

**A THEOREM ON THE GROWTH OF ENTIRE FUNCTIONS  
ON ASYMPTOTIC PATHS AND ITS APPLICATION TO  
THE OSCILLATION THEORY OF  $w'' + Aw = 0$**

Dedicated to Professor Nobuyuki Suita on the occasion of  
his sixtieth birthday

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**1. Introduction.**

Let  $A=A(z)$  be a transcendental entire function and let  $w_1, w_2$  be two linearly independent entire solutions of the differential equation

$$(1) \quad w'' + Aw = 0.$$

It is known that any non-zero solution of (1) is an entire function of infinite order ([1]). Put

$$E = w_1 w_2.$$

It then holds ([1], p. 354) that

$$(2) \quad 4A = (E'/E)^2 - 2E''/E - (c/E)^2,$$

where  $c$  is the Wronskian of  $w_1$  and  $w_2$ , which is a non-zero constant in this case.

For an entire function  $f$  we denote the order of  $f$  by  $\rho(f)$ , the lower order of  $f$  by  $\mu(f)$  and the order of  $N(r, 1/f)$  by  $\lambda(f)$ .

S. B. Bank and I. Laine ([1], Theorem 2, (A)) proved from (2) that  $\rho(A) < 1/2$  implies  $\lambda(E) = +\infty$ . They also gave examples of (1) with two linearly independent entire solutions each having no zeros, in each case of which,  $\rho(A)$  is either a positive integer or  $+\infty$  ([1], p. 356).

It is conjectured that if  $\rho(A)$  is finite and not a positive integer, then we always have  $\lambda(E) = +\infty$  (see [2], p. 164). In this direction J. Rossi ([12]) and L.-C. Shen ([13]) proved some results which contains that  $\rho(A) \leq 1/2$  implies  $\lambda(E) = +\infty$ . Recently, C.-Z. Huang ([9]) proved the following result which generalizes them.

**THEOREM A.** *If  $\mu(A) < 1$ , then either  $\lambda(E) = +\infty$  or*

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$$\mu(A)^{-1} + \lambda(E)^{-1} \leq 2 \quad ([9], \textit{Theorem 1}).$$

One of our main purpose of this paper is to give a result which contains Theorem A. To prove it, we need a growth property of  $A(z)$  in the set

$$\{z : |A(z)| > 1\}$$

and so we shall first give a result on the growth of entire functions along asymptotic paths. We shall assume that the reader is familiar with the standard notation of the Nevanlinna theory of meromorphic functions ([5]).

**2. Growth of entire functions along asymptotic paths.**

A few years ago J. Rossi and A. Weitsman ([11]) proved the following.

**THEOREM B.** *Let  $f(z)$  be a transcendental entire function. Suppose that for some constant  $K$  the set*

$$\{z : |f(z)| > K\}$$

*contains at least two components. Then there exists a path  $\Gamma$  from 0 to  $\infty$  such that for  $z \in \Gamma$*

$$(3) \quad \log |f(z)| > |z|^{\rho(f)/(2\rho(f)-1)-\varepsilon(z)} \quad (0 \leq \varepsilon(z) \rightarrow 0 \text{ as } z \rightarrow \infty).$$

(We consider  $\rho(f)/(2\rho(f)+k)=1/2$  when  $\rho(f)=+\infty$  and  $k$  is finite.)

Examples showing that Theorem B is sharp are given in [4]. Besides this result we can find interesting results on the growth of entire and subharmonic functions along asymptotic paths ([3], [4], [10], [11], [14], [15] and Chapter 8 in [8]).

The purpose of this section is to improve Theorem B and to give a subharmonic analogue, which is an improvement of Theorem 1 in [4].

**2-1. Lemmas.**

We shall give some lemmas for later use. Let  $D$  be an unbounded regular plane domain. We put

$$E(r) = \{\theta \in [0, 2\pi) : re^{i\theta} \in D\}$$

and

$$(4) \quad \theta(r) = \begin{cases} +\infty & \text{if } \{|z|=r\} \subset D \\ \text{the measure of } E(r) & \text{otherwise.} \end{cases}$$

It is clear that there is a positive number  $a$  such that  $\theta(r) > 0$  for all  $r \geq a$ .

**LEMMA 1.** *If*

$$\liminf_{r \rightarrow \infty} (\log r)^{-1} \pi \int_a^r \frac{dt}{t\theta(t)} = \mu \quad (1/2 \leq \mu < \infty),$$

then there exists  $u > 0$  harmonic in  $D$  such that for all  $z \in D$

$$u(z) \geq |z|^{\mu - \varepsilon(z)} \quad (0 \leq \varepsilon(z) \rightarrow 0 \text{ as } z \rightarrow \infty)$$

([11], Lemma 1 and its correction).

LEMMA 2. Let  $g(z)$  be regular in  $D$  and continuous on the closure of  $D$  such that

$$|g(z)| \leq 1 \quad (z \in \partial D).$$

If there exists one point  $z_0$  in  $D$  such that

$$|g(z_0)| > 1,$$

then

$$\log \log M(r, g) \geq \pi \int_a^{r/2} \frac{dt}{t\theta(t)} + O(1),$$

where  $M(r, g) = \sup\{|g(z)| : (|z|=r) \cap D\}$  ([17], p. 117).

LEMMA 3. Let  $v(z)$  be a non-constant subharmonic function in  $|z| < \infty$ . Then there exists a path  $\Gamma$  tending to  $\infty$  such that

$$v(z) \rightarrow +\infty \quad \text{as } z \rightarrow \infty \text{ on } \Gamma$$

([14], Theorem 1).

**2-2. Theorem.**

We shall give a result generalizing Theorem B.

THEOREM 1. Let  $f(z)$  be a transcendental entire function with  $\mu(f) < +\infty$ . Suppose that for some constant  $K$  the set

$$\{z : |f(z)| > K\}$$

contains at least  $N$  components  $D_1, \dots, D_N$ , where  $N \geq 2$ . Then for each  $j (= 1, \dots, N)$  there exists a path  $\Gamma_j$  tending to  $\infty$  in  $D_j$  such that on  $\Gamma_j$

$$(5) \quad \log |f(z)| > |z|^{\rho(f)/(2\rho(f)+1-N) - \varepsilon_j(z)} \quad (0 \leq \varepsilon_j(z) \rightarrow 0 \text{ as } z \rightarrow \infty).$$

*Proof.* It is clear that  $D_1, \dots, D_N$  are mutually disjoint unbounded regular domains in  $|z| < \infty$  and there exists an  $a > 0$  such that for all  $r \geq a$

$$\{|z|=r\} \cap D_j \neq \emptyset \quad (j=1, \dots, N).$$

We here use  $\theta_j(r)$  for  $D_j$  instead of  $\theta(r)$  defined for  $D$  in (4). Then

$$\theta_j(r) > 0 \quad (r \geq a)$$

and

$$(6) \quad \sum_{j=1}^N \theta_j(r) \leq 2\pi.$$

From (6) we obtain the inequality

$$(7) \quad \sum_{j=1}^N \int_a^r \frac{\theta_j(t)}{t} dt \leq 2\pi \log \frac{r}{a}$$

and by the Cauchy-Schwarz inequality we have

$$(8) \quad \int_a^r \frac{\theta_j(t)}{t} dt \int_a^r \frac{dt}{t\theta_j(t)} \geq \left( \int_a^r \frac{dt}{t} \right)^2 = \left( \log \frac{r}{a} \right)^2.$$

From (7) and (8) we have

$$(9) \quad \sum_{j=1}^N \frac{1}{\{\log(r/a)\}^{-1} \pi \int_a^r \frac{dt}{t\theta_j(t)}} \leq 2.$$

Applying Lemma 2 to  $f(z)/K$  in  $D$ , we obtain the following inequalities:

$$(10) \quad \log \log M(2r, f) \geq \pi \int_a^r \frac{dt}{t\theta_j(t)} + O(1)$$

from which we have

$$(11) \quad \liminf_{r \rightarrow \infty} (\log r)^{-1} \pi \int_a^r \frac{dt}{t\theta_j(t)} \leq \mu(f) < +\infty.$$

From (9) and (10) we have for each  $j (=1, \dots, N)$

$$\frac{N-1}{\{\log(r/a)\}^{-1} \{\log \log M(2r, f) + O(1)\}} + \frac{1}{\{\log(r/a)\}^{-1} \pi \int_a^r \frac{dt}{t\theta_j(t)}} \leq 2$$

and hence

$$(12) \quad \frac{N-1}{\rho(f)} + \frac{1}{\liminf_{r \rightarrow \infty} (\log r)^{-1} \pi \int_a^r \frac{dt}{t\theta_j(t)}} \leq 2.$$

From (11) and (12) we have for each  $j (=1, \dots, N)$

$$(13) \quad \frac{\rho(f)}{2\rho(f)+1-N} \leq \liminf_{r \rightarrow \infty} (\log r)^{-1} \pi \int_a^r \frac{dt}{t\theta_j(t)} \leq \mu(f) < +\infty.$$

Since  $\rho(f)/(2\rho(f)+1-N) \geq 1/2$  in (13), there exists a positive harmonic function  $u_j$  in  $D_j$  such that for all  $z \in D_j$

$$(14) \quad u_j(z) \geq |z|^{\rho(f)/(2\rho(f)+1-N) - \epsilon_0(z)} \quad (0 \leq \epsilon_0(z) \rightarrow 0 \text{ as } z \rightarrow \infty)$$

by Lemma 1. We can find  $z_j$  in  $D_j$  for which

$$|f(z_j)| > K$$

and choose a positive constant  $\delta$  so small that

$$\log |f(z_j)| > \delta u_j(z_j) + \log K.$$

We then define

$$U_j(z) = \begin{cases} \max\{\log(|f(z)|/K) - \delta u_j(z), 0\} & (z \in D_j) \\ 0 & (z \notin D_j). \end{cases}$$

Since  $U_j(z_j) > 0$  and  $U_j(z) = 0$  for  $z \notin D_j$ , it is clear that  $U_j(z)$  is a non-constant subharmonic function in  $|z| < \infty$ . Hence by Lemma 3 there exists a path  $\Gamma_j$  tending to  $\infty$  such that

$$U_j(z) \longrightarrow +\infty \text{ as } z \longrightarrow \infty \text{ on } \Gamma_j.$$

We may assume without loss of generality that

$$U_j(z) > 0 \text{ on } \Gamma_j,$$

so that  $\Gamma_j$  lies in  $D_j$  and on  $\Gamma_j$

$$U_j(z) = \log |f(z)| - \delta u_j(z) - \log K > 0.$$

Thus we have by (14)

$$\log |f(z)| > |z|^{\rho(f)/(2\rho(f)+1-N) - \varepsilon_j(z)} \quad (0 \leq \varepsilon_j(z) \rightarrow 0 \text{ as } z \rightarrow \infty) \text{ on } \Gamma_j.$$

*Remark 1.* By a well-known Ahlfors' theorem (see [6], p. 255), it is known that  $N=1$  when  $\mu(f) < 1$  and  $N \leq 2\mu(f)$  when  $1 \leq \mu(f) < +\infty$ .

From (9) and (10) we obtain for  $r \geq a$

$$N \leq 2\{\log \log M(2r, f) + O(1)\} / \log(r/a),$$

which reduces to  $N \leq 2\mu(f)$  when  $N \geq 2$ .

*Example 1.* Let

$$f(z) = \cos hz^{N/2} \quad (N=2, 3, \dots).$$

Then,

$$M(r, f) = \frac{\exp(r^{N/2}) + \exp(-r^{N/2})}{2} \text{ and } \rho(f) = \mu(f) = N/2.$$

It is easily seen that for  $k=0, 1, \dots, N-1$  and for  $0 \leq t < +\infty$

$$|f(te^{(2k+1)\pi i/N})| \leq 1$$

and

$$\log |f(te^{2k\pi i/N})| > t^{N/2 - \varepsilon(t)} \quad (0 < \varepsilon(t) \rightarrow 0 \text{ as } t \rightarrow +\infty).$$

*Remark 2.* This example shows that Theorem 1 is sharp.

*Example 2.* Let  $f(z)$  be an entire function of finite lower order with  $N(\geq 2)$  distinct finite asymptotic values. Then, for a sufficiently large  $K$  the set

$$\{z : |f(z)| > K\}$$

has at least  $N$  components.

We can find a concrete example of  $f(z)$  with  $N$  distinct finite asymptotic values in [8], p. 562.

**2-3. Subharmonic analogue.**

Let  $v(z)$  be a non-constant subharmonic function in  $|z| < \infty$ . Put

$$B(r, v) = \sup_{|z|=r} v(z),$$

$$\rho = \limsup_{r \rightarrow \infty} \log B(r, v) / \log r \quad (\text{the order of } v),$$

$$\mu = \liminf_{r \rightarrow \infty} \log B(r, v) / \log r \quad (\text{the lower order of } v).$$

It is said that  $v(z)$  has at least  $N$  tracts in  $|z| < \infty$  if and only if

$$\{z : v(z) > K\}$$

has at least  $N$  components for all sufficiently large  $K$ , where  $N$  is a positive integer ([7], [8]). When  $N \geq 2$ , the following result is given ([8], p. 593).

**THEOREM C.** *Suppose that  $v(z)$  has at least  $N(\geq 2)$  tracts in the finite plane. Then there exist sectionally polygonal paths  $\gamma_1, \dots, \gamma_N$  from 0 to  $\infty$  such that*

- 1)  $\gamma_j \cap \gamma_k = \{0, \infty\}$  ( $j \neq k$ ),
- 2)  $\gamma_j$  and  $\gamma_{j+1}$  bound a domain  $D_j$  and  $D_j \cap \gamma_k = \emptyset$  ( $\gamma_{N+1} = \gamma_1$ ),
- 3)  $v(z)$  is bounded on the  $\gamma_j$  and not bounded above in the  $D_j$ .

Put

$$B_j(r, v) = \sup\{v(z) : (|z|=r) \cap D_j\}.$$

By 3) in Theorem C, there exists  $z_j \in D_j$  for each  $j$  such that

$$v(z_j) > 0$$

and for all sufficiently large  $r$

$$B_j(r, v) > 0.$$

Further there exists a positive number  $M$  such that

$$V(z) = v(z) - M$$

is negative on  $\gamma_1 \cup \dots \cup \gamma_N$ .

**LEMMA 4.**  $\log B_j(r, v) \geq \pi \int_1^{r^{1/2}} \frac{dt}{t\theta_j(t)} + O(1)$

where we use  $\theta_j(r)$  for  $D_j$  instead of  $\theta(r)$  defined for  $D$  in (4).

We can prove this lemma by applying Theorem 8.3 ([8], p. 548) to  $V_j(z) = \max\{V_j(z), 0\}$  if  $z \in D_j$ ,  $=0$  otherwise.

**THEOREM 2.** *Suppose that  $v(z)$  has at least  $N (\geq 2)$  tracts in the finite plane and  $\mu < +\infty$ . Then there exists a path  $\Gamma_j$  tending to  $\infty$  in  $D_j$  such that*

$$v(z) > |z|^{\rho/(2\rho+1-N)-\varepsilon_j(z)} \quad (0 \leq \varepsilon_j(z) \rightarrow 0 \text{ as } z \rightarrow \infty)$$

on  $\Gamma_j (j=1, \dots, N)$ .

We can prove this theorem as in the case of Theorem 1 using Lemma 4 instead of Lemma 2. We note that  $N \leq 2\mu$  as in Remark 1.

### 3. Application to the oscillation theory of $w'' + Aw = 0$ .

We shall first give some lemmas for later use. We use the same notation as in the section 1.

**LEMMA 5.** *If  $\rho(E) < +\infty$ , for a given  $\varepsilon > 0$  there exists a positive number  $d = d(\varepsilon)$  such that*

$$|(E'/E)^2(re^{i\theta}) - 2(E''/E)(re^{i\theta})| \leq r^d$$

for all  $r \geq r_0 > 1$  and all  $\theta \in J(r)$ , where the angular measure of  $J(r)$ ,  $m(J(r)) \leq \varepsilon\pi$  ([12], Lemma 1).

**LEMMA 6.** *If  $\lambda(E) < \rho(E)$ , then*

$$\mu(E) = \rho(E) = \mu(A) = \rho(A)$$

and these numbers are equal to an integer or  $+\infty$ .

*Proof.* From (2) we easily have

$$(15) \quad 2T(r, E) = 2N(r, 1/E) + T(r, A) + S(r, E).$$

Set

$$E(z) = \Pi(z)e^{P(z)},$$

where  $\Pi(z)$  is the Weierstrass product of the zeros of  $E$  and  $P(z)$  is an entire function. Then, it is known that  $\rho(\Pi) = \lambda(E)$  (see [5]).

a) The case  $\rho(E) = +\infty$ . In this case,  $P(z)$  is transcendental and it is easy to see that  $\mu(E) = +\infty$ . Let  $\alpha$  be any number such that  $\lambda(E) < \alpha < +\infty$ . Then from (15) we have

$$(16) \quad 2T_\alpha(r, E) = 2N_\alpha(r, 1/E) + T_\alpha(r, A) + S_\alpha(r, E),$$

where

$$T_\alpha(r, E) = \int_1^r \frac{T(t, E)}{t^{1+\alpha}} dt \text{ is of lower order } +\infty,$$

$$N_\alpha(r, 1/E) = \int_1^r \frac{N(t, 1/E)}{t^{1+\alpha}} dt \text{ is bounded,}$$

$$T_\alpha(r, A) = \int_1^r \frac{T(t, A)}{t^{1+\alpha}} dt \leq T(r, A)/\alpha$$

and

$$S_\alpha(r, E) = \int_1^r \frac{S(t, E)}{t^{1+\alpha}} dt = o(T_\alpha(r, E)) \quad (r \rightarrow \infty)$$

(see [16], Proposition 1 and Lemma 1), so that

$$+\infty = \liminf_{r \rightarrow \infty} \frac{\log T_\alpha(r, E)}{\log r} = \liminf_{r \rightarrow \infty} \frac{\log T_\alpha(r, A)}{\log r} \leq \mu(A).$$

We have  $\mu(A) = \rho(A) = +\infty$ .

b) The case  $\rho(E) < +\infty$ . In this case,  $P(z)$  must be a polynomial and it is easy to see that

$$(17) \quad \mu(E) = \rho(E) = \text{the degree of } P(z)$$

since  $\lambda(E) < \rho(E)$ . From (15) and (17) we have

$$\mu(A) = \rho(A) = \mu(E) = \rho(E) = \text{an integer.}$$

**THEOREM 3.** *Suppose that  $\mu(A) < +\infty$  and for a positive constant  $K$  not smaller than  $1/2$  the set*

$$\{z : |A(z)| > K\}$$

*has at least  $N$  components. Then, either  $\rho(E) = +\infty$  or*

$$\frac{N}{\mu(A)} + \frac{1}{\rho(E)} \leq 2.$$

*Proof.* Suppose that  $\rho(E) < +\infty$ . Let  $D_0$  be a component of the set

$$\{z : |E(z)| > |c|\},$$

which is a non-empty unbounded set since  $E$  is transcendental by (2).

The set  $\{z : |A(z)| > K\}$  has at least  $N$  components, and since  $A(z)$  is transcendental and Theorem 1 holds for  $N \geq 2$ , for any positive integer  $p$  and for  $K_1 = \max\{K, M(1, A)\}$  the set

$$\{z : \log |A(z)| - p \log |z| - \log K_1 > 0\}$$

has at least  $N$  unbounded components. Let  $D_1, \dots, D_N$  be those  $N$  unbounded components. For  $j=0, 1, \dots, N$ , put



$$E_j(r) = \{\theta \in [0, 2\pi) : re^{i\theta} \in D_j\}$$

and

$$\theta_j(r) = \begin{cases} +\infty & \text{if } \{|z|=r\} \subset D_j, \\ \text{the measure of } E_j(r) & \text{otherwise.} \end{cases}$$

Then there is a positive number  $a$  such that  $\theta_j(r) > 0$  for all  $r \geq a$  and for all  $j$ . By Lemma 2 we have

$$(18) \quad \log \log M(r, E) \geq \pi \int_a^{r/2} \frac{dt}{t\theta_0(t)} + O(1)$$

and

$$(19) \quad \log \{\log M(r, A) - p \log r - \log K_1\} \geq \pi \int_a^{r/2} \frac{dt}{t\theta_j(t)} + O(1)$$

for  $j=1, \dots, N$ .

For any fixed positive number  $\varepsilon < 1$ , let  $p$  be a positive integer such that  $p > d$ , where  $d$  is the constant given in Lemma 5. We define for  $j=0, 1, \dots, N$

$$l_j(t) = \begin{cases} 2\pi & \text{if } \theta_j(t) = +\infty \\ \theta_j(t) & \text{otherwise.} \end{cases}$$

Then applying Lemma 5 to (2) we obtain the inequality

$$(20) \quad \sum_{j=0}^N l_j(t) \leq (2 + \varepsilon)\pi$$

for all  $r \geq b = \max(a, r_0)$  from which we have

$$(21) \quad \sum_{j=0}^N \int_b^r \frac{l_j(t)}{t} dt \leq (2 + \varepsilon)\pi \log(r/b).$$

By the Cauchy-Schwarz inequality

$$(22) \quad \int_b^r \frac{l_j(t)}{t} dt \int_b^r \frac{dt}{tl_j(t)} \geq \left( \int_b^r \frac{dt}{t} \right)^2 = \left( \log \frac{r}{b} \right)^2.$$

From (21) and (22) we obtain the inequality

$$(23) \quad \sum_{j=0}^N \frac{\log(r/b)}{\pi \int_b^r \frac{dt}{tl_j(t)}} \leq 2 + \varepsilon.$$

Define

$$B_0 = \{r : \theta_0(r) = +\infty\}.$$

Then,  $B_0$  is a sum of intervals. Let

$$\chi_0(r) = \begin{cases} 1 & \text{if } r \text{ belongs to } B_0 \\ 0 & \text{otherwise.} \end{cases}$$

If  $r$  belongs to  $B_0$  and  $r \geq b$ , we have

$$\theta_j(r) = l_j(r) \quad \text{for } j=1, \dots, N$$

and

$$\theta_1(r) + \dots + \theta_N(r) \leq \varepsilon\pi$$

from (20). Thus, if we set

$$F_j = \{r; \theta_j(r) \leq \varepsilon\pi\},$$

then

$$(24) \quad B_0 \subset \bigcup_{j=1}^N F_j.$$

Define

$$\phi_j(r) = \begin{cases} 1 & \text{if } r \text{ belongs to } F_j, \\ 0 & \text{otherwise} \end{cases}$$

and put

$$M(r) = \log M(r, A) - p \log r - \log K_1$$

We then have from (24)

$$(25) \quad \int_b^r \frac{\chi_0(t)}{t} dt \leq \sum_{j=1}^N \int_b^r \frac{\phi_j(t)}{t} dt \leq N\varepsilon \log M(2r) + O(1)$$

since  $\varepsilon^{-1}\phi_j(t) \leq \pi/\theta_j(t)$  and so

$$\varepsilon^{-1} \int_b^r \frac{\phi_j(t)}{t} dt \leq \pi \int_b^r \frac{dt}{t\theta_j(t)} \leq \log M(2r) + O(1)$$

by (19).

(i) The case  $N \geq 2$ . In this case it is clear that for  $j=1, \dots, N$

$$0 < \theta_j(r) < 2\pi \quad \text{and} \quad \theta_j(r) = l_j(r) \quad (r \geq b).$$

Since

$$(26) \quad \pi \int_b^r \frac{dt}{t\theta_0(t)} = \pi \int_b^r \frac{dt}{tl_0(t)} - \frac{1}{2} \int_b^r \frac{\chi_0(t)}{t} dt,$$

from (18), (19), (23) and (25) we obtain for  $r \geq b$

$$(27) \quad \frac{N \log(r/b)}{\log M(2r) + O(1)} + \frac{\log(r/b)}{\log \log M(2r, E) + (N\varepsilon/2) \log M(2r) + O(1)} \leq 2 + \varepsilon.$$

Let  $\{r_n\}$  be a sequence tending to  $+\infty$  such that

$$\lim_{n \rightarrow \infty} \frac{\log \log M(2r_n, A)}{\log 2r_n} = \mu(A).$$

Put  $r=r_n$  in (27) and let  $n$  tend to  $+\infty$ . We then obtain

$$\frac{N}{\mu(A)} + \frac{1}{\rho(E) + N\varepsilon\mu(A)/2} \leq 2 + \varepsilon.$$

Tending  $\varepsilon \rightarrow 0$ , we have

$$(28) \quad \frac{N}{\mu(A)} + \frac{1}{\rho(E)} \leq 2.$$

(ii) The case  $N=1$ . Let

$$B_1 = \{r : \theta_1(r) = +\infty\}.$$

Then,  $B_1$  is a sum of intervals. Define

$$\chi_1(r) = \begin{cases} 1 & \text{if } r \text{ belongs to } B_1 \\ 0 & \text{otherwise.} \end{cases}$$

If  $r$  belongs to  $B_1$  and  $r \geq b$ , we have

$$\theta_0(r) \leq \varepsilon\pi$$

by (20). Put

$$F_0 = \{r : \theta_0(r) \leq \varepsilon\pi\}$$

and

$$\psi_0(r) = \begin{cases} 1 & \text{if } r \text{ belongs to } F_0 \\ 0 & \text{otherwise.} \end{cases}$$

We then have

$$(29) \quad \int_b^r \frac{\chi_1(t)}{t} dt \leq \int_b^r \frac{\psi_0(t)}{t} dt \leq \varepsilon \log \log M(2r, E) + O(1)$$

since  $B_1 \subset F_0$ ,  $\varepsilon^{-1}\psi_0(t) \leq \pi/\theta_0(t)$  and so

$$\varepsilon^{-1} \int_b^r \frac{\psi_0(t)}{t} dt \leq \pi \int_b^r \frac{dt}{t\theta_0(t)} \leq \log \log M(2r, E) + O(1)$$

by (18). Since

$$\pi \int_b^r \frac{dt}{t\theta_1(t)} = \pi \int_b^r \frac{dt}{t\theta_0(t)} - \frac{1}{2} \int_b^r \frac{\chi_1(t)}{t} dt,$$

from (18), (19), (23) and (25) for  $N=1$ , (26) and (29), we have

$$(30) \quad \frac{\log(r/b)}{\log M(2r) + \varepsilon \log \log M(2r, E) + O(1)} + \frac{\log(r/b)}{\log \log M(2r, E) + (\varepsilon/2) \log M(2r) + O(1)} \leq 2 + \varepsilon.$$

Then as in the case of  $N \geq 2$  where we obtained (28) from (27), we obtain the inequality

$$\frac{1}{\mu(A)} + \frac{1}{\rho(E)} \leq 2$$

from (30).

COROLLARY. Under the same assumption as in Theorem 3,

1) If  $\mu(A) < \rho(A) = +\infty$ , then  $\lambda(E) = +\infty$ .

2) When  $\rho(A) < +\infty$ , if  $\mu(A) < \rho(A)$  or if  $A$  is of regular growth and  $\rho(A)$  is not equal to an integer, then either  $\lambda(E) = +\infty$  or

$$(31) \quad \frac{N}{\mu(A)} + \frac{1}{\lambda(E)} \leq 2.$$

3) If  $\mu(A) \leq 1/2$  or if  $\mu(A) = N/2$  in case of  $N \geq 2$ , then

$$\lambda(E) = +\infty.$$

Proof. 1) We easily have

$$\rho(A) \leq \rho(E)$$

from (15) and since  $\mu(A) < \rho(A) = +\infty$  we have

$$\lambda(E) = \rho(E) = +\infty$$

by Lemma 6.

2) In this case, we have

$$\lambda(E) = \rho(E)$$

by Lemma 6. We obtain (31) from Theorem 3.

3) Noting the fact that

$$"N=1 \text{ if } \mu(A) < 1 \text{ and } N \leq 2\mu(A) \text{ if } 1 \leq \mu(A) < +\infty"$$

(see Remark 1), we easily obtain  $\lambda(E) = +\infty$  when  $\mu(A) \leq 1/2$  or  $\mu(A) = N/2$  in case  $N$  is odd from 2) of this corollary.

When  $N$  is even and positive,  $\mu(A) = N/2$  implies  $\rho(E) = +\infty$  by Theorem 3. If  $\lambda(E) < +\infty$ , then  $\mu(A) = \rho(E) = +\infty$  by Lemma 6. This is a contradiction.  $\lambda(E)$  must be equal to  $+\infty$ .

Remark 3. The functions of Examples 1 and 2 in the section 2 satisfy the conditions of Theorem 3 for  $N \geq 2$ .

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