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A TAUBERIAN CONSTANT FOR THE (S, μ_{n+1}) TRANSFORMATION

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1. Corresponding to any fixed sequence $\{\mu_n\}$, Ramanujan [17] introduced the summability method given by the sequence-to-sequence transformation

$$t_n = \sum_{k=0}^{\infty} \binom{n+k}{n} (\Delta^k \mu_{n+1}) s_k , \tag{1.1}$$

where

$$\Delta\mu_n = \mu_n - \mu_{n+1},\ \Delta^0\mu_n = \mu_n,\ \Delta^k\mu_n = \Delta(\Delta^{k-1}\mu_n) . \tag{1.2}$$

Writing

$$t_n = b_0 + b_1 + \cdots + b_n ;\ s_k = a_0 + a_1 + \cdots + a_k , \tag{1.3}$$

we shall see in §2 that (1.1) is formally the same as

$$\begin{cases} b_0 = \sum_{k=0}^{\infty} (\Delta^k \mu_0)\, a_k , \\ b_n = \sum_{k=1}^{\infty} \binom{n+k-1}{n} (\Delta^k \mu_n)\, a_k , \qquad (n \geqq 1) . \end{cases} \tag{1.4}$$

We shall also see that for those sequences for which, for every fixed n

$$\binom{n+k-1}{n} (\Delta^k \mu_n) s_k \to 0 , \qquad \text{as} \quad k \to \infty , \tag{1.5}$$

(1.1) and (1.4) are, in fact, equivalent in the following sense. Suppose that (1.5) holds. Then, if (1.1) converges for all n, (1.4) holds; and if (1.4) converges for all n, (1.1) holds.

It will be convenient to change Ramanujan's notation slightly, and to denote the summability method given by the series-to-series transformation (1.4) by

(S, μ_{n+1}). The necessary and sufficient conditions under which (1. 1) is regular have been given by Ramanujan [17]; and it follows from a well known result (see Cooke [7, pp 64-68]) that (S, μ_{n+1}) is *regular* if (1. 1) is.

The object of § 3.4 is to obtain the best estimate of $|t_n - s_m|$ where t_n is defined by (1. 4) and s_m is the partial sum of a series satisfying the Tauberian condition

$$a_n = O(1/n). \tag{1.6}$$

We remark that, if

$$\binom{n+k-1}{n}(\Delta^k \mu_n) = o\left(\frac{1}{\log k}\right), \quad \text{as } k \to \infty, \tag{1.7}$$

then (1. 5) holds for every series satisfying (1. 6); and thus the theorems of § 3 will apply also to the sequence-to-sequence transformation (1.1). In section 5, we obtain sufficient conditions under which (1. 7) holds.

We may observe the result of § 3,4 includes a "o" Tauberian theorem for the (S, μ_{n+1}) transformation. It includes also a "O" Tauberian result for the (D, α) transform defind by Ishiguro ([11], p. 15), since the (S, μ_{n+1}) transformation reduces to the (D, α) transformation when $\mu_n = \binom{n+\alpha}{n}^{-1}$.

Theorems of this type were first considered by Hadwiger [9], and have since been dealt with by various authors; see for example (Agnew [2], [3]), Anjaneyulu [5], Jakimovski [12], Rajagopal [15] and Tenenbaum [20].

The special case of our result in which $\mu_{n+1} = r^{n+1}$ $(0 < r < 1)$, and which is called the $s(\alpha)$-transform of Meyer-König [14], or what is called also the $F(\alpha)$ of Laurent-series continuation introduced by Vermes [21] has been dealt with by Biegert [6]. This special case has also been considered by Anjaneyulu [4], but with the weaker Tauberian Condition

$$a_n = O(1/\sqrt{n}).$$

Some similar theorems heve been obtained with (1. 6) replaced by the Tauberian condition

$$\gamma_n = O(1),$$

where we write

$$\gamma_n = \frac{1}{n+1}\sum_{\nu=1}^{n} \nu a_\nu,$$

see for example Delange [8], Rajagopal [16], Meir [13] and Sherif ([18], [19]).

Finally, I have much pleasure in expressing my gratitude to Dr. B. Kuttner for his criticisms and suggestions for improvements to present this paper.

2. Let us consider the transformation (1.1), such that (1.3) holds. Then formally

$$t_n = \sum_{r=0}^{\infty} \binom{n+r}{n} (\Delta^r \mu_{n+1}) \sum_{k=0}^{r} a_k$$
$$= \sum_{k=0}^{\infty} a_k \sum_{r=k}^{\infty} \binom{n+r}{n} (\Delta^r \mu_{n+1}).$$

Thus

$$b_n = t_n - t_{n-1} = \sum_{k=0}^{\infty} \alpha_{n,k} a_k , \tag{2.1}$$

where

$$\alpha_{n,k} = \sum_{r=k}^{\infty} \left\{ \binom{n+r}{n} \Delta^r \mu_{n+1} - \binom{n-1+r}{n-1} \Delta^r \mu_n \right\}. \tag{2.2}$$

Here $\binom{n-1+r}{n-1}$ must be taken to mean 0 for all r including $r=0$ when $n=0$. Now

$$\Delta^r \mu_{n+1} = \Delta^r \mu_n - \Delta^{r+1} \mu_n ,$$

so that the expression in curly brackets in (2.2) is equal to

$$\left\{ \begin{array}{l} \left\{ \binom{n+r}{n} - \binom{n-1+r}{n-1} \right\} \Delta^r \mu_n - \binom{n+r}{n} \Delta^{r+1} \mu_n \\ = \binom{n+r-1}{n} \Delta^r \mu_n - \binom{n+r}{n} \Delta^{r+1} \mu_n . \end{array} \right. \tag{2.3}$$

Here $\binom{n+r-1}{n}$ must be taken as meaning 1 for all r including $r=0$ when $n=0$. Assuming that, for fixed n,

$$\binom{n+r}{n} \Delta^{r+1} \mu_n \to 0$$

as $r \to \infty$, it follows that

$$\alpha_{n,k} = \binom{n+k-1}{n} \Delta^k \mu_n, \tag{2.4}$$

which gives (1. 4).

THEOREM 2.1. *If* (1. 5) *holds, then* (1. 1) *is equivalent to* (1. 4) (*in the sense that if* (1. 1) *converges for all* n *then so does* (1. 4), *and conversely, and that the sums are related by* (1. 3)).

For the proof of Theorem 2.1, we require the following lemma.

LEMMA 2.1. *Let any series-to-series transformation* b_n *be such that*

$$b_n = \sum_{k=0}^{\infty} \alpha_{n,k} a_k . \tag{2.5}$$

Suppose that

$$\alpha_{n,k} s_k \to 0, \quad \text{as} \quad k \to \infty . \tag{2.6}$$

Then (2. 5) *and the transformation*

$$t_n = \sum_{k=0}^{\infty} \gamma_{n,k} s_k , \tag{2.7}$$

where

$$\gamma_{n,k} = \sum_{\nu=0}^{n} (\alpha_{\nu,k} - \alpha_{\nu,k+1}) , \tag{2.8}$$

are equivalent (*in the sense that if* (2. 5) *converges for all* n *then so does* (2. 7), *and conversely*).

PROOF. The convergence of (2. 5) for all n is equivalent to the convergence for all n of

$$t_n = \sum_{k=0}^{\infty} \beta_{n,k} a_k , \tag{2.9}$$

where we write

$$\beta_{n,k} = \alpha_{0,k} + \alpha_{1,k} + \cdots + \alpha_{n,k} ; \tag{2.10}$$

(see Vermes [19]), and if (2. 5), (2. 9) as converge, then the first equality of (1. 3) holds.

[The convergence of (2. 5) for one particular n is not equivalent to the convergence of (2. 9) for that particular n.]

Now

$$\sum_{k=0}^{K} \beta_{n,k} a_k = \sum_{k=0}^{K} \beta_{n,k}(s_k - s_{k-1})$$

$$= \sum_{k=0}^{K-1} \gamma_{n,k} s_k + \beta_{n,K} s_K$$

where we write

$$\gamma_{n,k} = \beta_{n,k} - \beta_{n,k+1} . \tag{2. 11}$$

Thus the equation

$$\sum_{k=0}^{\infty} \beta_{n,k} a_k = \sum_{k=0}^{\infty} \gamma_{n,k} s_k$$

will be valid for a given n in the sense that if either side converges then the other side converges, and has the same sum if (and only if) for that n

$$\beta_{n,k} s_k \rightarrow 0, \quad \text{as} \quad k \rightarrow \infty . \tag{2. 12}$$

This establishes our result (when (2. 5), (2. 7) as converge then the first equality of (1. 3) holds) provided that, for all n (2. 12) holds.

We also note that by the definition of $\beta_{n,k}$ the assertion that (2. 12) holds for all n is equivalent to the assertion that for all n, (2. 6) holds.

We are now in a position to prove Theorem 2.1. In the special case considered in Theorem 2.1,

$$\alpha_{\nu,k} = \binom{\nu+k-1}{\nu} \Delta^k \mu_\nu ,$$

where, in the case $k=0$, we take

$$\binom{\nu+k-1}{\nu} = \begin{cases} 0 & (\nu \geqq 1) \\ 1 & (\nu = 0) . \end{cases}$$

Now, by an easy deduction similar to that of equation (2. 3) we can show that

$$\gamma_{n,k}=\binom{n+k}{n}\Delta^k\mu_{n+1}\,.$$

The result now follows from Lemma 2.1.

3.

THEOREM 3.1. *Let* $\{\mu_n\}$ *be a moment sequence generated by the function* $\chi(t)$ $(0\leqq t\leqq 1)$ *such that*

$$\mu_0=\int_0^1 d\chi(t)=1\,, \tag{3.1}$$

$$\mu_n=\int_0^1 t^n\,d\chi(t)\,, \tag{3.2}$$

$$\chi(0)=0,\quad \chi(1)=1,\quad \chi(t)\ \textit{of bounded variation}, \tag{3.3}$$

$$\int_0^1 \frac{|\chi(t)|}{t}\,dt<\infty\,, \tag{3.4}$$

and

$$\int_0^1 \frac{|1-\chi(t)|}{1-t}\,dt<\infty\,. \tag{3.5}$$

Let $\sum a_n$ *be a series satisfying the Tauberian condition* (1.6). *Then*

(i) *the series* (1.4) *converges for each* $n=0,1,2,\cdots$.

(ii) *If* s_m *and* t_n *denote respectively the partial sums of the series* $\sum a_n$ *and* $\sum b_n$, *there exists a finite constant* $A(\alpha)$ $(0<\alpha<\infty)$ *such that*

$$\limsup_{\substack{n\to\infty\\ m/n\to\alpha}}|t_n-s_m|\leqq A(\alpha)\limsup_{n\to\infty}|n\,a_n|\,; \tag{3.7}$$

(iii) *further, if the function* $\chi(t)$ *satisfies the additional condition*

$$0\leqq\chi(t)\leqq 1\,, \tag{3.8}$$

then $A(\alpha)$ *is given by*

$$\int_{1/(\alpha+1)}^1 \frac{1-\chi(t)}{t(1-t)}\,dt+\int_0^{1/(\alpha+1)}\frac{\chi(t)}{t(1-t)}\,dt\,, \tag{3.9}$$

and is the best possible constant in (3.7) *in the sense that there is a series* $\sum a_n$ *such that* $0 < \limsup_{n\to\infty} |n a_n| < \infty$ *and the members of* (3.7) *are equal.*

We remark that, since (3.4), (3.5) clearly require that

$$\chi(0+) = 0, \qquad \chi(1-) = 1,$$

it follows from the results of [17] that the hypotheses imply the regularity of (S, μ_{n+1}).

To prove the theorem we require the following lemmas.

LEMMA 3.1. *Suppose that the transformation*

$$t_n = \sum_{m=0}^{\infty} \alpha_{n,m} s_m$$

has the proporties

(i) $\alpha_{n,m} \to 0$ *as* $n \to \infty$ *for any fixed* m,

(ii) $\sum_{m=0}^{\infty} |\alpha_{n,m}|$ *is bounded.*

Let

$$A_n = \sum_{m=0}^{\infty} |\alpha_{n,m}|,$$

and let

$$A = \limsup_{n\to\infty} A_n .$$

Then

$$\limsup_{n\to\infty} |t_n| \leqq A \limsup_{m\to\infty} |s_m| . \tag{3.10}$$

This result is the best possible; that is to say, $\{s_n\}$ *can be chosen so that there is equality in* (3.10).

The result is essentially due to Agnew ([1], Lemma 3.1). Agnew gives the analogous result for sequence-to-function transforms, but only obvious modifications of Agnew's argument are required.

LEMMA 3.2. *Let* (1.4) *be written as*

$$b_n = \sum_{k=0}^{\infty} \alpha_{n,k} a_k, \tag{3.11}$$

and let $\beta_{n,k}$ *be defined by* (2.10). *Suppose that* $\{\mu_n\}$ *is a moment sequence generated by the function* $\chi(t)$ $(0 \leqq t \leqq 1)$. *Then*

$$\beta_{n,o} = 1 \quad \textit{for all } n; \tag{3.12}$$

$$\beta_{n,k} = \frac{(n+k)!}{n!(k-1)!} \int_0^1 t^n (1-t)^{k-1} \chi(t)\, dt, \quad \textit{for } (n \geqq 0, k \geqq 1). \tag{3.13}$$

PROOF. It is easily seen that

$$\alpha_{n,k} = \begin{cases} \displaystyle\int_0^1 (1-t)^k \, d\chi(t) & \text{for } (n = 0,\ k \geqq 0). \quad (3.14) \\ 0 & \text{for } (n \geqq 1,\ k = 0). \quad (3.15) \\ \displaystyle\frac{(n+k-1)!}{n!(k-1)!} \int_0^1 (1-t)^k t^n \, d\chi(t) & \text{for } (n \geqq 0,\ k \geqq 1). \quad (3.16) \end{cases}$$

Using (3.14) and (3.15), it follows that

$$\beta_{n,o} = \int_0^1 d\chi(t).$$

Using (3.1), (3.12) clearly follows.

For the proof of (3.13), we integrate equations (3.14) and (3.16) by parts. It follows from (3.14) that for $n = 0$, $k \geqq 1$,

$$\alpha_{o,k} = k \int_0^1 (1-t)^{k-1} \chi(t)\, dt. \tag{3.17}$$

It also follows from (3.16) that, for $n \geqq 1$, $k \geqq 1$,

$$\alpha_{n,k} = \frac{(n+k-1)!}{n!(k-1)!} \int_0^1 [k(1-t)^{k-1} t^n - nt^{n-1}(1-t)^k]\, \chi(t)\, dt,$$

$$= \frac{(n+k-1)!}{n!(k-1)!} \int_0^1 [(k+n)(1-t)^{k-1} t^n - nt^{n-1}(1-t)^{k-1}]\, \chi(t)\, dt.$$

i.e.

$$\alpha_{n,k} = \frac{(n+k)!}{n!(k-1)!}\int_0^1 t^n(1-t)^{k-1}\chi(t)\,dt - \frac{(n+k-1)!}{(n-1)!(k-1)!}\int_0^1 t^{n-1}(1-t)^{k-1}\chi(t)\,dt. \tag{3.18}$$

Substituting with (3. 17) and (3. 18) in (2. 10), it is clear that (3. 13) follows.

PROOF OF THEOREM 3.1. (i) Writing b_n as in Lemma 3.2, it follows that the convergence of (3. 11) for all n is equivalent to the convergence for all n of

$$t_n = \sum_{k=0}^{\infty}\beta_{n,k}a_k, \tag{3.19}$$

where $\beta_{n,k}$ is defined as in Lemma 3.2. Also, by using Lemma 3.2, we have

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{|\beta_{n,k}|}{k} &\leqq \int_0^1 \frac{t^n}{1-t}|\chi(t)|\left\{\sum_{k=1}^{\infty}\frac{(n+k)!}{n!\,k!}(1-t)^k\right\}dt \\ &= \int_0^1 \frac{|\chi(t)|}{t(1-t)}(1-t^{n+1})\,dt \\ &\leqq (n+1)\int_0^1 \frac{|\chi(t)|}{t}\,dt \\ &< \infty\end{aligned} \tag{3.20}$$

by (3. 4).

(ii) We now have by using (3. 12) that

$$t_n - s_m = \sum_{k=1}^{\infty}\beta_{n,k}a_k - \sum_{k=1}^{m}a_k,$$

where $\beta_{n,k}$ is defined by (3. 13). Applying Lemma 3.1 with $s_k = ka_k$, we can assert that the best possible value of $A(\alpha)$ for which we can assert that (3. 7) holds is given by

$$\limsup_{n\to\infty} A_n,$$

where

(3.21) $$A_n = \sum_{k=1}^{m} \frac{|\beta_{n,k}-1|}{k} + \sum_{k=m+1}^{\infty} \frac{|\beta_{n,k}|}{k}.$$

Now is follows (3.13) that, for $n \geqq 1$,

(3.22) $$1 - \beta_{n,k} = \frac{(n+k)!}{n!(k-1)!} \int_0^1 t^n(1-t)^{k-1}(1 - \chi(t))\, dt.$$

Thus

(3.23) $$\sum_{k=1}^{m} \frac{|\beta_{n,k}-1|}{k} \leqq \int_0^1 t^n \left\{ \sum_{k=1}^{m} \frac{(n+k)!}{n!\,k!} (1-t)^{k-1} \right\} |1-\chi(t)|\, dt,$$

(3.24) $$\sum_{k=m+1}^{\infty} \frac{|\beta_{n,k}|}{k} \leqq \int_0^1 t^n \left\{ \sum_{k=m+1}^{\infty} \frac{(n+k)!}{n!\,k!} (1-t)^{k-1} \right\} |\chi(t)|\, dt.$$

Hence, in virtue of (3.4), (3.5), it will be enough to establish (ii) if we prove that, uniformly in $0 < t < 1$,

(3.25) $$\phi_n(t) = t^n \sum_{k=1}^{m} \frac{(n+k)!}{n!\,k!} (1-t)^{k-1} = O\left(\frac{1}{1-t}\right),$$

(3.26) $$\psi_n(t) = t^n \sum_{k=m+1}^{\infty} \frac{(n+k)!}{n!\,k!} (1-t)^{k-1} = O\left(\frac{1}{t}\right).$$

Let constants T_1, T_2 be chosen so that

$$0 < T_1 < \frac{1}{1+\alpha} < T_2 < 1.$$

Since,

(3.27) $$t^n \sum_{k=0}^{\infty} \frac{(n+k)!}{n!\,k!} (1-t)^{k-1} = \frac{1}{t(1-t)},$$

it is clear that (3.25) holds uniformly in $T_1 \leqq t < 1$, and that (3.26) holds uniformly in $0 < t \leqq T_2$. Now

(3.28) $$\frac{d}{dt} \phi_n(t) = t^{n-1} \sum_{k=1}^{m} \frac{(n+k)!}{n!\,k!} (1-t)^{k-2} \{n-(n+k-1)\,t\}.$$

Thus, if

$$n > (n+m-1)t\,,$$

the expression on the right of (3. 28) is positive. Since $m/n \to \alpha$, we have, for all sufficiently large n

$$n > (n+m-1)\,T_1\,,$$

and hence $\phi_n(t)$ is increasing in $(0, T_1)$. It is clear that (3. 25) follows. Again

(3. 29) $$\frac{d}{dt}\psi_n(t) = t^{n-1}\sum_{k=m+1}^{\infty}\frac{(n+k)!}{n!\,k!}(1-t)^{k-2}\{n-(n+k-1)t\}\,.$$

Thus if

$$n < (n+m)t$$

the expression on the right of (3. 29) is negative. Again, for sufficiently large n

$$n < (n+m)\,T_2\,,$$

and that $\psi_n(t)$ is decreasing in $(T_2, 1)$. Hence (3. 26) follows.

(iii) It follows from (3. 8), (3. 13), (3. 21) and (3. 22) that

$$\begin{aligned} A_n &= \sum_{k=1}^{m}\frac{1-\beta_{n,k}}{k} + \sum_{k=m+1}^{\infty}\frac{\beta_{n,k}}{k} \\ &= \int_0^1 (1-\chi(t))\,\phi_n(t)\,dt + \int_0^1 \chi(t)\,\psi_n(t)\,dt\,. \end{aligned}$$

Applying Theorem 139 of Hardy [10], it follows that as $n\to\infty$, $m/n\to\alpha$; $\phi_n(t)$ tends to $1/t(1-t)$ if $1/(\alpha+1) < t \leqq 1$ and to 0 if $0 \leqq t < 1/(\alpha+1)$, and that $\psi_n(t)$ tends to 0 if $1/(\alpha+1) < t \leqq 1$ and to $1/t(1-t)$ if $0 \leqq t < 1/(\alpha+1)$. Also, we can apply Lebesgue's theorem on term-by-term integration of boundedly convergent series. We thus obtain (3. 9).

THEOREM 3.2. *If $\alpha=0$ or ∞ in Theorem 3.1 (ii), then there exists, in each case, a series $\sum a_n$ such that*

$$\limsup_{\substack{n\to\infty \\ m/n\to\alpha}} |t_n - s_m| = \infty$$

even when $\lim_{n\to 0} na_n = 0$.

For the proof of Theorem 3.2, we need the the following lemma given by Agnew [1. 3].

LEMMA 3.3. *If $F=f_{n,k}$ is an infinite matrix of real or complex number such that*

$$\sum_{k=0}^{\infty} |f_{n,k}| < \infty, \qquad (n = 0, 1, 2, \cdots),$$

but

$$\limsup_{n\to\infty} \sum_{k=0}^{\infty} |f_{n,k}| = \infty,$$

then there is a real sequence $\{x_k\}$ such that $\lim x_k = 0$ and

$$\limsup_{n\to\infty} \left| \sum_{k=0}^{\infty} f_{n,k} x_k \right| = \infty.$$

Theorem 3.2 follows from Lemma 3.3, since if we write

$$t_n - s_m = \sum_{k=1}^{\infty} f_{n,k} x_k,$$

the $f_{n,k}$ are in either case, such that

$$\sum_{k=1}^{\infty} |f_{n,k}| < \infty,$$

but by an easy modification of the proof of (ii) of Theorem 3.1, we can show that

$$\limsup_{\substack{n\to\infty \\ m/n\to\alpha}} \sum_{k=1}^{\infty} |f_{n,k}| = \infty,$$

4. Suppose that the function $\chi(t)$ $(0 \leqq t \leqq 1)$ satisfies the conditions of Theorem 3.1 (iii) and that $A(\alpha)$ is defined by equation (3. 9). Then $A(\alpha)$ is an indefinite integral (with respect to α) of a function, say $A'(\alpha)$, where

$$A'(\alpha) = \frac{1}{\alpha} \left\{ 1 - 2\chi\left(\frac{1}{\alpha+1}\right) \right\}. \tag{4. 1}$$

Now suppose that $\chi(t)$ is non-decreasing. Then there is a t_0 $(0 < t_0 \leqq 1)$ such that [1]

$$\chi(t) \leqq 1/2 \ (0 \leqq t < t_0); \quad \chi(t) \geqq 1/2 \ (t_0 < t \leqq 1).$$

It clearly follows, from (4. 1) that the minimum value $A(\alpha)$ occurs for $\alpha = (1/t_0) - 1$.

5. THEOREM 5.1. *Suppose that the conditions of Theorem* 3.1 *are satisfied. Suppose also that*

$$\chi_1(t) = \int_0^t \chi(u)\,du = o\left(\frac{t}{\log 1/t}\right), \quad \text{as} \quad t \to 0. \tag{5. 1}$$

Then (1. 7) *holds; and hence the results of Theorems* 3.1 *and* 3.2 *apply also to the sequence-to-sequence transformation* (1. 1).

PROOF. Taking n as fixed,

$$\binom{n+k-1}{n} \sim \frac{k^n}{n!}$$

so that (1. 7) is equivalent to

$$\int_0^1 (1-t)^k t^n \, d\chi(t) = o\left(\frac{1}{k^n \log k}\right), \quad \text{as} \quad k \to \infty. \tag{5. 2}$$

Taking $k \geqq 2$ and integrating by parts twice, the expression on the left of (5. 2) is equal to

$$\int_0^1 R_n(k, t)\,\chi_1(t)\,dt$$

where

$$R_n(k, t) = \frac{d^2}{dt^2}\{(1-t)^k t^n\}.$$

Now

1) Either t_0 is unique, or t_0 can have any value in some closed interval.

(5.3) $$R_n(k, t) = (1-t)^{k-2} t^{n-2} [n(n-1)(1-t)^2 + 2nkt(1-t) + k(k-1) t^2].$$

Now for $1/\sqrt{k} \leqq t < 1$

$$\log(1-t)^{k-2} < -(k-2)t \leqq -\sqrt{k} + O(1);$$

so that uniformly in this range

$$(1-t)^{k-2} = O(e^{-\sqrt{k}}).$$

Also, it is clear that the expression in square brackets in (5.3) is $O(k^2 t^2)$. Since $\chi_1(t)$ is bounded, it is clear that

$$\int_{1/\sqrt{k}}^{1} R_n(k, t)\chi_1(t)\,dt = O(k^2 e^{-\sqrt{k}}).$$

Thus, it is enough to prove that

(5.4) $$\int_0^{1/\sqrt{k}} \frac{t}{\log(1/t)} R_n(k, t)\,dt = O\left(\frac{1}{k^n \log k}\right).$$

Also, the expression in square brackets in (5.3) is $O(1+k^2 t^2)$. Thus, for $n \geqq 1$, the expression on the left of (5.4) is

$$O\left(\frac{1}{\log k}\int_0^{1/\sqrt{k}} t^{n-1}(1-t)^{k-2}(1+k^2 t^2)\,dt\right)$$

$$= O\left(\frac{1}{\log k}\int_0^{1} t^{n-1}(1-t)^{k-2}(1+k^2 t^2)\,dt\right)$$

$$= O\left[\frac{1}{\log k}\left(\frac{\Gamma(n)\Gamma(k-1)}{\Gamma(n+k-1)} + \frac{k^2\Gamma(n+2)\Gamma(k-1)}{\Gamma(n+k-1)}\right)\right]$$

$$= O\left(\frac{1}{k^n \log k}\right).$$

In the case $n=0$, we can apply a similar argument, but we use the result that in this case the expression in square brackets in (5.3) is $O(k^2 t^2)$.

Anjaneyulu [5] has considered Tauberian constants for the series-to-series transformation formally equivalent to the sequence-to-sequence quasi-Hausdorff transformation

$$t_n = \sum_{k=n}^{\infty} \binom{k}{n} (\Delta^{k-n} \mu_{n+1}) s_k . \tag{5.5}$$

Let us write (5. 5) in the form (2. 5). By an argument similar to the proof of Theorem 5.1 it can be shown that if, in addition to the conditions assumed by Anjaneyulu, we suppose that (5. 1) holds then, for every series satisfying (1. 6), (2. 6) holds. Thus it follows, just as in the case of our Theorem 5.1 that under the additional hypothesis (5. 1), the results of Anjaneyulu apply also to the sequence-to-sequence transformation (5. 5).

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