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# A THEOREM ON RECURSIVELY ENUMERABLE VECTOR SPACES

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This paper* is based on [1] and [2], but since we study only r.e. spaces, we prefer an exposition which is almost self-contained. Let $F$ be a countable field for which there is a one-to-one mapping $\phi$ from $F$ onto a recursive subset of $\varepsilon = (0, 1, \ldots)$ such that: $\phi(0_F) = 0$, $\phi(1_F) = 1$, $+_F$ and $\cdot_F$ correspond to partial recursive functions, $\phi(F) = (0, \ldots, q - 1)$, if $\mathrm{card}(F) = q$ and $\phi(F) = \varepsilon$, if $\mathrm{card}(F) = \aleph_0$. We write $\mathcal{U}_F$ for the vector space over $F$ which consists of all sequences of field elements with at most finitely many nonzero components, together with component-wise addition and scalar multiplication. Put

$$(1) \qquad \Phi\{x_n\} = \prod_{n \leqslant k} p_n{}^{\phi(x_n)} - 1, \text{ for } \{x_n\} \in \mathcal{U}_F,$$

where $p_0 = 2$, $p_n =$ the $n$'th odd prime, $k$ any number such that $x_n = 0_F$, for $n > k$. Then $\Phi$ maps $\mathcal{U}_F$ onto a vector space $\overline{U}_F = [\varepsilon_F, +, \cdot]$, where $\varepsilon_F$ is an infinite recursive set and $+$ and $\cdot$ are partial recursive functions. Note that the ordinary number 0 is also the zero element of $\overline{U}_F$. Set $\underline{e_n} = p_n - 1$, $\eta = (e_0, e_1, \ldots)$, then $\eta$ is an infinite recursive basis of $\overline{U}_F$, hence $\dim(\overline{U}_F) = \aleph_0$. The word "space" will be used in the sense of "subspace of $\overline{U}_F$". A space $\overline{V} = [\alpha, +, \cdot]$ is called *r.e.*, if the set $\alpha$ is r.e., *recursive*, if $\overline{V}$ is r.e. and has at least one r.e. complementary space, *decidable*, if $\alpha$ is a recursive set, i.e., if both $\alpha$ and $\varepsilon_F - \alpha$ are r.e.

The *purpose* of this paper is to examine the relationship between (I) $\overline{V}$ is a recursive space, and (II) $\overline{V}$ is a decidable space. We shall prove:

(a) *if $F$ is finite*, (I) $\Leftrightarrow$ (II),
(b) *if $F$ is infinite*, (I) $\Rightarrow$ (II), *but not conversely*.

A linearly independent subset of $\varepsilon_F$ is called a *repère*. According to [1], p. 2, there is an effective procedure which enables us to decide for any

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finite subset $\sigma$ of $\varepsilon_F$ whether $\sigma$ is a repère. It follows ([1], p. 3) that a space is r.e. if and only if it has a r.e. basis. If $\overline{\beta}$ is a r.e. basis of the r.e. space $\overline{V}$ we can (*cf.* [1], p. 5), given any $x \in \overline{V}$, test whether $x \in \overline{\beta}$; moreover, if $x \neq 0$ and $x \notin \overline{\beta}$, we can effectively express $x$ as a linear combination of elements in $\overline{\beta}$, i.e., find the nonzero elements $r_0, \ldots, r_k \in F$ and the distinct elements $b_0, \ldots, b_k \in \overline{\beta}$ such that

(2) $x = r_0 b_0 + \ldots + r_k b_k$.

For a subset $S$ of $\overline{U}_F$ we write $\mathrm{L}(S)$ for the span of $S$. A repère $\beta$ is *perfect*, if for $x \in \varepsilon_F$,

(3) $x \in \mathrm{L}(\beta) \Longleftrightarrow x \in \mathrm{L}\{y \in \varepsilon_F \mid y \in \beta \text{ and } y \leqslant x\}$.

A basis of a space $V$ is a *perfect* basis of $V$, if it is also a perfect repère. One can prove ([2], Prop. B) that every space $V$ has exactly one perfect basis, say $\pi_V$, and that $V$ is a decidable space if and only if $\pi_V$ is a recursive set. If $f(n)$ is a function from $\varepsilon$ into $\varepsilon$, we write $\rho f$ for the range of $f(n)$. Let $V$ and $W$ be spaces; then $W \leqslant V$ means that $W$ is a subspace of $V$ and $W < V$ that $W$ is a proper subspace of $V$.

Proposition A *Every recursive space is decidable.*

*Proof:* Let $\overline{V}$ be a recursive space with $\overline{W}$ as a r.e. complementary space. Suppose that $\overline{\beta}$, $\overline{\partial}$ are r.e. bases of $\overline{V}$, $\overline{W}$ respectively. An element $x \in \varepsilon_F$ belongs to $\overline{V}$, if either (i) $x = 0$, or (ii) $x \neq 0$ and relative to the r.e. basis $\overline{\beta} \cup \overline{\partial}$ of $\overline{U}_F$ all coordinates of $x$ with respect to elements in $\overline{\partial}$ are zero. Thus $\overline{V}$ is a decidable space.

Proposition B *If the field $F$ is finite, a* r.e. *space $\overline{V}$ is recursive if and only if it is decidable.*

*Proof:* Let $\overline{V}$ be a decidable space. Since every finite dimensional space is r.e., every r.e. space of finite codimension is recursive. We may therefore assume that $\mathrm{codim}(\overline{V}) = \aleph_0$. Put

$$(4) \quad \begin{aligned} c_0 &= (\mu x)[x \in \varepsilon_F \ \& \ x \neq 0 \ \& \ x \notin \overline{V}], \\ c_{n+1} &= (\mu x)[x \in \varepsilon_F \ \& \ x \notin L(c_0, \ldots, c_n) \ \& \ \overline{V} \cap \mathrm{L}(c_0, \ldots, c_n, x) = (0)], \end{aligned}$$

then $\overline{V} \oplus \mathrm{L}(\rho c) = \overline{U}_F$. The number $c_0$ can be computed from (the recursive characteristic function of) $\overline{V}$. Assume that $c_0, \ldots, c_n$ have been computed and that $\overline{V} \cap \mathrm{L}(c_0, \ldots, c_n) = (0)$. Then we can for every $x \in \varepsilon_F$ test whether

(i) $x \notin \mathrm{L}(c_0, \ldots, c_n)$, i.e., whether $x \notin (c_0, \ldots, c_n)$ and $(c_0, \ldots, c_n, x)$ is a repère,
(ii) in case (i) holds, whether $\overline{V} \cap \mathrm{L}(c_0, \ldots, c_n, x) = (0)$.

Note that (i) can be tested whether $F$ is finite or infinite. However, in (ii) we use the fact that $F$ is finite. For if $\mathrm{card}(F) = q$, we can for every $x \notin \mathrm{L}(c_0, \ldots, c_n)$ compute the $q^{n+2}$ elements of $\mathrm{L}(c_0, \ldots, c_n, x)$ and determine whether any belongs to $\overline{V}$. Hence the function $c_n$ defined by (4) is recursive and so is the space $\overline{V}$.

Proposition C *If $\overline{V}$ is a recursive space and $p \in \varepsilon_F$, then $\overline{V} + \mathrm{L}(p)$ is also a recursive space.*

*Proof:* We only need to show that

(6) $\overline{V}$ recursive & $p \notin \overline{V} \Rightarrow \overline{V} \oplus L(p)$ recursive.

Assume the hypothesis. Let $\overline{\beta}$ be a r.e. basis of $\overline{V}$, $\overline{\partial}$ a r.e. basis of some r.e. complementary space of $\overline{V}$ and $\overline{\delta} = \overline{\beta} \cup \overline{\partial}$. Let $p = r_0 d_0 + \ldots + r_n d_n$, where $r_0, \ldots, r_n \in F - (0)$ and $d_0, \ldots, d_n$ are distinct elements of $\overline{\delta}$. Since $p \notin V$ at least one of $d_0, \ldots, d_n$ belongs to $\overline{\partial}$; we may assume w.l.g. that $d_0 \in \overline{\partial}$. Define $\partial^* = [\overline{\partial} - (d_0)] \cup (p)$, then $L(\partial^*)$ is also a r.e. complementary space of $\overline{V}$. It follows that $\overline{\beta} \cup (p)$ is a r.e. basis of $\overline{V} \oplus L(p)$, while $\overline{\partial} - (d_0)$ is a r.e. basis of the r.e. complementary space $L[\overline{\partial} - (d_0)]$ of $\overline{V}$. Thus $\overline{V} \oplus L(p)$ is a recursive space.

Corollary *The sum of a recursive space and a finite dimensional space is again a recursive space.*

We say that the element $x \in F$ *can be computed*, if we can compute $\phi(x)$. Similarly, a function $f(n)$ from $\varepsilon$ into $F$ is *recursive*, if the function $\phi f(n)$ from $\varepsilon$ into $\varepsilon$ is recursive. These definitions become superfluous if one identifies $F$ with a subset of $\varepsilon$, but it remains important to distinguish the field operations of $F$, the vector space operations of $\overline{U}_F$, and ordinary addition and multiplication in $\varepsilon$. If $x > 0$ we write $x^-$ for $x - 1$; thus $e_n = p_n^-$, for $n \in \varepsilon$. Finally, for $r \in F$ we abbreviate the number $2^{\phi(r)}$ by $h(r)$. The next proposition plays the *key role* in our paper.

Proposition D *For every infinite field F and every one-to-one recursive function $s_n$ ranging over a subset of $(p_1, p_2, \ldots)$, there is a recursive function $m(n)$ from $\varepsilon$ into F such that*

(7) $\overline{D} = L[m(0) \cdot e_0 + s_0^-, m(1) \cdot e_0 + s_1^-, \ldots]$

*is a decidable space.*

*Proof:* Let the one-to-one recursive function $s_n$ be given. Define for every function $m(n)$ from $\varepsilon$ into $F$,

(8) $\overline{D}_n = L[m(0) \cdot e_0 + s_0^-, \ldots, m(n) \cdot e_0 + s_n^-]$,
(9) $q_0 = \min[\overline{D}_0 - (0)]$, $q_{n+1} = \min[\overline{D}_{n+1} - \overline{D}_n]$.

If we can define a recursive function $m(n)$ such that the function $q_n$ is strictly increasing and recursive, we are done. For then $(q_0, \ldots, q_n)$ is the perfect basis of $\overline{D}_n$, hence $\rho q$ the perfect basis of $\overline{D}$; moreover, $\rho q$ is a recursive set, hence $\overline{D}$ a decidable space. First of all, for *every* recursive function $m(n)$, the function $q_n$ defined by (8) and (9) is recursive. For if

$$a_n = [m(0) \cdot e_0 + s_0^-] + \ldots + [m(n) \cdot e_0 + s_n^-],$$

then $a_n$ is a recursive function such that

$$a_0 \in \overline{D}_0 - (0) \text{ and } a_{n+1} \in \overline{D}_{n+1} - \overline{D}_n.$$

Also,

$$q_0 = (\mu y \leq a_0)[y \in \overline{D}_0 - (0)],$$
$$q_{n+1} = (\mu y \leq a_{n+1})[y \in \overline{D}_{n+1} - \overline{D}_n].$$

Since we know a finite basis for each of $\overline{D}_0$, $\overline{D}_1$, . . . and given any finite repère $\beta$, we can for every $x \in \varepsilon_F$ test whether $x \in L(\beta)$, it follows that $q_n$ is a recursive function. All that remains is the definition of a recursive function $m(n)$ from $\varepsilon$ into $F$ such that the function $q(n)$ is strictly increasing. We put $m(0) = 1_F$. Assume as inductive hypothesis that field elements $m(0), \ldots, m(n)$ have been defined such that $q_0 < \ldots < q(n)$. As observed above, $q_0, \ldots, q_n$ can be computed from $m(0), \ldots, m(n)$, hence $q_n$ is known. We now examine how $m(n+1)$ and $q_{n+1}$ should be related in order that

(10) $q_{n+1} = \min[\overline{D}_{n+1} - \overline{D}_n] > q_n$.

An element $x \in \overline{D}_{n+1} - \overline{D}_n$ looks like

$$[t_0 m(0) \cdot e_0 + t_0 \cdot s_0^-] + \ldots + [t_{n+1} m(n+1) \cdot e_0 + t_{n+1} \cdot s_{n+1}^-],$$

where $t_0, \ldots, t_{n+1} \in F$ and $t_{n+1} \neq 0$. Thus, by (1),

(11) $$x = \left[ h\left( \sum_{i=0}^{n+1} t_i m(i) \right) \prod_{i=0}^{n+1} s_i^{\phi(t_i)} \right]^- ,$$

where the summation sign refers to addition in $F$ and the product sign to ordinary multiplication in $\varepsilon$. Replacing $m(n+1)$ by $v$, we can rewrite (11) as

(12) $$x = \left[ h\left( \sum_{i=0}^{n} t_i m(i) +_F t_{n+1} v \right) \prod_{i=0}^{n+1} s_i^{\phi(t_i)} \right]^- .$$

The expression between the brackets in (12) will be abbreviated by $\Delta_v$. Hence $x = \Delta_v^-$. Note that $\Delta_v$ is a function of $(t_0, \ldots, t_{n+1})$, for every $v \in F$. We wish to choose $v = m(n+1)$ in such a way that for all $(t_0, \ldots, t_{n+1})$,

(13) $(t_0, \ldots, t_{n+1}) \in F^{n+2}$ & $t_{n+1} \neq 0 \Rightarrow \Delta_v > q(n) + 1$.

For a specific ordered $(n+2)$-tuple satisfying the hypothesis of (13), each of the following two conditions will guarantee that the conclusions of (13) be true:

(14) $s_i^{\phi(t_i)} > q(n) + 1$, for some $i \leq n+1$,

(15) $$h\left[ \sum_{i=0}^{n} t_i m(i) +_F t_{n+1} v \right] > q(n) + 1.$$

We call an ordered $(n+2)$-tuple $(t_0, \ldots, t_{n+1})$ with $t_{n+1} \neq 0$, *bad*, if it does not satisfy (14); let $B$ denote the set of all bad $(n+2)$-tuples. If $B$ is empty, $\Delta_v > q(n) + 1$, for every $v$, hence $x > q(n)$ for every choice of $m(n+1)$; then we define $m(n+1) = 1_F$. From now on we assume that $B$ is nonempty. $B$ is finite, since for every $i \leq n+1$, there are only finitely many elements $t_i$, such that $s_i^{\phi(t_i)} \leq q(n) + 1$. Let $\mathrm{card}(B) = w + 1$, then $w$ can be computed and $B$ can be effectively generated in a finite sequence $\beta_0, \ldots, \beta_w$. With every $u \leq w$ we wish to associate a field element $r(u)$ such that for all $v \in F$,

(16) $\phi(v) > \phi r(u) \Rightarrow \Delta_v > q(n) + 1$.

Such an element $r(u)$ exists, for if we put

$$a = \sum_{i=0}^{n} t_i\, m(i),\ b = t_{n+1}\ ,$$

then $a$ and $b$ are constants (depending on $u$) and $\Delta_v$ is of the form $h[a \oplus_F bv]$, a one-to-one function of $v$. From $a$ and $b$ we can compute the set

$$\delta_u = \{v \in F \mid h[a \oplus_F bv] \leq q(n) + 1\},$$

i.e., find out whether it is empty and determine its elements and cardinality if it is nonempty. Put

$$(17)\quad r(u) = \begin{cases} 0_F, \text{ if } \delta_u \text{ is empty,} \\ y, \text{ if } \delta_u \text{ is nonempty and } \phi(y) = \max \phi(\delta_u). \end{cases}$$

It follows that ($a$ and $b$ being defined in terms of $u$, i.e., in terms of $\beta_u$), we have for all $v \in F$,

$$\phi(v) > \phi r(u) \Rightarrow v \notin \delta_u \Rightarrow h[a +_F bv] > q(n) + 1.$$

The set $(r(0), \ldots, r(w))$ of field elements can be computed from $B$, hence from $m(0), \ldots, m(n)$. Thus the element $c \in F$ such that

$$\phi(c) = 1 + \max(\phi r(0), \ldots, \phi r(w))$$

can be computed. Then we have for all $v \in F$,

$$\phi(v) \geq \phi(c) \Rightarrow v \notin \bigcup_{u=0}^{w} \delta_u \Rightarrow h[a +_F bv] > q(n) + 1,$$

and this holds for every $\beta_u \in B$. Thus $h[a +_F bc] > q(n) + 1$ and (12) will be true if we take $v = c$. We therefore define $m(n+1) = c$. Then all elements of $\overline{D}_{n+1} - \overline{D}_n$ exceed $q(n)$ by (11); in particular, $q_{n+1} > q_n$. This completes the proof.

Proposition E *For every infinite field $F$ there is a decidable, but not recursive space.*

*Proof:* Suppose $s_n$ is a one-to-one recursive function ranging over a subset of $(p_1, p_2, \ldots)$. Let $m(n)$ be a recursive function from $\varepsilon$ into $F$ such that the r.e. space $\overline{D}$ defined by (7) is decidable. Then $e_0 \notin \overline{D}$ and

(18) $\overline{D} \oplus \mathrm{L}(e_0) = \mathrm{L}(e_0, s_0^-, s_1^-, \ldots)$.

In fact, $(e_0, s_0^-, s_1^-, \ldots)$ is the perfect basis of $\overline{D} \oplus \mathrm{L}(e_0)$. We now choose $s_n$ in such a way that the r.e. set $\rho s$ is not recursive; then the perfect basis of $\overline{D} \oplus \mathrm{L}(e_0)$ is not recursive, hence $\overline{D} \oplus \mathrm{L}(e_0)$ is not decidable. If, however, $\overline{D}$ were a recursive space, $\overline{D} \oplus \mathrm{L}(e_0)$ would be recursive by (b) and decidable by Proposition A. We conclude that the space $\overline{D}$ is not recursive.

Remark. This proof implies that for every infinite field $F$ there is a r.e. space $\overline{V}$ and an element $p \in \varepsilon_F$ such that

(19) $\overline{V}$ decidable & $p \in \overline{V}$ & $\overline{V} \oplus \mathrm{L}(p)$ not decidable,

in striking contrast with (b).

## REFERENCES

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