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A theorem on metric polynomial structures

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Abstract. Let f be a metric polynomial structure with respect to a metric tensor g and let ∇ denote the Riemannian connection defined by g. The purpose of this paper is to give a necessary and sufficient condition for $\nabla f = 0$ to hold.

0. All objects considered in this paper are assumed to be C^∞.

The following theorem is well known [3]:

Theorem 1. *For an almost Hermitian manifold M with almost complex structure J and metric g, the following conditions are equivalent*:

1° $\nabla J = 0$, *where ∇ is the Riemannian connection defined by g*;

2° *The Nijenhuis product $[J, J]$ vanishes and the fundamental 2-form of the almost Hermitian manifold M is closed.*

The subject of this paper is to give and to prove an analogous theorem in the case where J is replaced by an arbitrary metric polynomial structure. At first we recall some facts about polynomial structures.

Let M be a manifold of dimension n. By a polynomial structure on M we mean a (1, 1) tensor field f on M satisfying a polynomial equation

$$P(f) = f^d + a_1 f^{d-1} + \ldots + a_d I = 0,$$

where I is the identity (1, 1) tensor field on M, $a_1, \ldots, a_d$ are real numbers and the polynomial $P(\xi) = \xi^d + a_1 \xi^{d-1} + \ldots + a_d$ is the minimal polynomial of f_x at every point $x \in M$. Decompose the polynomial $P(\xi)$ into the prime factors:

$$P(\xi) = R_1'(\xi) \ldots R_{r'}'(\xi) \cdot R_1''(\xi) \ldots R_{r''}''(\xi),$$

where

$$R_i'(\xi) = (\xi - b_i)^{k_i}, \quad k_i \geqslant 1, \; i = 1, \ldots, r',$$

$$R_j''(\xi) = (\xi^2 + 2c_j \xi + d_j)^{l_j}, \quad l_j \geqslant 1, \; j = 1, \ldots, r'', \; c_j^2 - d_j < 0.$$

Let $D = (D_1', \ldots, D_{r'}', D_1'', \ldots, D_{r''}'')$ be the almost product structure associated with the polynomial structure f, i.e. $D_i' = \ker R_i'(f)$ and $D_j'' = \ker R_j''(f)$. It is known that there exist polynomials P_i', P_j'' such that $P_i'(f)$,

$P_j''(f)$ are projectors, respectively, onto D_i' and D_j''. The following theorem is due to Kobayashi [2].

THEOREM 2. *Let f be a polynomial structure such that*

$$\deg R_j' = 1 \quad or \quad \dim D_i', \quad i = 1, \ldots, r',$$

$$\deg R_j'' = 2 \quad or \quad \dim D_j'', \quad j = 1, \ldots, r''.$$

Then f is integrable if the Nijenhuis product $[f, f] = 0$.

1. Let (M, g) be a Riemannian manifold and let f be a metric polynomial structure on M. In other words, suppose that f is a polynomial structure such that $g(fX, fY) = g(X, Y)$ for any tangent vectors X and Y. The following proposition is due to J. Bureš and J. Vanžura ([1]).

PROPOSITION 3. *There are exactly four types of metric polynomial structures, whose minimal polynomials are given by*

(I) $$P(\xi) = (\xi^2+2a_1\xi+1)\ldots(\xi^2+2a_s\xi+1),$$

(II) $$P(\xi) = (\xi-1)(\xi^2+2a_1\xi+1)\ldots(\xi^2+2a_{s-1}\xi+1),$$

(III) $$P(\xi) = (\xi+1)(\xi^2+2a_1\xi+1)\ldots(\xi^2+2a_{s-1}\xi+1),$$

(IV) $$P(\xi) = (\xi-1)(\xi+1)(\xi^2+2a_1\xi+1)\ldots(\xi^2+2a_{s-2}\xi+1),$$

where $a_i^2 < 1$ and $a_i \neq a_j$ for $i \neq j$.

Let $D = (D_1, \ldots, D_s)$ be the almost product structure associated with f. Projectors of this structure will be denoted by $P_1, \ldots, P_s$. It is easy to verify that if f is a metric polynomial structure of the first type, then a tensor field J defined by

$$J = \sum_{i=1}^{s} \frac{f+a_i I}{\sqrt{1-a_i^2}} P_i$$

is an almost complex structure on M. J is called the *almost complex structure associated with f.*

PROPOSITION 4. *If f is a metric polynomial structure of the first type and J is defined as above, then $g(JX, JY) = g(X, Y)$ for any tangent vectors X and Y.*

Proof. Since $f(D_i) \subset D_i$, we have $f^{-1}(D_i) \subset D_i$ for $i = 1, \ldots, s$. Since $f^2+2a_if+I = 0$ on D_i, we have $f(f+2a_iI) = -I$ on D_i. Hence

(1) $$f^{-1} = \sum_{i=1}^{s} (-f-2a_iI)P_i.$$

We set

$$J' = \sum_{i=1}^{s} \frac{f^{-1}+a_iI}{\sqrt{1-a_i^2}} P_i.$$

By equality (1) it is obvious that $J' = -J$. Given two vectors X and Y, we obtain

$$
\begin{aligned}
g(JX, Y) &= \frac{1}{\sqrt{1-a_i^2}} g(fX + a_i X, Y) \\
&= \frac{1}{\sqrt{1-a_i^2}} \{g(fX, Y) + a_i g(X, Y)\} \\
&= \frac{1}{\sqrt{1-a_i^2}} \{g(X, f^{-1} Y) + g(X, a_i Y)\} \\
&= g\left(X, \frac{f^{-1} + a_i I}{\sqrt{1-a_i^2}} Y\right) = -g(X, JY).
\end{aligned}
$$

Therefore

$$g(JX, JY) = -g(X, J(JY)) = g(X, Y)$$

and this together with the following proposition, proves our assertion.

PROPOSITION 5. *The almost product structure* $D = (D_1, \ldots, D_s)$ *associated with a metric polynomial structure* f *is orthogonal, i.e.* D_i *is orthogonal to* D_j *if* $i \neq j$

Proof. It is sufficient to give a proof for a metric polynomial structure of type (IV). We shall consider the following cases:

1° $X \in D_1$ and $Y \in D_2$. Then

$$g(X, Y) = g(fX, fY) = g(X, -Y) = -g(X, Y).$$

Thus $g(X, Y) = 0$.

2° $X \in D_1$, $Y \in D_j$, $j \geqslant 3$. We have

$$g(X, Y) = g(fX, fY) = g(X, fY) = g(fX, f^2 Y) = g(X, f^2 Y). \tag{2}$$

Since $f^2 Y + 2a_{j-2} fY + Y = 0$, we have $g(X, f^2 Y + 2a_{j-2} fY + Y) = 0$. By equalities (2) we obtain

$$
\begin{aligned}
0 = g(X, f^2 Y + 2a_{j-2} fY + Y) &= g(X, f^2 Y) + 2a_{j-2} g(X, fY) + g(X, Y) \\
&= g(X, Y) + 2a_{j-2} g(X, Y) + g(X, Y).
\end{aligned}
$$

It is known that $a_{j-2} \neq -1$, and so $g(X, Y) = 0$.

3° $X \in D_2$, $Y \in D_j$, $j \geqslant 3$. The following equalities are evident:

$$g(X, Y) = g(fX, fY) = -g(X, fY) = -g(fX, f^2 Y) = g(X, f^2 Y).$$

Analogously to case 2°, we have

$$g(X, Y)-2a_{j-2}g(X, Y)+g(X, Y) = 0.$$

But $a_{j-2} \neq 1$ and hence $g(X, Y) = 0$.

4° $X \in D_i$, $Y \in D_j$ and $i \neq j$, $i, j \geqslant 3$. In this case

$$\begin{aligned} g(fX, Y) = g(X, f^{-1}Y) &= -g(X, fY+2a_{j-2}Y) \\ &= -g(X, fY)-2a_{j-2}g(X, Y) \\ &= -g(f^{-1}X, Y)-2a_{j-2}g(X, Y) \\ &= -g(-fX-2a_{i-2}X, Y)-2a_{j-2}g(X, Y) \\ &= g(fX, Y)+2a_{i-2}g(X, Y)-2a_{j-2}g(X, Y). \end{aligned}$$

From Proposition 3 we know that $a_{i-2} \neq a_{j-2}$; hence $g(X, Y) = 0$. The proof is finished.

Let us define a 2-form Φ on M by

$$\Phi(X, Y) = g(X, fY)-g(fX, Y)$$

for all tangent vectors X and Y.

Note that if f is an almost complex structure, then $\Phi = 2\chi$, where χ is the fundamental 2-form of the almost complex structure f. The form Φ defined above will be called the *fundamental 2-form* of a metric polynomial structure f.

Let ∇ denote the Riemannian connection on M induced by g.

PROPOSITION 6. *Let $T = (T_1, \ldots, T_m)$ be an almost product structure on M such that all distributions $T_1, \ldots, T_m$ are parallel with respect to ∇. Then for any vector fields $X \in T_i$, $Y \in T_j$, $i \neq j$, $[X, Y] = 0$, we have $\nabla_X Y = 0$.*

Proof. Since the connection ∇ is without torsion, $\nabla_X Y - \nabla_Y X = [X, Y]$. This means that $\nabla_X Y = \nabla_Y X$. But T_i and T_j are parallel with respect to ∇, and so $T_i \ni \nabla_Y X = \nabla_X Y \in T_j$. Hence $\nabla_X Y = \nabla_Y X = 0$.

The main purpose of this paper is to prove the following theorem.

THEOREM 7. *Let (M, g) be a Riemannian manifold and let f be a metric polynomial structure on M with respect to g. Then the following conditions are equivalent:*

1° $\nabla f = 0$;

2° $[f, f] = 0$, *the fundamental 2-form Φ of f is closed and the distributions of the almost product structure associated with f on which f is a multiple of identity are parallel with respect to ∇.*

Proof. Assume 2°. At first we shall consider a metric polynomial structure of type (I).

Let us define

$$\Psi(X, Y) = g(X, JY) - g(JX, Y).$$

We shall show that $d\Psi = 0$. For any vector fields X, Y, Z the following formula holds:

$$3d\Psi(X, Y, Z) = X(\Psi(Y, Z)) + Y(\Psi(Z, X)) + Z(\Psi(X, Y)) - \\ - \Psi([X, Y], Z) - \Psi([Z, X], Y) - \Psi([Y, Z], X).$$

Obviously, it is sufficient to verify that $d\Psi(X, Y, Z) = 0$ for $X = \partial/\partial x^k$, $Y = \partial/\partial x^l$, $Z = \partial/\partial x^m$, where $(x^1, \ldots, x^n)$ is a chart on M. Since the Nijenhuis product $[f, f]$ vanishes on M, the polynomial structure f is integrable by Theorem 2. If $\varphi = (x^1, \ldots, x^n)$ is a chart associated with the integrable polynomial structure f, then this chart is also associated with the integrable almost product structure $D = (D_1, \ldots, D_s)$.

Let $\varphi = (x^1, \ldots, x^n)$ be a chart associated with the integrable tensor field f and let $X = \partial/\partial x^k$, $Y = \partial/\partial x^l$, $Z = \partial/\partial x^m$. Vector fields obtained in this way will be called *f-holonomic vector fields*. There are three cases:

(I) $X \in D_i$, $Y \in D_j$, $Z \in D_k$ and $i \neq j$, $j \neq k$, $i \neq k$,

(II) $X, Y \in D_i$, $Z \in D_j$, $i \neq j$,

(III) $X, Y, Z \in D_i$.

In case (I) the equality $d\Psi(X, Y, Z) = 0$ is an immediate consequence of the definition of Ψ and Proposition 5. As regards case (II), we have

$$3d\Psi(X,Y,Z) = Z\Psi(X,Y) = Z\left(g\left(X, \frac{f+a_i I}{\sqrt{1-a_i^2}} Y\right)\right) - Z\left(g\left(\frac{f+a_i I}{\sqrt{1-a_i^2}} X, Y\right)\right)$$

$$= \frac{1}{\sqrt{1-a_i^2}} Z\big(g(X, fY) + a_i g(X, Y)\big) - \\ - \frac{1}{\sqrt{1-a_i^2}} Z\big(g(fX, Y) + a_i g(X, Y)\big) \\ = \frac{1}{\sqrt{1-a_i^2}} Z\big(g(X, fY) - g(fX, Y)\big).$$

But

$$0 = 3d\Phi(X, Y, Z) = Z\Phi(X, Y) = Z\big(g(X, fY) - g(fX, Y)\big).$$

Hence $d\Psi(X, Y, Z) = 0$. If vector fields X, Y, Z are as in case (III), then

$$3d\Psi(X, Y, Z) = X(\Psi(Y, Z))+Y(\Psi(Z, X))+Z(\Psi(X, Y))$$
$$= X\left(g\left(Y, \frac{f+a_i I}{\sqrt{1-a_i^2}}Z\right)\right)-X\left(g\left(\frac{f+a_i I}{\sqrt{1-a_i^2}}Y, Z\right)\right)+$$
$$+Y\left(g\left(Z, \frac{f+a_i I}{\sqrt{1-a_i^2}}X\right)\right)-Y\left(g\left(\frac{f+a_i I}{\sqrt{1-a_i^2}}Z, X\right)\right)+$$
$$+Z\left(g\left(X, \frac{f+a_i I}{\sqrt{1-a_i^2}}Y\right)\right)-Z\left(g\left(\frac{f+a_i I}{\sqrt{1-a_i^2}}X, Y\right)\right)$$
$$= \frac{3}{\sqrt{1-a_i^2}}d\Phi(X, Y, Z) = 0.$$

It is clear that $[f, f] = 0$ implies $[J, J] = 0$ (see [2]). Applying Theorem 1, to the almost Hermitian manifold M with the almost complex structure J, we obtain $\nabla J = 0$. Since $f = \sum_{i=1}^{s} (\sqrt{1-a_i^2}\, J - a_i I) P_i$, $\nabla f = 0$ if and only if $\nabla J = 0$ and $\nabla P_i = 0$ for $i = 1, \ldots, s$. Now it is sufficient to show that $\nabla P_i = 0$ for $i = 1, \ldots, s$. In order to get this we shall show that for any f-holonomic vector fields X, Y, Z such that $X \in D_i$, $Z \in D_j$ and $i \neq j$ we have, $g(\nabla_X Y, Z) = 0$. On account of Proposition 5 this will prove our assertion.

Let $\varphi = (x^1, \ldots, x^n)$ be a chart associated with the integrable tensor field f and let $X = \partial/\partial x^l$, $Y = \partial/\partial x^k$, $Z = \partial/\partial x^m$. Then

$$(3) \qquad 2g(\nabla_X Y, Z) = X(g(Y, Z))+Y(g(X, Z))-Z(g(X, Y)).$$

Let $Y \in D_i$. If $X \in D_i$ and $Z \in D_j$, $i \neq j$; then $2g(\nabla_X Y, Z) = -Z(g(X, Y))$. Given $Y' = J^{-1}Y$, there exists a vector $c \in \boldsymbol{R}^n$ such that $Y' = J^{-1}Y$. This follows from the obvious fact that the chart φ is also associated with the integrable almost complex structure J.

We have already proved that $d\Psi(X, Y', Z) = 0$. Therefore

$$0 = 3d\Psi(X, Y', Z) = Z\Psi(X, Y') = 2Z(g(X, JY')) = 2Z(g(X, Y)).$$

If $X \in D_k$, $Z \in D_j$ and $i \neq k$, $k \neq j$, $i \neq j$, then the equality $g(\nabla_X Y, Z)$ set $Z' = J^{-1}Z$ and we obtain

$$0 = 3d\Psi(X, Y, Z') = -Y\Psi(X, Z') = -2Y(g(X, JZ')) = -2Y(g(X, Z)).$$

If $X \in D_k$, $Z \in D_j$ and $i \neq k$, $k \neq j$, $i \neq j$, then the equality $g(\nabla_X Y, Z) = 0$ is evident by formula (3) and Proposition 5. Thus the proof of the assertion in the first case is completed.

Returning to the general case, we shall show that $f\nabla_X Y = \nabla_X fY$ for any vector fields X, Y. We set $T_1 = \ker(f-I)$, $T_2 = \ker(f+I)$, $T_3 = D_{i_1} \oplus \ldots \oplus D_{i_k}$, where $D_{i_1}, \ldots, D_{i_k}$ are all distribution of the almost product structure D on which f is not a multiple of identity. Of course, it may happen that $T_1 = 0$ or $T_2 = 0$ or $T_3 = 0$, but in such a case we simply need not consider all possibilities which can occur. The projective of the almost product structure $T = (T_1, T_2, T_3)$ will be denoted by Q_1, Q_2, Q_3, respectively. Clearly, $\nabla Q_i = 0$ for $i = 1, 2, 3$.

At first notice that it suffices to prove, the equality $f\nabla_X Y = \nabla_X fY$ for f-holonomic vector fields X and Y. In fact, if $f\nabla_X Y = \nabla_X fY$, then

$$\begin{aligned}\nabla_X f(\alpha Y) &= \alpha(\nabla_X fY)+(X\alpha)fY = \alpha f\nabla_X Y+f(X\alpha)Y\\ &= f\{\alpha\nabla_X Y+(X\alpha)Y\} = f\nabla_X(\alpha Y).\end{aligned}$$

Let $\varphi = (x^1, \ldots, x^n)$ be a chart associated with the integrable tensor field f and let $X = \partial/\partial x^k$, $Y = \partial/\partial x^l$. Then we have one of the following cases:

1° $Y \in T_1$. Since $\nabla Q_1 = 0$, $\nabla_X Y \in T_1$. Therefore

$$f\nabla_X Y = \nabla_X Y = \nabla_X fY.$$

2° $Y \in T_2$. Since $\nabla Q_2 = 0$, $\nabla_X Y \in T_2$ and just as above we have

$$f(\nabla_X Y) = -\nabla_X Y = \nabla_X fY.$$

3° $Y \in T_3$, $X \in T_1 \oplus T_2$. Since $[X, Y] = 0$, $\nabla_X Y = 0$ by Proposition 6. Hence $f\nabla_X Y = 0$. φ is a chart associated with the integrable tensor field f, and so there exists a vector $c \in \boldsymbol{R}^n$ such that $fY = d\varphi^{-1}(c)$. Consequently $[X, fY] = 0$. Of course, $fY \in T_3$ and, by Proposition 6, $\nabla_X fY = 0$.

4° $X \in T_3$, $Y \in T_3$. Let $x \in M$ and let N be an integral manifold of distribution T_3 through x. We set $X' = X_x$, $Y' = Y|_N$, $f' = f|_N$, $g' = g|_N$, $(fY)' = (fY)|_N$. (N, g') is a Riemannian manifold and f' is a metric polynomial structure on N of the first type. If Φ' denote the fundamental 2-form of f', then the assumption that Φ is closed implies that the fundamental 2-form Φ' is closed. Vanishing of the Nijenhuis product $[f, f]$ implies vanishing of $[f', f']$. From the first part of our proof we have

$$f'\nabla'_{X'} Y' = \nabla'_{X'} f' Y',$$

where ∇' is the Riemannian connection on M defined by g'. Since the distribution T_3 is parallel with respect to ∇, we obtain

$$f\nabla_{X'} Y = f'\nabla'_{X'} Y' = \nabla'_{X'} f' Y' = \nabla'_{X'} (fY)' = \nabla_{X'} fY.$$

Assume 1°. Since $\nabla f = 0$ and the connection ∇ is torsion-free, f is integrable and hence $[f, f] = 0$. Since the projectors $P_1, \ldots, P_s$ of the almost product structure D are polynomials in f, $\nabla P_i = 0$. In other words, the distributions $D_1, \ldots, D_s$ are parallel with respect to ∇. Tensor fields g

and f are parallel with respect to ∇, and so is Φ, i.e., $\nabla\Phi = 0$. Since ∇ is torsion free, we have $d\Phi = A(\nabla\Phi)$, where A denotes the alternation of the covariant tensor $\nabla\Phi$ ([3], Chapter III, § 8). This means that Φ is closed and this finishes the proof.

Theorem 7 is not true without the assumption that the distributions on which f is a multiple of identity are parallel with respect to ∇. For example, let $M = \boldsymbol{R}^4$ and let (x^1, x^2, x^3, x^4) denote the canonical coordinate system in $\boldsymbol{R}^4$. Let $X_1 = \partial/\partial x^1$, $X_2 = \partial/\partial x^2$, $X_3 = \partial/\partial x^3$, $X_4 = \partial/\partial x^4$. We set $f(X_1) = X_1$, $f(X_2) = -X_2$, $f(X_3) = X_4$, $f(X_4) = -X_3$. Of course, f is an integrable polynomial structure and $D_1 = \boldsymbol{R}X_1$, $D_2 = \boldsymbol{R}X_2$, $D_3 = \boldsymbol{R}X_3 \oplus \boldsymbol{R}X_4$. If we define a metric tensor g on $\boldsymbol{R}^4$ by one of the following matrices:

$$\text{(a)} \quad \begin{bmatrix} e^{x^1x^2} & 0 & 0 & 0 \\ 0 & e^{x^1x^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \text{(b)} \quad \begin{bmatrix} e^{x^1x^2x^3} & 0 & 0 & 0 \\ 0 & e^{x^1x^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\text{(c)} \quad \begin{bmatrix} e^{x^1x^3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

then f is a metric polynomial structure with respect to g. It is also easy to check that the fundamental 2-form Φ is closed in each of cases (a), (b), (c). In case (b) neither of distributions D_1, D_2, D_3 is parallel with respect to the Riemannian connection ∇ defined by g. In particular, D_1 is not parallel with respect to ∇, because

$$2g(\nabla_{X_1} X_l, X_2) = -\frac{\partial}{\partial x^2} e^{x^1x^2} \neq 0 \quad \text{whenever } x_1 \neq 0.$$

In case (a) only the distribution D_3 is parallel with respect to ∇. In case (c) only D_2 is parallel with respect to ∇.

Therefore, example (a) means that in the case of metric polynomial structure of type (IV) it is not sufficient to assume that the distribution $D_1 \oplus D_2$ is parallel with respect to ∇. By example (c), it is seen that it is also not sufficient to assume that one of distributions on which f is a multiple of identity is parallel with respect to ∇.

References

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