

A Theorem on Higher Bruhat Orders

S. Felsner and H. Weil

Fachbereich Mathematik und Informatik,
Freie Universität Berlin,
Takustrasse 9, 14195 Berlin, Germany
{felsner,weil}@inf.fu-berlin.de

Abstract. We show that inclusion order and single-step inclusion coincide for higher Bruhat orders $B(n, 2)$, i.e., $B(n, 2) = B_{\subseteq}(n, 2)$.

1. Preliminaries

Higher Bruhat orders were introduced by Manin and Schechtman [5] as generalizations of the weak Bruhat order on the symmetric group S_n . Further investigators of the subject are Voevodskij and Kapranov [6], Ziegler [7], Edelman and Reiner [1], [2], and Felsner and Weil [3]. We review the definition.

The set $[n] = \{1, \dots, n\}$ is equipped with the natural linear order. The set of s -element subsets of $[n]$ is $\binom{[n]}{s}$. For $X \in \binom{[n]}{s}$ with $s \geq i \geq 1$ we let $X^{[i]}$ denote the set X minus the i th-largest element of X (e.g., $\{3, 5, 8, 9\}^{[2]} = \{3, 8, 9\}$). For a set $P \in \binom{[n]}{s+1}$ the set of its s -element subsets $\{P^{[1]}, P^{[2]}, \dots, P^{[s+1]}\}$ is called an s -packet, which we also denote by P , where this can be done unambiguously. Let \mathcal{S} be a system of finite sets. The *single-step inclusion* order on \mathcal{S} is the transitive closure of the relation \triangleleft on \mathcal{S} defined by $S \triangleleft S'$ iff $S \subset S'$ and $|S' \setminus S| = 1$.

Definition 1. A subset $A \subseteq \binom{[n]}{s}$ is called *consistent* if its intersection with any s -packet P is either a beginning or an ending segment with respect to the lexicographic ordering of P , or equivalently if for any such packet P and $1 \leq i < j < k \leq s + 1$ the intersection of A with $\{P^{[i]}, P^{[j]}, P^{[k]}\}$ is neither $\{P^{[i]}, P^{[k]}\}$ nor $\{P^{[j]}\}$.

The *higher Bruhat order* $B(n, s - 1)$ is the set of consistent subsets of $\binom{[n]}{s}$ ordered by single-step inclusion. The partial order on this set by ordinary inclusion will be denoted by $B_{\subseteq}(n, s - 1)$.

In what follows it is preferable to work with $B(n, s)$ rather than with $B(n, s - 1)$. To avoid confusion on the readers part we change letters from s to r (of course $s = r + 1$).

$B(n, r)$ is a graded poset with unique minimal and maximal elements \emptyset and $\binom{[n]}{r+1}$, respectively. The rank of a consistent set A is $|A|$.

Further structural properties of higher Bruhat orders have been studied, in particular by Ziegler [7, Section 4]. He characterizes the pairs (n, r) such that $B(n, r)$ is a lattice. Ziegler also shows that $B(n, r) = B_{\subseteq}(n, r)$ for $r = 1$ and for $n - r \leq 4$ while $B(8, 3) \neq B_{\subseteq}(8, 3)$. He left open the question whether $B(n, 2)$ is ordered by inclusion for $n > 6$. Our main result is the affirmative answer to this question.

In the remainder of this introductory section we give alternative definitions for higher Bruhat orders and relate consistent subsets of $\binom{[n]}{3}$ to arrangements of pseudolines and the notation customary in studies of such arrangements. Readers with an appropriate background may skip this part and proceed to Section 2.

Manin and Schechtman defined the higher Bruhat order $B(n, r)$ as equivalence classes of *admissible permutations* of $\binom{[n]}{r}$. A permutation π of the elements of $\binom{[n]}{r}$ is *admissible*, if the elements of every r -packet P occur in π in lexicographic or in reverse-lexicographic order. In the second case packet P is called an *inversion* of π . Two admissible permutations π and π' are *equivalent* if there is a sequence of admissible permutations $\pi = \pi_0, \pi_1, \dots, \pi_t = \pi'$ such that for $k = 1, \dots, t - 1$ permutations π_{k-1} and π_k only differ by an adjacent transposition of two elements X, X' which are not contained in a common packet, i.e., two elements X, X' with $|X \cap X'| < r - 1$.

Ziegler [7] shows that admissible permutations of $\binom{[n]}{r}$ are equivalent iff they have the same sets of inversions and that a subset of $\binom{[n]}{r+1}$ is consistent iff it is the set of inversions of an admissible permutation of $\binom{[n]}{r}$. Hence, the two definitions for higher Bruhat orders are equivalent.

An admissible permutation of $\binom{[n]}{1}$ is just a permutation of $[n]$ and consistency corresponds to transitivity of the inversion and the noninversion relation. Hence, the higher Bruhat order $B(n, 1)$ is the weak Bruhat order of permutations.

Admissible permutations of $\binom{[n]}{2}$ have shown up in different facings. In studies on Coxeter groups they are the reduced decompositions of the reverse permutation. They are also the sequences of moves of a half-period (beginning with the identity) of a simple allowable sequence as defined by Goodman and Pollack [4]. Allowable sequences were introduced as a combinatorial model for arrangements of pseudolines. Of interest to us is the following statement: Admissible permutations of $\binom{[n]}{2}$ encode simple marked arrangements of pseudolines and consistent subsets of $\binom{[n]}{3}$ are in bijection with these arrangements.

Informally, a simple marked arrangements of pseudolines consists of n curves which begin at the left side of the page and move across to the right side such that each pair of curves will cross and no three curves cross at a single point. Given an arrangement \mathcal{A} label the pseudolines (= curves) such that at the left side they enter the picture in the natural order from bottom to top. Consider a triple $\{l_i, l_j, l_k\}$ of pseudolines with $i < j < k$ in \mathcal{A} . This triple can induce two combinatorially different arrangements. Either the crossing of l_i and l_k is above l_j or below. In the first case the pairs $\{i, j\}$, $\{i, k\}$, and $\{j, k\}$ appear in lexicographic order in the corresponding admissible permutation, in the second case the triple $\{i, j, k\}$ is an inversion. If the region enclosed by l_i, l_j , and

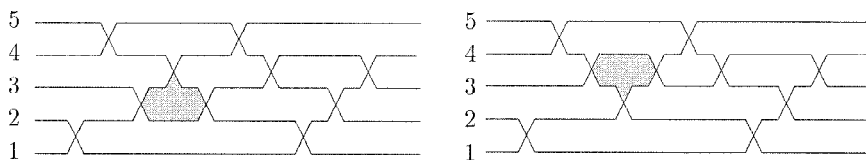


Fig. 1. Elementary flip at the shaded triangle.

l_k is a triangle, we can apply an elementary *flip* to obtain another arrangement which combinatorially differs from the original one only in the orientation of this one triangle. This corresponds to a single step in the higher Bruhat order, i.e., to two consistent sets that differ in just one triple. Figure 1 shows the wiring diagrams corresponding to the consistent sets $\{145, 234, 235, 245, 345\}$ on the left and $\{135, 145, 234, 235, 245, 345\}$ on the right.

Geometric interpretations of consistent sets have been the driving force for defining and investigating higher Bruhat orders (see [5]–[7] and [3]). Ziegler shows a bijection between consistent sets $A \in B(n, r)$ and uniform one-element extensions of the alternating oriented matroid $C^{n, n-r}$. By oriented matroid duality this also gives a bijection between consistent sets and one-element liftings of $C^{n, r}$.

Let $A \subseteq \binom{[n]}{r+1}$ be a consistent set and $X \in \binom{[n]}{r-1}$. Define an orientation \rightarrow_X of the complete graph with vertex set $[n] \setminus A$. For two vertices i, j with $i < j$ let

$$j \rightarrow_X i \iff X \cup \{i, j\} \in A.$$

Claim T. *The orientation \rightarrow_X is transitive, i.e., the graph is a transitive tournament.*

Proof. We argue by contradiction. Assume that the graph contains a cycle and hence a cycle on three vertices i, j, k , we assume $i < j < k$. We distinguish two cases:

- (1) $i \rightarrow_X j \rightarrow_X k \rightarrow_X i$.
- (2) $k \rightarrow_X j \rightarrow_X i \rightarrow_X k$.

The first case yields $X \cup \{i, j\} \notin A$, $X \cup \{i, k\} \in A$, and $X \cup \{j, k\} \notin A$, and the second case yields $X \cup \{i, j\} \in A$, $X \cup \{i, k\} \notin A$, and $X \cup \{j, k\} \in A$. Since $i < j < k$ it follows that in lexicographic order $X \cup \{i, j\} <_{\text{lex}} X \cup \{i, k\} <_{\text{lex}} X \cup \{j, k\}$. In both cases we obtain a contradiction to the consistency of A with respect to the packet $X \cup \{i, j, k\}$. \square

Transitive tournaments have a unique topological sorting, hence, we get a collection of linear orderings or permutations α_X of $[n] \setminus X$ for every $X \in \binom{[n]}{r-1}$ associated to $A \in B(n, r)$. We call them *local sequences*.

If $r = 2$, i.e., when we have an arrangement of pseudolines corresponding to A , a local sequence is a permutation $\alpha_{\{i\}}$ of $[n] \setminus \{i\}$. This permutation reports the order in which line l_i is crossed by the other lines. Basically these are the *local sequences of (un-)ordered switches* associated to an allowable sequence by Goodman and Pollack [4].

2. The Main Result

In this section we show that the single-step order and the inclusion order on consistent subsets of $\binom{[n]}{3}$ coincide.

Theorem 1. $B(n, 2) = B_{\subseteq}(n, 2)$ for all n .

To prove the result we show that for any two consistent sets $A \subset B \subseteq \binom{[n]}{3}$ there is a consistent set A' satisfying $A \triangleleft A'$ and $A' \subseteq B$. Iterating this argument we find a single-step chain $A = A_0, A_1, \dots, A_t = B$ connecting A and B in $B(n, 2)$.

Given consistent sets $A \subset B \subseteq \binom{[n]}{3}$ we call a triple $T \in B \setminus A$ a *difference triple*. From $A \subset B$ it follows that for all nondifference triples T' either $T' \in A \cap B$ or $T' \notin A \cup B$ holds. Let \mathcal{A} be an arrangement of pseudolines with inversion set A . We will show that in \mathcal{A} there is a triangular face F such that the three lines bounding F correspond to a difference triple. Call such a triple *elementary*. Given the triangle F we can apply an elementary flip to obtain an arrangement \mathcal{A}' such that its inversion set A' has the desired properties, i.e., $A \triangleleft A'$ and $A' \subseteq B$.

For $i < j < k$ the *basis* of the triple $\{i, j, k\}$ is the piece of line l_j between the intersections with lines l_i and l_k . Clearly an elementary triple has a basis which is an edge of the cell complex induced by \mathcal{A} . Call the basis of a triple which is an edge in the cell complex of \mathcal{A} an *elementary basis*.

Let α_i denote the local sequence of line l_i in \mathcal{A} , i.e., the permutation of $[n] \setminus \{i\}$ recording the order in which line l_i is crossed by other lines. For a triple $\{i_1, i_2, i_3\}$ and $i_1 < i_2$ recall the following equivalence:

$$\{i_1, i_2, i_3\} \in A \iff \{i_1, i_2\} \text{ is an inversion of } \alpha_{i_3}. \quad (*)$$

Lemma 2. *There is a difference triple T with an elementary basis.*

Proof. Among all difference triples $\{i, j, k\}$ with $i < j < k$ choose one of minimal width $k - i$. Let this triple be $T = \{i, j, k\}$. From $T \notin A$ and (*) we see that on line l_j the intersection with line l_i comes before the intersection with line l_k .

Claim A. *For every x between i and k in the local sequence α_j either $x < i$ or $x > k$.*

Proof. Suppose x with $i < x < k$ is between i and k on α_j denoted $i \prec x \prec k$. Now consider the order of i, x, k on the local sequence β_j of B . From $\{i, j, k\} \in B$ and (*) we obtain $k \prec i$ on β_j .

If $x < i$, on β_j we obtain from (*) that $\{i, j, x\}$ is a difference triple. If $i < x < j$ the width of this triangle is $j - i$, otherwise, if $i < j < x < k$ the width is $x - i$. In both cases this contradicts our choice of $\{i, j, k\}$ as a difference triangle of minimal width.

If $x \not< i$, then $k \prec x$ on β_j . In this case $\{x, j, k\}$ is a difference triangle of width either $k - x$ or $k - j$. Again, this contradicts our choice of $\{i, j, k\}$ as a difference triangle of minimal width. \square

Claim B. *There exists an elementary basis on the segment of l_j between the crossings with l_i and l_k .*

Proof. If i and k are adjacent elements of α_j we are done. Otherwise, by Claim A we can partition the elements between i and k into elements x with $x < i$ and elements y with $y > k$. For an x we note that from $i < x$ on α_j we obtain $\{x, i, j\} \in A$. Hence, $\{x, i, j\} \in B$, i.e., $i < x$ on β_j . Since $k < i$ on β_j the triple $\{x, j, k\}$ is a difference triple. For an element y we obtain by an analogous argument that $\{i, j, y\}$ is a difference triple.

If the element to the right of i on α_j is a y , the difference triple $\{i, j, y\}$ has an elementary basis and we are done. If the element to the left of k on α_j is an x , the difference triple $\{x, j, k\}$ has an elementary basis and we are again done. If both these conditions fail, then we find an adjacent pair (x, y) with $x < i$ and $y > k$ on α_j . On α_j we have $i < x < y < k$ while by the above considerations $y < k < i < x$ on β_j . This shows that $\{x, j, y\}$ is a difference triple with an elementary basis. □

This completes the proof of the lemma. □

We now consider a wiring diagram for \mathcal{A} . Wiring diagrams are closely related to allowable sequences. Informally, a wiring diagram is a drawing of \mathcal{A} in which the edges are associated to horizontal wires (see Fig. 1). For an edge e of \mathcal{A} we say e is on wire w if the horizontal portion of e is on wire w . Let $\{i, j, k\}$ be a difference triple with elementary basis such that the basis of $\{i, j, k\}$ is on the highest wire that contains elementary bases in the diagram.

Lemma 3. *The triple $\{i, j, k\}$ defined in the preceding paragraph is an elementary triple.*

Proof. Since the basis of $\{i, j, k\}$ is elementary any line l_x crossing the triangle of the three lines l_i, l_j, l_k enters the triangle through line l_i and leaves the triangle through line l_k . It follows that $i < x < k$.

If $i < x < j$, then $\{i, x, j\} \in A$, hence, $\{i, x, j\} \in B$ and on β_i we have $j < x$. With $k < j$ on β_i this shows that $\{i, x, k\}$ is a difference triple. Similarly, if $j < x < k$, then $\{j, x, k\} \in A$ and $\{j, x, k\} \in B$. Considering β_k we see that again $\{i, x, k\}$ is a difference triple.

Let F be the face of \mathcal{A} above the edge on l_j corresponding to the basis of $\{i, j, k\}$. The boundary of F consists of the basis b and edges e_0, \dots, e_t in clockwise order. Figure 2 shows a generic sketch of the situation. Note that in the wiring diagram of \mathcal{A} the edges e_0, \dots, e_t are all on the wire above the wire of b .

Claim C. *If $t > 1$, then one of the edges e_1, \dots, e_{t-1} is an elementary basis.*

Claim C gives a contradiction to the choice of the triple $\{i, j, k\}$ when $t > 1$. Therefore, $t = 1$ and face F is a triangle in \mathcal{A} . This shows that $\{i, j, k\}$ is an elementary triple. To prove the lemma it thus suffices to prove the claim.

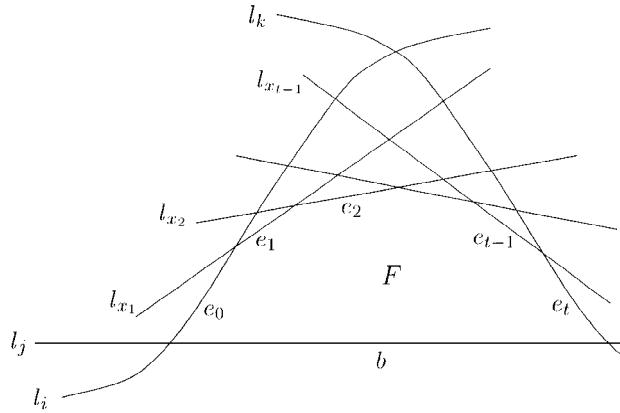


Fig. 2. Face F above the elementary basis b .

Proof. If $t = 2$ let l_x be the supporting line of e_1 . From the above considerations we know that $\{i, x, k\}$ is a difference triple. The basis of the triple is edge e_1 , hence is elementary.

If $t > 2$ let l_{x_s} be the supporting line of edge e_s for $s = 1, \dots, t - 1$. Note that $i < x_{s+1} < k$ on α_{x_s} and $k < i$ on β_{x_s} . Therefore, at least one of $\{i, x_s, x_{s+1}\}$ and $\{x_s, x_{s+1}, k\}$ is a difference triple. For $s = 1, \dots, t - 2$ let p_s be the vertex of $e_s \cap e_{s+1}$. Color p_s red if $\{i, x_s, x_{s+1}\}$ is a difference triple and blue otherwise.

If p_1 is a red vertex, then e_1 is an elementary basis. If p_{t-2} is a blue vertex, then e_{t-1} is an elementary basis. Now assume that p_1 is blue and p_{t-2} is red, then there is some s such that p_s is blue and p_{s+1} is red. Note that $x_s < x_{s+1} < x_{s+2}$ and $x_s < x_{s+2}$ on $\alpha_{x_{s+1}}$. From the definitions of red and blue vertices we obtain $i < x_s$ and $x_{s+2} < i$ on $\beta_{x_{s+1}}$. Hence, $\{x_s, x_{s+1}, x_{s+2}\}$ is a difference triple with elementary basis e_{s+1} . This proves the claim and completes the proof of the lemma. □

Lemmas 2 and 3 prove the theorem. □

3. Reorientations

If we fix a consistent set A_1 in $B(n, r)$ we can reorient this order to obtain $B^{A_1}(n, r)$. For $A_2 \in B(n, r)$ we define the corresponding *reoriented inversion set* $A_2^{A_1}$ in $B^{A_1}(n, r)$ as the symmetric difference $A_2 \Delta A_1$. The order relation of $B^{A_1}(n, r)$ is the single-step order on these reoriented inversion sets. Again we define $B_{\subseteq}^{A_1}(n, r)$ as the order on the same elements with inclusion as order relation. Ziegler [7] initiated the study of reoriented higher Bruhat orders. He showed that reorientations lack some of the structure of higher Bruhat orders. In particular he shows that while $B(6, 3)$ is ordered by inclusion there is a consistent set $A \in B(6, 3)$ such that $B^A(6, 3)$ is not ordered by inclusion. He shows that $B^A(6, 3)$ is not even bounded.

The following example shows a similar “bad” behavior for reorientations of $B(6, 2)$. Let \mathcal{A}_1 and \mathcal{A}_2 be the simple arrangements shown in Fig. 3. Both arrangements have

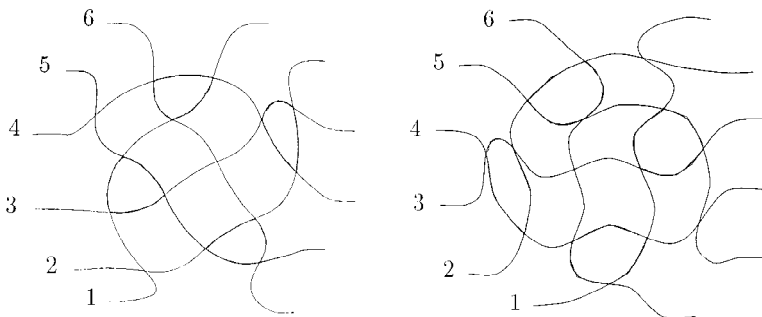


Fig. 3. Two arrangements \mathcal{A}_1 and \mathcal{A}_2 with corresponding triangular faces.

exactly four triangular faces determined by the following sets of lines $\{1, 3, 5\}$, $\{1, 4, 6\}$, $\{2, 3, 4\}$, and $\{2, 5, 6\}$, moreover, the orientation of these triangles is the same in both arrangements. It follows that starting from \mathcal{A}_1 every possible triangular flip leads to an arrangement with more 3-element sets of lines being oriented differently from their orientation in \mathcal{A}_2 . Hence, if we orient $B(6, 2)$ away from the consistent set \mathcal{A}_1 corresponding to \mathcal{A}_1 there is no single element step toward the consistent set \mathcal{A}_2 corresponding to \mathcal{A}_2 . Hence, every chain (in the inclusion order) from \mathcal{A}_1 to the complement $\overline{\mathcal{A}_1}$ through \mathcal{A}_2 has length less than $\binom{6}{3}$. This example shows:

- (1) Single step inclusion and inclusion are not identical for the reorientation $B^{A_1}(6, 2)$ of $B(6, 2)$ and hence $B(n, 2)$ for all $n \geq 6$.
- (2) Both, \mathcal{A}_1 and \mathcal{A}_2 admit no single step going down in $B^{A_1}(6, 2)$, hence the reorientation is unbounded.
- (3) An arrangement of pseudolines is not (necessarily) determined by the orientations of its triangular faces. Since the arrangements \mathcal{A}_1 and \mathcal{A}_2 are realizable the same holds for arrangements of lines.

References

1. P. Edelman and V. Reiner, Free arrangements and rhombic tilings, *Discrete Comput. Geom.*, **15** (1996), 307–340.
2. P. Edelman and V. Reiner, Erratum to free arrangements and rhombic tilings, *Discrete Comput. Geom.*, **17** (1997), 359.
3. S. Felsner and H. Weil, Sweeps, arrangements and signotopes, Technical Report, FU-Berlin, 1998.
4. J. E. Goodman and R. Pollack, Semispaces of configurations, cell complexes of arrangements, *J. Combin. Theory Ser. A*, **37** (1984), 257–293.
5. Y. Manin and V. Schechtman, Arrangements of hyperplanes, higher braid groups and higher Bruhat orders, in *Algebraic Number Theory - in Honour of K. Iwasawa*, J. C. et al., eds., Advanced Studies in Pure Mathematics, vol. 17, Kinokuniya Company/Academic Press, New York, 1989, pp. 289–308.
6. V. Voevodskij and M. Kapranov, Free n -category generated by a cube, oriented matroids, and higher Bruhat orders, *Functional Anal. Appl.*, **2** (1991), 50–52.
7. G. Ziegler, Higher Bruhat orders and cyclic hyperplane arrangements, *Topology*, **32** (1993), 259–279.

Received April 7, 1998.