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the Equation $|Ax| - |B|x| = b$**

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Technical report No. V-1092

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A Theorem of the Alternatives for the Equation $|Ax| - |B||x| = b$

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Abstract:

A theorem of the alternatives for the equation $|Ax| - |B||x| = b$ ($A, B \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$) is proved and several consequences are drawn. In particular, a class of matrices A, B is identified for which the equation has exactly 2^n solutions for each positive right-hand side b .

Keywords:

Absolute value equation, triple absolute value equation, alternatives, solution set, interval matrix, regularity.

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1 Introduction

We consider here the equation

$$|Ax| - |B||x| = b, \tag{1.1}$$

where $A, B \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$, which we call a *triple absolute value equation*. This equation could also be written in the form

$$\begin{aligned} |Ax| - C|x| &= b, \\ C &\geq 0, \end{aligned}$$

but we prefer the one-line expression (1.1). As far as known to us, nobody has studied this equation as yet.

In the main result of this paper we show that for each $A, B \in \mathbb{R}^{n \times n}$ exactly one of the following two alternatives holds: (i) for each $b > 0$ the equation (1.1) has exactly 2^n solutions and the set $\{Ax \mid |Ax| - |B||x| = b\}$ intersects interiors of all orthants of \mathbb{R}^n , (ii) the equation (1.1) has a nontrivial solution for some $b \leq 0$. In Corollary 2 we show that, even more, if the property mentioned in (i) holds for *some* $b_0 > 0$, then it is shared by *any* $b > 0$, and in Corollary 3 we prove that if A is nonsingular and the condition

$$\varrho(|A^{-1}||B|) < 1 \tag{1.2}$$

is satisfied, then (i) holds, so that for each $b > 0$ the equation (1.1) has exactly 2^n solutions. As it will be shown later, these results follow from necessary and/or sufficient conditions for regularity/singularity of interval matrices when applied to the interval matrix $[A - |B|, A + |B|]$. In turn, our results enable us to add three more such necessary and sufficient conditions to the list of forty of them surveyed in [11] (Proposition 7 below).

Nearest in form to the equation (1.1) is the *absolute value equation*

$$Ax + B|x| = b \tag{1.3}$$

which has been recently studied by Mangasarian [2], [3], [4], Mangasarian and Meyer [5], Prokopyev [7], and Rohn [10], [12]. There is, however, a big difference between these two equations: while the equation (1.3) has under the condition (1.2) exactly one solution for each b , the equation (1.1) under the same condition has exactly 2^n solutions for each $b > 0$. This sharp difference between both the equations is to be ascribed to the absence/presence of the absolute value of the term Ax .

The particular circumstances of discovery of the main theorem are briefly mentioned in the personal note in Section 8.

2 Notations

We use the following notations. Matrix inequalities, as $A \leq B$ or $A < B$, are understood componentwise. The absolute value of a matrix $A = (a_{ij})$ is defined by $|A| = (|a_{ij}|)$. The same notations also apply to vectors that are considered one-column matrices. For each $y \in \{-1, 1\}^n$ we denote

$$T_y = \text{diag}(y_1, \dots, y_n) = \begin{pmatrix} y_1 & 0 & \dots & 0 \\ 0 & y_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y_n \end{pmatrix},$$

and $\mathbb{R}_y^n = \{x; T_y x \geq 0\}$ is the orthant prescribed by the ± 1 -vector y . Notice that $T_y^{-1} = T_y$ for such a y . $\rho(A)$ stands for the spectral radius of A . Given $A, B \in \mathbb{R}^{n \times n}$, the set

$$[A - |B|, A + |B|] = \{S \mid |S - A| \leq |B|\}$$

is an interval matrix; it is called regular if each $S \in [A - |B|, A + |B|]$ is nonsingular, and it is said to be singular otherwise (i.e., if it contains a singular matrix).

3 Theorem of the alternatives

To simplify formulations, let us say that the equation (1.1) is *exponentially solvable* for a particular right-hand side b if it has exactly 2^n solutions and the set

$$\{Ax \mid |Ax| - |B||x| = b\} \quad (3.1)$$

intersects interiors of all orthants of \mathbb{R}^n . The following theorem is the main result of this paper.

Theorem 1. *For each $A, B \in \mathbb{R}^{n \times n}$ exactly one of the following two alternatives holds:*

- (i) *the equation (1.1) is exponentially solvable for each $b > 0$,*
- (ii) *the equation (1.1) has a nontrivial solution for some $b \leq 0$.*

Proof. Consider the following two options for the interval matrix $[A - |B|, A + |B|]$:

- (i') $[A - |B|, A + |B|]$ is regular,
- (ii') $[A - |B|, A + |B|]$ is singular.

We shall prove that the assertions (i), (ii) are equivalent to (i'), (ii'), respectively. Since exactly one of (i'), (ii') always holds, the same will be true for (i), (ii).

(i) \Rightarrow (i'). Let (i) hold. Take any $b_0 > 0$, then for each ± 1 -vector $y \in \mathbb{R}^n$ there exists a solution x_y of the equation $|Ax| - |B||x| = b_0$ such that $Ax_y \in \mathbb{R}_y^n$. Since x_y satisfies $|Ax_y| = |B||x_y| + b_0 > |B||x_y|$, the condition (v) of Theorem 3.1 in [9] is met and consequently the interval matrix $[A - |B|, A + |B|]$ is regular.

(i') \Rightarrow (i). If (i') holds, then for each ± 1 -vector y the interval matrix

$$[A - | -T_y|B||, A + | -T_y|B||] = [A - |B|, A + |B|]$$

is regular, hence by Proposition 4.2 in [10] the equation

$$Ax - T_y|B||x| = T_y b \quad (3.2)$$

has a unique solution x_y . This x_y then satisfies

$$T_y Ax_y - |B||x_y| = b, \quad (3.3)$$

which implies

$$T_y Ax_y = |B||x_y| + b \geq b > 0, \quad (3.4)$$

hence Ax_y belongs to the interior of \mathbb{R}_y^n and $T_y Ax_y = |Ax_y|$, which in view of (3.3) means that x_y is a solution of (1.1). Conversely, let x solve (1.1). Put $y_i = 1$ if $(Ax)_i \geq 0$ and $y_i = -1$ otherwise ($i = 1, \dots, n$), then $T_y Ax = |Ax|$, so that x is a solution of

$$T_y Ax - |B||x| = b$$

and thus also of (3.2). Because of the above-stated uniqueness of solution of (3.2), this implies that $x = x_y$. In this way we have proved that the solution set of (1.1) consists precisely of the points x_y for all possible ± 1 -vectors $y \in \mathbb{R}^n$. Thus to prove that (1.1) has exactly 2^n solutions, it will suffice to show that all the x_y 's are mutually different. To this end, take two ± 1 -vectors y and y' , $y \neq y'$. Then $y_i y'_i = -1$ for some i . From (3.4) it follows that $y_i (Ax_y)_i > 0$ and $y'_i (Ax_{y'})_i > 0$ and by multiplication $y_i (Ax_y)_i y'_i (Ax_{y'})_i > 0$, hence $(Ax_y)_i (Ax_{y'})_i < 0$, which clearly shows that $x_y \neq x_{y'}$.

(ii) \Leftrightarrow (ii'). Existence of a nontrivial solution of (1.1) for some $b \leq 0$ is equivalent to existence of a nontrivial solution of the inequality

$$|Ax| \leq |B||x|, \quad (3.5)$$

which, by Proposition 2.2 in [10], is in turn equivalent to singularity of the interval matrix $[A - |B|, A + |B|]$.

This proves the theorem. \square

4 Consequences

We can draw some consequences from Theorem 1 and its proof.

Corollary 2. *If the equation (1.1) is exponentially solvable for some $b_0 > 0$, then it is exponentially solvable for each $b > 0$.*

Proof. Indeed, in the proof of Theorem 1, implication “(i) \Rightarrow (i)”, we showed that exponential solvability of the equation (1.1) for some $b_0 > 0$ implies regularity of $[A - |B|, A + |B|]$ and thus, by “(i') \Rightarrow (i)”, also exponential solvability for each $b > 0$. \square

Corollary 3. *If A is nonsingular and*

$$\varrho(|A^{-1}||B|) < 1 \quad (4.1)$$

holds, then the equation (1.1) is exponentially solvable for each $b > 0$.

Proof. By the well-known Beeck's result in [1], the condition (4.1) implies regularity of the interval matrix $[A - |B|, A + |B|]$ and thus, by the equivalence “(i) \Leftrightarrow (ii)” established in the proof of Theorem 1, it also implies exponential solvability of (1.1) for each $b > 0$. \square

Corollary 4. *If A is nonsingular and*

$$\max_j (|A^{-1}||B|)_{jj} \geq 1 \quad (4.2)$$

holds, then the equation (1.1) is not exponentially solvable for any $b > 0$.

Proof. It follows from [8], Corollary 5.1, (iii) that the condition (4.2) implies singularity of the interval matrix $[A - |B|, A + |B|]$, which, by the proof of Theorem 1 and by Corollary 2, precludes exponential solvability of (1.1) for any $b > 0$. \square

For $A, B \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, denote

$$X(A, B, b) = \{x \mid |Ax| - |B||x| = b\},$$

i.e., the solution set of (1.1) (attention: not to be mixed with (3.1)). Observe that if $x \in X(A, B, b)$, then $-x \in X(A, B, b)$, hence the solutions appear in $X(A, B, b)$ in pairs $(x, -x)$. Thus, unless $b = 0$, the cardinality of $X(A, B, b)$, if finite, is even.

Corollary 5. *If the equation $|Ax| - |B||x| = b_0$ is exponentially solvable for some $b_0 > 0$, then for each $b > 0$ we have*

$$X(A, B, b) = \{x_y \mid y \in \{-1, 1\}^n\},$$

where for each $y \in \{-1, 1\}^n$, x_y is the unique solution of the absolute value equation

$$T_y Ax - |B||x| = b. \quad (4.3)$$

Proof. This has been proved in the “(i') \Rightarrow (i)” part of the proof of Theorem 1. \square

Corollary 6. *Under the assumptions of Corollary 5, we have $x_{-y} = -x_y$ for each $y \in \{-1, 1\}^n$.*

Proof. Since x_y is a solution of (4.3), it follows that $-x_y$ solves the equation

$$T_{-y} Ax - |B||x| = b,$$

and in view of the uniqueness of solution of this equation we have that $x_{-y} = -x_y$. \square

The equation (4.3) can be solved in a finite number of steps by a very efficient algorithm **absvaleqn** described in [12]. Corollary (6) reduces the number of x_y 's to be computed from 2^n to 2^{n-1} (e.g., it suffices to consider only the y 's with $y_n = 1$).

5 Example

The following computation was performed in MATLAB. Using the randomly generated data

```
>> n=3; rand('state',1); A=2*rand(n,n)-1, B=2*rand(n,n)-1, b=rand(n,1),
A =
    0.9056    0.1963    0.6736
    0.4081    0.6815    0.0374
    0.9078   -0.1144   -0.9556
```

```

B =
  -0.2482   -0.6009    0.8205
   0.7972   -0.3938    0.0506
  -0.1420    0.0766   -0.3863
b =
  0.0345
  0.7153
  0.7687

```

we obtain the solution set consisting of two solutions

```

X =
  18.4720  -18.4720
  27.7448  -27.7448
   6.0199   -6.0199

```

After scaling the matrix B by

```

>>B=0.1*B
B =
  -0.0248   -0.0601    0.0820
   0.0797   -0.0394    0.0051
  -0.0142    0.0077   -0.0386

```

we find that the solution set now consists of $2^3 = 8$ solutions

```

X =
  0.3921  -0.3921   0.1794  -0.1794   0.3679  -0.3679
  0.9521  -0.9521   1.0756  -1.0756  -1.3774   1.3774
 -0.5830   0.5830  -0.8066   0.8066  -0.3196   0.3196

  0.5740  -0.5740
 -1.5452   1.5452
 -0.0991   0.0991

```

and the set (3.1)

```

>> AX=A*X
AX =
  0.1492  -0.1492  -0.1697   0.1697  -0.1526   0.1526
  0.7870  -0.7870   0.7761  -0.7761  -0.8005   0.8005
  0.8041  -0.8041   0.8107  -0.8107   0.7968  -0.7968

  0.1497  -0.1497
 -0.8224   0.8224
  0.7925  -0.7925

```

intersects interiors of all the orthants. Hence, the equation $|Ax| - |0.1 \cdot B||x| = b$ is exponentially solvable for this right-hand side b (and thus it is exponentially solvable for each $b' > 0$).

6 Regularity conditions

Checking regularity of interval matrices is a co-NP-complete problem [6]. Forty necessary and sufficient regularity conditions were surveyed in [11]; the results of this paper enable us to add three more items to the list.

Proposition 7. *For a square interval matrix $[A - \Delta, A + \Delta]$, the following assertions are equivalent:*

(a) $[A - \Delta, A + \Delta]$ is regular,

(b) the equation

$$|Ax| - \Delta|x| = b \tag{6.1}$$

is exponentially solvable for each $b > 0$,

(c) the equation (6.1) is exponentially solvable for some right-hand side $b_0 > 0$,

(d) the equation

$$|Ax| - \Delta|x| = e$$

is exponentially solvable.

Proof. In the light of Theorem 1 and Proposition 7 we see that (a) \Rightarrow (b) \Rightarrow (d) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a) holds, which proves the mutual equivalence of all the assertions. \square

7 Conclusion

We have investigated the case of $b > 0$. For a general right-hand side b there seems not to be an easy clue to the cardinality of the solution set of (1.1). This should be a subject of further research.

8 Personal note

I am a little ashamed to admit that I discovered Theorem 1 during the Christmas Eve mass on December 24, 2006 in St Francis Church in Prague.

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