

A THEOREM OF UNIQUENESS FOR INFINITE CONVEX SURFACES (*)

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SOMMARIO. - *Si dimostra che una superficie strettamente convessa, infinita, completa, la cui immagine sferica appartiene ad una semisfera e le cui curvature principali soddisfano una condizione (4), è determinata a meno di traslazioni. Va notato che non viene fatta alcuna ipotesi sulla funzione supporto della superficie sul contorno della sua immagine sferica.*

SUMMARY. - *A theorem is proved, asserting that a strictly convex, infinite, complete surface whose spherical image belong to a half-sphere and whose principal curvatures satisfy a condition (4), is uniquely defined with an accuracy to the parallel transfer. To be noticed, that no boundary condition is laid down for the support function of the surface at the boundary of its spherical image.*

Let $f(k_1, k_2)$ be the continuous, positive, strictly increasing for each variable function, defined for $k_1 \geq 0$, $k_2 \geq 0$, $k_1 \geq k_2$, and satisfying the conditions:

- (1) $f(k_1, k_2) \rightarrow 0$ for $k_1^2 + k_2^2 \rightarrow 0$
- (2) $f(k_1, k_2) \rightarrow \infty$ for $k_1^2 + k_2^2 \rightarrow \infty$.

Let $\varphi(\nu)$ be the continuous positive function of the unit vector ν , given on the closed half-sphere $\bar{\omega}: x^2 + y^2 + z^2 = 1, z \leq 0$ satisfying the condition:

- (3) $\varphi(\nu) \geq c > 0$ on $\bar{\omega}$.

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We shall say that the twice differentiable convex surface F with the spherical image belonging to the half-sphere $\bar{\omega}$ satisfies the condition :

$$(4) \quad f(k_1, k_2) = \varphi(\nu)$$

if the principal curvatures $k_1, k_2, k_1 \geq k_2$ and the unit vector of the outer normal ν in each point of the surface satisfy this equation. The object of this paper is to prove the following theorem.

THEOREM. *Strictly convex, infinite, complete surface with the spherical image belonging to the half-sphere $\bar{\omega}$ and satisfying equation (4) is uniquely defined with an accuracy to the parallel transfer.*

This theorem differs from other theorems of uniqueness for the infinite convex surfaces by the absence of boundary condition which is usually laid down for the support function of the surface at the boundary of spherical image. Moreover, the theorem does not even imply any definite spherical image. It is only assumed that it belongs to the half-sphere $\bar{\omega}$, where the function φ is determined.

LEMMA 1. *The mean curvature of surface F , considered in the theorem, is restricted by the positive limits.*

The statement of lemma is easily obtained from the properties (1) and (2) of function f , strict positivity and boundary of function φ on $\bar{\omega}$.

LEMMA 2. *The spherical image of surface F , considered in the theorem, is an open half-sphere ω :*

$$x^2 + y^2 + z^2 = 1, \quad z < 0.$$

That the spherical image is open follows from the strict convexity of the surface. We shall show that it coincides with the half-sphere.

The set of rays coming from the point X_0 of surface F and going through the body bounded by the surface, represents a convex cone V . This cone is the limit of surfaces $\frac{1}{n}F$, obtained by similarity transformation with respect to the point X_0 with the coefficient of similarity $1/n$ when $n \rightarrow \infty$. The spherical image of cone V coincides with the closure of the spherical image of surface F . If the spherical image of the surface does not coincide with the half-sphere ω the cone V does not degenerate into a ray.

We take the tangential plane to surface F in point X_0 as the plane xy and direct z -axis so that the surface F is in half-space $z > 0$. We denote part of cone V , located in the half space $z \leq 1$ as V_1 .

We draw a sphere of radius R containing strictly inside it the cone V_1 . When n is large enough the part of surface F/n lying in the half-space $z \leq 1$ will be inside the sphere. From this the estimate $H < 4\pi R$ is received for the integral of mean curvature H of this part of the surface. For the corresponding part of the surface $FH < 4\pi Rn$. The area of this part of surface F is estimated through the area of the cone V_1 . Namely, $S > S(V_1)n^2$. From this we obtain the estimate for the specific mean curvature of the part of surface F lying in the half-space $z \leq n$

$$\frac{H}{S} < \frac{4\pi R}{S(V_1)n}.$$

We see that it is as little as you please when n is large enough. But this contradicts lemma 1.

LEMMA 3. *The surface F which is considered in the theorem is uniquely projected into the restricted convex region G on the plane xy .*

Let us draw the planes $z = n$ and $z = n - 1$, where n is large enough. From the surface they cut out a band F_n which is homeomorphic to a cylinder. Let us evaluate the specific mean curvature of this band, assuming that the region G is infinite. We shall denote the curves bounding F_n and lying in planes $z = n$ and $z = n - 1$ by γ_1 and γ_2 , respectively. Let l_1 and l_2 be the lengths of these curves. As the region G is infinite, l_1 and $l_2 \rightarrow \infty$ when $n \rightarrow \infty$. As the spherical image of surface F covers the whole half-sphere $x^2 + y^2 + z^2 = 1, z < 0$, the tangential planes to F_n when n is large enough, make angles with the plane xy as close to $\pi/2$ as you please. From this we conclude that $l_1/l_2 \rightarrow 1$ when $n \rightarrow \infty$.

Let \bar{F}_n be the closed convex surface made of F_n and the flat convex regions bounded by curves γ_1 and γ_2 . Let \bar{Z}_n be the closed cylinder with the bases in planes $z = n, z = n - 1$ and directrix γ_1 . The surface \bar{F}_n is contained in the cylinder \bar{Z}_n . Therefore the integral mean curvatures \bar{Z}_n and \bar{F}_n are connected by the inequality $H(\bar{F}_n) \leq H(\bar{Z}_n)$. The integral mean curvature of the surface \bar{F}_n is

$$H(\bar{F}_n) = H(F_n) + l_1 \left(\frac{\pi}{2} + \varepsilon_1 \right) + l_2 \left(\frac{\pi}{2} + \varepsilon_2 \right),$$

where $\varepsilon_1, \varepsilon_2 \rightarrow 0$ when $n \rightarrow \infty$. The integral mean curvature of cylinder \bar{Z}_n is

$$H(\bar{Z}_n) = \pi + \frac{\pi}{2} l_1 \cdot 2.$$

From this we obtain the estimate for the integral mean curvature of surface F_n

$$H(F_n) < l_1 |\varepsilon_1| + l_2 |\varepsilon_2| + (l_1 - l_2) \frac{\pi}{2} + \pi.$$

For the area of surface F_n we have an evident estimate $S(F_n) > l_2 \cdot 1$. Accordingly, for the specific mean curvature of the surface F_n we have

$$\frac{H(F_n)}{S(F_n)} < \frac{l_1}{l_2} |\varepsilon_1| + |\varepsilon_2| + \left(\frac{l_1}{l_2} - 1 \right) \frac{\pi}{2} + \frac{\pi}{S(F_n)}.$$

When $n \rightarrow \infty$ each term on the right hand of inequality tends to zero. So, $H(F_n)/S(F_n) \rightarrow 0$ which is impossible because of lemma 1. Thus, the region G in which the surface F is projected is finite.

LEMMA 4. *The region G in which the surface F is projected onto the plane xy is uniquely defined by the functions f and φ with the accuracy to a parallel transfer.*

Let γ be the closed convex curve bounding the region G . We shall show that the curve γ is smooth. Let Z denote the cylinder drawn through the curve γ with the generatrices parallel to z -axis. We shall move the surface F by n in the direction $z < 0$ and denote the part of the surface between the planes $z = 0$ and $z = 1$ by F_n . When $n \rightarrow \infty$ the surface F_n converges to the cylinder Z .

Let P be the angle point of curve γ and g the cylinder Z generatrix which goes through this point. Consider the part $F_{n,\varepsilon}$ of the surface F_n which is contained in ε -neighbourhood of the generatrix g . Because of the convergence F_n to Z the lower limit of the integral mean curvature of the surface $F_{n,\varepsilon}$ at the fixed ε and $n \rightarrow \infty$ is not less than $\vartheta \cdot 1$, where ϑ is the turn of curve γ in point P ($\vartheta > 0$). As to the area of surface $F_{n,\varepsilon}$ it is small together with ε . From this it follows that the specific mean curvature of surface $F_{n,\varepsilon}$ is as large as you please when ε is small and n large enough. But this contradicts lemma 1.

Let us show that the curvature of curve γ is a quite definite function of the outer normal ν and is determined from the equation $f(k_1, 0) = \varphi(\nu)$.

Let A be the arbitrary point of curve γ and ν_A the outer normal in this point. From z -axis let us draw two half-planes making a small angle α which contains the point A . The parts of surface F_n and cylinder Z which are inside this angle we denote by F'_n and Z' . When $n \rightarrow \infty$ the mean integral curvature F'_n converges to the mean integral curvature $Z' : H(F'_n) \rightarrow H(Z')$. We draw a middle half-plane from z -axis inside the angle α . It will intersect the surface F'_n along the curve γ' . When n is large enough the turn of curve γ' is as little as you please, $< \epsilon_n$. Accordingly, the measure of the set of points, where the curvature of curve γ' is more than $k > 0$, does not exceed ϵ_n/k . In the points of that set the smaller of the main curvatures (k_2) of surface F'_n is more than k .

Now we shall take the fixed small number k and a large enough number n . Then the smaller of the main curvatures of surface F'_n will be less than k everywhere but the set of measure m'_n , when $m'_n \rightarrow 0$ for $n \rightarrow \infty$. As to the larger curvature (k_1) it is certain to be bounded.

Because of the unique solution of the equation $f(k_1, k_2) = \varphi(\nu)$ with respect to k_1 and the slight change of ν on the surface F'_n , we can assert that on the part of surface F'_n outside the set m'_n k_1 differs little from the root k_1^0 of the equation $f(k_1, 0) = \varphi(\nu_A)$. Passing to the limit when $n \rightarrow \infty$ we conclude that the integral mean curvature is

$$H(Z') = \frac{1}{2} (k_1^0 + \epsilon') S(Z')$$

where $\epsilon' \rightarrow 0$ when the angle between the half-planes cutting out the surface F'_n decreases infinitely, i. e., $\alpha \rightarrow 0$. Tending the angle α to zero and noting that $2H(Z')/S(Z')$ tends to the curvature of curve γ in point A we conclude that the lemma is true.

PROOF OF THE THEOREM. Suppose that the theorem is not right and there exist two surfaces F_1 and F_2 satisfying the equation (4). Without restricting generality we can assume that the surfaces project into the same region G of plane xy . This region is finite and bounded by the closed, strictly convex curve γ . Let $H(x, y, z)$ and $H_2(x, y, z)$ be the support functions of surfaces. Consider the surface Φ given by the equation $z = H_1(x, y, -1) - H_2(x, y, -1)$. This surface projects uniquely on the whole plane xy . Because of the strict monotony of function $f(k_1, k_2)$ the surface Φ is a saddle-shaped one. As the support functions H_1 and H_2 on the boundary

of spherical image of surfaces F_1 and F_2 coincide it is easy to conclude that

$$\frac{H_1(x, y, -1) - H_2(x, y, -1)}{\sqrt{1 + x^2 + y^2}} \rightarrow 0 \text{ when } x^2 + y^2 \rightarrow \infty.$$

From this, according to the well known theorem of S. N. BERNSTEIN, we conclude that Φ is a cylindrical surface with the generatrices parallel to plane xy . From the strict convexity of region G where the surfaces F_1 and F_2 are projected on the plane xy it follows that $\partial(H_1 - H_2)/\partial x, \partial(H_1 - H_2)/\partial y \rightarrow 0$ when $x^2 + y^2 \rightarrow \infty$. This means that $H_1 - H_2 = \text{const}$. Consequently, the surfaces F_1 and F_2 in the position considered are superposed by the shift in the direction of z -axis. The theorem is proved.