# A TBR-based Trajectory Piecewise-Linear Algorithm for Generating Accurate Low-order Models for Nonlinear Analog Circuits and MEMS

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## ABSTRACT

In this paper we propose a method for generating reduced models for a class of nonlinear dynamical systems, based on truncated balanced realization (TBR) algorithm and a recently developed trajectory piecewise-linear (TPWL) model order reduction approach. We also present a scheme which uses both Krylov-based and TBR-based projections. Computational results, obtained for examples of nonlinear circuits and a micro-electro-mechanical system (MEMS), indicate that the proposed reduction scheme generates nonlinear macromodels with superior accuracy as compared to reduction algorithms based solely on Krylov subspace projections, while maintaining a relatively low model extraction cost.

## Categories and Subject Descriptors

B.7.2 [**Design Aids**]: Simulation; G.1.2 [**Approximation**]: Nonlinear approximation; I.6.5 [**Model Development**]: Modeling methodologies

## General Terms

Algorithms, Performance, Design

## Keywords

Model Order Reduction, nonlinear systems, Truncated Balanced Realization

## 1. INTRODUCTION

Model Order Reduction (MOR) plays an increasingly important role in system-level design automation, providing techniques for extracting low cost, easy to use macromodels for e.g. circuit interconnects or subsystems of a complicated mixed-signal, mixed-technology system-on-a-chip.

Most of the research effort so far focused on developing MOR techniques suitable for linear systems, with predominance of methods based on Krylov subspace projections [2], due to a low numerical cost associated with model generation. More recently, also methods based on Hankel norm approximants and Truncated Balanced Realization (TBR) gained much interest in the engineering community, due to their superior accuracy, as compared to Krylov-based methods, as well as guaranteed error bounds and a stability preservation property [3], [9].

Available MOR techniques for nonlinear systems are much scarcer and include methods based on linearization or bilinearization of the initial system around the equilibrium point [10], algorithms using Karhunen-Loève expansion (or Proper Orthogonal Decomposition) [13], and finally methods of balanced truncation [7], [12]. Unfortunately, the existing MOR algorithms for nonlinear systems based on balancing transformations, although accurate, either are characterized by high numerical cost of generating the models or overlook issues associated with the numerical cost of evaluating the final reduced order model (cf. Section 2).

In this paper we propose an approach which merges a recently developed trajectory piecewise-linear (TPWL) model order reduction technique [11], providing a cost-efficient representation of system's nonlinearity, with state space projection method based on TBR. As shown below, this method outperforms algorithms using solely Krylov subspace reduction, while maintaining a relatively low computational cost.

We start in the next section with briefly describing the Trajectory Piecewise-Linear (TPWL) approach toward modeling nonlinear systems, and then, in Section 3, we present an algorithm for generating the reduced order basis using the TBR algorithm. Section 4 gives examples of nonlinear systems, which served to test our approach, and Section 5 contains computational results. In Section 6 we present our conclusions.

## 2. TRAJECTORY PIECEWISE-LINEAR MACROMODELS

In this paper we consider a class of nonlinear dynamical systems which can be represented using the standard state space form:

$$\begin{cases} \dot{x}(t) = f(x(t)) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where $x(t) \in R^N$ is a vector of states at time $t$, $f : R^N \to R^N$ is a nonlinear vector-valued function, $B$ is an $N \times M$ input matrix, $u : R \to R^M$ is an input signal, $C$ is an $N \times K$ output matrix and $y : R \to R^K$ is the output signal.

In the context of nonlinear systems, the ultimate goal of model order reduction techniques is constructing macromodels capable of approximately simulating the input-output behavior of systems in form (1) at a significantly reduced numerical cost. One should note that, unlike in the linear case, construction of a macromodel using a simple order reduction, realized through basis projection, does *not* achieve this goal. Suppose we consider an $N \times q$ ($q \ll N$) projection basis $V$, i.e. we try to approximate every state ($x$) of the system with a linear combination of the columns of $V$. After performing the projection $x = Vz$ on system (1) and applying a 'testing' $N \times q$ basis $W$ ($W^T V = I$) we obtain the following reduced order system:

$$\left\{ \begin{array}{l} \dot{z} = W^T f(Vz(t)) + W^T Bu(t) \\ y(t) = CVz(t). \end{array} \right. \tag{2}$$

Although the order of the above system is reduced to $q$, its numerical solution remains costly. In particular, evaluation of $W^T f(Vz)$ term typically requires at least $O(N)$ operations, and makes system (2) as costly to evaluate as system (1). In order to overcome this difficulty, using the following quasi-piecewise-linear approximate representation of nonlinear function $f$ has been proposed in [11]:

$$f(x) \approx \sum_{i=0}^{s-1} \tilde{w}_i(x) \left( f(x_i) + A_i(x - x_i) \right), \tag{3}$$

where $x_i$'s ($i = 1, \ldots, (s-1)$) are some linearization points (states), $A_i$'s are the Jacobians of $f$ evaluated at states $x_i$, and $\tilde{w}_i(x)$'s are state-dependent weights ($\sum_{i=0}^{s-1} \tilde{w}_i(x) = 1$ for all $x$). Applying the above approximation, and performing a projection of system (1) with biorthonormal matrices $V$ and $W$ yields:

$$\left\{ \begin{array}{l} \dot{z} = (\sum_{i=0}^{s-1} w_i(z) A_{ir}) z + \gamma \cdot w(z) + B_r u \\ y = C_r z \end{array} \right. , \tag{4}$$

where:

$$\gamma = \left[ W^T(f(x_0) - A_0 x_0), \ldots, W^T(f(x_{s-1}) - A_{s-1} x_{s-1}) \right],$$

$w(z) = [w_0(z) \ldots w_{s-1}(z)]^T$ ($\sum_{i=0}^{s-1} w_i(z) = 1$ for all $z$) is a vector of weights, $A_{ir} = W^T A_i V$, $B_r = W^T B$, and $C_r = CV$. One should note that evaluation of the right hand side of equation (4) requires at most $O(sq^2)$ operations, where $s$ is the number of linearization points used.

As proposed in [11], linearization points $x_i$ used in system (4) are picked from a 'training trajectory' of the initial nonlinear system, corresponding to some appropriately selected 'training input'.

In order to obtain a reduced system in form (4) one also needs to pick suitable biorthonormal projection bases $V$ and $W$. This issue is addressed in more detail in the following section.

# 3. GENERATION OF THE REDUCED ORDER BASIS

In order to obtain a reduced system in form (4) we considered a few different strategies toward generating the projection bases $V$ and $W$. In the simplest approach we applied a standard TBR algorithm to obtain a balancing transformation for the linearization of system (1) at the initial state $x_0$. In an extended approach we included subsequent linearization points in the projection bases, as well as generated truncated balancing transformations for linearizations of system (1) at different states, located along the training trajectory. The multiple bases were then aggregated into a single basis using a biorthonormalization algorithm. In yet another approach we applied a two-step reduction: first we performed an intermediate reduction using a standard Krylov subspace method, and then applied a TBR-based projection to obtain the final macromodel. In the following sections the outlined reduction strategies are described in more detail.

## 3.1 Truncated Balanced Realization at the initial state

In the simplest projection strategy we consider linearization of (1) at the initial state $x_0$:

$$\left\{ \begin{array}{l} \dot{x} = A_0 x + f(x_0) - A_0 x_0 + Bu \\ y = Cx \end{array} \right. \tag{5}$$

In the above system, apart from a 'real' input term $Bu$, we also have a $(f(x_0) - A_0 x_0)$ term, which may be treated as an additional input that introduces a constant shift to the system. In case of Krylov-based reduction we may account for this constant shift by simply adding $(f(x_0) - A_0 x_0)$ vector to the projection basis $V$. If TBR-based reduction is used, the situation is more subtle. We have found, that for the examples of micromachined switch and nonlinear transmission line models (cf. Section 4) it is enough to use only the 'real' input term $\hat{B} = B$ in the balancing procedure (cf. below). However, for the op-amp example it is necessary to take $\hat{B} = [(f(x_0) - A_0 x_0)\ B]$ as an input term. Summing up, we compute balancing transformation for the matrices: $\hat{A} = A_0$, $\hat{C} = C$, and suitable $\hat{B}$ using the square-root TBR procedure [8]:

TBR($\hat{A}, \hat{B}, \hat{C}$)
**Input:** System matrices $\hat{A}$, $\hat{B}$, and $\hat{C}$.
**Output:** Projection bases $V$ and $W$.
(1) Find observability Grammian $P$:
$\hat{A}P + P\hat{A}^T = -\hat{B}\hat{B}^T$;
(2) Find controllability Grammian $Q$:
$\hat{A}^T Q + Q\hat{A} = -\hat{C}^T \hat{C}$;
(3) Using SVD compute Cholesky factors of $P$ and $Q$:
$P = Z_c Z_c^T, \quad Q = Z_o Z_o^T$;
(4) Compute SVD of Cholesky product: $U\Sigma V = Z_o^T Z_c$;
(5) Compute $V$ and $W$:
$V = Z_c V T \Sigma^{-1/2}, \quad W = Z_o U T \Sigma^{-1/2}$;
where $T = [I_{(q \times q)} 0]^T$ is an $N \times q$ truncation matrix.

The obtained projection bases $V$ and $W$ are then used to compute the reduced order Jacobians $A_{ir}$ (cf. (4)).

We have found that, in order to obtain an accurate reduced order model, it is often critical to include the linearization point $x_0$ in the reduced order basis (which ensures exact representation of this state in the reduced order state space). Suppose $V$ and $W$ are $N \times p$ bases computed with TBR algorithm and $x_0$ is not in the span of those bases. Then, we may form new bases $\tilde{W}$ and $\tilde{V}$ which include $x_0$ using a biorthonormalization algorithm. One should note that matrices $\tilde{V}$ and $\tilde{W}$ are no longer balancing transformations for system (5), although they still approximately capture the 'most controllable' and 'most observable' regions of the state space for this linearized system.

## 3.2 Aggregate TBR-based projection

In the discussed trajectory piecewise-linear MOR approach we consider a weighted combination of linearized models of system (1) in form (5). Clearly, each of those models has a different set of balancing transformations and a different 'most controllable' and 'most observable' part of the state space. In order to obtain a single, consistent reduced order representation which will include the most relevant parts of the state space for all the models, one may construct aggregate reduced bases $V_{agg}$ and $W_{agg}$ by merging the subsequent balanced realizations $[V_i, W_i]$ for different linearized models, and then biorthonormalizing.[1]

[1]During this process one typically needs to remove excessive vec-

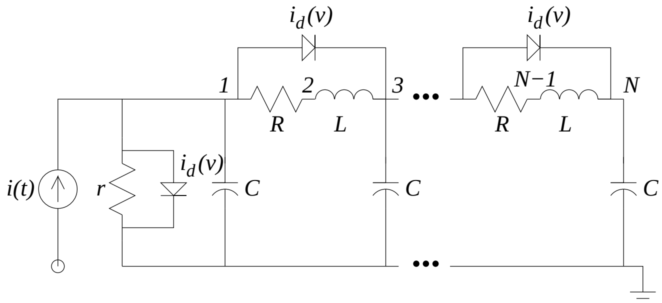


**Figure 1: An example of a nonlinear transmission line RLC circuit model.**

## 3.3 Two-step hybrid reduction

Due to high complexity of the TBR algorithm ($O(N^3)$, where $N$ is the initial size of the system) and problems with ill-conditioning of Lyapunov equations, reduction algorithms based solely on TBR are limited in practice to systems with a few hundred unknowns. In order to allow effective and efficient reduction of larger systems one may use a two-step hybrid reduction [5] in which a standard Krylov subspace method (cf. [10], [11]) is used to reduce the initial TPWL model to a medium-sized model (e.g. of order $q_i \approx 100$), and then TBR is used to further compress that model.

As shown in the following sections, the above approach provides excellent nonlinear reduced order models, generated with significantly reduced computational effort as compared to reduction schemes based solely on TBR.

# 4. EXAMPLES OF NONLINEAR SYSTEMS

In this Section we consider four examples of nonlinear systems which may be described by equations (1) and, due to their highly nonlinear dynamical behavior, are suitable to test the proposed reduction algorithms.

The first two examples (the first one was also examined in [1]) refer to a nonlinear transmission line circuit model shown in Figure 1. The first circuit consists of resistors, capacitors, and diodes with a constitutive equation $i_d(v) = \exp(40v) - 1$. For simplicity we assume that all the resistors and capacitors have unit resistance and capacitance, respectively ($R = 1$, $C = 1$) (In this case we assume that $L = 0$). The input is the current source entering node 1: $u(t) = i(t)$ and the (single) output is chosen to be the voltage at node 1: $y(t) = v_1(t)$. Consequently, if the state vector is taken as $x = [v_1, \ldots, v_N]$, where $v_i$ is the voltage at node $i$, the system has symmetric Jacobians at any linearization point, and $B = C$. In this example we considered the number of nodes $N = 400$ and $N = 1500$. In the second example (cf. Figure 1) we also consider inductors (with inductance $L = 10$), connected in series with the resistors. We apply the RL formulation in order to obtain a dynamical system in form (1) with voltages and currents at subsequent nodes (or branches) of the circuit as state variables. In this case the Jacobians of $f$ become nonsymmetric.

The third example is a micromachined switch (fixed-fixed beam) shown in Figure 2. Following Hung et al. [4], the dynamical behavior of this coupled electro-mechanical-fluid system can be modeled with 1D Euler's beam equation and 2D Reynolds' squeeze film damping equation [4]. Spatial discretization of those equations using a standard finite-difference scheme leads to a nonlinear dynamical system in form (1) with $N = 880$ states. For this system the

tors which are linearly dependent on the other columns of $V_{agg}$ or $W_{agg}$.

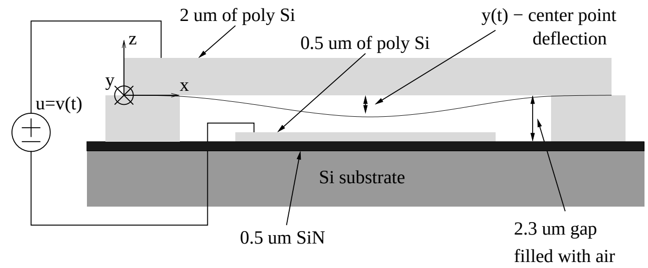


**Figure 2: Micromachined switch (following Hung et al. [4]).**

state vector ($x$) consists of heights of the beam above the substrate ($u$) computed at the grid points, values of $\partial(u^3)/\partial t$, and the values of pressure below the beam. For the considered example we select our output $y(t)$ as the deflection of the center of the beam from the equilibrium point (cf. Figure 2).

The last of the examples we consider in this paper is an operational amplifier with differential input and output, and consisting of 70 MOSFETs, 13 resistors and 9 linear capacitors connected to 51 circuit nodes. Nodal analysis yields a nonlinear model of the device in the form:

$$\begin{cases} E\dot{x} = f(x) + Bu \\ y = Cx \end{cases} \tag{6}$$

where $E$ is the capacitance matrix, and $x$ consists of voltages at the circuit nodes. (In this example we used transistor models with linearized capacitances, and consequently matrix $E$ is state-independent.) In order to transform system (6) to form (1), suitable for TBR-based reduction, we note that $E$ is a symmetric positive definite matrix. Consequently, we can perform Cholesky factorization on $E$: $E = R^T R$, where $R$ is an upper triangular matrix. Substituting change of variables $\tilde{x} = Rx$ to (6) yields the following system in form (1):

$$\begin{cases} \dot{\tilde{x}} = (R^T)^{-1} f(R^{-1}\tilde{x}) + (R^T)^{-1} Bu \\ y = CR^{-1}\tilde{x} \end{cases}$$

It has been found that the above representation provides linearized systems, which are better conditioned than e.g. a representation obtained by left-multiplying first of equations (6) by $E^{-1}$.

# 5. COMPUTATIONAL RESULTS

In this section we present results of numerical tests for the application examples described above.

## 5.1 Transmission line models

First, we considered nonlinear transmission line RLC circuit model. We applied the two-step hybrid projection algorithm proposed in Section 3.3. The initial problem size $N$ equaled 800. Using Krylov projection, the TPWL system (with 20 linearization points) was first reduced to a system of order order $q_i = 100$. Then we applied a simple TBR reduction (described in Section 3.1) which generated a truncated balancing transformation of order $q = 4$ (for the linearized system at the initial state $x_0 = 0$). Figure 3 compares a transient computed with the obtained reduced order model (denoted as Krylov-TBR TPWL model) with the transients obtained with full order nonlinear and linear models. One may note that Krylov-TBR TPWL reduced order model provides an excellent approximation of the transient for the initial system. It is also apparent that the model is substantially more accurate than a full order linear model of the transmission line.

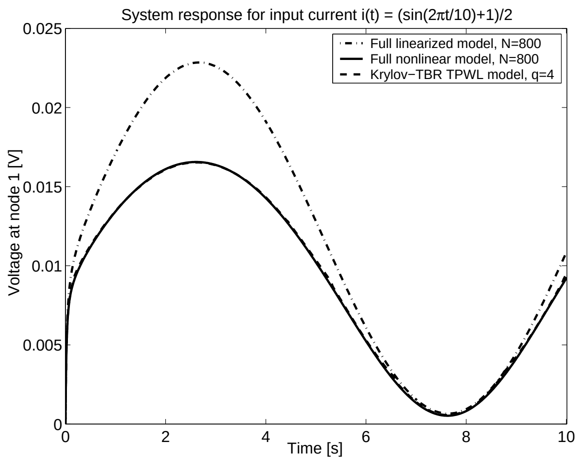


**Figure 3: Comparison of system response (nonlinear transmission line RLC circuit) computed with nonlinear and linear full-order models, as well as Krylov-TBR TPWL reduced order model (20 models of order $q = 4$) for the input current $i(t) = (\sin(2\pi t/10) + 1)/2$. The TPWL model was generated using a unit step input current. Intermediate Krylov model order $q_i = 100$.**

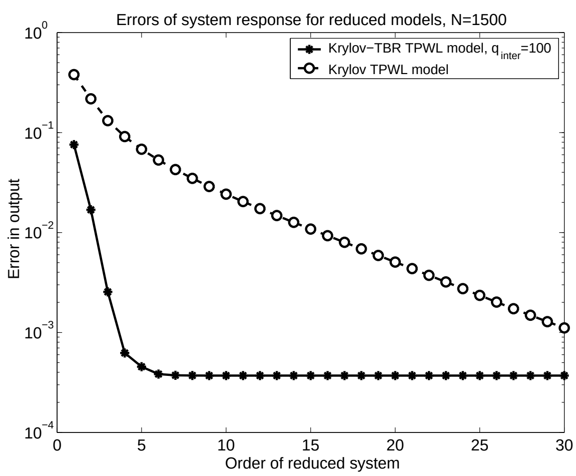


**Figure 4: Errors in output computed with Krylov and Krylov-TBR TPWL reduced order models (nonlinear transmission line RC circuit). Both training and testing inputs were unit step voltages. Initial order of system $N = 1500$; intermediate Krylov model order $q_i = 100$.**

Figure 4 shows the error in the output signal $\|y_r - y\|_2$ computed for different values of $q$, where $y_r$ is the output signal obtained with Krylov-TBR TPWL reduced order model, and $y$ is computed with full order nonlinear model (in this example $\|y\|_2 = 0.44$). Analogous errors were also computed for reduced order TPWL models obtained with pure Krylov-based reduction. The results on the graph show that Krylov-TBR TPWL models are significantly more accurate than Krylov TPWL models of the same size. Also, the discussed Krylov-TBR TPWL model achieves its best accuracy (limited by the quality of TPWL approximation to $f$) at a much lower order than the TPWL model based solely on Krylov subspace reduction.

It follows from Fig. 4 that the total error of TPWL reduced order approximation of a full nonlinear model consists of two components: the error due to projection procedure, and the error associated with piecewise-linear approximation of nonlinear function $f$. The first component is dominant when the order of the reduced model is small. One may note that for Krylov-TBR approach this error component becomes negligible as soon as the order of the reduced model is greater than 5. Further considerations on error estimates for TPWL models may be found in [11].

**Table 1: Comparison of model extraction times for Krylov, TBR and Krylov-TBR TPWL MOR techniques for different initial problem sizes.**

| Initial model size $N$ | TBR TPWL, $q = 6$ | Krylov-TBR TPWL, $q = 6$ | Krylov TPWL, $q = 30$ |
|---|---|---|---|
| 1500 | 1268 s | 30.57 s | 26.34 s |
| 800 | 181.8 s | 8.57 s | 7.75 s |
| 400 | 23.75 s | 2.73 s | 3.03 s |

In order to estimate how intermediate Krylov reduction affects total accuracy of a TPWL reduced order model, we used pure TBR reduction to construct TPWL models for the example of a nonlinear transmission line RC circuit model with $N = 400$ nodes. It has been found that using an intermediate Krylov reduction has a vanishingly small effect on the accuracy of the final TPWL reduced order model.

We also compared performance of different MOR strategies for the example of a nonlinear transmission line RC circuit model. Table 1 shows a comparison of model extraction times for MATLAB implementations of TBR, Krylov-TBR, and Krylov TPWL MOR algorithms. One may note that, while the pure TBR reduction quickly becomes expensive as $N$ grows, the cost of generating a reduced model (of order $q = 6$) with the two-step hybrid Krylov-TBR strategy remains low and matches the cost of generating an equally accurate model (of order $q = 30!$) with a pure Krylov-based reduction. This clearly follows from the fact that model extraction cost for pure TBR-based projection is $O(N^3)$, while hybrid Krylov-TBR projection can be performed at $O(Nq_i^2) + O(q_i^3)$ cost, provided matrix $A$ is sparse and structured.

## 5.2 Micromachined switch example

We also applied the Krylov-TBR TPWL model order reduction strategy to generate macromodels for the micromachined switch example described in Section 4. In this case, the reduced basis was generated using the linearized model of system (1) only at the initial state $x_0$, and the initial state was included in the bases $V$ and $W$.

Surprisingly, unlike in the previous example, we did *not* observe a monotonically decreasing output error behavior for the growing order $q$ of the reduced system. Instead, we found that macromodels with odd orders behave very differently than the macromodels with even orders. On one hand, models of even order are substantially more accurate than models of the same order generated by Krylov reduction – cf. Figure 5. On the other hand, if $q$ is odd we obtain inaccurate and unstable reduced order models. This phenomenon is reflected in the error plot shown in Figure 5.

The ‘even-odd’ phenomenon can be viewed in more detail by examining eigenvalues of the reduced order Jacobians from different linearization points. For the considered example, the initial nonlinear system is stable. Consequently, Jacobians of $f$ at all lin-

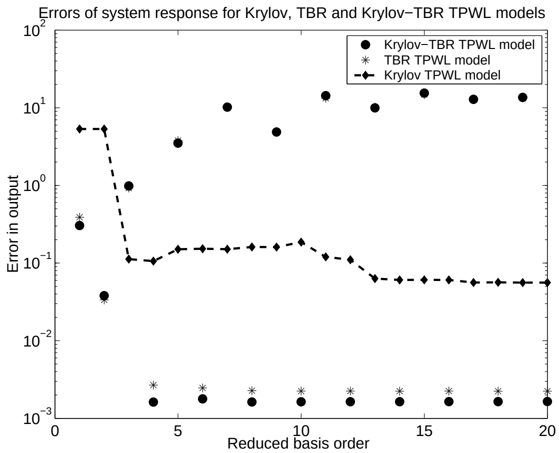


**Figure 5: Errors in output computed by TPWL models generated with different MOR procedures (micromachined switch example); $N = 880$; intermediate Krylov model order $q_i = 96$; 5.5-volt step testing and training input voltage.**

earization points are also stable. Nevertheless, in this example, the generated reduced order basis provides a truncated balancing transformation only for the linearized system from the initial state $x_0$. Therefore, only the reduced Jacobian from $x_0$ is guaranteed to be stable. Other Jacobians, reduced with the same set of bases, may develop eigenvalues with positive real parts.

Figure 6 shows spectra of the reduced order Jacobians for models of order $q = 7$ and $q = 8$. One may note that, for $q = 8$, the spectra of the Jacobians from a few first linearizations points are very similar. They also follow the same pattern: two of the eigenvalues are real, and the rest forms complex-conjugate pairs. If we increase or decrease the order of the model by 2 a new complex-conjugate pair of stable eigenvalues will add to the spectra of the Jacobians. On the other hand, if the order of the model is increased or decreased by 1 (cf. Figure 6 (top)) a certain complex-conjugate pair will be broken, and a real eigenvalue will form. At the first linearization point this eigenvalue is a relatively small negative number. As we move to the next linearization point, the corresponding eigenvalue shifts significantly to the right half-plane to form an unstable mode of the system. An obvious workaround for this problem in the considered example is to generate models of even order. Nevertheless, a true solution to this problem would involve investigating a much deeper issue, namely, that of robustness of stability preservation by balancing transformations applied to perturbed system matrices (corresponding, in the context of TPWL approximations, to Jacobians from different linearization points along a trajectory of the system).

Though the issue of TBR stability for perturbed systems requires further investigation to develop robust strategies, the proposed method is extremely effective. For example, the plot in Figure 7 demonstrates that a 4th order Krylov-TBR TPWL model generates transient behavior that is indistinguishable from the transient for the reference model.

## 5.3 Op-amp example

As the last example, we considered the operational amplifier example, described in Section 4. The examined circuit had $N = 51$ nodes and eight inputs: 1) the differential input with input signals $v_{in1}$ and $v_{in2}$, 2) the auxiliary inputs $v_{cmmrst}$, $v_{gnd}$, $v_{intn}$, $v_{intp}$, $v_{rst}$, and $v_{cmmin}$ used in common mode rejection testing.

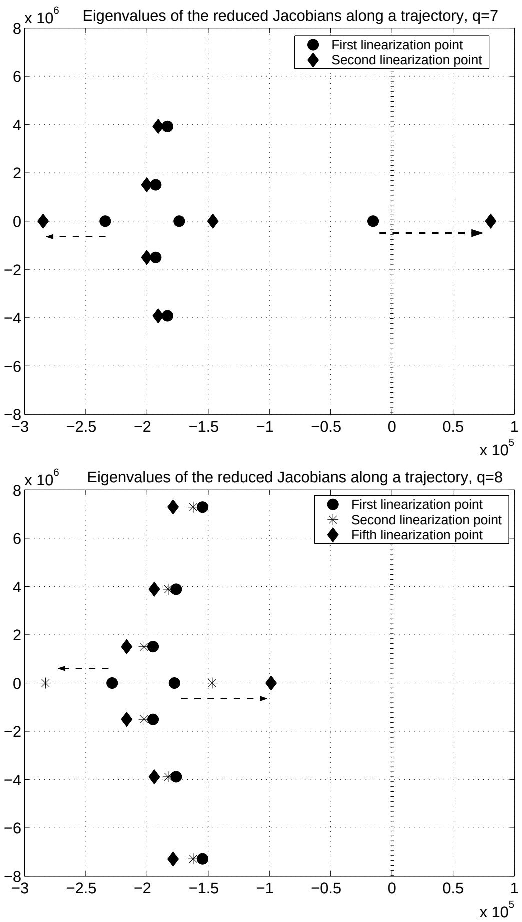


**Figure 6: Eigenvalues of the Jacobians from the first few linearization points (micromachined switch example, Krylov-TBR TPWL reduction). Order of the reduced system $q = 7$ (top), $q = 8$ (bottom).**

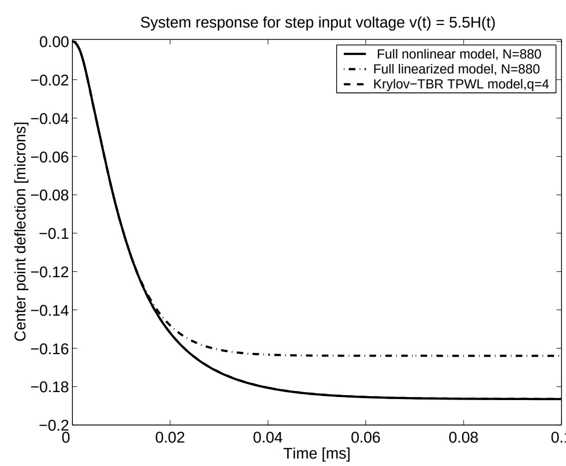


**Figure 7: Comparison of system response (micromachined switch example) computed with both nonlinear and linear full-order models, and Krylov-TBR TPWL reduced order model (7 models of order $q = 4$); intermediate Krylov model order $q_i = 96$; 5.5-volt step testing and training input voltage.**

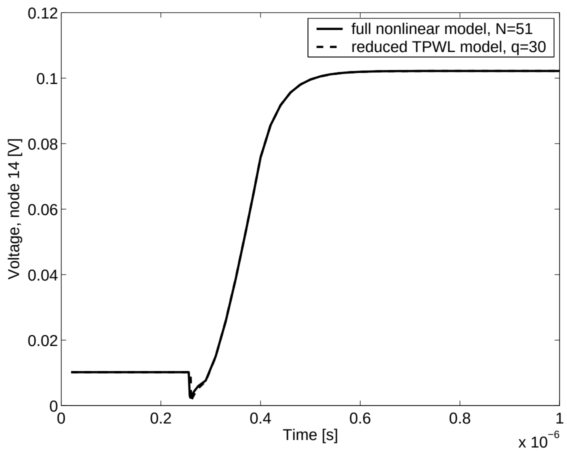


**Figure 8: Comparison of the output voltage (op-amp example, linearized capacitance models), computed with TBR TPWL reduced order model and NITSWIT circuit simulator.**

The full order nonlinear simulations were performed using the NITSWIT circuit simulator [6]. In order to generate the reduced order TPWL models we applied the following set of training inputs:

$$v_{in1}(t) = \begin{cases} 0 & t < 290 \\ 12.5 \cdot 10^{-3}(t-290)/10 & 290 \le t < 300 \\ 12.5 \cdot 10^{-3} & t \ge 300 \end{cases}$$

$v_{in2} = -v_{in1}$ and auxiliary PWL input signals – cf. [11].

We applied the MOR algorithm described in Section 3.2 (using multiple balancing bases from different linearization points), to generate a TPWL model of order $q = 30$ (with $s = 35$ linearized models and 8 inputs). The obtained reduced order TPWL model was then tested for the input:

$$v_{in1}(t) = \begin{cases} 0 & t < 290 \\ 11.5 \cdot 10^{-3}(t-290)/110 & 290 \le t < 400 \\ 11.5 \cdot 10^{-3} & t \ge 400 \end{cases}$$

($v_{in2} = -v_{in1}$). Figure 8 shows a comparison of the transients computed with NITSWIT and with the reduced order TBR TPWL model for one (of the two) output nodes of the amplifier. One may note very good agreement of the results. The relative error, computed as $e = \|y_r - y\|_1/\|y\|_1 \cdot 100\%$, where $y_r$ is the transient obtained with reduced order TPWL model and $y$ is the reference result, equals only $e = 0.71\%$. This error includes: 1) error due to the piecewise-linear approximation, 2) error due to reduced order projection. As computed in a separate test, the error due to piecewise-linear approximation equaled in this case 0.37%.

The results indicate that the MOR method proposed in Section 3.2, based on using aggregated truncated balancing transformations from different linearization points, may be effectively applied to model complicated nonlinear systems with multiple inputs. It is important to point out that not only do the TPWL models have a lower order than the original system, but also they are much easier to use. Since a TPWL model consists of a weighted combination of linear models, the time-stepping is very straightforward.

# 6. CONCLUSIONS AND ACKNOWLEDGMENTS

In this paper we proposed an algorithm for generating accurate reduced models for nonlinear dynamical systems, merging truncated balanced realization (TBR) algorithm and a trajectory piecewise-linear (TPWL) model order reduction approach. A two-step hybrid reduction algorithm based on Krylov subspace projection and TBR has also been examined. Results of numerical tests show that, for the considered application examples, the discussed MOR algorithm efficiently generates very low order TPWL models, characterized by superior accuracy over models extracted solely with Krylov subspace projection methods.

Summing up, the main advantages of the proposed MOR strategy include accuracy, performance, and applicability to strongly nonlinear systems. A fundamental limitation of the discussed algorithm is that TBR (unlike Krylov projection) may be computed only for stable linear systems, therefore our TBR-based TPWL procedures may approximate $f$ only in the region of the state space where linearizations of $f$ yield stable linear systems. One also needs a stable intermediate model for Krylov-TBR reduction (which Krylov projection does not guarantee).

There are also a number of issues yet to be addressed about the proposed approach, such as investigating applicability of the truncated balancing transformations to reducing system matrices corresponding to a number of different linearization points, or examining the problem of robustness of stability preservation for the proposed TBR-based TPWL model order reduction strategy.

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