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A TECHNIQUE IN PERTURBATION THEORY

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Introduction. A study of the effect of perturbations of differential equations depends on the method employed and on the nature of perturbations. One of the most used techniques is that of Lyapunov method and the other is the nonlinear variation of parameters formula [3]. These methods dictate that we measure the perturbations by means of a norm and thus destroy the ideal nature, if any, of the perturbing terms. Recently an effort was made to correct this unpleasant situation [1, 2].

In this paper, we wish to develop a new comparison theorem that connects the solutions of perturbed and unperturbed differential systems in a manner useful in the theory of perturbations. This comparison result blends, in a sense, the two approaches mentioned earlier and consequently provides a flexible mechanism to preserve the nature of perturbations. Our results will show that the usual comparison theorem in terms of Lyapunov function is imbedded as a special case in our present theorem and that the perturbation theory could be studied in a more fruitful way. An example is worked out to illustrate the results.

1. **A new comparison result.** We consider the two differential systems

$$y' = f(t, y), \ y(t_0) = x_0, \tag{1.1}$$

and

$$x' = F(t, x), \ x(t_0) = x_0, \tag{1.2}$$

where $f, F \in C[R^+ \times S(p), R^n]$. Here R^+ denotes the nonnegative real line, R^n the Euclidian n-space, $C[R^+ \times S(p), R^n]$ the class of continuous functions from $R^+ \times S(p)$ to R^n and $S(p) = [x \in R^n : \|x\| < p]$ where $\|\cdot\|$ is any convenient norm in R^n. Relative to the system (1.1), assume that

(H) the solutions $y(t, t_0, x_0)$ of (1.1) exist for all $t \geqq t_0$, are unique, continuous with respect to the initial data and $y(t, t_0, x_0)$ is locally Lipschitzian in x_0.

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Since (H) implies that $\|y(t, t, x)\| < p$ for $t \geqq t_0$, for any $V \in C[R^+ \times S(p), R^+]$ and any fixed $t \in (t_0, \infty)$, we define

$$D^+V(s, y(t, s, x)) \equiv \limsup_{h \to 0^+} \frac{1}{h} [V(s + h, y(t, s + h, x + hF(s, x))) - V(s, y(t, s, x))], \tag{1.3}$$

for $t_0 \leqq s < t$ and $x \in S(p)$.

The following comparison result which relates the solutions of (1.2) to the solutions of (1.1) is an important tool in our discussion.

THEOREM 1.1. *Assume that*

(i) *the hypothesis* (H) holds;

(ii) $V \in C[R^+ \times S(p), R^+]$, $V(s, x)$ *is locally Lipschitzian in* x *and for* $t_0 \leqq s < t, x \in S(p)$,

$$D^+V(s, y(t, s, x)) \leqq g(s, V(s, y(t, s, x))); \tag{1.4}$$

(iii) $g \in C[R^+ \times R^+, R]$ *and the maximal solution* $r(t, t_0, u_0)$ of

$$u' = g(t, u), u(t_0) = u_0 \geqq 0, \tag{1.5}$$

exists for $t \geqq t_0$.

Then if $x(t) = x(t, t_0, x_0)$ *is any solution of* (1.2), *we have*

$$V(t, x(t, t_0, x_0)) \leqq r(t, t_0, u_0), t_0 \leqq t < \infty, \tag{1.6}$$

provided $V(t_0, y(t, t_0, x_0)) \leqq u_0$.

PROOF. Let $x(t) = x(t, t_0, x_0)$ be any solution of (1.2) existing on $[t_0, \infty)$. We set $m(s) = V(s, y(t, s, x(s)))$ for $t_0 \leqq s \leqq t$ so that $m(t_0) = V(t_0, y(t, t_0, x_0))$. Then using the assumptions (H) and (ii), it is easy to obtain the differential inequality

$$D^+m(s) \leqq g(s, m(s)), t_0 \leqq s < t,$$

which yields by comparison theorem [3, Th. 1.4.1] the estimate

$$m(s) \leqq r(s, t_0, u_0), t_0 \leqq s < t, \tag{1.7}$$

provided $m(t_0) \leqq u_0$. Since $m(t) = V(t, y(t, t, x(t))) = V(t, x(t, t_0, x_0))$, the desired result (1.6) follows from (1.7) by setting $s = t$. The proof is complete.

Taking $u_0 = V(t_0, y(t, t_0, x_0))$, the inequality (1.6) becomes

$$V(t, x(t, t_0, x_0)) \leqq r[t, t_0, V(t_0, y(t, t_0, x_0))], t_0 \leqq t < \infty, \tag{1.8}$$

which shows the connection between the solutions of systems (1.1) and (1.2) in terms of the maximal solution of (1.5).

Remarks. A number of comments are in order:

(i) The trivial function $f(t, y) \equiv 0$ is admissible in Theorem 1.1 to yield the estimate (1.6) provided $V(t_0, x_0) \leqq u_0$. In this case, $y(t_0, x_0) = x_0$ and thus the hypothesis (H) is trivially verified. Since $y(t, s, x) = x$, the definition (1.3) reduces to

$$(1.9) \qquad D^+V(s, x) \equiv \limsup_{h \to 0^+} \frac{1}{h} [V(s + h, x + hF(s, x)) - V(s, x)],$$

which is the usual definition of the generalized derivative of the Lyapunov function relative to the system (1.2). Consequently, Theorem 1.1 reduces, in this special case, to a well-known comparison theorem (see Th. 3.1.1 in [3]). Thus it is clear that Theorem 1.1 is a natural extension of known comparison results.

(ii) Suppose that $f(t, y) = A(t)y$ where $A(t)$ is $n \times n$ continuous matrix. The solutions $y(t, t_0, x_0)$ of (1.1) then satisfy the relation $y(t, t_0, x_0) = U(t, t_0)x_0$ where $U(t, t_0)$ is the fundamental matrix solution of $y' = A(t)y$, $U(t_0, t_0) =$ Identity matrix. The hypothesis (H) is clearly satisfied. Suppose also that $g(t, u) \equiv 0$. Then (1.8) yields

$$(1.10) \qquad V(t, x(t, t_0, x_0)) \leqq V(t_0, U(t, t_0)x_0), \; t \geqq t_0.$$

If, in addition, $V(t, x) = \|x\|$, (1.10) leads to

$$(1.11) \qquad \|x(t, t_0, x_0)\| \leqq \|U(t, t_0)x_0\|, \; t \geqq t_0.$$

If, on the other hand, $g(t, u) = -\alpha u$, $\alpha > 0$, then we get a sharper estimate

$$(1.12) \qquad V(t, x(t, t_0, x_0)) \leqq V(t_0, U(t, t_0)x_0)\, e^{-\alpha(t-t_0)}, \; t \geqq t_0,$$

which, in the special case $V(t, x) = \|x\|$, reduces to

$$(1.13) \qquad \|x(t, t_0, x_0)\| \leqq \|U(t, t_0)x_0\| e^{-\alpha(t-t_0)}, \; t \geqq t_0.$$

Clearly, the relation (1.13) helps in improving the behavior of solutions of (1.2) relative to the behavior of solutions of (1.1). That this is an asset in the perturbation theory can be seen by setting $F(t, x) = f(t, x) + R(t, x)$, where $R(t, x)$ is the perturbed term.

(iii) Suppose that $f(t, y)$ is nonlinear and $f_y(t, y)$ exists and is continuous for $(t, y) \in R^+ \times S(p)$. Then it is known [3] that the solutions $y(t, t_0, x_0)$ are differentiable with respect to (t_0, x_0) and

$$(1.14)\qquad \begin{cases} \dfrac{\partial y}{\partial t_0}(t, t_0, x_0) = -\Phi(t, t_0, x_0)\, f(t_0, x_0), \\ \dfrac{\partial y}{\partial x_0}(t, t_0, x_0) = \Phi(t, t_0, x_0), \end{cases}$$

where $\Phi(t, t_0, x_0)$ is the matrix solution of the variational equation $z' = f_y(t, y(t, t_0, x_0))z$. If $V(s, x)$ is also assumed to be differentiable, then by (1.14) we have, for a fixed t,

$$(1.15)\qquad \begin{aligned} D_- V(s, y(t, s, x)) \equiv\; & V_s(s, y(t, s, x)) \\ & + V_x(s, y(t, s, x)) \cdot \Phi(t, s, x) \cdot [F(s, x) - f(s, x)]. \end{aligned}$$

The relation (1.15) gives an intuitive feeling of the definition (1.3). If, in addition, $V(t, x) = \|x\|^2$ and $F(t, x) = f(t, x) + R(t, x)$, (1.15) yields $D_- V(s, y(t, s, x)) \equiv 2y(t, s, x) \cdot \Phi(t, s, x)R(s, x)$ which shows how the perturbation terms $R(t, x)$ are involved in the computations.

(iv) When the solutions of (1.1) are known, a possible Lyapunov function for (1.2) is

$$(1.16)\qquad W(s, x) = V(s, y(t, s, x)),$$

where $V(s, x)$ and $y(t, s, x)$ are as before. One could take a convenient $V(t, s)$ like $V(s, x) = \|x\|$ so that $W(s, x) = \|y(t, s, x)\|$ is a candidate for Lyapunov function for (1.2). In case $y(t, s, x) \equiv x$, condition (1.4) reduces to

$$\liminf_{h \to 0^-} \frac{1}{h} [\|x + hF(t, x)\| - \|x\|] \leqq g(t, \|x\|),$$

which is an often used assumption.

2. **Stability and asymptotic behavior.** As an application of the comparison theorem 1.1, we shall consider in this section some results on stability and asymptotic behavior of solutions of (1.2).

THEOREM 2.1. *Assume that*

(1) *the conditions* (i) *and* (ii) *of Theorem* 1.1 *hold*;

(2) $g \in C[R^+ \times R^+, R]$, $g(t, 0) \equiv 0$, $f(t, 0) \equiv 0$ *and* $F(t, 0) \equiv 0$;

(3) *the trivial solution of* (1.1) *is uniformly stable and* $u = 0$ *of* (1.5) *is uniformly asymptotically stable*;

(4) $b(\|x\|) \leqq V(t, x) \leqq a(\|x\|)$, $(t, x) \in R^+ \times S(p)$, *where* $a, b \in C[[0, p), R^+]$, $a(u)$, $b(u)$ *are increasing and* $a(0) = 0$. *Then the trivial solution of* (1.2) *is uniformly asymptotically stable.*

PROOF. Let $0 < \epsilon < p$, $t_0 \in R^+$ be given. The uniform stability of $u = 0$ of (1.5) implies that given $b(\epsilon) > 0$, $t_0 \in R^+$ there exists a $\delta_1 = \delta_1(\epsilon)$ such that

$$u(t, t_0, u_0) < b(\epsilon), t \geqq t_0 \text{ if } u_0 \leqq \delta_1. \tag{2.1}$$

Let $\delta_2 = a^{-1}(\delta_1)$. Since $x = 0$ of (1.1) is uniformly stable, given $\delta_2 > 0$, $t_0 \in R^+$ there exists a $\delta = \delta(\epsilon) > 0$ such that

$$\|y(t, t_0, x_0)\| < \delta_2, t \geqq t_0, \text{ if } \|x_0\| < \delta. \tag{2.2}$$

We claim that $\|x_0\| < \delta$ also implies that $\|x(t, t_0, x_0)\| < \epsilon$, $t \geqq t_0$, where $x(t, t_0, x_0)$ is any solution of (1.2). If this is not true, there would exist a solution $x(t, t_0, x_0)$ of (1.2) with $\|x_0\| < \delta$, a $t_1 > t_0$ such that $\|x(t_1, t_0, x_0)\| = \epsilon$, and $\|x(t, t_0, x_0)\| < \epsilon$, $t_0 \leqq t < t_1$. Then, by Theorem 1.1 we have

$$V(t, x(t, t_0, x_0)) \leqq r(t, t_0, V(t_0, y(t, t_0, x_0))), t_0 \leqq t \leqq t_1.$$

Consequently, by (2.1), (2.2) and (4), we get

$$\begin{aligned} b(\epsilon) &\leqq V(t_1, x(t_1, t_0, x_0)) \leqq r(t_1, t_0, a(\|y(t_1, t_0, x_0)\|)) \\ &\leqq r(t_1, t_0, a(\delta_2)) \leqq r(t_1, t_0, \delta_1) < b(\epsilon). \end{aligned}$$

This contradiction proves that $x = 0$ of (1.2) is uniformly stable.

To show uniform asymptotic stability, we set $\epsilon = p$ and $\delta(p) = \delta_0$. Then from the foregoing argument, we have

$$b(\|x(t, t_0, x_0)\|) \leqq V(t, x(t, t_0, x_0)) \leqq r(t, t_0, V(t_0, y(t, t_0, x_0)))$$

for all $t \geqq t_0$ if $\|x_0\| < \delta_0$. From this it follows that

$$b(\|x(t, t_0, x_0)\|) \leqq r(t, t_0, \delta_1(p)), t \geqq t_0,$$

which implies the stated result by (3).

Remark. Setting $F(t, x) = f(t, x) + R(t, x)$ shows that although the unperturbed system (1.1) is only uniformly stable, the perturbed system (1.2) is uniformly asymptotically stable, an improvement caused by the perturbing term.

THEOREM 2.2. *Assume that*

(1) *the conditions* (i) *and* (ii) *of Theorem* 1.1 *hold with* $p = \infty$;

(2) $b(\|x\|) \leqq V(t, x) \leqq a(t, \|x\|)$, $(t, x) \in R^+ \times R^n$, *where* $a \in C[R^+ \times R^+, R^+]$, $a(t, u)$ *is increasing in* u, $b \in C[R^+, R^+]$, $b(u)$ *is increasing in* u, $b(0) = 0$ *and* $b(u) \to \infty$ *as* $u \to \infty$;

(3) *the system* (1.1) *is equibounded for* $0 \leqq \alpha < \eta(t_0)$ *and all solutions* $u(t, t_0, u_0)$ *of* (1.5) *satisfy* $\lim_{t\to\infty} u(t, t_0, u_0) = 0$.

Then every solution $x(t, t_0, x_0)$ *of* (1.2) *satisfies* $\lim_{t\to\infty} x(t, t_0, x_0) = 0$.

Proof. Since the system (1.1) is equibounded, given $0 < \alpha < \eta(t_0)$, $t_0 \in R^+$, there exists a $\beta = \beta(t_0, \alpha)$ such that

$$\|y(t, t_0, x_0)\| < \beta, \ t \geqq t_0 \text{ if } \|x_0\| \leqq \alpha.$$

Let $\|x_0\| \leqq \alpha$ so that $a(t_0, \|y(t, t_0, x_0)\|) \leqq a(t_0, \beta) \equiv \gamma(t_0, \alpha)$. This, in view of Theorem 1.1 and (2), implies that

$$\begin{aligned} b(\|x(t, t_0, x_0)\|) &\leqq V(t, x(t, t_0, x_0)) \leqq r(t, t_0, V(t_0, y(t, t_0, x_0))) \\ &\leqq r(t, t_0, \gamma(t_0, \alpha)), \ t \geqq t_0. \end{aligned}$$

Now the assumption (3) assures the stated conclusion and the proof is complete.

We may also approach the problem in a more fruitful way by introducing a new concept of stability. Let $y(t, t_0, x_0)$, $u(t, t_0, u_0)$ be solutions of (1.1) and (1.5) respectively. Then we define

$$v(t, t_0, x_0) = u(t, t_0, V(t_0, y(t, t_0, x_0))), \tag{2.3}$$

and note that $v(t_0, t_0, x_0) = V(t_0, x_0)$.

Definition. The differential equations (1.1) and (1.5) are said to be connectively quasi-equi-asymptotically stable if given $\alpha \geqq 0$, $\epsilon > 0$ and $t_0 \in R^+$ there exists a $T = T(t_0, \alpha, \epsilon)$ such that $\|x_0\| \leqq \alpha$ implies $v(t, t_0, x_0) < \epsilon$, $t \geqq t_0 + T$.

Notice that for the foregoing definition to hold it is not necessary that equations (1.1) and (1.5) have the trivial solutions. If, on the other hand, one assumes that (1.1) and (1.5) have unique trivial solutions and also that $V(t, 0) \equiv 0$, then it is clear from (2.3) that $v(t, t_0, 0) \equiv 0$. Consequently, in that case, we have the following definition.

Definition. The trivial solutions $x = 0$ and $u = 0$ of (1.1) and (1.5) are said to be connectively equi-stable if given $\epsilon < 0$, $t_0 \in R^+$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that $\|x_0\| < \delta$ implies that

$$v(t, t_0, x_0) < \epsilon, \ t \geqq t_0.$$

Other definitions may similarly be formulated.

Remark. It is important to realize that the connective stability concept is a joint property of the equations (1.1) and (1.5) and it does not necessarily imply that each of the equations (1.1) and (1.5) possesses the same kind of stability property. In fact, it is this ingredient that offers more flexibility in applications.

One can easily prove results which are consequences of connective stability notions using standard arguments. We merely state a typical result.

THEOREM 2.3. *Suppose that the conditions* (1) *and* (2) *of Theorem* 2.2 *are satisfied. Then the connective quasi-equi-asymptotic stability of equations* (1.1) *and* (1.5) *implies that the system* (1.2) *is quasi-equi-asymptotically stable.*

3. **An example.** Here we shall present a simple but illustrative example. Consider

$$y' = e^{-t}y^2, y(t_0) = x_0 \geqq 0, \tag{3.1}$$

whose solutions are given by $y(t, t_0, x_0) = x_0/[1 + x_0(e^{-t} - e^{-t_0})]$. Thus the fundamental matrix solution of the corresponding variational equation is $\Phi(t, t_0, x_0) = 1/[1 + x_0(e^{-t} - e^{-t_0})]^2$. Consequently, choosing $V(t, x) = x^2$, we see that

$$D^+V = 2y(t, s, x)\Phi(t, s, x)R(s, x), \tag{3.2}$$

where $R(t, x)$ is the perturbation. Let $R(t, x) = -x^2/2$ so that the perturbed equation is

$$x' = e^{-t}x^2 - (x^2/2), x(t_0) = x_0 \geqq 0. \tag{3.3}$$

Accordingly, it is easily seen that $g(t, u) = -u^{3/2}$ and hence the solutions of

$$u' = -u^{3/2}, u(t_0) = u_0 \geqq 0, \tag{3.4}$$

are $u(t, t_0, u_0) = 4u_0/[2 + u_0^{1/2}(t - t_0)]^2$. Thus, by Theorem 1.1, we get the relation

$$|x(t, t_0, x_0)|^2 \leqq \frac{|x_0|^2}{[1 + x_0(e^{-t} - e^{-t_0} + (t - t_0)/2)]^2}, t \geqq t_0, \tag{3.5}$$

which shows that all solutions $x(t, t_0, x_0) \to 0$ as $t \to \infty$, although only some solutions $y(t, t_0, x_0)$ are bounded. For example, setting $t_0 = 0$ and $x_0 = 1$ shows that e^t is the corresponding solution of (3.1) whereas for the same initial conditions the solution of (3.3) is $2/(2 + t + 2e^{-t})$.

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