

## Article

# A Theoretical Analysis of a Fractional Multi-Dimensional System of Boundary Value Problems on the Methylpropane Graph via Fixed Point Technique

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**Abstract:** Few studies have investigated the existence and uniqueness of solutions for fractional differential equations on star graphs until now. The published papers on the topic are based on the assumption of existence of one junction node and some boundary nodes as the origin on a star graph. These structures are special cases and do not cover more general non-star graph structures. In this paper, we state a labeling method for graph vertices, and then we prove the existence results for solutions to a new family of fractional boundary value problems (FBVPs) on the methylpropane graph. We design the chemical compound of the methylpropane graph with vertices specified by 0 or 1, and on every edge of the graph, we consider fractional differential equations. We prove the existence of solutions for the proposed FBVPs by means of the Krasnoselskii's and Schaefer's fixed point theorems, and further, we study the Ulam–Hyers type stability for the given multi-dimensional system. Finally, we provide an illustrative example to examine our results.

**Keywords:** fractional differential equation; boundary value problem; methylpropane graph; the Caputo fractional derivative; stability

## 1. Introduction

Some natural phenomena throughout the world have been studied using initial and BVPs for many years. Due to the broad diversity of BVPs, many academics have turned to mathematical tools and computer simulation software to explore a variety of practical processes. It has recently been shown that fractional differential equations may be used to describe many diverse applications in the applied sciences (see for examples, [1–10]). One of our purposes in this work is to extend the theoretical aspects of some applied concepts in chemistry to increase our abilities to model some processes in chemical reactions. Once that is accomplished, computer engineers will be able to build software that allow everyone to conduct chemical experiments without utilizing real materials. This will allow for the most effective management of the environmental resources that are available. We may utilize fractional calculus for this, which is useful in similar situations. It is possible to solve

fractional differential-equation-based mathematical models, derive the solution functions, then examine the qualitative behavior of solutions under specific boundary constraints using certain approaches. As may be shown in several published publications, some problems have been addressed before (see for examples, [11–23]).

Water pipes, gas transmission lines, population increase and computer network expansion are only a few examples of industrial and medical areas where new applied models have been developed to analyze the processes devised by engineers in these disciplines. There has been some interest in studying mathematical models that are expressed on graphs via ordinary or fractional differential equations, due to their graph representation. In fact, on a graph, BVPs are specified on every edge as a set of differential equations, along with particular boundary conditions at every vertex of the graph.

In 1980, Lumer’s work on differential equations defined on graphs is the preliminary stage for this theory [24]. The local operators established on ramification spaces were used to study generic evolution equations. Nicaise explored the propagation of nerve impulses using a similar system [25]. With respect to a geometric graph with linear differential equations, Zavgorodnij investigated those solutions that were coordinated on their inner vertices in 1989 [26]. An adjoint boundary problem was formulated by him, and the adjointness criteria were established [26]. On the basis of graph representation, Gordeziani et al. have studied the uniqueness-existence of solutions to ODEs in 2008 [27] and solved the given BVP using the double sweep method and provided a numerical methodology.

In the majority of the referenced studies, differential equations were evaluated on a graph and their solutions were determined using numerical as well as computational techniques. The existence aspects of solutions to fractional BVPs on graphs is, however, only established in a few publications using approaches from fixed point theory [28,29].

Graef et al. were the first researchers who used fixed point theory to explore the existing results [28]. They presented a three-vertex-star graph  $G = \mathcal{V}(G) \cup \mathcal{E}(G)$ , in which  $\mathcal{V}(G) = \{p_0, p_1, p_2\}$  and  $\mathcal{E}(G) = \{e_1 = \overrightarrow{p_1 p_0}, e_2 = \overrightarrow{p_2 p_0}\}$ , the vertex  $p_0$  regarded as junction node, the edge  $e_i = \overrightarrow{p_i p_0}$  connecting the vertices  $p_i$  to  $p_0$  having length  $l_i = |\overrightarrow{p_i p_0}|$  for  $i = 1, 2$  (Figure 1). At every  $e_i = \overrightarrow{p_i p_0}$ , by considering the origin at boundary vertices  $p_1$  and  $p_2$ , a local coordinate system is made together with the coordinate  $t \in (0, l_i)$ . A nonlinear fractional differential equation system was developed by Graef et al. on each edge  $e_i = \overrightarrow{p_i p_0}$  as follows ( $i = 1, 2$ )

$$-D_0^\alpha y_i(t) = g_i(t)h_i(t, y_i(t)),$$

in which,  $t \in (0, l_i)$  via the BCs

$$y_1(0) = y_2(0) = 0, \quad y_1(l_1) = y_2(l_2), \quad D_0^\beta y_1(l_1) + D_0^\beta y_2(l_2) = 0,$$

where  $\beta \in (0, \alpha)$ ,  $\alpha \in (1, 2]$ ,  $g_i \in C([0, l_i], \mathbb{R})$  and  $h_i \in C([0, l_i] \times \mathbb{R}, \mathbb{R})$  with  $g_i(t) \neq 0$  on  $[0, l_i]$ ,  $D_0^\alpha$  and  $D_0^\beta$  regarded as RL-fractional derivatives. Their desired results are obtained by applying Schauder’s theorem and the contraction principle.

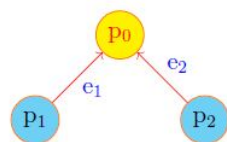


Figure 1. Star graph  $G$  having two edges.

Mehandiratta et al. in 2019 [29] presented an extension of star graph having  $n$  edges and  $n + 1$  vertices. Authors made a generic star graph  $G$  that including  $\mathcal{V}(G) = \{p_0, p_1, \dots, p_n\}$  and  $\mathcal{E}(G) = \{e_1 = \overrightarrow{p_1 p_0}, e_2 = \overrightarrow{p_2 p_0}, \dots, e_n = \overrightarrow{p_n p_0}\}$ , in which  $l_i = |\overrightarrow{p_i p_0}|$  is considered as the length of  $e_i$  joining vertices  $p_i$  to  $p_0$  ( $i \in \mathbb{N}_1^n$ ).

The following system consists of the BVPs that Mehandiratta et al. defined them on the edges of the above graph  $\mathbb{G}$  as

$$\begin{cases} {}^c\mathcal{D}_0^\alpha y_i(t) = f_i(t, y_i(t), {}^c\mathcal{D}_0^\beta y_i(t)), & (t \in (0, l_i)), \\ y_i(0) = 0, y_i(l_i) = y_k(l_k) \ (i \neq k), \quad \sum_{i=1}^n y'_i(l_i) = 0, \end{cases} \tag{1}$$

in which  $0 < \beta \leq \alpha - 1$ ,  $\alpha \in (1, 2]$ ,  $f_i \in C([0, l_i] \times \mathbb{R}^2, \mathbb{R})$ , and  ${}^c\mathcal{D}_0^\gamma$  is the  $\gamma^{th}$ -Caputo derivative. The authors presented the transforms  $v = \frac{t}{l_i} \in [0, 1]$  and  $r(v) = y(t) = y(l_i v)$  for  $t \in [0, l_i]$ , and showed that the equality  ${}^c\mathcal{D}_0^\alpha y(t) = l_i^{-\alpha} ({}^c\mathcal{D}_0^\alpha r(v))$  holds. Next, they transformed the system of FBVPs on the graph (1) to the succeeding system of FBVPs over the interval  $[0, 1]$  by performing these transformations

$${}^c\mathcal{D}_0^\alpha r_i(v) = l_i^\alpha h_i(v, r_i(v), l_i^{-\beta} {}^c\mathcal{D}_0^\beta r_i(v)), \quad (v \in [0, 1])$$

with  $r_i(0) = 0$ ,  $r_i(1) = r_k(1)$  for  $i \neq k$  and  $\sum_{i=1}^n l_i^{-1} r'_i(1) = 0$ , with  $r_i(v) = u_i(l_i v)$  and  $h_i(v, y, w) = f_i(l_i v, y, w)$  for  $i \in \mathbb{N}_1^n$ .

These works have inspired us to expand their BVPs to a novel BVP on the methylpropane graph. In fact, it is a generalized graph compared to star graphs. In the future, new scholars may generalize these notions to different graphs as new ideas.

**2. Preliminaries**

The Graph in the context of methylpropane compound is introduced in this section for defining a new category of fractional BVPs on it. Let us first bring the attention of readers to two crucial aspects regarding the methodologies employed in [28,29].

(1) As shown in Figures 1 and 2, the authors of both articles assume that  $\mathbb{G}$  is a star graph with one junction node  $p_0$ , but in general, graph  $G$  may not be a star graph and may have more than one junction node. Figure 3 shows five junction nodes, as an example.

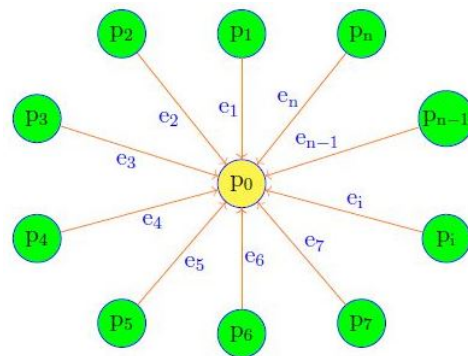


Figure 2. Star graph  $\mathbb{G}$  having  $n$  edges.

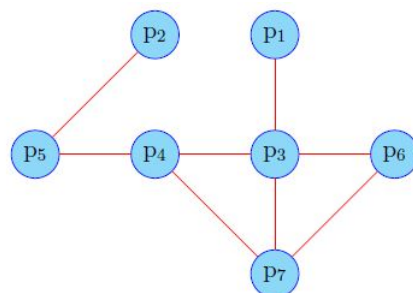


Figure 3. A non-star graph  $\mathbb{G}$  with five junction nodes.

(2) It is important to note that both articles treat the length of every edge as a variable  $|e_i| = l_i$  for  $i \in \{1, 2, \dots, n\}$ , where  $n$  represents how many vertices there are in the graph  $\mathbb{G}$ . Then, for the normalization of all edges, they use change of variable and transform  $[0, l_i]$  to interval  $[0, 1]$ , whereas in the beginning, one might assume that all edges are the same length  $|e_i| = 1$  without determining boundary vertex as the origin on each edge. A novel approach for labeling the vertices is proposed as a means of achieving this goal. This allows us to label each vertex of the graph with either 0 or 1. To put it as another way, each vertex has a different name (label) based on the direction of related edge. A vertex's label and its ending vertex's label are treated as 0 and 1, respectively, when we move along an edge arbitrarily. Hence some labels are 0, some are 1, some are both 0 and 1 and each origin in every edge is not fixed; it is changed with each change in direction of the corresponding edge. It is not necessary to normalize the length of all edges utilizing the mentioned transforms; therefore, we are permitted to select one of the two vertices of each edge as its origin. Figure 4 shows an example of how labeling may be used in this situation. We begin to walk along the edges of this graph in the first step from the blue vertex.

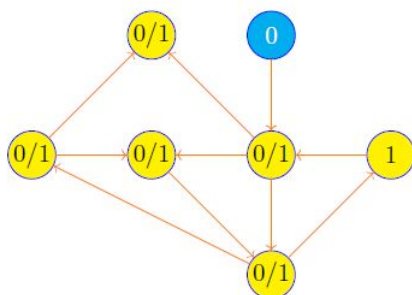


Figure 4. A general graph  $\mathbb{G}$  with labels 0 or 1.

In this paper, on the methylpropane graph, we investigate several desirable existence theorems for solutions of a new class of FBVPs (Figure 5).

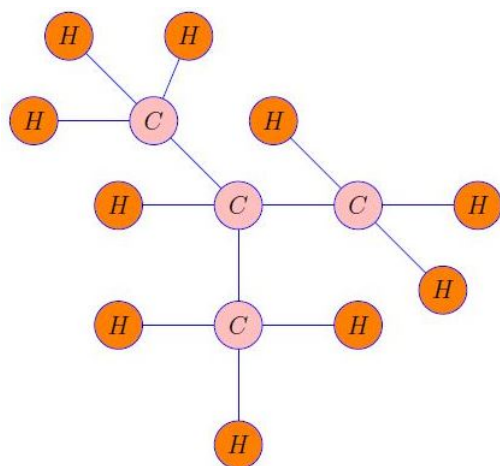


Figure 5. A sketch of methylpropane  $HC(CH_3)_3$  in the graph settings.

The chemical compound of methylpropane or isobutane has the molecular formula  $HC(CH_3)_3$ , where (H) and (C) are hydrogen and carbon. Methylpropane is an isomer of Butane, which is used as a precursor molecule in the petrochemical industry. This compound consists of three carbon–carbon bonds and three sub-branches  $CH_3$ . Inspired by this structure, Hydrogen atoms and carbon atoms are regarded as the graph' vertices. Graph's edges stand for the chemical bonds that occur between atoms. This molecular graph is not a star graph, thus the technique applied in [28,29] which assigns the origin at boundary nodes except the junction node  $p_0$ , will not work here because there exist more

than one junction node on this graph (four junction nodes). Consequently, we must adopt a new approach. We may label the vertices of this graph with either 0 or 1, and we can regard unit value  $l_j = 1$  as the length of each edge (see Figure 6).

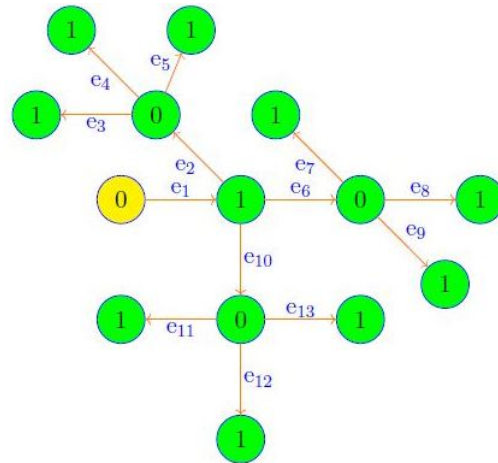


Figure 6. The graph of  $\text{HC}(\text{CH}_3)_3$  with labels 0 or 1.

The existence of solutions in relation to the multi-dimensional nonlinear multi-term FBVPs are studied in this manuscript by defining the graph of methylpropane labeled with 0 or 1, as shown above:

$$\begin{cases} {}^c\mathcal{D}_0^\vartheta z_j(t) = h_j(t, z_j(t), z'_j(t)) + k_j(t, z_j(t), z'_j(t)), & (t \in [0, 1]) \\ z_j(0) + {}^c\mathcal{D}_0^\beta z_j(0) + z'_j(1) = 0, \\ \int_0^1 [z_j(\xi) + {}^c\mathcal{D}_0^\beta z_j(\xi) + z'_j(\xi)] d\xi = 0, \end{cases} \quad (2)$$

where  $\vartheta \in (1, 2)$ ,  $\beta \in (0, 1)$ ,  $h_j, k_j \in C([0, 1] \times \mathbb{R}^2, \mathbb{R})$  for  $j \in \mathbb{N}_1^{13} := \{1, 2, \dots, 13\}$ , in which  $n = 13$  stands for the number of edges of methylpropane graph s.t.  $|e_j| = 1$ , and  ${}^c\mathcal{D}_0^\vartheta$  regards the  $\vartheta^{\text{th}}$ -Caputo derivative. Here in the boundary conditions, we consider the values of the unknown functions  $z_j$ , and their  $\beta^{\text{th}}$ -derivatives ( ${}^c\mathcal{D}_0^\beta z_j$ ) and their first-order derivatives ( $z'_j$ ) in the terminal points, and also we consider the integral of sum of them in the mid-points of the time domain  $[0, 1]$ .

Regarding the relationship between the above system of FBVPs and the molecular graph of methylpropane, it should be noted that the main reason for this study is the growing importance of chemical graph theory. This field of mathematics relates graph theory to chemistry, and investigates the chemical changes resulting from interatomic bonds along bond lines and their effects, and these studies are performed in the form of various models of fractional differential equations. Instances of these new applications can be found in bio-macromolecules and chemical kinetics. As the main idea of this work, the results of the given fractional BVPs (2) about the existence of solutions can be interpreted in various meanings in relation to organic chemistry. As a result, any solution  $z_j(t)$  at every edge  $e_j$  might represents the bond polarity, bond strength, bond energy, etc. Moreover, the integer and fractional order derivatives of the unknown functions  $z_j$ , i.e.,  $z'_j(t)$  and  ${}^c\mathcal{D}_0^\beta z_j(t)$  can interpret some chemical notions in this direction. Because  ${}^c\mathcal{D}_0^\beta z_j(t)$  has a nonlocal nature, it can show some useful properties of the curves of solutions over a large extent of a time interval  $[0, 1]$  during a chemical interaction, while  $z'_j(t)$  has a local nature and it can interprets the velocity of the same chemical interactions at a specific time  $t \in [0, 1]$ . Further,

$h_j$  and  $k_j$  are defined as functions of these quantities with respect to the time  $t$  on each edge  $e_j, j \in \mathbb{N}_1^{13}$ .

On the other hand, as we said above,  $n = 13$  shows the number of edges of the methylpropane graph and this index can be changed by increasing or decreasing the number of edges in different molecular structures. In other words, we can generalize this number to each arbitrary value  $k \in \mathbb{N}$  and this change will not affect the computational disorder. We are sure that in the future, the theory of chemical reactions and also chemical graph theory may benefit from some aspects of these developments. Similarly, see some recent works in this regard [30–33].

Now, we recall some basic definitions and properties of fractional calculus. Assume  $\vartheta > 0$ . Then we define RL-fractional integral of the continuous function  $z : [a, b] \rightarrow \mathbb{R}$  by

$$\mathcal{I}_0^\vartheta z(t) = \frac{1}{\Gamma(\vartheta)} \int_0^t (t - \xi)^{\vartheta-1} z(\xi) d\xi,$$

such that the above integral exists [34,35].

Let  $r - 1 < \vartheta < r$ . Then  $r = [\vartheta] + 1$ . We define the Caputo derivative of the function  $z \in C^{(r)}([a, b], \mathbb{R})$  as

$${}^c\mathcal{D}_0^\vartheta z(t) = \frac{1}{\Gamma(r - \vartheta)} \int_0^t (t - \xi)^{r-\vartheta-1} z^{(r)}(\xi) d\xi,$$

such that the above integral exists [34,35]. It is known that for  $n - 1 < \vartheta < n$ , the solution of  ${}^c\mathcal{D}_0^\vartheta z(t) = 0$  is  $z(t) = m_0^* + m_1^*t + m_2^*t^2 + \dots + m_{n-1}^*t^{n-1}$ ; also, we have

$$\mathcal{I}_0^\vartheta ({}^c\mathcal{D}_0^\vartheta z(t)) = z(t) + \sum_{j=0}^{n-1} m_j^* t^j = z(t) + m_0^* + m_1^*t + m_2^*t^2 + \dots + m_{n-1}^*t^{n-1}, \quad (3)$$

where  $m_0^*, \dots, m_{n-1}^* \in \mathbb{R}$  [36]. The Krasnoselskii and Scheafer fixed point theorems are main tools which we use for our analysis.

**Lemma 1.** [37] We assume  $\mathcal{E} \neq \emptyset$  as bounded, convex closed set in the Banach space  $\mathcal{M}$ . Consider  $Y_1$  and  $Y_2$  on  $\mathcal{E}$  so that  $Y_1z + Y_2w \in \mathcal{E}$  whenever  $z, w \in \mathcal{E}$ , and (a) the operator  $Y_1$  is compact and continuous, (b) the operator  $Y_2$  is a contraction. Then  $\exists y \in \mathcal{E}$  with  $y = Y_1y + Y_2y$ .

**Lemma 2.** [37] Consider  $\mathcal{M}$  as a Banach space and a completely continuous mapping  $Y$  on  $\mathcal{M}$ . Then  $\{z \in \mathcal{M} : z = \nu Yz, \nu \in (0, 1)\}$  is not bounded or there exist a fixed point of  $Y$  in  $\mathcal{M}$ .

### 3. Main Theorems Regarding the Existence

The present section is assigned to the results of existence theory on the graph of methylpropane (Figure 6). To begin it, we consider the Banach spaces  $\mathcal{M}_j = \{z_j : z_j, z'_j \in C[0, 1]\}$  with  $\|z_j\|_{\mathcal{M}_j} = \sup_{t \in [0, 1]} |z_j(t)| + \sup_{t \in [0, 1]} |z'_j(t)|$  for  $j \in \mathbb{N}_1^{13}$ . Then the product space  $\mathcal{M} = (\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_{13})$  equipped with the norm  $\|z = (z_1, z_2, \dots, z_{13})\|_{\mathcal{M}} = \sum_{j=1}^{13} \|z_j\|_{\mathcal{M}_j}$  is a Banach space.

**Lemma 3.** Let  $\varphi_j \in C[0, 1], j \in \mathbb{N}_1^{13}, \vartheta \in (1, 2)$  and  $\beta \in (0, 1)$ . Then  $z_j$  is a solution of the BVPs

$$\begin{cases} {}^c\mathcal{D}_0^\vartheta z_j(t) = \varphi_j(t), \\ z_j(0) + {}^c\mathcal{D}_0^\beta z_j(0) + z'_j(1) = 0, \\ \int_0^1 [z_j(\xi) + {}^c\mathcal{D}_0^\beta z_j(\xi) + z'_j(\xi)] d\xi = 0, \end{cases} \quad (4)$$

which is given by

$$\begin{aligned}
 z_j(t) = & \int_0^t \frac{(t-\xi)^{\vartheta-1}}{\Gamma(\vartheta)} \varphi_j(\xi) d\xi + \frac{t-\Delta}{\Delta-1} \int_0^1 \frac{(1-\xi)^{\vartheta-2}}{\Gamma(\vartheta-1)} \varphi_j(\xi) d\xi \\
 & + \frac{1-t}{\Delta-1} \left[ \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-1}}{\Gamma(\vartheta)} \varphi_j(m) dm d\xi + \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-2}}{\Gamma(\vartheta-1)} \varphi_j(m) dm d\xi \right. \\
 & \left. + \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-\beta-1}}{\Gamma(\vartheta-\beta)} \varphi_j(m) dm d\xi \right],
 \end{aligned} \tag{5}$$

where  $\Delta = \frac{3\Gamma(3-\beta)+2}{2\Gamma(3-\beta)} \neq 1$  and  $j \in \mathbb{N}_1^{13}$ .

**Proof.** Consider  $z_j$  as the solution of the problems (4) ( $j \in \mathbb{N}_1^{13}$ ). So by (3), there are constants  $a_0^{(j)}, a_1^{(j)} \in \mathbb{R}$  such that  $z_j(t) = \mathcal{I}_0^\vartheta \varphi_j(t) + a_0^{(j)} + a_1^{(j)}t$ , i.e.,

$$z_j(t) = \int_0^t \frac{(t-\xi)^{\vartheta-1}}{\Gamma(\vartheta)} \varphi_j(\xi) d\xi + a_0^{(j)} + a_1^{(j)}t. \tag{6}$$

Thus, we obtain  $z_j'(t) = \int_0^t \frac{(t-\xi)^{\vartheta-2}}{\Gamma(\vartheta-1)} \varphi_j(\xi) d\xi + a_1^{(j)}$  and

$${}^c\mathcal{D}_0^\beta z_j(t) = \int_0^t \frac{(t-\xi)^{\vartheta-\beta-1}}{\Gamma(\vartheta-\beta)} \varphi_j(\xi) d\xi + a_1^{(j)} \frac{t^{1-\beta}}{\Gamma(2-\beta)},$$

and so

$$\begin{aligned}
 \int_0^1 z_j(\xi) d\xi &= \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-1}}{\Gamma(\vartheta)} \varphi_j(m) dm d\xi + a_0^{(j)} + \frac{1}{2}a_1^{(j)}, \\
 \int_0^1 z_j'(\xi) d\xi &= \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-2}}{\Gamma(\vartheta-1)} \varphi_j(m) dm d\xi + a_1^{(j)}, \\
 \int_0^1 {}^c\mathcal{D}_0^\beta z_j(\xi) d\xi &= \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-\beta-1}}{\Gamma(\vartheta-\beta)} \varphi_j(m) dm d\xi + a_1^{(j)} \frac{1}{\Gamma(3-\beta)}.
 \end{aligned}$$

By using the first boundary condition, we obtain

$$a_0^{(j)} + a_1^{(j)} + \int_0^1 \frac{(1-\xi)^{\vartheta-2}}{\Gamma(\vartheta-1)} \varphi_j(\xi) d\xi = 0. \tag{7}$$

Further, by the linearity of the integral in the second boundary condition, we have

$$\int_0^1 z_j(\xi) d\xi + \int_0^1 {}^c\mathcal{D}_0^\beta z_j(\xi) d\xi + \int_0^1 z_j'(\xi) d\xi = 0,$$

and it gives

$$\begin{aligned}
 \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-1}}{\Gamma(\vartheta)} \varphi_j(m) dm d\xi + a_0^{(j)} + \frac{1}{2}a_1^{(j)} + \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-\beta-1}}{\Gamma(\vartheta-\beta)} \varphi_j(m) dm d\xi \\
 + a_1^{(j)} \frac{1}{\Gamma(3-\beta)} + \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-2}}{\Gamma(\vartheta-1)} \varphi_j(m) dm d\xi + a_1^{(j)} = 0.
 \end{aligned} \tag{8}$$

By some simple calculations on two latter Equations (7) and (8), and by considering the constant  $\Delta = \frac{3\Gamma(3-\beta)+2}{2\Gamma(3-\beta)} \neq 1$ , we obtain

$$a_1^{(j)} = \frac{1}{(\Delta-1)} \left[ \int_0^1 \frac{(1-\xi)^{\vartheta-2}}{\Gamma(\vartheta-1)} \varphi_j(\xi) d\xi - \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-1}}{\Gamma(\vartheta)} \varphi_j(m) dm d\xi - \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-2}}{\Gamma(\vartheta-1)} \varphi_j(m) dm d\xi - \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-\beta-1}}{\Gamma(\vartheta-\beta)} \varphi_j(m) dm d\xi \right].$$

Then by substituting  $a_1^{(j)}$  into (7), we obtain

$$a_0^{(j)} = - \int_0^1 \frac{(1-\xi)^{\vartheta-2}}{\Gamma(\vartheta-1)} \varphi_j(\xi) d\xi - \frac{1}{(\Delta-1)} \left[ \int_0^1 \frac{(1-\xi)^{\vartheta-2}}{\Gamma(\vartheta-1)} \varphi_j(\xi) d\xi - \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-1}}{\Gamma(\vartheta)} \varphi_j(m) dm d\xi - \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-2}}{\Gamma(\vartheta-1)} \varphi_j(m) dm d\xi - \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-\beta-1}}{\Gamma(\vartheta-\beta)} \varphi_j(m) dm d\xi \right].$$

Next, we substitute the values of  $a_0^{(j)}$  and  $a_1^{(j)}$  into Equation (6). In this case, we see that  $z_j$  satisfies (5). This completes the proof.  $\square$

Here by considering (3), define the operator  $\mathcal{T}$  on  $\mathcal{M}$  by

$$\mathcal{T}(z_1, z_2, \dots, z_{13})(t) := \left( (\mathcal{T}_1 z_1)(t), \dots, (\mathcal{T}_{13} z_{13})(t) \right), \tag{9}$$

where  $(\mathcal{T}_j : \mathcal{M}_j \rightarrow \mathcal{M}_j)$ ,

$$\begin{aligned} (\mathcal{T}_j z_j)(t) &= \int_0^t \frac{(t-\xi)^{\vartheta-1}}{\Gamma(\vartheta)} [h_j(\xi, z_j(\xi), z'_j(\xi)) + k_j(\xi, z_j(\xi), z'_j(\xi))] d\xi \\ &+ \frac{t-\Delta}{\Delta-1} \int_0^1 \frac{(1-\xi)^{\vartheta-2}}{\Gamma(\vartheta-1)} [h_j(\xi, z_j(\xi), z'_j(\xi)) + k_j(\xi, z_j(\xi), z'_j(\xi))] d\xi \\ &+ \frac{1-t}{\Delta-1} \left[ \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-1}}{\Gamma(\vartheta)} [h_j(m, z_j(m), z'_j(m)) + k_j(m, z_j(m), z'_j(m))] dm d\xi \right. \\ &+ \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-2}}{\Gamma(\vartheta-1)} [h_j(m, z_j(m), z'_j(m)) + k_j(m, z_j(m), z'_j(m))] dm d\xi \\ &\left. + \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-\beta-1}}{\Gamma(\vartheta-\beta)} [h_j(m, z_j(m), z'_j(m)) + k_j(m, z_j(m), z'_j(m))] dm d\xi \right], \end{aligned}$$

for all  $t \in [0, 1]$  and  $z_j \in \mathcal{M}_j$ . For convenience in writing, put

$$\mathcal{K}_0^* := \frac{1}{|\Delta-1|\Gamma(\vartheta+2)} + \frac{|\Delta-1|+1}{|\Delta-1|\Gamma(\vartheta+1)} + \frac{1+|\Delta|}{|\Delta-1|\Gamma(\vartheta)} + \frac{1}{|\Delta-1|\Gamma(\vartheta-\beta+2)},$$



$$\mathcal{K}_1^* := \frac{|\Delta - 1| + 1}{|\Delta - 1|\Gamma(\vartheta)} + \frac{1}{|\Delta - 1|\Gamma(\vartheta + 2)} + \frac{1}{|\Delta - 1|\Gamma(\vartheta - \beta + 2)} + \frac{1}{|\Delta - 1|\Gamma(\vartheta + 1)}. \tag{10}$$

We now analyze the fractional BVP (2) with different conditions by terms of the existence criterion.

**Theorem 1.** *Let  $h_1, \dots, h_{13} : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $k_1, \dots, k_{13} : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  be continuous functions. Suppose that there exist continuous maps  $\delta_1, \dots, \delta_{13} : [0, 1] \rightarrow \mathbb{R}^+$  and  $\mu_1, \dots, \mu_{13} : [0, 1] \rightarrow \mathbb{R}^+$ , and continuous non-decreasing maps,  $\phi_1, \dots, \phi_{13} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  and  $\psi_1, \dots, \chi_{13} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  s.t.*

$$|h_j(t, z_1, z_2)| \leq \delta_j(t)\phi_j\left(\sum_{i=1}^2 |z_i|\right), \quad |k_j(t, z_1, z_2)| \leq \mu_j(t)\chi_j\left(\sum_{i=1}^2 |z_i|\right).$$

In addition, assume that continuous maps  $\sigma_1, \dots, \sigma_{13} : [0, 1] \rightarrow \mathbb{R}$  exist s.t.

$$|k_j(t, z_1, z_2) - k_j(t, q_1, q_2)| \leq \sigma_j(t)(|z_1 - q_1| + |z_2 - q_2|),$$

for all  $t \in [0, 1]$ ,  $z_1, z_2, q_1, q_2 \in \mathbb{R}$  and  $j \in \mathbb{N}_1^{13}$ . If  $\lambda^* := (\mathcal{K}_0^* + \mathcal{K}_1^*) \sum_{j=1}^{13} \|\sigma_j\| < 1$ , then the fractional BVPs (2) has a solution, where  $\|\sigma_j\| = \sup_{t \in [0, 1]} |\sigma_j(t)|$ , and  $\mathcal{K}_0^*$  and  $\mathcal{K}_1^*$  are given in (10).

**Proof.** Set  $\|\delta_j\| = \sup_{t \in [0, 1]} |\delta_j(t)|$  and  $\|\mu_j\| = \sup_{t \in [0, 1]} |\mu_j(t)|$ . We select the appropriate constant  $\rho$  so that

$$\rho \geq \sum_{j=1}^{13} (\phi_j(\|z_j\|_{\mathcal{M}_j})\|\delta_j\| + \chi_j(\|z_j\|_{\mathcal{M}_j})\|\mu_j\|) \{\mathcal{K}_0^* + \mathcal{K}_1^*\}, \tag{11}$$

where  $\mathcal{K}_i^*$ 's are given in (10). Define subsets

$$\mathcal{B}_\rho := \{z = (z_1, z_2, \dots, z_{13}) \in \mathcal{M} = (\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_{13}) : \|z\|_{\mathcal{M}} \leq \rho\},$$

where  $\rho$  is given in (11). Define  $\mathcal{T}_1$  and  $\mathcal{T}_2$  on  $\mathcal{B}_\rho$  as

$$\begin{aligned} \mathcal{T}_1(z_1, z_2, \dots, z_{13})(t) &:= \left( (\mathcal{T}_1^{(1)} z_1)(t), (\mathcal{T}_1^{(2)} z_2)(t), \dots, (\mathcal{T}_1^{(13)} z_{13})(t) \right), \\ \mathcal{T}_2(z_1, z_2, \dots, z_{13})(t) &:= \left( (\mathcal{T}_2^{(1)} z_1)(t), (\mathcal{T}_2^{(2)} z_2)(t), \dots, (\mathcal{T}_2^{(13)} z_{13})(t) \right), \end{aligned}$$

where

$$\begin{aligned} (\mathcal{T}_1^{(j)} z_j)(t) &= \int_0^t \frac{(t - \xi)^{\vartheta - 1}}{\Gamma(\vartheta)} h_j(\xi, z_j(\xi), z_j'(\xi)) d\xi \\ &+ \frac{t - \Delta}{\Delta - 1} \int_0^1 \frac{(1 - \xi)^{\vartheta - 2}}{\Gamma(\vartheta - 1)} h_j(\xi, z_j(\xi), z_j'(\xi)) d\xi \\ &+ \frac{1 - t}{\Delta - 1} \left[ \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta - 1}}{\Gamma(\vartheta)} h_j(m, z_j(m), z_j'(m)) dm d\xi \right. \\ &+ \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta - 2}}{\Gamma(\vartheta - 1)} h_j(m, z_j(m), z_j'(m)) dm d\xi \\ &\left. + \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta - \beta - 1}}{\Gamma(\vartheta - \beta)} h_j(m, z_j(m), z_j'(m)) dm d\xi \right], \end{aligned}$$

and

$$\begin{aligned}
 (\mathcal{T}_2^{(j)} z_j)(t) &= \int_0^t \frac{(t-\xi)^{\vartheta-1}}{\Gamma(\vartheta)} k_j(\xi, z_j(\xi), z'_j(\xi)) \, d\xi \\
 &+ \frac{t-\Delta}{\Delta-1} \int_0^1 \frac{(1-\xi)^{\vartheta-2}}{\Gamma(\vartheta-1)} k_j(\xi, z_j(\xi), z'_j(\xi)) \, d\xi \\
 &+ \frac{1-t}{\Delta-1} \left[ \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-1}}{\Gamma(\vartheta)} k_j(m, z_j(m), z'_j(m)) \, dm d\xi \right. \\
 &+ \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-2}}{\Gamma(\vartheta-1)} k_j(m, z_j(m), z'_j(m)) \, dm d\xi \\
 &\left. + \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-\beta-1}}{\Gamma(\vartheta-\beta)} k_j(m, z_j(m), z'_j(m)) \, dm d\xi \right],
 \end{aligned}$$

for each  $t \in [0, 1]$  and  $z_j \in \mathcal{M}_j$ . Assume that  $\phi_j^* = \sup_{z_j \in \mathcal{M}_j} \phi_j(\|z_j\|_{\mathcal{M}_j})$  and  $\chi_j^* = \sup_{z_j \in \mathcal{M}_j} \chi_j(\|z_j\|_{\mathcal{M}_j})$ . Now for every  $z = (z_1, \dots, z_{13}), q = (q_1, \dots, q_{13}) \in \mathcal{B}_\rho$ , we have

$$\begin{aligned}
 |(\mathcal{T}_1^{(j)} z_j + \mathcal{T}_2^{(j)} q_j)(t)| &\leq \int_0^t \frac{(t-\xi)^{\vartheta-1}}{\Gamma(\vartheta)} \left[ |h_j(\xi, z_j(\xi), z'_j(\xi))| + |k_j(\xi, q_j(\xi), q'_j(\xi))| \right] \, d\xi \\
 &+ \frac{t+|\Delta|}{|\Delta-1|} \int_0^1 \frac{(1-\xi)^{\vartheta-2}}{\Gamma(\vartheta-1)} \left[ |h_j(\xi, z_j(\xi), z'_j(\xi))| + |k_j(\xi, q_j(\xi), q'_j(\xi))| \right] \, d\xi \\
 &+ \frac{1-t}{|\Delta-1|} \left( \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-1}}{\Gamma(\vartheta)} \left[ |h_j(m, z_j(m), z'_j(m))| + |k_j(m, q_j(m), q'_j(m))| \right] \, dm d\xi \right. \\
 &+ \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-2}}{\Gamma(\vartheta-1)} \left[ |h_j(m, z_j(m), z'_j(m))| + |k_j(m, q_j(m), q'_j(m))| \right] \, dm d\xi \\
 &\left. + \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-\beta-1}}{\Gamma(\vartheta-\beta)} \left[ |h_j(m, z_j(m), z'_j(m))| + |k_j(m, q_j(m), q'_j(m))| \right] \, dm d\xi \right) \\
 &\leq \int_0^t \frac{(t-\xi)^{\vartheta-1}}{\Gamma(\vartheta)} [\delta_j(\xi) \phi_j(\|z_j\|_{\mathcal{M}_j}) + \mu_j(\xi) \chi_j(\|q_j\|_{\mathcal{M}_j})] \, d\xi \\
 &+ \frac{t+|\Delta|}{|\Delta-1|} \int_0^1 \frac{(1-\xi)^{\vartheta-2}}{\Gamma(\vartheta-1)} [\delta_j(\xi) \phi_j(\|z_j\|_{\mathcal{M}_j}) + \mu_j(\xi) \chi_j(\|q_j\|_{\mathcal{M}_j})] \, d\xi \\
 &+ \frac{1-t}{|\Delta-1|} \left( \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-1}}{\Gamma(\vartheta)} [\delta_j(m) \phi_j(\|z_j\|_{\mathcal{M}_j}) + \mu_j(m) \chi_j(\|q_j\|_{\mathcal{M}_j})] \, dm d\xi \right. \\
 &\left. + \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-2}}{\Gamma(\vartheta-1)} [\delta_j(m) \phi_j(\|z_j\|_{\mathcal{M}_j}) + \mu_j(m) \chi_j(\|q_j\|_{\mathcal{M}_j})] \, dm d\xi \right)
 \end{aligned}$$

$$\begin{aligned}
 &+ \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta - \beta - 1}}{\Gamma(\vartheta - \beta)} [\delta_j(m)\phi_j(\|z_j\|_{\mathcal{M}_j}) + \mu_j(m)\chi_j(\|q_j\|_{\mathcal{M}_j})] \, dm d\xi \Big) \\
 &\leq \left( \|\delta_j\|\phi_j^* + \|\mu_j\|\chi_j^* \right) \left[ \frac{1}{\Gamma(\vartheta + 1)} + \frac{1 + |\Delta|}{|\Delta - 1|\Gamma(\vartheta)} + \frac{1}{|\Delta - 1|\Gamma(\vartheta + 2)} \right. \\
 &+ \left. \frac{1}{|\Delta - 1|\Gamma(\vartheta + 1)} + \frac{1}{|\Delta - 1|\Gamma(\vartheta - \beta + 2)} \right] \\
 &= \left( \|\delta_j\|\phi_j^* + \|\mu_j\|\chi_j^* \right) \mathcal{K}_0^*,
 \end{aligned}$$

and

$$\begin{aligned}
 |((\mathcal{T}_1^{(j)} z_j)'(t) + (\mathcal{T}_2^{(j)} q_j)'(t))| &\leq \int_0^t \frac{(t - \xi)^{\vartheta - 2}}{\Gamma(\vartheta - 1)} \left[ |h_j(\xi, z_j(\xi), z_j'(\xi))| + |k_j(\xi, q_j(\xi), q_j'(\xi))| \right] \, d\xi \\
 &+ \frac{1}{|\Delta - 1|} \int_0^1 \frac{(1 - \xi)^{\vartheta - 2}}{\Gamma(\vartheta - 1)} \left[ |h_j(\xi, z_j(\xi), z_j'(\xi))| + |k_j(\xi, q_j(\xi), q_j'(\xi))| \right] \, d\xi \\
 &+ \frac{1}{|\Delta - 1|} \left( \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta - 1}}{\Gamma(\vartheta)} \left[ |h_j(m, z_j(m), z_j'(m))| + |k_j(m, q_j(m), q_j'(m))| \right] \, dm d\xi \right. \\
 &+ \left. \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta - 2}}{\Gamma(\vartheta - 1)} \left[ |h_j(m, z_j(m), z_j'(m))| + |k_j(m, q_j(m), q_j'(m))| \right] \, dm d\xi \right. \\
 &+ \left. \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta - \beta - 1}}{\Gamma(\vartheta - \beta)} \left[ |h_j(m, z_j(m), z_j'(m))| + |k_j(m, q_j(m), q_j'(m))| \right] \, dm d\xi \right) \\
 &\leq \left( \|\delta_j\|\phi_j^* + \|\mu_j\|\chi_j^* \right) \left[ \frac{1}{\Gamma(\vartheta)} + \frac{1}{|\Delta - 1|\Gamma(\vartheta)} + \frac{1}{|\Delta - 1|\Gamma(\vartheta + 2)} \right. \\
 &+ \left. \frac{1}{|\Delta - 1|\Gamma(\vartheta + 1)} + \frac{1}{|\Delta - 1|\Gamma(\vartheta - \beta + 2)} \right] \\
 &= \left( \|\delta_j\|\phi_j^* + \|\mu_j\|\chi_j^* \right) \mathcal{K}_1^*.
 \end{aligned}$$

This yields

$$\|\mathcal{T}_1 z + \mathcal{T}_2 q\|_{\mathcal{M}} = \sum_{j=1}^{13} \|\mathcal{T}_1^{(j)} z_j + \mathcal{T}_2^{(j)} q_j\|_{\mathcal{M}_j} \leq \sum_{j=1}^{13} (\|\delta_j\|\phi_j^* + \|\mu_j\|\chi_j^*) (\mathcal{K}_0^* + \mathcal{K}_1^*) \leq \rho,$$

and so  $\|\mathcal{T}_1 z + \mathcal{T}_2 q\|_{\mathcal{M}} \leq \rho$  and  $\mathcal{T}_1 z + \mathcal{T}_2 q \in \mathcal{B}_\rho$ . The continuity of  $\mathcal{T}_1$  is proved from the continuity of the functions  $h_j$ . We now prove the uniform boundedness of the operator  $\mathcal{T}_1$ . We have

$$\begin{aligned}
 |(\mathcal{T}_1^{(j)} z_j)(t)| &\leq \int_0^t \frac{(t - \xi)^{\vartheta - 1}}{\Gamma(\vartheta)} |h_j(\xi, z_j(\xi), z_j'(\xi))| \, d\xi \\
 &+ \frac{t + |\Delta|}{|\Delta - 1|} \int_0^1 \frac{(1 - \xi)^{\vartheta - 2}}{\Gamma(\vartheta - 1)} |h_j(\xi, z_j(\xi), z_j'(\xi))| \, d\xi
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1-t}{|\Delta-1|} \left[ \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-1}}{\Gamma(\vartheta)} |h_j(m, z_j(m), z'_j(m))| \, dmd\xi \right. \\
 & + \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-2}}{\Gamma(\vartheta-1)} |h_j(m, z_j(m), z'_j(m))| \, dmd\xi \\
 & \left. + \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-\beta-1}}{\Gamma(\vartheta-\beta)} |h_j(m, z_j(m), z'_j(m))| \, dmd\xi \right] \\
 & \leq \|\delta_j\| \phi_j(|z_j(t)| + |z'_j(t)|) \left[ \frac{1+|\Delta|}{|\Delta-1|\Gamma(\vartheta)} + \frac{1}{|\Delta-1|\Gamma(\vartheta+2)} \right. \\
 & \left. + \frac{|\Delta-1|+1}{|\Delta-1|\Gamma(\vartheta+1)} + \frac{1}{|\Delta-1|\Gamma(\vartheta-\beta+2)} \right] \\
 & = \|\delta_j\| \phi_j(\rho) \mathcal{K}_0^*.
 \end{aligned}$$

Further, note that

$$|(\mathcal{T}_1^{(j)} z_j)'(t)| \leq \|\delta_j\| \phi_j(\rho) \mathcal{K}_1^*,$$

for all  $z$  in  $\mathcal{B}_\rho$ . Thus,

$$\|\mathcal{T}_1 z\|_{\mathcal{M}} = \sum_{j=1}^{13} \|\mathcal{T}_1^{(j)} z_j\|_{\mathcal{M}_j} \leq \sum_{j=1}^{13} \|\delta_j\| \phi_j(\rho) (\mathcal{K}_0^* + \mathcal{K}_1^*).$$

So  $\mathcal{T}_1$  is uniformly bounded on  $\mathcal{B}_\rho$ . Now, we are going to show the compactness of the operator  $\mathcal{T}_1$  on  $\mathcal{B}_\rho$ . To do this, let  $0 \leq t_1, t_2 \leq 1$  with  $t_1 < t_2$ ; therefore, we conclude

$$\begin{aligned}
 |(\mathcal{T}_1^{(j)} z_j)(t_2) - (\mathcal{T}_1^{(j)} z_j)(t_1)| & \leq \int_0^{t_1} \frac{[(t_2-\xi)^{\vartheta-1} - (t_1-\xi)^{\vartheta-1}]}{\Gamma(\vartheta)} |h_j(\xi, z_j(\xi), z'_j(\xi))| \, d\xi \\
 & + \int_{t_1}^{t_2} \frac{(t_2-\xi)^{\vartheta-1}}{\Gamma(\vartheta)} |h_j(\xi, z_j(\xi), z'_j(\xi))| \, d\xi \\
 & + \frac{(t_2-t_1)}{|\Delta-1|} \int_0^1 \frac{(1-\xi)^{\vartheta-2}}{\Gamma(\vartheta-1)} |h_j(\xi, z_j(\xi), z'_j(\xi))| \, d\xi \\
 & + \frac{(t_2-t_1)}{|\Delta-1|} \left[ \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-1}}{\Gamma(\vartheta)} |h_j(m, z_j(m), z'_j(m))| \, dmd\xi \right. \\
 & + \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-2}}{\Gamma(\vartheta-1)} |h_j(m, z_j(m), z'_j(m))| \, dmd\xi \\
 & \left. + \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-\beta-1}}{\Gamma(\vartheta-\beta)} |h_j(m, z_j(m), z'_j(m))| \, dmd\xi \right] \\
 & \leq \frac{\|\delta_j\| \phi_j(\rho)}{\Gamma(\vartheta+1)} [- (t_2-t_1)^\vartheta + (t_2-t_1)^\vartheta + t_2^\vartheta - t_1^\vartheta] + \frac{\|\delta_j\| \phi_j(\rho) (t_2-t_1)}{|\Delta-1|\Gamma(\vartheta)} \\
 & + \frac{\|\delta_j\| \phi_j(\rho) (t_2-t_1)}{|\Delta-1|} \left[ \frac{1}{\Gamma(\vartheta+2)} + \frac{1}{\Gamma(\vartheta+1)} + \frac{1}{\Gamma(\vartheta-\beta+2)} \right].
 \end{aligned}$$

Hence,  $|(\mathcal{T}_1^{(j)} z_j)(t_2) - (\mathcal{T}_1^{(j)} z_j)(t_1)| \rightarrow 0$  as  $t_1 \rightarrow t_2$ . Similarly, one can see that

$$|(\mathcal{T}_1^{(j)} z_j)'(t_2) - (\mathcal{T}_1^{(j)} z_j)'(t_1)| \rightarrow 0,$$

as  $t_1 \rightarrow t_2$ ; therefore, we obtain  $\|(\mathcal{T}_1 z_j)(t_2) - (\mathcal{T}_1 z_j)(t_1)\|_{\mathcal{X}}$  tends to zero as  $t_1 \rightarrow t_2$ . Thus,  $\mathcal{T}_1$  will be equi-continuous, and is compact on  $\mathcal{B}_\rho$  due to the Arzela–Ascoli property. Further, take  $z, q \in \mathcal{B}_\rho$ . Therefore

$$\begin{aligned} |(\mathcal{T}_2^{(j)} z_j)(t) - (\mathcal{T}_2^{(j)} q_j)(t)| &\leq \int_0^t \frac{(t-\xi)^{\vartheta-1}}{\Gamma(\vartheta)} |k_j(\xi, z_j(\xi), z_j'(\xi)) - k_j(\xi, q_j(\xi), q_j'(\xi))| d\xi \\ &+ \frac{t+|\Delta|}{|\Delta-1|} \int_0^1 \frac{(1-\xi)^{\vartheta-2}}{\Gamma(\vartheta-1)} |k_j(\xi, z_j(\xi), z_j'(\xi)) - k_j(\xi, q_j(\xi), q_j'(\xi))| d\xi \\ &+ \frac{1-t}{|\Delta-1|} \left( \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-1}}{\Gamma(\vartheta)} |k_j(m, z_j(m), z_j'(m)) - k_j(m, q_j(m), q_j'(m))| dm d\xi \right. \\ &+ \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-2}}{\Gamma(\vartheta-1)} |k_j(m, z_j(m), z_j'(m)) - k_j(m, q_j(m), q_j'(m))| dm d\xi \\ &+ \left. \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-\beta-1}}{\Gamma(\vartheta-\beta)} |k_j(m, z_j(m), z_j'(m)) - k_j(m, q_j(m), q_j'(m))| dm d\xi \right) \\ &\leq \int_0^t \frac{(t-\xi)^{\vartheta-1}}{\Gamma(\vartheta)} \sigma_j(\xi) (|z_j(\xi) - q_j(\xi)| + |z_j'(\xi) - q_j'(\xi)|) d\xi \\ &+ \frac{t+|\Delta|}{|\Delta-1|} \int_0^1 \frac{(1-\xi)^{\vartheta-2}}{\Gamma(\vartheta-1)} \sigma_j(\xi) (|z_j(\xi) - q_j(\xi)| + |z_j'(\xi) - q_j'(\xi)|) d\xi \\ &+ \frac{1-t}{|\Delta-1|} \left( \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-1}}{\Gamma(\vartheta)} \sigma_j(m) (|z_j(m) - q_j(m)| + |z_j'(m) - q_j'(m)|) dm d\xi \right. \\ &+ \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-2}}{\Gamma(\vartheta-1)} \sigma_j(m) (|z_j(m) - q_j(m)| + |z_j'(m) - q_j'(m)|) dm d\xi \\ &+ \left. \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-\beta-1}}{\Gamma(\vartheta-\beta)} \sigma_j(m) (|z_j(m) - q_j(m)| + |z_j'(m) - q_j'(m)|) dm d\xi \right) \\ &\leq \|\sigma_j\| \mathcal{K}_0^* \|z_j - q_j\|_{\mathcal{M}_j}, \end{aligned}$$

and in the similar manner,

$$|(\mathcal{T}_2^{(j)} z_j)'(t) - (\mathcal{T}_2^{(j)} q_j)'(t)| \leq \|\sigma_j\| \mathcal{K}_1^* \|z_j - q_j\|_{\mathcal{M}_j}.$$

Therefore, we have that

$$\|\mathcal{T}_2 z - \mathcal{T}_2 q\|_{\mathcal{M}} = \sum_{j=1}^{13} \|\mathcal{T}_2^{(j)} z_j - \mathcal{T}_2^{(j)} q_j\|_{\mathcal{M}_j} \leq (\mathcal{K}_0^* + \mathcal{K}_1^*) \sum_{j=1}^{13} \|\sigma_j\| \|z_j - q_j\|_{\mathcal{M}_j},$$

and so

$$\|\mathcal{T}_2 z - \mathcal{T}_2 q\|_{\mathcal{M}} \leq \lambda^* \|z - q\|_{\mathcal{M}}.$$

Since  $\lambda^* < 1$ ,  $\mathcal{T}_2$  is a contraction on  $\mathcal{B}_\rho$ . Utilizing Lemma 1, we deduced that a fixed point of  $\mathcal{T}$  exists that satisfies the fractional BVPs (2).  $\square$

In order to demonstrate the validity of the aforementioned conclusion, we give an abstract example.

**Example 1.** Consider the multi-term multi-dimensional DEs

$$\begin{cases} {}^c\mathcal{D}_0^{1.38} z_1(t) = h_1(t, z_1(t), z'_1(t)) + k_1(t, z_1(t), z'_1(t)), \\ {}^c\mathcal{D}_0^{1.38} z_2(t) = h_2(t, z_2(t), z'_2(t)) + k_2(t, z_2(t), z'_2(t)), \\ {}^c\mathcal{D}_0^{1.38} z_3(t) = h_3(t, z_3(t), z'_3(t)) + k_3(t, z_3(t), z'_3(t)), \\ {}^c\mathcal{D}_0^{1.38} z_4(t) = h_4(t, z_4(t), z'_4(t)) + k_4(t, z_4(t), z'_4(t)), \end{cases} \tag{12}$$

under the integral BCs

$$\begin{cases} z_j(0) + {}^c\mathcal{D}_0^{0.47} z_j(0) + z'_j(1) = 0, \quad (j = 1, 2, 3, 4) \\ \int_0^1 [z_j(\xi) + {}^c\mathcal{D}_0^{0.47} z_j(\xi) + z'_j(\xi)] d\xi = 0, \end{cases} \tag{13}$$

where  $\vartheta = 1.38$ ,  $\beta = 0.47$ ,  $n = 4$ , and  ${}^c\mathcal{D}_0^{1.38}$  and  ${}^c\mathcal{D}_0^{0.47}$  are derivatives of order  $\gamma \in \{1.38, 0.47\}$  of the Caputo type. Set  $h_j \in C([0, 1] \times \mathbb{R}^3, \mathbb{R})$ ,  $(j = 1, \dots, 4)$  by

$$\begin{cases} h_1(t, z(t), \ell(t)) = \frac{|\sin z(t)|}{2+t} + \frac{|\ell(t)|}{t+2}, \\ h_2(t, z(t), \ell(t)) = \frac{|z(t)|}{5+5|z(t)|} + \frac{0.4|\sin \ell(t)|}{2+2|\sin \ell(t)|}, \\ h_3(t, z(t), \ell(t)) = \frac{(t+1)}{700} |\arctan z(t)| + \frac{(t+1)|\ell(t)|}{700+700|\ell(t)|}, \\ h_4(t, z(t), \ell(t)) = \frac{e^t}{8} |z(t)| + 0.125|\ell(t)|e^t, \end{cases}$$

and also  $k_j \in C([0, 1] \times \mathbb{R}^3, \mathbb{R})$ ,  $(j = 1, \dots, 4)$  by

$$\begin{cases} k_1(t, z(t), \ell(t)) = \frac{t}{1000} \left( \frac{|z(t)|}{|z(t)|+1} + |\sin \ell(t)| \right), \\ k_2(t, z(t), \ell(t)) = \frac{|\sin z(t)|}{200+t} + \frac{|\arctan \ell(t)|}{200+t}, \\ k_3(t, z(t), \ell(t)) = 0.01t|z(t)| + \frac{t}{100} |\arctan \ell(t)|, \\ k_4(t, z(t), \ell(t)) = \frac{4t}{100} \frac{|z(t)|}{|z(t)|+1} + 0.04t \frac{|\ell(t)|}{1+|\ell(t)|}. \end{cases}$$

Define  $\phi_j : \mathbb{R}^+ \rightarrow \mathbb{R}$  as  $\phi_m(z) = z$  for all  $z \in \mathbb{R}^+$  and  $j = 1, \dots, 4$ . Further, we know that for each nonnegative real number  $z$ ,  $\sin z \leq z$  and  $\arctan z \leq z$ ; therefore, we have

$$|h_1(t, z(t), z'(t))| \leq \frac{1}{2+t} (|\sin z| + |z'|)$$

$$\begin{aligned}
 &\leq \frac{1}{2+t}(|z| + |z'|) = \frac{1}{2+t}\phi_1(|z| + |z'|), \\
 |h_2(t, z(t), z'(t))| &\leq \frac{1}{5}(|z| + |\sin z'|) \\
 &\leq \frac{1}{5}(|z| + |z'|) = \frac{1}{5}\phi_2(|z| + |z'|), \\
 |h_3(t, z(t), z'(t))| &\leq \frac{t+1}{700}(|\arctan z| + |z'|) \\
 &\leq \frac{t+1}{700}(|z| + |z'|) = \frac{t+1}{700}\phi_3(|z| + |z'|), \\
 |h_4(t, z(t), z'(t))| &\leq \frac{e^t}{8}(|z| + |z'|) \\
 &\leq \frac{e^t}{8}(|z| + |z'|) = \frac{e^t}{8}\phi_4(|z| + |z'|),
 \end{aligned}$$

in which for  $j = 1, \dots, 4$ ,  $\delta_j \in C([0, 1], \mathbb{R}^+)$  are given as  $\delta_1(t) = \frac{1}{2+t}$ ,  $\delta_2(t) = \frac{1}{5}$ ,  $\delta_3(t) = \frac{t+1}{700}$ , and  $\delta_4(t) = \frac{1}{8}e^t$ . Further, we define nondecreasing and continuous mappings  $\chi_j : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  by  $\chi_j(z) = z$  for all  $z \in \mathbb{R}^+$  and  $j = 1, \dots, 4$ . Then we obtain

$$\begin{aligned}
 |k_1(t, z(t), z'(t))| &\leq \frac{t}{1000}(|z| + |\sin z'|) \\
 &\leq \frac{t}{1000}(|z| + |z'|) = \frac{t}{1000}\chi_1(|z| + |z'|), \\
 |k_2(t, z(t), z'(t))| &\leq \frac{1}{200+t}(|\sin z| + |\arctan z'|) \\
 &\leq \frac{1}{200+t}(|z| + |z'|) = \frac{1}{200+t}\chi_2(|z| + |z'|), \\
 |k_3(t, z(t), z'(t))| &\leq \frac{t}{100}(|z| + |\arctan z'|) \\
 &\leq \frac{t}{100}(|z| + |z'|) = \frac{t}{100}\chi_3(|z| + |z'|), \\
 |k_4(t, z(t), z'(t))| &\leq \frac{4t}{100}(|z| + |z'|) \\
 &\leq \frac{4t}{100}(|z| + |z'|) = \frac{4t}{100}\chi_4(|z| + |z'|),
 \end{aligned}$$

in which for  $j = 1, \dots, 4$ ,  $\mu_j \in C([0, 1], \mathbb{R}^+)$  are given as  $\mu_1(t) = \frac{t}{1000}$ ,  $\mu_2(t) = \frac{1}{200+t}$ ,  $\mu_3(t) = \frac{t}{100}$ , and  $\mu_4(t) = \frac{4t}{100}$ . On the other hand, by the mean value theorem, we know that  $|\sin a - \sin b| \leq |a - b|$  and  $|\arctan a - \arctan b| \leq |a - b|$  for all  $a, b \in \mathbb{R}$ . Thus, for  $z_1, z_2, \ell_1, \ell_2 \in \mathbb{R}$ , we estimate

$$\begin{aligned}
 &|k_1(t, z_1(t), z_2(t)) - k_1(t, \ell_1(t), \ell_2(t))| \\
 &\leq \frac{t}{1000}(|z_1(t) - \ell_1(t)| + |\sin z_2(t) - \sin \ell_2(t)|)
 \end{aligned}$$

$$\leq \frac{t}{1000} (|z_1(t) - \ell_1(t)| + |z_2(t) - \ell_2(t)|),$$

$$\begin{aligned} &|k_2(t, z_1(t), z_2(t)) - k_2(t, \ell_1(t), \ell_2(t))| \\ &\leq \frac{1}{200+t} (|\sin z_1(t) - \sin \ell_1(t)| + |\arctan z_2(t) - \arctan \ell_2(t)|) \\ &\leq \frac{1}{200+t} (|z_1(t) - \ell_1(t)| + |z_2(t) - \ell_2(t)|), \end{aligned}$$

$$\begin{aligned} &|k_3(t, z_1(t), z_2(t)) - k_3(t, \ell_1(t), \ell_2(t))| \\ &\leq \frac{t}{100} (|z_1(t) - \ell_1(t)| + |\arctan z_2(t) - \arctan \ell_2(t)|) \\ &\leq \frac{t}{100} (|z_1(t) - \ell_1(t)| + |z_2(t) - \ell_2(t)|), \end{aligned}$$

and

$$\begin{aligned} &|k_4(t, z_1(t), z_2(t)) - k_4(t, \ell_1(t), \ell_2(t))| \\ &\leq \frac{4t}{100} (|z_1(t) - \ell_1(t)| + |z_2(t) - \ell_2(t)|) \\ &\leq \frac{4t}{100} (|z_1(t) - \ell_1(t)| + |z_2(t) - \ell_2(t)|). \end{aligned}$$

Hence,  $\sigma_1(t) = \frac{t}{1000}$ ,  $\sigma_2(t) = \frac{1}{200+t}$ ,  $\sigma_3(t) = \frac{t}{100}$ , and  $\sigma_4(t) = \frac{4t}{100}$ , where  $\|\sigma_1\| = \frac{1}{1000}$ ,  $\|\sigma_2\| = \frac{1}{200}$ ,  $\|\sigma_3\| = \frac{1}{100}$ ,  $\|\sigma_4\| = \frac{4}{100}$ , and so  $\sum_{j=1}^4 \|\sigma_j\| \simeq 0.056$ . According to the obtained values, we obtain  $\Delta \simeq 2.2364$ ,  $\mathcal{K}_0^* \simeq 5.1371$ ,  $\mathcal{K}_1^* \simeq 3.4112$ , and so  $\mathcal{K}_0^* + \mathcal{K}_1^* \simeq 8.5483$ . Hence,

$$\lambda^* := (\mathcal{K}_0^* + \mathcal{K}_1^*) \sum_{j=1}^4 \|\sigma_j\| = (\mathcal{K}_0^* + \mathcal{K}_1^*) (\|\sigma_1\| + \|\sigma_2\| + \|\sigma_3\| + \|\sigma_4\|) \simeq 0.4787048 < 1.$$

Application of Theorem 1 leads us to infer that the fractional BVPs (12) and (13) possesses a solution.

In the following, we investigate a special case of the fractional BVPs (2) on the methylpropane graph (Figure 6). Assume that  $k_j(t, z_j(t), z'_j(t)) = 0$  are constant functions. Then the fractional problems (2) are converted into the following BVPs on the methylpropane graph under the integral BCs

$$\begin{cases} {}^c\mathcal{D}_0^\theta z_j(t) = h_j(t, z_j(t), z'_j(t)), & (j \in \mathbb{N}_1^3) \\ z_j(0) + {}^c\mathcal{D}_0^\beta z_j(0) + z'_j(1) = 0, \\ \int_0^1 [z_j(\xi) + {}^c\mathcal{D}_0^\beta z_j(\xi) + z'_j(\xi)] d\xi = 0, \end{cases} \tag{14}$$



where  $\vartheta \in (1, 2)$ , and  $h_j : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  are continuous functions. By utilizing the Schaefer’s theorem, we prove our next existence criterion. Before that, with adequate regard for Lemma 3, define  $Y$  on  $\mathcal{M}$  as

$$Y(z_1, z_2, \dots, z_{13})(t) := \left( (Y_1 z_1)(t), \dots, (Y_{13} z_{13})(t) \right), \tag{15}$$

where

$$\begin{aligned} (Y_j z_j)(t) = & \int_0^t \frac{(t - \xi)^{\vartheta-1}}{\Gamma(\vartheta)} h_j(\xi, z_j(\xi), z'_j(\xi)) \, d\xi \\ & + \frac{t - \Delta}{\Delta - 1} \int_0^1 \frac{(1 - \xi)^{\vartheta-2}}{\Gamma(\vartheta - 1)} h_j(\xi, z_j(\xi), z'_j(\xi)) \, d\xi \\ & + \frac{1 - t}{\Delta - 1} \left[ \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta-1}}{\Gamma(\vartheta)} h_j(m, z_j(m), z'_j(m)) \, dm d\xi \right. \\ & + \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta-2}}{\Gamma(\vartheta - 1)} h_j(m, z_j(m), z'_j(m)) \, dm d\xi \\ & \left. + \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta-\beta-1}}{\Gamma(\vartheta - \beta)} h_j(m, z_j(m), z'_j(m)) \, dm d\xi \right], \end{aligned}$$

for all  $t \in [0, 1]$ ,  $z_j \in \mathcal{M}_j$ , and  $j \in \mathbb{N}_1^{13}$ . In the following, the existence property for FBVPs (14) are proved under the boundedness assumption for the continuous functions  $h_j$ .

**Theorem 2.** *Regard the continuous functions  $h_1, \dots, h_{13} : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ . Assume that the constants  $\varrho_j > 0$  exist with  $|h_j(t, z_1, z_2)| \leq \varrho_j$  for all  $z_1, z_2 \in \mathbb{R}$ , ( $j \in \mathbb{N}_1^{13}$ ) and  $\forall t \in [0, 1]$ . Then the FBVPs (14) have solutions on each edge of the methylpropane graph.*

**Proof.** Regarding the definition of operator  $Y$ , clearly, the fractional BVPs (14) have solutions if and only if there exist a fixed point of  $Y$  on  $\mathcal{M} = \mathcal{M}_1 \times \dots \times \mathcal{M}_{13}$ . Initially, we check that  $Y$  is complete continuous. Continuity of  $h_1, \dots, h_{13}$  implies that of  $Y : \mathcal{M} \rightarrow \mathcal{M}$ . Assume a bounded set  $\mathcal{O}$  in  $\mathcal{M}$  and  $z_j \in \mathcal{M}_j$ . Then we have

$$\begin{aligned} |(Y_j z_j)(t)| \leq & \int_0^t \frac{(t - \xi)^{\vartheta-1}}{\Gamma(\vartheta)} |h_j(\xi, z_j(\xi), z'_j(\xi))| \, d\xi \\ & + \frac{t + |\Delta|}{|\Delta - 1|} \int_0^1 \frac{(1 - \xi)^{\vartheta-2}}{\Gamma(\vartheta - 1)} |h_j(\xi, z_j(\xi), z'_j(\xi))| \, d\xi \\ & + \frac{1 - t}{|\Delta - 1|} \left[ \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta-1}}{\Gamma(\vartheta)} |h_j(m, z_j(m), z'_j(m))| \, dm d\xi \right. \\ & + \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta-2}}{\Gamma(\vartheta - 1)} |h_j(m, z_j(m), z'_j(m))| \, dm d\xi \\ & \left. + \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta-\beta-1}}{\Gamma(\vartheta - \beta)} |h_j(m, z_j(m), z'_j(m))| \, dm d\xi \right] \end{aligned}$$

$$\begin{aligned} &\leq \varrho_j \left[ \frac{1 + |\Delta|}{|\Delta - 1|\Gamma(\vartheta)} + \frac{1}{|\Delta - 1|\Gamma(\vartheta + 2)} + \frac{|\Delta - 1| + 1}{|\Delta - 1|\Gamma(\vartheta + 1)} + \frac{1}{|\Delta - 1|\Gamma(\vartheta - \beta + 2)} \right] \\ &= \varrho_j \mathcal{K}_0^*. \end{aligned}$$

$\forall t \in [0, 1]$ , and  $\mathcal{K}_0^*$  is introduced in (10). Similarly,

$$|(Y_j z_j)'(t)| \leq \varrho_j \left[ \frac{|\Delta - 1| + 1}{|\Delta - 1|\Gamma(\vartheta)} + \frac{1}{|\Delta - 1|\Gamma(\vartheta + 1)} + \frac{1}{|\Delta - 1|\Gamma(\vartheta - \beta + 2)} + \frac{1}{|\Delta - 1|\Gamma(\vartheta + 2)} \right] = \varrho_j \mathcal{K}_1^*,$$

$\forall t \in [0, 1]$ , in which  $\mathcal{K}_1^*$  is as (10). Thus  $\|Y_j z_j(t)\|_{\mathcal{M}_j} \leq \varrho_j(\mathcal{K}_0^* + \mathcal{K}_1^*)$ , and  $\|Yz(t)\|_{\mathcal{M}} = \sum_{j=1}^{13} \|Y_j z_j(t)\|_{\mathcal{M}_j} \leq \sum_{j=1}^{13} \varrho_j(\mathcal{K}_0^* + \mathcal{K}_1^*) < \infty$ . From this we deduced that  $Y$  is uniformly bounded. To confirm the equi-continuity of operator  $Y$ , assume  $z = (z_1, z_2, \dots, z_{13}) \in \mathcal{O}$  and  $0 \leq t_1, t_2 \leq 1$  with  $t_1 < t_2$ . Then

$$\begin{aligned} |(Y_j z_j)(t_2) - (Y_j z_j)(t_1)| &\leq \int_0^{t_1} \frac{[(t_2 - \xi)^{\vartheta-1} - (t_1 - \xi)^{\vartheta-1}]}{\Gamma(\vartheta)} |h_j(\xi, z_j(\xi), z_j'(\xi))| d\xi \\ &\quad + \int_{t_1}^{t_2} \frac{(t_2 - \xi)^{\vartheta-1}}{\Gamma(\vartheta)} |h_j(\xi, z_j(\xi), z_j'(\xi))| d\xi \\ &\quad + \frac{(t_2 - t_1)}{|\Delta - 1|} \int_0^1 \frac{(1 - \xi)^{\vartheta-2}}{\Gamma(\vartheta - 1)} |h_j(\xi, z_j(\xi), z_j'(\xi))| d\xi \\ &\quad + \frac{(t_2 - t_1)}{|\Delta - 1|} \left[ \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta-1}}{\Gamma(\vartheta)} |h_j(m, z_j(m), z_j'(m))| dm d\xi \right. \\ &\quad \left. + \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta-2}}{\Gamma(\vartheta - 1)} |h_j(m, z_j(m), z_j'(m))| dm d\xi \right. \\ &\quad \left. + \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta-\beta-1}}{\Gamma(\vartheta - \beta)} |h_j(m, z_j(m), z_j'(m))| dm d\xi \right] \\ &\leq \frac{\varrho_j}{\Gamma(\vartheta + 1)} \left[ -(t_2 - t_1)^\vartheta + (t_2 - t_1)^\vartheta + t_2^\vartheta - t_1^\vartheta \right] + \frac{\varrho_j(t_2 - t_1)}{|\Delta - 1|\Gamma(\vartheta)} \\ &\quad + \frac{\varrho_j(t_2 - t_1)}{|\Delta - 1|} \left[ \frac{1}{\Gamma(\vartheta + 2)} + \frac{1}{\Gamma(\vartheta + 1)} + \frac{1}{\Gamma(\vartheta - \beta + 2)} \right]. \end{aligned}$$

The right-side converges to zero (no depending upon  $z \in \mathcal{O}$ ) as  $t_1 \rightarrow t_2$ . Similarly, we have

$$|(Y_j z_j)'(t_2) - (Y_j z_j)'(t_1)| \rightarrow 0,$$

as  $t_1 \rightarrow t_2$ . Hence,  $\|Yz(t_2) - Yz(t_1)\|_{\mathcal{M}} \rightarrow 0$  as  $t_1 \rightarrow t_2$ . This gives that  $Y$  is equi-continuous on  $\mathcal{M}$ . Hence  $Y$  is completely continuous by Arzela–Ascoli theorem. Assume a subset

$$\Omega := \{(z_1, z_2, \dots, z_{13}) \in \mathcal{M} : (z_1, z_2, \dots, z_{13}) = vY(z_1, z_2, \dots, z_{13}), v \in (0, 1)\}$$

of  $\mathcal{M}$ . We prove the boundedness of  $\Omega$ . Let  $(z_1, z_2, \dots, z_{13}) \in \Omega$ . Then

$$(z_1, z_2, \dots, z_{13}) = vY(z_1, z_2, \dots, z_{13}),$$

and so  $z_j(t) = \nu(Y_j z_j)$  for all  $t \in [0, 1]$  and  $j \in \mathbb{N}_1^{13}$ . Thus,

$$\begin{aligned}
 |z_j(t)| &\leq \nu \left( \int_0^t \frac{(t-\xi)^{\vartheta-1}}{\Gamma(\vartheta)} |h_j(\xi, z_j(\xi), z'_j(\xi))| d\xi \right. \\
 &\quad + \frac{t+|\Delta|}{|\Delta-1|} \int_0^1 \frac{(1-\xi)^{\vartheta-2}}{\Gamma(\vartheta-1)} |h_j(\xi, z_j(\xi), z'_j(\xi))| d\xi \\
 &\quad + \frac{1-t}{|\Delta-1|} \left[ \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-1}}{\Gamma(\vartheta)} |h_j(m, z_j(m), z'_j(m))| dm d\xi \right. \\
 &\quad + \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-2}}{\Gamma(\vartheta-1)} |h_j(m, z_j(m), z'_j(m))| dm d\xi \\
 &\quad \left. + \int_0^1 \int_0^\xi \frac{(\xi-m)^{\vartheta-\beta-1}}{\Gamma(\vartheta-\beta)} |h_j(m, z_j(m), z'_j(m))| dm d\xi \right] \Big) \\
 &\leq \nu \varrho_j \left[ \frac{1+|\Delta|}{|\Delta-1|\Gamma(\vartheta)} + \frac{1}{|\Delta-1|\Gamma(\vartheta+2)} + \frac{|\Delta-1|+1}{|\Delta-1|\Gamma(\vartheta+1)} + \frac{1}{|\Delta-1|\Gamma(\vartheta-\beta+2)} \right] \\
 &= \nu \varrho_j \mathcal{K}_0^*,
 \end{aligned}$$

and

$$|z'_j(t)| \leq \nu \varrho_j \left[ \frac{|\Delta-1|+1}{|\Delta-1|\Gamma(\vartheta)} + \frac{1}{|\Delta-1|\Gamma(\vartheta+2)} + \frac{1}{|\Delta-1|\Gamma(\vartheta-\beta+2)} + \frac{1}{|\Delta-1|\Gamma(\vartheta+1)} \right] = \nu \varrho_j \mathcal{K}_1^*.$$

These inequalities follow that

$$\|z\|_{\mathcal{M}} = \sum_{j=1}^{13} \|z_j\|_{\mathcal{M}_j} \leq \nu \sum_{j=1}^{13} \varrho_j (\mathcal{K}_0^* + \mathcal{K}_1^*) < \infty,$$

therefore  $\Omega$  is bounded. Utilizing Lemmas 2 and 3, we find out that in  $\mathcal{M}$ ,  $Y$  possesses a fixed point, which is a solution of (14).  $\square$

#### 4. Ulam–Hyers Stability

Since the concept of stability is so important in determining the solutions to many dynamical systems, due to this importance, we here establish two Ulam–Hyers and the generalized version of stabilities in relation to the fractional multi-dimensional BVP on the methylpropane graph (14). For more information, see [38–43]. In consistence with [41], we define the following notions:

**Definition 1 ([41]).** *The fractional multi-dimensional BVP on the methylpropane graph (14) is Ulam–Hyers stable if  $\exists d_{h_j}^* \in \mathbb{R}_+$  s.t.  $\forall \varepsilon_j > 0$  and  $\forall z_j^*(t) \in C([0, 1], \mathbb{R})$  as a solution function satisfying*

$$\left| {}^c \mathcal{D}_0^\vartheta z_j^*(t) - h_j(t, z_j^*(t), z_j^{*'}(t)) \right| < \varepsilon_j, \tag{16}$$

$\exists z_j(t) \in C([0, 1], \mathbb{R})$  fulfilling the multi-dimensional BVP on the methylpropane graph (14) with

$$|z_j^*(t) - z_j(t)| \leq \varepsilon_j d_{h_j}^*, \quad t \in [0, 1].$$

**Definition 2 ([41]).** The fractional multi-dimensional BVP on the methylpropane graph (14) is generalized Ulam–Hyers stable if  $\exists G_j \in C(\mathbb{R}^+, \mathbb{R}^+)$  with  $G_j(0) = 0$  s.t.  $\forall \varepsilon_j > 0$  and  $\forall z_j^*(t) \in C([0, 1], \mathbb{R})$  fulfilling

$$\left| {}^c\mathcal{D}_0^\vartheta z_j^*(t) - h_j(t, z_j^*(t), z_j^{*\prime}(t)) \right| < \varepsilon_j,$$

$\exists z_j(t) \in C([0, 1], \mathbb{R})$  satisfying the multi-dimensional BVP on the methylpropane graph (14) with

$$|z_j^*(t) - z_j(t)| \leq G_j(\varepsilon_j), \quad t \in [0, 1].$$

**Remark 1.** Definition 1  $\Rightarrow$  Definition 2.

**Remark 2.**  $z_j^*(t) \in C([0, 1], \mathbb{R})$  is named as a solution for (16) iff  $\exists H_j \in C([0, 1], \mathbb{R})$  depending on  $z_j^*$  s.t.

- (1)  $|H_j(t)| < \varepsilon_j, t \in [0, 1].$
- (2)  ${}^c\mathcal{D}_0^\vartheta z_j^*(t) = h_j(t, z_j^*(t), z_j^{*\prime}(t)) + H_j(t), t \in [0, 1].$

We now study the above stabilities for the fractional multi-dimensional BVP on the methylpropane graph (14).

**Theorem 3.** Assume that the assertion

$$|h_j(t, z_j^*(t), z_j^{*\prime}(t)) - h_j(t, z_j(t), z_j'(t))| \leq \mathcal{L}_j^* (|z_j^* - z_j| + |z_j^{*\prime} - z_j'|),$$

for some  $\mathcal{L}_j^* > 0$ . Then the fractional multi-dimensional BVP on methylpropane graph (14) is Ulam–Hyers stable on  $[0, 1]$  and is generalized Ulam–Hyers stable if

$$\mathcal{L}_j^* \mathcal{K}_0^* < 1, \quad j \in \mathbb{N}_1^{13},$$

for which  $\mathcal{K}_0^*$  is given in (10).

**Proof.**  $\forall \varepsilon_j > 0$  and  $\forall z_j^* \in C([0, 1], \mathbb{R})$  fulfilling

$$\left| {}^c\mathcal{D}_0^\vartheta z_j^*(t) - h_j(t, z_j^*(t), z_j^{*\prime}(t)) \right| < \varepsilon_j,$$

one can find  $H_j \in C([0, 1], \mathbb{R})$ , which satisfies

$${}^c\mathcal{D}_0^\vartheta z_j^*(t) = h_j(t, z_j^*(t), z_j^{*\prime}(t)) + H_j(t),$$

with  $|H_j(t)| \leq \varepsilon_j$ . It gives

$$\begin{aligned} z_j^*(t) &= \int_0^t \frac{(t - \xi)^{\vartheta-1}}{\Gamma(\vartheta)} \left( h_j(\xi, z_j^*(\xi), z_j^{*\prime}(\xi)) + H_j(\xi) \right) d\xi \\ &+ \frac{t - \Delta}{\Delta - 1} \int_0^1 \frac{(1 - \xi)^{\vartheta-2}}{\Gamma(\vartheta - 1)} \left( h_j(\xi, z_j^*(\xi), z_j^{*\prime}(\xi)) + H_j(\xi) \right) d\xi \\ &+ \frac{1 - t}{\Delta - 1} \left[ \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta-1}}{\Gamma(\vartheta)} \left( h_j(m, z_j^*(m), z_j^{*\prime}(m)) + H_j(m) \right) dm d\xi \right. \\ &\left. + \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta-2}}{\Gamma(\vartheta - 1)} \left( h_j(m, z_j^*(m), z_j^{*\prime}(m)) + H_j(m) \right) dm d\xi \right] \end{aligned}$$

$$+ \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta - \beta - 1}}{\Gamma(\vartheta - \beta)} \left( h_j(m, z_j^*(m), z_j^{*'}(m)) + H_j(m) \right) dm d\xi \Big].$$

Consider  $z_j(t) \in C(J, \mathbb{R})$  to be the solution of (14), which is

$$\begin{aligned} z_j(t) = & \int_0^t \frac{(t - \xi)^{\vartheta - 1}}{\Gamma(\vartheta)} h_j(\xi, z_j(\xi), z_j'(\xi)) d\xi + \frac{t - \Delta}{\Delta - 1} \int_0^1 \frac{(1 - \xi)^{\vartheta - 2}}{\Gamma(\vartheta - 1)} h_j(\xi, z_j(\xi), z_j'(\xi)) d\xi \\ & + \frac{1 - t}{\Delta - 1} \left[ \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta - 1}}{\Gamma(\vartheta)} h_j(m, z_j(m), z_j'(m)) dm d\xi \right. \\ & + \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta - 2}}{\Gamma(\vartheta - 1)} h_j(m, z_j(m), z_j'(m)) dm d\xi \\ & \left. + \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta - \beta - 1}}{\Gamma(\vartheta - \beta)} h_j(m, z_j(m), z_j'(m)) dm d\xi \right]. \end{aligned}$$

We estimate

$$\begin{aligned} |z_j^*(t) - z_j(t)| \leq & \int_0^t \frac{(t - \xi)^{\vartheta - 1}}{\Gamma(\vartheta)} |h_j(\xi, z_j^*(\xi), z_j^{*'}(\xi)) - h_j(\xi, z_j(\xi), z_j'(\xi))| d\xi \\ & + \frac{t + |\Delta|}{|\Delta - 1|} \int_0^1 \frac{(1 - \xi)^{\vartheta - 2}}{\Gamma(\vartheta - 1)} |h_j(\xi, z_j^*(\xi), z_j^{*'}(\xi)) - h_j(\xi, z_j(\xi), z_j'(\xi))| d\xi \\ & + \frac{1 - t}{|\Delta - 1|} \left[ \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta - 1}}{\Gamma(\vartheta)} |h_j(m, z_j^*(m), z_j^{*'}(m)) - h_j(m, z_j(m), z_j'(m))| dm d\xi \right. \\ & + \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta - 2}}{\Gamma(\vartheta - 1)} |h_j(m, z_j^*(m), z_j^{*'}(m)) - h_j(m, z_j(m), z_j'(m))| dm d\xi \\ & \left. + \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta - \beta - 1}}{\Gamma(\vartheta - \beta)} |h_j(m, z_j^*(m), z_j^{*'}(m)) - h_j(m, z_j(m), z_j'(m))| dm d\xi \right] \\ & + \int_0^t \frac{(t - \xi)^{\vartheta - 1}}{\Gamma(\vartheta)} |H_j(\xi)| d\xi + \frac{t + |\Delta|}{|\Delta - 1|} \int_0^1 \frac{(1 - \xi)^{\vartheta - 2}}{\Gamma(\vartheta - 1)} |H_j(\xi)| d\xi \\ & + \frac{1 - t}{|\Delta - 1|} \left[ \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta - 1}}{\Gamma(\vartheta)} |H_j(m)| dm d\xi + \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta - 2}}{\Gamma(\vartheta - 1)} |H_j(m)| dm d\xi \right. \\ & \left. + \int_0^1 \int_0^\xi \frac{(\xi - m)^{\vartheta - \beta - 1}}{\Gamma(\vartheta - \beta)} |H_j(m)| dm d\xi \right] \\ \leq & \mathcal{L}_j^* \|z_j^* - z_j\| \left[ \frac{1 + |\Delta|}{|\Delta - 1| \Gamma(\vartheta)} + \frac{1}{|\Delta - 1| \Gamma(\vartheta + 2)} \right. \\ & \left. + \frac{|\Delta - 1| + 1}{|\Delta - 1| \Gamma(\vartheta + 1)} + \frac{1}{|\Delta - 1| \Gamma(\vartheta - \beta + 2)} \right] \end{aligned}$$

$$\begin{aligned}
 & + \varepsilon_j \left[ \frac{1 + |\Delta|}{|\Delta - 1|\Gamma(\vartheta)} + \frac{1}{|\Delta - 1|\Gamma(\vartheta + 2)} + \frac{|\Delta - 1| + 1}{|\Delta - 1|\Gamma(\vartheta + 1)} + \frac{1}{|\Delta - 1|\Gamma(\vartheta - \beta + 2)} \right] \\
 & = \varepsilon_j \mathcal{K}_0^* + \mathcal{K}_0^* \mathcal{L}_j^* \|z_j^* - z_j\|.
 \end{aligned}$$

Hence

$$\|z_j^* - z_j\| \leq \varepsilon_j \mathcal{K}_0^* + \mathcal{K}_0^* \mathcal{L}_j^* \|z_j^* - z_j\|,$$

in which  $\mathcal{K}_0^*$  is known in (10). From above, we obtain that

$$\|z_j^* - z_j\| \leq \frac{\varepsilon_j \mathcal{K}_0^*}{1 - \mathcal{K}_0^* \mathcal{L}_j^*},$$

and thus by taking  $d_{h_j}^* = \frac{\mathcal{K}_0^*}{1 - \mathcal{K}_0^* \mathcal{L}_j^*}$ , we find that the multi-dimensional methylpropane

FBVP (14) is Ulam–Hyers stable ( $\mathcal{K}_0^* \mathcal{L}_j^* < 1$ ). Further, for  $G_j(\varepsilon_j) = \frac{\varepsilon_j \mathcal{K}_0^*}{1 - \mathcal{K}_0^* \mathcal{L}_j^*}$  with  $G_j(0) = 0$ , we have the generalized Ulam–Hyers stability for the mentioned multi-dimensional methylpropane FBVP (14). □

**Example 2.** Consider the multi-term multi-dimensional DEs

$$\begin{cases}
 {}^c\mathcal{D}_0^{1.38} z_1(t) = h_1(t, z_1(t), z_1'(t)), \\
 {}^c\mathcal{D}_0^{1.38} z_2(t) = h_2(t, z_2(t), z_2'(t)), \\
 {}^c\mathcal{D}_0^{1.38} z_3(t) = h_3(t, z_3(t), z_3'(t)), \\
 {}^c\mathcal{D}_0^{1.38} z_4(t) = h_4(t, z_4(t), z_4'(t)),
 \end{cases} \tag{17}$$

under the integral BCs

$$\begin{cases}
 z_j(0) + {}^c\mathcal{D}_0^{0.47} z_j(0) + z_j'(1) = 0, \quad (j = 1, 2, 3, 4) \\
 \int_0^1 [z_j(\xi) + {}^c\mathcal{D}_0^{0.47} z_j(\xi) + z_j'(\xi)] d\xi = 0,
 \end{cases} \tag{18}$$

where  $\vartheta = 1.38$ ,  $\beta = 0.47$ ,  $n = 4$ , and  ${}^c\mathcal{D}_0^{1.38}$  and  ${}^c\mathcal{D}_0^{0.47}$  are derivatives of order  $\gamma \in \{1.38, 0.47\}$  of the Caputo type. Set  $h_j \in C([0, 1] \times \mathbb{R}^3, \mathbb{R})$ ,  $(j = 1, \dots, 4)$  by

$$\begin{cases}
 h_1(t, z(t), \ell(t)) = \frac{|\sin z(t)|}{8} + \frac{|\ell(t)|}{8} + e^t, \\
 h_2(t, z(t), \ell(t)) = \frac{0.1e|z(t)|}{5 + 5|z(t)|} + \frac{e|\sin \ell(t)|}{50(1 + |\sin \ell(t)|)} + 2 \sin(t), \\
 h_3(t, z(t), \ell(t)) = 2 + \frac{1}{700} |\arctan z(t)| + \frac{|\ell(t)|}{700 + 700|\ell(t)|}, \\
 h_4(t, z(t), \ell(t)) = \frac{e}{80} |z(t)| + 0.0125e|\ell(t)| - 21t.
 \end{cases}$$

By the mean value theorem, we know that  $|\sin a - \sin b| \leq |a - b|$  and  $|\arctan a - \arctan b| \leq |a - b|$  for all  $a, b \in \mathbb{R}$ . Thus, for  $z_1, z_2, \ell_1, \ell_2 \in \mathbb{R}$ , we estimate

$$|h_1(t, z_1(t), z_2(t)) - h_1(t, \ell_1(t), \ell_2(t))|$$

$$\begin{aligned} &\leq \frac{1}{8} (|\sin z_1(t) - \sin \ell_1(t)| + |z_2(t) - \ell_2(t)|) \\ &\leq \frac{1}{8} (|z_1(t) - \ell_1(t)| + |z_2(t) - \ell_2(t)|), \end{aligned}$$

$$\begin{aligned} &|h_2(t, z_1(t), z_2(t)) - h_2(t, \ell_1(t), \ell_2(t))| \\ &\leq \frac{0.1e}{5} (|z_1(t) - \ell_1(t)| + |\sin z_2(t) - \sin \ell_2(t)|) \\ &\leq \frac{0.1e}{5} (|z_1(t) - \ell_1(t)| + |z_2(t) - \ell_2(t)|), \end{aligned}$$

$$\begin{aligned} &|h_3(t, z_1(t), z_2(t)) - h_3(t, \ell_1(t), \ell_2(t))| \\ &\leq \frac{1}{700} (|\arctan z_1(t) - \arctan \ell_1(t)| + |z_2(t) - \ell_2(t)|) \\ &\leq \frac{1}{700} (|z_1(t) - \ell_1(t)| + |z_2(t) - \ell_2(t)|), \end{aligned}$$

and

$$\begin{aligned} &|h_4(t, z_1(t), z_2(t)) - h_4(t, \ell_1(t), \ell_2(t))| \\ &\leq \frac{e}{80} (|z_1(t) - \ell_1(t)| + |z_2(t) - \ell_2(t)|) \\ &\leq \frac{e}{80} (|z_1(t) - \ell_1(t)| + |z_2(t) - \ell_2(t)|). \end{aligned}$$

Hence,  $\mathcal{L}_1^* = \frac{1}{8}$ ,  $\mathcal{L}_2^* = \frac{0.1e}{5}$ ,  $\mathcal{L}_3^* = \frac{1}{700}$ , and  $\mathcal{L}_4^* = \frac{e}{80}$ . According to the obtained values, we obtain  $\Delta \simeq 2.2364$  and  $\mathcal{K}_0^* \simeq 5.1371$ . Since,

$$\begin{aligned} \mathcal{L}_1^* \mathcal{K}_0^* &\simeq 0.64213 < 1, & \mathcal{L}_2^* \mathcal{K}_0^* &\simeq 0.27843 < 1, \\ \mathcal{L}_3^* \mathcal{K}_0^* &\simeq 0.007338 < 1, & \mathcal{L}_4^* \mathcal{K}_0^* &\simeq 0.17401 < 1. \end{aligned}$$

Then by taking the constants

$$\begin{aligned} d_{h_1}^* &= \frac{\mathcal{K}_0^*}{1 - \mathcal{K}_0^* \mathcal{L}_1^*} \simeq 14.354653, & d_{h_2}^* &= \frac{\mathcal{K}_0^*}{1 - \mathcal{K}_0^* \mathcal{L}_2^*} \simeq 7.119337, \\ d_{h_3}^* &= \frac{\mathcal{K}_0^*}{1 - \mathcal{K}_0^* \mathcal{L}_3^*} \simeq 5.175085, & d_{h_4}^* &= \frac{\mathcal{K}_0^*}{1 - \mathcal{K}_0^* \mathcal{L}_4^*} \simeq 6.219324, \end{aligned}$$

the conclusion of Theorem 3 confirms that the fractional system of BVPs (17) and (18) is Ulam–Hyers stable. Moreover, by defining the functions

$$\begin{aligned} G_1(\varepsilon_1) &= \frac{\varepsilon_1 \mathcal{K}_0^*}{1 - \mathcal{K}_0^* \mathcal{L}_1^*} \simeq 14.354653 \varepsilon_1, & G_2(\varepsilon_2) &= \frac{\varepsilon_2 \mathcal{K}_0^*}{1 - \mathcal{K}_0^* \mathcal{L}_2^*} \simeq 7.119337 \varepsilon_2, \\ G_3(\varepsilon_3) &= \frac{\varepsilon_3 \mathcal{K}_0^*}{1 - \mathcal{K}_0^* \mathcal{L}_3^*} \simeq 5.175085 \varepsilon_3, & G_4(\varepsilon_4) &= \frac{\varepsilon_4 \mathcal{K}_0^*}{1 - \mathcal{K}_0^* \mathcal{L}_4^*} \simeq 6.219324 \varepsilon_4, \end{aligned}$$

so that  $G_j(0) = 0$ , ( $j = 1, 2, 3, 4$ ), we see that the conclusion of Theorem 3 implies that the fractional system of BVPs (17)–(18) is the generalized Ulam–Hyers stable.

## 5. Conclusions

In this study, we extended the technique applied in [28,29] to the non-star methylpropane graph and studied a new family of multi-term multi-dimensional FBVPs on every edge of the graph by introducing a labeling method for vertices. We discussed the existence of solutions for aforesaid FBVPs with the aid of Krasnoselskii and Schaefer fixed point theorems. Moreover, the stability analysis in the sense of Ulam–Hyers and generalized Ulam–Hyers was investigated. In addition, we prepared an abstract example to explain one of our results. In view of the importance of differential equations in chemical graph theory, we designed our FBVPs (2) on the molecular graph representation of methylpropane as an example of a non-star graph. This research is of pivotal nature for fractional mathematical modeling on different chemical compounds of molecules in the framework of complicated non-star graphs. Our suggested FBVPs are applicable on a vast range of graph structures, especially digraphs, which are usually utilized in medical technologies in relation to protein networks. In our future projects, we plan to study more problems on the graph representation of different molecular structures by using analytic and numerical methods.

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## Abbreviations

The following abbreviations are used in this manuscript:

FBVP	Fractional Boundary Value Problem,
ODEs	Ordinary Differential Equations,
BCs	Boundary Conditions,
RL	Riemann–Liouville,
DEs	Differential Equations.

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