

$$\begin{aligned}
 S_2 &< \frac{1}{\beta_n} \sum_{\nu=n+1}^{\infty} \left(\log \frac{\beta_n + \varepsilon(\nu-n)}{\varepsilon(\nu-n)} - \frac{\beta_n}{\beta_n + \varepsilon(\nu-n)} \right) \\
 &= \frac{1}{\beta_n} \sum_{\nu=1}^{\infty} \left(\log \frac{\beta_n + \varepsilon\nu}{\varepsilon\nu} - \frac{\beta_n}{\beta_n + \varepsilon\nu} \right) \\
 (5) \quad &< \frac{1}{\beta_n} \int_0^{\infty} \left(\log \frac{\beta_n + \varepsilon x}{\varepsilon x} - \frac{\beta_n}{\beta_n + \varepsilon x} \right) dx \\
 &= \frac{1}{\varepsilon} \int_0^{\infty} \left(\log \frac{1+t}{t} - \frac{1}{1+t} \right) dt = \frac{1}{\varepsilon}.
 \end{aligned}$$

From (3), (4) and (5) follows (2). From (2) and (i) follows (1) which completes the proof.

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A theorem on bounded moments

by

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1. The well known theorem of MÜNTZ [6] can be formulated as follows:

(I) If β_1, β_2, \dots is an increasing sequence such that $\sum_{n=1}^{\infty} 1/\beta_n = \infty$ and $f(x)$ a function integrable in $[a, b]$ (where $a \geq 0$) such that

$$\int_a^b x^{\beta_n} f(x) dx = 0 \quad (n=1, 2, \dots),$$

then $f(x)=0$ almost everywhere in $[a, b]$.

If particularly $\beta_n=n$, this theorem reduces itself to the well known theorem of LEBESGUE [1]. On the other hand, the following theorem holds [2]:

(II) If $f(x)$ is integrable in $[1, b]$ and there exists a number M such that

$$(1) \quad \left| \int_1^b x^n f(x) dx \right| < M \quad (n=1, 2, \dots).$$

then $f(x)=0$ almost everywhere in $[1, b]$.

It is easy to see that the lower bound of the integral cannot be diminished. Indeed, all moments of any function which vanishes for $x>1$ are always commonly bounded.

The theorem (II) can be generalized by replacing the natural sequence of exponents n by any sequence $\{n^{\alpha}\}$ where $0 < \alpha \leq 1$ [4]. The question arises if the sequence of exponents may be replaced

by more general sequences. It is easy to show that the condition $\sum_{n=1}^{\infty} 1/\beta_n = \infty$ alone does not suffice. In fact, there exists a function $f(x)$, continuous and non-vanishing identically in $[1, 2]$ such that $\int_1^2 x^{n^2} f(x) dx = 0$. Its transform

$$F(\beta) = \int_1^2 x^\beta f(x) dx$$

is a continuous function which vanishes for $\beta = n^2$ ($n=1, 2, \dots$). Thus, if β is near to n^2 , we have $|F(\beta)| < 1$. Consequently we can complete the sequence $\{1/n^2\}$ by so much terms, that the new sequence should have the property $\sum_{n=1}^{\infty} 1/\beta_n = \infty$ and that the inequalities $|F(\beta_n)| < 1$ hold.

On the other hand, the theorem on bounded moments will be still true if we add, to the condition $\sum_{n=1}^{\infty} 1/\beta_n = \infty$, a supplementary condition $\beta_{n+1} - \beta_n > \varepsilon > 0$ ($n=1, 2, \dots$). This is the chief result of our paper. It can be explicitly written as follows:

Theorem. *If β_1, β_2, \dots is a sequence of positive numbers such that*

$$\sum_{n=1}^{\infty} \frac{1}{\beta_n} = \infty \quad \text{and} \quad \beta_{n+1} - \beta_n > \varepsilon > 0 \quad \text{for} \quad n=1, 2, \dots$$

and $f(x)$ is a function, integrable in $[1, b]$, such that

$$\left| \int_1^b x^{\beta_n} f(x) dx \right| < M \quad \text{for} \quad n=1, 2, \dots,$$

then $f(x) = 0$ almost everywhere in $[1, b]$.

The proof will be based on a *discontinuity factor*, analogous to that used by PHRAGMÉN [7]

$$\varphi(x) = \lim_{n \rightarrow \infty} \exp(-e^{nx}).$$

This method was extended by PICONE [8] and MIKUSIŃSKI [4]. In the sequel, we shall use a very general form of discontinuity factor, by replacing the function $\exp(-e^{nx})$ by a suitable sequence of generalized exponential functions.

2. Before the proof we shall give some corollaries of the Theorem.

Corollary 1. *If the sequence β_1, β_2, \dots satisfies the conditions of the Theorem and $f(x)$ is a function, integrable in $[0, b]$, such that*

$$\left| \int_0^b x^{\beta_n} f(x) dx \right| < M q^{\beta_n} \quad \text{for} \quad n=1, 2, \dots,$$

then $f(x) = 0$ a. e.¹⁾ in $q < x < b$.

Indeed, we have

$$q^{\beta_n} \int_0^{b/q} x^{\beta_n} q f(qx) dx = \int_0^b x^{\beta_n} f(x) dx.$$

Thus, if $0 < q < b$,

$$\left| \int_1^{b/q} x^{\beta_n} q f(qx) dx \right| \leq M + \int_0^1 q |f(qx)| dx$$

and, by the Theorem, $q f(qx) = 0$ a. e. in $[1, b/q]$, that is $f(x) = 0$ a. e. in $[q, b]$.

Corollary 2. *If the sequence β_1, β_2, \dots satisfies the conditions of the Theorem and $g(t)$ is a function, integrable in $[0, T]$, such that*

$$\left| \int_0^T e^{\beta_n t} g(t) dt \right| < M$$

then $g(t) = 0$ a. e. in $[0, T]$.

This Corollary follows from the Theorem by the substitution

$$x = e^t, \quad b = e^T, \quad f(x) = g(t).$$

3. Now, we approach the proof. Write

$$\varphi_m(x) = 1 - \alpha_1^{(m)} x^{\beta_m} + \alpha_2^{(m)} x^{\beta_{2m}} - \dots,$$

where

$$\alpha_n^{(m)} = \frac{1}{e} \prod_{r=1}^{\infty} \frac{\beta_{nr}}{|\beta_{nr} - \beta_{mn}|} \exp\left(-\frac{\beta_{mn}}{\beta_{nr}}\right).$$

By the preceding paper [5], the functions $\varphi_m(x)$ have the following properties:

¹⁾ a. e. = almost everywhere.

1° $\varphi_m(x)$ decreases in the interval $0 < x < \infty$ from 1 to 0;

$$2^\circ \int_0^\infty \varphi_m(x) dx = \prod_{v=1}^{\infty} \frac{\beta_{mv}}{\beta_{mv} + 1} \exp\left(\frac{1}{\beta_{mv}}\right);$$

$$3^\circ \frac{\log \alpha_n^{(m)}}{\beta_{mv}} < \frac{2}{m\varepsilon} \quad (m, n=1, 2, \dots).$$

We are going to show that

$$(2) \quad \lim_{m \rightarrow \infty} \varphi_m(x) = \begin{cases} 1 & \text{for } 0 \leq x < 1, \\ 0 & \text{for } 1 < x < \infty. \end{cases}$$

Let $0 \leq \theta < \lambda < 1$. Since $\alpha_n^{(m)} < \exp \frac{2\beta_{mn}}{m\varepsilon}$, we have

$$(3) \quad |1 - \varphi_m(\theta)| \leq \sum_{n=1}^{\infty} \alpha_n^{(m)} \theta^{\beta_{mn}} \\ \leq \sum_{n=1}^{\infty} \left(\theta \exp \frac{2}{m\varepsilon} \right)^{\beta_{mn}} < \sum_{n=1}^{\infty} \lambda^{\varepsilon(mn-1)} = \frac{\lambda^{\varepsilon(m-1)}}{1 - \lambda^{\varepsilon}}$$

for sufficiently large m .

This proves that

$$\lim_{m \rightarrow \infty} \varphi_m(x) = 1 \quad \text{for } 0 \leq x < 1.$$

To prove

$$\lim_{m \rightarrow \infty} \varphi_m(x) = 0 \quad \text{for } 1 < x < \infty,$$

it suffices to show that

$$\lim_{m \rightarrow \infty} \int_0^\infty \varphi_m(x) dx = 1,$$

for $\varphi_m(x)$ are positive and decreasing. But this follows from the convergence of the infinite product

$$\prod_{v=1}^{\infty} \frac{\beta_v}{\beta_v + 1} \exp\left(\frac{1}{\beta_v}\right).$$

Thus, the formula (2) is proved.

The formula (2) enables us to use the sequence $\varphi_m(x)$ as a discontinuity factor; to this purpose, we write

$$\int_1^b [1 - \varphi_m(\theta x)] f(x) dx = \sum_{n=1}^{\infty} (-1)^{n+1} \alpha_n^{(m)} \theta^{\beta_{mn}} \int_1^b x^{\beta_{mn}} f(x) dx.$$

If $1 < \theta^{-1} < b$, the left member approaches the limit $\int_{\theta^{-1}}^b f(x) dx$.

On the other hand, the right member tends to 0, for its absolute value is, by (1), less than

$$M \sum_{n=1}^{\infty} \alpha_n^{(m)} \theta^{\beta_{mn}}$$

and the last expression tends to 0, by (3). In this way we have

$$\int_{\theta^{-1}}^b f(x) dx = 0$$

and, as θ^{-1} can be fixed arbitrarily in $(1, b)$, $f(x) = 0$ a. e. in $[1, b]$.

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