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A THEOREM ON A REPRESENTATION OF $*$–REGULARLY VARYING SEQUENCES

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Abstract. In this paper we shall prove a theorem on a representation of $*$–regularly varying sequences in the sense of Karamata [1].

1. Introduction and results

Consider the sequences (c_n), $c_n > 0$ $(n \in N)$ which are nondecreasing and satisfy the following asymptotic condition

$$\lim_{\lambda\to 1+} \overline{\lim_{n\to+\infty}} \frac{c_{[n\lambda]}}{c_n} = 1. \tag{1}$$

Such sequences are called $*$–regularly varying, and they have an important role in the analysis of divergent sequential processes (see e.g. [6]).

Condition (1) is weaker than the Karamata condition of regular variability and stronger than the condition of $\mathcal{O}$–regular variability (see e.g. [2], [5], [8] and [9]).

Relation (1) obviously means that for any such sequence, the function

$$k_0(\lambda) = \overline{\lim_{n\to+\infty}} \frac{c_{[n\lambda]}}{c_n}$$

is right continuous at $\lambda = 1$.

We shall first define an important class of functions.

CRV is the class of all measurable functions $F : [a, +\infty) \mapsto (0, +\infty)$ $(a > 0)$ such that $F(x(t)) \sim F(y(t))$ as $t \to +\infty$, for any two functions x, y with the properties

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} y(t) = +\infty,$$

and $x(t) \sim y(t)$ as $t \to +\infty$.

This class is investigated in detail in the papers [3] and [4].

Proposition 1. *If (c_n) is an arbitrary nondecreasing sequence of positive numbers, then the following assertions are equivalent*:

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(a) $c_{[x]}$ $(x \geq 1)$ *belongs to the class* CRV;

(b) (c_n) *is a* $*$*–regularly varying sequence.*

Corollary 1. *The function* $k_0(\lambda)$ *is defined for every* $\lambda > 0$.

Corollary 2. *For any* $*$*–regularly varying sequence* (c_n) *we have*

$$\lim_{\lambda \to 1} k_0(\lambda) = 1.$$

Corollary 3. *For any* $*$*–regularly varying sequence* (c_n) *and arbitrary positive* s, t *we have* $k_0(st) \leq k_0(s)k_0(t)$.

Corollary 4. *For an arbitrary* $*$*–regularly varying sequence* (c_n)*, the function* $k_0(\lambda)$ *is continuous in* $\lambda > 0$.

Corollary 5. *If* (c_n) *is an arbitrary nondecreasing sequence of positive numbers, then the following assertions are equivalent:*

(a) (c_n) *is a* $*$*–regularly varying sequence*;

(b) *For arbitrary mappings* $K_1, K_2 : \mathbb{N} \mapsto \mathbb{N}$ *with the properties*

$$\lim_{n \to +\infty} K_1(n) = \lim_{n \to +\infty} K_2(n) = +\infty$$

and $K_1(n) \sim K_2(n)$ *as* $n \to +\infty$*, one has* $c_{K_1(n)} \sim c_{K_2(n)}$ *as* $n \to +\infty$;

(c) $\lim_{\substack{\lambda \to 1 \\ n \to +\infty}} \dfrac{c_{[\lambda n]}}{c_n} = 1$.

The asymptotic condition (c) is in fact the Schmidt convergence condition (see e.g. [7]).

The next theorem is a representation theorem for $*$–regularly varying sequences.

Theorem 1. *Let* (c_n) *be an arbitrary nondecreasing sequence of positive numbers. Then the following assertions are equivalent:*

(a) (c_n) *is a* $*$*–regularly varying sequence*;

(b) *There is an* $n_0 \in N$ *such that*

$$c_n = \exp\left\{\tilde{\mu}_n + r_1(\log n) + \sum_{k=n_0}^{n} \frac{\delta_k}{k}\right\}$$

for every $n \geq n_0$, where the sequence $\tilde{\mu}_n \to 0$ as $n \to +\infty$, r_1 is a bounded and uniformly continuous function on the interval $[\log n_0, +\infty)$, and (δ_n) is a bounded sequence.

Remark. It can be proved that all previous results remain true for sequences (c_n) which are not necessarily nondecreasing if the condition $\lambda \to 1+$ in (1) is replaced by $\lambda \to 1$.

2. Proofs of results

Proof of Proposition 1. By some results from [4,p.454] the implication (a) $\Longrightarrow$ (b) is trivial.

(b) $\Longrightarrow$ (a). Let (c_n) be a $*$–regularly varying sequence. From (1) we find that

$$\overline{\lim_{n\to+\infty}} \frac{c_{[\lambda n]}}{c_n} < +\infty$$

for some $a > 1$ and all $\lambda \in [1, a)$. Hence, the function $k_0(\lambda)$ is finite for all $\lambda \in [1, a)$. For an arbitrary fixed $\lambda \in [1, a)$ define

$$k(\lambda) = \overline{\lim_{x\to+\infty}} \frac{c_{[\lambda x]}}{c_{[x]}}.$$

Then

$$\begin{aligned} k(\lambda) &\leq \overline{\lim_{x\to+\infty}} \frac{c_{[\lambda[x]]}}{c_{[x]}} \cdot \overline{\lim_{x\to+\infty}} \frac{c_{[\lambda x]}}{c_{[\lambda[x]]}} \leq \\ &\leq k_0(\lambda) \cdot \overline{\lim_{x\to+\infty}} \frac{c_{[\lambda x]}}{c_{[\lambda[x]]}}. \end{aligned}$$

Since $(\lambda x)/[\lambda[x]] \to 1+$ as $x \to +\infty$, we find that for a fixed λ and all $x \geq x_0$, $(\lambda x)/[\lambda[x]] \in [1, 1+\delta]$. Hence,

$$\overline{\lim_{x\to+\infty}} \frac{c_{[\lambda x]}}{c_{[\lambda[x]]}} = \overline{\lim_{x\to+\infty}} \frac{c_{\left[\frac{\lambda x}{[\lambda[x]]}[\lambda[x]]\right]}}{c_{[\lambda[x]]}} \leq k_0(1+\delta)$$

for all $\delta > 0$. Thus for a given λ and all $\delta \in (0, a-1)$ we have $k(\lambda) \leq k_0(\lambda) \cdot k_0(1+\delta)$. Since the function k_0 is right continuous at $\lambda = 1$, we have that $k(\lambda) \leq k_0(\lambda)$ for a fixed $\lambda \in [1, a)$. Therefore $\lim_{\lambda\to1+} k(\lambda) \leq \lim_{\lambda\to1+} k_0(\lambda) \leq 1$. Since $c_{[x]}$ is an nondecreasing function, we have that $\lim_{\lambda\to1+} k(\lambda) = 1$ and $\lim_{\lambda\to1-} k(\lambda) \leq 1$. Finally, since the function $k(\lambda)$ is defined on an interval $[1-a', 1+a'']$ $(a', a'' > 0)$, we get that $c_{[x]}$ belongs to the class of $\mathcal{O}$–regularly varying functions ([1]), that is we have

$$k(\lambda) = \overline{\lim_{x\to+\infty}} \frac{c_{[\lambda x]}}{c_{[x]}} < +\infty$$

for all $\lambda > 0$.

Hence we get $1 = k(1) \le k(\lambda) \cdot k(1/\lambda)$, and consequently $k(1/\lambda) \to c \ge 1$ as $\lambda \to 1+$. This gives that $\lim_{\lambda \to 1-} k(\lambda) = 1$, that is $\lim_{\lambda \to 1} k(\lambda) = 1$, which by some results from [4,p.454] yields that $c_{[x]}$ belongs to the class CRV. $\square$

Remark. Since $k_0(\lambda) \le k(\lambda)$ for $\lambda > 0$, and $k(\lambda) \le k_0(\lambda) \cdot k_0(1+\delta)$ for all $\delta > 0$, we find that $k_0(\lambda) = k(\lambda)$ for all $\lambda > 0$. Therefore, using the properties of the index function of CRV functions, Corollaries 2, 3 and 4 follow immediately.

Proof of Corollary 5. (a) $\Longrightarrow$ (b). If (c_n) is a $*$–regularly varying sequence, then the function $F(x) = c_{[x]}$ $(x \ge 1)$ belongs to the class CRV. Next, let K_1 and K_2 be arbitrary functions with the properties in (b). Then, it is easily seen that the functions k_1, k_2 defined by

$$k_i(x) = K_i(n) \qquad (x \in [n, n+1);\ n \in \mathbb{N};\ i = 1,2)$$

have the properties $\lim_{x \to +\infty} k_1(x) = \lim_{x \to +\infty} k_2(x) = +\infty$, and $k_1(x) \sim k_2(x)$ as $x \to +\infty$. Hence $\lim_{x \to +\infty} \big(F(k_1(x))/F(k_2(x))\big) = 1$, and consequently $c_{K_1(n)} \sim c_{K_2(n)}$ as $n \to +\infty$.

(b) $\Longrightarrow$ (c). Assume (b), and let (λ_k) be an arbitrary sequence such that $\lim\limits_{k \to +\infty} \lambda_k = 1$. Then

$$S_k = \frac{[\lambda_k k] + 1}{k} \ge \lambda_k \ge \overline{S}_k = \frac{[\lambda_k k]}{k}$$

for every $k \in \mathbb{N}$, and $S_k, \overline{S}_k \to 1$ as $k \to +\infty$. Putting $n_1(k) = [\lambda_k k]$, $n_2(k) = [\lambda_k k] + 1$, $n_3(k) = k$, we have that $n_1, n_2, n_3 : \mathbb{N} \mapsto \mathbb{N}$, $n_1(k), n_2(k)$, $n_3(k) \to +\infty$ as $k \to +\infty$ and

$$\lim_{k \to +\infty} \frac{n_1(k)}{n_3(k)} = \lim_{k \to +\infty} \frac{n_2(k)}{n_3(k)} = 1.$$

Since

$$1 = \lim_{k \to +\infty} \frac{c_{\frac{n_1(k)}{n_3(k)} n_3(k)}}{c_{n_3(k)}} = \lim_{k \to +\infty} \frac{c_{[\lambda_k k]}}{c_k} \le$$
$$\le \lim_{k \to +\infty} \frac{c_{\frac{n_2(k)}{n_3(k)} n_3(k)}}{c_{n_3(k)}} = 1,$$

we get condition (c).

(c) $\Longrightarrow$ (a). Assuming (c), and taking an arbitrary $\epsilon > 0$, we find some $k_0 \in \mathbb{N}$ and a $\delta > 0$ such that $1 - \epsilon \le c_{[\lambda k]}/c_k \le 1 + \epsilon$ whenever $k \ge k_0$ and $|1 - \lambda| < \delta$. Hence

$$1 - \epsilon \le \overline{\lim_{k \to +\infty}} \frac{c_{[\lambda k]}}{c_k} \le 1 + \epsilon,$$

thus $|k_0(\lambda) - 1| \le \epsilon$ if $|1 - \lambda| < \delta$. This means that

$$\lim_{\lambda \to 1} \overline{\lim_{k \to +\infty}} \frac{c_{[\lambda k]}}{c_k} = 1,$$

so that we have (a). $\square$

Proof of Theorem 1. (a) $\Longrightarrow$ (b). Let (c_n) be a $*$–regularly varying sequence. Then, by Proposition 1, the function $F(x) = c_{[x]}$ $(x \ge 1)$ belongs to the class CRV. By [4] there is a $B > 0$ such that for any $n \ge B$ one has

$$c_n = F(n) = \exp\Big\{\tilde{\mu}(n) + r(\log n) + \int_B^n \frac{\epsilon(t)}{t}\, dt\Big\},$$

where the functions $\epsilon(x)$ and $\tilde{\mu}(x)$ are bounded measurable functions in $[B, +\infty)$, $r(x)$ is a uniformly continuous bounded function in $[\log B, +\infty)$, and we have $\lim_{x \to +\infty} \tilde{\mu}(x) = 0$. Putting $n_0 = [B] + 1$ and $s = \int_B^{n_0} \frac{\epsilon(t)}{t}\, dt \in \mathbb{R}$, we get that the function $r_1(t) = r(t) + s$ is bounded and uniformly continuous in $t \in [\log n_0, +\infty)$. Hence, for $n \ge n_0$ we have

$$c_n = \exp\Big\{\tilde{\mu}_n + r_1(\log n) + \sum_{k=n_0}^{n} \frac{\delta_k}{k}\Big\},$$

where $\lim_{n \to +\infty} \tilde{\mu}_n = 0$, r_1 is a bounded and uniformly continuous function on the interval $[\log n_0, +\infty)$, $\delta_k = k \int_{k-1}^{k} \frac{\epsilon(t)}{t}\, dt$ $(k \ge n_0 + 1)$ and $\delta_{n_0} = 0$. Finally, we find that

$$\begin{aligned}|\delta_k| = k \cdot \Big| \int_{k-1}^{k} \frac{\epsilon(t)}{t}\, dt \Big| &\le k \cdot \sup_{t \ge k-1} |\epsilon(t)| \cdot \log\Big(1 + \frac{1}{k-1}\Big) \le \\ &\le 2 \sup_{t \ge k-1} |\epsilon(t)| < M,\end{aligned}$$

for any $k \ge n_0 + 1$, since the function $\epsilon(t)$ is bounded on $[B, +\infty)$.

(b) $\Longrightarrow$ (a). Assuming (b), let $\lambda > 1$ and $n \ge n_0$. Then

$$\frac{c_{[\lambda\, n]}}{c_n} = \exp\Big\{\tilde{\mu}_{[\lambda\, n]} - \tilde{\mu}_n + r_1(\log{[\lambda\, n]}) - r_1(\log n) + \sum_{k=n+1}^{[\lambda\, n]} \frac{\delta_k}{k}\Big\}.$$

Since

$$\lim_{\lambda\to 1+}\overline{\lim_{n\to+\infty}}\left(\tilde{\mu}_{[\lambda\, n]}-\tilde{\mu}_n\right)=0,$$

$$\left|\sum_{k=n+1}^{[\lambda\, n]}\frac{\delta_k}{k}\right|\le\sup_{k\ge n+1}|\delta_k|\cdot\int_{n+1}^{[\lambda\, n]+1}\frac{dt}{t-1}=\sup_{k\ge n+1}|\delta_k|\cdot\log\frac{[\lambda\, n]}{n},$$

that is $\lim_{\lambda\to 1+}\overline{\lim}_{n\to+\infty}\left|\sum_{k=n+1}^{[\lambda\, n]}\frac{\delta_k}{k}\right|=0$, and r_1 is a uniformly continuous function on the interval $[\log n_0,+\infty)$, we get that

$$\lim_{\lambda\to 1+}\overline{\lim_{n\to+\infty}}\frac{c_{[\lambda\, n]}}{c_n}=1.$$

In other words, (c_n) is a $*$–regularly varying sequence. $\square$

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