A theorem on the structure of linear operations

by

A. ALEXIEWICZ (Poznań)

In a previous paper [1] I proved a general theorem concerning linear operations depending on a parameter[1]). This theorem contains as particular cases some theorems of Saks [7], [8] concerning the structure of the sequences of operations with values in the space of measurable functions, *viz.* those which deal with the behaviour of the sequences at individual points. Some new theorems of Saks's type were also obtained in [1] as applications. However, the theorems of Saks, dealing with the behaviour of the sequences in the mean, were not obtainable from the results of [1].

It is the purpose of this paper to generalize the results of [1] so as to fill the above-mentioned gap.

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1. Preliminary definitions

T will denote an abstract set in which a σ-algebra $\mathfrak{E}$ of subsets (Halmos [3], p. 28) is defined. We suppose that μ is a σ-measure in $\mathfrak{E}$, such that $\mu(T)<\infty$. Under these circumstances the measure space $(T,\mathfrak{E},\mu)$ is defined, namely on introducing the distance of two sets $e_1, e_2 \epsilon \mathfrak{E}$ by the formula

$$\varrho(e_1,e_2)=\mu\big((e_1+e_2)-e_1e_2\big)$$

we get a pseudometric space. Identifying two sets $e_1,e_2\epsilon\mathfrak{E}$ if $\varrho(e_1,e_2)=0$ we get a metric space which is also denoted by $(T,\mathfrak{E},\mu)$; this space is complete. We shall suppose in the sequel that the space $(T,\mathfrak{E},\mu)$ is separable (it is usual to call the measure *separable* in this case). By e, h,

[1]) This paper has many points in common with the paper [6] of Orlicz which also aims at deducing some theorems of Saks' type from a general theorem. The methods of Orlicz are different from ours and are suitable for systems of operations depending only on a discrete denumerable parameter. Professor Orlicz has been the first to introduce the operation $U(x|e)$ (see below).

$e_n, k_n, e', e'', e''', h'$ we shall denote sets of $\mathfrak{E}$. L will denote an arbitrary linear space.

By Y we shall denote an F-space (Banach [2], p. 35) which has the following properties: the elements of Y are functions from T to L, and the addition and multiplication by scalars are defined in the usual way. Denoting by y_e, for every $y \in Y$ and $e \in \mathfrak{E}$, the function defined by the equations

$$y_e(t) = \begin{cases} y(t) & \text{for } t \in e, \\ 0 & \text{elsewhere,} \end{cases}$$

we suppose the following postulates to be satisfied:

(a_1) $y \in Y$, $e \in \mathfrak{E}$ implies $y_e \in Y$;

(a_2) $y \in Y$, $e \in \mathfrak{E}$ implies $\|y_e\| \leqslant \|y\|$;

(a_3) $\mu(e_n) \to 0$ implies $\|y_{e_n}\| \to 0$.

These postulates give to the elements of Y a certain character of measurability. We deduce easily

(a_4) $\mu(e) = 0$ implies $\|y_e\| = 0$,

(a_5) $y_e \in Y$, $y_h \in Y$ implies $y_{e+h} \in Y$.

Indeed, (a_4) being trivial, we prove only (a_5). By (a_1) $y_{h-e} = (y_h)_{h-e} \in Y$, hence $y_{e+h} = y_e + y_{h-e}$ belongs to Y as the sum of two elements of the space.

X will denote a separable F-space.

B will stand for an analytic set[2]) in Y satisfying the following condition:

(b_1) $y \in B_1$, $e \in \mathfrak{E}$ implies $y_e \in B$.

In sections 3, 4 we shall postulate that the set B is linear; in this case it will satisfy the following condition:

(b_2) $y_e \in B$, $y_h \in B$ implies $y_{e+h} \in B$

(the proof is analogous to this of (a_5)).

In Theorem 2 the following condition will be needed:

(b_3) $y_{e_n} \in B$ (for $n = 1, 2, \ldots$), and $e = \sum_{n=1}^{\infty} e_n$ implies $y_e \in B$.

Let $U(x)$ be an arbitrary linear operation from X to Y. We set

$$U(x|e) = U(x)_e.$$

[2]) By an analytic set we mean any set which is the result of the operation A (Kuratowski [4], p. 4) performed upon open sets.

The main result of this paper (Theorem 1) states that in the case of the operation U being linear the set T may be decomposed, $T = e + h$, in such a manner that for every x the element $U(x|e)$ is "nearly contained" in B and that no set $h' \subset h$ of positive measure has this property, unless x belongs to a set of the first category.

2. Properties of the operation $U(x|e)$

LEMMA 1. *The operation $U(x|e)$ is continuous in the space $X \times (T, \mathfrak{E}, \mu)$.*

Proof. The operation $U(x|e)$ is continuous for fixed x. This follows from (a_3). Let $x_n \to x_0$, $e_n \to e_0$ and write $V_n(x) = U(x|e_n)$. Then $V_n(x)$ is a sequence of linear operations from X to Y, convergent everywhere, whence by a theorem of Mazur and Orlicz ([5], p. 153-154)

$$\|V_n(x_n) - V_n(x_0)\| \to 0,$$

which implies

$$U(x_n|e_n) \to U(x_0|e_0).$$

The following condition (B, h, ε) will be needed:

There is a set e such that $\mu(h-e) < \varepsilon$ and $U(x|e) \in B$.

The set of the elements x for which this condition is satisfied will be written $P(B, h, \varepsilon)$.

LEMMA 2. *For every h and $\varepsilon > 0$ the set $P(B, h, \varepsilon)$ is analytic.*

Proof. The set $\mathfrak{H}$ of the elements $e \in (T, \mathfrak{E}, \mu)$ for which $\mu(h-e) < \varepsilon$, is obviously open. The set Q of the couples (x, e) such that $U(x|e) \in B$ is analytic, for it is the inverse image under U of the analytic set B. Using the symbolical notation (Kuratowski [4], p. 1-13) we can write

$$P(B, h, \varepsilon) = \underset{x}{E}\{\sum_{e} [(x, e) \in Q(X \times \mathfrak{H})]\},$$

that is, $P(B, h, \varepsilon)$ is the projection on X of the analytic set $Q(X \times \mathfrak{H})$; hence it is analytic too.

LEMMA 3. *Let the set B be linear. If the set $P(B, h, \varepsilon)$ is of the second category, then the set $P(B, h, 2\varepsilon)$ is residual.*

Proof. Since $P(B, h, \varepsilon)$ is an analytic set, it fulfils the condition of Baire; hence it contains a sphere except a set of the first category. Hence the set W of the differences of the elements of $P(B, h, \varepsilon)$ contains a set $S = K - N$ where K is a sphere with centre 0 and N is of the first category. We notice now that $x_1, x_2 \in P(B, h, \varepsilon)$ implies $x_1 - x_2 \in P(B, h, 2\varepsilon)$, for there are sets e_1, e_2 such that $\mu(h-e_1) < \varepsilon$, $\mu(h-e_2) < \varepsilon$, $U(x_1|e_1) \in B$, $U(x_2|e_2) \in B$, hence for the set $e = e_1 e_2$ we have $\mu(h-e) < 2\varepsilon$ and $U(x_1|e) = U(x_1|e_1)_{e_2} \in B$,

$U(x_2|e)=U(x_2|e_2)_{e_1}\epsilon B$. It follows now that $K-N\subset P(B,h,2\varepsilon)$. Finally, it is obvious that $x\epsilon P(B,h,2\varepsilon)$ implies $\lambda x\epsilon P(B,h,2\varepsilon)$, whence

$$\sum_{n=1}^{\infty} n(K-N)\subset P(B,h,2\varepsilon), \qquad X-\sum_{n=1}^{\infty} nN\subset P(B,h,2\varepsilon),$$

nA denoting in these formulae the set of the elements na with $a\epsilon A$. The set nN is obviously of the first category.

3. Decomposition theorems

In this paragraph the set B will be supposed to be linear.

THEOREM 1. *There exists a decomposition $T=e+h$ and a residual set R in X such that*

(i) *for every x and every $\varepsilon>0$ there exists a set e' such that $\mu(e-e')<\varepsilon$ and $U(x|e')\epsilon B$,*

(ii) *for every $x\epsilon R$ and every set $h'\subset h$ of positive measure $U(x|h')$ non ϵB.*

Proof. Let $\mathfrak{F}$ be the class of the sets h for which the condition (B,h,ε) is satisfied for every x and every $\varepsilon>0$, and let σ denote the supremum of the measures of the sets in $\mathfrak{F}$. There exist sets $e_n\epsilon\mathfrak{F}$ such that $\sigma-1/n\leqslant\mu(e_n)$. Let us write

$$e=\sum_{n=1}^{\infty} e_n, \qquad h=T-e.$$

The condition (i) is then evidently satisfied.

Now consider the following condition:

(n) there exists a set $h'\subset h$ such that $\mu(h')>0$ and $U(x|h')\epsilon B$.

To prove (ii) it suffices to show that the set Z of the elements x satisfying the condition (n) is of the first category. Suppose the contrary, and denote by Q_n the set of the elements x for which there exists a set $h'\subset h$ such that

$$\mu(h')>1/n \qquad \text{and} \qquad U(x|h')\epsilon B.$$

Clearly

$$Z=\sum_{n=1}^{\infty} Q_n,$$

whence one of the sets Q_n, say Q_r, must also be of the second category. In the class $\mathfrak{G}$ of the sets $k\subset h$ of measure not less than $a=1/r$ there exists a sequence k_n composing a dense set. Let us write $X_{mn}=P(B,k_n,2^{-m})$, then

$$Q_r\subset\sum_{n=1}^{\infty} X_{mn},$$

hence for every m there is an n_m such that the set X_{mn_m} is of the second category. By Lemma 3 the set $P(B,k_{n_m},2^{-m+1})$ is residual. Now write

$$W=\prod_{m=1}^{\infty} P(B,k_{n_m},2^{-m+1}), \qquad e'=\overline{\lim_{m\to\infty}}\, k_{n_m}.$$

Then the set W is residual, and

$$\mu(e')\geqslant\overline{\lim_{m\to\infty}}\,\mu(k_{n_m})\geqslant a.$$

Let $x\epsilon W$, then for every m there exists a set $e_m(x)$ such that

$$\mu\big(k_{n_m}-e_m(x)\big)<2^{-m+1} \qquad \text{and} \qquad U\big(x|e_m(x)\big)\epsilon B.$$

We may suppose freely that $e_1(x)\subset e_2(x)\subset\ldots$ Then

$$\mu\big(e'-e_m(x)\big)=\lim_{m\to\infty}\mu\big(e'-e_m(x)\big)=0,$$

for we have

$$\mu\big(e'-e_m(x)\big)\leqslant\mu\Big(\sum_{m=p}^{\infty}k_{n_m}-e_m(x)\Big)\leqslant\sum_{m=p}^{\infty}\mu\big(k_{n_m}-e_m(x)\big)\leqslant 2^{-p+2}.$$

Thus $W\subset P(B,e',\varepsilon)$ for every $\varepsilon>0$, whence

$$W\subset\prod_{n=1}^{\infty}P(B,e',1/n)=V.$$

The set V is evidently linear, it satisfies the condition of Baire (being analytic) and is residual since it includes the set W. This implies $X=V$.

Now for every $x\epsilon V$, $\varepsilon>0$ the condition (B,e',ε) is satisfied, hence $e'\epsilon\mathfrak{F}$. This, however, leads to a contradiction, since $(e'+e)\epsilon\mathfrak{F}$ and $\mu(e+e')=\mu(e)+\mu(e')=\sigma+a>\sigma$, contrarily to the definition of the number σ.

Now the question arises whether or not the set e' in the assertion (i) of Theorem 1 might be chosen independently of x, *i.e.* whether (i) might be replaced by the following assertion:

(i') for every $\varepsilon>0$ there exists a set e' such that $\mu(e-e')<\varepsilon$ and $U(x|e')\epsilon B$ for every x.

We shall show by a counterexample that the answer is negative. Let $X=Y$ be the well-known space $\boldsymbol{L}$ of the Lebesgue measurable functions in $[a,b]$, $U(x)=x$. By B we shall denote the subset of Y composed of essentially bounded functions. This set is linear and of F_σ type. By well-known theorems (i) is true with $e=[a,b]$; however, a set of positive measure on which all functions of $\boldsymbol{L}$ are simultaneously essentially bounded, does not exist.

THEOREM 2. *Let the set B satisfy the condition* (b_3). *Then there exists a decomposition $T=e+h$ and a residual set $R\subset X$ such that*

(i′) $U(x|e)\epsilon B$ *for every* x,

(ii′) $U(x|h')$ non ϵB *for every* $x\epsilon R$ *and every set* $h'\subset h$ *of positive measure.*

The proof is obvious.

4. Applications

Now we shall present some applications of the above theorems.

Let us denote by $\mathfrak{S}$ the space of the sequences $y=\{\eta_n(t)\}$ of real valued μ-measurable functions defined on T. The elements of this space may be considered as functions defined in T, with values in the space $\boldsymbol{s}$ of the sequences of real numbers (Banach [2], p. 10). We define the addition and multiplication by scalars in $\mathfrak{S}$ as usual, and the norm as

$$\|y\|=\sum_{n=1}^{\infty}\frac{1}{2^n}\int_T\frac{|\eta_n(t)|}{1+|\eta_n(t)|}\,dt;$$

then $\mathfrak{S}$ becomes an F-space. Upon setting $L=\boldsymbol{s}$, $Y=\mathfrak{S}$ we see that the conditions (a_1)-(a_3) are satisfied. A sequence $y_k=\{\eta_{nk}(t)\}_{n=1,2,\ldots}$ of elements of $\mathfrak{S}$ converges to $y=\{\eta_n(t)\}$ if and only if

$$\lim_{k\to\infty}\text{as}\ \eta_{nk}(t)=\eta_n(t)\qquad\text{for}\quad n=1,2,\ldots$$

Denote by $B_1,\ldots,B_5$ the sets of the elements $y=\{\eta_n(t)\}$ of $\mathfrak{S}$ for which the following conditions are satisfied respectively:

(1) the sequence $\{\eta_n(t)\}$ is *asymptotically bounded* (*i.e.* $\lambda_n\to0$ implies $\lim\limits_{n\to\infty}\text{as}\ \lambda_n\eta_n(t)=0$); this is equivalent to the following condition: for every $\varepsilon>0$ there exists a K such that for any n

$$\mu\big(\underset{t}{E}\{|\eta_n(t)|>K\}\big)<\varepsilon,$$

(2) the sequence $\{\eta_n(t)\}$ converges asymptotically,

(3) the sequence $\{\eta_n(t)\}$ is bounded a. e. (almost everywhere),

(4) the sequence $\{\eta_n(t)\}$ converges a. e.,

(5) $\sum\limits_{n=1}^{\infty}|\eta_n(t)|^\alpha<\infty$ a. e. $(\alpha>0)$,

(6) $\sup\limits_{n}\int_T|\eta_n(t)|^\alpha dt<\infty$ $(\alpha>0)$,

(7) the sequence $\{\eta_n(t)\}$ converges in L^α $(\alpha>0)$,

(8) $\sum\limits_{n=1}^{\infty}\int_T|\eta_n(t)|^\alpha dt<\infty$ $(\alpha>0)$.

These sets are obviously linear; we shall prove that they are measurable (B).

Ad B_1. The set

$$A_{nmp}=\underset{y}{E}\{y=\{\eta_i(t)\},\mu\big(\underset{t}{E}\{|\eta_n(t)|>m\}\big)\leqslant1/p\}$$

is evidently closed and

$$B_1=\prod_{p=1}^{\infty}\sum_{m=1}^{\infty}\prod_{n=1}^{\infty}A_{nmp}.$$

Ad B_2. The sets

$$B_{nmpq}=\underset{y}{E}\{y=\{\eta_i(t)\},\mu\big(\underset{t}{E}\{|\eta_m(t)-\eta_n(t)|>1/p\}\big)\leqslant1/q\}$$

are closed and

$$B_2=\prod_{p=1}^{\infty}\prod_{q=1}^{\infty}\sum_{r=1}^{\infty}\prod_{m=r}^{\infty}\prod_{n=r}^{\infty}B_{nmpq}.$$

Ad B_3. Given any element $y=\eta_n(t)$ let us write

$$\omega_n(y)=\omega_n(y,t)=\max_{i=1,\ldots,n}|\eta_i(t)|.$$

Then $\|y_n-y\|\to0$ implies

$$\lim_{s\to\infty}\text{as}\ \omega_s(y_s,t)=\omega_n(y_s,t)\qquad\text{for every}\quad n.$$

The sequence $\{\eta_n(t)\}$ is bounded if and only if the sequence $\{\omega_n(y,t)\}$ is asymptotically bounded, hence

$$B_3=\prod_{p=1}^{\infty}\sum_{m=1}^{\infty}\prod_{n=1}^{\infty}A^*_{nmp},$$

where

$$A^*_{nmp}=\underset{y}{E}\{\mu\big(\underset{t}{E}\{\omega_n(y,t)>m\}\big)\leqslant1/p\}.$$

We shall prove that the sets A^*_{nmp} are closed. Let $y_k\epsilon A^*_{nmp}$, $y_k\to y$, and write

$$e_k=\underset{t}{E}\{\omega_n(y_k,t)>m\},$$

then $\mu(e_k)\leqslant1/p$. Since

$$\lim_{k\to\infty}\text{as}\ \omega_n(y_k,t)=\omega_n(y,t),$$

there exists a sequence $\{k_i\}$ such that $\omega_n(y_{k_i},t)\to\omega_n(y,t)$ a. e.

Let us write

$$e_0=\underline{\lim_i}\,e_{k_i},$$

then $\mu(e_0)\leqslant 1/p$ and $t\in T-e_0$ implies $t\in T-e_{k_i'}$ (the sequence $\{k_i'\}$ being extracted from $\{k_i\}$) whence $\omega_n(y_{k_i},t)\leqslant m$. It follows that $\omega_n(y,t)\leqslant m$ a. e. in $T-e_0$, thus

$$\underset{t}{E}\{\omega_n(y,t)>m\}\subset e_0$$

and $y\in A^*_{nmp}$.

Ad B_4. Write

$$\omega_{pq}(y,t)=\max_{p\leqslant i<j\leqslant q}|\eta_i(t)-\eta_j(t)|$$

and

$$D_{pqmn}=\underset{y}{E}\{\mu(\underset{t}{E}\{\omega_{pq}(y,t)>1/m\})\leqslant 1/n\}.$$

We can prove, as above, that the sets D_{pqmn} are closed; then we apply the formula

$$B_4=\prod_{n=1}^{\infty}\prod_{m=1}^{\infty}\sum_{k=1}^{\infty}\prod_{p=k}^{\infty}\prod_{q=k}^{\infty}D_{pqmn}.$$

Ad B_6. We write

$$E_{mn}=\underset{y}{E}\Big\{y=\{\eta_i(t)\},\ \int_T|\eta_i(t)|^\alpha dt\leqslant m\Big\}.$$

This set is closed. For, if $y_k=\{\eta_{kp}\}_{p=1,2,\ldots}\in E_{nm}$, $y_k\to y$, then there exists a sequence $\{k_i\}$ such that $\eta_{k_ip}(t)\to\eta_p(t)$ a. e., whence by Fatou's lemma

$$\int_T|\eta_p(t)|^\alpha dt\leqslant\lim_{i\to\infty}\int_T|\eta_{k_ip}(t)|^\alpha dt,$$

i. e. $y\in E_{nm}$. (B)-measurability of the set B_6 follows by formula

$$B_6=\sum_{m=1}^{\infty}\prod_{n=1}^{\infty}E_{nm}.$$

The proofs in the remaining cases are similar.

All the sets B_1-B_8 have the properties (b_1) and (b_2). The sets B_1, B_2,B_3,B_4,B_5 have also the property (b_3).

Let us denote by $\mathfrak{S}$ the space of measurable functions $\eta=\eta(t)$ defined in T, with the norm

$$\|\eta\|=\int_T\frac{|\eta(t)|}{1+|\eta(t)|}dt.$$

It is an F-space (Banach [2], p. 9). An operation $V(x)=V(x,t)$ from X to S is *linear* if it is additive and $x_n\to 0$ implies

$$\lim_{n\to\infty}\text{as } V(x_n,t)=0.$$

We shall consider any sequence $\{V_n(x,t)\}$ of linear operations from X to S as an operation $U(x,t)=\{V_n(x,t)\}$ from X to $\mathfrak{S}$. Taking as Y the space $\mathfrak{S}$ we derive from Theorems 1 and 2 the

THEOREM 3. *Given any sequence $\{V_n(x,t)\}$ of linear operations from X to S there exist decompositions $T=e_1+h_1=\ldots=e_8+h_8$ and a residual set R such that*[3]*):*

(i_1) *$V_n(x,t)$ is asymptotically bounded on e_1 for every x,*

(ii_1) *$V_n(x,t)$ is not asymptotically bounded on every set $h\subset h_1$ of positive measure and every $x\in R$,*

(i_2) *$V_n(x,t)$ converges asymptotically on e_2 for every x,*

(ii_2) *$V_n(x,t)$ does not converge asymptotically on every set $h\subset h_2$ of positive measure and every $x\in R$,*

(i_3) *$V_n(x,t)$ is bounded a. e. in e_3 for every x,*

(ii_3) *$V_n(x,t)$ is unbounded a. e. in h_3 for every $x\in R$,*

(i_4) *$V_n(x,t)$ converges a. e. in e_4 for every x,*

(ii_4) *$V_n(x,t)$ diverges a. e. in h_4 for every $x\in R$,*

(i_5) *$\sum_{n=1}^{\infty}|V_n(x,t)|^\alpha<\infty$ a. e. in e_5 for every x,*

(ii_5) *$\sum_{n=1}^{\infty}|V_n(x,t)|^\alpha=\infty$ a. e. in h_5 for every $x\in R$.*

Moreover, for every x and $\varepsilon>0$ there exist sets e',e'',e''' such that $\mu(e_6-e')<\varepsilon$, $\mu(e_7-e'')<\varepsilon$, $\mu(e_8-e''')<\varepsilon$ and

(i_6) $\sup_n\int_{e'}|V_n(x,t)|^\alpha dt<\infty$,

(ii_6) $\sup_n\int_h|V_n(x,t)|^\alpha dt=\infty$ *for every set $h\subset h_6$ of positive measure and every $x\in R$,*

(i_7) $\lim_{n,m\to\infty}\int_{e''}|V_n(x,t)-V_m(x,t)|^\alpha dt=0$,

(ii_6) $\overline{\lim_{n,m\to\infty}}\int_h|V_n(x,t)-V_m(x,t)|^\alpha dt>0$ *for every set $h\subset h_7$ of positive measure and every $x\in R$,*

(i_8) $\sum_{n=1}^{\infty}\int_{e'''}|V_n(x,t)|^\alpha dt<\infty$

(ii_8) $\sum_{n=1}^{\infty}\int_h|V_n(x,t)|^\alpha dt=\infty$ *for every set $h\subset h_8$ of positive measure and every $x\in R$.*

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[3]) Orlicz [6] deduces also all the cases considered here from a general theorem.

Now denote by $\mathfrak{S}_1$ the space of the functions $y=\eta_\lambda(t)$ depending on the parameter $\lambda\epsilon[a,b)$, which are continuous in λ for fixed t, and μ-measurable for fixed λ. The norm is defined as

$$\|y\|=\sum_{n=0}^{\infty}\frac{1}{2^n}\int_T\frac{\max\limits_{a\leqslant\lambda\leqslant a_n}|\eta_\lambda(t)|}{1+\max\limits_{a\leqslant\lambda\leqslant a_n}|\eta_\lambda(t)|}dt,$$

where $a_n\to b-$; this space is complete. The elements of $\mathfrak{S}_1$ will be regarded as sequences depending on the continuous parameter λ. Choosing as L the space of the functions which are continuous in $[a,b)$ we easily see that we can consider $\mathfrak{S}_1$ as the space Y of type described in section 1. Denote by $B_1,\ldots,B_5$ the sets of the elements of $\mathfrak{S}_1$ for which respectively

(1) the sequence $\eta_\lambda(t)$ is asymptotically bounded when $\lambda\to b-$,

(2) the sequence $\eta_\lambda(t)$ converges asymptotically when $\lambda\to b-$,

(3) $\int\limits_a^{b-}d\lambda\int\limits_T\eta_\lambda(t)\,dt$ [4] exists,

(4) $\lim\limits_{\lambda,\mu\to b-}\int\limits_T|\eta_\lambda(t)-\eta_\mu(t)|^\alpha dt=0\quad(\alpha>0)$,

(5) $\int\limits_T[\operatorname*{var}\limits_{a\leqslant\lambda<b}\eta_\lambda(t)]^\alpha dt<\infty$ [5] $\quad(\alpha>0)$.

All these sets are linear, measurable (B), and satisfy the conditions (b_1), (b_2); the sets B_1,B_2 satisfy also the condition (b_3).

Similarly to Theorem 3 we can deduce now

THEOREM 4. *Let $V_\lambda(x,t)$ denote for fixed $\lambda\epsilon[a,b)$ a linear operation from X to $\boldsymbol{S}$; suppose it to be continuous in λ for fixed x and t. Then there exist decompositions $T=e_1+h_1=\ldots=e_5+h_5$ and a residual set R such that*

(i_1) *the sequence $V_\lambda(x,t)$ is asymptotically bounded on e_1 for every x, as $\lambda\to b-$,*

(ii_1) *for every set $h\subset h_1$ of positive measure and every $x\epsilon R$ the sequence $V_\lambda(x,t)$ is not asymptotically bounded on h, as $\lambda\to b-$,*

(i_2) $\lim\limits_{\lambda\to b-}\text{as}\ V_\lambda(x,t)$ *exists on e_2 for every x,*

(ii_2) $\lim\limits_{\lambda\to b-}\text{as}\ V_\lambda(x,t)$ *does not exist on every set $h\subset h_2$ of positive measure and every $x\epsilon R$.*

Moreover, for every x and $\varepsilon>0$ there exist sets e',e'',e''' such that $\mu(e_3-e')<\varepsilon$, $\mu(e_4-e'')<\varepsilon$, $\mu(e_5-e''')<\varepsilon$ and

[4] $\int\limits_a^{b-}\varphi(\lambda)d\lambda=\lim\limits_{t\to b-}\int\limits_a^t\varphi(\lambda)d\lambda.$

[5] $\operatorname*{var}\limits_{a\leqslant\lambda<b}\varphi_\lambda=\sup\limits_{a\leqslant\lambda<b}\operatorname*{var}\limits_{a\leqslant\mu\leqslant\lambda}\varphi_\mu.$

(i_3) $\int\limits_a^{b-}d\lambda\int\limits_{e'}V_\lambda(x,t)\,dt$ *exists,*

(ii_3) $\int\limits_a^{b-}d\lambda\int\limits_h V_\lambda(x,t)\,dt$ *does not exist for every set $h\subset h_3$ of positive measure and every $x\epsilon R$,*

(i_4) $\lim\limits_{\lambda,\mu\to b-}\int\limits_{e''}|V_\lambda(x,t)-V_\mu(x,t)|^\alpha dt=0$,

(ii_4) $\overline{\lim\limits_{\lambda,\mu\to b-}}\int\limits_h|V_\lambda(x,t)-V_\mu(x,t)|^\alpha dt>0$ *for every set $h\subset h_4$ of positive measure and every $x\epsilon R$,*

(i_5) $\int\limits_{e'''}[\operatorname*{var}\limits_{a\leqslant\lambda<b}V_\lambda(x,t)]^\alpha dt<\infty$,

(ii_5) $\int\limits_h[\operatorname*{var}\limits_{a\leqslant\lambda<b}V_\lambda(x,t)]^\alpha dt=\infty$ *for every set $h\subset h_5$ of positive measure and every $x\epsilon R$.*

Now let us denote by X a separable F-space composed of functions $x=x(\zeta)$ of the complex variable ζ, defined for $|\zeta|<1$, continuous on every radius $\arg\zeta=\text{const}$, and measurable for $|\zeta|=\text{const}$. Suppose further that $\|x_n\|\to0$ implies $\lim\limits_{n\to\infty}\text{as}\ x_n(re^{it})=0$ for fixed r. Suppose that the addition and multiplication are defined in X as usual and that $x(\zeta)\epsilon X$, $h\epsilon\mathfrak{C}$ implies $c_h(\varphi)x(re^{i\varphi})\epsilon X$, where $c_h(\varphi)$ stands for the characteristic function (of the variable φ) of the set h. Then setting $V_\lambda(x,t)=x(\lambda e^{it})$ we deduce immediately from Theorem 4 the

THEOREM 5. *There exist decompositions $T=e_1+h_1=e_2+h_2$ and a residual set R such that*

(i_1) *the sequence $x(\lambda e^{it})$ is asymptotically bounded on e_1 as $\lambda\to1-$,*

(ii_1) *the sequence $x(\lambda e^{it})$ is not asymptotically bounded on every set $h\subset h_1$ of positive measure and every $x\epsilon R$,*

(i_2) $\lim\limits_{\lambda\to1-}\text{as}\ x(\lambda e^{it})$ *exists on e_2 for every x,*

(ii_2) $\lim\limits_{\lambda\to1-}\text{as}\ x(\lambda e^{it})$ *does not exist on every set $h\subset h_2$ of positive measure and every $x\epsilon R$.*

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INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

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Sur les fonctionnelles multiplicatives

par

T. LEŻAŃSKI (Warszawa)

Introduction

Ce travail est une continuation de mon article précédent [2]. Nous y considérons un sous-espace linéaire fermé Ξ de l'espace $\bar{X}$ conjugué à un espace X du type B; un espace linéaire fermé $\mathfrak{K}$ d'opérations linéaires de Ξ à Ξ; enfin un espace linéaire $\mathfrak{M}$ de fonctionnelles linéaires dans $\mathfrak{K}$, qui satisfont à l'axiome qui était désigné dans [2] par (F). Cet axiome sera cité plus loin sous la condition (12). A toute fonctionnelle F qui appartient à $\mathfrak{M}$, nous faisons correspondre une opération T_F linéaire de Ξ à Ξ, notamment

$$T_F\varphi x \underset{df}{=} F_{\psi\varphi}\{\psi x\cdot\varphi y\} \qquad (\varphi\in\Xi,\ x\in X)$$

(voir [2], Introduction).

Nous étudions ensuite l'équation $\varphi+T_F\varphi=\psi$ $(\varphi,\psi\in\Xi)$, en faisant correspondre à l'opération $I+T_F$ un nombre $D(F)$ qu'on appelle le *déterminant* de cette équation.

En général, on ne peut pas demander que le nombre correspondant à l'équation $(I+T_{F_1})(I+T_{F_2})\varphi=\psi$ soit égal à $D(F_1)\cdot D(F_2)$, vu que la fonctionnelle F et, par conséquent, $D(F)$ ne sont pas déterminées par T_F.

Nous introduisons ici une sorte de „multiplication" des éléments de $\mathfrak{M}$, de manière que l'on ait

$$T_{(F^{(1)}\cdot F^{(2)})}=T_{F^{(1)}}\cdot T_{F^{(2)}} \quad \text{pour} \quad F^{(i)}\in\mathfrak{M} \qquad (i=1,2);$$

nous démontrerons que la fonctionnelle $D(F)$ vérifie l'équation

$$D(F^{(1)})\cdot D(F^{(2)})=D(F^{(1)}+F^{(2)}+F^{(1)}F^{(2)})$$

pour tout couple $F^{(1)},F^{(2)}$ d'éléments permutables de $\mathfrak{M}$ [1]).

I. Considérations générales

Soit $\mathfrak{A}$ un anneau du type (B), c'est-à-dire un anneau linéaire avec une norme homogène $\|A\|$ satisfaisant à l'inégalité $\|A\cdot B\|\leqslant\|A\|\cdot\|B\|$ pour $A\in\mathfrak{A}$, $B\in\mathfrak{A}$; regardé comme espace linéaire, cet anneau est un es-

[1]) M. R. Sikorski a remplacé la condition de permutabilité d'éléments $F^{(1)}$, $F^{(2)}$ par une autre, moins restrictive.