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A THEORETICAL APPROACH TO THE PROBLEM OF THE MOST DANGEROUS INITIAL DEFLECTION SHAPE IN STABILITY TYPE STRUCTURAL PROBLEMS

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1. INTRODUCTION

The known solutions of nonlinear stability type problems show that the initial deflection w_0 influences the values of various quantities characterising the state of stress and strain of the structure markedly already at small values of load. In general the influence of w_0 has its maximum near the critical load while for loads exceeding several times the critical load it is negligible. The importance of solving the problem of the most dangerous initial deflection shape is therefore especially urgent in the load interval ranging from zero to at about twice the value of the critical load. The theory presented in this paper treats the problem in the range of loads from zero to the critical load. Since the theory is based on an examination of linearized equations and their corresponding functionals, the notions such as potential energy or differential equation of equilibrium throughout the paper should be understood to be quadratic or linear, respectively, if not stated otherwise.

The first important starting step of the theoretical analysis is to determine a set of functions from which the most dangerous initial deflection should be specified. This is done by defining a measure for admissible initial deflection functions w_0. Unlike the classical local measure – the amplitude, a global measure is introduced given by the value of the bending strain energy of structure having the deflection w_0. Now we can formulate a definition:

Definition 1.1. *For the given value of the load the most dangerous initial deflection w_0 is that w_0 from the set of admissible functions having the same global measure (the same value of the corresponding bending strain energy functional), for which the potential energy of structure attains the minimum value.*

Definition 1.1 is in a certain sense connected with the definition of a buckled state which a "real" plate prefers, proposed in [1] on the basis of numerical solutions to the nonlinear problem of compressed initially flat thin plate. The authors [1] assume that the plate always jumps from any state to the state with the smallest value of the corresponding potential energy functional. Numerous numerical solutions concerning the nonlinear problem of thin initially deflected plate in shear obtained by the author [2] show, that the definition of the prefered state of a plate is suitable with regard to the strength of plate.

The substantial feature of the presented theory is the introduction of global measure of initial deflection. This allowed to prove that the use of the minimum of potential energy as a criterion for the determination of the most dangerous initial deflection is justified from the point of view of bending strain energy of a structure. The bending strain energy is being understood as a global stress state measure of buckled structure (GSSM). The solvability of the formulated minimization problem is established and its critical points are found. The corresponding equilibrium configurations are compared from the view-point of potential energy and of bending strain energy values.

For some nonlinear problems, numerical solutions are presented showing the influence of the chosen imperfection shapes on the GSSM and on the formulated local stress state measure of the buckled structure (LSSM). For the mentioned special cases, the initial deflections from the set of eigenvectors with the same amplitude were also investigated. The paper in this part extends the results of I. Hlaváček [3].

The author publishes in the paper the material essentially contained in Chapter 3 of the research report [4] but for the section 3.4.4. A talk on the present theory was also delivered on the 17th Polish Solid Mechanics Conference in Szczyrk in 1975.

2. MODEL PROBLEM

The nonlinear problem of a rectangular thin elastic plate given by the Föppl-Kármán-Marguerre's partial differential equations

$$(2.1)\qquad \begin{aligned} &D\Delta\Delta(w - w_0) - t[(\Phi_{xx} + \lambda\Phi_{0,xx})\, w_{yy} + (\Phi_{yy} + \lambda\Phi_{0,yy})\, w_{xx} - \\ &\qquad - 2(\Phi_{xy} + \lambda\Phi_{0,xy})\, w_{xy}] = 0\,, \\ &-\frac{t}{E}\Delta\Delta\Phi - t[w_{xx}w_{yy} - w_{xy}^2 - w_{0,xx}w_{0,yy} + w_{0,xy}^2] = 0 \quad \text{in } \Omega \end{aligned}$$

and by the boundary conditions

$$(2.2)\qquad w = w_{nn} = w_0 = w_{0,nn} = 0|_\Gamma$$

and

$$(2.3)\qquad \Phi = \Phi_n = 0|_\Gamma$$

will be used as a model problem. Here $\Delta\Delta$ is the biharmonic operator $\Omega = (0, a) \times (0, b)$ is a rectangular domain, $D = Et^3/(12(1 - \mu^2))$ the flexural, rigidity of the plate, t – thickness, E – modulus of elasticity, μ – Poisson's ratio, w_0 the initial and w the overall deflection. Parameter λ is the measure of the load, Φ_0 a biharmonic function giving the form of the membrane load of the plate, $\Phi + \lambda\Phi_0$ – the Airy stress function. n denotes the independent variable in the direction to the normal to the boundary Γ.

We linearize the given nonlinear problem neglecting the third and higher order terms in w, w_0 in the potential energy functional. The corresponding equation of equilibrium has the form

$$(2.4) \qquad D\Delta\Delta(w - w_0) - \lambda t[w_{xx}\Phi_{0,yy} + w_{yy}\Phi_{0,xx} - 2w_{xy}\Phi_{0,xy}] = 0 \quad \text{in } \Omega .$$

The potential energy of the linearized problem is up to an additive constant

$$C_0(\lambda\Phi_0) = \int_0^b\int_0^a \left\{- \lambda^2 \frac{t}{2E}(\Delta\Phi_0)^2 + \lambda^2 \frac{t(1 + \mu)}{E}(\Phi_{0,xx}\Phi_{0,yy} - \Phi_{0,xy}^2)\right\} \mathrm{d}x\,\mathrm{d}y$$

equal to the functional

$$(2.5) \quad \Pi^L = \int_0^b\int_0^a \left\{\frac{D}{2}[\Delta(w - w_0)]^2 + \frac{t}{2}\lambda(\Phi_{0,yy}w_x^2 + \Phi_{0,xx}w_y^2 - 2\Phi_{0,xy}w_xw_y) - \right.$$
$$\left. - \frac{t}{2}\lambda(\Phi_{0,yy}w_{0,x}^2 + \Phi_{0,xx}w_{0,y}^2 - 2\Phi_{0,xy}w_{0,x}w_{0,y})\right\} \mathrm{d}x\,\mathrm{d}y ,$$

where Δ is the Laplace operator.

According to the definition, we minimize (2.5) while introducing an auxiliary condition

$$(2.6) \qquad C - \frac{D}{2}\int_0^b\int_0^a (\Delta w_0)^2\,\mathrm{d}x\,\mathrm{d}y = 0 .$$

Using the method of Lagrange multipliers with a multiplier χ we get the functional

$$\tilde{\Pi}^L = \Pi^L + \chi\left[C - \frac{D}{2}\int_0^b\int_0^a (\Delta w_0)^2\,\mathrm{d}x\,\mathrm{d}y\right].$$

The condition that $\tilde{\Pi}^L$ should be stationary implies formally (Eqs. 2.4), (2.6) and equation

$$(2.7)$$
$$-D\Delta\Delta(w - w_0) - \chi D\Delta\Delta w_0 + \lambda t[w_{0,xx}\Phi_{0,yy} + w_{0,yy}\Phi_{0,xx} - 2w_{0,xy}\Phi_{0,xy}] = 0 .$$

Let $\dot{W}_2^2(\Omega)$ denote a Sobolev space defined as the closure in the norm of $W_2^2(\Omega)$ of the set of smooth functions defined in $\bar{\Omega}$ and vanishing on the boundary. We introduce in $\dot{W}_2^2(\Omega)$ the scalar product

$$(w, \psi) = D \int_0^b \int_0^a \Delta w \Delta \psi \, dx \, dy \,, \qquad w, \psi \in \dot{W}_2^2(\Omega)$$

which generates in $\dot{W}_2^2(\Omega)$ a norm $\|.\|$ equivalent to the norm of $W_2^2(\Omega)$. In the usual manner we define now the variational solution w, $w_0 \in \dot{W}_2^2(\Omega)$ to the problem (2.4), (2.7), (2.2), (2.6) by the identities

$$(2.8) \qquad \int_0^b \int_0^a \{ D\Delta(w - w_0) \Delta\psi + \lambda t[(\Phi_{0,xx} w_y - \Phi_{0,xy} w_x) \psi_y + (\Phi_{0,yy} w_x - \Phi_{0,xy} w_y) \psi_x]\} \, dx \, dy = 0 \,,$$

$$(2.9) \qquad \int_0^b \int_0^a \{ -D\Delta(w - w_0) \Delta\psi_0 - \chi D \Delta w_0 \Delta\psi_0 - \lambda t[(\Phi_{0,xx} w_{0,y} - \Phi_{0,xy} w_{0,x}) \psi_{0,y} + (\Phi_{0,yy} w_{0,x} - \Phi_{0,xy} w_{0,y}) \psi_{0,x}]\} \, dx \, dy = 0$$

which should be satisfied for all ψ, $\psi_0 \in \dot{W}_2^2(\Omega)$ and (2.6). Adopting the procedure used in [5] we form by means of Riesz representation theorem operator equations equivalent to the variational identities (2.8), (2.9). The resulting equations are

$$(2.10) \qquad w - w_0 - \lambda A w = 0 \,,$$

$$(2.11) \qquad -w + w_0 - \chi w_0 + \lambda A w_0 = 0 \,.$$

A is a linear selfadjoint compact operator acting from $\dot{W}_2^2(\Omega)$ into itself [6]. Now (2.5), (2.6) can be written in the form

$$(2.12) \qquad \Pi^L = \frac{1}{2} \|w - w_0\|^2 - \frac{\lambda}{2} (Aw, w) + \frac{\lambda}{2} (Aw_0, w_0) \,,$$

$$(2.13) \qquad C - \tfrac{1}{2} \|w_0\|^2 = 0 \,.$$

3. GENERALIZED PROBLEM

Let us assume that the expressions (2.10)–(2.13) are written in a certain real separable Hilbert space H, i.e., w, $w_0 \in H$, $(.,.)$ and $\|.\|$ denote now the scalar product and the corresponding norm in H, A is a linear selfadjoint compact operator acting from H into itself. In such a way we extend our investigations to the class of problems, whose bending strain energy can be represented by one half of the squared norm in H and the potential energy is given by (2.12) but for the constant.

In the sequel, the notation $\varrho_1, \varrho_2, \ldots$ will be used for the sequence of eigenvalues and $\varphi_1, \varphi_2, \ldots$ for the corresponding orthogonalized sequence of eigenvectors of the eigenvalue problem

$$(3.1) \qquad Aw - \varrho w = 0\,.$$

It is well known that (3.1) has a countable number of real eigenvalues ϱ_i, every eigenvalue $\varrho_i \neq 0$ has a finite dimensional eigensubspace and zero is the only limit point of the sequence $\{\varrho_i\}$. We order $\{\varrho_i\}$ so that $|\varrho_1| \geqq |\varrho_2| \geqq \dots$ and if $|\varrho_i| = |\varrho_j|$ and $\varrho_i > \varrho_j$ then $i < j$. The reciprocal values of ϱ_i are denoted λ_i. The smallest positive λ_i which is the value of load parameter λ corresponding to the critical load is denoted by λ_{cr}.

Theorem 3.1. *For every $\lambda \in (0, \lambda_{cr})$ there exists at least one absolute minimum w, $w_0 \in H$ of the functional* (2.12) *under the constraint* (2.13).

Proof: For $0 < \lambda < \lambda_{cr}$ we have (note that $1/\lambda_{cr} = \max\limits_{\|w\|=1} (Aw, w)$)

$$\|w\|^2 - \lambda(Aw, w) \geqq \|w\|^2 - \frac{\lambda}{\lambda_{cr}} \|w\|^2 = \varepsilon\|w\|^2$$

and from this further the inequality

$$(3.2) \qquad 2\Pi^L(w, w_0, \lambda) = \varepsilon\|w\|^2 - 2\|w\|\,\|w_0\| + \text{const}\,.$$

Using the space $H \times H$ of couples $\{w, w_0\}$ we can easily show that the functional (2.12) is on the set $G = \{\{w, w_0\} \in H \times H : \|w\| \leqq R,\ R > 0,\ \frac{1}{2}\|w_0\|^2 = C\}$ weakly lower semicontinuous. This follows from the compactness of A. According to Theorem 9.2 [7] the functional (2.12) attains its minimum value on the weakly closed set G. We choose the value of R using (3.2) in such a way that w, w_0 with $\|w\| > R$, $\frac{1}{2}\|w_0\|^2 \leqq C$ satisfy $\Pi^L(w, w_0, \lambda) > 0$. So the point of minimum of Π^L on $\{w, w_0\} \in H \times H$, $\frac{1}{2}\|w_0\|^2 \leqq C$ is from G.

Now we show that the minimum is attained for w_0 satisfying the condition (2.13). As $0 < \lambda < \lambda_{cr}$, Eq. (2.10) with w_0 on the right hand side may be solved. It is

Now we show that the minimum is attained for w_0 satisfying the condition (2.13). As $0 < \lambda < \lambda_{cr}$. Eq. (2.10) with w_0 on the right hand side may be solved. It is

$$(3.3) \qquad w = (I - \lambda A)^{-1}\, w_0\,.$$

Let us denote by $w(w_0)$ the dependence of w on w_0 according to (3.3). Clearly, for a fixed w_0, $w(w_0)$ minimizes the potential energy functional. We can easily find such w_0 that $\Pi^L(w, w_0, \lambda) < 0$ (see the following explanation), which excludes $w_0 = 0$ as a possible point of minimum of Π^L. From the equality

$$\Pi^L(w(kw_0), kw_0, \lambda) = k^2\, \Pi^L(w(w_0), w_0, \lambda)$$

we then deduce the validity of (2.13) for the point of minimum of Π^L on G, which completes the proof of the theorem.

We have seen that the method of Lagrange multipliers leads in our case to Eqs. (2.10), (2.11) and (2.13). Correctness of the procedure used is ensured by the known Ljusternik theorem. Inserting (3.3) into (2.11) we get the eigenvalue problem

$$[-(I - \lambda A)^{-1} + I + \lambda A]\, w_0 - \chi w_0 = 0 \tag{3.4}$$

with a selfadjoint and bounded operator $-(I - \lambda A)^{-1} + I + \lambda A$ ($(I - \lambda A)^{-1}$ is bounded according to Banach theorem). Moreover, with respect to an obvious equality

$$I = (I - \lambda A)^{-1} - \lambda A(I - \lambda A)^{-1}$$

the operator of the eigenvalue problem (3.4) may be written in the form

$$-\lambda A(I - \lambda A)^{-1} + \lambda A\,,$$

which proves its compactness.

The solution of (2.10), (2.11), (2.13) or of its equivalent eigenvalue problem (3.4), (2.13) with Eq. (3.3) can be sought in the form $w_0 = \varphi_i$, $w = K_i\varphi_i$ (K_i is a real number). In what follows we assume that the dimension of the subspace of the elements $w \in H$, $Aw = 0$ is zero. This simplifies the argument and does not alter the statements of the following theorems. Substituting for w and w_0 into (2.10) we get

$$K_i\varphi_i = \varphi_i - \lambda K_i A\varphi_i = 0\,,$$

which yields

$$w = \frac{\lambda_i}{\lambda_i - \lambda}\,\varphi_i\,. \tag{3.5}$$

From (2.11) we have, when writing χ_i instead of χ,

$$\left(-\frac{\lambda_i}{\lambda_i - \lambda} + 1 + \frac{\lambda}{\lambda_i} - \chi_i\right)\varphi_i = 0$$

and further

$$\chi_i = -\frac{\lambda^2}{\lambda_i(\lambda_i - \lambda)}$$

(evidently $\chi_i < 0$ for $0 < \lambda < \lambda_{cr}$). Thus the eigenvectors of the linear stability problem (3.1) satisfying (2.13) are for every $\lambda \in (0, \lambda_{cr})$ solutions of the eigenvalue problem (3.4), (2.13) and together with (3.5) solutions of the conditions of stationariness (2.10), (2.11), (2.13). Since the system of eigenvectors $\{\varphi_i\}$ is complete in H, $\chi_i(\lambda_i, \lambda)$ are all the eigenvalues of (3.4). Consequently, further solutions (different from eigenvectors of (3.1)) can be found only in the eigensubspaces of multiple χ_i

when $\chi_i(\lambda_i, \lambda) = \chi_j(\lambda_j, \lambda)$, $\lambda_i\lambda_j < 0$. The correponding eigenvectors of (3.4) are then the combinations of $\tilde{\varphi}_i$, $\tilde{\varphi}_j$ which represent an arbitrary eigenvector from the eigensubspaces of ϱ_i, ϱ_j, respectively. Clearly this occurs when $\lambda_i(\lambda_i - \lambda) = \lambda_j(\lambda_j - \lambda)$ for $\lambda \in (0, \lambda_{cr})$ which is possible only if (3.1) possesses both positive and negative eigenvalues. The set of such points λ has zero Lebesgue measure.

Let us evaluate the functional (2.12) at the stationary points $w_0 = \tilde{\psi}_i$, $\frac{1}{2}\|\tilde{\psi}_i\|^2 = C$ (generally $\tilde{\psi}_i = c_i\tilde{\varphi}_i + c_j\tilde{\varphi}_j$). Using (2.10) we have

$$\Pi^L(w(\tilde{\psi}_i), \tilde{\psi}_i, \lambda) = -\tfrac{1}{2}(w - \tilde{\psi}_i, \tilde{\psi}_i) + \frac{\lambda}{2}(A\tilde{\psi}_i, \tilde{\psi}_i)$$

and then

$$\Pi^L(w(\tilde{\psi}_i), \tilde{\psi}_i, \lambda) = \frac{1}{2}\left[-\left(\frac{\lambda_i}{\lambda_i - \lambda} - 1\right) + \frac{\lambda}{\lambda_i}\right]\|c_i\tilde{\varphi}_i\|^2 +$$
$$+\frac{1}{2}\left[-\left(\frac{\lambda_j}{\lambda_j - \lambda} - 1\right) + \frac{\lambda}{\lambda_j}\right]\|c_j\tilde{\varphi}_j\|^2 = \frac{1}{2}\chi_i\|\tilde{\psi}_i\|^2 .$$

The result shows that the same values of Π^L correspond to the equilibrium configurations corresponding to w_0 in the shapes of eigenvectors from the eigensubspace of χ_i satisfying (2.13). Comparing the positive expressions $\lambda_i(\lambda_i - \lambda)$ we easily show that from the set of χ_i corresponding to positive λ_i, denoted λ_{ip}, the lowest value of Π^L is attained for $\lambda_{1p} = \min_i \lambda_{ip} = \lambda_{cr}$. From the set of χ_i corresponding to negative λ_i, denoted λ_{in}, the lowest value of Π^L is attained for $\lambda_{1n} = \max_i \lambda_{in}$. Thus the most dangerous initial deflection is in the shape of eigenvectors from the eigensubspace of $\chi(\lambda_{1p}, \lambda)$ or $\chi(\lambda_{1n}, \lambda)$ or their combinations.

Theorem 3.2. *There exists a real number $C_1 > 0$ such that for λ, $0 < \lambda < C_1 \leqq \leqq \lambda_{cr}$ the couple $(-s)$ $w_0 = \tilde{\varphi}_i$, $w(w_0)$, $\frac{1}{2}\|\tilde{\varphi}_1\|^2 = C$ ($\tilde{\varphi}_1$ representing an arbitrary element of the eigensubspace of $\varrho_1 = 1/\lambda_1$), is a point of absolute minimum of the functional* (2.12) *under the constraint* (2.13). *For $\lambda_1 > 0$ we have $C_1 = \lambda_1 = \lambda_{cr}$, for $\lambda_1 < 0$ we have $C_1 = \lambda_{cr} + \lambda_1$.*

Note 3.1. If $\lambda_1 < 0$ and $\lambda \in (\lambda_1 + \lambda_{cr}, \lambda_{cr})$, the point of absolute minimum of (2.12), (2.13) is the couple $(-s)$ $w_0 = \tilde{\varphi}_{cr}$, $w(w_0)$, $\frac{1}{2}\|\tilde{\varphi}_{cr}\|^2 = C$. If $\lambda_1 < 0$ and $\lambda = \lambda_1 + \lambda_{cr}$, the absolute minimum of the problem is attained for w_0 in the shape of any combination of $\tilde{\varphi}_1$, $\tilde{\varphi}_{cr}$ satisfying (2.13).

Theorem 3.3. *There exists a real number C_2, $0 < C_2 \leqq C_1 \leqq \lambda_{cr}$ such that for $\lambda \in (0, C_2)$ the inequality*

$$\|w(w_0) - w_0\| < \|w(\tilde{\varphi}_1) - \tilde{\varphi}_1\|$$

is true for every $w_0 \neq \tilde{\varphi}_1$, $\|w_0\| = \|\tilde{\varphi}_1\|$ ($\tilde{\varphi}_1$ *represents an arbitrary element of the eigensubspace of* ϱ_1). *For* $\lambda_1 > 0$ *we have* $C_2 = \lambda_1 = \lambda_{cr}$ *and for* $\lambda_1 < 0$ *we have* $C_2 = \frac{1}{2}(\lambda_1 + \lambda_{cr})$. *Further, if for* $w_{0,1}, w_{0,2}$ *with* $\|w_{0,1}\| = \|w_{0,2}\|$ *there exists an interval of values* λ, $0 < \lambda < C_3 \leqq \lambda_{cr}$ *on which*

$$0 < \Pi^L(w(w_{0,2}), w_{0,2}, \lambda) - \Pi^L(w(w_{0,1}), w_{0,1}, \lambda) = o(\lambda^2)\,, \tag{3.6}$$

then there exists such $C_4 \leqq \lambda_{cr}$ *that for* λ, $0 < \lambda < C_4$

$$\|w(w_{0,1}) - w_{0,1}\| > \|w(w_{0,2}) - w_{0,2}\|\,.$$

Proof. Let us have an element $w_0 \in H$, $w_0 \neq \tilde{\varphi}_1$, $\|w_0\| = \|\tilde{\varphi}_1\|$ and let $\|\varphi_i\| = \|w_0\|$, $i = 1, 2, \ldots$. We can write

$$w_0 = \sum_{i=1}^{\infty} c_i \varphi_i \tag{3.7}$$

with

$$\sum_{i=1}^{\infty} c_i^2 = 1 \tag{3.8}$$

and

$$w(w_0) = \sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i - \lambda}\, c_i \varphi_i\,. \tag{3.9}$$

It may be easily shown that there exists a number C_2, $C_2 = \lambda_1 = \lambda_{cr}$ if $\lambda_1 > 0$ and $C_2 = \frac{1}{2}(\lambda_{cr} + \lambda_1)$ if $\lambda_1 < 0$, such that

$$(\lambda_1 - \lambda)^2 < (\lambda_j - \lambda)^2 \quad \text{for} \quad \lambda_1 \neq \lambda_j\,, \quad 0 < \lambda < C_2\,. \tag{3.10}$$

Without a loss of generality we assume $\|w_0\| = 1$, then using (3.7), (3.9), (3.10) and (3.8) we have for $0 < \lambda < C_2$

$$\|w(w_0) - w_0\|^2 = \sum_{i=1}^{\infty} \left\| \left(\frac{\lambda_i}{\lambda_i - \lambda} - 1 \right) c_i \varphi_i \right\|^2 = \sum_{i=1}^{\infty} \frac{\lambda^2}{(\lambda_i - \lambda)^2}\, c_i^2 <$$
$$< \frac{\lambda^2}{(\lambda_1 - \lambda)^2} = \|w(\tilde{\varphi}_1) - \tilde{\varphi}_1\|^2\,,$$

which proves the first part of the theorem.

Let $w_{0,1} = \sum_{i=1}^{\infty} c_{1,i} \varphi_i$, $w_{0,2} = \sum_{i=1}^{\infty} c_{2,i} \varphi_i$ ($\|\varphi_i\| = 1$) be the elements of H satisfying (3.6) on $0 < \lambda < C_3 \leqq \lambda_{cr}$ and $\|w_{0,1}\| = \|w_{0,2}\|$. The orthogonality of φ_i in H and

$$(A\varphi_i, \varphi_j) = 0\,, \quad i \neq j$$

imply that

$$\Pi^L(w(w_{0,1}), w_{0,1}, \lambda) = \tfrac{1}{2} \sum_{i=1}^{\infty} \chi_i c_{1,i}^2 .$$

With a similar arrangement of $\Pi^L(w(w_{0,2}), w_{0,2}, \lambda)$ we rewrite (3.6) as

$$0 < \frac{1}{2} \lambda^2 \sum_{i=1}^{\infty} \frac{1}{\lambda_i(\lambda_i - \lambda)} (c_{1,i}^2 - c_{2,i}^2) = o(\lambda^2) .$$

Using the tools of the classical analysis of function series we get that the series $\sum_{i=1}^{\infty} (c_{1,i}^2 - c_{2,i}^2)/(\lambda_i(\lambda_i - \lambda))$ converges uniformly to a continuous function $S_1(\lambda)$ on every interval $[0, K]$, $K < \lambda_{cr}$ and $S_1(0) = \sum_{i=1}^{\infty} (c_{1,i}^2 - c_{2,i}^2)/\lambda_i^2$. (3.6) yields $S_1(0) >$ > 0. Now

$$\|w(w_{0,1}) - w_{0,1}\|^2 - \|w(w_{0,2}) - w_{0,2}\|^2 = \lambda^2 \sum_{i=1}^{\infty} \frac{1}{(\lambda_i - \lambda)^2} (c_{1,i}^2 - c_{2,i}^2) .$$

Again we can easily deduce that the series $\sum_{i=1}^{\infty} (c_{1,i}^2 - c_{2,i}^2)/(\lambda_i - \lambda)^2$ converges uniformly to a continuous function $S_2(\lambda)$ on every interval $[0, K]$, $K < \lambda_{cr}$ and $S_2(0) = S_1(0)$ which proves the existence of the constant C_4 from the second part of the theorem.

Note 3.2. In the case of a positive operator A the inequality $\lambda_i - \lambda_j$ yields

$$\Pi^L(w(\tilde{\varphi}_i), \tilde{\varphi}_i, \lambda) < \Pi^L(w(\tilde{\varphi}_j), \tilde{\varphi}_j, \lambda)$$

and

$$\|w(\tilde{\varphi}_i) - \tilde{\varphi}_i\| > \|w(\tilde{\varphi}_j) - \tilde{\varphi}_j\|$$

on the whole interval $(0, \lambda_1 = \lambda_{cr})$.

Note 3.3. If $\lambda_1 < 0$ and $\lambda \in (\frac{1}{2}(\lambda_1 + \lambda_{cr}), \lambda_{cr})$ the inequality

$$\|w(w_0) - w_0\| < \|w(\tilde{\varphi}_{cr}) - \tilde{\varphi}_{cr}\|$$

is true for every $w_0 \neq \tilde{\varphi}_{cr}$, $\|w_0\| = \|\tilde{\varphi}_{cr}\|$. For $\lambda_1 < 0$ and $\lambda = \frac{1}{2}(\lambda_1 + \lambda_{cr})$, the inequality

$$\|w(w_0) - w_0\| < \|w(c_1\tilde{\varphi}_1 + c_2\tilde{\varphi}_{cr}) - c_1\tilde{\varphi}_1 - c_2\tilde{\varphi}_{cr}\|$$

holds for every $w_0 \in \{c_1\tilde{\varphi}_1 + c_2\tilde{\varphi}_{cr}\}$, $\|w_0\| = \|c_1\tilde{\varphi}_1 + c_2\tilde{\varphi}_{cr}\|$, where c_1, c_2 are real constants. The statements can be easily proved following the proof of the first part of Theorem 3.3 with simple inequalities

$$(\lambda_{cr} - \lambda)^2 < (\lambda_j - \lambda)^2 , \quad \lambda_j \neq \lambda_{cr} , \quad \tfrac{1}{2}(\lambda_1 + \lambda_{cr}) < \lambda < \lambda_{cr} ,$$

or

$$(\lambda_1 - \lambda)^2 = (\lambda_{cr} - \lambda)^2 < (\lambda_j - \lambda)^2 , \quad \lambda_j \neq \lambda_1 \cup \lambda_{cr} ,$$
$$\lambda = \tfrac{1}{2}(\lambda_1 + \lambda_{cr}) ,$$

respectively, used instead of (3.10).

It can be shown that the extremization of the values of bending strain energy in the equilibrium configurations $w(w_0)$, $w_0 \in H$ leads under the condition (2.13) to the eigenvalue problem

$$(I - \lambda A)^{-1} [(I - \lambda A)^{-1} - I - \lambda A] w_0 - \vartheta w_0 = 0$$

with a selfadjoint and compact operator (compare (3.4)). The eigenvectors of (3.1) satisfying (2.13) are the stationary points of this problem on the whole interval $(0, \lambda_{cr})$ and $\vartheta_i(\lambda_i, \lambda) = \lambda^2/(\lambda_i - \lambda)^2$ represent all of its eigenvalues. Similarly as in the case of (3.4), (2.13) further solutions can be found only in the eigensubspaces of multiple ϑ_i when $\vartheta_i(\lambda_i, \lambda) = \vartheta_j(\lambda_j, \lambda)$, $\lambda_i\lambda_j < 0$ and $\lambda \in (0, \lambda_{cr})$, as combinations of $\tilde{\varphi}_i$, $\tilde{\varphi}_j$ satisfying (2.13). This occurs now if $(\lambda_i - \lambda)^2 = (\lambda_j - \lambda)^2$, $\lambda \in (0, \lambda_{cr})$ (only possible if (3.1) possesses both positive and negative eigenvalues) and the set of such points λ has zero Lebesgue measure. Note that these new solutions are no more solutions of the problem (3.4), (2.13).

For pointing out the possibility of defining inverse variational problem the author is indebted to Dr. V. Horák*). In this case the functional of bending strain energy corresponding to w_0 ($U_B = \frac{1}{2}\|w_0\|^2$) is extremized under the condition of constant potential energy.

4. SPECIAL CASES

4.1. Compressed column simply supported

The differential equation of the compressed column is

$$EIw_{xxxx} + \lambda P w_{xx} = EIw_{0,xxxx} , \quad x \in (0, a) ,$$

where I is the moment of inertia, $P = \pi^2 EI/a^2$ and a denotes the length of the column. Let us have

$$w = w_{xx} = w_0 = w_{0,xx} = 0|_{x=0,a} .$$

The functional Π^L of the problem is

$$\Pi^L = \tfrac{1}{2}EI \int_0^a (w_{xx} - w_{0,xx})^2 \, dx - \tfrac{1}{2}\lambda P \int_0^a w_x^2 \, dx + \tfrac{1}{2}\lambda P \int_0^a w_{0,x}^2 \, dx$$

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*) Stavební ústav ČVUT, Praha

and the subsidiary condition on w_0 has the form

$$C - \tfrac{1}{2}EI \int_0^a w_{0,xx}^2 \, dx = 0 .$$

Using the energy space H with the norm

$$\|w\| = \left[\tfrac{1}{2}EI \int_0^a w_{xx}^2 \, dx\right]^{1/2} ,$$

we easily obtain the desired operator expressions (2.12), (2.13) with a strictly positive operator A ($(Aw, w) > 0$, $w \neq 0$). According to Section 3 the most dangerous initial deflection from the set $w_0 \in H$, $\|w_0\| = \text{const}$ has the shape of the critical eigenvector φ_1 for every load parameter $0 < \lambda < \lambda_1 = \lambda_{cr} = 1$. For the same λ – s the bending strain energy corresponding to $w_0 = \varphi_1$ attains the largest value.

Let us now investigate the initial deflections in the form of eigenvectors φ_i of a perfect column having equal amplitudes. In order to find the most unfavourable initial deflection we shall try to use the criterion of the minimum of potential energy. Since $\varphi_i \approx \sin i\pi(x/a)$ (φ_i has the shape of $\sin i\pi(x/a)$) we get using a suitable nondimensional form $\bar{\Pi}^L$ of Π^L that

$$\bar{\Pi}^L(w(\varphi_i), \varphi_i, \lambda) = \frac{\lambda^2}{\lambda - i^2}\, i^2 \xrightarrow[i \to \infty]{} -\lambda^2$$

is an increasing function of i with the minimum $\lambda^2/(\lambda - 1)$ at $i = 1$. Further,

$$\|w(\varphi_i) - \varphi_i\| = \text{const}\, \frac{\lambda i^2}{i^2 - \lambda} \xrightarrow[i \to \infty]{} \text{const}\, \lambda \tag{4.1}$$

is a decreasing function with the maximum const $\lambda/(1 - \lambda)$ at $i = 1$. The absolute value of the maximum moment used as the LSSM but for the multiplication by a constant is equal to the norm of the resulting deflection – the formula (4.1). We see that the used criterion showed in this case correctly the most dangerous shape of the initial deflection, both from the points of view of GSSM and LSSM.

Note that from the set of eigenvectors $\{\varphi_i\}$ having the same value of global measure we get for $w_0 = \varphi_1$ the largest value of the maximum moment, too.

4.2. Simply supported rectangular plate in compression

Starting from the equation of equilibrium (2.4) we express the case of compression choosing $\Phi_0 = -\sigma_E y^2/2$, $\sigma_E = \pi^2 E t^2/(12(1 - \mu^2)\, b^2)$. Deflections w and w_0 satisfy the conditions (2.2). It can be shown that the operator A of the corresponding operator form of functional Π^L (Eq. (2.12)) is strictly positive. Thus, the initial deflection in the shape of the critical eigenvector $(-s)\ \tilde{\varphi}_1$ is the most dangerous initial deflection among all $w_0 \in H$, $\|w_0\| = \text{const}$ on the whole interval $0 < \lambda < \lambda_1 = \lambda_{cr}$ giving here the largest value of GSSM of the corresponding equilibrium state.

As in the Subsection 4.1 we investigate now the set of eigenvectors of a perfect compressed plate which have the same amplitudes. Let us denote in this case the eigenvector functions and their corresponding reciprocal eigenvalues by φ_{mn} and λ_{mn}, $m, n, = 1, 2, \ldots$.

It is well known that

$$\varphi_{mn} \approx \sin m\pi \frac{x}{a} \sin n\pi \frac{y}{b} .$$

Having chosen a suitable nondimensional form $\bar{\Pi}^L$ of Π^L we have $(\alpha = a/b)$

$$\bar{\Pi}^L(w(\varphi_{mn}), \varphi_{mn}, \lambda) = - \frac{\lambda^2 m^2}{\left(\frac{m}{\alpha} + \alpha \frac{n^2}{m}\right)^2 - \lambda},$$

and further

$$\|w(\varphi_{mn}) - \varphi_{mn}\| = \text{const} \frac{\lambda\left(\frac{1}{\alpha} + \alpha \frac{n^2}{m^2}\right)}{\left(\frac{1}{\alpha} + \alpha \frac{n^2}{m^2}\right)^2 - \frac{\lambda}{m^2}} .$$

For n fixed it follows:

$$\lim_{m\to\infty} \bar{\Pi}^L = -\lambda^2\alpha^2 ,$$

$$\lim_{m\to\infty} \|w(\varphi_{mn}) - \varphi_{mn}\| = \text{const}\ \lambda\alpha .$$

Comparison of the limits with the m, n-terms yields

$$\inf_{m,n} \bar{\Pi}^L = \lim_{m\to\infty} \bar{\Pi}^L = -\lambda^2\alpha^2 , \quad 0 < \lambda \leqq 2$$

$$\sup_{m,n} \|w(\varphi_{mn}) - \varphi_{mn}\| = \lim_{m\to\infty} \|w(\varphi_{mn}) - \varphi_{mn}\| = \text{const}\ \lambda\alpha , \quad 0 < \lambda \leqq 1 .$$

$\bar{\Pi}^L$ is for $0 < \lambda \leqq 2$ $(\min_{\alpha} \lambda_{cr} = 4)$ a decreasing function of m and the bending strain energy is for $0 < \lambda \leqq 1$ an increasing function of m.

We see that for small λ $(0 < \lambda \leqq 1)$ there is a good correlation between Π^L and the bending strain energy (GSSM) values. Despite of this for $0 < \lambda \leqq 1$ it is not possible to determine from the given set the initial deflection for which GSSM and potential energy attain their maximum and minimum values, respectively.

Let us now illustrate the usefulness of theoretical predictions of the most unfavourable imperfection shape on numerical solutions to the nonlinear problem (2.1), (2.2) with static boundary conditions assumed to give the zero membrane shear stress along the boundary and to maintain the edges straight, and having the aspect ratio $a/b = 2$. The solution of the boundary value problem is carried out by the method of Papkovich. Representing the deflections w and w_0 by linear combinations

(4.2) $$w = \sum_{m,n} w_{mn} \sin m\pi \frac{x}{a} \sin n\pi \frac{y}{b},$$

(4.3) $$w_0 = \sum_{m,n} w_{0,mn} \sin m\pi \frac{x}{a} \sin n\pi \frac{y}{b},$$

Tab. 4.1

GSSM LSSM $\overline{Q}$	$\left[C_N \frac{D}{2} \int_0^b \int_0^a (\Delta w_0)^2 \, dx \, dy\right]^{1/2} = 0{\cdot}4$				
λ	$w_0 \approx \sin \pi \frac{x}{a} \sin \pi \frac{y}{b}$ $\lvert w_0 \rvert / t = 0{\cdot}32$	$w_0 \approx \sin 2\pi \frac{x}{a} \sin \pi \frac{y}{b}$ $\lvert w_0 \rvert / t = 0{\cdot}2$	$w_0 \approx \sin 3\pi \frac{x}{a} \sin \pi \frac{y}{b}$ $\lvert w_0 \rvert / t = 0{\cdot}123$	$w_0 \approx \sin 4\pi \frac{x}{a} \sin \pi \frac{y}{b}$ $\lvert w_0 \rvert / t = 0{\cdot}08$	$w_0 \approx \sin 5\pi \frac{x}{a} \sin \pi \frac{y}{b}$ $\lvert w_0 \rvert / t = 0{\cdot}055$
0·1	0·00593 1·0310 −0·000008	0·00997 1·0416 −0·000020	0·00860 1·0247 −0·000015	0·00646 1·0128 −0·000008	0·00480 1·0068 −0·000005
1·0	0·06694 1·0371 −0·00087	0·12634 1·0604 −0·00258	0·10599 1·0342 −0·00183	0·07550 1·0162 −0·00098	0·05375 1·0080 −0·00052
2·0	0·15485 1·0455 −0·00410	0·34894 1·1039 −0·01456	0·28339 1·0547 −0·00988	0·18533 1·0224 −0·00482	0·12403 1·0100 −0·00239
4·0	0·45851 1·0560 −0·02426	1·3163 1·3656 −0·14709	1·2548 1·2315 −0·10145	0·65908 1·0591 −0·03496	0·35694 1·0181 −0·01381
8·0	4·6467*) 2·0270 −1·4122	3·4240 1·8833 −1·7755	4·5516 2·1037 −2·1647	4·4274 1·7202 −1·1258	2·6310 1·1992 −0·25928
	$w_{11}, w_{13}, w_{15}, w_{31}, w_{33}, w_{51}$	$w_{21}, w_{23}, w_{25}, w_{61}, w_{63}, w_{10,1}$	w_{31}, w_{33}, w_{35}	w_{41}, w_{43}, w_{45}	w_{51}, w_{53}, w_{55}

*) Another branch of solutions.

Tab. 4.2

GSSM LSSM $\bar{Q}$	$\max_{x,y} \lvert w_0 \rvert / t = 0.2$				
λ	$w_0 \approx \sin \pi \frac{x}{a} \sin \pi \frac{y}{b}$	$w_0 \approx \sin 2\pi \frac{x}{a} \sin \pi \frac{y}{b}$	$w_0 \approx \sin 3\pi \frac{x}{a} \sin \pi \frac{y}{b}$	$w_0 \approx \sin 4\pi \frac{x}{a} \sin \pi \frac{y}{b}$	$w_0 \approx \sin 5\pi \frac{x}{a} \sin \pi \frac{y}{b}$
0·1	0·00392 1·0129 −0·000003	0·00997 1·0416 −0·000020	0·01370 1·0641 −0·000038	0·01566 1·0777 −0·000051	0·01674 1·0858 −0·000059
1·0	0·04509 1·0160 −0·00037	0·12634 1·0604 −0·00258	0·16658 1·0872 −0·00470	0·18033 1·0968 −0·00587	0·18536 1·1005 −0·00649
2·0	0·10771 1·0209 −0·00177	0·34894 1·1039 −0·01456	0·43082 1·1333 −0·02469	0·43040 1·1289 −0·02826	0·41943 1·1224 −0·02953
4·0	0·34132 1·0345 −0·01147	1·3163 1·3656 −0·14709	1·5123 1·3836 −0·20428	1·2945 1·2637 −0·18021	1·1101 1·1916 −0·16004
8·0	4·6405*) 2·0087 −1·4403	3·4240 1·8833 −1·7755	4·4929 2·1772 −2·5692	4·6250 1·9502 −2·0703	3·9507 1·5987 −1·4056

*) Another branch of solutions.

the equation of compatibility is solved exactly and the first of Eqs. (2.1) is then treated by the Bubnov-Galerkin's method. The nondimensional values of GSSM chosen as $\|w - w_0\| \sqrt{C_N}$, $C_N = 24ab(1 - \mu^2)/(E\pi^4 t^5)$ and of the nondimensional energy $\bar{Q} = 2C_N Q/\pi^2$ are given in Tabs. 4.1 and 4.2. It is $Q = \Pi - C_0(\lambda\Phi_0)$, where Π is the nonlinear potential energy of the plate. $C_0(\lambda\Phi_0)$ is a constant given in Section 2. The initial deflections having the shapes of eigenvectors $\varphi_{11}, \varphi_{21}, \ldots, \varphi_{51}$ are assumed to have the same global measures ($\|w_0\| = \text{const}$) – Tab. 4.1 or the same amplitudes ($\max_{x,y} |w_0(x, y)| = \text{const.}$) – Tab. 4.2.

The values of LSSM defined as the rate of increase of the maximum membrane stress intensity of a buckled plate in comparison to its ideal flat equilibrium configuration are presented, too.

Noting that $0 < \lambda_{21} < \lambda_{31} < \lambda_{11} = \lambda_{41} < \lambda_{51}$ we see from Tab. 4.1 that GSSM values behave in a fairly good accord with the predictions of the theory in the whole undercritical range of load ($\lambda_{cr} = \lambda_{21} = 4$). At the same time, the comparison of energy gives us also a useful information about the maximum value of LSSM. When the chosen w_0 have the same amplitudes (Tab. 4.2), the predictions of the linear analysis were confirmed, too. Note that for $0 < \lambda \leqq 1$ the values of LSSM behave with the increasing number of sine waves of w_0 like the values of GSSM. In the last row of Tab. 4.1 the coefficients w_{mn} are listed indicating the coordinate functions used in (4.2). The values presented in Tab. 4.2 were computed with the same approximation of deflection w.

4.3. Simply supported rectangular plate in shear

A special case of the problem (2.4), (2.2) is being investigated when $\Phi_0 = -\sigma_E xy$. As can be shown, the coefficients λ_i of the corresponding eigenvalue problem of a perfect plate occur in couples having the same value except for the signs. This implies that $\lambda_1 = \lambda_{cr}$ and then according to Section 3, the critical eigenvector $(-s)$ $\tilde{\varphi}_1$ represents the most dangerous initial deflection shape from all $w_0 \in \dot{W}_2^2(\Omega)$, $\|w_0\| = \text{const}$ on $0 < \lambda < \lambda_{cr}$ maximizing here the bending strain energy value.

Unfortunately, we have no explicit forms of eigenvectors for an analysis of the equiamplitude set of initial deflections having the shapes of eigenvectors of the perfect plate problem. Thus using the Bubnov-Galerkin's method with w in the form (4.2), approximate solutions to the eigenvalue problems were computed. The eigenvectors φ_i of a square plate in each of the four classes of symmetry were approximated by 72 coordinate functions. Then the values of $\Pi^L(w(\varphi_i), \varphi_i, \lambda)$ were computed for various values of λ. The case of a rectangular plate with the aspect ratio $a/b = 2$

Tab. 4.3

$a/b = 1, \quad \lambda_1 = \lambda_{cr} \doteq 9{\cdot}325, \quad \max\limits_{x,y} |\varphi_i|/t = 1$

Number of eq. / Number of coor. func.	λ_i	Type of φ_i	$C_N \Pi^L(w(\varphi_i), \varphi_i, \lambda)$ $\lambda = 0{\cdot}1$	$\lambda = 1{\cdot}0$
42/72	9·325	$m + n =$ even n., $w_{mn} = w_{nm}$	$-0{\cdot}23731_{10^{-3}}$	$-0{\cdot}26297_{10^{-1}}$
42/72	24·81	$m + n =$ even n., $w_{mn} = w_{nm}$	$-0{\cdot}18838_{10^{-3}}$	$-0{\cdot}19550_{10^{-1}}$
36/72	32·27	$m + n =$ even n., $w_{mn} = -w_{nm}$	$-0{\cdot}38521_{10^{-3}}$	$-0{\cdot}39630_{10^{-1}}$
36/72	60·95	$m + n =$ even n., $w_{mn} = -w_{nm}$	$-0{\cdot}22331_{10^{-3}}$	$-0{\cdot}22666_{10^{-1}}$
36/72	11·55	$m + n =$ odd n., $w_{mn} = -w_{nm}$	$-0{\cdot}60434_{10^{-3}}$	$-0{\cdot}65591_{10^{-1}}$
36/72	26·79	$m + n =$ odd n., $w_{mn} = -w_{nm}$	$-0{\cdot}28019_{10^{-3}}$	$-0{\cdot}28997_{10^{-1}}$
36/72	44·19	$m + n =$ odd n., $w_{mn} = -w_{nm}$	$-0{\cdot}20900_{10^{-3}}$	$-0{\cdot}21336_{10^{-1}}$
36/72	30·66	$m + n =$ odd n., $w_{mn} = w_{nm}$	$-0{\cdot}19723_{10^{-3}}$	$-0{\cdot}20322_{10^{-1}}$

Tab. 4.4

$a/b = 2,\ \lambda_1 = \lambda_{cr} \doteq 6{\cdot}547,\ \max_{x,y} |\varphi_i|/t = 1$

Number of eq. / Number of coor. func.	λ_i	Type of φ_i	$C_N \Pi^L(w(\varphi_i), \varphi_i, \lambda)$ $\lambda = 0{\cdot}1$	$\lambda = 1{\cdot}0$
33/33	6·547	$m + n =$ even n.	$-0{\cdot}76803_{10^{-3}}$	$-0{\cdot}89264_{10^{-1}}$
33/33	9·941	$m + n =$ even n.	$-0{\cdot}83088_{10^{-3}}$	$-0{\cdot}91452_{10^{-1}}$
33/33	17·18	$m + n =$ even n.	$-0{\cdot}78989_{10^{-3}}$	$-0{\cdot}83384_{10^{-1}}$
33/33	25·17	$m + n =$ even n.	$-1{\cdot}08219_{10^{-3}}$	$-1{\cdot}12248_{10^{-1}}$
33/33	27·84	$m + n =$ even n.	$-0{\cdot}53580_{10^{-3}}$	$-0{\cdot}55377_{10^{-1}}$
33/33	29·68	$m + n =$ even n.	$-1{\cdot}28383_{10^{-3}}$	$-1{\cdot}32412_{10^{-1}}$
33/33	40·36	$m + n =$ even n.	$-0{\cdot}70911_{10^{-3}}$	$-0{\cdot}72533_{10^{-1}}$
35/35	6·575	$m + n =$ odd. n.	$-1{\cdot}86487_{10^{-3}}$	$-2{\cdot}16594_{10^{-1}}$
35/35	11·325	$m + n =$ odd. n.	$-1{\cdot}56387_{10^{-3}}$	$-1{\cdot}70018_{10^{-1}}$
35/35	18·790	$m + n =$ odd n.	$-1{\cdot}18368_{10^{-3}}$	$-1{\cdot}24356_{10^{-1}}$
35/35	25·560	$m + n =$ odd n.	$-1{\cdot}01131_{10^{-3}}$	$-1{\cdot}04836_{10^{-1}}$
35/35	28·460	$m + n =$ odd n.	$-0{\cdot}97549_{10^{-3}}$	$-1{\cdot}00746_{10^{-1}}$
35/35	30·205	$m + n =$ odd n.	$-0{\cdot}85410_{10^{-3}}$	$-0{\cdot}88042_{10^{-1}}$

was treated while using 33 and 35 coordinate functions. The most important of the results are presented in Tabs. 4.3, 4.4. We see that in both cases the smallest value of Π^L is attained for w_0 in the shape of the eigenvector φ_3.

For a square plate with $w_0 \approx \varphi_1$, $w_0 \approx \varphi_3$ and

$$w_0 = w_{0,11} \sin \pi \frac{x}{a} \sin \pi \frac{y}{b} \tag{4.4}$$

the nonlinear problem (2.1), (2.2), (2.3) was approximately solved. Assuming the deflections w, w_0 in the forms (4.2), (4.3) and the function Φ according to [2]

(4.5)

$$\Phi = \sum_{r,s \geqq 2} \Phi_{rs} \left[\cos r\pi \frac{x}{a} - \cos\left(1 - (-1)^r\right) \frac{\pi x}{2a}\right] \left[\cos s\pi \frac{y}{b} - \cos\left(1 - (-1)^s\right) \frac{\pi y}{2b}\right],$$

the unknown coefficients w_{mn}, Φ_{rs} were determined from the conditions of orthogonality of the coordinate functions used in (4.2), (4.5) to the first and the second equation (2.1), respectively. In the case of w_0 given by (4.4) and $w_0 \approx \varphi_1$, 25 + 28 and in the case $w_0 \approx \varphi_3$, 19 + 25 coordinate functions were used. The eigenvectors φ_1, φ_3 were approximated by the same functions as w. The results are shown in Tabs. 4.5, 4.6. LSSM and $\bar{Q}$ are defined like in Section 4.2. Further solutions to the

Tab. 4.5

GSSM LSSM $\bar{Q}$	$\left[C_N \frac{D}{2} \int_0^b \int_0^a (\Delta w_0)^2 \, \mathrm{d}x \, \mathrm{d}y\right]^{1/2} = 0{\cdot}7$		
λ	$w_0 \approx$ $\sin \pi \frac{x}{a} \sin \pi \frac{y}{b}$ $\lvert w_0 \rvert / t = 0{\cdot}7$	$w_0 \approx \varphi_1$ $\lvert w_0 \rvert / t = 0{\cdot}49$	$w_0 \approx \varphi_3$ $\lvert w_0 \rvert / t = 0{\cdot}248$
0·932	0·06072 1·0178 −0·00074	0·07292 1·0417 −0·00103	0·05967 1·0226 −0·00068
4·66	0·39986 1·0404 −0·02288	0·56538 1·0825 −0·04140	0·43649 1·0411 −0·02534
9·32	1·6675 1·1555 −0·20223	2·0926 1·2281 −0·37629	1·6363 1·1201 −0·21296
13·98	4·4746 1·4132 −1·3175	4·8481 1·4848 −1·9251	4·1813 1·3161 −1·1911
18·64	8·1737 1·7370 −5·1221	8·3762 1·7996 −6·3454	7·7651 1·5944 −4·4466

nonlinear problem of a rectangular plate in shear are given in [2] ($a/b = 1, 2, 3$). The results underline the role of the minimum potential energy criterium.

5. CONCLUSION

The above presented theoretical results show that the set of eigenvectors of the linear stability problem is characteristic set of initial deflection shapes of the corresponding imperfect problem. The initial deflections having these shapes and the given value of global measure (the given value of energy norm) represent the common stationary points of potential energy and of bending strain energy functionals extremized on the set of admissible initial deflections $w_0 \in H$, $\|w_0\| = \text{const}$ in the corresponding equilibrium configurations for $\lambda \in (0, \lambda_{cr})$. The most dangerous initial deflec-

Tab. 4.6

GSSM LSSM $\bar{Q}$	$\max_{x,y} \lvert w_0\rvert/t = 0{\cdot}7$		
λ	$w_0 \approx \sin \pi \frac{x}{a} \sin \pi \frac{y}{b}$	$w_0 \approx \varphi_1$	$w_0 \approx \varphi_1$
0·932	0·06072 1·0178 −0·00074	0·09894 1·0771 −0·00200	0·14392 1·1448 −0·00464
4·66	0·39986 1·0404 −0·02288	0·71648 1·1357 −0·07518	0·91294 1·2080 −0·15134
9·32	1·6675 1·1555 −0·20223	2·3159 1·2912 −0·56902	2·4701 1·3327 −0·88310
13·98	4·4746 1·4132 −1·3175	4·9967 1·5375 −2·4201	4·8790 1·5173 −2·9570
18·64	8·1737 1·7370 −5·1221	8·4397 1·8421 −7·2108	8·1824 1·7674 −7·6941

tion defined from the standpoint of stability of structure in the sense of minimum of the potential energy contains also the standpoint of strength in the sense of maximum of the bending strain energy. The theory was applied to the column and plate problems and illustrated by numerical results. The case of compressed cylindrical panel was treated in Section 3.4.4 of the research report [4].

In special cases, the investigation of the equiamplitude set of initial deflections having the shapes of eigenvectors of the perfect problem was carried out. The results confirmed that the critical eigenvector is often not the most unfavourable initial deflection from this set (Hlaváček [3]). However, some cases were shown in which for a sufficiently small value of the load it may be even impossible to determine from the given set the most unfavourable initial deflection from the view-point of minimum of the potential energy value or of maximum of the bending strain energy value.

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Súhrn

TEORETICKÉ RIEŠENIE PROBLÉMU NAJNEBEZPEČNEJŠIEHO TVARU ZAČIATOČNÉHO PRIEHYBU PRI ÚLOHÁCH STABILITNÉHO TYPU

ZOLTÁN SADOVSKÝ

Zavádza sa globálna miera začiatočného priehybu w_0 daná energetickou normou. Na stanovenie najnebezpečnejšieho tvaru w_0 sa formuluje minimalizačný problém s vedľajšou podmienkou. Teoretické výsledky zahrňujú široký okruh stabilitných úloh stavebnej mechaniky.

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