrace{[x,y,z]}_{\alpha}}^{1} + \overbrace{\big(\underbrace{[y,x,z]}_{\beta} + \underbrace{y(xz)}_{\lambda}\big)}^{2} + \overbrace{\big(\underbrace{-[z,x,y]}_{\gamma} + \underbrace{(zx)y}_{\mu}\big)}^{3} - \overbrace{\underbrace{[z,y,x]}_{\delta}}^{4}$$

$$+ \overbrace{\big(\underbrace{[x,z,y]}_{\epsilon} \underbrace{-(xz)y}_{\mu}\big)}^{5} + \overbrace{\big(\underbrace{-[y,z,x]}_{\kappa} \underbrace{-y(zx)}_{\lambda}\big)}^{6}$$

$$= \overbrace{[x,y,z]}^{\alpha} + \overbrace{[y,x,z]}^{\beta} - \overbrace{[z,x,y]}^{\gamma} - \overbrace{[z,y,x]}^{\delta} + \overbrace{[x,z,y]}^{\epsilon} - \overbrace{[y,z,x]}^{\kappa}$$

$$+ \overbrace{y(xz - zx)}^{\lambda} - \overbrace{(xz - zx)y}^{\mu}.$$

(2) We compute $[w,[x,y,z]] - [w,x,yz] - [w,y,zx] - [w,z,xy] = w[x,y,z] + \big(-[x,y,z]w - [x,yz,w] + [xy,z,w]\big) - [w,y,zx]$ [by alternativity] $= w[y,z,x] + \big(x[y,z,w] - [x,y,zw]\big) - [w,y,zx]$ [by the Teichmüller Identity II.21.1.1(3)] $= 0$ [linearizing $x \mapsto x, w$ in the Left Bumping Formula II.21.1.1(2) $x[y,z,x] = [x,y,zx]$]. ☐

Matrix Associator Facts Lemma C.1.2 *In the algebra $\mathcal{M}_n(\mathrm{A})$ of $n \times n$ matrices over an arbitrary linear algebra* A*, the matrix associator for $A = (a_{ij}), B = (b_{k\ell}), C = (c_{pq})$, reduces to algebra associators for the entries $a_{ij}, b_{k\ell}, c_{pq} \in \mathrm{A}$:*

$$(1.2.0) \qquad [A,B,C] = \textstyle\sum_{r,s} \Big(\sum_{j,k} [a_{rj}, b_{jk}, c_{ks}]\Big) E_{rs}.$$

In the algebra $\mathcal{H}_3(\mathrm{D},-)$ of 3×3 hermitian matrices $X = \overline{X}^{tr}$ over an alternative algebra D *with nuclear involution, the diagonal rr-coordinates and off-diagonal rs-coordinates (r,s,t a cyclic permutation of $1,2,3$) of associators are given by:*

$$(1.2.1) \qquad [A,B,C]_{rr} = [a_{rs}, b_{st}, c_{tr}] + [c_{rs}, b_{st}, a_{tr}] = [C,B,A]_{rr};$$

$$(1.2.2) \qquad [A^2, A]_{rr} = [A,A,A]_{rr} = 2a \qquad (a := [a_{12}, a_{23}, a_{31}]);$$

$$(1.2.3) \qquad [A,B,C]_{rs} = -[a_{rs}, b_{rs}, c_{rs}] - [a_{rs}, b_{st}, c_{st}] - [a_{tr}, b_{tr}, c_{rs}];$$

$$(1.2.4) \qquad [A,A,A]_{rs} = 0.$$

PROOF. Writing $A = \sum_{ij} a_{ij}E_{ij}$, $B = \sum_{k\ell} b_{k\ell}E_{k\ell}$, $C = \sum_{pq} c_{pq}E_{pq}$ in terms of the standard (associative) matrix units E_{rs}, we calculate the associator

$$[A,B,C] = \sum_{i,j,k,\ell,p,q}[a_{ij}, b_{k\ell}, c_{pq}]E_{ij}E_{k\ell}E_{pq}$$
$$= \sum_{i,j,p,q}[a_{ij}, b_{jp}, c_{pq}]E_{iq} = \sum_{r,s}\big(\sum_{j,k}[a_{rj}, b_{jk}, c_{ks}]\big)E_{rs}.$$

establishing (1.2.0).

Now assume $n = 3$ and $\mathrm{A} = \mathrm{D}$. For hermitian matrices we have $a_{ji} = \overline{a_{ij}}$, and since by hypothesis the symmetric elements $\bar{a} = a$, in particular all traces $a + \bar{a} \in \mathcal{H}(\mathrm{D}, -)$, lie in the nucleus of D, they vanish from associators: any associator involving a_{ii} vanishes, and any a_{ji} in an associator may be replaced by $-a_{ij}$ because $[\ldots, \bar{a}, \ldots] = [\ldots, (a + \bar{a}) - a, \ldots] = -[\ldots, a, \ldots]$.

For the diagonal entry, we may (by symmetry in the indices) assume $r = 1$. Then by (1.2.0) $[A,B,C]_{11} = \sum_{j,k}[a_{1j}, b_{jk}, c_{k1}]$; any terms with $j = 1, j = k$, or $k = 1$ vanish because associators with diagonal terms vanish, so $1, j, k$ are distinct indices from among $1, 2, 3$, leading to only two possibilities: $j = 2, k = 3$ and $j = 3, k = 2$. Thus $[A,B,C]_{11} = [a_{12}, b_{23}, c_{31}] + [a_{13}, b_{32}, c_{21}] = [a_{12}, b_{23}, c_{31}] + [-a_{31}, -b_{23}, -c_{12}]$ [replacing a_{ji} by $-a_{ij}$ etc.] $= [a_{12}, b_{23}, c_{31}] - [a_{31}, b_{23}, c_{12}] = [a_{12}, b_{23}, c_{31}] + [c_{12}, b_{23}, a_{31}]$ [by alternativity] as claimed in (1.2.1), and this expression is clearly symmetric in A, C. In particular, when $A = B = C$ we get $2[a_{12}, a_{23}, a_{31}] = 2a$ as in (1.2.2) [note that a is invariant under cyclic permutation, so $a = [a_{rs}, a_{st}, a_{tr}]$ for any cyclic permutation r, s, t of $1, 2, 3$].

For the off-diagonal entry, we can again assume $r = 1, s = 2$ by symmetry. Then by (1.2.0) $[A,B,C]_{12} = \sum_{j,k}[a_{1j}, b_{jk}, c_{k2}]$, where again we can omit terms with $j = 1, j = k$, or $k = 2$. When $j = 2$ then $k \neq j, 2$ allows two possibilities $k = 1, 3$, but when $j = 3$ then $k \neq j, 2$ allows only one possibility $k = 1$. Thus the sum reduces to $[a_{12}, b_{21}, c_{12}] + [a_{12}, b_{23}, c_{32}] + [a_{13}, b_{31}, c_{12}] = [a_{12}, -b_{12}, c_{12}] + [a_{12}, b_{23}, -c_{23}] + [-a_{31}, b_{31}, c_{12}]$ (again replacing a_{ji} by $-a_{ij}$) $= -[a_{12}, b_{12}, c_{12}] - [a_{12}, b_{23}, c_{23}] - [a_{31}, b_{31}, c_{12}]$ as claimed in (1.2.3). In particular, when $A = B = C$ we get $-[a_{12}, a_{12}, a_{12}] - [a_{12}, a_{23}, a_{23}] - [a_{31}, a_{31}, a_{12}] = 0$ as in (1.2.4), since by alternativity each associator with a repeated term vanishes. □

Now we can finally establish the general theorem creating a functor from the category of alternative algebras with nuclear involutions to the category of Jordan algebras.

3×3 **Coordinate Theorem** C.1.3 *A matrix algebra $\mathcal{H}_3(\mathrm{D}, -)$ under the bullet product $A \bullet B = \frac{1}{2}\{A, B\}$ is a Jordan algebra iff the coordinate algebra* $(\mathrm{D}, -)$ *is alternative with nuclear involution.*

PROOF. We saw in the Jordan Coordinates Theorem II.14.1.1 that the condition on the coordinate algebra was *necessary*; the sticking point is showing that it is also *sufficient*. Since commutativity (JAX1) of the bullet product is obvious, it remains to prove the Jordan Identity (JAX2) for the brace associator: $[A, B, A^2]^+ = 0$ for all hermitian A, B. To prove that its diagonal entries $[A, B, A^2]_{rr}$ and off-diagonal entries $[A, B, A^2]_{rs}$ all vanish, it suffices by symmetry in the indices to prove these for $r = 1, s = 2$.

Using the formula for the brace associator in Alternative Associator Fact (1.1.1), for $C = A^2$ we have

$$(1.3.1)\quad [A, B, C]^+ = ([A, B, C] - [C, B, A]) + ([B, A, C] - [C, A, B]) + ([A, C, B] - [B, C, A]) + [B, [A, C]].$$

For the 11-component, the paired associator terms in (1.3.1) disappear by the symmetry in Matrix Associator Fact (1.2.1), and since by Matrix Associator Facts (1.2.2), (1.2.4) the matrix $[A, C] = [A, A^2] = -[A^2, A]$ is diagonal with associator entries,

$$(1.3.1)\qquad [A, C] = [A, A^2] = -2a\mathbb{1}_3, \qquad a = [a_{12}, b_{23}, c_{31}].$$

It follows that the 11-component is $[A, B, A^2]^+_{11} = [B, [A, C]]_{11} = [b_{11}, -2a] \in [\mathcal{N}uc(\mathrm{D}), [\mathrm{D}, \mathrm{D}, \mathrm{D}]] = \mathbf{0}$ by Nuclear Slipping II.21.2.1.

For the 12-component in (1.3.1), let us separate the positive and negative summands in the above. From Matrix Associator Fact (1.2.3),

$$\begin{aligned}
&[A, B, C]_{12} + [B, A, C]_{12} + [A, C, B]_{12} \\
&= -\big(\overbrace{[a_{12}, b_{12}, c_{12}]}^{1} + \overbrace{[a_{12}, b_{23}, c_{23}]}^{2} + \overbrace{[a_{31}, b_{31}, c_{12}]}^{3}\big) \\
&\quad -\big(\overbrace{[b_{12}, a_{12}, c_{12}]}^{1} + \overbrace{[b_{12}, a_{23}, c_{23}]}^{\alpha} + \overbrace{[b_{31}, a_{31}, c_{12}]}^{3}\big) \\
&\quad -\big(\overbrace{[a_{12}, c_{12}, b_{12}]}^{\beta} + \overbrace{[a_{12}, c_{23}, b_{23}]}^{2} + \overbrace{[a_{31}, c_{31}, b_{12}]}^{\gamma}\big) \\
&= -\overbrace{[b_{12}, a_{23}, c_{23}]}^{\alpha} - \overbrace{[b_{12}, a_{12}, c_{12}]}^{\beta} - \overbrace{[b_{12}, a_{31}, c_{31}]}^{\gamma} =: \quad -t_{12}
\end{aligned}$$

(by alternativity). This is clearly skew in A and C, so the negative terms sum to $+t_{12}$, so their difference is $-2t_{12}$. The remaining term is the commutator $[B, [A, C]]_{12} = [b_{12}, -2a]$ (recall (1.3.2)). So far we have $[A, B, A^2]^+_{12} = -2([b_{12}, a] + t_{12})$ reducing to $-2\big([b_{12}, [a_{12}, a_{23}, a_{31}]] + [b_{12}, a_{23}, c_{23}] + [b_{12}, a_{12}, c_{12}] + [b_{12}, a_{31}, c_{31}]\big)$. Now we have to look more closely at the entries c_{ij} of $C = A^2$. We have $c_{ij} = a_{ii}a_{ij} + a_{ij}a_{jj} + a_{ik}a_{kj} = a_{ii}a_{ij} + a_{ij}a_{jj} + \overline{a_{jk}a_{ki}}$. Since nuclear a_{ii} slip out of associators by Nuclear Slipping, and associators with repeated terms a_{ij}, a_{ij} vanish by alternativity, only the $\overline{a_{jk}a_{ki}}$ terms survive; replacing $\bar{x}$ in the associators by $-x$ shows that $[A, B, A^2]^+_{12}$ reduces to $-2\big([b_{12}, [a_{12}, a_{23}, a_{31}]] - [b_{12}, a_{23}, a_{31}a_{12}] - [b_{12}, a_{12}, a_{23}a_{31}] - [b_{12}, a_{31}, a_{12}a_{23}]\big)$. But this vanishes by Alternative Associator Fact (1.1.2). Thus all entries of the brace associator $[A, B, A^2]^+$ vanish, and the Jordan identity (JAX2) holds. □

This establishes the functor $(\mathrm{D},-) \longrightarrow \mathcal{H}_3(\mathrm{D},-)$ from the category of unital alternative algebras with nuclear involution to the category of unital Jordan algebras mentioned after Hermitian Matrix Theorem II.3.2.4. Now we turn to the Jordan matrix algebras coordinatized by alternative algebras with scalar involution, where the Jordan algebra is a cubic factor determined by a cubic norm form.

C.2 The General Construction

We begin by verifying that the general construction $\mathcal{J}ord(N,\#,c)$ creates unital degree–3 Jordan algebras from any *sharped cubic form* $(N,\#,c)$ on a module over a general ring of scalars. Let us recall bygone concepts and results that we have already defined or established in Chapter 4 of Part II for the general setting, independent of nondegeneracy.

Bygone Definitions C.2.1 (1) *A* basepoint $N(c) = 1$ *for a cubic form determines* trace linear *and* bilinear forms *and* spur quadratic form:

$$\begin{aligned} T(x) &:= N(c;x), \quad S(x) := N(x;c) && \text{(Trace, Spur Definition)},\\ T(x,y) &:= T(x)T(y) - N(c,x,y) && \text{(Bilinear Trace Definition)},\\ S(x,y) &:= N(x,y,c) = T(x)T(y) - T(x,y) && \text{(Spur–Trace Formulas)},\\ T(x) &= T(x,c) && \text{(c–Trace Formula)},\\ T(c) &= S(c) = 3 && \text{(Unit Values)}. \end{aligned}$$

(2) *A* sharp mapping *for a cubic form N is a quadratic map on X strictly satisfying the following*:

$$\begin{aligned} T(x^{\#},y) &= N(x;y) && \text{(Trace–Sharp Formula)},\\ x^{\#\#} &= N(x)x && \text{(Adjoint Identity)},\\ c\#y &= T(y)c - y && \text{(c–Sharp Identity)}. \end{aligned}$$

A sharped cubic form $(N,\#,c)$ *consists of a cubic form N with basepoint c together with a choice of sharp mapping $\#$.*

Cubic Consequences Proposition C.2.2 *We have the following Cubic Consequences.*

(1) *The c–Sharp Identity, Unit Values, and $\frac{1}{2} \in \Phi$ imply*

$$c^{\#} = c,$$

while the Trace–Sharp Formula implies:

$$\begin{aligned} S(x) = T(x^{\#}),\ S(x,y) &= T(x\#y) && \text{(Spur Formulas)},\\ T(x\#z,y) = T(x,z\#y) &= N(x,y,z) && \text{(Sharp Symmetry)},\\ T(x\bullet y,z) &= T(y,x\bullet z) && \text{(Bullet Symmetry)},\\ T(U_xy,z) &= T(y,U_xz) && \text{(U Symmetry)}. \end{aligned}$$

(2) *The condition that the Adjoint Identity holds strictly is equivalent (assuming the Trace–Sharp Formula) to the following linearizations*:

$$x^\#\#(x\#y) = N(x)y + T(x^\#, y)x \qquad \text{(Adjoint}' \text{ Identity)},$$
$$(x\#y)^\# + x^\#\#y^\# = T(x^\#, y)y + T(y^\#, x)x \qquad \text{(Adjoint}'' \text{ Identity)}.$$

PROOF. The first equality in (1) follows from $2c^\# = c\#c = T(c)c - c = 3c - c = 2c$ by Bygones C.2.1 c-Sharp Identity and Unit Values. Sharp Symmetry, Spur Formulas, and Bullet Symmetry were all established in the Little Reassuring Argument of II.4.3.3. We cannot deduce U Symmetry from Bullet Symmetry, since we do not yet know that the U–operator $T(x,y)x - x^\#\#y$ coincides with the usual $2x \bullet (x \bullet y) - x^2 \bullet y$. We argue instead that $T(U_x y, z) = T(x,y)T(x,z) - T(x^\#\#y, z) = T(x,y)T(x,z) - T(x^\#, y\#z)$ [by Sharp Symmetry] is symmetric in y and z.

In (2), the Adjoint$'$ and Adjoint$''$ Identities are just linearizations of the Adjoint Identity, replacing $x \mapsto x + ty$ and equating coefficients of t and t^2 respectively. □

Now we go through the General Cubic Construction Theorem II.4.2.2 in full detail. Since we have been introduced by name twice to these identities and formulas, we will henceforth call them by their first name and refrain from using their last names C.2.1(1)–(2) and C.2.2(1)–(2).

Cubic Construction Theorem C.2.3 *Any sharped cubic form $(N, \#, c)$ gives rise to a unital Jordan algebra $\mathcal{J}ord(N, \#, c)$ with unit element c and U–operator*

$$\text{(2.3.1)} \qquad U_x y := T(x,y)x - x^\#\#y.$$

The square and bilinear product are defined by

$$\text{(2.3.2)} \qquad x^2 := U_x c, \quad \{x,y\} := U_{x,y}c, \quad x \bullet y := \tfrac{1}{2}U_{x,y}c.$$

The sharp map and sharp product are related to the square and bilinear product by the Sharp Expressions

$$\text{(2.3.3)} \quad x^\# = x^2 - T(x)x + S(x)c, \quad x\#y = \{x,y\} - T(x)y - T(y)x + S(x,y)c,$$

and all elements satisfy the Degree–3 Identity

$$\text{(2.3.4)} \quad x^3 - T(x)x^2 + S(x)x - N(x)c = 0 \quad (x^3 := U_x x = x \bullet x^2),$$

$$\text{(2.3.5)} \qquad \textit{equivalently,} \qquad x \bullet x^\# = N(x)c.$$

PROOF. Note that here we start with U as the primary operation, and derive the bullet from it. The formulas (2.3.1), (2.3.2) hold by definition. Since $x^2 := U_x c = T(x,c)x - x^\#\#c = T(x)x - \big(T(x^\#)c - x^\#\big)$ [by c–Trace Formula and c–Sharp Identity] $= T(x)x - S(x)c + x^\#$ [by Spur Formulas] we get the first Sharp Expression in (2.3.3); the second results from linearization, and the bullet by dividing by 2.

Before we can show that the two versions of the Degree–3 Identity hold, and that the two definitions of the cube coincide, we need a few more formulas. The c–Trace and c–Sharp Identities and the Cubic Consequences (1) are precisely the conditions that c be the unit of the Jordan algebra $\mathcal{J}ord(N,\#,c)$: for U-unitality $U_c y := T(c,y)c - c^{\#}\#y = T(y)c - c\#y = y$, and similarly for brace unitality $\{c,y\} := U_{c,y}c = T(y,c)c + T(c,c)y - (c\#y)\#c = T(y)c + 3y - \big(T(y)c - y\big)\#c$ [by c–Trace, c–Sharp] $= T(y)c + 3y - 2T(y)c^{\#} + y\#c = 2y$; for bullet unitality, simply scale by $\frac{1}{2}$. Thus we have

$$(2.3.6)\qquad U_c y = y, \quad \{c,y\} = 2y, \quad c \bullet y = y \qquad \text{(Unitality)}.$$

Setting $z = c$ in Bullet Symmetry and using c–Trace and Unitality (2.3.6), we can recover the bilinear from the linear trace:

$$(2.3.7)\qquad T(x,y) = T(x \bullet y) \qquad \text{(Bullet Trace)}.$$

The Degree–3 Identity, the Jordan axiom, and the composition rules in the next theorem will require a series of $(x^{\#},x)$**-Identities** for the various products of $x^{\#}$ and x.

$$(2.3.8)\qquad x\#x^{\#} = [S(x)T(x) - N(x)]c - T(x)x^{\#} - T(x^{\#})x;$$
$$(2.3.9)\qquad S(x,x^{\#}) = S(x)T(x) - 3N(x);$$
$$(2.3.10)\qquad x \bullet x^{\#} = N(x)c, \qquad T(x,x^{\#}) = 3N(x);$$
$$(2.3.11)\qquad U_x x^{\#} = N(x)x, \qquad U_x U_{x^{\#}} = N(x)^2 \mathbb{1}_J;$$
$$(2.3.12)\qquad \{x,x^{\#},y\} = 2N(x)y.$$

(Recall that in the nondegenerate case II.4.3.2 we carefully avoided (2.3.9) by moving to the other side of an inner product!) For (2.3.8), using c–Sharp twice we have $x^{\#}\#x = x^{\#}\#[T(x)c - x\#c] = T(x)[T(x^{\#})c - x^{\#}] - x^{\#}\#(x\#c) = T(x)S(x)c - T(x)x^{\#} - [N(x)c + T(x^{\#},c)x]$ [using Spur Formula and Adjoint′] $= T(x)S(x)c - T(x)x^{\#} - N(x)c - T(x^{\#})x$ [by c–Trace]. (2.3.9) follows by taking the trace of (2.3.8) (in view of the Spur Formula). For (2.3.10), we cancel 2 from $2x^{\#} \bullet x = \{x^{\#},x\} = x^{\#}\#x + T(x)x^{\#} + T(x^{\#})x - S(x,x^{\#})c$ [by Sharp Expressions] $= [S(x)T(x) - N(x)]c - [S(x)T(x) - 3N(x)]c$ [by (2.3.8),(2.3.9)] $= 2N(x)c$ for the first part; the second part follows by taking traces [using Bullet Trace (2.3.7)], or directly from Trace–Sharp and Euler. For the first part of (2.3.11), $U_x x^{\#} = T(x,x^{\#})x - x^{\#}\#x^{\#} = 3N(x)x - 2(x^{\#})^{\#}$ [by the second part of (2.3.10)] $= 3N(x)x - 2N(x)x = N(x)x$ by the Adjoint Identity. For the second part of (2.3.11), $U_x U_{x^{\#}} y = U_x\big(T(x^{\#},y)x^{\#} - x^{\#\#}\#y\big) = N(x)\big(T(x^{\#},y)x - U_x(x\#y)\big)$ [by the first part of (2.3.11) and the Adjoint Identity] $= N(x)\big(T(x^{\#},y)x - T(x,x\#y)x + x^{\#}\#(x\#y)\big) = N(x)\big(T(x^{\#},y)x - T(x\#x,y)x + [N(x)y + T(x^{\#},y)x]\big)$ [by Sharp Symmetry and Adjoint′] $= N(x)N(x)y$. (2.3.12) follows from $\{x,x^{\#},y\} = T(x,x^{\#})y + T(x^{\#},y)x -$

$x^\#\#(x\#y) = [3N(x) - N(x)]y$ [by the last part of (2.3.10) and Adjoint′] $= 2N(x)y$.

This completes our verification of (2.3.8)–(2.3.12). Now we apply them to establish (2.3.4)–(2.3.5) and the Jordan identity. (2.3.10) includes (2.3.5), which by the Sharp Expressions yields the Degree–3 Identity (2.3.4) in terms of $x^3 := x \bullet x^2$. To show that (2.3.4) also holds for $x^3 = U_x x$, by the definition of U and the Sharp Expressions we have

$$\begin{aligned} &U_x x - T(x)x^2 + S(x)x - N(x)c \\ &= \big(\underbrace{T(x,x)x}_{1} - \underbrace{x^\#\#x}_{2}\big) - T(x)\big(\underbrace{x^\#}_{3} + \underbrace{T(x)x}_{4} - \underbrace{S(x)c}_{5}\big) + \underbrace{S(x)x}_{6} - \underbrace{N(x)c}_{7} \\ &= \big(\overbrace{T(x,x)}^{1} - \overbrace{T(x)T(x)}^{4} + \overbrace{S(x,x)}^{6}\big)x \\ &\quad + \big(-\overbrace{x^\#\#x}^{2} + [\overbrace{S(x)T(x)}^{5} - \overbrace{N(x)}^{7}]c - \overbrace{T(x)x^\#}^{3} - \overbrace{S(x)x}^{6}\big) \\ &= 0 + 0 = 0 \end{aligned}$$

by Spur–Trace and (2.38). Thus $\mathcal{J}ord(N, \#, c)$ is a unital degree 3 algebra, and the two definitions of cube agree.

The key to *Jordanity* is the identity

$$\{x, x, y\} = \{x^2, y\}, \tag{2.3.13}$$

which metamorphoses out of the linearization of (2.3.8) using Trace–Sharp as follows:

$$\begin{aligned} 0 &= \big(\underbrace{x\#(x\#y)}_{1} + \underbrace{y\#x^\#}_{2}\big) - \big(\underbrace{S(x)T(y)}_{3} + \underbrace{S(x,y)T(x)}_{4} - \underbrace{T(x^\#, y)}_{5}\big)c \\ &\quad + \big(\underbrace{T(y)x^\#}_{6} + \underbrace{T(x)x\#y}_{7}\big) + \big(\underbrace{S(x,y)x}_{8} + \underbrace{S(x)y}_{3}\big) \\ &= + \underbrace{\overbrace{x\#(x\#y)}^{1}}_{\alpha} + \underbrace{\big(\overbrace{x^\#}^{2} + \overbrace{T(x)x}^{7} - \overbrace{S(x)c}^{3}\big)}_{\beta}\#y - \underbrace{\overbrace{S(T(x)x, y)c}^{4}}_{\gamma} \\ &\quad + \overbrace{\big(\underbrace{T(x^\#)T(y)}_{\delta} - \underbrace{S(x^\#, y)}_{\epsilon}\big)}^{5}c + \underbrace{\overbrace{T(y)x^\#}^{6}}_{\eta} + \overbrace{\big(\underbrace{T(x)T(y)}_{\kappa} - \underbrace{T(x,y)}_{\lambda}\big)}^{8}x \\ &\quad + \overbrace{\big(\underbrace{T(x^2)}_{\mu} - \underbrace{T(x,x)}_{\nu}\big)}^{00}y \end{aligned}$$

[using the Spur–Trace Formula twice, once in each direction, and Bullet Trace]

$$= \overbrace{\underbrace{x^2\#y}_{a}}^{\beta} + T(y)\big(\underbrace{\overbrace{x^\#}^{\eta} + \overbrace{T(x)x}^{\kappa} - \overbrace{S(x)c}^{\delta}}_{b}\big) + \overbrace{\underbrace{T(x^2)y}_{c}}^{\mu}$$

$$- S(\underbrace{\overbrace{x^\#}^{\epsilon} + \overbrace{T(x)x}^{\gamma} - \overbrace{S(x)c}^{\delta}, y}_{d})c + \big(\underbrace{\overbrace{x\#(x\#y)}^{\alpha} - \overbrace{T(x,y)x}^{\lambda} - \overbrace{T(x,x)y}^{\nu}}_{e}\big)$$

[using Sharp Expressions for β, for δ using Spur Formula and $S(c,y) = T(c)T(y) - T(c,y) = 2T(y)$ by Spur–Trace and c–Trace], which then becomes

$$= \big(\overbrace{x^2\#y}^{a} + \overbrace{T(y)x^2}^{b} + \overbrace{T(x^2)y}^{c} - \overbrace{S(x^2,y)\,c}^{d}\big) + \big(\overbrace{(x\#y)\#x - T(y,x)x - T(x,x)y}^{e}\big)$$
$$= \big(\{x^2,y\}\big) + \big(-U_{x,y}x\big) \;=\; \{x^2,y\} - \{x,x,y\}$$

using the Sharp Expressions for (a,b,c,d) and the definition of U for (e).

This identity (2.3.13) has some immediate consequences relating the U–operator and triple product to the bilinear products:

(2.3.14) $\{\{x,z\},y\} = \{x,z,y\} + \{z,x,y\}$;

(2.3.15) $2U_x = V_x^2 - V_{x^2}, \quad U_x = 2L_x^2 - L_{x^2}$;

(2.3.16) $\{x^\#,x,y\} = 2N(x)y$;

(2.3.17) $x\#(x^\#\#y) = N(x)y + T(x,y)x^\#$ (Dual Adjoint′).

(2.3.14) is simply the linearization $x \mapsto x,z$ of (2.3.12), and (2.3.15) follows from this by interpreting $\{x,y,x\} = -\{y,x,x\} + \{\{y,x\},x\}$ as an operator on y; note that this means that the U defined from the sharp is the usual Jordan U–operator II.1.8.1(2). (2.3.16) follows via (2.3.14): $\{x^\#,x,y\} = \{\{x,x^\#\},y\} - \{x,x^\#,y\} = \{2N(x)c,y\} - 2N(x)y$ [by (2.3.10) and (2.3.12)] $= 4N(x)y - 2N(x)y = 2N(x)y$ by unitality (2.3.6). For (2.3.17), $(x^\#\#y)\#x = -\{x^\#,x,y\} + T(x^\#,x)y + T(y,x)x^\#$ [by definition of U] $= -2N(x)y + 3N(x)y + T(x,y)x^\#$ [by (2.3.16), (2.3.10)] $= N(x)y + T(x,y)x^\#$.

Now we are over the top, and the rest is downhill. To establish the Jordan identity (JAX2) that V_x and V_{x^2} commute, it suffices to prove that V_x and $V_{x^\#}$ commute, because V_{x^2} is a linear combination of $V_{x^\#}, V_x, V_c = 2\mathbb{1}_J$ by the Sharp Expressions and V–unitality (2.3.6). But using linearized (2.3.13) twice we see that

$$\begin{aligned}[V_x, V_{x^\#}]y &= \{x,\{x^\#,y\}\} - \{x^\#,\{x,y\}\}\\ &= \big(\{x,x^\#,y\} + \{x,y,x^\#\}\big) - \big(\{x^\#,x,y\} + \{x^\#,y,x\}\big)\\ &= \{x,x^\#,y\} - \{x^\#,x,y\} = 0\end{aligned}$$

by (2.3.12), (2.3.16). Thus we have a unital degree 3 Jordan algebra $\mathcal{J}ord(N,\#,c)$ for any sharped cubic form. □

The reader should pause and breathe a sigh of relief, then take another deep breath, because there is more tough slogging to establish the composition formula in this general situation.

Cubic Composition Theorem C.2.4 *If N is a sharped cubic form, its sharp mapping is always multiplicative,*

$$(1) \qquad \big(U_x y\big)^{\#} = U_{x^\#} y^{\#}.$$

If Φ is a faithful ring of scalars (e.g., if Φ has no 3–torsion or no nilpotents), then N permits composition with the U–operator and the sharp:

$$(2) \qquad N(1) = 1, \qquad N(U_x y) = N(x)^2 N(y), \qquad N(x^\#) = N(x)^2.$$

PROOF. The multiplicative property of the sharp follows by direct calculation, using the various Adjoint Identities: by the definition C.2.2(1) of U,

$$\begin{aligned}
\big(U_x y\big)^{\#} - U_{x^\#} y^{\#} &= \big[T(x,y)x - x^\# \# y\big]^{\#} - \big[T(x^\#, y^\#)x^\# - x^{\#\#} \# y^\#\big] \\
&= \big[T(x,y)^2 x^\# - T(x,y)x\#(x^\#\#y) + (x^\#\#y)^\#\big] - T(x^\#, y^\#)x^\# + x^{\#\#}\#y^\# \\
&= \overbrace{T(x,y)^2 x^\#}^{1} - T(x,y)\big[\overbrace{N(x)y}^{2} + \overbrace{T(x,y)x^\#}^{1}\big] \\
&\quad + \big[-\overbrace{x^{\#\#}\#y^\#}^{3} + \overbrace{T(x^{\#\#},y)y}^{2} + \overbrace{T(y^\#,x^\#)x^\#}^{4}\big] - \overbrace{T(x^\#,y^\#)x^\#}^{4} + \overbrace{x^{\#\#}\#y^\#}^{3} \\
&= 0
\end{aligned}$$

[by the Dual Adjoint′, Adjoint″, and the Adjoint Identities].

When Φ has no 3–torsion, all is smooth as silk: the norm permits composition with sharp and U, since [using (2.3.10) twice] $3N(x^\#) = T([x^\#]^\#, x^\#) = N(x)T(x, x^\#) = 3N(x)^2$ and $3N(U_x y) = T((U_x y)^\#, U_x y) = T(U_{x^\#} y^\#, U_x y)$ [by the above multiplicativity of sharp] $= T(U_x U_{x^\#} y^\#, y)$ [by U Symmetry] $= N(x)^2 T(y^\#, y)$ [by the second part of (2.3.11)] $= 3N(x)^2 N(y)$.

It is not true in general that $N(x^\#) = N(x)^2$, only that the difference is a scalar which kills c, hence $\alpha \mathrm{J} = (\alpha c) \bullet \mathrm{J} = 0$, so this scalar will be condemned to vanish by our faithful scalar hypothesis. To witness the murder:

$$\begin{aligned}
0 &= \big(x^{\#\#} - N(x)x\big)\#x^\# && \text{[by the Adjoint Identity]} \\
&= \big([S(x^\#)T(x^\#) - N(x^\#)]c - T(x^\#)x^{\#\#} - T(x^{\#\#})x^\#\big) && \text{[applying (2.3.8)} \\
&\quad - N(x)\big([S(x)T(x) - N(x)]c - T(x)x^\# - S(x)x\big) && \text{to } x \text{ and to } x^\#] \\
&= \big([N(x)T(x)S(x) - N(x^\#)]c - S(x)N(x)x - N(x)T(x)x^\#\big) \\
&\quad - N(x)\big([S(x)T(x) - N(x)]c - T(x)x^\# - S(x)x\big)
\end{aligned}$$

[using the Adjoint Identity, noting that $S(x^\#) = T(x^{\#\#}) = N(x)T(x)$ and $T(x^\#) = S(x)$ by the Spur Formula]

$$= \big(N(x)^2 - N(x^\#)\big)c.$$

Since this holds strictly, it shows that $\alpha(x) = N(x)^2 - N(x^\#)$ and all its linearizations kill c. In general, whenever $\alpha c = 0$ we have $3\alpha = \alpha^3 = 0$ (taking traces and norms; note this implies Φ is automatically faithful if it has no 3–torsion or nilpotents) and $\alpha \mathrm{J} = \mathbf{0}$, so in the faithful case $\alpha = 0$ and we have strict sharp composition.

We claim that whenever sharp composition holds strictly, it implies U composition. To see this, first equate coefficients of t^3 in $N([x+ty]^\#) = N(x+ty)^2$. On the left side $N(x^\# + tx\#y + t^2y^\#)$ produces $N(x\#y) + N(x^\#, x\#y, y^\#)$ (the only combinations of three powers $1, t, t^2$ having total degree 3 are t, t, t and $1, t, t^2$) $= N(x\#y) + T(x^\#\#(x\#y), y^\#)$ [by linearized Trace–Sharp] $= N(x\#y) + T([N(x)y + T(x^\#, y)x], y^\#)$ [by Adjoint′ and Trace–Sharp], which reduces by (2.3.10) to

$$\text{(3L)} \qquad N(x\#y) + 3N(x)N(y) + T(x^\#, y)T(y^\#, x).$$

On the right side, $\big(N(x) + tN(x; y) + t^2N(y; x) + t^3N(y)\big)^2$ produces the term $2N(x; y)N(y; x) + 2N(x)N(y)$ (the only combinations of two powers of $1, t, t^2, t^3$ having total degree 3 are t, t^2 and $1, t^3$), which reduces by Trace–Sharp to

$$\text{(3R)} \qquad 2T(x^\#, y)T(y^\#, x) + 2N(x)N(y).$$

Subtracting these two sides gives

$$\text{(3)} \qquad N(x\#y) = T(x^\#, y)T(y^\#, x) - N(x)N(y).$$

From this we can derive U composition directly (using Trace–Sharp repeatedly, and setting $n = N(x)$, $t = T(x, y)$ for convenience):

$$\begin{aligned}
N(U_xy) &= N(tx - x^\#\#y) \\
&= t^3N(x) - t^2N(x; x^\#\#y) + tN(x^\#\#y; x) - N(x^\#\#y) \\
&= t^3N(x) - t^2T(x^\#, x^\#\#y) + tT([x^\#\#y]^\#, x) - N(x^\#\#y) \\
&= t^3N(x) - t^2T(x^\#\#x^\#, y) - [T(x^{\#\#}, y)T(y^\#, x^\#) - N(x^\#)N(y)] \\
&\qquad + tT([-x^{\#\#}\#y^\# + T(x^{\#\#}, y)y + T(y^\#, x^\#)x^\#], x) \\
&\qquad\qquad \text{[using Adjoint}'' \text{ and (3) with } x \text{ replaced by } x^\#] \\
&= t^3n - t^2T(2nx, y) - [nT(x, y)T(y^\#, x^\#) - n^2N(y)] \\
&\qquad + t[-nT(x\#x, y^\#) + nT(x, y)T(y, x) + T(y^\#, x^\#)T(x^\#, x)] \\
&\qquad\qquad \text{[using Adjoint, Sharp Symmetry, and sharp composition],} \\
&= t^3n - 2nt^3 + [n^2N(y) - ntT(x^\#, y^\#)] \qquad \text{[by (2.3.10)]} \\
&\qquad - 2ntT(x^\#, y^\#) + nt^3 + 3ntT(y^\#, x^\#) \\
&= n^2N(y) = N(x)^2N(y).
\end{aligned}$$

□

This finishes the involved calculations required to establish the general cubic construction. We now turn to the Freudenthal and Tits Constructions, and show that they produce Jordan algebras because they are special cases of the general construction,

C.3 The Freudenthal Construction

Our first, and most important, example of a sharped cubic form comes from the algebra of 3×3 hermitian matrices over an alternative algebra with central involution. Replacing the original scalars by the $*$–center, we may assume that the original involution is actually a *scalar* involution. For the octonions with standard involution this produces the reduced Albert algebras. We already know that the resulting matrix algebra will be Jordan by the 3×3 Coordinate Theorem C.1.3, but in this section we will give an independent proof showing that it arises as a $\mathcal{J}ord(N, \#, c)$: the scalar-valued norm $n(a)$ on the coordinates D allows us to create a Jordan norm $N(x)$ on the matrix algebra.

Recall the results of the Central Involution Theorem II.21.2.2 and Moufang Lemma II.21.1.1(2): If D is a unital alternative algebra over Φ with involution $a \mapsto \bar{a}$ such that there are quadratic and linear norm and trace forms $n, t : \mathrm{D} \to \Phi$ with $n(a)1 = a\bar{a},\ t(a)1 = a + \bar{a} \in \Phi 1$, then we have

(D1) $\quad t(\bar{a}) = t(a), \quad n(\bar{a}) = n(a) \quad$ (Bar Invariance),

(D2) $\quad t(ab) = n(\bar{a}, b) = t(ba) \quad$ (Trace Commutativity),

(D3) $\quad t((ab)c) = t(a(bc)) \quad$ (Trace Associativity),

(D4) $\quad n(ab) = n(a)n(b) \quad$ (Norm Composition),

(D5) $\quad \bar{a}(ab) = n(a)b = (ba)\bar{a} \quad$ (Kirmse Identity),

(D6) $\quad a(bc)a = (ab)(ca) \quad$ (Middle Moufang).

We will establish the Freudenthal construction for the untwisted case $\mathcal{H}_3(\mathrm{D}, -)$; there is no point, other than unalloyed masochism, in proving the twisted case $\mathcal{H}_3(\mathrm{D}, \Gamma)$, since this arises as the Γ–isotope of the untwisted form, and we know in general (by Cubic Factor Isotopes II.7.4.1) that isotopes of cubic factors are again cubic. Thus we can avoid being distracted by swarms of gammas. We will only establish that the Freudenthal construction furnishes a sharp norm form, hence by the Cubic Construction Theorem C.2.2 a Jordan algebra. In the Freudenthal Construction Theorem II.4.4.1 we verified that the Jordan structure determined by the sharp norm form coincides with that of the matrix algebra $\mathcal{H}_3(\mathrm{D}, -)$, hence this too is a Jordan algebra.

Freudenthal Construction C.3.1 *Let* D *be a unital alternative algebra over* Φ *with scalar involution. Then the hermitian matrix algebra* $\mathcal{H}_3(\mathrm{D},-)$ *is a cubic factor* $\mathcal{J}ord(N,\#,c)$ *whose Jordan structure is determined by the basepoint* c*, cubic form* N*, trace* T*, and sharp* $\#$ *defined as follows for elements* $x=\sum_{i=1}^3\alpha_ie_i+\sum_{i=1}^3 a_i[jk]$, $y=\sum_{i=1}^3\beta_ie_i+\sum_{i=1}^3 b_i[jk]$ *with* $\alpha_i,\beta_i\in\Phi$, $a_i,b_i\in\mathrm{D}$ *in Jacobson box notation* ($e_i:=E_{ii}$, $d[jk]:=dE_{jk}+\bar{d}E_{kj}$, *where* (ijk) *is always a cyclic permutation of* (123))*:*

$$\begin{aligned}
c &:= 1 = e_1+e_2+e_3,\\
N(x) &:= \alpha_1\alpha_2\alpha_3-\sum_k \alpha_k n(a_k)+t(a_1a_2a_3),\\
T(x) &= \sum_k \alpha_k, \quad T(x,y)=\sum_k\alpha_k\beta_k+\sum_k t(\overline{a_k}b_k),\\
x^{\#} &:= \sum_k\big(\alpha_i\alpha_j-n(a_k)\big)e_k+\sum_k\big(\overline{a_ia_j}-\alpha_ka_k\big)[ij],\\
x\#y &:= \sum_k\big(\alpha_i\beta_j+\beta_i\alpha_j-n(a_k,b_k)\big)e_k\\
&\quad+\sum_k\big(\overline{(a_ib_j+b_ia_j)}-\alpha_kb_k-\beta_ka_k\big)[ij].
\end{aligned}$$

PROOF. The first two assertions are definitions; note that c is a basepoint, since $N(c)=1$. We verify the conditions C.2.1(2) for a sharped cubic form. We begin by identifying the trace linear and bilinear forms:

$$\begin{aligned}
N(x;y) &= \alpha_1\alpha_2\beta_3+\alpha_1\beta_2\alpha_3+\beta_1\alpha_2\alpha_3-\textstyle\sum_k\alpha_kn(a_k,b_k)-\sum_k\beta_kn(a_k)\\
&\quad+t(a_1a_2b_3+a_1b_2a_3+b_1a_2a_3)\\
&= \textstyle\sum_k[\alpha_i\alpha_j-n(a_k)]\beta_k+\sum_k t([a_ia_j-\alpha_k\overline{a_k}]b_k)
\end{aligned}$$

by trace commutativity and associativity (D2), (D3). In this relation, if we set $x=c$, $\alpha_k=1$, $a_k=0$ we get $N(c;y)=\sum_k[1\,1\,\beta_k-0]+\sum_k 0$, so

$$T(y)=\textstyle\sum_k\beta_k.$$

Thus the linear trace form is what we claimed it is. For the trace bilinear form, if instead we linearize $x\mapsto x,c$, we get $N(x,c,y)=\sum_k(\alpha_i+\alpha_j-0)\beta_k+\sum_k t([0-1\,\overline{a_k}]b_k)=\sum_k(\alpha_i+\alpha_j+\alpha_k)\beta_k-\sum_k\alpha_k\beta_k-\sum_k t(\overline{a_k}b_k)=T(x)\sum_k\beta_k-\sum_k\alpha_k\beta_k-\sum_k t(\overline{a_k}b_k)=T(x)T(y)-\sum_k\alpha_k\beta_k-\sum_k t(\overline{a_k}b_k)$. Thus

$$T(x,y):=T(x)T(y)-N(x,y,c)=\textstyle\sum_k\alpha_k\beta_k+\sum_k t(\overline{a_k}b_k)$$

as claimed.

Comparing this formula for $T(x, y)$ with the above formula for $N(x; y)$ shows that the *Trace–Sharp Formula* $N(x; y) = T(x^\#, y)$ holds for the adjoint $x^\# := \sum_k \big(\alpha_i\alpha_j - n(a_k)\big)e_k + \sum_k \big(\overline{a_i a_j} - \alpha_k a_k\big)[ij]$.

To verify the *Adjoint Identity* $x^{\#\#} = N(x)x$, let us write $x^\# = \sum_{i=1}^3 \beta_i e_i + \sum_{i=1}^3 b_i[jk]$ for convenience. Then by definition of sharp we have $x^{\#\#} = \sum \delta_i e_i + \sum d_i[jk]$, where the diagonal entries are given by

$$\begin{aligned}
\delta_k &= \beta_i\beta_j - n(b_k) = \big[\alpha_j\alpha_k - n(a_i)\big]\big[\alpha_k\alpha_i - n(a_j)\big] - n\big(\overline{a_i a_j} - \alpha_k a_k\big)\\
&= \big[\alpha_j\alpha_k - n(a_i)\big]\big[\alpha_k\alpha_i - n(a_j)\big] - \big[n(a_i a_j) - \alpha_k n(a_k, \overline{a_i a_j}) + \alpha_k^2 n(a_k)\big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{[by (D1)]}\\
&= \alpha_k\big[\alpha_k\alpha_i\alpha_j - \alpha_j n(a_j) - \alpha_i n(a_i) - \alpha_k n(a_k) + n(\overline{a_i a_j}, a_k)\big]\\
&\quad + n(a_i)n(a_j) - n(a_i a_j)\\
&= N(x)\alpha_k \qquad\qquad\qquad\qquad \text{[by (D1), (D2), (D4)]},
\end{aligned}$$

and the off-diagonal entries are given by

$$\begin{aligned}
d_k &= \overline{b_i b_j} - \beta_k b_k\\
&= \big[a_k a_i - \alpha_j\overline{a_j}\big]\big[a_j a_k - \alpha_i\overline{a_i}\big] - \big[\alpha_i\alpha_j - n(a_k)\big]\big[\overline{a_i a_j} - \alpha_k a_k\big]\\
&= (a_k a_i)(a_j a_k) - \alpha_j\overline{a_j}(a_j a_k) - \alpha_i(a_k a_i)\overline{a_i} + \alpha_i\alpha_j\,\overline{a_j}\,\overline{a_i}\\
&\quad -\alpha_i\alpha_j\overline{a_i a_j} - \alpha_k n(a_k)a_k + \alpha_i\alpha_j\alpha_k a_k + n(a_k)\overline{a_i a_j}\\
&= \big[\alpha_k\alpha_i\alpha_j - \alpha_j n(a_j) - \alpha_i n(a_i) - \alpha_k n(a_k) + t(a_i a_j a_k)\big]a_k\\
&= N(x)a_k
\end{aligned}$$

by Kirmse (D5), involution $\overline{ab} = \bar{b}\,\bar{a}$, Middle Moufang (D6), and $a(bc)a + \overline{(bc)}n(a) = [a(bc) + \overline{(bc)}\,\bar{a}]a$ [by the Flexible Law and Kirmse (D5)] $= t(abc)a = t(bca)a$ [by Trace Commutativity and Associativity (D2)–(D3)]. This establishes the Adjoint Identity.

Finally, to establish the *c–Sharp Identity* $c\#y = T(y)c - y$, linearize $x \mapsto c, y$ in the definition of $\#$ to get

$$\begin{aligned}
c\#y &= \textstyle\sum_k(1\beta_j + \beta_i 1 - 0)e_k + \sum_k(\bar{0} - 1b_k - 0)[ij]\\
&= \textstyle\sum_k(\beta_j + \beta_i)e_k - \sum_k b_k[ij]\\
&= \textstyle\sum_k(\beta_j + \beta_i + \beta_k)e_k - \big[\sum_k \beta_k e_k + \sum_k b_k[ij]\big]\\
&= \textstyle\sum_k T(y)e_k - y = T(y)c - y.
\end{aligned}$$

Thus by the Cubic Construction Theorem C.2.2 $\mathcal{J}ord(N, \#, c)$ is a Jordan algebra, which coincides with the Jordan matrix algebra $\mathcal{H}_3(\mathrm{D}, -)$. □

C.4 The Tits Constructions

The second example of a sharp norm form occurs in two constructions, due to Jacques Tits, of Jordan algebras of Cubic Form Type out of degree–3 associative algebras. These Jordan algebras need not be reduced, and they provide our first explicit examples of exceptional Jordan division algebras. A.A. Albert was the first to construct a (very complicated) example of an exceptional division algebra. Tits's beautiful method is both easy to understand, and completely general: *all* Albert algebras over a field arise by the Tits First or Second Construction. We begin by describing the associative setting for these constructions.

Associative Degree–3 Definition C.4.1 *An associative algebra* A *of* **degree** 3 *over* Φ *is one with a* **cubic norm form** n *satisfying the following three axioms. First, the algebra strictly satisfies the generic degree–3 equation*

$$\text{(A1)} \qquad a^3 - t(a)a^2 + s(a)a - n(a)1 = 0$$
$$(t(a) := n(1; a), \quad s(a) := n(a; 1), \quad n(1) = 1).$$

In terms of the usual adjoint $a^{\#} := a^2 - t(a)a + s(a)1$, *this can be rewritten as*

$$\text{(A1}'\text{)} \qquad aa^{\#} = a^{\#}a = n(a)1.$$

Second, the adjoint is a sharp mapping for n,

$$\text{(A2)} \qquad n(a; b) = t(a^{\#}, b) \qquad (t(a, b) := t(a)t(b) - n(1, a, b)).$$

Finally, the trace bilinear form is the linear trace of the associative product,

$$\text{(A3)} \qquad t(a, b) = t(ab).$$

Axiom (*A*3) *can be rewritten in terms of the linearization of the quadratic form* s,

$$\text{(A3}'\text{)} \qquad s(a, b) = t(a)t(b) - t(ab),$$

since by symmetry always $s(a, b) = n(a, b, 1) = n(1, a, b) = t(a)t(b) - t(a, b)$.

From the theory of generic norms, it is known that these conditions are met when A is a finite-dimensional semisimple associative algebra of degree 3 over a field Φ; in that case the algebras are just the forms of the split algebras $\Omega \boxplus \Omega \boxplus \Omega \cong \begin{pmatrix} \Omega & 0 & 0 \\ 0 & \Omega & 0 \\ 0 & 0 & \Omega \end{pmatrix} \subseteq \mathcal{M}_3(\Omega)$, $\Omega \boxplus \mathcal{M}_2(\Omega) \cong \begin{pmatrix} \Omega & 0 & 0 \\ 0 & \Omega & \Omega \\ 0 & \Omega & \Omega \end{pmatrix} \subseteq \mathcal{M}_3(\Omega)$, and $M_3(\Omega)$ for Ω the algebraic closure of Φ, where the axioms (A1), (A2), (A3) are immediate consequences of the Hamilton–Cayley Theorem and properties of the usual adjoint, trace, and determinant. We want to be able to construct Jordan algebras of Cubic Form Type over general rings of scalars, and the above axioms are what we will need. These immediately imply other useful relations.

Associative Cubic Consequences Lemma C.4.2 *Any cubic form on a unital associative algebra with basepoint* $n(1) = 1$*, sharp mapping* $a^\# := a^2 - t(a)a + s(a)1$*, and sharp product* $a\#b := (a+b)^\# - a^\# - b^\#$ *automatically satisfies*

$$\text{(A4)} \qquad t(1) = s(1) = 3, \qquad\qquad 1^\# = 1,$$
$$\text{(A5)} \qquad s(a,1) = 2t(a), \qquad\qquad 1\#a = t(a)1 - a.$$

If (A2)–(A3) *hold, then*

$$\text{(A6)} \qquad s(a) = t(a^\#), \qquad\qquad 2s(a) = t(a)^2 - t(a^2).$$

When n *is a cubic norm satisfying* (A1)–(A3) *and* $\frac{1}{2} \in \Phi$ *then we have*

$$\text{(A7)} \qquad a^{\#\#} = n(a)a,$$
$$\text{(A8)} \qquad aba = t(a,b)a - a^\#\#b,$$
$$\text{(A9)} \qquad (ab)^\# = b^\# a^\#,$$
$$\text{(A10)} \qquad n(ab)1 = n(a)n(b)1, \qquad\qquad t(ab) = t(ba), \quad t(abc) = t(bca).$$

PROOF. For (A4), $t(1) = s(1) = n(1;1) = 3n(1) = 3$ by Euler's Equation, so $1^\# = 1^2 - t(1)1 + s(1)1 = 1 - 3 + 3 = 1$. For (A5), by symmetry $s(a,1) = n(a,1,1) = n(1,1,a) = 2n(1;a) = 2t(a)$. Then $1\#a = 1a + a1 - t(1)a - t(a)1 + s(a,1)1 = 2a - 3a - t(a)1 + 2t(a)1 = -a + t(a)1$. For (A6), when (A2)–(A3) hold, setting $b = 1$ in (A2) yields $s(a) := n(a;1) = t(a^\#, 1) = t(a^\#)$, so taking the trace of the definition of the adjoint gives $s(a) = t(a^2) - t(a)^2 + 3s(a)$.

The next three formulas require a bit of effort. It will be convenient to use the *bar mapping*

$$\bar{a} := t(a)1 - a$$

(though, in contrast to the quadratic form case, this map is not an algebra anti-isomorphism, nor of period 2), so we may abbreviate the adjoint by

$$a^\# = s(a)1 - a\bar{a}.$$

We will make use twice of the tracial condition

$$z = t(z)1 \Longrightarrow z = 0,$$

which follows by taking traces to get $t(z) = 3t(z)$ hence $2t(z) = 0$, hence the existence of $\frac{1}{2}$ insures that $t(z) = 0$ and $z = 0$.

Turning to (A7), the element $z := a^{\#\#} - n(a)a$ has

$$\begin{aligned} z &= (a^\#)^2 - t(a^\#)a^\# + s(a^\#)1 - n(a)a \\ &= a^\#[s(a)1 - a\bar{a}] - s(a)a^\# + t(a^{\#\#})1 - n(a)a \qquad \text{[using (A6) twice]} \end{aligned}$$

$$= -n(a)[\bar{a}+a] + t(a^{\#\#})1 \qquad \text{[using (A1}')]$$
$$= t(a^{\#\#} - n(a)a)1 = t(z)1,$$

so $z = 0$ by our tracial condition.

For (A8) we linearize the degree–3 equation (A1) and use (A2) to get (writing $\{a,b\} := ab + ba$ as usual)

$$aba = \Big(-\{a^2, b\} + t(a)\{a,b\}\Big) + t(b)a^2 - s(a)b - s(a,b)a + t(a^{\#}, b)1$$
$$= \Big(\underbrace{-\{[a^2 - t(a)a + s(a)1], b\}}_{1} + \underbrace{2s(a)b}_{2}\Big) + t(b)\Big(\underbrace{a^{\#}}_{3} + \underbrace{t(a)a}_{4} - \underbrace{s(a)1}_{5}\Big)$$
$$- \underbrace{s(a)b}_{2} - \Big(\underbrace{t(a)t(b)}_{4} - \underbrace{t(a,b)}_{6}\Big)a + \Big(\underbrace{t(a^{\#})t(b)}_{5} - \underbrace{s(a^{\#}, b)}_{7}\Big)1$$

[using (A3), (A3′) to switch between the bilinear forms s, t]

$$= \Big(-\overbrace{\{a^{\#}, b\}}^{1} + \overbrace{t(a^{\#})b}^{2} + \overbrace{t(b)a^{\#}}^{3} - \overbrace{s(a^{\#}, b)1}^{7}\Big) + \overbrace{t(a,b)a}^{6} \quad \text{[by (A6) for (5)]}$$
$$= -a^{\#} \# b + t(a,b)a.$$

For (A9),we compute

$$(ab)^{\#} = (ab)(ab) - t(ab)ab + s(ab)1 \qquad \text{[by definition]}$$
$$= [aba - t(a,b)a]b + s(ab)1 \qquad \text{[by (A3)]}$$
$$= [-a^{\#}\#b]b + t((ab)^{\#})1 \qquad \text{[by (A8),(A6)]}$$
$$= [b^{\#}a^{\#} - t(b^{\#}, a^{\#})1] + t((ab)^{\#})1$$

[linearizing $b \mapsto b, a^{\#}$ in (A1′) $b^{\#}b = n(b)1$, and using (A2)]

$$= [b^{\#}a^{\#} - t(b^{\#}a^{\#})1] + t((ab)^{\#})1 \qquad \text{[by (A3)]}.$$

Thus the element $z = (ab)^{\#} - b^{\#}a^{\#}$ has $z = t(z)1,$ and again $z = 0$.

The first part of (A10) follows from (A1′) and (A9): $n(ab)1 = (ab)(ab)^{\#} = (ab)(b^{\#}a^{\#}) = a(bb^{\#})a^{\#} = a(n(b)1)a^{\#} = n(b)aa^{\#} = n(a)n(b)1$, while the last two parts of (A10) follow from (A3) and the symmetry of the trace bilinear form. □

Now we have the associative degree–3 preliminaries out of the way, and are ready to construct a Jordan algebra.

First Tits Construction C.4.3 *Let n be the cubic norm form on a degree–3 associative algebra* A *over* Φ*, and let* $\mu \in \Phi$ *be an invertible scalar. From these ingredients we define a module* $\mathrm{J} = \mathrm{A}_{-1} \oplus \mathrm{A}_0 \oplus \mathrm{A}_1$ *to be the direct sum of three copies of* A*, and define a basepoint c, norm N, trace T, and sharp $\#$ on* J *by*

$$\begin{aligned}
c &:= 0 \oplus 1 \oplus 0 = (0,1,0),\\
N(x) &:= \mu^{-1} n(a_{-1}) + n(a_0) + \mu n(a_1) - t(a_{-1}a_0a_1),\\
&= \sum_{i=-1}^{1} \mu^i n(a_i) - t(a_{-1}a_0a_1),\\
T(x) &:= t(a_0),\\
T(x,y) &:= \sum_{i=-1}^{1} t(a_{-i}, b_i),\\
[x^\#]_{-i} &:= \mu^i a_i^\# - a_{[i+1]}a_{[i-1]},
\end{aligned}$$

for elements $x = (a_{-1}, a_0, a_1)$, $y = (b_{-1}, b_0, b_1)$*, where the indices* $[j]$ *are read modulo* 3*. Then* $(N, \#, c)$ *is a sharped cubic form, and the algebra* $\mathcal{J}ord(\mathrm{A}, \mu) := \mathcal{J}ord(N, \#, c)$ *is a Jordan algebra.*

PROOF. It will be convenient to read subscripts n modulo 3, and always to choose coset representative $[n] = -1, 0, 1$:

$$[n] := \begin{cases} +1 \text{ if } n \equiv 1 \bmod 3;\\ \;\;\,0 \text{ if } n \equiv 0 \bmod 3;\\ -1 \text{ if } n \equiv 2 \bmod 3. \end{cases}$$

From (A2) and the definition of the norm we compute the linearization

$$\begin{aligned}
N(x;y) &= \textstyle\sum_i \mu^i t(a_i^\#, b_i) - t(b_{-1}a_0a_1) - t(a_{-1}b_0a_1) - t(a_{-1}a_0b_1)\\
&= \textstyle\sum_i \mu^i t(a_i^\#, b_i) - t(a_0a_1b_{-1}) - t(a_1a_{-1}b_0) - t(a_{-1}a_0b_1)\\
&= \textstyle\sum_{i=-1}^1 t([\mu^i a_i^\# - a_{[i+1]}a_{[i-1]}], b_i) && \text{[by cyclicity (A10)]},\\
N(c;y) &= t(b_0) = T(y) && [x = c \text{ has } a_0 = 1 = a_0^\#, a_{-1} = a_1 = 0].
\end{aligned}$$

Thus the trace coming from the norm is that given in the theorem. Linearizing $x \mapsto x, c$ gives

$$\begin{aligned}
N(x,c,y) &= \textstyle\sum_i t([\mu^i a_i \# c_i - a_{[i+1]}c_{[i-1]} - c_{[i+1]}a_{[i-1]}], b_i)\\
&= -t(1a_1, b_{-1}) + t(a_0\#1, b_0) - t(a_{-1}1, b_1)\\
&= -t(a_1, b_{-1}) + t(t(a_0)1 - a_0, b_0) - t(a_{-1}, b_1) && \text{[by (A5)]}\\
&= t(a_0)t(b_0) - t(a_1, b_{-1}) - t(a_0, b_0) - t(a_{-1}, b_1)
\end{aligned}$$

$$= T(x)T(y) - \textstyle\sum_{i=-1}^{1} t(a_{-i}, b_i) \qquad \text{[by (A3)]},$$

$$T(x,y) = T(x)T(y) - N(x,y,c) = \textstyle\sum_{i=-1}^{1} t(a_{-i}, b_i).$$

Therefore the trace bilinear form coming from the norm is also that given in the theorem. Then from the above we see that $N(x;y) = \sum_{i=-1}^{1} t([\mu^i a_i^{\#} - a_{[i+1]}a_{[i-1]}], b_i) = T(x^{\#}, y)$ for the sharp given in the theorem, verifying the *Trace–Sharp Formula.*

Now we are ready to verify the *Adjoint Identity*, $x^{\#\#} = N(x)x$. Set $x = (a_0, a_1, a_2)$, $y = x^{\#} = (b_0, b_1, b_2)$ as usual; then we have as ith component

$$\begin{aligned}
[x^{\#\#}]_i &= [y^{\#}]_{-(-i)} = \mu^{-i} b_{-i}^{\#} - b_{[-i+1]}b_{[-i-1]} && \text{[by definition]} \\
&= \mu^{-i} b_{-i}^{\#} - b_{-[i-1]}b_{-[i+1]} \\
&= \mu^{-i}\big(\mu^i a_i^{\#} - a_{[i+1]}a_{[i-1]}\big)^{\#} \\
&\qquad -\big(\mu^{[i-1]}a_{[i-1]}^{\#} - a_i a_{[i+1]}\big)\cdot\big(\mu^{[i+1]}a_{[i+1]}^{\#} - a_{[i-1]}a_i\big) \\
&= \big(\mu^i a_i^{\#\#} - a_i^{\#}\#a_{[i+1]}a_{[i-1]} + \mu^{-i}(a_{[i+1]}a_{[i-1]})^{\#}\big) - \mu^{[i-1]}\mu^{[i+1]}a_{[i-1]}^{\#}a_{[i+1]}^{\#} \\
&\qquad +\mu^{[i+1]}a_i a_{[i+1]}a_{[i+1]}^{\#} + \mu^{[i-1]}a_{[i-1]}^{\#}a_{[i-1]}a_i - a_i a_{[i+1]}a_{[i-1]}a_i \\
&= \mu^i n(a_i)a_i - t(a_i, a_{[i+1]}a_{[i-1]})a_i + \mu^{[i+1]}a_i n(a_{[i+1]}) + \mu^{[i-1]}n(a_{[i-1]})a_i \\
&\qquad \text{[using (A1$'$) twice, (A7), (A8), (A9), and } \mu^{[i-1]}\mu^{[i+1]} = \mu^{-i}] \\
&= \big(\textstyle\sum_{j=-1}^{1} \mu^j n(a_j) - t(a_{-1}a_0a_1)\big)a_i && \text{[from (A3), (A10)]} \\
&= N(x)a_i = N(x)[x]_i && \text{[from the definition of } N].
\end{aligned}$$

Since n continues to be a cubic norm in all scalar extensions, the Adjoint Identity continues to hold, and thus holds strictly.

Finally, we verify the *c–Sharp Identity* by comparing components on both sides: linearizing $x \mapsto c, y$ in the sharp mapping and using (A5) gives

$$[c\#y]_0 = \mu^0 c_0\#b_0 - b_1c_{-1} - c_1b_{-1} = 1\#b_0 - 0 - 0 = t(b_0)1 - b_0 = [T(y)c - y]_0,$$

$$[c\#y]_{-1} = \mu^1 c_1\#b_1 - c_{-1}b_0 - b_{-1}c_0 = 0 - 0 - b_{-1} = [T(y)c - y]_{-1},$$

$$[c\#y]_1 = \mu^{-1} c_{-1}\#b_{-1} - c_0b_1 - b_0c_1 = 0 - b_1 - 0 = [T(y)c - y]_1.$$

This completes the verification that $(N, \#, c)$ is a sharped cubic form, and therefore produces a Jordan algebra $\mathcal{J}ord(N, \#, c) =: \mathcal{J}ord(\mathrm{A}, \mu)$. □

For the second construction, let A be an associative algebra of degree 3 over Ω, and let $*$ be an involution **of second kind** on A, meaning that it is not Ω-linear, it is only *semi-linear* $(\omega a)^* = \omega^* a^*$ for a nontrivial involution on

Ω with fixed ring $\Phi := \mathcal{H}(\Omega, *) < \Omega$. (For example, the conjugate-transpose involution on complex matrices is only real-linear). The involution is **semi-isometric** with respect to a norm form if $n(a^*) = n(a)^*$.

Second Tits Construction C.4.4 *Let n be the cubic norm of a degree–3 associative algebra* A *over Ω as in C.4.3, with a semi-isometric involution of second kind over Φ. Let $u = u^* \in A$ be a hermitian element with norm $n(u) = \mu\mu^*$ for an invertible scalar $\mu \in \Omega$. From these ingredients we define a Φ-module $\mathrm{J} = \mathcal{H}(\mathrm{A}, *) \oplus \mathrm{A}$ to be the direct sum of a copy of the hermitian elements and a copy of the whole algebra, and define a basepoint c, norm N, trace T, and sharp $\#$ on elements $x = (a_0, a)$, $y = (b_0, b)$ for $a_0, b_0 \in \mathcal{H}(A, *)$, $a, b \in \mathrm{A}$, by the following formulas*:

$$\begin{aligned} c &:= 1 \oplus 0 = (1, 0), \\ N(x) &:= n(a_0) + \mu n(a) + \mu^* n(a^*) - t(a_0 a u a^*), \\ T(x) &:= t(a_0), \\ T(x, y) &:= t(a_0, b_0) + t(ua^*, b) + t(au, b^*), \\ x^{\#} &= (a_0^{\#} - aua^*, \mu^*(a^*)^{\#} u^{-1} - a_0 a). \end{aligned}$$

Then $(N, \#, c)$ is a sharped cubic form over Φ, and the algebra $\mathcal{J}ord(\mathrm{A}, u, \mu, *)$ $:= \mathcal{J}ord(N, \#, c)$ is a Jordan Φ-algebra. Indeed, the map $(a_0, a) \xrightarrow{\varphi} (ua^*, a_0, a)$ identifies $\mathcal{J}ord(N, \#, c)$ with the fixed points $\mathcal{H}(\mathcal{J}ord(\mathrm{A}, \mu), \widetilde{*})$ of the algebra $\mathrm{J}(\mathrm{A}, \mu)$ over Ω obtained by the First Construction relative to the semi-linear involution $(ua_{-1}, a_0, a_1) \mapsto (ua_1^*, a_0^*, a_{-1}^*)$. If Ω contains invertible skew elements (e.g., if it is a field), then the second construction is a form of the first: $\mathcal{J}ord(\mathrm{A}, u, \mu, *)_\Omega \cong \mathcal{J}ord(\mathrm{A}, \mu)$.

PROOF. We could verify directly that $\mathrm{J} := \mathcal{J}ord(\mathrm{A}, u, \mu, *)$ is Jordan by verifying that $(N, \#, c)$ is a sharped cubic, but Jordanity will follow from the more precise representation of J as $\mathcal{H}(\widetilde{\mathrm{J}}, \widetilde{*})$ for $\widetilde{\mathrm{J}} := \mathcal{J}ord(\mathrm{A}, \mu) = \mathcal{J}ord(\widetilde{N}, \widetilde{\#}, \tilde{c})$ obtained by the First Tits Construction over Ω.

The first thing to do is verify that $\widetilde{*}$ is indeed an involution of Jordan algebras. Since the products are built out of the basepoint $\tilde{c}$, norm $\widetilde{N}$, and sharp $\widetilde{\#}$ in the Cubic Construction, we need only prove that $\widetilde{*}$ preserves these. It certainly preserves the basepoint,

$$\tilde{c}^{\widetilde{*}} = (u0, 1, 0)^* = (u0^*, 1^*, 0^*) = (u0, 1, 0) = \tilde{c}.$$

Since the original $*$ is semi-isometric on A, it respects norms, traces, and sharps:

$$n(a^*) = n(a)^*, \quad t(a^*) = t(a)^*, \quad s(a^*) = s(a)^*, \quad (a^*)^{\#} = (a^{\#})^*.$$

From this we see that the involution $\widetilde{*}$ interacts smoothly with the norm $\widetilde{N}$: for any element $\tilde{x} = (ua_{-1}, a_0, a_1)$ we compute

$$\begin{aligned}
&\widetilde{N}(\tilde{x}^{\widetilde{*}}) = \widetilde{N}(ua_1^*, a_0^*, a_{-1}^*) \\
&= \mu^{-1}n(ua_1^*) + n(a_0^*) + \mu n(a_{-1}^*) - t(ua_1^* a_0^* a_{-1}^*) \\
&= \mu^{-1}n(u)n(a_1)^* + n(a_0)^* + \mu n(a_{-1})^* - t(a_{-1}a_0a_1u)^* \\
&= \mu^* n(a_1)^* + n(a_0)^* + (\mu^*)^{-1}n(u)^* n(a_{-1})^* - t(a_{-1}a_0a_1u)^* \\
&= \big(\mu n(a_1) + n(a_0) + \mu^{-1}n(ua_{-1}) - t(ua_{-1}a_0a_1)\big)^* \\
&= \widetilde{N}(ua_{-1}, a_0, a_1)^* = \widetilde{N}(\tilde{x})^*.
\end{aligned}$$

using, respectively, (1) the definition C.4.3 of $\widetilde{*}$; (2) the definition of $\widetilde{N}$; (3) (A10) and $u^* = u$; (4) $n(u) = \mu\mu^*$; (5) (A10) and the fact that $*$ is an involution on A; (6) the definition of $\widetilde{N}$. The involution interacts less smoothly with the adjoint: for any element $\tilde{x}$, we have

$$\begin{aligned}
&(\tilde{x}^{\widetilde{*}})^{\widetilde{\#}} = \big(ua_1^*, a_0^*, a_{-1}^*\big)^{\widetilde{\#}} \\
&= \big(\mu(a_{-1}^*)^\# - (ua_1^*)(a_0^*),\ (a_0^*)^\# - (a_{-1}^*)(ua_1^*),\ \mu^{-1}(ua_1^*)^\# - (a_0^*)(a_{-1}^*)\big) \\
&= \big(\mu(a_{-1}^*)^\# - ua_1^*a_0^*,\ (a_0^*)^\# - a_{-1}^*ua_1^*,\ \mu^{-1}(a_1^*)^\# u^\# - a_0^*a_{-1}^*\big) \\
&= \big(u[(\mu^*)^{-1}u^\#(a_{-1}^\#)^* - a_1^*a_0^*],\ (a_0^\#)^* - a_{-1}^*ua_1^*,\ \mu^*(a_1^\#)^*u^{-1} - a_0^*a_{-1}^*\big) \\
&= \big(u[\mu^{-1}a_{-1}^\# u^\# - a_0a_1]^*,\ [a_0^\# - a_1ua_{-1}]^*,\ [\mu u^{-1}a_1^\# - a_{-1}a_0]^*\big) \\
&= \big(u[\mu u^{-1}a_1^\# - a_{-1}a_0],\ [a_0^\# - a_1ua_{-1}],\ [\mu^{-1}a_{-1}^\# u^\# - a_0a_1]\big)^{\widetilde{*}} \\
&= \big(\mu a_1^\# - (ua_{-1})a_0,\ a_0^\# - a_1(ua_{-1}),\ \mu^{-1}(ua_{-1})^\# - a_0a_1\big)^{\widetilde{*}} \\
&= \big((ua_{-1},\ a_0,\ a_1)^{\widetilde{\#}}\big)^{\widetilde{*}} = \big(\tilde{x}^{\widetilde{\#}}\big)^{\widetilde{*}}
\end{aligned}$$

using, respectively, (1) the definition of $\widetilde{*}$; (2) the definition C.4.3 of $\widetilde{\#}$; (3) (A9); (4) $uu^\# = n(u)1 = \mu\mu^*$ by (A1$'$), and $(a^*)^\# = (a^\#)^*$; (5) $(u^\#)^* = u^\#$, $u^* = u$; (6) the definition of $\widetilde{*}$; (7) (A9); (8) the definition of $\widetilde{\#}$. Thus $\widetilde{*}$ preserves all the ingredients of the Jordan structure, and is a semilinear involution on $\mathcal{J}ord(\widetilde{N}, \widetilde{\#}, \tilde{c})$.

From this it is clear that the space $\mathcal{H}(\widetilde{\mathrm{J}}, \widetilde{*})$ of symmetric elements is precisely all (ua_{-1}, a_0, a_1), where $a_0^* = a_0, a_{-1} = a_1^*$, so we have a Φ-linear bijection $\varphi :\ \mathrm{J} \to \mathrm{H} := \mathcal{H}(\widetilde{\mathrm{J}}, \widetilde{*})$ given by

$$\varphi(a_0, a) = (ua^*, a_0, a).$$

To show that this is an isomorphism of Jordan algebras, it suffices to prove that the map preserves the basic ingredients for both cubic constructions, the basepoints, norms, and sharps. As usual the basepoint is easy:

$$\varphi(c) = \varphi(1, 0) = (0, 1, 0) = \tilde{c}.$$

The norm, as usual, presents no difficulties:

$$
\begin{aligned}
\widetilde{N}(\varphi(a_0,a)) &= \widetilde{N}(ua^*,a_0,a)\\
&= \mu^{-1}n(ua^*) + n(a_0) + \mu n(a) - t(ua^*a_0a)\\
&= n(a_0) + \mu n(a) + \mu^{-1}n(u)n(a)^* - t(a_0aua^*)\\
&= n(a_0) + \mu n(a) + \mu^* n(a)^* - t(a_0aua^*)\\
&= N(a_0,a),
\end{aligned}
$$

using, respectively, (1) the definition of φ; (2) the definition C.4.4 of $\widetilde{N}$; (3) (A10); (4) $n(u) = \mu\mu^*$; (5) the definition of N. Once more the sharp is messier:

$$
\begin{aligned}
\varphi(a_0,a)^{\widetilde{\#}} &= (ua^*,a_0,a)^{\widetilde{\#}}\\
&= (\mu a^\# - (ua^*)a_0,\ a_0^\# - a(ua^*),\ \mu^{-1}(ua^*)^\# - a_0a)\\
&= (u[\mu u^{-1}a^\# - a^*a_0],\ a_0^\# - a(ua^*),\ \mu^{-1}(a^*)^\# u^\# - a_0a)\\
&= (u[\mu^*(a^*)^\# u^{-1} - a_0a]^*,\ [a_0^\# - aua^*],\ [\mu^*(a^*)^\# u^{-1} - a_0a])\\
&= \varphi\big(a_0^\# - aua^*,\ \mu^*(a^*)^\# u^{-1} - a_0a\big)\\
&= \varphi\big((a_0,a)^\#\big).
\end{aligned}
$$

using, respectively, (1) the definition of φ; (2) the definition C.4.4 of $\widetilde{\#}$; (3) (A9); (4) $\mu^{-1}u^\# = \mu^* u^{-1}$ by (A1′); (5) the definition of φ again; (6) the definition of $\#$. This completes the verification that φ induces an isomorphism of J with H.

Since the original involution $*$ on A is of the second kind, there exist scalars $\omega \in \Omega$ with $\omega^* \neq \omega$. Then there exist nonzero skew elements $\lambda = \omega - \omega^*$. Assume that some such λ is invertible; then $\mathrm{Skew}(\Omega,*) = \Phi\lambda$ and $\mathrm{Skew}(\widetilde{\mathrm{J}},\tilde{*}) = \mathrm{H}\lambda$ (if $s \in \widetilde{\mathrm{J}}$ is skew, then by commutativity $h := \lambda^{-1}s$ is symmetric, $h^{\tilde{*}} = (\lambda^{-1})^* s^{\tilde{*}} = (-\lambda^{-1})(-s) = \lambda^{-1}s$, so $s = \lambda h$ (heavily using invertibility of λ). The presence of $\frac{1}{2}$ guarantees that all Φ-modules with involution are the direct sum of their symmetric and skew parts, $\Omega = \mathcal{H}(\Omega,*) \oplus \mathcal{S}k(\Omega,*) = \Phi \oplus \Phi\lambda$, $\widetilde{\mathrm{J}} = \mathcal{H}(\mathrm{J},\tilde{*}) \oplus \mathcal{S}k(\mathrm{J},\tilde{*}) = \mathrm{H} \oplus \mathrm{H}\lambda$, and the Φ-isomorphism $\varphi : \mathrm{J} \to \mathrm{H}$ extends to an Ω-isomorphism of J_Ω with H_Ω, which by standard properties of tensor products is isomorphic to $\widetilde{\mathrm{J}}$. (Recall that $\mathrm{H}_\Omega = \mathrm{H} \otimes_\Phi (\Phi \oplus \Phi\lambda) = (\mathrm{H} \otimes_\Phi \Phi) \oplus (\mathrm{H} \otimes_\Phi \Phi\lambda) \cong \mathrm{H} \oplus \mathrm{H}\lambda = \widetilde{\mathrm{J}}$.) □

Notice that our verification that the Freudenthal Construction produces degree–3 Jordan algebras involves only basic facts (D1)–(D6) about alternative algebras with scalar involution, and the verification for the Tits Constructions involves only basic properties (A1)–(A10) about cubic norm forms for associative algebras. In both cases the hard part is discovering a *recipe,* and once the blueprints are known, any good mathematical carpenter can assemble the algebra from the ingredients.

C.5 Albert Division Algebras

We mentioned in II.6.1.6 that the Tits Construction easily yields Albert division algebras, i.e., 27-dimensional Jordan algebras of anisotropic cubic forms.

Tits Division Algebra Criterion C.5.1 *The Jordan algebra* $\mathcal{J}ord(\mathrm{A}, \mu)$ *and* $\mathcal{J}ord(\mathrm{A}, u, \mu, *)$ *constructed from a degree–3 associative algebra* A *over a field* Φ *will be Jordan division algebras iff* A *is an associative division algebra and* $\mu \notin n(\mathrm{A})$ *is not a norm.*

PROOF. We know that $\mathcal{J}ord(\mathrm{A}, \mu)$ will be a division algebra precisely when its cubic norm form is anisotropic. A direct attempt to prove anisotropy in terms of the norm $N(x) := \mu^{-1}n(a_{-1}) + n(a_0) + \mu n(a_1) - t(a_{-1}a_0a_1)$ for $x = (a_{-1}, a_0, a_1)$ is difficult; as Macbeth's witches said, "Equation cauldron boil and bubble, two terms good, three terms trouble." *Necessity* of the conditions is easy: The algebra A must be a division algebra (anisotropic norm n) because it is a subalgebra with the same norm $N(0, a, 0) = n(a)$, and the scalar μ must not be a norm because $\mu = n(u)$ would imply $N(0, u, -1) = 0 + n(u) + \mu n(-1) = n(u) - \mu = 0$ and N would be isotropic. The hard part is showing that these two conditions are *sufficient*. We finesse the difficulty of the three-term norm equation by replacing it by two-term adjoint equations:

$$N(y) = 0 \text{ for some } y \neq 0 \Longrightarrow x^{\#} = 0 \text{ for some } x \neq 0.$$

Indeed, either already $x = y^{\#}$ is zero, or else $x \neq 0$ but $x^{\#} = (y^{\#})^{\#} = N(y)y = 0$ by the fundamental Adjoint Identity. But the vector equation $x^{\#} = 0$ for $x = (a_{-1}, a_0, a_1)$ implies three two-term equations $0 = [x^{\#}]_{-i} = \mu^i a_i^{\#} - a_{[i+1]}a_{[i-1]}$ for $i = -1, 0, 1$. Then $\mu^i a_i^{\#} = a_{[i+1]}a_{[i-1]}$, so multiplying on the left or right by a_i and using $a^{\#}a = aa^{\#} = n(a)1$ yields $\mu^i n(a_i)1 = a_{[i+1]}a_{[i-1]}a_i = a_i a_{[i+1]}a_{[i-1]}$. Since $x \neq 0$, some $a_i \neq 0$, and therefore $n(a_i) \neq 0$ by anisotropy of n. Then $a_{[i+1]}a_{[i-1]}a_i \neq 0$, and *none* of the a_i vanishes. Hence $\mu^0 n(a_0)1 = a_1a_{-1}a_0 = \mu^1 n(a_1)$ implies that $\mu = n(a_0a_1^{-1})$ is a norm. □

Albert Division Algebra Example C.5.2 *Let* A *be a* 9*-dimensional central-simple associative division algebra over a field* Φ. (*Such do not exist over* $\mathbb{R}$ *or* $\mathbb{C}$, *but do exist over* $\mathbb{Q}$ *and* p*-adic fields.*) *Then the extension* $\mathrm{A}(t) = \mathrm{A} \otimes_\Phi \Phi(t)$ *by the rational function field in one indeterminate remains a division algebra, but does not have the indeterminate* t *as one of its norms. Then* $\mathcal{J}ord(\mathrm{A}(t), t)$ *is an Albert division algebra over* $\Phi(t)$.

PROOF. Let $x_1, \ldots, x_9$ be a basis for A over Φ, with cubic norm form $n(x) = n(\sum \xi_i x_i) = \sum \alpha_{i,j,k}\xi_i\xi_j\xi_k$ for scalars $\alpha \in \Phi$. We first show that $\mathrm{A}(t)$ remains a division algebra, i.e., that the extended norm form remains anisotropic. Suppose instead that $n(x(t)) = 0$ for some nonzero $x(t) = \sum \xi_i(t)x_i$ for rational functions $\xi_i(t) \in \Phi(t)$. We can clear denominators to obtain a new isotropic $x(t)$ with *polynomial* coefficients $\xi_i(t) \in \Phi[t]$, and then divide through by the g.c.d. of the coefficients to get them relatively prime. If we substitute $t = 0$ into the polynomial relation $\sum \alpha_{i,j,k}\xi_i(t)\xi_j(t)\xi_k(t) = n(x(t)) = 0$ in $\Phi[t]$, we get $\sum \alpha_{i,j,k}\xi_i(0)\xi_j(0)\xi_k(0) = n(x(0)) = 0$ in Φ. Here $x(0) = \sum_i \xi_i(0)x_i$ has constant coefficients in Φ and so lies in A, yet $x(0)$ is not zero since relative primeness implies the polynomial coefficients $\xi_i(t)$ of the basis vectors x_i are not all divisible by t and hence do not all vanish at $t = 0$, contradicting anisotropy of n on A.

Next we check that t is not a norm on this division algebra over $\Phi(t)$. Suppose to the contrary that some $n(x(t)) := n(\sum \xi_i(t)x_i) = t$. We can write $\xi_i(t) = \frac{\eta_i(t)}{\delta(t)}$ for polynomials η_i, δ which are "relatively prime" in the sense that no factor of δ divides *all* the η_i. Again substituting $t = 0$ in $n(y(t)) := n\big(\sum \eta_i(t)x_i\big) = n\big(\delta(t)x(t)\big) = \delta(t)^3 t$ gives $n(y(0)) = 0$, so by anisotropy on A we have $y(0) = 0$, and by independence of the x_i each coefficient $\eta_i(0)$ is zero. But then each $\eta_i(t) = t\eta_i'(t)$ is divisible by t, $y(t) = ty'(t)$ for $y'(t) \in$ A$[t]$, and $t^3 n(y'(t)) = n(y(t)) = t\delta(t)^3$ in $\Phi[t]$. Then t^2 divides δ^3, hence the irreducible t divides δ, which contradicts $\delta(t)$ being relatively prime to the original $\eta_i(t) = t\eta_i'(t)$.

Thus A(t) and $\mu = t$ can be used in the Tits First Construction to produce a 27-dimensional Jordan Albert division algebra which is a form of the split Albert algebra $\mathcal{A}lb(\overline{\Phi})$ over the algebraic closure $\overline{\Phi}$, because $\mathrm{A}_{\overline{\Phi}} \cong \mathcal{M}_3(\overline{\Phi})$, and the Tits Construction applied to a split $\mathcal{M}_3(\Omega)$ produces a split Albert algebra $\mathcal{A}lb(\Omega)$ (see Problem C.2 below). □

C.6 Problems for Appendix C

PROBLEM C.1 Show that $\mathcal{J}ord(\mathrm{A}, \mu n(v)) \cong \mathcal{J}ord(\mathrm{A}, \mu)$ for any invertible scalar μ and invertible element $v \in \mathrm{A}$. In particular, the scalar can always be adjusted by an invertible cube (just as the scalar in the Cayley–Dickson Construction can always be adjusted by an invertible square).

PROBLEM C.2 (1) Verify that the First Tits Construction applied to a split matrix algebra produces (for any invertible scalar) a split Albert algebra, $\mathcal{J}ord(\mathcal{M}_3(\Phi), \mu) \cong \mathcal{A}lb(\Phi)$. [Hint: You can make your life easier by changing to $\mu = 1$, since every element of Φ is a norm in this case.] (2) Verify that when Φ is a field, the First Tits Construction is preserved under scalar extension: $\mathcal{J}ord(\mathrm{A}, \mu)_\Omega \cong \mathcal{J}ord(\mathrm{A}_\Omega, \mu \otimes 1)$ for any field extension Ω of Φ. (3) Show that for central-simple degree–3 algebras over a field Φ, the First Tits Construction always produces a *form* of a split Albert algebra. [Use the associative fact that every central-simple degree–3 algebra A over a field Φ is a form of 3×3 matrices, $\mathrm{A}_{\overline{\Phi}} \cong \mathcal{M}_3(\overline{\Phi})$ over the algebraic closure.]

QUESTION C.1 (1) Does the First Tits Construction respect scalar extension for arbitrary rings of scalars, i.e., if A is a degree–3 algebra over Φ and Ω is any scalar Φ-algebra, does A_Ω remain a degree–3 algebra over Ω, with $\mathcal{J}ord(\mathrm{A}, \mu)_\Omega \cong \mathcal{J}ord(\mathrm{A}_\Omega, \mu \otimes 1)$?

D

The Jacobson–Bourbaki Density Theorem

Since not all readers will have been exposed to the Density Theorem, we have made our exposition (cf. Strict Simplicity Theorem II.1.7.1, Prime Dichotomy Theorem III.9.2.1) independent of it. However, students should be aware of this fundamental associative result, so we include a brief treatment here.

Throughout this appendix we will consider left modules over a Φ-algebra R. This algebra need not be unital or commutative, and so is quite a different beast than the scalars we have dealt with in the rest of the book. Most of the interest here is the purely ring-theoretic case where $\Phi = \mathbb{Z}$ and the underlying Φ-module R is a mere abelian group under addition, but by allowing Φ to be dragged along unobtrusively we can handle those cases where we want a fixed underlying ground field (such as the reals or complexes).

D.1 Semisimple Modules

Recall that an R-module is called **simple** if it is non-trivial, $\mathrm{R} \cdot \mathrm{M} \neq \mathbf{0}$, and it has no proper submodules. A module is called **semisimple** if it is a direct sum of simple modules. These are frequently called **irreducible** and **completely reducible** modules respectively; we prefer to speak of irreducible and completely reducible *representations*, using the terms *simple* and *semisimple* to refer to algebraic structures (in keeping with the terminology of simple groups and rings).

The crucial fact we need about semisimple modules is their complementation property: as in vector spaces, every submodule $\mathrm{N} \subseteq \mathrm{M}$ is a direct summand and has a **complement** N': $\mathrm{M} = \mathrm{N} \oplus \mathrm{N}'$. Indeed, this property is equivalent to semisimplicity. In complete analogy with vector spaces, we have the following powerful equivalence.

Semisimplicity Equivalence Theorem D.1.1 *The following are equivalent for an* R*-module* M :

(1) M *is a direct sum of simple submodules* (*semisimple*) :

(2) M *is a sum of simple submodules*;

(3) *every submodule* N *has a complement which is a direct sum of simple submodules*;

(4) *every submodule* N *has a complement.*

These conditions are inherited by every submodule, every quotient module, and all sums. Moreover, even though R *need not be unital, we have* $\mathrm{RN} = \mathrm{N}$ *for all submodules* N*, in particular* $\mathrm{R}m = \widehat{\mathrm{R}}m$ *always contains* m*.*

PROOF. (1) $\Longrightarrow$ (2) is clear. (2) $\Longrightarrow$ (3) is highly nontrivial, depending on Zorn's Lemma. Let $\{\mathrm{M}_\sigma\}_{\sigma\in\mathcal{S}}$ be the collection of all simple submodules of M. Choose a subset $X \subseteq \mathcal{S}$ maximal among those having the two properties

(i) $\mathrm{S} := \sum_{\sigma\in X} \mathrm{M}_\sigma$ misses N, $\mathrm{S}\cap\mathrm{N} = \mathbf{0}$;

(ii) the submodules corresponding to the indices in X are independent, $\mathrm{M}_\sigma \cap \left(\sum_{\tau\neq\sigma\in X} \mathrm{M}_\tau\right) = \mathbf{0}$.

Such maximal X exist by Zorn's lemma since both properties are of "finite type": the union $X = \cup X_\alpha$ of a directed collection of such sets X_α again has the two properties, since *by directedness* any finite set of indices $\sigma_i \in X_{\alpha_i}$ are contained in some X_β, so any element m of $\mathrm{S} := \sum_{\sigma\in X} \mathrm{M}_\sigma$ and any (finite) dependence relation $\sum m_\sigma = 0$ lives in some S_β, which by hypothesis (i) misses N and by (ii) has the $\mathrm{M}_\sigma(\sigma\in X_\beta)$ independent. [It is important that we don't just take directed sets of semisimple submodules missing N: to make sure that the union is again semisimple, it is important to make sure that we have a directed set of "bases" X_α as well.]

We claim that this S is a complement for N. By independence (i) we certainly have $\mathrm{S}\oplus\mathrm{N} \subseteq \mathrm{M}$. To prove that it is all of M, it suffices by (2) to show that $\mathrm{S}\oplus\mathrm{N}$ contains each simple submodule M_τ of M. But if NOT, by simplicity $\mathrm{M}_\tau \not\subseteq \mathrm{S}\oplus\mathrm{N} \Longrightarrow \mathrm{M}_\tau\cap(\mathrm{S}\oplus\mathrm{N}) \neq \mathrm{M}_\tau \Longrightarrow \mathrm{M}_\tau\cap\mathrm{S} = \mathrm{M}_\tau\cap(\mathrm{S}\oplus\mathrm{N}) = \mathbf{0}$ [by simplicity of M_τ] $\Longrightarrow (\mathrm{S}\oplus\mathrm{M}_\tau)\cap\mathrm{N} = \mathbf{0}$ [if $s + m_\tau = n \neq 0$ then $m_\tau = -s + n \in \mathrm{M}_\tau \cap (\mathrm{S}\oplus\mathrm{N}) = \mathbf{0}$, hence $s = n \in \mathrm{S}\cap\mathrm{N} = \mathbf{0}$ implies $s = n = 0$, contrary to $n \neq 0$]. This would produce a larger independent direct sum $\mathrm{S}\oplus\mathrm{M}_\tau$ (larger set $X\cup\{\tau\}$) missing N, *contrary to maximality.* Thus S is the desired complement for N.

Note that (3) $\Longrightarrow$ (1) is clear by taking $\mathrm{N} = \mathbf{0}$, so we have a closed circle

$$(1) \Longleftrightarrow (2) \Longleftrightarrow (3).$$

We prefer to show the stronger result that complementation alone implies semisimplicity. Clearly (3) $\Longrightarrow$ (4), and we will complete the circle by showing (4) $\Longrightarrow$ (2).

To squeeze consequences from (4) we need to show that this property is inherited. If we have global complementation (4) in M then we have local complementation *in each submodule* $\mathrm{N} \lhd \mathrm{M}$: if $\mathrm{P} \lhd \mathrm{N}$ then $\mathrm{M} = \mathrm{P} \oplus \mathrm{P}'$ [by global complementation], hence $\mathrm{N} = \mathrm{P} \oplus (\mathrm{P}' \cap \mathrm{N})$ by Dedekind's Modular Law [$n = p + p' \Longrightarrow p' = n - p \in \mathrm{N}$ shows that $\mathrm{N} = \mathrm{P} + \mathrm{P}' \cap \mathrm{N}$, and certainly $\mathrm{P} \cap (\mathrm{P}' \cap \mathrm{N}) \subseteq \mathrm{P} \cap \mathrm{P}' = \mathbf{0}$]. Similarly, we have complementation *in each quotient* $\overline{\mathrm{M}} := \mathrm{M}/\mathrm{N}$: any submodule of $\overline{\mathrm{M}}$ has the form $\overline{\mathrm{P}}$ for $\mathrm{P} \supseteq \mathrm{N}$, so if $\mathrm{M} = \mathrm{P} \oplus \mathrm{P}'$ globally then $\overline{\mathrm{M}} = \overline{\mathrm{P}} \oplus \overline{\mathrm{P}'}$ in the quotient [the two span $\overline{\mathrm{M}}$, and they are independent in the quotient since $\overline{p} = \overline{p'} \Longrightarrow p' \in \mathrm{P} + \mathrm{N} = \mathrm{P} \Longrightarrow p' = 0$ by global independence of P, P' in M]. Once we establish that semisimplicity is equivalent to (2), it will be clear that semisimplicity *is preserved by arbitrary sums*, and this will establish the assertions about heredity at the end of the theorem.

We return to the problem of extracting the consequence (2) out of complementation (4). Let N be the sum of all the simple submodules of M (for all we know at this stage, there may not be any at all, in which case $\mathrm{N} = \mathbf{0}$), and suppose $\mathrm{N} \neq \mathrm{M}$, so some $m \notin \mathrm{N}$. But *any* "avoidance pair" (m, N) consisting of a submodule $\mathrm{N} \lhd \mathrm{M}$ and element $m \notin \mathrm{N}$ avoiding N (for example, $\mathrm{N} = \mathbf{0}$ and m any nonzero element) gives rise to a simple submodule. Indeed, choose any submodule P maximal among those containing N but missing m. An easy Zornification shows such P exist (a union of a chain of submodules missing a subset is always another submodule of the same sort). Then complementation (4) gives $\mathrm{M} = \mathrm{P} \oplus \mathrm{P}'$, and we claim *this complement* P' *is simple.* Indeed, if it had a proper submodule Q' then *by the above inheritance of complementation* it would have a local complement, $\mathrm{P}' = \mathrm{Q}' \oplus \mathrm{Q}$, hence $\mathrm{M} = \mathrm{P} \oplus \mathrm{Q} \oplus \mathrm{Q}'$ where $\mathrm{Q}, \mathrm{Q}' \neq \mathbf{0} \Longrightarrow \mathrm{P} \oplus \mathrm{Q}, \mathrm{P} \oplus \mathrm{Q}' > \mathrm{P}$; by maximality these larger submodules cannot miss m, we must have $m \in \mathrm{P} \oplus \mathrm{Q}, m \in \mathrm{P} \oplus \mathrm{Q}'$ and hence $m \in (\mathrm{P} \oplus \mathrm{Q} \oplus \mathbf{0}') \cap (\mathrm{P} \oplus \mathbf{0} \oplus \mathrm{Q}') = \mathrm{P} \oplus \mathbf{0} \oplus \mathbf{0}'$, contrary to our hypothesis $m \notin \mathrm{P}$. But the particular N we chose was the sum of all simple submodules, so $\mathrm{P}' \subseteq \mathrm{N} \subseteq \mathrm{P}$, contradicting directness. This contradiction shows that there is no such m, and $\mathrm{N} = \mathrm{M}$.

This establishes the equivalence of (1)–(4), and we have already noted inheritance of semisimplicity by submodules. We have $\mathrm{M} \subseteq \mathrm{RM}$ trivially for simple M [recall $\mathrm{RM} = \mathbf{0}$ is explicitly ruled out in the definition of simplicity], hence is true for direct sum of simples, i.e., for all semisimple modules, hence all $\mathrm{N} \lhd \mathrm{M}$. In particular, for the submodule generated by an element m we have $\widehat{\mathrm{R}}m = \mathrm{R}\widehat{\mathrm{R}}m = \mathrm{R}m$, establishing the last assertions of the theorem. $\square$

D.2 The Jacobson–Bourbaki Density Theorem

Recall that an associative algebra R is **primitive** iff it has a faithful irreducible representation, i.e., a faithful simple module M. Then the left regular representation $r \mapsto L_r$ is (by faithfulness) an isomorphism of R with an algebra of Δ-linear transformations on a left vector space $V = \mathrm{M}$ over $\Delta = \mathcal{E}nd_{\mathrm{R}}(\mathrm{M})$ (which by Schur's Lemma is a division algebra). The Density Theorem describes approximately what this algebra of transformations looks like: it is "thick" or "dense" in the full ring $\mathcal{E}nd_\Delta(V)$. More precisely, we say an algebra of linear transformations is **dense** in $\mathcal{E}nd_\Delta(V)$ if for any Δ-independent $x_1, \ldots, x_n \in V$ and arbitrary $y_1, \ldots, y_n \in V$ there is an $r \in \mathrm{R}$ with $rx_i = y_i$ for $i = 1, \ldots, n$. (Density can be interpreted as ordinary topological density of L_{R} with respect to a certain topology on $\mathcal{E}nd_\Delta(V)$.)

Jacobson Density Theorem D.2.1 (1) *An algebra is primitive iff it is isomorphic to a dense ring of transformations on a vector space.* (2) *If* M *is a simple left* R*-module, then* $V = \mathrm{M}$ *is a left vector space over the division ring* $\Delta := End_{\mathrm{R}}(\mathrm{M})$ *and* L_{R} *is a dense algebra of linear transformations in* $\mathcal{E}nd_\Delta(V)$.

PROOF. Density in (1) is certainly sufficient for primitivity, because then $\mathrm{R} \subseteq \mathcal{E}nd_\Delta(V)$ has a faithful representation (the identity) on V, which is irreducible since $\mathrm{R}x = V$ for any nonzero vector $x \in V$: $x \neq 0$ is Δ-independent, hence for any $y \in V$ there is by density an $r \in \mathrm{R}$ with $rx = y$. Density in (1) is necessary for primitivity by (2). The result (2) about simple modules will follow from a more general density theorem for semisimple modules below. □

To describe the general setting, we need to reformulate density in terms of double centralizers. For any independent set of k vectors x_i and any k arbitrary vectors y_i in a left vector space V, there is a linear transformation $T \in \mathcal{E}nd_\Delta(V) = \mathcal{C}ent_V(L_\Delta) = \mathcal{C}ent_V(\mathcal{C}ent_V(L_{\mathrm{R}})) \subseteq \mathcal{E}nd_\Phi(V)$ taking each x_i to the corresponding y_i, $T(x_i) = y_i$. Thus the above density will follow if every linear transformation $T \in \mathcal{C}ent_V(\mathcal{C}ent_V(L_{\mathrm{R}}))$ in the double centralizer agrees at least locally with an element of R: for every finite-dimensional Δ-subspace $W \subseteq \mathrm{M}$ we can find an element $r \in \mathrm{R}$ with $L_r|_W = T|_W$. Such double-centralizer theorems play an important role in group representations as well as ring and module theory. Moreover, as Bourbaki emphasized, Jacobson's somewhat tricky computational proof becomes almost a triviality when the situation is generalized from simple to semisimple modules.

Jacobson–Bourbaki Density Theorem D.2.2 *If* M *is a semisimple left* R*-module, then* M *is a left* R'*-module and left* R''*-module for the centralizer* $\mathrm{R}' := \mathcal{C}ent_{\mathrm{M}}(L_{\mathrm{R}})$ *and double-centralizer* $\mathrm{R}'' := \mathcal{C}ent_{\mathrm{M}}(\mathrm{R}') = \mathcal{C}ent_{\mathrm{M}}(\mathcal{C}ent_{\mathrm{M}}(L_{\mathrm{R}}))$, *and for any finite set* $m_1, \ldots, m_k \in \mathrm{M}$ *and any* R'*-linear transformation* $T \in \mathrm{R}''$, *there is an element* $r \in \mathrm{R}$ *with* $L_r m_i = T(m_i)$ *for* $i = 1, \ldots, k$.

PROOF. The case $k = 1$ is easy (for both Jacobson and Jacobson–Bourbaki Density): by complementation D.1.1.4 we have $\mathrm{M} = \mathrm{R}m \oplus \mathrm{P}$ for a complementary R-submodule P, and the projection $E(tm \oplus p) := tm$ on $\mathrm{R}m$ along P belongs to $\mathrm{R}' = \mathcal{E}nd_{\mathrm{R}}(\mathrm{M})$ [projection of a direct sum of R-modules on the first factor is always R-linear], so $T \in \mathrm{R}''$ commutes with E and $T(m) = T(E(m))$ [remember that $m \in \mathrm{R}m$ by D.1.1] $= E(T(m)) \in E(\mathrm{M}) \subseteq \mathrm{R}m$ implies $T(m) = rm$ for some r.

Now comes the elegant leap to the general case, where k elements are fused into one. For any finite set $m_1, \ldots, m_k \in \mathrm{M}$ set $\widetilde{m} = m_1 \oplus \cdots \oplus m_k \in \widetilde{\mathrm{M}} := \mathrm{M} \oplus \cdots \oplus \mathrm{M}$, and for any $T \in \mathrm{R}''$ set $\widetilde{T} := T \oplus \cdots \oplus T$ on $\widetilde{\mathrm{M}}$. As a direct sum $\widetilde{\mathrm{M}}$ remains a left module over $\widetilde{\mathrm{R}} := \mathrm{R}$, hence a left module over $\widetilde{\mathrm{R}}' := \mathcal{C}ent_{\widetilde{\mathrm{M}}}(L_{\widetilde{\mathrm{R}}})$ and left module over $\widetilde{\mathrm{R}}'' := \mathcal{C}ent_{\widetilde{\mathrm{M}}}(\mathcal{C}ent_{\widetilde{\mathrm{M}}}(L_{\widetilde{\mathrm{R}}}))$. [Warning: while $\widetilde{\mathrm{R}} = \mathrm{R}$, the centralizers $\widetilde{\mathrm{R}}', \widetilde{\mathrm{R}}''$ are NOT the same as $\mathrm{R}', \mathrm{R}''$ as they are computed in $\mathcal{E}nd_\Phi(\widetilde{\mathrm{M}})$ instead of $\mathcal{E}nd_\Phi(\mathrm{M})$. For this reason we have re-christened R as $\widetilde{\mathrm{R}}$ to denote its new elevated role on $\widetilde{\mathrm{M}}$.] We will show that $\widetilde{T} \in \widetilde{\mathrm{R}}''$, i.e., $\widetilde{T}$ commutes with all $S \in \widetilde{\mathrm{R}}' = \mathcal{E}nd_{\mathrm{R}}(\widetilde{\mathrm{M}})$. If we denote the i^{th} copy of M in $\widetilde{\mathrm{M}}$ by $[\mathrm{M}]_i$ we have $S([n]_i) = \sum_j [S_{ji}(n)]_j$ (for $S_{ji} := F_j^{-1} \circ E_j \circ S \circ F_i \in \mathcal{E}nd_R(\mathrm{M}) = R'$ as the composition of the canonical R-linear insertions $F_k : \mathrm{M} \to [\mathrm{M}]_k$ and projections $E_k : \widetilde{\mathrm{M}} \to [\mathrm{M}]_k$) since $S([n]_i) = SF_i(n) = (\sum_j E_j)SF_i(n) = \sum_j (F_j F_j^{-1})E_j S F_i(n) = \sum_j F_j S_{ji}(n) = \sum_j [S_{ji}(n)]_j$. Thus we have $\widetilde{T}(S(\widetilde{n}) = \widetilde{T}\big(S(\sum_{i=1}^k [n_i]_i)\big) = \widetilde{T}\big(\sum_{i,j} [S_{ji}(n_i)]_j\big) = (T \oplus \cdots \oplus T)\big(\bigoplus_j [\sum_i S_{ji}(n_i)]_j\big) = \bigoplus_j [T\big(\sum_i S_{ji}(n_i)\big)]_j = \bigoplus_j [\sum_i S_{ji}(T(n_i))]_j$ [since $T \in \mathrm{R}''$ commutes with $S_{ji} \in \mathrm{R}'$] $= S\big(\bigoplus_i [T(n_i)]_i\big) = S\big((T \oplus \cdots \oplus T)(n_1 \oplus \cdots \oplus n_k)\big) = S\big(\widetilde{T}(\widetilde{n})\big)$.

Since $\widetilde{\mathrm{M}}$ is still semisimple by closure under sums in D.1.1, by the case $k = 1$ there is $r \in \widetilde{\mathrm{R}}$ with $r\widetilde{m} = \widetilde{T}(\widetilde{m})$, i.e., an $r \in \mathrm{R}$ with $rm_i = T(m_i)$ for each i. This finishes the proof of both density theorems. □

EXERCISE D.2.2 In the case of modules over a noncommutative ring, if M is a *left* R-module it is preferable to write the linear transformations $T \in \mathrm{R}' = \mathcal{E}nd_{\mathrm{R}}(\mathrm{M})$ *on the right* of the vectors in M, $(x)T$ [so we have the "associativity" condition $(rm)T = r(mT)$ instead of the "commutativity" condition $T(rm) = rT(m)$], and dually when M is a right R'-module we write transformations in $\mathrm{R}'' = \mathcal{E}nd_{\mathrm{R}'}(\mathrm{M})$ on the left [so $T(mr') = (Tm)r'$]. Because most beginning students have not developed sufficient ambidexterity to move smoothly between operators on the left and the right, we have kept our discussion left-handed. Try out your right hand in the following situations. (1) Show that if V is finite-dimensional, the only dense ring of linear transformations on V is the full ring $\mathcal{E}nd_\Delta(V)$. (2) Explain why in the n-dimensional case $V \cong \Delta^n$ we have $\mathcal{E}nd_\Delta(V) \cong \mathcal{M}_n(\Delta^{op})$ instead of $\mathcal{M}_n(\Delta)$. (3) Show that the dual space $V^* = \mathcal{H}om_\Delta(V, \Delta)$ of a left vector space carries a natural structure of a *right* vector space over Δ via $f\delta = R_\delta \circ f$, but no natural structure of a *left* vector space [$L_\delta \circ f$ is not in general in V^*]. (4) Show that for a left vector space V there is a natural bilinear pairing on $V \times V^* \to \Delta$ [it is additive in each variable with $< \delta x, f >= \delta < x, f >$, $< x, f\delta > =< x, f > \delta$ for all $x \in V, f \in V^*, \delta \in \Delta$] via $< x, f >= (x)f$.

E

Hints

Here we give hints (varying from unhelpful to mildly helpful to outright answers) to the starred exercises at the end of each chapter in Parts II and III.

E.1 Hints for Part II

Chapter 1. The Category of Jordan Algebras

Exercise 1.3.1: (1) This is clearly a homomorphism of non-unital algebras, corresponding to the imbedding of a subalgebra in a larger algebra. (2) There is no unital homomorphism $\mathcal{M}_2(\Phi) \to \mathcal{M}_3(\Phi)$ when Φ is a field. Indeed, E_{11}, E_{22} are supplementary orthogonal idempotents in $\mathcal{M}_2(\Phi)$, conjugate under the inner automorphism $\varphi(X) = AXA^{-1}$ for $A = E_{12} + E_{21}$, so their images e, e' would have to be two conjugate supplementary idempotents in $\mathcal{M}_3(\Phi)$, so they would both have the same integral rank $r = 1$ or 2, and $3 = \text{rank}(1) = \text{rank}(e) + \text{rank}(e') = 2r$, which is impossible. [In characteristic 0 we can use traces instead of ranks of idempotents, since $\text{trace}(e) = r1$ and $0 \neq n \in \mathbb{N} \implies 0 \neq n1 \in \Phi$.]

Exercise 1.6.1B: Compute directly, heavily using the commutativity of c, or use the identity $[x, y, z] - [x, z, y] + [z, x, y] = [z, x]y + x[z, y] - [z, xy]$ valid in all linear algebras.

Exercise 1.6.3: (1) Reduce q.i. to inverse, then cancel T from $T[T^{-1}, S]T = [S, T] = 0$. (2) $\text{Ker}(T)$, $\text{Im}(T)$ are S-invariant.

Exercise 1.7.1: The relation is shorter because the commutator and associators vanish for $i = 1$, where $M(x_1) = 1$ is central, so the relation must vanish entirely; by independence of the ω_i, this implies that the associators vanish for each i. Once the $M(x_i)$ are scalars in Φ, they can be moved to the other side of the tensor product, forming $\omega = \sum \gamma_i \omega_i$.

Exercise 1.8.5: (1) $\lambda = 1$ yields $u + v = 0$; hence $\lambda^2 u + \lambda^2 v = 0$ for any λ, so subtracting $\lambda u + \lambda^2 v = 0$ gives $\mu u = 0$ for all $\mu = \lambda - \lambda^2$. If $\lambda = \frac{1}{2}$, then $\mu = \frac{1}{4}$ is invertible with inverse 4. (2) $f(x + \lambda z; y) - f(x; y) - \lambda^3 f(z; y) = \lambda a + \lambda^2 b$ for $a = f(x; y; z), b = f(z; y; x)$.

Problem 1.1: Consult Barry Mitchell *The Theory of Categories*, Academic Press, New York, 1965, page 2. (1) Show that the left unit is unique; (2) identify the objects X with the identity maps e, $e = \mathbb{1}_X$; (3) define $\mathcal{M}or(X, Y) := \mathbb{1}_X \mathcal{M} \mathbb{1}_Y$; (4) show that $\mathcal{M}$ is the disjoint union of the $\mathcal{M}or(X, Y)$'s, (5) show that fg is defined iff $f \in \mathcal{M}or(X, Y), g \in \mathcal{M}or(Y, Z)$, i.e., the domain of f is the codomain of g.

Problem 1.3: (1) By the universal property, any hull $\mathrm{A}^1 = \Phi 1 + \mathrm{A}$ is an epimorphic image of $\widehat{\mathrm{A}}$, and a quotient $\widehat{\mathrm{A}}/\mathrm{I}'$ faithfully contains A iff the kernel I' is disjoint from A. (2) Tightening is possible: maximal disjoint ideals $\mathrm{M}' \triangleleft \widehat{\mathrm{A}}$ exist by Zorn's Lemma; A remains faithfully imbedded in the resulting quotient $\widehat{\mathrm{A}}/\mathrm{M}'$, and all nonzero ideals of the quotient hit A [by maximality of M']. (3) $\mathcal{A}nn_{\mathrm{A}}(\mathrm{A})$ is a pathological ideal, which is not permitted in semiprime algebras. (4) Any disjoint ideal I' has $\mathrm{I}'\mathrm{A} + \mathrm{A}\mathrm{I}' \subseteq \mathrm{I}' \cap \mathrm{A}$ [since both I' and A are ideals of $\widehat{\mathrm{A}}] = \mathbf{0}$ [by disjointness]. Then I' is a subset of the ideal $\mathrm{M}' = \mathcal{A}nn_{\widehat{\mathrm{A}}}(\mathrm{A})$, which is a disjoint ideal in the robust case ($\mathrm{M}' \cap \mathrm{A} = \mathcal{A}nn_{\mathrm{A}}(\mathrm{A}) = \mathbf{0}$), so M' is the unique maximal disjoint ideal.

Problem 1.4: (1) We noted that $\widehat{\mathrm{A}} = \Phi(\hat{1} - 1) \boxplus \mathrm{A}$, and $\Phi(\hat{1} - 1) = \mathcal{A}nn_{\widehat{\mathrm{A}}}(\mathrm{A})$. Here $\widehat{\mathrm{A}}/\mathrm{M}' = \mathrm{A}$, so as expected $\widetilde{\mathrm{A}} = \mathrm{A}$ is its own tight unital hull. (2) $\widehat{\mathrm{A}} = \mathbb{Z}1 \oplus 2\mathbb{Z}, \mathrm{M}' = \mathbb{Z}(2 \oplus -2)$ with $\widehat{\mathrm{A}}/\mathrm{M}'$ consisting of all $\overline{n1 \oplus 2m} = \overline{(n + 2m) \oplus 0} \cong n + 2m$, so as expected the tight unital hull is $\mathbb{Z}$.

Problem 1.5: (1) A is always an ideal in any unital hull. (2) Nonzero orthogonal ideals in $\widetilde{\mathrm{A}}$ would have nonzero orthogonal traces on A. In $\widetilde{\mathrm{A}}$ we always have $\mathrm{M}'\mathrm{A} = \mathbf{0}$. Always M' consists of all $\omega 1 \oplus -z(\omega)$ for $z(\omega) \in \mathrm{A}$ with $za = \omega a = az$ for all $a \in \mathrm{A}$; the set of such $\omega \in \Phi$ forms a Φ-subspace (i.e., ideal) $\Omega \triangleleft \Phi$. In infinite matrices these z's are just all $\omega 1_\infty$. (3) Nonzero self-orthogonal ideals in $\widetilde{\mathrm{A}}$ would have nonzero self-orthogonal traces on A. An $\omega 1 \oplus -z(\omega) \in \mathrm{M}'$ as above squares to 0 iff $\omega^2 \oplus -\omega z(\omega) = 0$, which means that $\omega^2 = 0$ in $\Phi 1 \subset \widehat{\mathrm{A}}$; then $\Phi\omega$ is a trivial ideal in Φ, and generates a trivial ideal $\omega\mathrm{A}$ in A *if* Φ *acts faithfully on* A. If $\Phi' = \Phi \oplus \Omega$ for a trivial Φ-ideal Ω, then A becomes a Φ'-algebra via $\Omega\mathrm{A} = \mathbf{0}$, and $\widehat{\mathrm{A}}$ has a trivial ideal $\Omega 1$ no matter how nice A is as a Φ-algebra. (4) If $z(\omega)$ acts like $\omega 1$, then $\omega^{-1} z(\omega)$ is already a unit for A. (5) If $\mathcal{A}nn_\Phi(\mathrm{A})1 + \mathrm{M}'$ were a larger ideal, then it would hit A, so some $\alpha 1 + m = a \in \mathrm{A}$, thus $ax = \alpha x + mx = mx \in \mathrm{M}' \cap \mathrm{A} = \mathbf{0}$ and dually, therefore $a \in \mathcal{A}nn_{\mathrm{A}}(\mathrm{A}) = \mathbf{0}$ by robustness.

Problem 1.6: (2) The maximum number of mutually orthogonal idempotents in $\mathcal{M}_k(\Phi)$ over a field is k, achieved when all the idempotents are of "rank one" ($e\mathrm{A}e = \Phi e$). If $\mathcal{M}_n(\Phi) \to \mathcal{M}_m(\Phi)$ is a unital homomorphism when Φ is a field, then the mutually conjugate supplementary orthogonal idempotents $E_{11}, \ldots, E_{nn}$ in $\mathcal{M}_n(\Phi)$ would be carried to mutually conjugate supplementary orthogonal idempotents $e_1, \ldots e_n$ in $\mathcal{M}_m(\Phi)$; by conjugacy each e_i would have the same decomposition into r completely primitive idempotents, hence would have integral rank r. But then we would have $m = \text{rank}(1) = \text{rank}(e_1) + \cdots + \text{rank}(e_n) = nr$, which is impossible unless n divides m.

QUESTION 1.1: Think of the example in Problem 1.5.

QUESTION 1.2: Almost never. It suffices if Ω itself is a Boolean ring of scalars, and this is also necessary if B is free as a Φ-module.

QUESTION 1.3: Outer ideals $U_{\hat{J}}I \subseteq I$ reduce to ordinary ideals in the presence of $\frac{1}{2}$.

Chapter 2. The Category of Alternative Algebras

EXERCISE 2.5.1: $\mathcal{KD}(\mathrm{A}, \alpha^2\mu) \to \mathcal{KD}(\mathrm{A}, \mu)$ via $\varphi(a + bm) = a + \alpha bm'$ has $\varphi(1) = 1,\ N'(\varphi(a + bm)) = n(a) - \mu n(\alpha b) = n(a) - \mu\alpha^2 n(b) = N(a + bm)$, $\varphi\big((a_1+b_1m)(a_1+b_1m)\big) = \varphi\big((a_1a_2+\alpha^2\mu\bar{b}_2)+(b_2a_1+b_1\bar{a}_2)\big) = (a_1a_2+\alpha^2\mu\bar{b}_2)+ \alpha(b_2a_1 + b_1\bar{a}_2)m' = (a_1 + \alpha b_1m')(a_2 + \alpha b_2m') = \varphi(a_1 + \alpha b_1m)\varphi(a_2 + \alpha b_2m)$.

EXERCISE 2.6.1A: (2) Construct $\mathcal{Z}orn(\mathbb{R}) = \mathrm{A} \oplus \mathrm{A}\ell$ using $\mathrm{A} = \begin{pmatrix} \mathbb{R} & \mathbb{R}\vec{e}_1 \\ \mathbb{R}\vec{e}_1 & \mathbb{R} \end{pmatrix}$ spanned by $E_{11} := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right),\ E_{22} := \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right),\ E_{12} := \left(\begin{smallmatrix} 0 & \vec{e}_1 \\ 0 & 0 \end{smallmatrix}\right),\ E_{21} := \left(\begin{smallmatrix} 0 & 0 \\ \vec{e}_1 & 0 \end{smallmatrix}\right)$ [which is clearly isomorphic to $\mathcal{M}_2(\mathbb{R})$], and take $\ell = \begin{pmatrix} 0 & \vec{e}_2 \\ -\vec{e}_2 & 0 \end{pmatrix}$ so $N(\ell) = 1$. Show that $\ell \perp \mathrm{A}$. Compute the space $\mathrm{A}\ell$ as the span of

$$E_{11}\ell = \left(\begin{smallmatrix} 0 & \vec{e}_2 \\ 0 & 0 \end{smallmatrix}\right), \qquad E_{22}\ell = \left(\begin{smallmatrix} 0 & 0 \\ -\vec{e}_2 & 0 \end{smallmatrix}\right)$$

$$E_{12}\ell = \left(\begin{smallmatrix} 0 & \vec{e}_1 \\ 0 & 0 \end{smallmatrix}\right)\begin{pmatrix} 0 & \vec{e}_2 \\ -\vec{e}_2 & 0 \end{pmatrix} = \left(\begin{smallmatrix} 0 & 0 \\ \vec{e}_1\times\vec{e}_2 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ \vec{e}_3 & 0 \end{smallmatrix}\right), \qquad E_{21}\ell = \left(\begin{smallmatrix} 0 & 0 \\ \vec{e}_1 & 0 \end{smallmatrix}\right)\begin{pmatrix} 0 & \vec{e}_2 \\ -\vec{e}_2 & 0 \end{pmatrix} = \left(\begin{smallmatrix} 0 & -\vec{e}_1\times(-\vec{e}_2) \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & \vec{e}_3 \\ 0 & 0 \end{smallmatrix}\right).$$

Take B spanned by 1, $i = \begin{pmatrix} 0 & \vec{e}_1 \\ -\vec{e}_1 & 0 \end{pmatrix},\ j = \begin{pmatrix} 0 & \vec{e}_2 \\ -\vec{e}_2 & 0 \end{pmatrix},\ k = \begin{pmatrix} 0 & -\vec{e}_3 \\ \vec{e}_3 & 0 \end{pmatrix}$ [beware the minus sign in k!]; show that B is a copy of $\mathbb{H}$, and $\mathcal{Z}orn(\mathbb{R}) = \mathrm{B} + \mathrm{B}m$ for $m = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ with $m \perp \mathrm{B},\ N(m) = -1$.

EXERCISE 2.6.1B: In $\mathcal{C} = \mathcal{KD}(\mathrm{A}, \mu)$ itself we have an element $m' := \alpha m$ with $\mathcal{C} = A + Am',\ N(A, m') = 0,\ \mu' = -N(m') = -\alpha^2 N(m) = \alpha^2\mu$, so $\mathcal{C}$ also *equals* $\mathcal{KD}(\mathrm{A}, \mu')$.

Chapter 3. Three Special Examples

EXERCISE 3.1.D: By imbedding in the unital hull, we may assume that A is unital, so the left regular representation $x \mapsto L_x$ is by the left alternative law an injective specialization $\mathrm{A}^+ \to \mathrm{End}_\Phi(\mathrm{A})^+$.

EXERCISE 3.2.2A: (3) Take A' to be the algebra direct sum of A with a space N with trivial product $\mathrm{A}'\mathrm{N} = \mathrm{N}\mathrm{A}' = 0$ and skew involution $n^* = -n$ for all

n; if A is unital and you want the extension A′ to remain unital with the same unit, take N to be a direct sum of copies of the regular bimodule A_i with skew involution and trivial products $A_i A_j = 0$.

Exercise 3.3.2: (1) To show that a basepoint-preserving isometry φ is an algebra isomorphism, show that it preserves traces $T'(\varphi(x)) = T(x)$ and hence involutions $\varphi(\bar{x}) = \overline{\varphi(x)}$; then use the U–formula 3.3.1 to show that φ preserves products (or use the Degree–2 Identity to show that it preserves squares). To show that an algebra isomorphism is a basepoint-preserving isometry, note that first that φ must preserve unit elements $\varphi(c) = c'$; then apply φ to the Degree–2 Identity in J and subtract off the Degree–2 Identity applied to $\varphi(x)$ in J′, to get $[T'(\varphi(x)) - T(x)]\varphi(x) + [Q(x) - Q'(\varphi(x))]c' = 0$. If $\varphi(x)$ is linearly independent of c' in J′, show that $Q(x) = Q'(\varphi(x))$. If $\varphi(x)$ is dependent on c', show that $x = \lambda c,\ Q(x) = Q'(\varphi(x)) = \lambda^2$.

Problem 3.2: (1) The measures $\Delta(x, y) := \varphi(xy) - \varphi(x)\varphi(y),\ \Delta^*(x, y) := \varphi(xy) - \varphi(y)\varphi(x)$ of homomorphicity and anti-homomorphicity satisfy the relations $\Delta(x, y)\Delta^*(x, y) = \varphi\big((xy)^2 + U_x y^2 - \{xy, y, x\}\big) = 0$.

Problem 3.4: $u = \alpha\lambda 1 - e, v = \alpha 1 + \lambda e, \lambda \neq \pm i, \lambda \notin \mathbb{R}$. Answer: $\alpha = -\frac{1+\lambda\bar{\lambda}}{1+\lambda^2}$.

Problem 3.7: (1) Linearizing $x \mapsto x, c$ in $Q(D(x), x) = 0$ yields $T(D(x)) = 0$, so $\overline{D(x)} = D(\bar{x}) = -D(x)$. Thus $D(U_x\bar{y}) - U_x(D(\bar{y})) - U_{D(x),x}(\bar{y}) = Q(D(x), x)y - \big(Q(D(x), y) + Q(x, D(y))\big)x$, and the two conditions suffice. Conversely, for a derivation the vanishing of this last linear combination for y independent of x forces each coefficient to vanish [or setting $y = x$ forces $Q(D(x), x)x = 0$; applying $Q(D(x), \cdot)$ to this yields $Q(D(x), x)^2 = 0$, so absence of nilpotents compels the term to vanish]. (2) $T(1) = 1 \Leftrightarrow \varepsilon D(1) = 0,\ Q(T(x)) = Q(x)$ for all $x \in$ J $\Leftrightarrow \varepsilon Q(D(x), x) = 0$.

Chapter 4. Jordan Algebras of Cubic Forms

Exercise 4.1.0: (1) We have intrinsically determined $A := N(x+y) - N(x) - N(y) = N(x; y) + N(y; x),\ B := N(x + \lambda y) - N(x) - \lambda^3 N(y) = \lambda N(x; y) + \lambda^2 N(y; x),\ C := \lambda^2 A - B = -\lambda(1 - \lambda)N(x; y)$.

Exercise 4.3.3A: (1) Use the c–Sharp Identity and $T(c) = 3$. (2) Use Trace–Sharp, the c–Trace Formula, and the definition of T.

Exercise 4.3.3B: From Bullet Symmetry deduce $V_x^* = V_x,\ U_x^* = U_x$ [assuming you know that $U_x = 2L_x^2 - L_{x^2}$, or else directly from the definition of U_x using Sharp Symmetry], $V_{x,y}^* = (V_x V_y - U_{x,y})^* = V_y V_x - U_{x,y} = V_{y,x}$ [or use Sharp Symmetry and the recipe 4.2.2(2) for the triple product to see that $T(\{x, y, z\}, w) - T(z, \{y, x, w\}) = T(x, y)T(z, w) + T(z, y)T(x, w) - T(x\#z, y\#w) - T(y, x)T(z, w) - T(w, x)T(z, y) + T(z\#x, y\#w) = 0$].

Exercise 4.3.3D: (1) $x\#U_x y = x\#(T(x, y)x - x^\#\#y) = 2T(x, y)x^\# - [N(x)y + T(x, y)x^\#]$. (2) $\{U_x y, z, x\} - U_x\{y, x, z\} = \big(T((2 - 1)U_x y, z)x +$

$T(x,z)[T(x,y)x - x^\#\#y] - [x\#U_x y]\#z\big) - \big(T(x,V_{y,x}z)x - x^\#\#\{y,x,z\}\big) = T([2U_xy - T(x,y)x + x^\#\#y + T(x,y)x - V_{x,y}x], z)x - [T(x,y)x^\# - N(x)y]\#z - x^\#\#[T(x,z)y - \{y,x,z\}] = T(x^\#\#y,z)x + N(x)y\#z - x^\#\#[T(x,y)z + T(x,z)y - \{y,x,z\}] = T(x^\#, y\#z)x + N(x)y\#z - x^\#\#[x\#(y\#z)] = 0$.

Problem 4.1: (1): Set $x = 1$ in the VU-Commuting Identity to get $V_{1,y} = V_{y,1}$ or (acting on x) using $U_1 = 1\!\!1_X$. (2) Set $y = 1$ in the VU-Commuting Identity. (3): Apply the VU-Commuting Identity to 1 to get $U_{x,x^2} = U_xV_x$, linearize $x \mapsto x, 1$ to get $U_{1,x^2} + 2U_{x,x} = U_{x,1}V_x + U_xV_1$, and use $U_{x,x} = 2U_x, V_1 = 2 1\!\!1_X$.

Problem 4.2: (1): Use the c-Sharp Identity and $\frac{1}{2}$ for the first, the c-Sharp Identity for the second. (2) For Adjoint′, linearize the strict Adjoint Identity; for U–x–Sharp, use Sharp Symmetry and Adjoint′ $U_x(x\#y) = T(x, x\#y)x - x^\#\#(x\#y) = 2T(x^\#, y)x - [N(x)y + T(x^\#, y)x] = T(x^\#, y)x - N(x)y$. (3) For Dual Adjoint′ use Sharp Symmetry to move $T(\text{Adjoint}'(y), z) = T(y, \text{Dual Adjoint}'(z))$, then use nondegeneracy. (4) Compute $\big(V_{x,y}U_x - U_xV_{y,x}\big)z = \{x,y,U_xz\} - U_x\{y,x,z\} = \big(T(x,z)\{x,y,x\} - \{x,y,x^\#\#z\}\big) - U_x\big(T(z,x)y + T(y,x)z - x\#(y\#z)\big) = \big(2T(x,z)U_xy - T(x^\#\#z,y)x - T(x,y)x^\#\#z + y\#[N(x)z + T(x,z)x^\#]\big) - \big(T(x,z)U_xy + T(x,y)U_xz - [T(x^\#, y\#z)x - N(x)y\#z]\big)$ [using the above formulas for $x\#(x^\#\#z)$ and $U_x(x\#(y\#z))$] $= -T(x,y)[U_xz + x^\#\#z] + T(x,z)[U_xy + y\#x^\#$ [by Sharp Symmetry for $T(x^\#\#z,y)$] $= -T(x,y)[T(x,z)x] + T(x,z)[T(x,y)x] = 0$.

Problem 4.4: (2) Show that (A3) implies that $\Delta(x)\varphi(x) = 0$ for $\Delta(x) := N(\varphi(x)) - N(x)$, and (A1), (A2) imply $\Delta(x+ty) = \Delta(x) + t^3\Delta(y)$. These continue to hold in any extension, so linearized Adjoint in $\mathrm{J}[t]$ yields $\Delta(x)\varphi(y) = 0$ for all x, y.

Problem 4.5: (1) If $\alpha c = 0$, then $\alpha^3 = N(\alpha c) = 0$, and take $\varepsilon := \alpha^2$ (unless α^2 is already zero, in which case take α). (2) $T(c) = 3 = 0$ guarantees that $N'(c) = 1$, $T(x+\lambda y)^3 = T(x)^3 + \lambda^3 T(y)^3$ guarantees that $N'(x;y) = N(x;y)$, so $T' = T, S' = S, \#' = \#$ remains a sharp mapping, and the Adjoint Identity holds because $\varepsilon\mathrm{J} = \mathbf{0}$. (3) $x = e_1 + e_2$, $y = e_3 = x^\# = y^2$ have $T(x) = 2, T(y) = T(x^\#) = T(y^2) = 1, N(x) = N(y) = 0$, so $N'(x) = -\varepsilon, N'(y) = \varepsilon$, and therefore $N'(x)^2 = N'(y)^2N'(c) = 0$ but $N'(x^\#) = N'(U_yc) = \varepsilon \neq 0$.

Chapter 5. Two Basic Principles

Exercise 5.3.3: $U_{T(z)} = TU_zT^*$ vanishes on $\widehat{\mathrm{J}}$ or J, respectively, since U_z does.

Problem 5.2: (1) If $U_z\mathrm{I} = \mathbf{0}$, then any $U_z\hat{a}$ is trivial in J by $U_{U_z\hat{a}}\widehat{\mathrm{J}} = U_zU_{\hat{a}}U_z \subseteq U_z(U_{\widehat{\mathrm{J}}}U_{\mathrm{I}}\widehat{\mathrm{J}}) \subseteq U_z\mathrm{I} = \mathbf{0}$. (2) Show that if $z \in \mathrm{K}$, then all $w \in U_z(\widehat{\mathrm{J}})$ are trivial, using the Fundamental Formula to show that $U_w(\widehat{\mathrm{J}}) \subseteq U_z\mathrm{K} \subseteq U_{\mathrm{K}}\mathrm{K} = \mathbf{0}$; either some $w \neq 0$ is trivial or else all $w = 0$ and z itself is trivial.

Problem 5.3: By induction, with $M_1 = \mathbb{1}_{\mathrm{J}}, M_2 = V_x$, find a formula for M_{n+2}. [Answer: $U_x M_n + V_{x,x^n}$.]

Problem 5.4: (3) For an example where nondegeneracy of $\mathcal{H}(\mathrm{A}, *)$ doesn't force nondegeneracy of A, choose A_2 to be trivial and skew, so that it contributes nothing to $\mathcal{H}$.

Problem 5.5: (1) If I is a trivial ideal, $\mathrm{I}\,\mathrm{I} = \mathbf{0}$, then also $\mathrm{I}^*\,\mathrm{I}^* = \mathbf{0}^* = \mathbf{0}$, so $\mathrm{I} + \mathrm{I}^*$ is a nilpotent $*$-ideal: $(\mathrm{I} + \mathrm{I}^*)^2 = (\mathrm{I})(\mathrm{I}^*) + (\mathrm{I}^*)(\mathrm{I}) \subseteq \mathrm{I} \cap \mathrm{I}^*$ and $(\mathrm{I} + \mathrm{I}^*)^3 = (\mathrm{I} + \mathrm{I}^*)(\mathrm{I} \cap \mathrm{I}^*) = \mathbf{0}$. (2) A nonzero trivial $*$-ideal I would have $\mathrm{I} \cap \mathrm{J}$ a nonzero trivial Jordan ideal. (3) A skew ideal certainly misses $\mathcal{H}$ when $\frac{1}{2} \in \Phi$. Conversely, if z lies in a $*$-ideal I missing $\mathcal{H}$, then all $z + z^*, \in \mathcal{H} \cap \mathrm{I} = \mathbf{0}$ implies $z^* = -z$, and all elements of I are skew. (4) For $\mathrm{J} = \mathcal{H}$, there is a unique maximal skew ideal (the sum of all such).

Question 5.3: $[x)$ is only a weak inner ideal since it needn't contain $x^2 = U_x\hat{1} \in U_{[x)}\widehat{\mathrm{J}}$. We have inclusions $U_{[x)}\widehat{\mathrm{J}} \subseteq (x]$, $U_{[x)}\mathrm{J} \subseteq (x)$ by $U_{\alpha x + U_x a} = U_x B_{\alpha, -a, x}$, (x) is contained in both $[x)$ and $(x]$, which in turn are both contained in $[x]$.

Question 5.4: Yes, yes. The associative algebra A can't have trivial multiplication since already the Jordan algebra inside it has nontrivial products; any nonzero ideal I has a nonzero intersection with J, which is itself an ideal in J. Then this intersection $\mathrm{J} \cap \mathrm{I}$ must be all of J, so $\mathrm{J} \subseteq \mathrm{I}$. By definition of cover, J generates all of A as associative algebra, so $\mathrm{A} \subseteq \mathrm{I}$, thus $\mathrm{I} = \mathrm{A}$ and A is simple. [Similarly for $*$-simplicity.]

Chapter 6. Inverses

Exercise 6.1.4: (1) $x = \alpha\hat{1} + \beta\varepsilon$ has $U_x\widehat{\mathrm{J}} = 0$ if $\alpha = 0$; otherwise, $x = \alpha(\hat{1} + \alpha^{-1}\beta\varepsilon)$ is invertible with $x^{-1} = \alpha^{-1}(\hat{1} - \alpha^{-1}\beta\varepsilon)$. (2) Let $b \neq 0$ be an arbitrary nonzero element of A. Show that (i) $b\widehat{\mathrm{A}}b \neq 0$ so $b\widehat{\mathrm{A}}b = \mathrm{A}$; (ii) $b = bcb$ for some $c \in \mathrm{A}$, so $b\mathrm{A}b = \mathrm{A}$; (iii) $bca = a = acb$ for all a, hence $bc = cb = 1$ is a unit element for A; (iv) conclude that A is unital and all nonzero elements are invertible. [Subhints for (i)–(iv): (i) else $\mathrm{B} := \Phi b$ would be inner, $\Phi b = \mathrm{A}$, and $\mathrm{AA} = \Phi bb = 0$; (ii) $b = b\hat{a}b \Leftarrow b = bcb$ for $c = \hat{a}b\hat{a} \in \mathrm{A}$; (iii) write $a = ba'b$; (iv) if A has a left unit and a right unit, they coincide and form the (unique) unit.]

Exercise 6.1.7: (1) U_x is invertible from (Q2) and the Fundamental Formula. (2) $U_x(x^n \bullet y) = U_x(x^{n-1})$ for $n = 1, 2$ [using (Qn)] since L_{x^n} commutes with U_x. (3) (L1)–(L2) $\Longrightarrow U_x y = x$ as in (Q1); $x^2 \bullet y^2 = 1$ from $[x^2, y, y] = -2[x \bullet y, y, x]$ [linearized Jordan identity (JAX2)$'$] $= 0$; then $x \bullet y^2 = (x^2 \bullet y) \bullet y^2 = (x^2 \bullet y^2) \bullet y = y$, so $U_x y^2 = 1$ as in (Q2) (or $0 = 2[y^2, x, x] + 4[y \bullet x, x, y] - [x^2, y, y] - 2[x \bullet y, y, x] = U_x y^2 - 1$).

Problem 6.1: (1) $u^2 = 1$ iff $e := \frac{1}{2}(u + 1) = \frac{2}{4}(u + 1) = \frac{1}{4}(2u + u^2 + 1) = \left(\frac{u+1}{2}\right)^2 =: e^2$.

PROBLEM 6.2: Cancel U_x from $V_{x,x^{-1}}U_x = U_{U_x x^{-1},x} = 2U_x$ to get $V_{x,x^{-1}} = 2\mathbb{1}_J$. Then compute $D(x) = D(U_x x^{-1}) = 2D(x) + U_x D(x^{-1})$.

PROBLEM 6.3: (1) $D(y) = D(y^2 x) = L_{y^2}D(x) + V_y R_x D(y)$, where $D(y)x = D(yx) - yD(x) = -L_y D(x)$. (2) For nuclear y we have $L_y V_y - L_{y^2} = L_y R_y$.

QUESTION 6.1: (1) If x satisfies $q(t)$, we can scale so the constant term is 1, then write $q(t) = 1 - tp(t)$; try $y = p(x)$. (2) No — all commutative Jordan algebras are ncJa's, and elements x of Jordan algebras of quadratic forms have many "inverses" $x \bullet y = 1$.

Chapter 7. Isotopes

EXERCISE 7.2.1A: The Inverse Condition requires only the Fundamental Formula, but the Linear Inverse Condition requires the formula $\{y, x^{-1}, U_x z\} = \{y, z, x\}$.

PROBLEM 7.2: (1) Example of orthogonal non-automorphism: think of $T = -\mathbb{1}_J$. [This is an automorphism of the Jordan triple system, but not of the Jordan algebra.] (7) If $T(1) = x^2$, try U_x.

PROBLEM 7.3: For associativity, $[x, y, z]^u = u(uxy)z - ux(uyz) = u[u, x](yz) + u[x, y, z]$ vanishes identically iff A is associative and u lies in the center of A. For unitality, the element u^{-1} is always a *left unit*, but is a *right unit* iff u lies in the center: $x^u u^{-1} = uxu^{-1} = [u, x]u^{-1} + x$.

QUESTION 7.1: (1) Verify, either directly from the definitions or in terms of $J^{(\tilde{u})}$ as a subalgebra of $\widetilde{J}^{(\tilde{u})}$, that Proposition 7.2 still holds. (2) Let J be the finite matrices in the algebra of all infinite matrices, or the transformations of finite rank in all linear transformations on an infinite-dimensional vector space, and u an invertible diagonal matrix (the case $u = 1$ gives the original J, $u = \alpha 1$ gives an "α-isotope," and non-constant diagonal u give complicated isotopes).

Chapter 8. Peirce Decompositions

EXERCISE 8.1.2: Use the formulas $U_x = 2L_x^2 - L_{x^2}, U_{x,y} = 2(L_x L_y + L_y L_x - L_{x\bullet y})$ for $x, y = e, e'$. [Answer: $E_2 = -L_e + 2L_e^2 = -L_e(\mathbb{1}_J - 2L_e)$, $E_1 = 4L_e - 4L_e^2 = 4L_e(\mathbb{1}_J - L_e)$, $E_0 = \mathbb{1}_J - 3L_e + 2L_e^2 = (\mathbb{1}_J - L_e)(\mathbb{1}_J - 2L_e)$.]

EXERCISE 8.1.3: (3) $e \leftrightarrow u$ is a bijection with involutory *elements*, but $u \leftrightarrow \mathcal{U} = U_u$ is *not* bijective between involutory elements and involutory *automorphisms*: e and $1 - e$ have the same u up to sign, $u(1 - e) = -u(e)$, and $\pm u$ have the same $\mathcal{U}$.

Chapter 9. Off-Diagonal Rules

EXERCISE 9.2.2A: (1) Use the Peirce U–Product Rules 8.2.1 and $x_1^2 = U_{x_1}e_i + U_{x_1}e_j$. (2) Use (1), the Commuting Formula (FFII), and the Eigenspace Laws 8.1.4, since $J_1(e) = J_1(1 - e)$. (3) Use the Peirce U–Formulas, Triple Switch, and the Peirce U–Product Rules. (4) Use (1) and (FFI). (5) Use the Peirce U–Product Rules to show that $0 = E_i(2U_{x_1}(a_i)) = E_i(\{x_1, \{x_1, a_i\}\} - \{x_1^2, a_i\}) = q_i(x_1, \{x_1, a_i\}) - \{q_i(x_1), a_i\}$, so that we have (5a) $q_i(x_1, \{x_1, a_i\}) = V_{a_i} q_i(x_1)$; from this show that we have (5b) $2U_{a_i} q_i(x_1) = \left(V_{a_i}^2 - V_{a_i^2}\right) q_i(x_1) = V_{a_i} q_i(\{a_i, x_1\}, x_1) - q_i(\{a_i^2, x_1\}, x_1) = \big(q_i(\{a_i, x_1\}, \{a_i, x_1\}) + q_i(\{a_i, \{a_i, x_1\}\}, x_1)\big) - q_i(\{a_i^2, x_1\}, x_1) = 2q_i(\{a_i, x_1\})$.

Exercise 9.2.3: (1) One way is to use 9.2.2(1) on the element $z_1 + tx_1$ in $\mathrm{J}[t]$ for $z_1 \in \mathcal{R}ad(q_j)$, $x_1 \in \mathrm{J}_1$ to get $q_i(z_1) = 0$, $q_i(z_1, x_1)^2 = U_{z_1} q_j(x_1)$, $z_1^2 = 0$, $q_i(z_1, x_1)^4 = 0$. (2) Use 9.2.2(2) in $\widehat{\mathrm{J}}$ to show that a radical z_1 is trivial; to show that a trivial z_1 is radical, $q_i(z_1, y_1) = 0$, use the general identity $\{y, z\}^2 = \{y, U_z(y)\} + U_y(z^2) + U_z(y^2)$.

Chapter 11. Spin Coordinatization

Exercise 11.4.1: (1) Use the Commuting Formula to show that $U_x V_{x^2} = U_{x^3,x}$. Then apply U_x^{-1} on the left and right, and use the Fundamental Formula.

Chapter 12. Hermitian Coordinatization

Exercise 12.1.1: (1) $\{a_2, v\} = \{q_0(v), a_0, v\}$ for $a_0 := U_v^{-1} a_2$. (2) $v^{-1} \in \mathcal{D}_2(v) + \mathcal{D}_0(v)$ because $v^{-1} = v^{-2} \bullet v \in \mathcal{D}_2(v) + \mathcal{D}_0(v)$.

Exercise 12.2.1A: (1) For any $x_1 \in \mathrm{J}_1$ set $y_1 := U_v^{-1} x_1$. Then we have that $\{U_v a_i, \{u_j, U_v y_1\}\} = \{U_v a_i, u_j, U_v y_1\} = U_v\{a_i, U_v u_j, y_1\} = U_v\{a_i, e_i, y_1\} = U_v V_{a_i} y_1$. (2) $T \mapsto U_v T U_v^{-1}$ is an (inner) automorphism of linear transformations on J_1, and maps the generators (hence all) of $\mathcal{D}_i$ into $\mathcal{D}_j$; the reverse inclusion holds by applying this to the invertible element v^{-1}.

Exercise 12.2.1B: (1) $V_{U_v(a_i)}(v) = V_{q_i(v)} V_{a_i}(v)$.

Chapter 13. Multiple Peirce Decompositions

Exercise 13.1.3A: Multiply $EA + AE = 0$ on the left and right to get $EA = -EAE = AE$; then get $2EA = 2AE = 0$. From the linearizations (i) $U_{x^2} = U_x^2$, (ii) $U_{x^2,\{x,y\}} = U_x U_{x,y} + U_{x,y} U_x$, (iii) $U_{x^2,y^2} + U_{\{x,y\}} = \{U_x, U_{x,y}\} + U_{x,y}^2$, (iv) $U_{x^2,\{y,z\}} + U_{\{x,y\},\{x,z\}} = \{U_{x,y}, U_{x,z}\} + \{U_x, U_{y,z}\}$ show that $(i) \Longrightarrow$ (1), $(ii) \Longrightarrow$ (4), $(iii) \Longrightarrow [(2) \Longleftrightarrow (3)]$, $(iv) \Longrightarrow [(5) \Longleftrightarrow (6)]$. Show that $(3) \Longrightarrow (2), (5), (7)$ because $e \perp f, g, f+g; e, f, e+f \perp g, h, g+h$. Multiply (iii) by U_e and use (4) and $U_e f = 0$ to get (3).

The general identity follows from the Macdonald Principle; (2) follows directly since $V_{e_i,e_j} = 0, U_{e_i} e_j = 0$ by Peirce Orthogonality, where we have $e_i \in \mathrm{J}_2, e_j \in \mathrm{J}_0$ relative to $e = e_i$.

Exercise 13.1.3B: (1) In the expansion of $E(\mathbf{t})E(\mathbf{t}^3) = E(\mathbf{t}^4)$ (i) one variable t_i^8 arises only for $\{i, j\} = \{k, \ell\} = \{i\}$ equal sets of size 1; (ii) two variables $t_i^2 t_k^6$ arise only if $\{i, j\} = \{i\}, \{k, \ell\} = \{k\}$ are disjoint sets of size 1; (iii) two variables $t_i t_j^7$ arise only if $\{i,j\}$ of size 2 contains $\{k,\ell\} = \{j\}$ of size 1; (iv) two variables $t_k^5 t_\ell^3$ arise only if $\{i,j\} = \{k\}$ of size 1 is contained in $\{k,\ell\}$ of size 2; (v) three variables $t_i^2 t_k^3 t_\ell^3$ arise only if $\{i,j\} = \{i\}$ of size 1 is disjoint from $\{k,\ell\}$ of size 2; (vi) three variables $t_i t_j t_k^6$ arise only if $\{i,j\}$ of size 2 is disjoint from $\{k,\ell\} = \{k\}$ of size 1; (vii) two variables $t_i^4 t_j^4$ arise only if $\{i,j\} = \{k,\ell\}$ are equal of size 2; (viii) three variables $t_i t_j^4 t_\ell^3$ arise only if $\{i,j\}$ of size 2 overlaps $\{k,\ell\} = \{j,\ell\}$ of size 2 in precisely one element; (ix) four variables $t_i t_j t_k^3 t_\ell^3$ arise only if $\{i,j\}, \{k,\ell\}$ are disjoint sets of size 2.

Exercise 13.6.0: When $n = 0, \Delta = 1$; $\Delta_n = \prod_{n\geq i>j\geq 1}[t_i - t_j] = \prod_{n>j\geq 0}[t_n - t_j]\cdot\Delta_{n-1}$; check that the extremely monic $t_n - t_j$ are nonsingular.

Problem 13.1: (1) If we use $S \leftrightarrow T$ to indicate that the operators S, T commute, show that $V_x, V_{x^2}, U_x \leftrightarrow V_x, V_{x^2}, U_x$; show that $U_x y = x^2 \bullet y$; show that $U_{U_x y, y} = V_{x^2} U_y$; show that $U_{\{x,y,1\}} = (2U_x + V_x^2 - V_{x^2})U_y$. (3) Macdonald. (4) $L_e = L_{e^2}$ commutes with $L_f = L_{f^2}$ because $e \in \mathrm{J}_2(e), f \in \mathrm{J}_0(e)$, and use Peirce Associativity 9.1.3 and Orthogonality 8.2.1(4).

Problem 13.2: (4) Try any $x, y \in \mathrm{A} := \varepsilon\mathrm{A}'$ for $\varepsilon \in \Phi$ with $\varepsilon^3 = 0$.

Chapter 14. Multiple Peirce Consequences

Exercise 14.4.1: (1) By Peirce Orthogonality 13.3.1(3) $x \bullet y = \sum_i x_{ii} \bullet y_{ii} = \sum_i e_i = 1$, $x^2 \bullet y = \sum_i x_{ii}^2 \bullet y_{ii} = \sum_i x_{ii} = x$. (2) Taking the component in J_{ii} of $1 = x \bullet y = \sum_i x_{ii} \bullet y_{ii} + \sum_{i<j}(x_{ii} + x_{jj}) \bullet y_{ij}$, $x = x^2 \bullet y = \sum_i x_{ii}^2 \bullet y_{ii} + \sum_{i<j}(x_{ii}^2 + x_{jj}^2) \bullet y_{ij}$ gives $e_i = x_{ii} \bullet y_{ii}$, $x_{ii} = x_{ii}^2 \bullet y_{ii}$.

Chapter 15. Hermitian Symmetries

Exercise 15.1.3: (1) In 15.1.1, the Supplementary Rule (1) and $h_{ii}^2 = h_{ii}$ in (2) hold by definition; show that $h_{ij} \in \mathrm{J}_{ij}$ so $\{h_{ii}, h_{ij}\} = h_{ij}$. (2) Deduce Hermitian Orthogonality (3) from Peirce Orthogonality. (3) Establish $h_{ij}^2 = h_{ii} + h_{jj}$ $(i \neq j)$ by considering the cases (1) i or j equal to 1, (2) for $i, j, 1$ distinct. (5) Establish $\{h_{ij}, h_{jk}\} = h_{ik}$ by considering separately the cases (i) $(j = 1)$ $\{v_{1j}, v_{1k}\}$, (ii) $(i = 1 \neq j, k)$ $\{h_{ij}, h_{jk}\} = \{v_{1j}, \{v_{1j}, v_{1k}\}\}$, (iii) (dually if $i, j \neq 1 = k$), (iv) $(i, j, k, 1 \neq)$ $\{h_{ij}, h_{jk}\} = \{\{v_{1i}, v_{1j}\}, \{v_{1j}, v_{1k}\}\}$.

Exercise 15.2.1A: The explicit actions in (2) [except for $\mathcal{U}_{ij}$ on J_{ij}] and (3) involve only the h_{ij} and $x_{k\ell}$ in distinct Peirce spaces.

Exercise 15.2.1B: For Index Permutation (3), if $\{k,\ell\} \cap \{i,j\}$ has size 0, we have $\mathcal{U}_{(ij)} = 1$ on $h_{k\ell}$ by Action (2); if $\{k,\ell\} \cap \{i,j\} = \{i,j\}$ has size 2, we have $\mathcal{U}_{(ij)} h_{k\ell} = U_{h_{ij}} h_{ij}$ [by (2)] $= h_{ij}$; and if $\{k,\ell\} \cap \{i,j\} = \{j\}$ has size 1 (dually if $\{k,\ell\} \cap \{i,j\} = \{i\}$), then by (2) $\mathcal{U}_{(ij)} h_{k\ell} = V_{h_{ij}} h_{j\ell} = \{h_{ij}, h_{j\ell}\} = h_{i\ell}$.

Chapter 16. The Coordinate Algebra

Exercise 16.1.3A: $1_L = e$ being a left unit implies $1_R = e^*$ is a right unit, and *always* if $1_L, 1_R$ exist they are equal and are the unit, so $e = e^*$.

EXERCISE 16.1.3B: (1) $\overline{\delta_0(a_{11})} = U_{h_{12}}V_{h_{12}}(a_{11}) = U_{h_{12}^2,h_{12}}(a_{11}) = \{a_{11},h_{12}\} = \delta_0(a_{11})$. (2) $d \cdot \bar{d} = \{\mathcal{U}_{(23)}(d),\mathcal{U}_{(13)}\mathcal{U}_{(12)}(d)\} = \{\mathcal{U}_{(23)}(d),\mathcal{U}_{(23)}\mathcal{U}_{(13)}(d)\}$ becomes $\mathcal{U}_{(23)}\{d,\mathcal{U}_{(13)}(d)\} = \mathcal{U}_{(23)}\{d,\{d,h_{13}\}\} = \mathcal{U}_{(23)}\{d^2,h_{13}\} = \mathcal{U}_{(23)}\{q_{11}(d),h_{13}\} = \{q_{11}(d),\mathcal{U}_{(23)}(h_{13})\} = \{q_{11}(d),h_{12}\} = \delta_0(q_{11}(d))$. (3) $\delta_0(a_{11}) \cdot d = \{\mathcal{U}_{(23)}(\{a_{11},h_{12}\}),\mathcal{U}_{(13)}(d)\} = \{\{a_{11},h_{13}\},\mathcal{U}_{(13)}(d)\} = \{a_{11},\{h_{13},\mathcal{U}_{(13)}(d)\}\} = \{a_{11},\mathcal{U}_{(13)}\mathcal{U}_{(13)}(d)\} = \{a_{11},d\}$. (4) $\delta_0(a_{11}) \cdot \delta_0(a_{11}) = \{a_{11},\delta_0(a_{11})\} = \{a_{11},\{a_{11},h_{12}\}\} = \{a_{11}^2,h_{12}\} = \delta_0(a_{11}^2)$. (5) $\delta_0(a_{11}) = 0 \iff 0 = q_{11}(h_{12},\{h_{12},a_{11}\}) = \{a_{11},q_{11}(h_{12})\} = \{a_{11},h_{11}\} = 2a_{11} \iff 0 = a_{11}$.

Chapter 18. Von Neumann Regularity

EXERCISE 18.1.2: (1) $U_xy := U_x(U_zx) = U_xU_zU_xz = U_{U_xz}z = U_xz = x$ and $U_yx := U_z(U_xU_zx) = U_zx = y$. (2) vNr 18.1.2 shows that x is vNr iff $[x] \subseteq (x)$. Always $(x) \subseteq (x]$ and $(x) \subseteq [x) \subseteq [x]$.

QUESTION 18.1: Yes: $v = T^{-1}(1)$ has $\mathbb{1}_J = U_{T(v)} = TU_vT^*$ so $U_vT^* = T^{-1}$ invertible implies that U_v is surjective, therefore v invertible, hence U_v invertible, so $T^* = U_v^{-1}T^{-1}$ is invertible too. Then $u = T(1)$ has $U_u = TT^*$ invertible, so u is invertible.

QUESTION 18.2: (1) Expand $U_{\widehat{T}(\alpha 1+x)} = \widehat{T}\big(\alpha^2 1 + \alpha V_x + U_x\big)\widehat{T}^*$ evaluated on $\beta 1 + y$, and collect coefficients of like $\alpha^i\beta^j$. $\big[$Answer: (1) $(\alpha^0\beta^0) : TU_xT^*y = U_{T(x)}y$ (i.e., the given structural linkage), $(\alpha^1\beta^0) : TV_xT^* = U_{\hat{t},T(x)}$ $[= \tau V_{T(x)} + U_{t,T(x)}]$, $(\alpha^2\beta^0) : TT^* = U_{\hat{t}}$ $[= \tau^2\mathbb{1}_J + \tau V_t + U_t]$, $(\alpha^0\beta^1) : T(x)^2 = T(U_x\hat{t}^*)$ $[= \tau T(x^2) + T(U_xt^*)]$, $(\alpha^1\beta^1) : V_{\hat{t}}T = TV_{\hat{t}^*}$ $[V_tT = TV_{t^*}]$, $(\alpha^2\beta^1) : \widehat{T}(\hat{t}^*) = \hat{t}^2$ $[T(t^*) = \tau t + t^2]\big]$. (2) Combine the conditions of (1) for T and for T^*.

Chapter 19. Inner Simplicity

EXERCISE 19.2.1B: (1) The element b is paired with $c = (1+d)b(1+d)$ because $bcb = b(1+d)b(1+d)b = bdbdb$ (by *nilpotence* $b^2 = 0$) $= b$ (by *pairing* $bdb = b$) and $cbc = (1+d)b(1+d)b(1+d)b(1+d) = (1+d)b(1+d)$ (by the above) $= c$. So far we have used only $b^2 = 0$, but to get $c^2 = 2c$ we need to assume that $d^2 = 0$ too: $c^2 = (1+d)\big(b(1+d)^2b\big)(1+d) = (1+d)\,(b(1+2d)b)\,(1+d)$ (by *nilpotence* $d^2 = 0$) $= (1+d)\,(2bdb)\,(1+d)$ (by *nilpotence* $b^2 = 0$) $= 2(1+d)\,(b)\,(1+d) = 2c$. Therefore $e = \frac{1}{2}c$ is idempotent with $(e] = (c]$ structurally paired with $(b]$ by Principal Pairing 18.2.5, and again e is a simple idempotent.

Chapter 20. Capacity

PROBLEM 20.1: (1) $x \in J_0(e + e_{n+1}),\ e \in J_2(e + e_{n+1}) \implies \{e,x\} = 0$ by Peirce Orthogonality. (2) $e_{n+1} \in J_0(e) \cap J_2(e + e_{n+1})$.

PROBLEM 20.2: (1) Replace J by $J_2(e_1+e_2)$, which remains nondegenerate by Diagonal Inheritance 10.1.1(1). (2) Use 19.2.1(2). (3) Use 9.2.2(1), $q_i(x_1)^2 = U_{x_1}q_j(x_1)$. (4) Use nondegeneracy. (5) Any v with $q_2(v) \neq 0$ automatically has $q_0(v) \neq 0$ as well [using (3)], hence $q_i(v)$ are invertible in the *division algebras* J_i, so v^2 [hence v] is invertible.

PROBLEM 20.3: (3) Use the identity $v^2 = \{v_{ij}, v_{jk}\}^2 = \{v_{ij}, U_{v_{jk}}(v_{ij})\} + U_{v_{jk}}(v_{ij}^2) + U_{v_{ij}}(v_{jk}^2)$ [by Macdonald].

PROBLEM 20.4: (1) $U_{e_i}\mathrm{K} \neq 0 \Longrightarrow e_i \in \mathrm{K} \Longrightarrow$ all $e_j \in \mathrm{K} \Longrightarrow 1 \in \mathrm{K}$. (2) Work with $e = e_i$ in the nondegenerate subalgebra $\mathrm{J}' = \mathrm{J}_2(e_i + e_j)$ where $U_{e_i,e_j}\mathrm{K} = \mathrm{K}_1$; show that (1) $\Longrightarrow$ all $q_r(\mathrm{K}_1) = 0 \Longrightarrow U_{\mathrm{K}_1}(\mathrm{J}') = \mathbf{0}$ [using the q-Properties Proposition 9.2.2(3)] $\Longrightarrow \mathrm{K}_1 = 0$ [using nondegeneracy].

Chapter 21. Herstein–Kleinfeld–Osborn Theorem

EXERCISE 21.1.1: (2) Left alternativity implies that $x \mapsto L_x$ is a monomorphism $\mathrm{A}^+ \to \mathcal{E}nd(\widehat{\mathrm{A}})^+$ of linear Jordan algebras, hence automatically of quadratic Jordan algebras when $\frac{1}{2} \in \Phi$, where $2U_xy := x(xy + yx) + (xy + yx)x - (x^2y + yx^2) = 2x(yx)$ by Left and linearized Left Moufang.

EXERCISE 21.2.1B: Its square b^2 is symmetric and still invertible, hence with symmetric inverse, so $b^{-1} = (b^2)^{-1}b \in \mathcal{H}\mathrm{B} \subseteq \mathrm{B}$.

EXERCISE 21.2.2: For norm composition (4), $\big(n(xy) - n(x)n(y)\big)1 = n(xy)1 - (n(x)1)\,(n(y)1) = (xy)\overline{(xy)} - (x\bar{x})(n(y)1) = \big([x, y, \overline{xy}] + x(y(\bar{y}\,\bar{x}))\big) - x(n(y)\bar{x})$ [since $\overline{}$ is an involution and $n(y)$ is a scalar] $= -[x, y, xy] - x[y, \bar{y}, \bar{x}]$ [removing a bar] $= -[x, y, xy] - x[y, y, x]$ [removing two bars] $= [x, xy, y] - 0$ [by alternativity] $= (x^2y)y - x(xy^2)$ [by alternativity] $= x^2y^2 - x^2y^2$ [by alternativity again] $= 0$. For Kirmse (5), $n(x)y - \bar{x}(xy) = [\bar{x}, x, y] = -[x, x, y]$ [removing a bar] $= 0$ [by left alternativity], and dually on the right.

PROBLEM 21.1: (I)(3) Some commutator $\gamma = [\alpha, \beta] \in \Delta$ is invertible, so $\mathcal{H}$ generates $(\gamma^{-1}, \gamma^{-1})\,((\alpha, \alpha)(\beta, \beta) - (\beta\alpha, \beta\alpha)) = (1, 0)$, also $(1, 1) - (1, 0) = (0, 1)$, so all $(\delta, \delta)(1, 0) = (\delta, 0)$ and $(\delta, \delta)(0, 1) = (0, \delta)$, hence all of $(\Delta, 0) + (0, \Delta) = (\Delta, \Delta) = \mathrm{D}$. (II) An associative division algebra Δ with *non-central involution* is symmetrically generated by Step 4 of the proof of Herstein–Kleinfeld–Osborn 21.3.1. (III) $\mathcal{H}(\mathrm{D}, -)$ equals the $*$-center Ω, and the subalgebra generated by $\mathcal{H}$ is Ω, so D is symmetrically generated iff $\mathrm{D} = \mathcal{H}$, i.e., iff the involution is trivial.

PROBLEM 21.2: (2) By (1) we know $z\mathrm{D} = \overline{\mathrm{D}}z = \mathrm{D}z$. For right idealness $(zx)y = z(yx)$ of $z\mathrm{D}$ (dually left idealness of $\mathrm{D}z$), argue that $(zx)y = \bar{y}(zx) = -z(\bar{y}x) + (\bar{y}z + z\bar{y})x = -z(\bar{y}x) + (z(y + \bar{y}))x = z(-\bar{y}x + (y + \bar{y})x) = z(yx)$. For triviality, use the Middle Moufang Identity $(zx)(yz) = z(xy)z$, valid in all alternative algebras, or argue directly that $(zx)(yz) = z((yz)x) = z((z\bar{y})x) = z(z(x\bar{y})) = z^2(x\bar{y}) = 0$.

PROBLEM 21.3: (4) Use (3) to show that $[\mathcal{N}, \mathcal{N}] = \mathbf{0}$: either recall 21.2.1(4), or prove it from scratch using a hiding track [hide m in an associator till the coast is clear: $nm[x, y, z] = n[mx, y, z] = [mx, y, z]n = m[x, y, z]n = mn[x, y, z]$.

Problem 21.4: (5) For $z\bar{z} = \bar{z}z = 0$, note that they can't both be invertible by (4), and if one is zero, so is the other by (1) with x, y taken to be $z, \bar{z}$. For $z\mathcal{H}\bar{z} = 0$, show that it lies in $\mathcal{H}$ and kills z, or is not invertible by (1) with $x = hz, y = \bar{z}$. (6) Otherwise, if $xy = 0$ then $\mathrm{I} := \hat{D}x\hat{D}, \mathrm{K} := \hat{D}y\hat{D}$ would be nonzero ideals which kill each other, while if $xy \neq 0$ then $\mathrm{I} := \hat{D}xy\hat{D}$ would be a nonzero ideal which kills itself.

Problem 21.6: (1) It suffices to prove L; from Left and Middle Moufang $L_x L_y^2 L_x = L_{xy^2x} = L_{(xy)(yx)} = L_1 = \mathbb{1}_{\mathrm{A}}$ shows that L_x has both a left and right inverse, hence is invertible, so from $L_x L_y L_x = L_{xyx} = L_x$ we can cancel to get $L_x L_y = L_y L_x = \mathbb{1}_{\mathrm{A}}$.

Chapter 22. Osborn's Capacity–Two Theorem

Exercise 22.1.1B: (1) Use the $U1q$-Rule 9.2.2(2) twice to show that $q_0([V_{a_2}, V_{b_2}]x_1, y_1) = q_0(x_1, [V_{b_2}, V_{a_2}]y_1)$. (2) Use the $U1q$-Rule 9.5(2) again, together with Peirce Specialization 9.1.1 on the Jordan product $[a, b]^2$ to compute (setting $z_1 := [[a_2, b_2], x_1]$) that $2q_0(z_1) = q_0(z_1, z_1) = q_0([V_{a_2}, V_{b_2}]x_1, z_1) = q_0(x_1, [V_{b_2}, V_{a_2}]z_1) = q_0\left(x_1, -[V_{a_2}, V_{b_2}]^2(x_1)\right) = -q_0\left(x_1, V_{[a_2,b_2]^2}(x_1)\right) = -\{x_1, [a_2, b_2]^2, x_1\} = -2U_{x_1}[a_2, b_2]^2$.

Chapter 23. Classical Classification

Exercise 23.1.2: In II.7. Problem 7.2 you were to show that structural transformations are just the isomorphism between isotopes, so isomorphism classes of isotopes correspond to conjugacy classes of invertible elements under the structure group. Here we are concerned with the orbit of 1: $T(1) = 1^{(u)} = u^{-1}$ imples $\mathrm{J} \cong \mathrm{J}^{(u)}$. (1) Here T is structural with $T^*(y) = \alpha a^* y a$. (2) If $v = u^{-1}$ has $v = bb^*$, then $u^{-1} = \alpha a a^*$ for $\alpha = 1, a = b$. (3) If $\gamma_i = d_i \overline{d_i}$ then $\Gamma^{-1} = aa^*$ for $a = diag(\overline{d_1}^{-1}, \ldots, \overline{d_n}^{-1})$. (4) In (i) $\mathrm{D} = \mathcal{E}x(\mathrm{B})$, all symmetric $\gamma = (\beta, \beta) = (\beta, 1)\,\overline{(\beta, 1)^{ex}}$ are norms; in (ii) $\mathrm{D} = \mathcal{M}_2(\Omega)$, all symmetric $\left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & \omega \\ -1 & 0 \end{smallmatrix}\right) \overline{\left(\begin{smallmatrix} 0 & \omega \\ -1 & 0 \end{smallmatrix}\right)}$.

E.2 Hints for Part III

Chapter 1. The Radical

Exercise 1.3.2: Try $z = -1$.

Exercise 1.4.3: Check that the given proof of 1.4.3(2) uses only weak structurality.

Exercise 1.5.1: (2) Show that γ is structural with $\gamma^* = \gamma$.

Exercise 1.7.3: By nondegeneracy, J has no nonzero *weakly-trivial* elements z ($(z) = U_z \mathrm{J} = \mathbf{0}$), so run the argument with principal $(z]$'s replaced by open principal (z)'s. Note that we wind up with true vNrity, $y \in (y)$, instead of the ("weak" but equivalent) vNrity $y \in (y]$.

PROBLEM 1.1: (1) The infinite series converges in the complete vector space, since $\|x\| = \rho < 1$ implies $\|x^n\| \leq \rho^n$ where $\sum_0^\infty \rho^n = \frac{1}{1-\rho} < \infty$. (2) If $\|x\| < \alpha$ then $\|y\| < 1$ for $y := \alpha^{-1}x$ implies $1 - y$ and $\alpha(1-y) = \alpha 1 - x$ are invertible. (3) It suffices if $1 - u^{-1}v$ is invertible, and here $\|1 - u^{-1}v\| = \|u^{-1}(u-v)\| \leq \|u^{-1}\|\|u - v\| < 1$.

PROBLEM 1.2: Since the quasi-inverse is unique, it suffices to check that $\sum_{n=0} t^n x^n$ is indeed the inverse of $\hat{1} - tx$ in the formal power-series algebra $\widehat{\mathrm{J}}[[t]]$.

PROBLEM 1.4: Show that (x, y) q.i. in $\mathrm{J}^{(z)}$ iff x is q.i. in $(\mathrm{J}^{(z)})^{(y)}$ by $\mathrm{J}^{(U_z y)} = (\mathrm{J}^{(z)})^{(y)}$, and also iff $\{x, y\}^{(z)} - U_x^{(z)} y^{(2,z)}$ q.i. in J by 1.4.2(4)(vi).

PROBLEM 1.5: Set $v := u^{-1}$ for typographical convenience. (1) $u - x = 1^{(v)} - x$ is invertible in J iff it is in $\mathrm{J}^{(v)}$, i.e., iff x is q.i. in $\mathrm{J}^{(v)}$. (2) Apply (1) to $u = \hat{1} - y$ in the unital hull.

PROBLEM 1.7: (1) If f is homogeneous of degree d_i in x_i, set $\widetilde{\mathrm{J}} = \widehat{\mathrm{J}} \otimes \Phi[\varepsilon_1, \ldots, \varepsilon_n]$, where $\varepsilon_i^{d_i+1} = 0$ [realized as $\widehat{\mathrm{J}}[t_1, \ldots, t_n]/\langle t_1^{d_1+1}, \ldots, t_n^{d_n+1}\rangle$], so that $u_i := 1 + \varepsilon_i x_i$ is invertible in $\widetilde{\mathrm{J}}$. Then the coefficient of $\varepsilon_1^{d_1} \cdots \varepsilon_n^{d_n}$ in $f(u_1, \ldots, u_n) = 0$ is $f(x_1, \ldots, x_n)$. (2) Show that the invertible elements are Zariski-dense.

PROBLEM 1.8: (2) $B_{\alpha,x,y} = U^{(y)}_{\alpha 1^{(y)} - x}$, $U_{B_{\alpha,x,y}(z)} U_y = U^{(y)}_{U^{(y)}(\alpha 1^{(y)} - x)(z)}$, $U_z B_{\alpha,y,x} U_y = U_z U_y B_{\alpha,x,y} = U_z^{(y)} U^{(y)}_{\alpha 1^{(y)} - x}$. (3) $U_{B_{\alpha,x,y}(U_x w)} = U_{U_{\alpha x - U_x(y)}(w)} = U_{\alpha x - U_x(y)} U_w U_{\alpha x - U_x(y)} = \big(B_{\alpha,x,y} U_x\big) U_w \big(U_x B_{\alpha,y,x}\big) = B_{\alpha,x,y} U_{U_x(w)} B_{\alpha,y,x}$.

PROBLEM 1.9: (1) The only minimal inner ideals B in a nondegenerate algebra are of Idempotent Type II or Nilpotent Type III, and those of Nilpotent Type II are regularly (not just structurally) paired with a D of Idempotent Type. If $\mathrm{R} := \mathcal{R}ad(\mathrm{J}) \neq \mathbf{0}$, we can find a minimal $\mathrm{B} \neq \mathbf{0}$ inside, hence some $b \in \mathrm{B}$ is regularly paired with a nonzero idempotent e; but then $b \in \mathrm{R} \implies e = U_e b \in \mathrm{R}$, contradicting 1.7.2(1). (2) By nondegeneracy, $0 \neq b \in (c] \implies \mathbf{0} \neq (b] \subseteq (c] \implies (b] = (c] \ni b \implies b$ is a vNr. [Dually for open principal inners (x).] (3) Any nonzero inner I contains nonzero inner $(c]$'s, hence minimal ones, and therefore contains vNrs. (4) Since R can't contain nonzero vNrs, it must vanish.

QUESTION 1.3: $\widehat{\mathrm{J}} = \Phi\hat{e} \boxplus \mathrm{J}$ for $\hat{e} := \hat{1} - 1$, $\hat{1} - x = \hat{e} \boxplus (1 - x)$ is invertible in $\widehat{\mathrm{J}}$ iff $1 - x$ is invertible in J.

QUESTIONS 1.4: The closed case is heritable, because inclusion $[x] \subseteq [y]$ is equivalent to the elemental condition $x \in [y]$, and strict inclusion $[x] < [y]$ is equivalent to $x \in [y], y \notin [x]$. I don't know the answer for open or ordinary principal inner ideals.

Chapter 2. Begetting and Bounding Idempotents

EXERCISE 2.1.1A: (2) If $e = ax$ is idempotent, then $f = xea$ is idempotent with $afx = e$.

EXERCISE 2.1.1B: Every nonzero inner ideal contains a nonzero trivial element or a nonzero open (b), so all non-nilpotent B will contain an idempotent iff all these (b) do.

EXERCISE 2.2.2: $(x^m) = (x^{4m+2}) \Longrightarrow x^{2m+1} = U_{x^m}x \in (x^m) = (x^{4m+2}) = U_{x^{4m+2}}(\mathrm{J}) = U_{x^{2m+1}}U_{x^{2m+1}}(\mathrm{J})$ is a double vNr.

EXERCISE 2.4.1: $f < g \Longleftrightarrow 1-f > 1-g \Longleftrightarrow f \bullet g = f \Longleftrightarrow (1-f)\bullet(1-g) = 1 - f - g + f \bullet g = 1 - g$.

PROBLEM 2.2: Each element $x = \alpha_1 e_1 + \ldots + \alpha_n e_n$ has $(x] \subseteq \mathrm{I}_n := \boxplus_{k=1}^{n} \Phi e_k$ finite-dimensional, but the ideals $\mathrm{I}'_n := \boxplus_{k>n} \Phi e_k$ decrease strictly.

Chapter 3. Bounded Spectra Beget Capacity

EXERCISE 3.1.1A: (4) Let $\mathrm{J} = \mathrm{J}_1 \boxplus \mathrm{J}_2$ for unital subalgebras J_i; for $z \in \mathrm{I} = \mathrm{J}_1$ show that $\mathcal{R}es_{\Phi,\mathrm{J}}(z) = \mathcal{R}es_{\mathrm{J}_1}(z) \setminus \{0\}$, $\mathcal{S}pec_{\Phi,\mathrm{J}}(z) = \mathcal{S}pec_{\Phi,\mathrm{J}_1}(z) \cup \{0\}$, so in particular the unit e_1 of J_1 has $\mathcal{S}pec_{\Phi,\mathrm{J}_1}(e_1) = \{1\}$ but $\mathcal{S}pec_{\Phi,\mathrm{J}}(e_1) = \{1, 0\}$.

PROBLEM 3.3: (1) is just vector space theory. (2) Follows from $|\Gamma| \geq |\Omega| \geq |\Phi| > [\mathrm{J} : \Phi] \geq [\mathrm{J} : \Omega] \geq [\mathrm{J} : \Gamma]$.

PROBLEM 3.4: Repeat the Jordan proof, using in (1) $\mathrm{C}_i := \cap_{k\neq i} \mathrm{B}_k$ in place of C_{ij}, and in (2) $\mathrm{B}_\lambda := R_{\lambda\tilde{1}-z}\mathrm{A} = \mathrm{A}(\lambda\tilde{1} - z)$ in place of $U_{\lambda\tilde{1}-z}\mathrm{J}$. (3) If $\lambda \in f\text{-}\mathcal{S}pec_{\Phi,\mathrm{A}}(z(s))$ then, in particular, f vanishes at $\lambda\tilde{1} - z(s)$, and the polynomial $g(t) = f(t - z(s))$ vanishes at $\lambda \in \Phi \subseteq \Omega$, where $g(t)$ of degree N in $\Omega[t]$ ($\Omega := \Phi[s]$ an integral domain) cannot have more than N roots in Ω.

Chapter 4. Absorbers of Inner Ideals

EXERCISE 4.1.4: The intersection $I := \cap \mathrm{I}_\sigma$ is co-whatever-special because $\mathrm{J}/\mathrm{I} \hookrightarrow \prod_\sigma \mathrm{J}/\mathrm{I}_\sigma$ is isomorphic to a subalgebra of a direct product, and subalgebras and direct products inherit whatever-speciality (it is *quotients* that preserve i-speciality but not speciality).

EXERCISE 4.2.1A: $V_{r,s}(U_z x) = (U_{V_{r,s}z,z} - U_z V_{s,r})x \subseteq q(\mathrm{B})$ [by Fundamental Lie (FFII) and double absorption by z] and $U_r(U_z x) = (U_{\{r,z\}} + U_{U_r z,z} + U_z U_r - U_{\{r,z\},z} V_r + U_z V_{r^2})x$ [by Macdonald, or taking coefficients of λ^2 in the linearized Fundamental Formula] $\subseteq U_{\mathrm{B}}(\mathrm{J}) \subseteq \mathrm{B}$ [by innerness and absorption $\{r, z\}, U_r z \in \mathrm{B}$.]

EXERCISE 4.2.1B: See the proof of (2) in the Quadratic Absorber Theorem 4.2.1.

EXERCISE 4.3.3: Write $(-\frac{1}{2})(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-i+1) = (-\frac{1}{2})^i(1)(3)\cdots(2i-1)$ where $(1)(3)\cdots(2i-1) = (2i-1)!/\big((2)(4)\ldots(2i-2)\big)$ and $(2)(4)\cdots(2i-2) = 2^{i-1}(1)(2)\cdots(i-1) = 2^{i-1}(i-1)!$.

Chapter 5. Primitivity

EXERCISE 5.1.1: (2) $U_{\hat{1}-c}\hat{1}$ is never in $\mathrm{B} \subseteq \mathrm{J}$. (3) c modularity of the hull $\widehat{\mathrm{B}}^c$ requires $(\hat{1}-c)^2 = (\hat{1}-c) + (c^2 - c)$ and $\{\hat{1}-c, \widehat{\mathrm{J}}, \hat{1}-c\} = 2U_{\hat{1}-c}\widehat{\mathrm{J}}$ to lie in $\widehat{\mathrm{B}}^c$. (4) The modularity terms (Mod 1)–(Mod 3) for the contraction fall in J and are in B' by hypothesis. Note that if $c - c^2$ belongs to B' as well as $U_{\hat{1}-c}\hat{1}$, then $\hat{1}-c$ must belong too, so $\widehat{\mathrm{B}}^c \subseteq \mathrm{B}'$. Conversely, if $b' = \alpha\hat{1} + x \in \mathrm{B}'$ for $x \in \mathrm{J}$, then $b' - \alpha(\hat{1}-c) = \alpha c + x \in \mathrm{B}' \cap \mathrm{J} = \mathrm{B}$, so that $b' = \alpha(\hat{1}-c) + b \in \widehat{\mathrm{B}}^c$.

EXERCISE 5.4.1: Note that $z^{(2n,\tilde{y})} = \left(z^{(n,\tilde{y})}\right)^{(2,\tilde{y})}$ and also all $z^{(2n+k,\tilde{y})} = U_{z^{(n,\tilde{y})}} U_y z^{(k,\tilde{y})}$ for $k \geq 1$ vanish.

PROBLEM 5.2: (1) Note that $\{z^2, y^2\} \in \mathcal{R}ad(\mathrm{J}) \subseteq \mathcal{C}ore(\mathrm{B}) \subseteq \mathrm{B}$.

PROBLEM 5.3: Consider inverses of $1 - tz$ in the power series algebra $\widehat{\mathrm{A}}[[t]]$ and the polynomial subalgebra $\widehat{\mathrm{A}}[t]$.

PROBLEM 5.4: (1) If the result holds for $\overline{\mathrm{J}} := \mathrm{J}/\mathrm{I}$, then z p.n.b.i. in $\widetilde{\mathrm{J}} \Longrightarrow \bar{z}$ p.n.b.i. in $\overline{\mathrm{J}}[T] = \widetilde{\mathrm{J}}/\widetilde{\mathrm{I}} \Longrightarrow \bar{z} = \bar{0} \Longrightarrow z \in \widetilde{\mathrm{I}} \cap \mathrm{J} = \mathrm{I}$. (3) Choose (by infiniteness of T) a t which doesn't appear in the polynomial $\tilde{y}$; then identify coefficients of t in $z^{(2n-2,\tilde{y}+tx)} = 0$ to conclude that $U_{z^{(n-1,\tilde{y})}} x = 0$ for all $x \in \mathrm{J}$, so $U_{z^{(n-1,\tilde{y})}} \widetilde{\mathrm{J}} = 0$, forcing $z^{(n-1,\tilde{y})} = 0$ by (2).

Chapter 6. The Primitive Heart

QUESTION 6.1: The heart (associative and Jordan) consists of all finite matrices (those having only a finite number of nonzero entries whatsoever). The heart is not unital, but it is simple; it is better known as being the *socle* of A, the sum of all minimal left (equivalently, right or inner) ideals.

QUESTION 6.2: (1) The heart consists of all linear transformations of finite rank (finite-dimensional range). Again, this is the same as the socle, and the heart is simple but not unital. (2) is just the special case of (1) where V is a countable-dimensional right vector space over Δ with basis $\mathcal{B} = \{e_1, e_2, \dots\}$, where a linear transformation T is determined by its values $T(e_i)$; these are finite linear combinations of e_j's, and the finite collection of coefficients forms the ith column of the matrix $[T]_{\mathcal{B}\mathcal{B}}$ of T with respect to this basis.

QUESTION 6.3: You need to know a little functional analysis to come up with the answer. The heart is the space of *compact operators*, those operators that take the unit ball to a compact set; this is just the norm-closure of the ideal of operators of finite rank.

QUESTION 6.4: H will be the heart if it satisfies any of the following equivalent conditions: (i) its left (respectively, right) annihilator is zero, $a\mathrm{H} = \mathbf{0}$ (respectively, $\mathrm{H}a = \mathbf{0}) \Longrightarrow a = 0$; (ii) for all $a \neq 0$ there is $h \in \mathrm{H}$ with $ah \neq 0$ (respectively, $ha \neq 0$); (iii) H separates elements of A, $a_1 \neq a_2 \Longrightarrow \exists h \in \mathrm{H}$ with $a_1 h \neq a_2 h$ (respectively, $ha_1 \neq ha_2$); (iv) there is no disjoint ideal $\mathbf{0} \neq \mathrm{I} \triangleleft \mathrm{A}$, $\mathrm{I} \cap \mathrm{H} = \mathbf{0}$.

Chapter 7. Filters and Ultrafilters

PROBLEM 7.1: Y has finite or bounded complement iff $Y = X \setminus W$ with complement W a finite set F or a bounded set B, respectively. If $Y_i = X \setminus W_i$ then $Y_1 \cap Y_2 = X \setminus (W_1 \cup W_2)$. Any set with finite complement consisting of $n < \cdots < N$ or bounded complement $\leq N$ contains an interval (N, ∞).

PROBLEM 7.2: If $F = \{x_1\} \cup \cdots \cup \{x_n\} \in \mathcal{F}$ then some $\{x_{i_0}\} \in \mathcal{F}$ and $\mathcal{F} = \{Y \mid x_{i_0} \in Y\}$ is the principal ultrafilter $\mathcal{F}_{x_{i_0}}$.

PROBLEM 7.3: (Filt 1) $Y_1 \cap Y_2 \supseteq U_1 \cap U_2$ is still open containing x_0; (Filt 2) $Y' \supseteq Y \supseteq U \Longrightarrow Y' \supseteq U$; (Filt 3) $x_0 \notin \emptyset$ so $\emptyset \notin \mathcal{F}$.

Chapter 8. Ultraproducts

EXERCISE 8.2.3 (1) $z(y) \neq 0 \Longleftrightarrow z(y) \notin \mathcal{R}ad(Q_y) \Longleftrightarrow$ either (i) $Q_y(z(y)) \neq 0$ or (ii) there exists $a_(y)$ with $Q_y(z(y), a(y)) \neq 0$, i.e., $y \in Y_1 \cup Y_2$. By the basic property of ultrafilters, $\mathcal{S}upp(z) = Y = Y_1 \cup Y_2 \in \mathcal{F} \Longleftrightarrow Y_1 \in \mathcal{F}$ or $Y_2 \in \mathcal{F} \Longleftrightarrow Q'(z') \neq 0'$ or some $Q'(z', a') \neq 0'$. (2) Similarly, $z' \in \mathcal{R}ad(Q') \Longleftrightarrow Zer(Q(z)) = \{x \in X \mid Q_x(z(x)) = 0\}$ and $Zer(Q(z, \cdot)) = \{x \in X \mid Q_x(z(x), V_x) = \mathbf{0}\}$ belongs to $\mathcal{F} \Longleftrightarrow Zer(Q(z)) \cap Zer(Q(z, \cdot)) \in \mathcal{F} \Longleftrightarrow \{x \mid z(x) \in \mathcal{R}ad(Q_x)\} \in \mathcal{F} \Longleftrightarrow z \equiv_{\mathcal{F}} w \in \prod \mathcal{R}ad(Q_x)$.

PROBLEM 8.1. (1) Thinking of the imbedding as inclusion, the kernels $\mathrm{K}_j := \mathrm{Ker}(\pi_i) = \mathrm{A}_0 \cap \mathrm{A}_j$ are orthogonal ideals in A_0 because any of their products falls in $\mathrm{K}_1 \cap \mathrm{K}_2 \subseteq \mathrm{A}_1 \cap \mathrm{A}_2 = \mathbf{0}$. (2) For $i = 1, 2$ set $\mathrm{A}_i := \prod_{x_i \in X_i} \mathrm{A}_{x_i}$ and check $\mathrm{A} = \mathrm{A}_1 \boxplus \mathrm{A}_2$. For the case of n summands or unions, use induction.

QUESTION 8.1. All answers except to the first question are *Yes.*

Chapter 9. The Final Argument

PROBLEM 9.1: Note that characteristic p is finitely defined ($p \cdot 1 = 0$), whereas characteristic 0 is not (for all $n > 0$, $n \cdot 1 \neq 0$). In $\big(\prod_p \mathbb{Z}_p\big)/\mathcal{F}$, the characteristic will always be 0 *unless* the filter is the principal ultrafilter determined by some particular p, in which case the ultraproduct is $\mathbb{Z}_p$ again. In general, the ultraproduct $\prod \Phi_x/\mathcal{F}$ will have characteristic 0 unless the filter is "homogeneous," containing the set $X_p := \{x \in X \mid \Phi_x \text{ has characteristic } p\}$.

E.3 Hints for Part IV

Appendix B. Macdonald's Theorem

EXERCISE B.1.0B: The involution is never mentioned because it is the identity involution.

Part V

Indexes

Introduction

I will have no true index for this book, but instead five separate specialized indexes. The first is an *index of collateral readings.* This is not meant to be a formal Bibliography, nor even a Guide to Further Reading. It is primarily a guide to parallel readings in other textbooks. I list several standard textbooks on Jordan theory, and several articles of historical interest. I use the opportunity to inject a few miscellaneous thoughts on the historical development of Jordan structure theory.

The second index is a *pronouncing index of names*, indicating the pages where a mathematician's work is discussed. I do not include references to every occurrence of a result or object named after a given mathematician (so I do not list every time Albert algebras appear, or Macdonald's Theorem is used); the references are primarily to the mathematician's work as it appears in an historical context. A novel feature of this index is that I attempt to instruct American readers in the proper pronunciation of the names of the foreign mathematicians listed.

The third index is an *index of notations*, which contains symbols other than words which are used in the text, with a brief description of their meaning as a convenience for the reader, and a reference to a location where the notation is explained in detail.

The fourth index is an alphabetical list of each *named statement* (result or formula), giving a reference to the location of its statement or definition, but I do not attempt to list each of its occurrences in the text, nor do I include a restatement of result itself. The fifth and final index is a list of *definitions*, terms that have been defined in boldfaced type throughout the text, where again reference is given only to their page of definition.

There is no index which indicates where *topics* are *discussed.* Thus there is no reference to all places in the text where I discuss *semisimplicity* or *semisimple algebras* or *Albert algebras.*

A

Index of Collateral Readings

In this Index I list a few works on Jordan algebras that can be useful supplements to this book for beginning students. It is primarily a guide to other textbooks at the same level. I do list a few articles explicitly mentioned in the text for their historical value, though I must stress that I am not writing a history of Jordan theory, and do not cite a reference for each theorem I mention.

A.1 Foundational Readings

In the past, when graduate students inquired about research in the field of Jordan and nonassociative algebras, I recommended that they first look at the Carus Monograph *Studies in Modern Algebra* [1963A], especially the survey by Lowell Page of good old-fashioned finite-dimensional Jordan structure theory (using linear algebra arguments such as associative trace forms) and Charley Curtis's nice historical sketch of composition algebras and Jacobson's bootstrap proof of Hurwitz's Theorem, as well as Dick Schafer's *Introduction to Nonassociative Algebra* [1966S], a very readable account of general nonassociative, power associative, and Jordan algebras, with a full treatment of finite-dimensional alternative algebras. Recently, John Baez's eminently readable survey *The Octonions* appeared in the Bulletin [2002B]. For an overall view of developments I also referred students to three of my own survey articles, an article *Jordan algebras and their applications* [1973M], a talk *Quadratic methods in nonassociative algebra* [1975M] propagandizing on behalf of the new quadratic methods (not just in Jordan theory), and a survey of *The Russian revolution in Jordan algebras* [1984M]. One of my main purposes in writing this present book is to provide one general survey that covers all this ground.

Once students had gotten a taste of Jordan and nonassociative algebra, there were several texts where they could learn the foundations of the subject. Of course, the standard text where I learned my Jordan algebra was Jake's Colloquium volume *Structure and Representations of Jordan Algebras* [1968J], giving the definitive treatment of Jordan rings with d.c.c. Most subsequent books have simply quoted Jacobson when the proofs got long and technical. An exception was Hel Braun and Max Koecher's *Jordan Algebren* [1966BK], featuring a novel approach to Jordan algebras coming from number theory and differential geometry, which during the next ten years was to help reshape the Jordan landscape. An appendix to Jake's book sketched the brand-new quadratic theory, and his Tata Lecture Notes *Lectures on Quadratic Jordan Algebras* [1969J] were the first detailed exposition of that theory; unfortunately, these notes are almost unreadable due to typesetting (Jake never proofread the printed manuscript).

Tonny Springer's book *Jordan algebras and Algebraic Groups* [1973Sp] reduced Jordan algebras to algebraic groups: the algebraic structure of a Jordan algebra resided in its inversion map $j(x) = -x^{-1}$ satisfying the Hua Identity at $x = 1$ [in the beginning was the j, and Koecher said "Let there be U," and there was U, and the U was $U_x = \left(\partial j|_x\right)^{-1}$], and the gospel of inversion was carried from 1 to the dense set of invertible x's by means of a transitive structure group. This presents Jordan theory in a form congenial to researchers in group theory and geometry. In particular, the classification of simple Jordan algebras is derived from the classification of semisimple algebraic groups, and Peirce decompositions arise from representiations of tori (Peircers in disguise).

Ottmar Loos's elegant Springer Lecture notes *Jordan Pairs* [1975L] developed the entire structure theory for quadratic Jordan pairs with minimum condition on inner ideals; the triple and algebra structure could, with effort, be read off from this. (Loos, like Zel'manov after him, did not atempt to give an independent proof of the finite-dimensional algebra classification, particularly coordinatization and Albert algebras.) Erhard Neher's Springer Lecture Notes *Jordan Triple Systems by the Grid Approach* [1987N] is the definitive treatment of grids, the Peirce decompositions needed in Jordan triple systems; these have a rich combinatorial structure related to the root systems that are at the heart of Lie structure theory.

The book *Rings that are Nearly Associative* [1978S] by K. A. Zhevlakov, Arkadi Slin'ko, Ivan Shestakov, and Anatoli Shirshov (which appeared in English in 1982), gave a very detailed account of alternative and Jordan algebras on the eve of the Russian Revolution, with much emphasis on radicals, identities, and free algebras, but again left the final coordinatization stages of the classical theory of Jordan algebras with d.c.c. to Jake's book. Jacobson's Arkansas Lecture notes [1981J] contains the final version of the Classical structure theory, for quadratic Jordan algebras with capcacity. It also proves Zel'manov's Exceptional Theorem for quadratic Jordan algebras, and gives Zel'manov's classification of linear Jordan division algebras. This presents the state of Jordan structure theory just before Zel'manov's complete structure theory (in particular, the tetrad eaters) had been fully digested.

None of these works gives an accessible and comprehensive account of our general post-Zel'manov understanding of Jordan structure theory. Efim

Zel'manov's path-breaking papers [1979Z]-[1983Z] are, even today, a rich mine of insights and novel techniques, but (like most diamond deposits) they lie buried deep and are not easy to extract. Another of my purposes in writing this text is to present an overview of Zel'manov's revolutionary work, and a full exposition of his Exceptional Theorem. (To fully exposit his classification of *all* prime algebras would take another 200 pages and another semester-long graduate course.)

Suggested Foundational Readings

[1963A] A.A. Albert (ed.): *Studies in Modern Algebra,* M.A.A. Studies in Mathematics v. 2, Mathematical Association of America, 1963.

[2000B] J.C. Baez: *The Octonions,* Bull. Amer. Math. Soc. 39 (2002), 145-205.

[1966BK] H. Braun and M. Koecher: *Jordan–Algebren,* Grundl. der Math. v. 128, Springer Verlag, Berlin, 1966.

[1968J] N. Jacobson: *Structure and Representations of Jordan Algebras,* Amer. Math. Soc. Colloq. Publ. v. 39, Providence, 1968.

[1969J] — : *Lectures on Quadratic Jordan Algebras,* Lecture Notes, Tata Institute of Fundamental Research, Bombay, 1969.

[1981J] — : *Structure Theory of Jordan Algebras,* U. Arkansas Lecture Notes in Mathematics, v. 5, Fayetteville, 1981.

[1975L] O. Loos: *Jordan Pairs,* Lecture Notes in Mathematics v. 460, Springer Verlag, Berlin, 1975.

[1975M] K. McCrimmon: *Quadratic methods in nonassociative algebra,* Proc. Int. Cong. Math. 1974 v.1 (325-330), Canad. Math. Congress, Montreal 1975.

[1978M] — : *Jordan algebras and their applications,* Bull. Amer. Math. Soc. 84 (1978), 612-627.

[1984M] — : *The Russian revolution in Jordan algebras,* Algebra, Groups, and Geometries 1 (1984), 1-61.

[1987N] E. Neher: *Jordan Triple Systems by the Grid Approach,* Lecture Notes in Mathematics v. 1280, Springer Verlag, Berlin, 1987.

[1966S] R.D. Schafer: *An Introduction to Nonassociative Algebras,* Academic Press, New York, 1966.

[1973Sp] T.A. Springer: *Jordan Algebras and Algebraic Groups,* Ergebnisse der Math. v. 75, Springer Verlag, Berlin, 1973.

[1978S] K.A. Zhevlakov, A.M. Slin'ko, I.P. Shestakov, A.I. Shirsov: *Rings that are Nearly Associative*, Academic Press, New York, 1982 (Nauka, 1978).

A.2 Readings in Applications

The Colloquial Survey gives only sketchy outlines of certain applications, and is not intended to prepare students for work in those areas. Students interested in pursuing those subjects more deeply (here I deviate from my stated limitation to *collateral* reading) are encouraged to consult the following books and articles. (This list covers only the applications discussed in the Survey, so leaves out important applications to statistics, probability theory, differential equations, and other areas.) A good place to start is the collection of survey lectures in several areas given at the 1994 Oberwolfach Conference:

W. Kaup, K. McCrimmon, H. Petersson (eds.): *Jordan Algebras*, de Gruyter, Berlin, 1994.

Another collection of lectures on applications as well as theory is the Conference Proceedings from Santos González's big conference in Oviedo, symbolizing the strong new Spanish contributions to Jordan theory:[1]

S. González (ed.): *Nonassociative Algebra and its Applications*, Kluwer, Dordrecht, 1994.

The applications of Jordan algebras and octonion algebras to Lie algebras date from the inception of both theories, the study of Lie algebras as independent entities (not infinitesimal Lie groups) initiated by Hermann Weyl, and the study of abstract Jordan algebras (not formally real) by Adrian Albert in 1949. One detailed treatment of the ways that Jordan algebras arise in the construction of the exceptional Lie algebras is Jake's small book [1971J]. A nice survey article by John Faulkner and Joe Ferrar [1977FF] shows linkages between exceptional Jordan objects, Lie algebras, and projective geometries.

Applications to finite-dimensional differential geometry can be found in the books of Ottmar Loos [1969L], [1977L] and Wolfgang Bertram [2000B], and Max Koecher's Minnesota notes (as updated with notes by Alois Krieg and Sebastian Walcher in 1999)[2] [1962K]; applications to infinite-dimensional

[1] Warning: most of the articles here are pretty hard-core. I am told that a brief part of my survey lecture was chosen to appear on Spanish national television, though less because of popular enthusiasm for my subject of local Jordan algebras than because of the fact that I was one of the few speakers to be wearing a suit.

[2] If I may be permitted a personal remark, my own Yale Ph.D. came about quite accidently from Koecher's Minnesota notes. Each student in the 1964 Algebra Seminar had to talk on a chapter from the mimeographed notes; I was assigned Chapter 2, where a Jordan algebra was constructed from an ω-domain. Being weak in differential geometry, I presented an "algebraic" version of Koecher's proof using just the differential calculus (which works as well for rational maps over arbitrary fields as well as it does for real analytic ones). Jake said I should write up the result as my thesis. Luckily, I was quite unprepared to graduate in '64, so I remained a graduate student for another year, and got to go along with Jake's students when he visited the University of Chicago Algebra Year for the Fall 1964 semester. This proved to be a very exciting time, where I first saw the giants of the field (Albert, Kaplansky, Herstein, Amitsur, Cohn, Martindale, Taft, Maclane, Topping, and many others).

differential geometry are found in Harald Upmeier's book [1985U]. A good survey of the classical finite-dimensional theory and the transition to the infinite-dimensional theory is Wilhelm Kaup's survey article [2002K]. The Jordan triple approach to symmetric spaces and bounded symmetric domains in finite-dimensions is slowly gaining recognition on a par with the Lie approach (as witnessed in the books [1980S] of Ichiro Satake and [1994F] of Jacques Faraut and Adam Korányi), and is dominant in infinite-dimensions (where the Lie theory is not fully developed). The study of domains in infinite dimensions requires tools from functional analysis; good discussions of the importance of Jordan C^* algebras are Harald Hanche-Olsen and Erling Störmer's book [1978AS] giving a detailed treatment leading up to the Gelfand-Naimark theorem for Jordan algebras, and Harald Upmeier's book [1987U].

A treatment of the approach to octonion geometry in characteristic not 2 or 3 of Hans Freudenthal, Tonny Springer, and the Dutch school of geometers can be found in an updated version [2000SV] of the original lectures. The first application of quadratic Jordan algebras to this geometry appears in John Faulkner's Memoir [1970F].

Suggested Readings in Applications

[2000B] W. Bertram, *The Geometry of Jordan and Lie Structures*, Lecture Notes in Math. vol. 1754, Springer Verlag, Berlin, 2000.

[1994F] J. Faraut, A. Korányi: *Analysis in Symmetric Cones*, Oxford U. Press, 1994.

[1970F] J. Faulkner: *Octonion Planes Defined by Quadratic Jordan Algebras,* Memoirs of Amer. Math. Soc. v. 104, Providence, 1970.

[1977FF] J. Faulkner, J.C. Ferrar: *Exceptional Lie Algebras and related algebraic and geometric structures*, Bull. London Math. Sooc. 9 (1977), 1-35.

[1971J] N. Jacobson: *Exceptional Lie Algebras*, M. Dekker, New York, 1971.

[2002] W. Kaup: *Bounded symmetric domains and derived geometric structures*, lecture at conference *Harmonic Analysis on Complex Homogeneous Domains and Lie Groups*, to appear in Geometria.

[1962K] M. Koecher: *The Minnesota Notes on Jordan Algebras and their Applications,* Lecture Notes in Math. 1710, Springer Verlag, Berlin, 1999.

[1969L] O. Loos: *Symmetric Spaces I,II,* Benjamin, New York, 1969.

[1977L] — : *Bounded Symmetric Domains and Jordan Pairs,* Lecture Notes, U. Cal. Irvine, 1977.

[1984OS] H. Hanche–Olsen, E. Størmer: *Jordan Operator Algebras,* Monographs in Mathematics v. 21, Pitman, 1984.

[1980S] I. Satake: *Algebraic Structures of Symmetric Domains*, Princeton U. Press, 1980.

[2000SV] T. Springer, F. Veldkamp: *Octonions, Jordan Algebras, and Exceptional Groups,* Monographs in Math., Springer Verlag, Berlin, 2000.

[1985U] H. Upmeier: *Symmetric Banach manifolds and Jordan C^* Algebras,* North Holland Mathematics Studies 104, Elsevier, 1985.

[1987U] H. Upmeier: *Jordan Algebras in Analysis, Operator Theory, and*

Quantum Mechanics, C.B.M.S. Conference Report, Amer. Math. Soc, Providence, 1987.

A.3 Historical Perusals

Here I will just indicate a very few of the articles mentioned in the text connected with the historical development of Jordan structure theory; I will not list papers on applications except when they directly affected the algebraic theory itself. These papers are not meant to be read in their entirety; in keeping with the concept of collateral reading, these are primarily meant for the reader to skim through and notice the original form in which familiar concepts and results appeared. One good way to help get the big picture of a subject is to look back at its earlier stages, to see it arising from the mist of partial comprehension and then going on to take its final mature form.

Pascual Jordan set forth his program for finding an alternate algebraic foundation for quantum mechanics in [1933J], and the fruition (or stillbirth) of the program is found in the classical Jordan–von Neumann–Wigner paper [1934J] the next year. Topping's offhand introduction of inner ideals in [1965T] was followed quickly by the full Artin-Wedderburn-Jacobson theory in [1966J], followed rapidly in turn by the introduction of quadratic Jordan algebras in [1966M]. The subject was undergoing a growth spurt at this point, 1965-1970, only to be overshadowed later by the growth spurt 1978-1983 during the Russian revolution. A high point in this exciting time was the first of a long series of international conferences on Jordan theory held in Oberwolfach, Germany, in 1967.[3]

Linear Jordan triple systems were introduced by Kurt Meyberg in [1969My], and quadratic triples in [1972My]. The germ of Jordan pairs ("verbundene Paare") is found in [1969My], but these later came forth full-grown in Ottmar Loos's book [1975L]. Jordan superalgebras were a scene of feverish activity in the late 1970s, with the Victor Kac's classification appearing in [1977K]. The Gelfand-Naimark Theorem for JB-algebras proved in [1978AS] was a portent of things to come, but its true significance went unrecognized. The flood of papers during the Russian Revolution 1978-1983 is too vast to list in its entirety. I list here only the articles of Efim Zel'manov on the structure of Jordan algebras (not triples or pairs).

[3] If I may be permitted one last personal remark, when at this meeting I gave my first lecture on quadratic Jordan algebras (entitled "Was Sind und was Sollen die Jordan-Algebren", where "sind" meant linear and "sollen" meant quadratic), I had as yet no way to show that the hermitian 3×3 octonion matrices satisfied the quadratic axioms. Tonny Springer took me aside after lunch one day and carefully explained the methods he and Freudenthal used to describe the algebra and geometry of Albert algebras entirely in terms of the norm form and the adjoint, and suggested how the method should work equally well in characteristic 2. I went back home after the conference and worked through his suggestions, resulting in the paper [1969MS], which forms the basis for the treatment of degree–3 algebras over general rings of scalars in this book.

A Few Historical Articles

[1978AS] E.M. Alfsen, F.W. Shultz, E. Størmer: *A Gelfand-Neumark theorem for Jordan algebras,* Adv. Math. 28 (1978), 11-56.

[1966J] N. Jacobson: *Structure theory for a class of Jordan algebras,* Proc. Nat. Acad. Sci. U.S.A. 55 (1966), 243-251.

[1933J] P. Jordan: *Uber Verallgemeinerungsmöglichkeiten des Formalismus der Quantenmechanik,* Nachr. Ges. Wiss. Göttingen (1933), 29-64.

[1934J] P. Jordan, J. von Neumann, E. Wigner: *On an algebraic generalization of the quantum mechanical formalism,* Ann. Math. 36 (1934), 29-64.

[1977K] Victor Kac: *Classification of simple $\mathbb{Z}$-graded Lie superalgebras and simple Jordan superalgebras*, Comm. in Alg. 13 (1977), 1375-1400.

[1966M] K. McCrimmon: *A general theory of Jordan rings.* Proc. Nat. Acad. Sci. U.S.A. 56 (1966), 1071-1079.

[1969MS] K. McCrimmon: *The Freudenthal–Springer–Tits constructions of exceptional Jordan algebras,* Trans. Amer. Math. Soc. 139 (1969), 495-510.

[1969My] K. Meyberg: *Jordan-Tripelsysteme und die Koecher-Konstruktion von Lie Algebren*, Math. Z. 115 (1970), 115-132.

[1972My] K. Meyberg: *Lectures on algebras and triple systems*, Lecture notes, University of Virginia, Charlottesville, 1972.

[1965T] D.M. Topping: *Jordan Algebras of Self-Adjoint Operators,* Memoirs of Amer. Math. Soc. v.53, Providence, 1965.

[1978Z] E.I. Zel'manov: *Jordan algebras with finiteness conditions,* Alg. i Logika 17 (1978), 693-704.

[1979Z] — : *On prime Jordan algebras,* Alg. i Logika 18 (1979), 162-175.

[1979Z] — : *Jordan division algebras,* Alg. i Logika 18 (1979), 286-310.

[1983Z] — : *On prime Jordan algebras II,* Siber. Math. J. 24 (1983), 89-104.

B

Pronouncing Index of Names

Here I give references to the places in the book where a mathematician's work is discussed; these are divided into entries for: (O) Introduction (Colloquial Survey), (I) Part 1 (Historical Survey), (II) Part 2 (The Classical Theory), (III) Part 3 (Zel'manov's Exceptional Theorem), (ABCD) Appendixes.

I also give in each case a rough approximation of how that mathematician (alive or dead) would like to have his or her name pronounced. If I were delivering these lectures in person, you would hear an approximately correct pronunciation. But in aural absence, I include this guide to prevent readers (especially Americans) from embarrassing mistakes. The names of most American mathematicians in our story are pronounced in the good-old American way, and won't merit much explication. I have tried to make the pronunciations as phonetic as possible. A few sounds are hard to represent because they do not occur in English: I use (rr) to indicate a slightly trilled r; (ll) indicates a slightly liquid ell as found in French, Spanish, and to some extent Russian (soft ell); (y) indicates a brief y (voiced consonant) before the Russian vowel e.

A

B

J

K

L

M

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C

Index of Notations

In addition to the *Standard Notations* listed in the Preface, I gather here for reference a list of notations, symbols, and abbreviations introduced in the book. For each entry I give a brief definition or formula which usually suffices, but I include page references to places where the notation is explained in more detail. Since many of these notations are not standard in the literature, the ability to recognize them is important primarily in the context of this book. The reader will need to know what $\text{Abspec}_J(x)$ means in reading Part III Section 6, but will survive comfortably the rest of his or her life without that knowledge.

I have resisted the temptation to list every single symbol that occurs in the book; I list only those that appear several times, far away from their home of definition, and thus run the risk of not being recognized.

Elemental Products

$x \cdot y$	product in any linear algebra A
$x \bullet y$	symbol for basic product in a linear Jordan algebra; in special algebras it is the Jordan product or quasi-multiplication or anticommutator $\frac{1}{2}(xy+yx)$ [cf. 41]
x^n	nth power of element x in any power-associative algebra; defined recursively in linear algebras by $x \cdot x^{n-1}$, in quadratic algebras by $U_x(x^{n-2})$ [cf. 200]
x^{-1}	inverse $U_x^{-1}(x)$ of an invertible element x in a Jordan or alternative algebra [cf. 211]
$qi(x)$	quasi-inverse of x $((\hat{1}-x)^{-1} = \hat{1} - qi(x))$ [cf. 366]
$qi(x,y)$	quasi-inverse of pair (x,y) (quasi-inverse of x in the y-homotope) [cf. 369]
x^y	quasi-inverse $qi(x,y)$ of x in the y-homotope [cf. 369]
$[x,y]$	commutator or Lie bracket $xy - yx$ in linear algebras; symbol for basic product in a Lie algebra [cf. 55]
$[x,y,z]$	associator $(xy)x - x(yz)$ in linear algebras [cf. 56]
$[x,y,z]^+$	associator taken in a plus algebra A^+ [cf. C.1.1]
$[x,y,z]^{(u)}$	associator taken in a homotope $A^{(u)}$

$\{x,y\}$	anticommutator or brace product $\{x,1,y\}$ or $2x \bullet y$; reduces to $xy+yx$ in associative algebras [cf.147]
$\{x,y,z\}$	triple product or 3-tad; in a linear Jordan algebra, given by $2\big(x\bullet(y\bullet z)+(x\bullet y)\bullet z-(x\bullet z)\bullet y\big)$; in a quadratic Jordan algebra it is $U_{x,y}(z)$; it reduces to $xyz+zyx$ in associative algebras [cf. 5, 147]
$\{x,y,z,w\}$	tetrad $xyzw+wzyx$ in any associative algebra (NOT a Jordan product!) [cf. 449]
$\{x_1,\ldots,x_n\}$	n-tad product $x_1\cdots x_n+x_n\cdots x_1$ in any associative algebra (NOT a Jordan product if $n>3$) [cf. 449]
U_xz	basic U-product of quadratic Jordan algebras $(2x\bullet(x\bullet y)-x^2\bullet y)$; reduces to xzx in special algebras [cf. 5, 7, 81, 147]
$[[x,y],z]$	double commutator product $D_{x,y}z=\{x,y,z\}-\{y,x,z\}$; reduces to $[x,y]z-z[x,y]$ in special algebras [cf. 349]
$[x,y]^2$	square of commutator $U_{[x,y]}1$ [cf. 349]
$U_{[x,y]}z$	U operator of commutator $\big(U_{\{x,y\}}-2\{U_x,U_y\}\big)z$; reduces to $[x,y]z[x,y]$ in special algebras [cf. 349]
$[[x,y]^3,z]$	commutator of cube of commutator $\big(D^3_{x,y}+3D_{x,y}U_{[x,y]}\big)z$ in any Jordan algebra; reduces to $[x,y]^3z-z[x,y]^3$ in special algebras [cf. 349]
$x\bullet^{(u)}y$	bullet product in a Jordan homotope $\mathrm{J}^{(u)}$ $\big(x\bullet(u\bullet y)+(x\bullet u)\bullet y-(x\bullet y)\bullet u=\frac{1}{2}(\{x,u,y\}\big)$ [cf. 71, 86, 223]
$\{x,y\}^{(u)}$	brace product $\{x,u,y\}$ in a Jordan homotope $\mathrm{J}^{(u)}$ [cf. 86, 223]
$\{x,y,z\}^{(u)}$	triple product $\{x,U_uy,z\}$ in a Jordan homotope $\mathrm{J}^{(u)}$ [cf. 86, 223]
$U_x^{(u)}z$	U-product U_xU_uz in a Jordan homotope $\mathrm{J}^{(u)}$ [cf. 86, 223]
$1^{(u)}$	unit u^{-1} in the Jordan isotope $\mathrm{J}^{(u)}$ (if u invertible) [cf. 71, 86, 223]
x_uy	product xuy in a nuclear u-homotope A_u [cf. 72, 220]
1_u	unit u^{-1} in a nuclear isotope A_u (only if u invertible) [cf. 72, 220]
$*_u$	nuclear u-isotope of an involution $*$ $(x^{*_u}:=ux^*u^{-1})$ [cf. 72, 221]
s_x	symmetry of a symmetric space at x, multiplication by x in Loos' algebraic formulation of symmetric spaces [cf. 16]

Multiplication Operators

$ad_{x,y}$	Jordan derivation $D_{x,y}$ [cf. 349]
$ad_{[x,y]^3}$	adjoint map of the cube of a commutator $D^3_{x,y}+3D_{x,y}U_{[x,y]}$ in any Jordan algebra; reduces truly to $ad_{[x,y]^3}$ in special algebras [cf. 349]
$B_{x,y}$	Bergmann operator $\mathbb{1}_{\mathrm{J}}-V_{x,y}+U_xU_y$ in any Jordan system [cf. 205]
$B_{\alpha,x,y}$	generalized Bergmann operator $\alpha^2\mathbb{1}_{\mathrm{J}}-\alpha V_{x,y}+U_xU_y$ in any Jordan system [cf. 205]
$D_{x,y}$	inner Jordan derivation $V_{x,y}-V_{y,x}=[V_x,V_y]=4[L_x,L_y]$; reduces to $ad([x,y])$ in special algebras [cf. 349]. Also denotes the analogue of $V_{x,y}$ in Jordan pairs [cf. 9]
E_i $E_i(e)$	Peirce projections of J on Peirce subspaces $E_i(\mathrm{J})=\mathrm{J}_i$ relative to an idempotent e $(E_2=U_e,\ E_1=U_{1-e,e},\ E_0=U_{1-e})$ [cf. 236]
$E(t)$	Peircer with respect to idempotent e $(E(t)=U_{e(t)},e(t)=te+(1-e))$ [cf. 237]

E_{ij} Peirce projections of J on Peirce subspaces $E_{ij}(\mathrm{J}) = \mathrm{J}_{ij}$ relative to a
$E_{ij}(\mathcal{E})$ supplementary orthogonal family $\mathcal{E}$ of idempotents
($E_{ii} = U_{e_i}, E_{ij} = U_{e_i,e_j}$) [cf. 278]
$E(\mathbf{t})$ Peircer with respect to orthogonal family $\mathcal{E}$ ($E(\mathbf{t}) = U_{e(\mathbf{t})}, e(\mathbf{t}) = \sum_i t_i e_i$) [cf. 279]
L_x left multiplication operator $L_x(y) = x \cdot y$ in any linear algebra [cf. 55]
$L_{x,y}$ analogue of $V_{x,y}$ in Jordan triple systems [cf. 8]
$M_{p,q}$ multiplication operator on $\mathcal{FJ}[x, y, z]$ parameterized by monomials p, q in the variables x, y [cf. 462]
P_x analogue of U_x in Jordan triple systems [cf. 8]
Q_x analogue of U_x in Jordan pairs [cf. 9]
R_x right multiplication operator $R_x(y) = y \cdot x$ in any linear algebra [cf. 55]
$\mathcal{U}_{ij}$ hermitian symmetry automorphisms of J determined by a supplementary
$\mathcal{U}_\pi$ family of n hermitian matrix units (hermitian involutions $\mathcal{U}_{ij} = U_{u_{ij}}$, $u_{ij} = 1 - (h_{ii} + h_{jj}) + h_{ij}$, $\mathcal{U}_\pi = \mathcal{U}_{i_1j_1} \cdots \mathcal{U}_{i_rj_r}$ if $\pi = (i_1j_1) \cdots (i_rj_r)$) [cf. 303]
U_x basic U-operator of quadratic Jordan algebras ($2L_x^2 - L_{x^2}$); reduces to L_xR_x in special algebras [cf. 5, 7, 81, 147]
$U_{x,y}$ linearization $U_{x+y} - U_x - U_y$ of the U-operator, $U_{x,y}z = \{x, z, y\}$ [cf. 7, 81, 147]
$U_x^{(u)}$ U-operator U_xU_u in a homotope $\mathrm{J}^{(u)}$ [cf. 71, 86, 223]
$U_{[x,y]}$ U operator of a commutator $U_{\{x,y\}} - 2\{U_x, U_y\}$ [cf. 349]
V_x left brace multiplication $z \mapsto \{x, z\}$; in Jordan algebras it has equivalent forms $V_x = U_{x,1} = V_{x,1} = V_{1,x} = 2L_x$ [cf. 7 81, 147]
$V_x^{(u)}$ V-operator $V_{x,u}$ in a homotope $\mathrm{J}^{(u)}$ [cf. 71, 86, 223]
$V_{x,y}$ left triple multiplication $z \mapsto \{x, y, z\}$ by x, y in a Jordan triple product [cf. 7, 81, 147]
$V_{x,y}^{(u)}$ V-operator V_{x,U_uy} in a homotope $\mathrm{J}^{(u)}$ [cf. 71, 86, 223]

Operations with Idempotents

$e \leq f$ ordering of idempotents in a Jordan algebra ($e \bullet f = e, e \in \mathrm{J}_2(f)$) [cf. 384]
$e \perp f$ orthogonality of idempotents in an algebra ($e \bullet f = 0, e \in \mathrm{J}_0(f)$) [cf. 384]
$e_i \sim e_j$ orthogonal idempotents are connected or strongly connected (there is $v_{ij} \in \mathrm{J}_{ij}$ invertible or an involution in $\mathrm{J}_2(e_i + e_j)$) [cf. 297, 298]
$\{e_{ij}\}$ family of associative matrix units ($e_{ij}e_{k\ell} = \delta_{jk}e_{i\ell}$) [cf. 302]
$\{h_{ij}\}$ family of hermitian matrix units $h_{ij} = h_{ji}$ ($h_{ii}^2 = h_{ii}, h_{ij}^2 = h_{ii} + h_{jj}$, $\{h_{ij}, h_{jk}\} = h_{ik} (k \neq i), \{h_{ij}, h_{k\ell}\} = 0$ if $\{i, j\} \cap \{k, \ell\} = \emptyset$) [cf. 301]
σ_i Peirce specialization of J_i on J_1 ($\sigma_i(a) = V_{a_i}|_{\mathrm{J}_1}$) [cf. 248]
$\sigma\kappa(a_i)$ skewtrace action $V_{sktr(a_i)} = V_{a_i} - V_{\overline{a_i}} = \sigma_i(a_i) - \sigma_j(\overline{a_i})$ on J_{ij} [cf. 350]
q_i Peirce quadratic forms on J_1 ($q_i(x_1) = E_i(x_1^2)$) [cf. 250]
σ_{ij} Peirce specialization of J_{ij} on J_{ik} ($\sigma_{ij}(a_{ij}) = V_{a_{ij}}|_{\mathrm{J}_{ik}+\mathrm{J}_{jk}}$) [cf. 294]
q_{ii} Peirce quadratic forms on J_{ij} ($q_{ii}(x_{ij}) = E_{ii}(x_{ij}^2)$) [cf. 295]

Alphabetical Constructs

$\mathcal{A}\!f\!f(V)$ affine plane of a 2-dimensional vector space V or Δ^2 over a division
$\mathcal{A}\!f\!f(\Delta)$ ring Δ (points are vectors of V, lines are 1-dimensional affine subspaces, incidence is membership) [cf. 30]
$\mathcal{A}lb(\Phi)$ split albert algebra $\cong \mathcal{H}(\mathcal{O}(\Phi)) \cong \mathcal{H}_3(\mathcal{Z}orn(\Phi))$ over Φ [cf. 66]

$\mathcal{A}nn_{\mathrm{R}}(S)$	annihilator from R of a set S (elements $x \in \mathrm{R}$ which annihilate the set $S \subseteq M$ in a left R-module, $xS = 0$) [cf. 151, 307]
$\mathcal{A}ut(\mathrm{A})$	group of automorphisms of the algebra A (invertible linear T with $T(xy) = T(x)T(y)$) [cf. 134]
$\Gamma(\mathrm{A})$	centroid of A (the linear operators commuting with all multiplications) [cf. 142]
$\mathcal{C}ent(\mathrm{A})$	center of an algebra A (x with $[x, \mathrm{A}] = [x, \mathrm{A}, \mathrm{A}] = [\mathrm{A}, x, \mathrm{A}] = [\mathrm{A}, \mathrm{A}, x] = \mathbf{0}$) [cf. 56, 141]
$\mathcal{C}ent(\mathrm{A}, *)$	$*$-center of a $*$-algebra A (central elements with $x^* = x$) [cf. 141]
$\mathcal{C}ent_{\mathrm{A}}(S)$	centralizer of a set S in an algebra A ($x \in \mathrm{A}$ with $[x, S] = \mathbf{0}$) [cf. 142]
$\mathcal{C}ore(S)$	core of subset $S \subseteq \mathrm{A}$ (largest two-sided A-ideal contained in it) [cf. 413]
$\mathcal{D}eg(\mathrm{J})$	degenerate radical of a Jordan algebra (smallest ideal with nondegenerate quotient) [cf. 92]
$\mathcal{D}er(\mathrm{A})$	Lie algebra of derivations of the algebra A (linear D with $D(xy) = D(x)y + xD(y)$) [cf. 0.1, 11; 134]
$D(\mathrm{J})$	open unit disc of a formally real Jordan algebra (imbedded as the unit ball of $\mathrm{J}_{\mathbb{C}}$); bounded symmetric domain arising as the open unit ball $\{x \in \mathrm{J} \mid \mathbb{1}_{\mathrm{J}} - \frac{1}{2}L_{x,x} > 0\}$ of the positive hermitian triple system J [cf. 24]
$\mathcal{E}x(\mathrm{A})$	exchange $*$-algebra ($\mathrm{A} \boxplus \mathrm{A}^{op}$ with exchange involution $ex(a, b) = (b, a)$) [cf. 140]
$\mathcal{FA}[X]$	free unital associative algebra on a set X [cf. 447]
$\mathcal{FA}_0[X]$	free unitless associative algebra on a set X [cf. 454]
$\mathcal{FJ}[X]$	free unital Jordan algebra on a set X [cf. 455]
$\mathcal{FJ}_0[X]$	free unitless Jordan algebra on a set X [cf. 475]
$\mathcal{FSJ}[X]$	free special unital Jordan algebra on a set X [cf. 448]
$\mathcal{FSJ}_0[X]$	free special unitless Jordan algebra on a set X [cf. 454, 475]
$Half(\mathrm{J})$	Koecher's upper half space of formally real Jordan algebra J [cf. 23]
$\mathcal{H}(\mathrm{A})$	Jordan algebra of hermitian elements $x = x^*$ of a $*$-algebra A when the involution is understood, e.g., composition algebras [cf. 171]
$\mathcal{H}(\mathrm{A}, *)$	algebra of hermitian elements $x = x^*$ of a linear algebra A with involution [cf. 3, 46, 58, 168, 46, 58]
$\mathcal{H}_n(\mathrm{D}, -)$	Jordan matrix algebra $\mathcal{H}(\mathrm{A}, *)$ of hermitian elements of $\mathrm{A} = \mathcal{M}_n(\mathrm{D})$ for $*$ the standard matrix involution (conjugate transpose) [cf. 46, 59]
$\mathcal{H}_n(\mathrm{D}, \Gamma)$	"twisted" Jordan matrix algebra $\mathcal{H}(\mathrm{A}, *_\Gamma)$ of hermitian elements of $\mathrm{A} = \mathcal{M}_n(\mathrm{D})$ for $*_\Gamma$ the canonical involution (the Γ-isotope $x^{*\Gamma} = \Gamma x^* \Gamma^{-1}$ of the standard involution by a diagonal matrix Γ with diagonal entries invertible in the nucleus); it is isomorphic to the isotope $\mathcal{H}_n(\mathrm{D}, -)^{(\Gamma)}$ of the untwisted matrix algebra [cf. 73, 229]
$\mathcal{H}_3(\mathrm{D}, \Gamma)$	twisted matrix algebra; it is Jordan for alternative D whose hermitian elements are all nuclear [cf. 78, 230]
$\mathcal{I}_{\mathrm{A}}(S)$	ideal of the algebra A generated by a set S [cf. 403]
i-Specializer(A)	i-Specializer of a Jordan algebra (smallest ideal whose quotient is i-special) [cf. 116]
$\mathcal{J}ord(\mathrm{A}, \mu)$	cubic Jordan algebra of the First Tits Construction from a degree–3 associative Φ-algebra A [cf. 196]
$\mathcal{J}ord(\mathrm{A}, u, \mu, *)$	cubic Jordan algebra of the Second Tits Construction from a degree–3 associative Ω-algebra A with involution $*$ of second kind over Φ [cf. 196]

$\mathcal{J}ord(N, c)$	the Springer Construction of a degree–3 Jordan algebra $\mathcal{J}ord(N, c)$ from a Jordan cubic form N with basepoint c; it is just $\mathcal{J}ord(N, \#, c)$ for the adjoint $\#$ derived from the Jordan cubic N by $T(x^{\#}, y) = N(x; y)$, using nondegeneracy of $T(\cdot, \cdot)$ [cf. 77, 191]
$\mathcal{J}ord(N, \#, c)$	Jordan algebra of a sharp cubic form N with basepoint c and sharp mapping $(2x \bullet y = x\#y + T(x)y + T(y)x - S(x, y)c)$ [cf. 190, 481]
$\mathcal{J}ord(Q, c)$	degree–2 Jordan algebra of the quadratic form Q with basepoint c $(2x \bullet y = T(x)y + T(y)x - Q(x, y)c)$ [cf. 176]
$\mathcal{J}ord(Q^{(u)}, c^{(u)})$	u-isotope of the quadratic form $(Q^{(u)}(x) = Q(x)Q(u),\ T^{(u)}(x) = Q(x, \bar{u}),\ c^{(u)} = u^{-1})$ [cf. 76, 225]
$\mathcal{J}Spin(M, \sigma)$	Jordan spin factor $\Phi 1 \oplus M$ determined by a symmetric bilinear form σ on a Φ-module M $(v \bullet w = \sigma(v, w)1)$ [cf. 74, 178]
$\mathcal{J}Spin_n(\Phi)$	the original Jordan spin factor of Zorn, $\mathcal{J}Spin(V, \sigma)$ for $V = \Phi^n$, σ the ordinary dot product [cf. 3, 47, 58, 179]
$\mathcal{KD}(\mathrm{A}, \mu)$	the Cayley-Dickson algebra obtained by doubling the algebra A by the Cayley-Dickson Recipe [cf. 64, 160]
$\mathcal{L}oc(\mathrm{A})$	locally nilpotent (Levitzki) radical (maximal ideal which is locally nilpotent, all finitely-generated subalgebras are nilpotent) [cf. 107]
$\mathcal{M}[X]$	free monad on the set X (nonassociative monomials in the elements of X) [cf. 456]
$\mathcal{M}_n(\mathrm{D})$	linear algebra of $n \times n$ matrices with entries from a coordinate algebra D [cf. 456]
$\mathcal{M}_\infty(\mathrm{D})$	all $\infty \times \infty$ matrices with only finitely many nonzero entries from D [cf. 425]
$\mathcal{M}or_{\mathcal{C}}(X, Y)$	all morphisms from X to Y in the category $\mathcal{C}$ [cf. 132]
$\mathcal{M}ouf(\mathrm{O})$	the Moufang projective plane coordinatized by the octonion division algebra O [cf. 33]
$\mathcal{M}ult(\mathrm{A})$	multiplication algebra of A (unital subalgebra of $\mathcal{E}nd(\mathrm{A})$ generated by $\mathbb{1}_{\mathrm{A}}$ and all left and right multiplications L_a, R_a; in the Jordan case it is also generated by the V_x, U_x) [cf. 142]
$\mathcal{N}il(\mathrm{A})$	nil (Köthe) radical (maximal nil ideal) of the algebra A [cf. 109, 367]
$\mathcal{N}uc(\mathrm{A})$	nucleus of the algebra A (x with $[x, \mathrm{A}, \mathrm{A}] = [\mathrm{A}, x, \mathrm{A}] = [\mathrm{A}, \mathrm{A}, x] = \mathbf{0}$) [cf. 56, 141]
$\mathcal{PNBI}(\mathrm{A})$	set of properly–nilpotent–of–bounded–index elements of A [cf. 421]
$\mathcal{P}nil(\mathrm{A})$	set of properly nilpotent elements of A (nilp. in every homotope) [cf. 370]
$\mathcal{PQI}(\mathrm{A})$	set of properly quasi-invertible elements of A (q.i. in every homotope) [cf. 369]
$\mathcal{P}rime(\mathrm{A})$	prime or semiprime (Baer) radical of A (smallest ideal with semiprime quotient)
$\mathcal{P}roj(\mathrm{J})$	octonion projective plane determined by a reduced Albert algebra $\mathrm{J} = \mathcal{H}_3(\mathrm{O})$ (points are 1-dimensional inner ideals, lines are 10-dimensional inner ideals, incidence is inclusion) [cf. 34]
$\mathcal{P}roj(T)$	isomorphism $\mathrm{Proj}(\mathrm{J}) \to \mathrm{Proj}(\mathrm{J}')$ of projective planes induced by a structural $T : \mathrm{J} \to \mathrm{J}'$ of reduced Albert algebras [cf. 35]
$\mathcal{P}roj(V)$ $\mathcal{P}roj(\Delta)$	projective plane of a 3-dimensional vector space V or Δ^3 over an associative division ring Δ (points are 1-dimensional subspaces, lines are 2-dimensional subspaces, incidence is inclusion) [cf. 29]
$\mathcal{QI}(\mathrm{A})$	set of quasi-invertible elements of A ($\hat{1} - x$ invertible in $\widehat{\mathrm{A}}$) [cf. 366]

$\mathcal{R}ad(\mathrm{A})$ semiprimitive (Jacobson) radical of A (THE radical; smallest ideal with semiprimitive quotient, largest q.i. ideal, consists precisely of all p.q.i. elements) [cf. 89, 366]
$\mathcal{R}ad(Q)$ radical of the quadratic form Q (z with $Q(z) = Q(z, V) = 0$) [cf. 63]
$\mathcal{R}ed\mathcal{S}pin(q)$ reduced spin algebra of quadratic form q on Φ-module M ($\Phi e_1 \oplus M \oplus \Phi e_2$ with square $(\alpha, w, \beta)^2 := (\alpha^2 + q(w), (\alpha + \beta)w, \beta^2 + q(w))$) [cf. 181]
$\mathcal{S}eq(\mathrm{A})$ algebra of all sequences from A (same as $\prod_1^\infty \mathrm{A}$) [cf. 418]
$\mathcal{S}kew(\mathrm{A}, *)$ set of elements $x^* = -x$ skew with respect to the involution $*$ [cf. 171]
$\mathcal{S}pecializer(\mathrm{J})$ specializer of a Jordan algebra J (smallest ideal whose quotient is special) [cf. 399]
$\mathcal{S}trg(\mathrm{J})$ structure group $U(\mathrm{J})Aut(\mathrm{J})$ of a Jordan algebra J [cf. 12]
$\mathcal{S}trl(\mathrm{J})$ structure Lie algebra $L(\mathrm{J}) + Der(\mathrm{J})$ of a Jordan algebra J [cf. 12]
$\mathcal{TKK}(\mathrm{J})$ Tits–Kantor–Koecher Lie algebra $\mathrm{J}_{-1} \oplus Inder(\mathrm{J}) \oplus \mathrm{J}_1$ of J [cf. 13]
$\mathcal{Z}ann(S)$ Zel'manov annihilator in the Jordan algebra J of a set S (z with $\{z, S, \widehat{\mathrm{J}}\} = \mathbf{0}$) [cf. 108]
$\mathcal{Z}orn(\Phi)$ Zorn vector-matrix algebra over Φ (split octonion algebra) [cf. 158]

Symbolic Constructs

A^{op} opposite linear algebra, same space but opposite product ($x \cdot_{op} y = y \cdot x$) [cf. 140]
A^+ plus algebra of any linear algebra (A under the Jordan product $\frac{1}{2}(xy + yx)$); it is Jordan if A is associative or alternative [cf. 3, 46, 58, 168]
A^- minus algebra of any linear algebra (A under the Lie bracket $xy - yx$); it is Lie if A is associative
$\overline{\mathrm{A}}$ quotient or factor algebra A/I [cf. 53, 135, 148]
A_Ω scalar extension $\mathrm{A}_\Omega := \Omega \otimes_\Phi \mathrm{A}$ of the Φ-algebra [cf. 69, 135]
A^1 general unital hull $\mathrm{A}^1 = \Phi 1 + \mathrm{A}$ of a linear algebra [cf. 138]
$\widehat{\mathrm{A}}$ (formal) unital hull $\Phi 1 \oplus \mathrm{A}$ [cf. 52, 138]
$\widehat{\mathrm{A}}^\Omega$ (formal) unital hull $\Omega 1 \oplus \mathrm{A}$ of A considered as an Ω-algebra [cf. 138]
$\mathrm{A}[\varepsilon]$ algebra of dual numbers $\mathrm{A}[\varepsilon] = \mathrm{A} \otimes_\Phi \Phi[\varepsilon]$ for $\varepsilon^2 = 0$ [cf. 134]
A_{ij} associative Peirce spaces $e_i \mathrm{A} e_j$ [cf. 241, 285]
$\mathrm{A}[T]$ formal polynomials in indeterminates T with coefficients from A [cf. 417]
$\mathrm{A}[[T]]$ formal power series in indeterminates T with coefficients from A [cf. 417]
$\mathrm{A}(t)$ rational functions in t over A ($\mathrm{A}_{\Phi(t)}$ for rational function field $\Phi(t)$ over a field Φ)
A_u nuclear u-homotope of the algebra A (product $x_u y = xuy$) [cf. 72]
$\boxplus_I \mathrm{A}_i$ infinite direct sum of algebras [cf. 54, 135]
$\prod_I \mathrm{A}_i$ direct product of algebras [cf. 54, 135]
$\precapprox \prod_I \mathrm{A}_i$ subdirect product of algebras. [cf. 135]
$\heartsuit(\mathrm{A})$ heart of A (smallest nonzero ideal, if such exists) [cf. 119, 422]
$\mathrm{J}_i, \mathrm{J}_i(e)$ Peirce subspaces of J relative to idempotent e or supplementary
$\mathrm{J}_{ij}, \mathrm{J}_{ij}(\mathcal{E})$ orthogonal family $\mathcal{E}$ of idempotents (eigenspace for Peircer with eigenvalue t^i or $t_i t_j$) [cf. 99, 100 236, 280]
J^t Jordan triple system built out of a Jordan algebra J by taking triple
J^{t*} product $\{x, y, z\}$ or $\{x, y^*, z\}$ for an involution $*$ on J [cf. 8]

$J^{(u)}$	Jordan u-homotope of the algebra J (product $x \bullet^{(u)} y := \frac{1}{2}\{x,u,y\}$) [cf. 71, 223]
$J_{[u]}$	Jordan isotope $J^{(u^{-1})}$ with unit $1_{[u]} = u$ [cf. 15]
$J[T,T^{-1}]$	algebra $J \otimes \Phi[T,T^{-1}]$ of polynomials in T, T^{-1} [cf. 238, 283]
$\ell a(B)$ $\ell a^n(B)$	linear, higher linear absorber of an inner ideal B (the set of all $b \in B$ with $V^n_{\widehat{J}}(b) \subseteq B$) [cf. 116, 398]
$(N, \#, c)$	sharped cubic form [cf. 190]
$(N^{(u)}, \#^{(u)}, c^{(u)})$	u-isotope $(N(x)N(u), N(u)^{-1}U_{u^\#}x^\#, u^{-1})$ of a sharped cubic form [cf. 226]
$qa(B)$ $qa^n(B)$	quadratic, higher quadratic absorber of inner ideal B (the set of all $b \in B$ with $\left(V_{J,\widehat{J}} + U_{\widehat{J}}\right)^n(b) \subseteq B$) [cf. 116, 398]
$\Phi[T]_S$	localization of the polynomial ring at a monoid S [cf. 289]
$S^\perp$	orthogonal complement of the set S relative to given bilinear form σ (x with $\sigma(x,S) = 0$)
Y'	complement $Y' = X \setminus Y$ of a subset Y in ambient space X [cf. 427]

Elemental Constructs

$\mathcal{A}bs\,\mathcal{S}pec_{\Phi,J}(x)$ $\mathcal{A}bs\,\mathcal{S}pec(x)$	absorber spectrum of an element x (λ with $qa(U_{\lambda\hat{1}-x}J) = \mathbf{0}$) [cf. 402]
$d[ij]$	Jacobson box notation $dE_{ij} + \bar{d}E_{ji}$ for off-diagonal entries in Hermitian matrix algebra $\mathcal{H}_n(D)$ ($d \in D$ arbitrary) [cf. 174]
$d[ij]_\Gamma$	Jacobson Γ-box notation $\gamma_i dE_{ij} + \gamma_j \bar{d}E_{ji} \in \mathcal{H}_n(D,\Gamma)$ for $d \in D$ [cf. 229]
$\delta[ii]$	Jacobson box notation δE_{ii} for diagonal entries in Hermitian matrix algebra $\mathcal{H}_n(D)$ ($\delta \in \mathcal{H}(D,-)$) [cf. 174]
$\delta[ii]_\Gamma$	Jacobson Γ-box notation $\gamma_i \delta E_{ii} \in \mathcal{H}_n(D,\Gamma)$ for $\delta \in \mathcal{H}(D,-)$ [cf. 229]
$\mathcal{E}ig(x)$	eigenvalues of an element x (λ with $U_{\lambda\hat{1}-x}z = 0$ for some $z \neq 0$ in J) [cf. 121]
$f\text{-}\mathcal{S}pec(x)$	f-spectrum of x (λ with $f(U_{\lambda\hat{1}-x}J) = \mathbf{0}$, where $f(J) \neq \mathbf{0}$) [cf. 121, 390]
$N(x), n(a)$	cubic norm form on a degree–3 Jordan or associative algebra [cf. 76, 189, 195]
$\mathcal{R}es(x)$ $\mathcal{R}es_{\Phi,J}(x)$	resolvent of an element x (complement of spectrum) [cf. 389]
$\mathcal{S}pec(x)$ $\mathcal{S}pec_{\Phi,J}(x)$	Φ–spectrum of an element x (all scalars λ with $U_{\lambda\hat{1}-x}$ not invertible on J) [cf. 121, 388]
$sktr(x)$	skewtrace $x - x^*$ of x in a $*$-algebra [cf. 349]
$S(x), s(a)$	quadratic spur form of a cubic N or n [cf. 77, 189, 195]
$S(x,y)$, $s(a,b)$	bilinearization of the quadratic spur form of a cubic [cf. 189, 195]
$t(x), n(x)$	trace $x + x^*$, norm $n(x) = xx^*$ of x in a $*$-algebra [cf. 139, 349]
$T(x)$	trace form $T(x) = Q(x,c)$ of a quadratic form Q with basepoint [cf. 75, 156]
$T(x)$, $t(a)$	trace form $T(x) = N(c;x)$ of a cubic form N or n with basepoint [cf. 77, 189, 195]
$T(x,y)$, $t(a,b)$	trace bilinear form $T(x,y) = T(x)T(y) - N(c,x,y)$ of a cubic form N or n [cf. 77, 189, 195]

$\bar{x}$ often denotes coset $[x]_{\mathrm{I}} = \pi(x)$ of $x \in \mathrm{A}$ mod I (but it also indicates the image of x under an involution $\overline{}$) [cf. 53]
$x^{\#}, a^{\#}$ quadratic sharp mapping for a cubic form N or n [cf. 77]
$x\#y$ bilinearization of the sharp mapping [cf. 77]
(x) open principal inner ideal $U_x\mathrm{J}$ [cf. 205]
$(x]$ principal inner ideal $U_x(\widehat{\mathrm{J}}) = \Phi x^2 + U_x\mathrm{J}$ determined by x [cf. 87,205]
$[x]$ closed principal inner ideal $\Phi x + \Phi x^2 + U_x\mathrm{J}$ [cf. 205]
$x \bowtie y$ regularly paired elements $(U_x y = x, U_y x = y)$ [cf. 318]

Filters on Index Set of a Direct Product

$agree(x, y)$ agreement set for elements of a direct product (indices i where $x_i = y_i$) [cf. 126, 433]
$\mathcal{F}$ filter on a set X [cf. 427]
$\mathcal{F}|_Y$ restriction filter $\mathcal{F} \cap \mathcal{P}(Y)$ on $Y \in \mathcal{F}$ for a filter $\mathcal{F}$ on X [cf. 428]
$\mathcal{F} \cap Y$ intersection filter $\mathcal{F} \cap Y = \{Z \cap Y | Z \in \mathcal{F}\}$ on $Y \in \mathcal{F}$ for a filter $\mathcal{F}$ on X [cf. 428]
$\overline{\mathcal{F}_0}$ enlargement filter of downward-directed collection $\mathcal{F}_0$ of subsets (all subsets containing some $Y \in \mathcal{F}_0$) [cf. 428]
$\mathcal{F}(\mathrm{A}_0)$ support filter of a prime subalgebra A_0 of a direct product (generated by the support sets of nonzero elements of A_0, which are downward-directed by primeness) [cf. 127, 429]
$\equiv_{\mathcal{F}}$ equivalence relation induced on direct product by a filter ($x \equiv_{\mathcal{F}} y$ iff $x_i = y_i$ for all i in some set $Z \in \mathcal{F}$) [cf. 433]
$(\prod \mathrm{A}_x)/\mathcal{F}$ filtered product of algebras $((\prod \mathrm{A}_x)/\equiv_{\mathcal{F}})$ [cf. 433]
$\mathrm{I}(\mathcal{F})$ filter ideal (ideal of elements equivalent to 0 mod the filter) [cf. 126, 433]
$\mathcal{S}upp(x)$ support set of a function or element of a direct product (indices i where $x_i \neq 0$) [cf. 126, 429]
$\mathcal{S}upp(\mathrm{A}_0)$ collection of all support sets $\mathcal{S}upp(a_0)$ of nonzero elements in the prime subalgebra A_0 of a direct product [cf. 429]
$\mathcal{Z}ero(x)$ zero set of a function or element of a direct product (indices i where $x_i = 0$) [cf. 126, 429]

Lists

(AInv1), (AInv2): Associative Inverse conditions (AInv1) $xy = 1$, (AInv2) $yx = 1$ [cf. 213].
(AltAX1)–(AltAX3): Alternative axioms (AltAX1) left alternative $x(xy) = x^2y$, (AltAX2) right alternative $(yx)x = yx^2$, (AltAX3) flexible $(xy)x = x(yx)$ (a consequence of the first two) [cf. 60, 153]
(A1)–(A4): axioms for a cubic norm form on an associative algebra (A1) Degree–3 Identity; (A2) Trace-Sharp Formula; (A3) Trace–Product Formula; (A4) Adjoint Identity [cf. 194]
$A_n, B_n, C_n, D_n; G_2, F_4, E_6, E_7, E_8$ 4 Great Classes and 5 Sporadic Exceptions of simple Lie algebras (respectively, groups): A_n is matrices of trace 0 (resp. determinant 1), B_n, D_n are the skew matrices $T^* = -T$ (respectively, isometric $T^* = T^{-1}$) with respect to a nondegenerate symmetric, C_n with respect to a nondegenerate skew-symmetric bilinear form [cf. 12]
(FAQ1)–(FAQ6): Frequently Asked Questions settled by the Russian Revolution [cf. 107]
(FFI)–(FFV): fundamental operator formulas (FFI) Fundamental Formula; (FFII) Commuting Formula; (FFIII) Triple Shift Formula; (FFIV) Triple Switch Formula; (FFV) Fundamental Lie Formula; (FFI)′ Alternate Fundamental Formula, (FFIII)′ Specialization Formulas; (FFV)′ 5-Linear Identity [cf. II.5.2.3, 202 for full details]
(FFIe)–(FFVe): fundamental element identities (FFIe) Fundamental Identity; (FFIIe) Commuting Identity; (FFIIIe) Triple Shift Identity; (FFIVe) Triple Switch Identity; (FFVe)′ 5-Linear Identity [cf. II.5.2.3, 202 for full details]

$G_8, G_9, \text{Ш}_8, \text{Ш}_9, T_{10}, T_{11}$ s-Identities: Glennie's identities $H_n(x,y,z) = H_n(y,x,z)$ where $H_8(x,y,z) := \{U_xU_yz, z, x \bullet y\} - U_xU_yU_z(x \bullet y)$, $H_9(x,y,z) := 2U_x(z) \bullet U_{y,x}U_z(y^2) - U_xU_zU_{x,y}U_y(z)$; Shestakov's identities $[[x,y]^3, z^2] = \{z, [[x,y]^3, z]\}$, $[[x,y]^3, z^3] = \{z^2, [[x,y]^3, z]\} + U_z[[x,y]^3, z]$; Thedy's identities $T_{11}(x,y,z,w) = T_{10}(x,y,z)(w)$, $T_{10}(x,y,z) = U_{U_{[x,y]}z} - U_{[x,y]}U_zU_{[x,y]}$ [cf. 5, 469]

(Filt 1)–(Filt 3): conditions for a filter on X (closed under intersection; enlargement; does not contain $\emptyset$) [cf. 125, 427]

(Filt 1)$'$, (Filt 2)$'$, (Filt 2)$''$, (Filt 3)$'$: auxiliary filter conditions (closed under finite intersections; closed under unions with any subset; contains X; not $\mathcal{P}(X)$) [cf. 427]

(JAX1), (JAX2): Jordan axioms (JAX1) commutativity $x \bullet y = y \bullet x$, (JAX2) Jordan identity $x^2 \bullet (y \bullet x) = (x^2 \bullet y) \bullet x$ [cf. 45, 57, 146]

(JAX2)$'$, (JAX2)$''$: first, second linearizations of (JAX2) (of degree 2, 1 in x) [cf. 148].

(KD0)–(KD4) Cayley-Dickson product rules for the Cayley-Dickson Recipe: (KD0) $am = m\bar{a}$; (KD1) $ab = ab$; (KD2) $a(bm) = (ba)m$; (KD3) $(am)b = (a\bar{b})m$; (KD4) $(am)(bm) = \mu\bar{b}a$ [cf. 160]

(LAX1), (LAX2): Lie axioms (LAX1) skewness $[x,y] = -[y,x]$, (LAX2) Jacobi identity $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ [cf. 57]

(LJInv1), (LJInv2): Linear Jordan Inverse conditions (LJInv1) $x \bullet y = 1$, (LJInv2) $x^2 \bullet y = x$ [cf. 70, 215]

(Mod 1)–(Mod 3), (Mod 3a), (Mod3b): conditions for c to be modulus for inner ideal B in J (Mod1) $U_{1-c}\mathrm{J} \subseteq \mathrm{B}$; (Mod2) $c - c^2 \in \mathrm{B}$; (Mod3) $\{1-c, \widehat{\mathrm{J}}, \mathrm{B}\} \subseteq \mathrm{B}$; (Mod3a) $\{1-c, \mathrm{J}, \mathrm{B}\} \subseteq \mathrm{B}$; (Mod3b) $\{c, \mathrm{B}\} \subseteq \mathrm{B}$ [cf. 118; 411]

q.i., *p.q.i.*, *p.n.*, *p.n.b.i.*: property of an element being quasi-invertible, properly quasi-invertible, properly-nilpotent, properly nilpotent of bounded index ($\hat{1} - x$ is invertible in $\widehat{\mathrm{J}}$; x is q.i. in each homotope; x is nilpotent in each homotope; x is p.n. of bounded index (there is n with $x^{(n,y)} = 0$ for all $y \in \mathrm{J}$)) [cf. 366; 369; 369; 369]

(QInv1), (QInv2): Quasi-Inverse conditions (QInv1) $U_{\hat{1}-z}w = z^2 - z$; (QInv2) $U_{\hat{1}-z}w^2 = z^2$ [cf. 368].

(QInvP1), (QInvP2): Quasi-Inverse Pair conditions (QInvP1) $B_{z,y}w = U_zy - z$, (QInvP2) $B_{z,y}U_wy = U_zy$ [cf. 370]

(QJAX1)–(QJAX3): unital quadratic Jordan axioms (QJAX1) $U_1 = \mathbb{1}_{\mathrm{J}}$, (QJAX2) $V_{x,y}U_x = U_xV_{y,x}$, (QJAX3) $U_{U_xy} = U_xU_yU_x$ [cf. 0.1, 7; 83]

(QJInv1), (QJInv2): Quadratic Jordan Inverse conditions (QJInv1) $U_xy = x$; (QJInv2) $U_xy^2 = 1$ [cf. 85, 211, 363].

$q(x,y), N(x;y), N(x,y,z)$: linearization of a quadratic map q, first, complete linearization of a homogeneous cubic N [cf. 74/187]

$\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{K}; \mathbb{A}; \mathcal{M}_{12}(\mathbb{K})$ reals, complexes, Hamilton's quaternions, Cayley's octonions (real composition division algebras); Albert algebra $\mathbb{A} = \mathcal{H}_3(\mathbb{K})$ of hermitian 3×3 matrices over Cayley algebra (27-dimensional simple exceptional Jordan algebra); bi-Cayley Jordan triple (16-dimensional simple exceptional triple system) [cf. 4, 8]

(RFilt): restriction filter [cf. 125]

(UFilt 1)–(UFilt 4) equivalent conditions for an ultrafilter: (UFilt 1) maximal filter; (UFilt 2)/(UFilt 3) if a union of two/finite number of sets belongs to $\mathcal{F}$ then one is already in $\mathcal{F}$; (UFilt 4) for any subset Y, either Y or its complement lies in $\mathcal{F}$ [cf. 126, 431]

$\mathcal{U}(\Phi), \mathcal{B}(\Phi), \mathcal{Q}(\Phi), \mathcal{O}(\Phi)$ split composition algebras: split unarions, binarions, quaternions, octonions of dimension 1,2,4,8 over Φ [cf. 66, 157]

D

Index of Statements

The following list contains an entry for each Statement that has been christened with a proper name in the text. Named statements include Theorems, Propositions, Lemmas, Examples, Definitions, and named formulas. For each entry I give the proper name of the entry, its numerical tag, and a reference to the page(s) where the statement is made, but I don't repeat its statement. The citation gives only the page(s) where the statement is made, not to all the places it is discussed or used. The numerical tag lists Part, Chapter, Section, and Statement number, with parts of statements indicated by parentheses (e.g., I.2.3.4 or I.2.3.4(5)); the Colloquial Survey has no Part and no Chapter, it is designated by O and the section number (e.g., O.1).

Besides being a reference for locating statements, the list provides a useful review and summary of the basic *results* of Jordan theory. Going through the list will test the student's ability to formulate and recall these facts, and reinforce them in memory. Of course, many of the entries are of lesser importance, useful only for students intending to pursue research in specific aspects of Jordan structure theory; they can stay safely buried in this index, to be looked up rather than stored at the tip of the tongue.

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Index of Definitions

The following is a list of definitions, terms that appeared in bold-faced type in the book. The citation gives for each entry only the page(s) where the term is defined, not all the places it is discussed or used, nor a repetition of the definition. As with the Index of Statements, this can provide a useful review and summary of the basic *terms* of Jordan theory. Going through the list will test the student's ability to formulate and recall these facts, and reinforce them in memory.

A

B

C