

A TECHNIQUE IN STABILITY THEORY OF
DELAY-DIFFERENTIAL EQUATIONS

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In the study of stability theory for delay-differential equations using Lyapunov functions and the theory of differential inequalities, it becomes necessary to choose an appropriate minimal class of functions relative to which the derivative of the Lyapunov function is estimated. This approach has recently been recognized as a very natural method in the study of the qualitative behavior of delay equations.

Consider a system of functional differential equations of the form

$$(0.1) \quad x'(t) = f(t, x_t), \quad x_{t_0} = \psi_0.$$

Let V be a Lyapunov function and define, as usual,

$$D^+V(t, \psi(0), \psi) \equiv \limsup_{h \rightarrow 0^+} \frac{1}{h} [V(t+h, \psi(0) + hf(t, \psi)) - V(t, \psi(0))].$$

It is well known [3] that if

$$(0.2) \quad D^+V(t, \psi(0), \psi) \leq g(t, V(t, \psi(0)))$$

for $\psi \in \Omega$ where

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$$(0.3) \quad \Omega = [\psi: V(t+s, \psi(s)) \leq V(t, \psi(0)), \quad -\tau \leq s \leq 0],$$

the stability of the null solution of

$$(0.4) \quad u' = g(t, u),$$

implies the stability of the null solution of (0.1).

In this method, the following comparison result and its variants are employed [2,3].

Theorem 0.1. If $D^+m(t) \leq g(t, m(t))$ for $t \in I$ where

$$I = [t > t_0: m(t+s) \leq m(t), \quad -\tau \leq s \leq 0],$$

and $g \geq 0$, then $m(t) \leq r_R(t, t_0, u_0)$, $t \geq t_0$, provided $m(t_0+s) \leq u_0$, $-\tau \leq s \leq 0$, $r_R(t, t_0, u_0)$ being the right maximal solution of (0.4).

If $g \geq 0$ is not demanded, the assumptions of Theorem 0.1 have to be strengthened to

$$D^+m(t) \leq g(t, m(t)), \quad \text{for } t \in I_0, \quad \text{where}$$

$$I_0 = [t > t_0: m(t+s) \leq r_L(t+s, t, m(t)), \quad -\tau \leq s \leq 0],$$

$r_L(t, t_0, u_0)$ being the left maximal solution of (0.4) which is assumed to exist [3].

If we apply Theorem 0.1 to the example

$$(0.5) \quad x'(t) = -b(x(t) + x^3(t)) - \alpha \int_{t-\tau}^t (x(s) + x^3(s)) ds, \quad a, b, \tau > 0$$

with $V(t, x) = x^2$, $g(t, u) \equiv 0$, and $\Omega = [\psi: |\psi(s)|^2 \leq |\psi(0)|^2]$, it

easily follows that the trivial solution of (0.5) is stable if $a\tau \leq b$.

However, for the equation

$$(0.6) \quad x'(t) = -a \int_{t-\tau}^t (x(s) + x^3(s)) ds,$$

this method offers no information at all.

In this paper, we develop new comparison results which offer better mechanism for the study of functional differential equations. The idea is to use upper and lower comparison estimates simultaneously together with auxiliary functions that are constructed so as to sandwich the growth of the function to be compared in a best possible way. Our work is influenced by the recent work of Ansari [1], who introduces the idea of using four comparison functions to obtain sharper results for the scalar functional differential equation. Our comparison results are through arbitrary cones and are consequently in terms of systems of differential inequalities. However, we do not assume the quasimonotone nondecreasing property as is commonly assumed in discussing systems of differential inequalities. Our assumptions are weaker and are implied by quasimonotone property.

As an application of the comparison results obtained, we prove a typical result on stability and demonstrate how our technique yields a larger region of stability for (0.5) and implies stability for the equation (0.6).

I. COMPARISON RESULTS

A proper subset K of R^n is called a solid cone if (i) $\lambda K \subset K$,

$\lambda \geq 0$, (ii) $K + K \subset K$, (iii) $K = \bar{K}$, (iv) $K \cap (-K) = \{0\}$ and (v) K^0 is nonempty. Here \bar{K} is the closure of K and K^0 is the interior of K .

The cone K induces the order relation on R^n defined by

$$x \leq y \text{ iff } y - x \in K \text{ and } x < y \text{ iff } y - x \in K^0.$$

The set K^* defined by $K^* = \{\phi \in R^n: \phi(x) \geq 0 \text{ for all } x \in K\}$ satisfies the properties (i) to (v) and is called the adjoint cone.

We note that $K = (K^*)^*$, $x \in K^0$ iff $\phi(x) > 0$ for all $\phi \in K_0^*$ and $x \in \partial K$, iff $\phi(x) = 0$ for some $\phi \in K_0^*$, where $K_0 = K - \{0\}$ and ∂K is the boundary of K .

For any $\tau > 0$, let $C^n = C[[-\tau, 0], R^n]$ with $\|\psi\|_0 = \max_{-\tau \leq \theta \leq 0} \|\psi(\theta)\|$ and for any $t \geq 0$, let $x_t \in C^n$ be defined by $x_t = x_t(\theta) = x(t+\theta)$, $-\tau \leq \theta \leq 0$, where $x \in C[[t_0 - \tau, \infty), R^n]$. Let us list the following assumptions for convenience.

(A₁) for $i = 1, 2$, $r_i, \rho_i \in C[[t_0 - \tau, \infty) \times [t_0, \infty), R^n]$, $t_0 \geq 0$, $\tau > 0$ such that

$$(a) \quad r_2(\theta, t_0) \leq \rho_i(\theta, t) \leq r_1(\theta, t_0), \quad \theta \in (t_i, t), \text{ where}$$

$$t_i = t_i(t) \text{ is chosen such that } t_0 \leq t_i \leq t;$$

$$(b) \quad r_i(t, t_0) = \rho_i(t, t), \quad t \geq t_0;$$

$$(c) \quad r_1(\theta, t_0) = \rho_2(\theta, t), \quad t - \tau \leq \theta \leq t_2,$$

$$r_2(\theta, t_0) = \rho_1(\theta, t), \quad t - \tau \leq \theta \leq t_1.$$

(A₂) $m \in C[[t_0 - \tau, \infty), R^n]$, $g_i \in C[[t_0, \infty) \times R^n \times C^n, R^n]$, $i = 1, 2$ and

$$(-1)^{i+1} D^+ m(t) \leq (-1)^{i+1} g_i(t, m(t), m_t), \quad t > t_0.$$

(A₃) for $i = 1, 2$, $\phi \in K^*$,

$$(i) \quad (-1)^{i+1} D^+ \phi(r_i(t, t_0)) > (-1)^{i+1} \phi(g_i(t, u(t), u_t)),$$

for all functions $u(t)$ such that $\phi(r_i(t, t_0)) = \phi(u(t))$,

$$(-1)^{i+1} \phi(\rho_i(s, t)) \leq (-1)^{i+1} \phi(u(s)), \quad t-\tau \leq s \leq t \quad \text{and}$$

$$r_2(s, t_0) \leq u(s) \leq r_1(s, t_0), \quad t-\tau \leq s \leq t,$$

and

$$(ii) \quad (-1)^{i+1} D^+ \phi(\rho_i(\xi, t)) > (-1)^{i+1} \phi(g_i(\xi, u(\xi), u_\xi)), \quad t_i < \xi \leq t,$$

for all functions $u(t)$ such that $\phi(u(\xi)) = \phi(\rho_i(\xi, t))$ and

$$r_2(s, t_0) \leq u(s) \leq r_1(s, t_0), \quad \xi-\tau \leq s \leq \xi.$$

We are now in a position to prove the following comparison theorem

Theorem 1.1. If the assumptions (A₁), (A₂) and (A₃) hold, then

$$(1.1) \quad r_2(t, t_0) < m(t) < r_1(t, t_0), \quad t \geq t_0,$$

provided that

$$(1.2) \quad r_2(s, t_0) < m(s) < r_1(s, t_0), \quad t_0-\tau \leq s \leq t_0.$$

Proof. Suppose that the assertion (1.1) is false. Then there exists

a $t^* > t_0$ such that

$$\text{either } r_1(t^*, t_0) - m(t^*) \in \partial K \quad \text{or} \quad m(t^*) - r_2(t^*, t_0) \in \partial K$$

and

$$r_1(t, t_0) - m(t) \in K^0, \quad m(t) - r_2(t, t_0) \in K^0 \quad \text{for } t \in (t_0, t^*).$$

Hence there exists a $\phi \in K_0^*$ such that

$$(1.3) \quad \phi(m(t^*)) = \phi(r_i(t^*, t_0)), \quad i = 1 \text{ or } 2.$$

Also, we have

$$(1.4) \quad r_2(s, t_0) \leq m(s) \leq r_1(s, t_0), \quad t_0 - \tau \leq s \leq t^*.$$

In view of (b) of (A_1) , it then follows that

$$(1.5) \quad \phi(m(t^*)) = \phi(\rho_i(t^*, t^*)), \quad i = 1 \text{ or } 2$$

We shall first consider the case $i = 1$. Let us show that $\phi(\rho_1(t, t^*)) \leq \phi(m(t))$, $t \in [t_1, t^*]$ so that by (c) of (A_1) , we then have

$$(1.6) \quad \phi(\rho_1(s, t^*)) \leq \phi(m(s)), \quad t^* - \tau \leq s \leq t^*.$$

For this purpose, we let $v(t) = \phi(m(t) - \rho_1(t, t^*))$ and note that $D^+v(t^*) < 0$ because of (1.4), (1.5), (ii) of (A_3) and (A_2) . It then follows by (1.5) that $v(t)$ is increasing for $t < t^*$ in a sufficiently small interval $t^* - \epsilon \leq t \leq t^*$, $\epsilon > 0$, which implies that

$$\phi(m(t^* - \epsilon)) > \phi(\rho_1(t^* - \epsilon, t^*)).$$

We now wish to show that $\phi(m(s)) > \phi(\rho_1(s, t^*))$ for $t_1 < s < t^* - \epsilon$.

If this is not true, there would exist a $s^* \in (t_1, t^* - \epsilon)$ such that

$$(1.7) \quad \phi(m(s^*)) = \phi(\rho_1(s^*, t^*)),$$

and
$$\phi(m(s)) > \phi(\rho_1(s, t^*)), \quad s \in (s^*, t^* - \epsilon).$$

We therefore have $D^+\phi(m(s^*)) \geq D^+\phi(\rho_1(s^*, t^*))$. However, in view of

(1.4) and (1.7), the assumption (ii) of (A_3) and (A_2) show that

$$D^+\phi(\rho_1(s^*, t^*)) > \phi(g_1(s^*, m(s^*), m_{s^*})) \geq D^+\phi(m(s^*)),$$

a contradiction. Thus (1.6) is true and consequently, the relations (1.3), (1.4), (1.6) and the assumptions (i) of (A_3) and (A_2) imply

$$D^+\phi(r_1(t^*, t_0)) > \phi(g_1(t^*, m(t^*), m_{t^*})) \geq D^+\phi(m(t^*))$$

which shows [2] $D_-\phi(r_1(t^*, t_0)) > D_-\phi(m(t^*))$. On the other hand, (1.3) and (1.4) yield $D_-\phi(r_1(t^*, t_0)) \leq D_-\phi(m(t^*))$, which is a contradiction.

In case $i = 2$, we proceed similarly to show first that $\phi(m(s)) \leq \phi(\rho_2(s, t^*))$, $t^* - \tau \leq s \leq t^*$ which leads to a contradiction as before. Thus the assertion (1.1) is true and the proof is complete.

The following corollary is useful in obtaining upper and lower bounds on the solutions of functional differential equations.

Corollary 1.1. Suppose that the assumptions (A_1) and (A_3) hold with $g_1 \equiv g_2 \equiv g$. Let $m(t)$ be any solution of the functional differential equation

$$(1.8) \quad m'(t) = g(t, m(t), m_t), \quad m_{t_0} = \psi_0.$$

Then $r_2(t_0 + s, t_0) < m(t_0 + s) < r_1(t_0 + s, t_0)$, $-\tau \leq s \leq 0$ implies

$$r_2(t, t_0) < m(t) < r_1(t, t_0), \quad t \geq t_0.$$

If the cone $K = R_+^n$ so that the inequalities between two vectors are component-wise inequalities, we have a further specialization of

Corollary 1.1. We list below a comparison result that simplifies the assumptions (A_1) and (A_3) even more, at the cost of stronger requirements.

Corollary 1.2. Assume that

- (i) $K = \mathbb{R}_+^n$, $g_1 \equiv g_2 \equiv g$ and $m(t)$ is any solution of (1.8);
(ii) $r_i \in C[[t_0 - \tau, \infty), \mathbb{R}^n]$, $t_0 \geq 0$, $\tau > 0$ such that for $1 \leq k \leq n$,

$$D_- r_2^k(t) < g_k(t, u(t), u_t) \quad \text{if } u_k(t) = r_2^k(t) \quad \text{and} \\ r_2(\theta) \leq u(\theta) \leq r_1(\theta), \quad t - \tau \leq \theta \leq t,$$

$$D_- r_1^k(t) > g_k(t, u(t), u_t) \quad \text{if } u_k(t) = r_1^k(t) \quad \text{and} \\ r_2(\theta) \leq u(\theta) \leq r_1(\theta), \quad t - \tau \leq \theta \leq t.$$

Then $r_2(t_0 + \theta) < m(t_0 + \theta) < r_1(t_0 + \theta)$, $-\tau \leq \theta \leq 0$ implies

$$r_2(t) < m(t) < r_1(t), \quad t \geq t_0.$$

The scalar version of Theorem 1.1 is itself important in applications which we list separately.

Theorem 1.2. Assume that

- (i) (A_1) and (A_2) hold with $n = 1$;
(ii) for $i = 1, 2$,

$$(-1)^{i+1} D^+ r_i(t, t_0) > (-1)^{i+1} g_i(t, u(t), u_t),$$

for all functions $u(t)$ such that $r_i(t, t_0) = u(t)$ and

$$(-1)^{i+1} \rho_i(s, t) \leq (-1)^{i+1} m(s) \leq (-1)^{i+1} r_i(s, t_0), \quad t-\tau \leq s \leq t.$$

(iii) for $i = 1, 2,$

$$(-1)^{i+1} D^+ \rho_i(\xi, t) > (-1)^{i+1} g_i(\xi, u(\xi), u_\xi), \quad t_i < \xi \leq t,$$

for all functions $u(t)$ such that $u(\xi) = \rho_i(\xi, t)$ and

$$r_2(s, t_0) \leq u(s) \leq r_1(s, t_0), \quad \xi-\tau \leq s \leq \xi.$$

Then $r_2(t_0+s, t_0) < m(t_0+s) < r_1(t_0+s, t_0), \quad -\tau \leq s \leq 0$ implies

$$r_2(t, t_0) < m(t) < r_1(t, t_0), \quad t \geq t_0.$$

Remark 1.1. In particular, if $g_1 \equiv g_2 \equiv g$ and $m(t)$ is a solution of (1.8), assumption (A_2) can be dispensed with in Theorem 1.2.

II. STABILITY RESULTS

Let us consider the functional differential system

$$(2.1) \quad x'(t) = f(t, x(t), x_t), \quad x_{t_0} = \psi_0,$$

where $f \in C[[t_0, \infty) \times S_\rho \times C_\rho, \mathbb{R}^n]$. Here $S_\rho = \{x \in \mathbb{R}^n: \|x\| < \rho\}$ and $C_\rho = \{\psi \in C^0: \|\psi\|_0 < \rho\}$. Let $V \in C[[t_0-\tau, \infty) \times S_\rho, \mathbb{R}^1]$ and define for $(t, \psi) \in [t_0, \infty) \times C_\rho$,

$$(2.2) \quad D^+ V(t, \psi(0), \psi) \equiv \limsup_{h \rightarrow 0^+} \frac{1}{h} [V(t+h, \psi(0) + hf(t, \psi(0), \psi)) - V(t, \psi(0))].$$

We shall now prove a typical result on stability as an application of Theorem 1.1. Other types of stability results can be formulated based on this result.

Theorem 2.1. Assume that

- (i) the hypotheses (A_1) and (A_3) hold;
 (ii) $V \in C[[t_0 - \tau, \infty) \times S_{\rho}, R^N]$, $V(t, x)$ is locally Lipschitzian in x and

$$(-1)^{i+1} D^+ V(t, \psi(0), \psi) \leq (-1)^{i+1} g_i(t, V(t, \psi(0)), V_t), \quad i = 1, 2,$$

where $g_i \in C[[t_0, \infty) \times R^N \times C^N, R^N]$ and $V_t = V(t+s, \psi(s))$,

$$-\tau \leq s \leq 0;$$

- (iii) for some $\phi_0 \in K_0^*$ and $(t, x) \in [t_0, \infty) \times S_{\rho}$, $|\phi_0(V(t, x))|$ is positive definite and decrescent;

- (iv) given $\epsilon > 0$, $t_0 \in R_+$, there exists a $\delta(\epsilon) > 0$ such that

$$(-1)^{i+1} \phi_0(r_2(t, t_0)) < (-1)^{i+1} \epsilon, \quad t \geq t_0,$$

provided $(-1)^{i+1} \phi_0(r_2(t_0+s, t_0)) < (-1)^{i+1} \delta(\epsilon)$, $-\tau \leq s \leq 0$.

Then the trivial solution of (2.1) is stable.

Proof. Let $x(t_0, \psi_0)(t)$ be any solution of (2.1). Set $m(t) = V(t, x(t_0, \psi_0)(t))$. By (ii), we then obtain

$$(-1)^{i+1} D^+ m(t) \leq (-1)^{i+1} g_i(t, m(t), m_t), \quad t > t_0,$$

which shows that (A_2) holds. Hence by Theorem 1.1, we get

$$r_2(t, t_0) < V(t, x(t_0, \psi_0)(t)) < r_1(t, t_0), \quad t > t_0,$$

provided $r_2(t_0+s, t_0) < V(t_0+s, \psi_0(s)) < r_1(t_0+s, t_0)$, $-\tau \leq s \leq 0$. We can now use standard technique to prove the stability of the trivial

solution of (2.1) in view of (iii) and (iv). We omit the details.

Let us now apply our method to the example (0.5). Let us first note that we do not have to assume α to be positive in which case we get $|\alpha\tau| \leq b$ by the method described in the introduction.

To demonstrate our method, let us choose the functions $r_i, \rho_i, i = 1, 2$, as follows:

$$r_1(t, t_0) = -r_2(t, t_0) = \varepsilon, \quad -\tau \leq t \leq \infty,$$

$$\rho_2(s, t) = -\rho_1(s, t) \quad \text{for } t > 0, \quad t - \tau \leq s \leq t,$$

$$\rho_1(s, t) = \begin{cases} -\varepsilon, & t - \tau \leq s \leq t_1(t), \\ \varepsilon - (t-s) \frac{A\varepsilon}{2}, & t_1(t) \leq s \leq t, \end{cases}$$

where $t_1(t)$ is chosen such that

$$t_1(t) = \max\left[0, t - \frac{4}{A}\right].$$

Here $A = 2(b + \alpha\tau)(1 + \varepsilon^2)$. One can easily check that these functions are admissible in Theorem 1.2 with $g_1 \equiv g_2 \equiv g$, where $g(t, x, x_t) = -b(x(t) + x^3(t)) - \alpha \int_{t-\tau}^t (x(s) + x^3(s)) ds$ and $x(t)$ is any solution of (0.5). We obtain

$$x'(t) \leq -b(\varepsilon + \varepsilon^3) + \alpha(\varepsilon + \varepsilon^3)(t_1 - t + \tau) - \alpha \int_{t_1}^t [\rho_1(s, t) + \rho_1^3(s, t)] ds$$

which yields, after simple computation,

$$(2.3) \quad (\varepsilon + \varepsilon^3) \left[-b + \alpha\left(\tau - \frac{4}{A}\right)\right] - \frac{2\alpha}{A} \left(\varepsilon + \frac{\varepsilon^3}{2}\right) < 0.$$

This shows that it is enough to assume (for convenience of illustration) that

$$-b + a\left(\tau - \frac{4}{A}\right) \leq 0;$$

which implies that

$$(2.4) \quad a \leq \frac{1 + \sqrt{1 + b^2 \tau^2 (1 + \epsilon^2)^2}}{\tau^2 (1 + \epsilon^2)}$$

If $b = 0$, we have (for $\epsilon = 1$),

$$(2.5) \quad a\tau^2 \leq \frac{2}{1 + \epsilon^2} = 1.$$

Consequently, if a and τ satisfy the inequality (2.5), the null solution of (0.6) is stable.

Remark. After this paper was written, we learnt from Professor W. Walter, the existence of the important result of Müller [4,5] for ordinary differential equations. It so happens that our Corollary 1.2 is an extension of Müller's [4] result for functional differential equations.

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