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A THEORY FOR THE APPROXIMATION OF  
SOLUTIONS OF BOUNDARY VALUE PROBLEMS  
ON INFINITE INTERVALS

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ABSTRACT

An ad hoc method to solve boundary value problems which are posed on infinite intervals is to reduce the infinite interval to a finite but large one and to impose additional boundary conditions at the far end. These boundary conditions should be posed in a way so that they express the asymptotic behaviour of the actual solution well. In this paper a rigorous theory is derived which defines classes of appropriate additional boundary conditions. Appropriate is to be understood in the sense that the solutions of the approximate problems converge to the actual solution of the 'infinite' problem as the length of the finite interval tends to infinity. Moreover boundary conditions which produce convergence with the largest expectable order are devised.

AMS (MOS) Subject Classifications: 34B15, 34C05, 34E05, 41A60

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SIGNIFICANCE AND EXPLANATION

A boundary value problem on an infinite interval consists of a system of ordinary differential equations, some boundary conditions at a finite point and a continuity condition at infinity, for example it is required that the solution converge to a finite limit as the independent variable converges to infinity. This condition is problematic when solutions are sought computationally. Therefore it is useful to cut the infinite interval at a point which is far out and to impose some suitable, so called asymptotic, boundary conditions at that far end and to solve the resulting two-point boundary value problem which is now posed on a finite but large interval by any appropriate code. Difficulties occur in finding appropriate asymptotic boundary conditions in the sense that the solutions of the approximating two-point boundary value problems converge to the 'infinite' solution as the length of the interval converges to infinity. This paper devises suitable asymptotic boundary conditions which produce a fast - in most practical problems - exponential - order of convergence.

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A THEORY FOR THE APPROXIMATION OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS

ON INFINITE INTERVALS

Peter A. Markowich

1. Introduction

Boundary value problems on infinite intervals, which are posed in the following way

$$(1.1) \quad y' = t^\alpha f(t, y), \quad 1 < t < \infty, \quad \alpha \in \mathbb{N} \cup \{0\},$$

$$(1.2) \quad y \in C([1, \infty]) : \Leftrightarrow y \in C([1, \infty)) \text{ and } \lim_{t \rightarrow \infty} y(t) \text{ exists,}$$

$$(1.3) \quad b(y(1)) = 0$$

where  $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$  are often solved numerically by restricting the infinite interval to a finite but large one and by imposing additional suitable boundary conditions at the right end. The resulting two-point boundary value problem has the following form:

$$(1.4) \quad x_T' = t^\alpha f(t, x_T), \quad 1 < t < T, \quad T \gg 1,$$

$$(1.5) \quad b(x_T(1)) = 0,$$

$$(1.6) \quad S(x_T(T), T) = 0,$$

and can be solved by any appropriate code. The questions this paper answers are the following:

1) What class of asymptotic boundary conditions  $S(x_T(T), T) = 0$  imply convergence in the following sense

$$(1.7) \quad \|x_T - y\|_{[1, T]} \rightarrow 0 \text{ as } T \rightarrow \infty$$

where  $\|z\|_{[a, b]} := \sup_{t \in [a, b]} |z(t)|$  and

2) which asymptotic boundary conditions yield a reasonably fast order of convergence.

It will be shown that the admissible boundary conditions have to be constructed with regard to the invariant subspaces and eigenspaces of the matrix

$$(1.8) \quad A_0(y_\infty) := f_y(\infty, y(\infty))$$

and that the order of convergence depends on the decay properties of the solution  $y$  of (1.1), (1.2) and (1.3) and on the largest algebraic multiplicity of the eigenvalues of (1.8) which have real part zero. Calling this number  $\bar{r}$  we will show that for all admissible boundary conditions (1.6)

$$(1.9) \quad \|x_T - y\|_{[1,T]} \leq \text{const. } T^{(\alpha+1)\bar{r}} (\ln T)^{2j} \|S(y(T), T)\|, \quad 0 \leq j \leq n$$

holds if the solution  $y$  of the problem (1.1), (1.2), (1.3) is isolated and decays sufficiently fast.

Work on the solvability and asymptotic behavior of the solutions, of problems of the kind (1.1), (1.2) and (1.3) has been done by M. Lentini and H. B. Keller (1980), F. de Hoog and Richard Weiss (1980a,b) and P. Markowich (1980a,b).

The paper is organized as follows: in Chapter 2 linear constant coefficient problems are treated, Chapter 3 is concerned with linear time varying problems which have the property that the matrix describing the system evaluated at infinity has distinct eigenvalues, in Chapter 4 this assumption is neglected, Chapter 5 deals with nonlinear problems and a practical problem from fluid dynamics is dealt with in Chapter 6.

The novelty of this paper in comparison to the above mentioned ones is that no severe assumptions on  $f_y(t, y(\infty))$  are made while Lentini and Keller (1980) require a certain convergence behaviour of this matrix as  $t$  approaches infinity.

The used techniques are similar to those used by de Hoog and Weiss (1980b) who treated the case where  $f_y(\infty, y(\infty))$  does not have an eigenvalue with a zero real part.

## 2. Linear Constant Coefficient Problems

The problem

$$(2.1) \quad y' - t^\alpha Ay = t^\alpha f(t), \quad 1 < t < \infty, \quad \alpha \in \mathbb{R}, \quad \alpha > -1,$$

$$(2.2) \quad y \in C([1, \infty]),$$

$$(2.3) \quad By(1) \approx \beta$$

shall be approximated by the 'finite' problem

$$(2.4) \quad x_T' - t^\alpha Ax_T = t^\alpha f(t), \quad 1 < t < T,$$

$$(2.5) \quad Bx_T(1) \approx \beta,$$

$$(2.6) \quad S(T)x_T(T) = \gamma(T)$$

as  $T$  approaches infinity.  $A$  is assumed to be a real  $n \times n$  matrix with the Jordan form  $J$ :

$$(2.7) \quad A = EJE^{-1} \neq \theta$$

and  $J$  has the block diagonal form

$$(2.8) \quad J = \text{diag}(J^+, J^0, J^-)$$

where  $J^+$  contains the eigenvalues of  $A$  with positive real part,  $J^0$  the eigenvalues of  $A$  with a zero real part and  $J^-$  the eigenvalues of  $A$  with a negative real part.

The dimension of these three matrices are  $r_+$ ,  $r_0$  resp.  $r_-$  and the geometrical multiplicity of the eigenvalue zero will be called  $\bar{r}_0$ .

The projection like matrices  $G_+$ ,  $G_0$ ,  $G_-$  and  $\bar{G}_0$  are obtained by taking the matrices  $D_+$ ,  $D_0$ ,  $D_-$  and  $\bar{D}_0$ , which are the projections onto the direct sums of invariant subspaces of  $J$  belonging to eigenvalues with positive, zero, negative real part resp. onto the direct sums of eigenspaces belonging to zero eigenvalues of  $J^0$ , and by cancelling all columns of these matrices which have only zero entries. So  $G_+$  is  $n \times r_+$ ,  $G_0$  is  $n \times r_0$ ,  $G_-$  is  $n \times r_-$  and  $\bar{G}_0$  is  $n \times \bar{r}_0$ .

By substituting

$$(2.9) \quad u = E^{-1}y$$

we get the problem

$$(2.10) \quad u' - t^\alpha Ju = t^\alpha E^{-1}f(t), \quad 1 < t < \infty,$$

$$(2.11) \quad u \in C([1, \infty)) ,$$

$$(2.12) \quad BEu(1) = \beta .$$

The general solution of (2.10), (2.11) is

$$(2.13) \quad u(t) = [\phi(t)\bar{G}_0, \phi(t)G_-] \eta + (Hf)(t), \quad \eta \in C^{\bar{r}_0 + r_-}$$

where

$$(2.14) \quad \phi(t) = \exp\left(\frac{J}{\alpha + 1} t^{\alpha+1}\right)$$

and  $H$  is a solution operator for the inhomogeneous problem:

$$Hf = H_+f + H_0f + H_-f$$

where

$$(2.15) \quad (H_+f)(t) = \phi(t) \int_{\infty}^t D_+ \phi^{-1}(s) E^{-1} f(s) s^\alpha ds ,$$

$$(2.16) \quad (H_0f)(t) = \phi(t) \int_{\infty}^t D_0 \phi^{-1}(s) E^{-1} f(s) s^\alpha ds ,$$

$$(2.17) \quad (H_-f)(t) = \phi(t) \int_{\delta}^t D_- \phi^{-1}(s) E^{-1} f(s) s^\alpha ds$$

holds for  $\delta > 1$ .  $f$  is assumed to be in  $C([1, \infty))$  and in order to make the integral in

(2.16) exist we assume that

$$(2.18) \quad \|D_0 E^{-1} f(t)\| = O(t^{-(\alpha+1)r-\epsilon}), \quad \epsilon > 0 .$$

Here  $r$  is the maximal dimension of the invariant subspaces of  $J$  associated with imaginary eigenvalues. We assume that  $r > 0$  because the case  $r = 0$  has been treated by de Hoog and Weiss (1980b). An analysis of the operator  $H$  can be found in de Hoog and Weiss (1980a) and Lentini and Keller (1980a). Markowich (1980b) has shown the following estimates, which hold for  $t > \delta > 1$

$$(2.19) \quad \|(H_+f)(t)\| \leq \text{const.} \|D_+ E^{-1} f\|_{[t, \infty)} ,$$

$$(2.20) \quad \|(H_0f)(t)\| \leq \text{const.} t^{-\epsilon} \max_{s>t} \|s^{(\alpha+1)r+\epsilon} D_0 E^{-1} f(s)\| ,$$



$$(2.21) \quad \| (H_f)(t) \| \leq \text{const.} \cdot t^{-\gamma} \max_{\delta < s < t} \| s^{\gamma} D_E^{-1} f(s) \| \text{ for } \gamma > 0.$$

All constants are independent of  $f$  and  $\delta$ . Moreover we assume that  $B$  is a  $(\bar{r}_0 + r_-) \times n$ -matrix, that  $\beta \in R^{\bar{r}_0 + r_-}$  and that the  $(\bar{r}_0 + r_-) \times (\bar{r}_0 + r_-)$  matrix

$$(2.22) \quad BE[\phi(1)\bar{G}_0, \phi(1)G_-] \text{ is regular}$$

so that (2.3) defines  $\bar{r}_0 + r_-$  independent boundary conditions. According to Markowich (1980a,b) these propositions are necessary and sufficient for the unique solvability of the problems (2.1), (2.2) and (2.3) for all appropriate  $f$ 's. Therefore  $S(T)$  has to be a  $(r_+ + (r_0 - \bar{r}_0)) \times n$ -matrix and  $\gamma(T) \in R^{\bar{r}_+ + (r_0 - \bar{r}_0)}$  so that (2.5) and (2.6) set up  $n$  boundary conditions.

At first we prove a stability estimate for (2.4), (2.5), (2.6):

**Theorem 2.1:** We assume that (2.22) holds and that (A), (B), (C) which are defined as follows, are fulfilled.

- (A)  $\| S(T) \| \leq \text{const.}$  for  $T \rightarrow \infty$ ,
- (B)  $\| S(T) \bar{E} G_0 \| = o(T^{-(\alpha+1)(r-1)})$  for  $T \rightarrow \infty$ ,
- (C)  $\| [S(T) \bar{E} G_+, S(T) \bar{E} G_0]^{-1} \| \leq \text{const.}$  for  $T \rightarrow \infty$

where  $\bar{G}_0$  is the  $n \times (r_0 - \bar{r}_0)$ -matrix which is obtained by cancelling the columns of the matrix  $\bar{D}_0 = D_0 - \bar{D}_0$  which have only zero entries.

Then the problem (2.4), (2.5), (2.6) has a unique solution  $x_T$  for all  $T$  sufficiently large and  $x_T$  fulfills the stability estimate:

$$(2.23) \quad \| x_T \|_{[1,T]} \leq \text{const.} (\| B \| + T^{(\alpha+1)(r-1)} \| \gamma(T) \| + T^{(\alpha+1)r} \| f \|_{[1,T]})$$

if  $f \in C([1,T])$ ,  $\beta \in R^{\bar{r}_0 + r_-}$ ,  $\gamma(T) \in R^{n - (\bar{r}_0 + r_-)}$ .

In order to prove this we first reorder  $J^0$  by permuting its lines and columns so that

$$(2.24) \quad R J^0 R^{-1} = \left[ \begin{array}{c|c} J_1^0 & \theta \\ \hline J_2^0 & \underbrace{\begin{matrix} 0 & \dots & 0 \end{matrix}}_{\bar{r}_0} \end{array} \right] = \tilde{J}^0$$

where  $R$  is an appropriate permutation matrix. This corresponds to a reordering of the columns of  $E$ . The reordered matrix will be called  $\tilde{E}$ . Now

$$(2.25) \quad (a) \quad \bar{G}_0 = \begin{matrix} \left. \begin{matrix} \theta \\ I_{r_0} \\ \theta \end{matrix} \right\} r_+ + (r_0 - \bar{r}_0) \\ \left. \begin{matrix} \theta \\ I_{r_0} \\ \theta \end{matrix} \right\} r_- \\ r_0 \end{matrix}, \quad (b) \quad \bar{G}_0 = \begin{matrix} \left. \begin{matrix} \theta \\ I_{r_0 - \bar{r}_0} \\ \theta \end{matrix} \right\} r_+ \\ \left. \begin{matrix} \theta \\ I_{r_0} \\ \theta \end{matrix} \right\} \bar{r}_0 + r_- \\ r_0 - \bar{r}_0 \end{matrix}$$

holds. From (2.24) it is easily concluded that

$$(2.26) \quad \exp\left(\frac{J^0}{\alpha+1} (t^{\alpha+1} - 1)\right) = \begin{matrix} \left. \begin{matrix} e_1(t^{\alpha+1} - 1) & \theta \end{matrix} \right\} r_0 - \bar{r}_0 \\ \left. \begin{matrix} e_2(t^{\alpha+1} - 1) \\ I_{r_0} \end{matrix} \right\} \bar{r}_0 \\ r_0 - \bar{r}_0 \quad \bar{r}_0 \end{matrix}$$

We substitute

$$(2.27) \quad \tilde{v}_T = \tilde{E}^{-1} x_T$$

and get the problem

$$(2.28) \quad \tilde{v}_T' - t^{\alpha} \tilde{J} \tilde{v}_T = t^{\alpha} \tilde{E}^{-1} f(t), \quad 1 \leq t \leq T,$$

$$(2.29) \quad \tilde{B} \tilde{E} \tilde{v}_T(1) = \beta,$$

$$(2.30) \quad S(T) \tilde{E} \tilde{v}_T(T) = \gamma(T)$$

where  $\tilde{J}$  has the block structure

$$(2.31) \quad \tilde{J} = \text{diag}(J^+, \tilde{J}^0, J^-).$$

We write the general solution of (2.28) as follows:

$$(2.32) \quad \tilde{v}_T(t) = A(t, T) \xi_1 + C(t) \xi_2 + \tilde{v}_p(t, T)$$

where

$$(2.33) \quad A(t, T) = \left[ \begin{array}{cc} \exp\left(\frac{J^+}{\alpha+1} (t^{\alpha+1} - T^{\alpha+1})\right) & \theta \\ \theta & \theta \\ \theta & e_1(t^{\alpha+1} - 1) \\ \theta & e_2(t^{\alpha+1} - 1) \\ \theta & \theta \end{array} \right] \Bigg\}^n$$

$\underbrace{\hspace{10em}}_{r_+} \qquad \underbrace{\hspace{10em}}_{r_0 - \bar{r}_0}$

and

$$(2.34) \quad C(t) = \left[ \begin{array}{cc} \theta & \theta \\ \theta & \theta \\ I & \theta \\ \theta & \exp\left(\frac{J^-}{\alpha+1} t^{\alpha+1}\right) \\ \theta & \theta \end{array} \right] \Bigg\}^n$$

$\underbrace{\hspace{10em}}_{\bar{r}_0} \qquad \underbrace{\hspace{10em}}_{r_-}$

so that  $\xi_1 \in C^{r_+ + (r_0 - \bar{r}_0)}$  and  $\xi_2 \in C^{\bar{r}_0 + r_-}$  holds.  $\tilde{v}_p(t, T)$  is an appropriate particular solution which will be defined later.

From (2.26) we easily derive the following properties of  $e_1, e_2$ :

$$(2.35) \quad (a) \quad e_1(0) = I_{r_0 - \bar{r}_0}, \quad (b) \quad e_2(0) = \theta,$$

$$(2.36) \quad (a) \quad e_1(t^{\alpha+1} - 1)^{-1} = e_1(1 - t^{\alpha+1}),$$

$$(2.36) \quad (b) \quad e_2(t^{\alpha+1} - 1)e_1(1 - t^{\alpha+1}) = -e_2(1 - t^{\alpha+1})$$

for all  $t \in \mathbb{R}$ . A more general statement than (2.36) is

$$(2.37) \quad (a) \quad e_1(t_0^{\alpha+1} - 1)e_1(1 - t_1^{\alpha+1}) = e_1(t_0^{\alpha+1} - t_1^{\alpha+1}),$$

$$(2.37) \quad (b) \quad e_2(t_0^{\alpha+1} - 1)e_1(1 - t_1^{\alpha+1}) = e_2(t_0^{\alpha+1} - t_1^{\alpha+1}) - e_2(1 - t_1^{\alpha+1})$$

for all  $t_0, t_1 \in \mathbb{R}$ .

$$(2.38) \quad (a) \quad \|e(t^{\alpha+1} - 1)\| = O(t^{(\alpha+1)(r-1)}) \quad \text{for } t \rightarrow \infty,$$

$$(2.38) \quad (b) \quad \|e_2(t^{\alpha+1} - 1)\| = O(t^{(\alpha+1)(r-1)}) \quad \text{for } t \rightarrow \infty.$$

The estimate (2.38) is derived by using that

$$(2.39) \quad \exp\left(\frac{J_k}{\alpha+1} t^{\alpha+1}\right) = \exp\left(i \frac{\gamma}{\alpha+1} t^{\alpha+1}\right) F(t)$$

where  $J_k$  is an  $r_k$  dimensional Jordan block with the imaginary eigenvalue  $i\gamma$  and  $F(t)$  is a real matrix whose entries are polynomials of maximal degree  $(r_k - 1)(\alpha + 1)$ .

By inserting (2.32) into the boundary conditions (2.29), (2.30) we get the linear block system

$$(2.40) \quad \begin{bmatrix} A_1(T) & A_2 \\ A_3(T) & A_4(T) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \beta - \widetilde{B}\widetilde{E}\widetilde{V}_p(1, T) \\ \gamma(T) - S(T)\widetilde{E}\widetilde{V}_p(T, T) \end{bmatrix}$$

where

$$(2.41) \quad (a) \quad A_1(T) = [\widetilde{B}\widetilde{E}\widetilde{G}_+ \exp\left(\frac{J^+}{\alpha+1} (1 - T^{\alpha+1})\right), \widetilde{B}\widetilde{E}\widetilde{G}_0],$$

$$(2.41) \quad (b) \quad A_2 = [\widetilde{B}\widetilde{E}\widetilde{G}_0, \widetilde{B}\widetilde{E}\widetilde{G}_- \exp\left(\frac{J^-}{\alpha+1}\right)],$$

$$(2.41) \quad (c) \quad A_3(T) = [S(T)\widetilde{E}\widetilde{G}_+, S(T)\widetilde{E}\widetilde{G}_0 e_1 (T^{\alpha+1} - 1) + S(T)\widetilde{E}\widetilde{G}_0 e_2 (T^{\alpha+1} - 1)],$$

$$(2.41) \quad (d) \quad A_4(T) = [S(T)\widetilde{E}\widetilde{G}_0, S(T)\widetilde{E}\widetilde{G}_- \exp\left(\frac{J^-}{\alpha+1} T^{\alpha+1}\right)].$$

The system (2.40) is soluble iff the matrices  $A_2$  and  $(A_3(T) - A_4(T)A_2^{-1}A_1(T))$  are invertible.  $A_2$  is invertible by assumption (2.22) and the existence of  $(A_3 - A_4A_2^{-1}A_1)^{-1}$  has to be proven. We will show that

$$(2.42) \quad (A_3 - A_4A_2^{-1}A_1)^{-1} = \begin{bmatrix} I & \theta \\ \theta & e_1(1 - T^{\alpha+1}) \end{bmatrix} [S(T)\widetilde{E}\widetilde{G}_+, S(T)\widetilde{E}\widetilde{G}_0]^{-1} (I + o(1)) \quad \text{for } T \rightarrow \infty.$$

The existence of the right hand side of (2.42) is assured by proposition (C) of Theorem 2.1.

We split  $A_3(T)$  into:

$$(2.43) \quad A_3(T) = \underbrace{[S(T)\widetilde{E}\widetilde{G}_+, S(T)\widetilde{E}\widetilde{G}_0 e_1 (T^{\alpha+1} - 1)]}_{A_3^1(T)} + \underbrace{[\theta, S(T)\widetilde{E}\widetilde{G}_0 e_2 (T^{\alpha+1} - 1)]}_{A_3^2(T)}$$

and get

$$(2.44) \quad A_3^1(T) = (S(T)\tilde{E}G_+, S(T)\tilde{E}G_0) \begin{bmatrix} I & 0 \\ \theta & e_1(T^{\alpha+1} - 1) \end{bmatrix}.$$

From (C) and from (2.36)(a) we conclude

$$(2.45) \quad (A_3^1(T))^{-1} = \begin{bmatrix} I & 0 \\ \theta & e_1(1 - T^{\alpha+1}) \end{bmatrix} [S(T)\tilde{E}G_+, S(T)\tilde{E}G_0]^{-1}.$$

Moreover

$$(2.46) \quad \begin{aligned} A_3^2(T)(A_3^1(T))^{-1} &= [\theta, S(T)\tilde{E}G_0 e_2(T^{\alpha+1} - 1)e_1(1 - T^{\alpha+1})] [S(T)\tilde{E}G_+, S(T)\tilde{E}G_0]^{-1} = \\ &= [\theta, -S(T)\tilde{E}G_0 e_2(1 - T^{\alpha+1})] [S(T)\tilde{E}G_+, S(T)\tilde{E}G_0]^{-1} \end{aligned}$$

holds because of (2.36)(b). The proposition (B) and (2.38)(b) assure that

$$(2.47) \quad \|A_3^2(T)(A_3^1(T))^{-1}\| = o(1) \text{ for } T \rightarrow \infty.$$

Therefore  $A_3(T)^{-1}$  exists and can be written as

$$(2.48) \quad \begin{aligned} (A_3(T))^{-1} &= (A_3^1(T))^{-1} (I + \sum_{i=1}^{\infty} (-1)^i (A_3^2(T)(A_3^1(T))^{-1})^i) = \\ &= (A_3^1(T))^{-1} (I + o(1)) \text{ for } T \rightarrow \infty. \end{aligned}$$

Moreover

$$(2.49) \quad (a) \quad \|A_4(T)\| = o(T^{-(\alpha-1)(r-1)}) \text{ for } T \rightarrow \infty,$$

$$(2.49) \quad (b) \quad \|A_1(T)\| = O(1) \text{ for } T \rightarrow \infty$$

and therefore

$$(2.50) \quad \|A_4(T)A_2^{-1}A_1(T)(A_3(T))^{-1}\| = o(1) \text{ for } T \rightarrow \infty$$

because of (2.48), (2.49), (2.45). So

$$(2.51) \quad \begin{aligned} (A_3 - A_4A_2^{-1}A_1)^{-1} &= (A_3(T))^{-1} (I + \sum_{i=1}^{\infty} (A_4(T)A_2^{-1}A_1(T)(A_3(T))^{-1})^i) \\ &= (A_3^1(T))^{-1} (I + o(1)) \text{ for } T \rightarrow \infty \end{aligned}$$

and (2.42) follows immediately. The linear equation (2.40) can now be solved:

$$(2.52) \quad \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{bmatrix} -(A_3 - A_4 A_2^{-1} A_1)^{-1} A_4 A_2^{-1} & (A_3 - A_4 A_2^{-1} A_1)^{-1} \\ A_2^{-1} + A_2^{-1} A_1 (A_3 - A_4 A_2^{-1} A_1)^{-1} A_4 A_2^{-1} & -A_2^{-1} A_1 (A_3 - A_4 A_2^{-1} A_1)^{-1} \end{bmatrix} \begin{pmatrix} \beta - B \tilde{E} \tilde{V}_p(1, T) \\ \gamma(T) - S(T) \tilde{E} \tilde{V}_p(T, T) \end{pmatrix}.$$

Inserting  $\xi_1$  into (2.33) the term  $A(t, T)(A_3(T) - A_4(T)A_2^{-1}A_1(T))^{-1}$  appears. From (2.33) and (2.42) we get

$$(2.53) \quad A(t, T)(A_3 - A_4 A_2^{-1} A_1)^{-1} = \begin{bmatrix} \exp\left(\frac{J^+}{\alpha+1} (t^{\alpha+1} - T^{\alpha+1})\right) & 0 \\ 0 & e_1(t^{\alpha+1} - 1)e_1(1 - T^{\alpha+1}) \\ 0 & e_2(t^{\alpha+1} - 1)e_1(1 - T^{\alpha+1}) \\ 0 & 0 \end{bmatrix} O(1).$$

Using (2.37) we conclude that

$$(2.54) \quad \|A(\cdot, T)(A_3(T) - A_4(T)A_2^{-1}A_1(T))^{-1}\|_{[1, T]} = O(T^{(\alpha+1)(r-1)}),$$

holds.

From (2.52), (2.33) and (A) we derive:

$$(2.55) \quad \|\tilde{v}_T\|_{[1, T]} < \text{const.} (\|\beta\| + T^{(\alpha+1)(r-1)} \|\gamma(T)\| + \|\tilde{v}_p(1, T)\| + T^{(\alpha+1)(r-1)} \|\tilde{v}_p(T, T)\| + \|\tilde{v}_p(\cdot, T)\|_{[1, T]}).$$

Now  $\tilde{v}_p(\cdot, T)$  has to be defined. We set

$$(2.56) \quad \hat{f}(t, T) = \begin{cases} f(t), & 1 < t < T \\ f(T), & T < t < \infty \end{cases}$$

and

$$(2.57) \quad ({}_{T+} H f)(t) := \int_T^t \exp\left(\frac{\tilde{J}}{\alpha+1} (t^{\alpha+1} - s^{\alpha+1})\right) D_+ \tilde{E}^{-1} s^\alpha f(s) ds,$$

$$(2.58) \quad ({}_{T-} H f)(t) := \int_T^t \exp\left(\frac{\tilde{J}}{\alpha+1} (t^{\alpha+1} - s^{\alpha+1})\right) D_0 \tilde{E}^{-1} s^\alpha f(s) ds,$$

$$(2.59) \quad ({}_{T-} H f)(t) = ({}_{H-} \tilde{E}^{-1} f)(t), \quad \delta < t < T, \quad 1 < \delta < T$$

so that we can define:

$$(2.60) \quad \tilde{v}_p(\cdot, T) = H_T f = {}_T H_+ f + {}_T H_0 f + {}_T H_- f .$$

From the estimates (2.19) and (2.21) we conclude that

$$(2.61) \quad (a) \quad \| {}_T H_+ f + {}_T H_- f \|_{[\delta, T]} < \text{const.} \| f \|_{[\delta, T]}$$

because

$$(2.61) \quad (b) \quad \| {}_T H_+ f \|_{[\delta, T]} < \text{const.} \| H_+ \|_{[\delta, \infty]} \| f(\cdot, T) \|_{[\delta, \infty]}$$

holds. Moreover (2.58) can be estimated as follows

$$(2.62) \quad \begin{aligned} \| ({}_T H_0 f)(t) \| &< \text{const.} \| f \|_{[t, T]} \int_t^T (t^{\alpha+1} - s^{\alpha+1})^{(r-1)} s^\alpha ds < \\ &< \text{const.} T^{(\alpha+1)r} \| f \|_{[t, T]} . \end{aligned}$$

Altogether we get

$$(2.63) \quad \| H_T f \|_{[\delta, T]} < c T^{(\alpha+1)r} \| f \|_{[\delta, T]} ,$$

and

$$(2.64) \quad \| \tilde{v}_p(T, T) \| = \| ({}_T H_- f)(T) \| < \text{const.} \| f \|_{[1, T]}$$

holds because  $({}_T H_0 f)(T) = 0$ . From these estimates and from (2.55) the stability estimate (2.23) follows.

In order to derive a convergence statement we write the problem (2.1), (2.2), (2.3) as follows

$$(2.65) \quad y' - t^\alpha A y = t^\alpha f(t), \quad 1 < t < T, \quad y \in C([1, \infty)) ,$$

$$(2.66) \quad B y(1) = \beta ,$$

$$(2.67) \quad S(T) y(T) = S(T) y(1)$$

and subtract (2.4), (2.5), (2.6) from (2.65), (2.66), (2.67). We get

$$(2.68) \quad (y - x_T)' - t^\alpha A (y - x_T) = 0 ,$$

$$(2.69) \quad B (y - x_T)(1) = 0 ,$$

$$(2.70) \quad S(T) (y - x_T)(T) = S(T) y(T) - \gamma(T) .$$

If  $S(T)$  fulfills the assumptions (A), (B), (C) Theorem 2.1 can be applied giving:

$$(2.71) \quad \| y - x_T \|_{[1, T]} < \text{const.} T^{(\alpha+1)(r-1)} \| S(T) y(T) - \gamma(T) \| .$$

Setting  $\gamma(T) \equiv 0$  and using (2.9), (2.13) we get

$$(2.72) \quad \|y - x_T\|_{[1,T]} < \text{const. } T^{(\alpha+1)(r-1)} \|S(T)E[\phi(T)\bar{G}_0, \phi(T)G_-] \eta + S(T)E(Hf)(T)\|$$

for some  $\eta \in C^{r-\bar{r}_0}$ . Assumption (B) guarantees that all columns of the fundamental matrix which are constant are damped by an  $o(T^{-(\alpha+1)(r-1)})$ . All other appearing columns decay exponentially. Therefore the term which originates from the solution of the homogeneous problem converges as an  $o(1)$  as  $T \rightarrow \infty$ . So Assumption (B) is necessary for convergence for general  $\beta$  and  $f$ . Now let

$$(2.73) \quad \|f(t)\| = o(t^{-(\alpha+1)(2r-1)-\epsilon}), \quad \epsilon > 0.$$

From the estimates (2.19), (2.20), (2.21) we conclude that

$$(2.74) \quad \|(Hf)(T)\| = o(T^{-(\alpha+1)(r-1)-\epsilon})$$

holds. Altogether we get

$$(2.75) \quad \|y - x_T\|_{[1,T]} < \text{const. } T^{(\alpha+1)(r-1)} (\|S(T)E\bar{G}_0\| + \|S(T)E\phi(T)G_-\| + T^{-\epsilon}).$$

If  $f(t)$  contains an exponentially decreasing factor so that it has the asymptotic behaviour

$$(2.76) \quad \|f(t)\| = o(t^\beta \exp(-\frac{\omega}{\alpha+1} t^{\alpha+1})), \quad \omega > 0$$

then there is an operator  $\bar{H}$  so that  $y_p = E\bar{H}f$  is a particular solution of (2.65) and

$$(2.77) \quad \|(E\bar{H}f)(T)\| = o(T^{\beta+(\alpha+1)r} (\ln T) \exp(-\frac{\omega}{\alpha+1} T^{\alpha+1}))$$

holds. A proof for this can be found in Markowich (1980b). In this case  $T^{-\epsilon}$  in (2.75)

has to be substituted by  $T^{\beta+(\alpha+1)(2r-1)} \ln T \exp(-\frac{\omega}{\alpha+1} T^{\alpha+1})$ .

An optimal choice  $S(T) = S_D(T)$  would be so that

$$(2.78) \quad [S_D(T)E\bar{G}_0, S_D(T)EG_-] = \theta$$

holds. (2.78) is fulfilled for

$$(2.79) \quad S_D(T)E \equiv S_D E = \begin{bmatrix} (G_+)^T \\ (\bar{G}_0)^T \end{bmatrix} \iff E^T S_D^T = [G_+, \bar{G}_0].$$

These are linear equations for the rows of  $S_D$  which can be chosen independently of  $T$ . The asymptotic boundary condition (2.79), called projection condition fulfills (A), (B), (C) in Theorem 2.1 and is optimal in the sense that it makes the first two terms on the right hand side of (2.75) vanish.



### 3. Linear Variable Coefficient Problems - Distinct Eigenvalues

In this chapter we analyze the problem

$$(3.1) \quad y' - t^\alpha A(t)y = t^\alpha f(t), \quad \alpha \in \mathbb{N}_0, \quad 1 < t < \infty,$$

$$(3.2) \quad y \in C([1, \infty]),$$

$$(3.3) \quad By(1) = \beta$$

and we require the  $n \times n$  matrix  $A(t)$  to fulfill

$$(3.4) \quad A \in C([1, \infty]), \quad A(\infty) \neq 0,$$

$$(3.5) \quad A(t) = \sum_{i=0}^{\infty} A_i t^{-i} \text{ for } t \text{ sufficiently large.}$$

Moreover let  $J_0$  be the Jordan form of  $A_0$  obtained by

$$(3.6) \quad A_0 = EJ_0E^{-1}.$$

The following assumption is basic for this chapter:

$$(3.7) \quad J_0 = \text{diag}(\lambda_1, \dots, \lambda_n), \quad \lambda_i \neq \lambda_j \text{ for } i \neq j.$$

The substitution

$$(3.8) \quad u = E^{-1}y$$

gives the problem

$$(3.9) \quad u' - t^\alpha J(t)u = t^\alpha E^{-1}f(t), \quad 1 < t < \infty,$$

$$(3.10) \quad u \in C([1, \infty))$$

where

$$(3.11) \quad J(t) = E^{-1}A(t)E = \sum_{i=0}^{\infty} J_i t^{-i}, \quad J_i = E^{-1}A_i E.$$

The fundamental matrix of the homogeneous problem (3.9) can be represented as an asymptotic series (see Wasow (1965) and Coddington and Levinson (1955)):

$$(3.12) \quad \phi(t) = P(t)t^D e^{Q(t)}$$

where

$$(3.13) \quad P(t) \sim I + \sum_{i=1}^{\infty} P_i t^{-i},$$

$$(3.14) \quad D = \text{diag}(d_1, \dots, d_n),$$

$$(3.15) \quad Q(t) = J_0 \frac{t^{\alpha+1}}{\alpha+1} + Q_1 \frac{t^\alpha}{\alpha} + \dots + Q_\alpha t, \quad Q_i \text{ are diagonal matrices}$$

hold. The unknown coefficients  $P_i$ ,  $Q_i$  and  $D$  can be calculated by algebraic operation from the  $J_i$ 's. An algorithm for that is given in Markowich (1980b), and therefore the asymptotic behaviour of the basic solution can be determined knowing the  $Q_i$ 's and  $D$ .

Let  $\tilde{D}_0$  be the projection onto the direct sum of eigenspaces of  $J_0$  associated with those eigenvalues with a real part zero which produce a basic solution which is in  $C([1, \infty))$  and let  $\tilde{G}_0$  be the projection like matrix which is obtained by cancelling those columns of  $\tilde{D}_0$  which have only zero entries.  $\tilde{G}_0$  be a  $n \times \tilde{r}_0$  matrix. Then the general solution of the problem (3.9), (3.10) is

$$(3.16) \quad u(t) = [\phi(t)\tilde{G}_0, \phi(t)G_-] \eta + (Hf)(t), \quad \eta \in C^{\tilde{r}_0 + r_-}$$

where  $G_-$  is defined as in Chapter 2 and  $u_p(t) = (Hf)(t)$  is an appropriate particular solution, which has been described by Markowich (1980b). The operator  $H$  operates on the space of all functions fulfilling

$$(3.17) \quad f \in C([\delta, \infty)), \quad \delta > 1 \quad \text{and} \quad \|f(t)\| = O(t^{-\alpha-1-\epsilon}), \quad \epsilon > 0.$$

Then the estimate

$$(3.18) \quad \|(Hf)(t)\| \leq \text{const.} \cdot t^{-\epsilon} \ln t \cdot \max_{s > \delta} \|s^{\alpha+1+\epsilon} f(s)\|$$

has been proven by Markowich (1980b). The particular solution on  $[1, \infty)$  is obtained by continuation. The boundary value problems (3.1), (3.2), (3.3) is - under the given assumption on  $A(t)$  and  $f$  - for all  $\beta \in R^{\tilde{r}_0 + r_-}$  uniquely soluble iff the

$(\tilde{r}_0 + r_-) \times (\tilde{r}_0 + r_-)$  matrix

$$(3.19) \quad BE[\phi(1)\tilde{G}_0, \phi(1)G_-] \text{ is regular.}$$

Of course, B has to be a  $(\tilde{r}_0 + r_-) \times n$  matrix. We consider the approximating problems

$$(3.20) \quad x_T' - t^\alpha A(t)x_T = t^\alpha f(t), \quad 1 \leq t \leq T, \quad \alpha \in N_0,$$

$$(3.21) \quad Bx_T(1) = \beta,$$

$$(3.22) \quad S(T)x_T(T) = \gamma(T).$$

$S(T)$  is a  $(n - (\tilde{r}_0 + r_-)) \times n$  matrix and  $\gamma(T) \in R^{n - (\tilde{r}_0 + r_-)}$ . For the following  $G_+$  is defined as in Chapter 2 and  $\tilde{G}_0$  is the  $n \times (r_0 - \tilde{r}_0)$  matrix which is obtained by cancelling the zero columns of  $D_0 - \tilde{D}_0$ . Then the following stability theorem, which is the analogous to Theorem 2.1, holds:

**Theorem 3.1:** Assume that (3.19) and  $(A_1)$ ,  $(B_1)$ ,  $(C_1)$  which are defined as follows, hold:

$$(A_1) \quad \|S(T)\| \leq \text{const. as } T \rightarrow \infty,$$

$$(B_1) \quad \|S(T)EG_0\| = o(1) \text{ as } T \rightarrow \infty,$$

$$(C_1) \quad \|[S(T)EG_+, S(T)EG_0]^{-1}\| \leq \text{const. as } T \rightarrow \infty.$$

Then the problems (3.19), (3.21), (3.22) has a unique solution  $x_T$  for sufficiently large  $T$ .  $x_T$  fulfills the estimate

$$(3.23) \quad \|x_T\|_{[1,T]} \leq \text{const.} (\|\beta\| + \|\gamma(T)\| + T^{\alpha+1} \|nT\|_{[1,T]})$$

for  $f \in C([1,T])$ ,  $\beta \in R^{\tilde{r}_0 + r_-}$ ,  $\gamma(T) \in R^{n - (\tilde{r}_0 + r_-)}$ .

The substitution

$$(3.24) \quad v_T = E^{-1}x_T$$

gives the new problem

$$(3.25) \quad v_T' = t^\alpha J(t)v_T = t^\alpha E^{-1}f(t), \quad 1 \leq t \leq T,$$

$$(3.26) \quad BEv_T(1) = \beta,$$

$$(3.27) \quad S(T)Ev_T(T) = \gamma(T).$$

As the general solution of (3.25) we take for convenience

$$(3.28) \quad v_T(t) = \phi(t)e^{-Q(T)}T^{-D}[G_+, \tilde{G}_0]\xi_1 + \phi(t)[\tilde{G}_0, G_-]\xi_2 + v_p(t, T)$$

where  $\xi_1 \in C^{r_+ + (r_0 - \tilde{r}_0)}$ ,  $\xi_2 \in C^{\tilde{r}_0 + r_-}$  hold and  $v_p(\cdot, T)$  is an appropriate particular solution which will be defined later.  $\phi(t)$  is the fundamental matrix as of (3.12).

Evaluation of the boundary conditions (3.26), (3.27) gives the linear block system

$$(3.29) \begin{bmatrix} BE\phi(1)e^{-Q(T)}T^{-D}[G_+, \bar{G}_0] & BE\phi(1)[\tilde{G}_0, G_-] \\ S(T)EP(T)[G_+, \bar{G}_0] & S(T)E\phi(T)[\tilde{G}_0, G_-] \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \beta - BEv_p(1, T) \\ \gamma(T) - S(T)Ev_p(T, T) \end{bmatrix}.$$

The matrix in the (1.1) position is bounded because of the definition of  $G_+$ ,  $\bar{G}_0$  and because of the diagonal form of  $Q(T)$  and  $D$ . The matrix in the (2.2) position is bounded too, because  $\phi(t)[\tilde{G}_0, G_-]$  is the matrix, whose columns are the basic solution of the homogeneous problem which are in  $C([1, \infty))$  and because  $(A_1)$  holds. Moreover

$$(3.30) \quad \|S(T)E\phi(T)\| = o(1) \text{ as } T \rightarrow \infty$$

because of  $(B_1)$  and (3.12). The matrix in the (1.2) position is invertible because of (3.19) and its inverse is, as the matrix, independent of  $T$ . Finally

$$(3.31) \quad S(T)EP(T)[G_+, \bar{G}_0] = S(T)E[G_+, \bar{G}_0] + O(T^{-1})$$

because of the asymptotic expansion for  $P(t)$ ,  $(C_1)$  assures the bounded invertibility of the matrix in the (2.1) position.

From (2.40) and (2.52) we conclude immediately that the system (3.29) has a unique solution  $(\xi_1, \xi_2) \in C^n$  and the estimate

$$(3.32) \quad \|v_T\|_{[1, T]} \leq \text{const.} (\|\beta\| + \|\gamma(T)\| + \|v_p(\cdot, T)\|_{[1, T]})$$

follows.

The particular solution  $v_p(\cdot, T)$  has to be defined now. We set

$$(3.33) \quad (a) \quad v_p(\cdot, T) = H_T^{(1)}f = H_+^{(1)}f + H_0^{(1)}f + H_-^{(1)}f, \quad (b) \quad H_0^{(1)}f = \sum_{i=1}^{r_0} H_{0i}^{(1)}f$$

where

$$(3.34) \quad (H_+^{(1)}f)(t) = \phi(t) \int_T^t D_+ \phi^{-1}(s) s^{\alpha-1} E^{-1} f(s) ds,$$

$$(3.35) \quad (H_-^{(1)}f)(t) = \phi(t) \int_\delta^t D_- \phi^{-1}(s) s^{\alpha-1} E^{-1} f(s) ds, \quad \delta > 1$$

and

$$(3.36) \quad ({}_{T}H_{0i}^{(1)} f)(t) = \begin{cases} \phi(t) \int_T^t D_{0i} \phi^{-1}(s) s^{\alpha-1} f(s) ds, & \text{if (I) holds} \\ \phi(t) \int_{\delta}^t D_{0i} \phi^{-1}(s) s^{\alpha-1} f(s) ds, & \text{if (II) holds.} \end{cases}$$

$D_{0i}$  is the projection onto that eigenspace of  $J_0$  which belongs to the  $i$ th eigenvalue with real part zero and (I), (II) are defined as

$$(I) \quad \operatorname{Re}(Q(t)D_{0i}) \rightarrow +\infty \text{ or } (\operatorname{Re}(Q(t)D_{0i}) \equiv 0 \text{ and } \operatorname{Re}(DD_{0i}) > 0),$$

$$(II) \quad \operatorname{Re}(Q(t)D_{0i}) \rightarrow -\infty \text{ or } (\operatorname{Re}(Q(t)D_{0i}) \equiv 0 \text{ and } \operatorname{Re}(DD_{0i}) < 0).$$

From the considerations in Markowich (1980b), Chapter 3, we immediately conclude that

$$(3.37) \quad \|H_T^{(1)} f\|_{[\delta, T]} < T^{\alpha+1} \ln T \|f\|_{[\delta, T]}.$$

Therefore the estimate (3.23) follows and Theorem 3.1 is proven.

As in Chapter 2 the convergence estimate follows

$$(3.38) \quad \|y - x_T\|_{[1, T]} < \operatorname{const.} \|S(T)y(T) - \gamma(T)\|$$

for all  $(r_+ + (r_0 - \tilde{r}_0)) \times n$  matrices  $S(T)$  which fulfill  $(A_1)$ ,  $(B_1)$  and  $(C_1)$ .

Setting  $\gamma(T) \equiv 0$  and inserting (3.16) we conclude

$$(3.39) \quad \|y - x_T\|_{[1, T]} < \operatorname{const.} (\|S(T)E\phi(T)[\tilde{G}_0, G_-]\| + \|S(T)E(Hf)(T)\|).$$

The assumptions  $(A_1)$  and  $(B_1)$  guarantee convergence for all  $f$  fulfilling (3.17) because

(3.18) holds. If

$$(3.40) \quad S(T)E\tilde{G}_0 \equiv 0$$

convergence of the order  $T^{-1} + T^{-\epsilon} \ln T$  follows. In many practical cases all eigenvalues

with real part zero produce exponentially decaying solutions and  $f$  also decays

exponentially. The operator  $H$  can be changed to an operator  $\bar{H}$ , so that  $(\bar{H}f)(t)$

decays with the same exponential factor (see Markowich (1980b)). In this case exponential convergence follows from (3.39).

The optimal boundary condition is again the projection condition and it has to be calculated from the equation

$$(3.41) \quad S_D(T)EP(T) \equiv \begin{bmatrix} G_+^T \\ (\tilde{G}_0)^T \end{bmatrix}$$

which is uniquely soluble because of the regularity of  $E$  and  $P(T)$ .

The asymptotic boundary condition

$$(3.42) \quad S_D(T)x_T(T) = 0$$

would imply that

$$(3.43) \quad S_D(T)E\phi(T)[\tilde{G}_0, G_-] \equiv 0$$

because of the form of  $\phi(T)$ . However, we do not know  $P(T)$ , but we can calculate the coefficients  $P_i$  of its expansion recursively (for the algorithm see Markowich (1980b)).

Having calculated  $P_1, P_2, \dots, P_k$  we set

$$(3.44) \quad \bar{P}(T) = I + \sum_{i=1}^k P_i T^{-i}$$

and solve

$$(3.45) \quad \bar{S}_D(T)E\bar{P}(T) = \begin{bmatrix} G_+^T \\ (\tilde{G}_0)^T \end{bmatrix}$$

instead of (3.41). Because

$$(3.46) \quad P(T) = \bar{P}(T) + O(T^{-k-1}) \quad \text{as } T \rightarrow \infty$$

holds we get by a simple perturbation analysis

$$(3.47) \quad \bar{S}_D(T) = S_D(T) + O(T^{-k-1}) \quad \text{as } T \rightarrow \infty.$$

Therefore

$$(3.48) \quad \|\bar{S}_D(T)E\phi(T)[\tilde{G}_0, G_-]\| < \text{const. } T^{-k-1} \|\phi(T)[\tilde{G}_0, G_-]\|$$

holds and the boundary condition  $\bar{S}_D(T)x_T(T) = 0$  implies at least convergence of the order  $T^{-k-1}$  if  $f \equiv 0$  holds. More generally speaking the order of convergence is determined by inserting (3.48) into (3.39).

However, this rather work-intensive procedure does only make sense if some columns of  $\phi(t)[\tilde{G}_0, G_-]$  do not converge exponentially. Only in this case the projection conditions imply a significant improvement of the order of convergence.

#### 4. Linear Problems - The General Case

In this chapter we admit a general Jordan form of  $A_0$ . So we deal with problems of the form (3.1), (3.2), (3.3) with the assumption (3.4), (3.5), (3.6).

Again we perform the substitution (3.8) and get (3.9), (3.10).

The fundamental matrix  $\phi(t)$  of the homogeneous problem (3.9) can now be represented as an asymptotic log-exponential power series of the following form

$$(4.1) \quad \phi(t) = P(t)t^D e^{Q(t)}$$

where

$$(4.2) \quad P(t) \sim \sum_{i=0}^{\infty} P_i t^{-\frac{i}{p}} \text{ as } t \rightarrow \infty, \quad p \in \mathbb{N}_0,$$

(4.3)  $D$  is a constant matrix in Jordan form,

$$(4.4) \quad Q(t) = \text{diag}(J_0) \frac{t^{\alpha+1}}{\alpha+1} + Q_1 \frac{t^{\alpha+1-\frac{1}{p}}}{p(\alpha+1)-1} + \dots + Q_{p(\alpha+1)-1} t^{\frac{1}{p}}$$

and the  $Q_i$  are diagonal matrices.  $\text{diag}(J_0)$  is the matrix which has the same diagonal entries as  $J_0$  and all other entries zero. The matrices  $T^D$  and  $e^{Q(t)}$  commute because the diagonal elements of  $Q(t)$  which belong to a particular Jordan block of  $D$  are equal. Moreover  $P(t)$  can be split up into:

$$(4.5) \quad P(t) = P_{(1)}(t) \cdot P_{(2)}(t) \cdot P_{(3)}(t)$$

where

$$(4.6) \quad P_{(1)}(t) \sim I + \sum_{i=1}^{\infty} P_{(1)i} t^{-i},$$

$$(4.7) \quad P_{(k)}(t) \sim \sum_{i=0}^{\infty} P_{(k)i} t^{-\frac{i}{p}}, \quad k = 2, 3.$$

$P_{(2)}(t)$ ,  $P_{(3)}(t)$  are in block diagonal form too. The  $i$ -th diagonal block of  $P_{(2)}(t)$  corresponds to that block in  $J_0$  which is obtained by gathering all Jordan blocks belonging to the  $i$ -th eigenvalue of  $J_0$  and the  $j$ -th diagonal block of  $P_{(3)}(t)$

corresponds to the  $j$ -th eigenvalue of  $Q(t)$  where in both cases only different eigenvalues are counted. Markowich (1980b) has shown that

$$(4.8) \quad \|(P_{(2)}(t))^{-1} D_i\| \leq \text{const. } t^{(\alpha+1)(r_i-1)}$$

holds, where  $D_i$  is the projection onto the direct sum of invariant subspaces associated with the  $i$ -th eigenvalue of  $J_0$  and  $r_i$  is the algebraic multiplicity of that  $i$ -th eigenvalue. The statement (4.8) holds for the matrix  $P_{(2)}$  derived as in Markowich (1980b).

The matrix  $P_{(3)}(t)t^D$  is the fundamental matrix of the system

$$(4.9) \quad z' = \left(\frac{1}{x} B + \frac{1}{x^2} \tilde{B}(x)\right)z, \quad \tilde{B} \in C([1, \infty])$$

where

$$(4.10) \quad u = P_{(1)}(t)P_{(2)}(t)e^{Q(t)}z$$

has been set. (4.9) has a singularity of the first kind of  $t = \infty$ . Obviously  $P_{(3)}(t)$  and  $D$  are not uniquely defined, only their product is unique (neglecting multiplication with a constant matrix from the right side).  $\tilde{P}_{(3)}(t)t^{\tilde{D}} = (P_{(3)}(t)t^{-1})t^{D+I}$  would also be a way of splitting the product. The algorithm given by Wasow (1965) establishes a matrix  $\tilde{P}_{(3)}(t)$  which has a convergent power series expansion, but  $\tilde{P}_{(3)}(\infty)$  is not regular. We will show now that a representation can be given, so that  $\tilde{P}_{(3)}(\infty)$  is regular. Therefore we assume that  $B$  is in Jordan - canonical form:

$$(4.11) \quad B = \text{diag}(B_1, \dots, B_q)$$

and  $B_i$  has the only eigenvalue  $b_i$ , where  $\text{Re}(b_i) < 0$  for  $1 < i < s$  and  $\text{Re}(b_j) > 0$  for  $s+1 < j < n$ . We write (4.9) as

$$(4.12) \quad z' = \frac{1}{x} Bz + \frac{1}{x} (\tilde{B}(x)z), \quad \tilde{B}(x) = \frac{1}{x} \tilde{B}(x)$$

and set for  $1 < i < s$

$$(4.13) \quad z_i(t) = \begin{bmatrix} \theta \\ \cdot \\ \theta \\ t^{B_i} \\ \theta \\ \cdot \\ \theta \end{bmatrix} + (GBz_i)(t), \quad \delta < t < \infty$$

where  $G$  is the operator defined in Markowich (1980b), Chapter 4 which applied to a



function  $g$  defines an appropriate particular solution of the problem

$$(4.14) \quad z' = \frac{1}{x} Bz + \frac{1}{x} g, \quad \delta < x < \infty$$

where  $\delta > 1$ . Markowich (1980b) derived that

$$(4.15) \quad \|(Gg)(t)\| < \text{const.} (\ln t)^{j+k} t^{-\epsilon} \|g\|_{[\delta, \infty)}, \quad 0 < j < \max(\dim(B_i))$$

holds if

$$(4.16) \quad g(t) = t^{-\epsilon} (\ln t)^k \bar{g}(t), \quad g, \bar{g} \in C_b([\delta, \infty))$$

where  $C_b([\delta, \infty))$  is the space of all functions  $f \in C([\delta, \infty))$  which are bounded as

$t \rightarrow \infty$ . Setting

$$(4.17) \quad \sigma_i(t) = \|t\|^{B_i} = c \cdot t^{\text{Re}(b_i)} (\ln t)^{\dim(B_i)-1}, \quad 1 < i < S.$$

We want to show that (4.13) establishes a fixed-point equation for  $z_i \in A_{\sigma_i, \delta}$  where

$$(4.18) \quad (A_{\sigma_i, \delta} = \{u | u(t) = U(t)\sigma_i(t), U \in C_b([\delta, \infty))\}, \|u\|_i = \|U\|_{[\delta, \infty)}).$$

We want to show that the operator

$$(4.19) \quad (\psi_i(z_i))(t) = \begin{bmatrix} \theta \\ \cdot \\ \cdot \\ B_i \\ t^i \\ \cdot \\ \cdot \\ \theta \end{bmatrix} + (GBz_i)(t)$$

is a contraction on  $A_{\sigma_i, \delta}$  for  $\delta$  sufficiently large. From (4.13), (4.15), (4.17) we

conclude that  $\psi_i$  maps  $A_{\sigma_i, \delta}$  into  $A_{\sigma_i, \delta}$ . Moreover

$$(4.20) \quad \|\psi_i(z_{i1}) - \psi_i(z_{i2})\|_i \approx \|GB(z_{i1} - z_{i2})\|_i < \text{const.} \delta^{-1} (\ln \delta)^j \|z_{i1} - z_{i2}\|_i$$

holds, and therefore  $\psi_i$  is a contraction on  $A_{\sigma_i, \delta}$  for  $\delta$  sufficiently large. From

(4.13) we conclude

$$(4.21) \quad z_i(t) = (I + O(t^{-1} (\ln t)^{2j+\dim(B_i)})) \begin{bmatrix} \theta \\ \cdot \\ \theta \\ B_i \\ t^i \\ \theta \\ \cdot \\ \theta \end{bmatrix} \text{ as } t \rightarrow \infty.$$

Now let  $s+1 < j < n$  hold, so that  $\text{Re}(b_j) > 0$ . We substitute

$$(4.22) \quad z_j = \tilde{z}_j \cdot t^{b_j+1}$$

and (4.12) becomes

$$(4.23) \quad \tilde{z}_j' = \frac{1}{x} (B - (b_j + 1)I) \tilde{z}_j + \frac{1}{x} \bar{B}(x) \tilde{z}_j.$$

Now we set

$$(4.24) \quad \tilde{z}_j(t) = \begin{bmatrix} \theta \\ \cdot \\ \theta \\ t^{B_j - (b_j+1)I} \\ \cdot \\ \theta \end{bmatrix} + (GBz_j)(t)$$

so that

$$(4.25) \quad \sigma_j = \begin{bmatrix} B_j - (b_j+1)I & \\ & I \end{bmatrix} = c \cdot t^{-1}(\ln t)^{\dim(B_j)-1}$$

and (4.13), (4.21) implies that

$$(4.26) \quad \tilde{z}_j(t) = (I + O(t^{-1}(\ln t)^{2j+\dim(B_j)})) \begin{bmatrix} \theta \\ \cdot \\ \theta \\ t^{B_j - (b_j+1)I} \\ \cdot \\ \theta \end{bmatrix}$$

holds and from (4.22) we conclude

$$(4.27) \quad z_j(t) = (I + O(t^{-1}(\ln t)^{2j+\dim(B_j)})) \begin{bmatrix} \theta \\ \cdot \\ \theta \\ t^{B_j} \\ \cdot \\ \theta \end{bmatrix} \text{ as } t \rightarrow \infty.$$

Obviously the matrix

$$(4.28) \quad z(t) = [z_1(t), \dots, z_n(t)] = [I + O(t^{-1}(\ln t)^m)] t^B, \quad m \in \mathbb{N}_0$$

is a fundamental matrix of the system (4.9). Therefore  $P_{(3)}(t)$  and  $D$  in (4.5) and

(4.1) can be chosen so that

$$(4.29) \quad \|P_{(3)}(t)^{-1}\| = \|I + o(1)\| \leq \text{const.}$$

holds.

Knowing the fundamental matrix asymptotically we can sort out the basic solution  $\varphi_1$  fulfilling  $\varphi_1 \in C([1, \infty))$  so that the general solution of (3.9), (3.10) is

$$(4.30) \quad u(t) = \phi(t) [\tilde{G}_0, G_-] n + (Hf)(t), \quad n \in C^{\tilde{r}_0 + r_-}$$

where  $H$  is defining an appropriate particular solution  $Hf$  on  $[\delta, \infty)$  if

$$(4.31) \quad f(t) = t^{-(\alpha+1)\tilde{r}-\varepsilon} (\ln t)^k F(t), \quad F \in C_b([\delta, \infty))$$

where  $\tilde{r}$  is the maximal algebraic multiplicity of eigenvalues of  $J_0$  which have real part zero. Moreover

$$(4.32) \quad \|(Hf)(t)\| \leq \text{const. } t^{-\varepsilon} (\ln t)^{j_0+k} \|F\|_{[\delta, \infty)}, \quad 0 \leq j_0 < n.$$

The particular solution on  $[1, \infty)$  is obtained by continuation.

Again the boundary value problem on the infinite interval is uniquely soluble for all  $B \in R^{\tilde{r}_0 + r_-}$  and  $f$ 's which fulfill (4.31) iff the  $(\tilde{r}_0 + r_-) \times (\tilde{r}_0 + r_-)$  matrix

$$(4.33) \quad BE[\phi(1)\tilde{G}_0, \phi(1)G_-] \text{ is regular.}$$

Of course  $B$  is an  $(\tilde{r}_0 + r_-) \times n$  matrix.

The approximating problems have the form (3.20), (3.21), (3.22). We will again prove a stability theorem.

Theorem 4.1. Assume that (4.33) and  $(A_2)$ ,  $(B_2)$ ,  $(C_2)$  which are defined as follows, hold:

$$(A_2) \quad \|S(T)\| \leq \text{const. as } T \rightarrow \infty,$$

$$(B_2) \quad \|S(T)EP(T)\tilde{G}_0\| = o(T^{-(\alpha+1)(\tilde{r}-1)}),$$

$$(C_2) \quad \|(S(T)EP_{(1)}(T)[G_+, P_{(2)}(T)P_{(3)}(T)\tilde{G}_0]^{-1})\| = o(T^{(\alpha+1)(\tilde{r}-1)}).$$

Then there is a unique solution  $x_T$  of the problem (3.20), (3.21), (3.22) for  $T$  sufficiently large and the following estimate holds for all  $\beta \in R^{\tilde{r}_0 + r_-}$ ,

$$\gamma(T) \in R^{(\tilde{r}_0 - \tilde{r}_0) + r_+}, \quad f \in C([1, T]):$$

$$(4.34) \quad \|x_T\|_{[1, T]} \leq \text{const.} (\|\beta\| + T^{(\alpha+1)(\tilde{r}-1)} \|\gamma(T)\| + T^{(\alpha+1)\tilde{r}} (\ln T)^j \|f\|_{[1, T]}).$$

We substitute

$$(4.35) \quad x_T(t) = EP_{(1)}(t)w_T(t), \quad w_T = \begin{pmatrix} w_T^+ \\ w_T^0 \\ w_T^- \end{pmatrix}$$

and get three separate problems

$$(4.36) \quad \begin{bmatrix} w_T^+(t) \\ w_T^0(t) \\ w_T^-(t) \end{bmatrix}' = t^\alpha \begin{bmatrix} J^+(t) & & \\ & J^0(t) & \\ & & J^-(t) \end{bmatrix} \begin{bmatrix} w_T^+(t) \\ w_T^0(t) \\ w_T^-(t) \end{bmatrix} + \underbrace{t^{\alpha P-1}(t) E^{-1} f(t)}_{\tilde{f}(t)}$$

where

$$(4.37) \quad \begin{bmatrix} J^+(\infty) & & \\ & J^0(\infty) & \\ & & J^-(\infty) \end{bmatrix} = \begin{bmatrix} J_0^+ & & \\ & J_0^0 & \\ & & J_0^- \end{bmatrix} = J_0$$

and the eigenvalues of  $J_0^+$  have positive real part, the eigenvalues of  $J_0^0$  have a zero real part and the eigenvalues of  $J_0^-$  have a negative real part. This structure can always be obtained by reordering the columns of  $E$ . Now we rewrite the equation for  $w_T^+$ :

$$(4.38) \quad w_T^+(t)' = t^\alpha J_0^+ w_T^+(t) + (J^+(t) - J_0^+) w_T^+(t) + t^\alpha \tilde{f}_+(t).$$

We define the general solution of (4.38) as:

$$(4.39) \quad w_T^+(t) = \exp\left(\frac{J_0^+}{\alpha+1} (t^{\alpha+1} - T^{\alpha+1})\right) \xi^+ + ({}_{T}H_+(J^+ - J_0^+) w_T^+)(t) + ({}_{T}H_+ \tilde{f}_+)(t)$$

where  ${}_{T}H_+$  is defined in (2.57) with  $E = I$  and  $J^0 = J_0^+$ . We derive

$$(4.40) \quad ((I - {}_{T}H_+(J^+ - J_0^+)) w_T^+)(t) = \exp\left(\frac{J_0^+}{\alpha+1} (t^{\alpha+1} - T^{\alpha+1})\right) \xi^+ + ({}_{T}H_+ \tilde{f}_+)(t)$$

de Hoog and Weiss (1980b) have shown that  $(I - {}_{T}H_+(J^+ - J_0^+))$  is invertible as operator on  $C([\delta, T])$  with  $\delta$  and  $T$  sufficiently large, so that

$$(4.41) \quad w_T^+(t) = \psi_+(t, T) \xi^+ + \psi_+(\tilde{f}_+)(t) \in C([\delta, T])$$

where

$$(4.42) \quad \psi_+(\cdot, T) = (I - {}_{T}H_+(J^+ - J_0^+))^{-1} \exp\left(\frac{J_0^+}{\alpha+1} (h - T^{\alpha+1})\right), \quad h(t) = t^{\alpha+1}$$

and

$$(4.43) \quad {}_{T}\psi_+(\tilde{f}_+) = (I - {}_{T}H_+(J^+ - J_0^+))^{-1} {}_{T}H_+ \tilde{f}_+.$$

Moreover, they have shown that

$$(4.44) \quad (a) \quad \|\psi_+(\cdot, T)\|_{[\delta, T]} < \text{const.}, \quad (b) \quad \|\psi_+(\tilde{f}_+)\|_{[\delta, T]} < \text{const.} \|\tilde{f}_+\|_{[\delta, T]}$$

and from (2.57) and (4.42) we derive that

$$(4.45) \quad \psi_+(T, T) = I.$$

Now we define the general solution of (4.36) as

$$(4.46) \quad w_T(t) = \begin{bmatrix} \psi_+(t, T) & \theta \\ \theta & \theta \end{bmatrix} G_{+, P(2)}(t) P_{(3)}(t) e^{Q(t)-Q(T)} \begin{pmatrix} t \\ T \end{pmatrix}^D \begin{matrix} G_0 \\ G_0 \end{matrix} \xi_1 + \\ + P_{(2)}(t) P_{(3)}(t) e^{Q(t)} t^D \begin{matrix} \tilde{G}_0 \\ G_- \end{matrix} \xi_2 + \\ + \begin{pmatrix} \psi_+(\tilde{f}_+)(t) \\ w_P^0(t, T) \\ w_P^-(t, T) \end{pmatrix}, \quad \xi_1 \in C^{r_+(r_0 - \tilde{r}_0)}, \quad \xi_2 \in C^{\tilde{r}_0 + r_-}$$

where  $w_P^0, w_P^-$  are appropriate particular solutions. This solution is defined on  $[\delta, T]$  and the corresponding solution on  $[1, T]$  is obtained by continuing  $\psi_+(\cdot, T)$ . Resubstituting in (4.35) and evaluating at the boundaries sets up the linear block system for  $\xi_1, \xi_2$ :

$$(4.47) \quad \begin{bmatrix} \text{BEP}_{(1)}(1) \begin{bmatrix} \psi_+(1, T) & \theta \\ \theta & \theta \end{bmatrix} G_{+, P(2)}(1) P_{(3)}(1) e^{Q(1)-Q(T)} T^{-D} \begin{matrix} G_0 \\ G_0 \end{matrix} \\ \text{S}(T) \text{EP}_{(1)}(T) [G_{+, P(2)}(T) P_{(3)}(T) \begin{matrix} \tilde{G}_0 \\ G_0 \end{matrix}] \end{bmatrix} \begin{matrix} \text{BE}\phi(1) [\tilde{G}_0, G_-] \\ \text{S}(T) \text{E}\phi(T) [\tilde{G}_0, G_-] \end{matrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \\ = \begin{pmatrix} \beta - \text{BEP}_{(1)}(1) \begin{pmatrix} T \psi_+(\tilde{f}_+)(1) \\ w_P^0(1, T) \\ w_P^-(1, T) \end{pmatrix} \\ \gamma(T) - \text{S}(T) \text{EP}_{(1)}(T) \begin{pmatrix} T \psi_+(\tilde{f}_+)(T) \\ w_P^0(T, T) \\ w_P^-(T, T) \end{pmatrix} \end{pmatrix}.$$

The matrix in the (1.1) position is bounded,  $BE\phi(1)[\tilde{G}_0, G_-]$  is independent of  $T$  and invertible because of (4.33), the matrix in the (2.1) position fulfills  $(C_2)$  and

$$(4.48) \quad \|S(T)E\phi(T)[\tilde{G}_0, G_-]\| < o(1)T^{-(\alpha+1)(\bar{r}-1)}$$

holds because of  $(B_2)$ . From (2.40), (2.52) we conclude that

$$(4.49) \quad \|X_T\|_{[1,T]} < \text{const.} (\|B\| + T^{(\alpha+1)(\bar{r}-1)} \|Y(T)\| + \|v_p(\cdot, T)\| + \\ + T^{(\alpha+1)(\bar{r}-1)} \|S(T)E v_p(T, T)\| + \|v_p(\cdot, T)\|_{[1,T]})$$

where

$$(4.50) \quad v_p(t, T) := P_{(1)}(t) \begin{pmatrix} T\psi_+(\tilde{f}_+)(t) \\ w_p^0(t, T) \\ w_p^-(t, T) \end{pmatrix} = (H_T^{(2)} f)(t)$$

has been used. Splitting  $H_T^{(2)}$  into  $T H_+^{(2)}$ ,  $T H_0^{(2)}$ ,  $T H_-^{(2)}$  where  $T H_+^{(2)}$  is already defined by  $T\psi_+(\tilde{f}_+)$ , we can define  $T H_0^{(2)}$  as we have defined  $T H_0^{(1)}$  in (3.36) only the  $D_{01}$  have to be substituted by the projections onto the invariant subspaces of  $D$ . The estimate

$$(4.51) \quad \|T H_0^{(2)} f\|_{[1,T]} < \text{const.} T^{(\alpha+1)\bar{r}} (\lambda n T)^{j_0} \|f\|_{[1,T]}$$

results as in Chapter 3.

$T H_-^{(2)} f$  can be constructed by the same perturbation approach we used for the construction of  $T H_+^{(2)} f$ . We set

$$(4.52) \quad T\psi_-(\tilde{f}_-) = (I - T H_-(J^- - J_0^-))^{-1} T H_- \tilde{f}_- = w_p^-(\cdot, T)$$

where  $T H_-$  is defined in (2.59) with  $E = I$  and  $J = J_0^-$  and then

$$(4.53) \quad (T H_-^{(2)} f)(t) = P_{(1)}(t) \begin{pmatrix} \theta \\ \theta \\ T\psi_-(\tilde{f}_-)(t) \end{pmatrix}$$

holds. Moreover the estimate

$$(4.54) \quad \|T H_-^{(2)} f\|_{[1,T]} < \text{const.} \|f\|_{[1,T]}$$

is fulfilled. Because of  $(B_2)$  and (3.36) we get

$$(4.55) \quad \|S(T)E_{T H_0}^{(2)} f(T)\| < \text{const. } T^{\alpha+1} (\ln T)^j \|f\|_{[1,T]}$$

so that the estimate (4.34) follows.

Again the convergence estimate follows:

$$(4.56) \quad \|y - x_T\|_{[1,T]} < \text{const. } T^{(\alpha+1)(\bar{r}-1)} \|S(T)y(T) - \gamma(T)\|$$

for all matrices  $S(T)$  fulfilling  $(A_2)$ ,  $(B_2)$ ,  $(C_2)$ . Setting  $\gamma(T) \equiv 0$  and using (4.30)

we get the order of convergence as follows:

$$(4.57) \quad \|y - x_T\|_{[1,T]} < \text{const. } T^{(\alpha+1)(\bar{r}-1)} (\|S(T)E\phi(T)[\tilde{G}_0, G_-]\| + \|(Hf)(T)\|).$$

Assumption  $(C_2)$  guarantees convergence for all  $f$  which fulfill

$$(4.58) \quad \|f(t)\| = O(t^{-(\alpha+1)(2\bar{r}-1)-\epsilon}), \quad \epsilon > 0$$

because the columns in  $\phi(T)$  which may be constant as  $T \rightarrow \infty$  are damped by the factor  $O(T^{-(\alpha+1)(\bar{r}-1)})$ . Again if all columns of  $\phi(t)$  and  $f$  decay exponentially, the convergence is exponential, too.

Still the question has to be answered whether there is a matrix  $S(T)$  fulfilling the assumption of Theorem 4.1 and how it can be constructed. We set

$$(4.59) \quad S(T)EP_{(1)}(T) = \tilde{S}(T)$$

and choose  $\tilde{S}(T)$  so that:

$$(4.60) \quad \tilde{S}(T)[G_+, \theta, \theta] = \begin{bmatrix} (G_+)^T \\ \theta \end{bmatrix},$$

$$(4.61) \quad \tilde{S}(T)[\theta, \theta, G_-] = \theta$$

and

$$(4.62) \quad \tilde{S}(T)P_{(2)}(T)P_{(3)}(T)[\theta, G_0, \theta] = T^{-(\alpha+1)(\bar{r}-1)} \begin{bmatrix} \theta \\ (\tilde{G}_0)^T \end{bmatrix}$$

because of the block structure of  $P_{(2)}$ ,  $P_{(3)}$  (4.62) is equivalent to

$$(4.63) \quad \tilde{S}(T)[\theta, G_0, \theta] = T^{-(\alpha+1)(\bar{r}-1)} \begin{bmatrix} \theta \\ (\tilde{G}_0)^T \end{bmatrix} P_{(3)}^{-1}(T)P_{(2)}^{-1}(T).$$

The equations (4.60), (4.61), (4.63) determine  $\tilde{S}(T)$  and  $S(T)$  can be calculated from

$$(4.64) \quad S(T) = \tilde{S}(T)P_{(1)}^{-1}(T)E^{-1}.$$

$S(T)$  fulfills  $(B_2)$  because of (4.62), the proposition  $(A_2)$  follows from (4.8) and (4.29), and  $(C_2)$  is implied by (4.60), (4.62). This asymptotic boundary condition (with  $\gamma(T) \equiv 0$ ) is the projection condition fulfilling

$$(4.65) \quad S_D(T)E\phi(T)[\tilde{G}_0, G_-] = \theta.$$

In general we only can determine a finite number of coefficients of the expansion of  $P_{(1)}$ ,  $P_{(2)}$ ,  $P_{(3)}$ . An algorithm is given in Markowich (1980b) and it is shown that

$$(4.66) \quad P_{(2)}(t) = \prod_{i=1}^m S_i(t)E_i P_{(2)}^i(t)$$

where the matrices  $S_i$  are in block diagonal form and their diagonal blocks are

$$(4.67) \quad S_{ij}(t) = \text{diag}(1, t^{-g_{ij}}, \dots, t^{-(k_{ij}-1)g_{ij}}).$$

The  $E_i$ 's are regular and

$$(4.68) \quad P_{(2)}^i(t) \sim I + \sum_{j=1}^{\infty} P_{(2)}^{ij} t^{-\frac{j}{p}} \text{ as } t \rightarrow \infty$$

holds.

We denote by

$$(4.69) \quad P_{(2)0}(t) = \prod_{i=1}^m S_{i0}(t)E_i P_{(2)0}^i(t), \quad P_{(2)0}^i(t) \sim I + \sum_{j=1}^{\infty} P_{(2)0}^{ij} t^{-\frac{j}{p}}$$

that diagonal block of  $P_{(2)}$  which is associated with the zero-real part eigenvalue of  $J_0$ . Assume now that we know  $P_{(2)0}^{i1}, \dots, P_{(2)0}^{ik_2}$  for  $i = 1(1)m$  and  $k_2 > 2p(\alpha + 1)(\bar{r} - 1) - 1$  then we set

$$(4.70) \quad \bar{P}_{(2)0}(t) = \prod_{i=1}^m S_{i0}(t)E_i \bar{P}_{(2)0}^i(t), \quad \bar{P}_{(2)0}^i = I + \sum_{j=1}^{k_2} P_{(2)0}^{ij} t^{-\frac{j}{p}}$$



and assume that we know

$$(4.71) \quad \bar{P}_{(3)0}(t) = I + \sum_{j=1}^{k_3} P_{(3)0}^j t^{-j}, \quad k_3 > p(\alpha + 1)(\bar{r} - 1) - 1.$$

$\bar{P}_{(3)0}(t)$  consists of the first  $k_3 + 1$  summands of  $P_{(3)0}(t)$  which is that block of  $P_{(3)}(t)$  associated with the real part zero eigenvalues of  $J_0$ . Moreover we have to know

$$(4.72) \quad \bar{P}_{(1)}(t) = I + \sum_{j=1}^{k_1} P_{(1)j} t^{-j}, \quad k_1 > (\alpha + 1)(\bar{r} - 1) - 1.$$

Using  $\bar{P}_{(1)}, \bar{P}_{(2)0}, \bar{P}_{(3)0}$  instead of  $P_{(1)}, P_{(2)0}, P_{(3)0}$  we calculate a matrix  $\bar{S}(T)$  instead of  $S(T)$  using (4.60), (4.61), (4.63), (4.64) and a perturbation analysis shows that

$$(4.73) \quad \bar{S}(T) = S(T) + O(T^{-k-1})$$

where

$$(4.74) \quad k = \min\left\{k_1, \frac{k_2 + 1}{p} - (\alpha + 1)(\bar{r} - 1) - 1, \frac{k_3 + 1}{p} - 1\right\} > (\alpha + 1)(\bar{r} - 1).$$

Therefore

$$(4.75) \quad \bar{S}(T)\phi(T)[\tilde{G}_0, G_-] = O(T^{-k-1})\phi(T)[\tilde{G}_0, G_-]$$

and the order of convergence for homogeneous problems is at least  $T^{(\alpha+1)(\bar{r}-1)-k-1}$ .

The requirement that  $A(t)$  is analytical in  $t = \infty$  is very restricting, therefore we will now admit matrices  $A(t)$  fulfilling

$$(4.76) \quad A \in C([1, \infty)), \quad A\left(\frac{1}{\delta}\right) \in C^{(\alpha+1)\bar{l}+1}\left(\left[0, \frac{1}{\delta}\right]\right), \quad \delta > 1$$

where  $\bar{l}$  is the maximal algebraic multiplicity of an eigenvalue of  $A(\infty)$  with nonpositive real part. Therefore  $A$  can be expanded:

$$(4.77) \quad A(t) = A_0 + t^{-1}A_1 + \dots + t^{-(\alpha+1)\bar{l}}A_{(\alpha+1)\bar{l}} + \tilde{A}(t)$$

where

$$(4.78) \quad \tilde{A}(t) = a(t)t^{-(\alpha+1)\bar{l}-1-\beta}, \quad \beta > 0, \quad a \in C_b([1, \infty)).$$

The problem (3.1), (3.2) can now be rewritten as

$$(4.79) \quad y' - t^\alpha \sum_{i=0}^{(\alpha+1)\bar{\ell}} A_i t^{-i} y = t^{\alpha\tilde{A}(t)} y + t^\alpha f(t),$$

$$(4.80) \quad y \in C([1, \infty))$$

and can be regarded as a perturbed system of

$$(4.81) \quad \tilde{y}' - t^\alpha \sum_{i=0}^{(\alpha+1)\bar{\ell}} A_i t^{-i} \tilde{y} = t^\alpha f(t),$$

$$(4.82) \quad \tilde{y} \in C([1, \infty)).$$

Markowich (1980b) has proven that the  $n \times (\tilde{r}_0 + r_-)$  solution matrix  $E\psi_-^0$  of (4.79),

(4.80) fulfills for large  $t$ :

$$(4.83) \quad \|\psi_-^0(t) - \phi(t)[\tilde{G}_0, G_-]\| \leq \text{const. } t^{-1-\beta} (\ln t)^j \|\phi(t)[\tilde{G}_0, G_-]\|$$

where  $0 < j < 2n$  holds and  $\phi(t)[\tilde{G}_0, G_-]$  is the general solution of (4.81), (4.82).

Moreover a particular solution  $E\psi(f)$  of (4.79), (4.80) can be constructed if  $f$  fulfills

(4.31) and

$$(4.84) \quad \|\psi(f)(t) - (Hf)(t)\| \leq \text{const. } t^{-1-\beta} (\ln t)^j \|(Hf)(t)\|$$

holds. The problem (3.1), (3.2), (3.3) is for all  $\beta \in \mathbb{R}^{\tilde{r}_0 + r_-}$  and  $f$  fulfilling (4.31)

uniquely soluble iff the  $(\tilde{r}_0 + r_-) \times (\tilde{r}_0 + r_-)$  matrix

$$(4.85) \quad BE\psi_-^0(1) \text{ is regular.}$$

Of course,  $B$  is a  $(\tilde{r}_0 + r_-) \times n$  matrix.

For the existence theory and for the following stability theorem it is sufficient to

require that (4.77) holds with  $\bar{\ell} = \bar{r}$  and with  $\tilde{A}(t) = t^{-(\alpha+1)r-\epsilon} a(t)$ . This implies the

right hand sides of (4.83) and (4.84) to equal  $\text{const.} (\ln T)^j T^{-\epsilon}$ .

We consider the asymptotic boundary value problem

$$(4.86) \quad x_T' - t^\alpha A(t)x_T = t^\alpha f(t), \quad 1 < t < T,$$

$$(4.87) \quad Bx_T(1) = \beta,$$

$$(4.88) \quad S(T)x_T(T) = \gamma(T)$$

and show that the construction of  $S(T)$  and the stability estimate (4.34) depend only on

the validity of  $(A_2)$ ,  $(B_2)$ ,  $(C_2)$  for the perturbed problem

$$(4.89) \quad \tilde{x}_T - t^\alpha \left( \sum_{i=0}^{(\alpha+1)\bar{r}} A_i t^{-i} \right) \tilde{x}_T = t^\alpha f(t),$$

$$(4.90) \quad S(T) \tilde{x}_T(T) = \gamma(T)$$

if (4.85) holds.

Theorem 4.2. Let A fulfill (4.76) and let (4.85), (A<sub>2</sub>), (B<sub>2</sub>), (C<sub>2</sub>) of Theorem 4.1 hold where

$$\phi(t) = P_{(1)}(t)P_{(2)}(t)P_{(3)}(t)t^{RQ(t)}$$

is the fundamental matrix of the homogeneous problem (4.81). Then there is a unique

solution  $x_T$  of the problem (4.86), (4.87), (4.88) for T sufficiently large and for all  $\beta \in R_{0^+ + r^-}$ ,  $\gamma(T) \in R_{+ + (r_0^- - r_0)}$ ,  $f \in C([1, T])$ . This solution  $x_T$  fulfills the estimate

(4.34).

We write (4.87)

$$(4.91) \quad x_T' - t^\alpha \left( \sum_{i=0}^{(\alpha+1)\bar{r}} A_i t^{-i} \right) x_T = t^\alpha (\tilde{A}(t)x_T + f(t)), \quad 1 < t < T$$

and write the general solution after having set  $x_T = Ev_T$  as

$$(4.92) \quad v_T(t) = P_{(1)}(t) \left\{ \begin{bmatrix} \psi_+(t, T) & \theta \\ \theta & \theta \end{bmatrix} G_+, P_{(2)}(t)P_{(3)}(t)e^{Q(t)-Q(T)} \left( \frac{t}{T} \right)^{D_{\tilde{G}_0}} \right\} \xi_1 + \\ + P(t)e^{Q(t)} t^D [G_0^+, G_-] \xi_2 + (H_T^{(2)\tilde{A}} Ev_T)(t) + (H_T^{(2)} f)(t).$$

We restrict t to the interval  $[\delta, T]$  where  $\delta$  is sufficiently large, so that

$$(4.93) \quad H_T^{(2)\tilde{A}} : C([\delta, T]) \rightarrow C([\delta, T])$$

holds. Then the following estimate is fulfilled:

$$\begin{aligned}
(4.94) \quad \|H_T^{(2)\sim} \tilde{A}\|_{[\delta, T]} &= \max_{\|u\|_{[\delta, T]} \leq 1} \|H_T^{(2)\sim} \tilde{A}u\|_{[\delta, T]} < \max_{\|u\|_{[\delta, T]} \leq 1} (\|H_{T+}^{(2)\sim} \tilde{A}u\|_{[\delta, T]} + \\
&\quad + \|H_{T_0}^{(2)\sim} \tilde{A}u\|_{[\delta, T]} + \|H_{T-}^{(2)\sim} \tilde{A}u\|_{[\delta, T]}) < \\
&< \text{const.} (\|\tilde{A}\|_{[\delta, T]} + \delta^{-\epsilon} (\ln \delta)^j \|a\|_{[\delta, T]}) < \frac{1}{2\epsilon E}
\end{aligned}$$

for all  $T > \delta > \bar{\delta}$  sufficiently large. This follows from the estimates for  $H_{T+}^{(1)}$ ,  $H_{T-}^{(1)}$  given in Chapter 2 and from the estimates for  $H_0$  given in Markowich (1980b). Therefore

$(I - H_T \tilde{A} E)$  is invertible on  $C([\delta, T])$  and we get:

$$(4.95) \quad v_T(t) = {}_T \psi_+^0(t) \xi_1 + {}_T \psi_-^0(t) \xi_2 + {}_T \psi(f)(t), \quad t \in [\delta, T]$$

where  ${}_T \psi_+^0$ ,  ${}_T \psi_-^0$  fulfill the equations:

$$(4.96) \quad {}_T \psi_+^0 - H_T^{(2)\sim} \tilde{A} E_T \psi_+^0 = P_{(1)}(\cdot) \left[ \begin{array}{c} \left[ \begin{array}{cc} \psi_+(\cdot, T) & \theta \\ \theta & \theta \end{array} \right]_{G_+, P(2)}(\cdot) P_{(3)}(\cdot) e^{Q(\cdot) - Q(T)} \left( \frac{\cdot}{T} \right) \begin{array}{c} D \\ G_0 \end{array} \end{array} \right],$$

$$(4.97) \quad {}_T \psi_-^0 - H_T^{(2)\sim} \tilde{A} E_T \psi_-^0 = \phi[\tilde{G}_0, G_-]$$

and  ${}_T \psi(f)$  fulfills

$$(4.98) \quad {}_T \psi(f) - H_T^{(2)\sim} \tilde{A} E_T \psi(f) = H_T^{(2)} f.$$

By evaluating at the boundaries the following block system is generated:

$$(4.99) \quad \begin{bmatrix} B E_T \psi_+^0(1) & B E_T \psi_-^0(1) \\ S(T) E_T \psi_+^0(T) & S(T) E_T \psi_-^0(T) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \beta - B E_T \psi(f)(1) \\ \gamma(T) - S(T) E_T \psi(f)(T) \end{bmatrix}.$$

Obviously, the matrix in the (1.1) position is bounded as  $T \rightarrow \infty$ . From (4.97) we derive:

$$(4.100) \quad S(T) E_T \psi_-^0(T) = S(T) E \phi(T) [\tilde{G}_0, G_-] + S(T) E (H_T^{(2)\sim} \tilde{A} E_T \psi_-^0)(T).$$

From the definition of  $H_T^{(2)}$  we conclude that

$$\begin{aligned}
(4.101) \quad S(T) E (H_T^{(2)\sim} \tilde{A} E_T \psi_-^0)(T) &= S(T) E P(T) \tilde{D}_0 e^{Q(T)} T^D \int_{\delta}^T \tilde{D}_0 e^{-Q(s)} s^{-D} P^{-1}(s) E^{-1} \tilde{A} E_T \psi_-^0(s) ds + \\
&\quad + S(T) E (H_{T-}^{(2)\sim} \tilde{A} E_T \psi_-^0)(T)
\end{aligned}$$

where  $\bar{D}_0$  is the projection onto the direct sum of invariant subspaces of  $D$  belonging to solutions which decay to zero.

Now from Markowich (1980b), Chapter 4 we conclude

$$(4.102) \quad \| (H_T^{(2)} \tilde{A}u)(T) \| = \| (H_T \tilde{A}u)(T) \| = O(T^{-(\alpha+1)\bar{r} - \frac{\epsilon}{2}}), \quad u \in C([\delta, T]).$$

This and (B<sub>2</sub>) guarantee that

$$(4.103) \quad \| S(T)E_T \psi_-^0(T) \| = o(T^{-(\alpha+1)(\bar{r}-1)}),$$

holds. Similarly (4.96) implies

$$(4.104) \quad S(T)E_T \psi_+^0(T) = S(T)E P_{(1)}(T) [G_+, P_{(2)}(T) P_{(3)}(T) \tilde{G}_0^0] + o(T^{-(\alpha+1)(\bar{r}-1)}).$$

Markowich (1980b) has proven that

$$(4.105) \quad \psi_-^0 - \tilde{H}AE\psi_-^0 = \phi[\tilde{G}_0, G_-]$$

where  $\tilde{H}$  is defined as  $H_T^{(2)}$  only the integrals are stretched to  $\infty$  instead of  $T$  and  $C$  is a regular  $(\tilde{r}_0 + r_-) \times (\tilde{r}_0 + r_-)$  matrix. We subtract (4.97) from (4.105) getting

$$(4.106) \quad (\psi_-^0 - \psi_-^0) - \tilde{H}AE\psi_-^0 + H_T^{(2)}AE_T\psi_-^0 = 0$$

or

$$(4.107) \quad (\psi_-^0 - \psi_-^0) = H_T^{(2)}AE(\psi_-^0 - \psi_-^0) + (\tilde{H}AE\psi_-^0 - H_T^{(2)}AE\psi_-^0), \quad \delta \leq t \leq T$$

and therefore

$$(4.108) \quad \| \psi_-^0 - \psi_-^0 \|_{[\delta, T]} \leq \text{const.} \| \tilde{H}AE - H_T^{(2)}AE \|_{[\delta, T]}.$$

By continuation to  $[1, T]$  we get

$$(4.109) \quad \lim_{T \rightarrow \infty} \| \psi_-^0 - \psi_-^0 \|_{[1, T]} = 0$$

so that

$$(4.110) \quad \psi_-^0(1) = \psi_-^0(1)C + o(1) \quad \text{as } T \rightarrow \infty.$$

Also we derive

$$(4.111) \quad S(T)E_T \psi(f)(T) = S(T)E(H_T^{(2)}f)(T) + o(T^{-(\alpha+1)(\bar{r}-1)}) \| f \|_{[\delta, T]}.$$

Theorem 4.2 follows now from (4.103), (4.104), (4.110), (4.111) by considering the system

(4.99) as in the proof of Theorem 4.1. The convergence results change correspondingly to

(4.83), (4.84):

$$(4.112) \quad \| y-x_T \|_{[1, T]} \leq \text{const.} T^{(\alpha+1)(\bar{r}-1)} (\| S(T)E\phi(T) \| [\tilde{G}_0, G_-] + \| (Hf)(T) \| + T^{-\epsilon} (\ln T)^j).$$

## 5. Nonlinear Problems

Now we deal with problems of the following form:

$$(5.1) \quad y' = t^\alpha f(t, y), \quad 1 \leq t < \infty, \quad \alpha \in \mathbb{N},$$

$$(5.2) \quad y \in C([1, \infty]),$$

$$(5.3) \quad b(y(1)) = 0$$

$f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$  is supposed to be continuous in  $(\infty, y(\infty))$ . (5.1) and (5.2) imply that

$$(5.4) \quad f(\infty, y(\infty)) = 0$$

holds. So  $y(\infty)$  can be calculated a priori as solution of a system of  $n$  nonlinear equations. If

$$(5.5) \quad \text{rank}\left(\frac{\partial f(\infty, y(\infty))}{\partial y}\right) = n$$

then the solution manifold is discrete so that the possible values of  $y(\infty)$  are known a priori. This case has been treated by de Hoog and Weiss (1980b). We will assume that the rank of this matrix is smaller than  $n$ , so that we have to expect a continuous solution manifold  $y_\infty(\mu)$  with  $\mu \in S \subset \mathbb{R}^{n_1}$ ,  $n_1 < n$ . We assume that we have determined such a  $n_1$ -dimensional manifold and that  $f(t, \cdot) \in C^1(\mathbb{R}^n)$  for all  $t \in [1, \infty)$  and

$$(5.6) \quad A(t, \mu) = \frac{\partial f(t, y_\infty(\mu))}{\partial y} = \sum_{i=0}^{\infty} A_i(\mu) t^{-i} \quad \text{for } t > \tilde{t}$$

holds. We calculate the fundamental matrix  $E(\mu)\phi(t, \mu)$  of the linearized system

$$(5.7) \quad \tilde{y}' = t^\alpha A(t, \mu) \tilde{y}$$

as an asymptotic series.  $E(\mu)$  transforms  $A_0(\mu)$  to its Jordan canonical form  $J_0(\mu)$ .

Now we restrict  $\mu$  to subsets  $\tilde{S} \subset S$  so that  $r_+$ ,  $r_0$ ,  $r_-$ ,  $\bar{r}$  which are defined for  $J_0(\mu)$  as in the last chapters are independent of  $\mu$  in  $\tilde{S}$ . Moreover we require that there is a  $n \times \hat{r}_0$ -projection like matrix  $\hat{G}_0$  independent of  $\mu \in \tilde{S}$  so that

$$(5.8) \quad \|\phi(t, \mu)[\hat{G}_0, G_-]\| \leq C(\mu) t^{-(\alpha+1)\bar{r}-\epsilon_1(\mu)} (\ln t)^j, \quad \epsilon_1(\mu) > 0$$

and that

$$(5.9) \quad \|f(t, y_\infty(\mu))\| \leq C(\mu) t^{-2(\alpha+1)\bar{r}-\epsilon_2(\mu)}, \quad \epsilon_2(\mu) > 0, \quad \mu \in \tilde{S}$$

holds. Assuming that  $f_y$  is locally uniformly Lipschitz continuous around  $y_\infty(\mu)$ .

Markowich (1980b) showed that there are solutions  $y = y(\cdot, \mu, \eta)$  in the space of functions in  $C([\delta, \infty])$  which decay to a finite limit at least as fast as  $t^{-(\alpha+1)\bar{r}-\epsilon} \cdot (\ln t)^{2j}$

where  $\epsilon = \min(\epsilon_1, \epsilon_2)$  and  $\delta$  sufficiently large. These solutions fulfill the estimate:

$$(5.10) \quad \|y(t, \mu, \eta) - y_\infty(\mu) - E(\mu)\phi(t, \mu)[\hat{G}_0, G_-]\eta\| \leq \text{const}(\mu) \cdot (\ln t)^{2j} t^{-(\alpha+1)\bar{r}-\epsilon}$$

for  $\eta \in R^{r_0+r}$ . For many important applications

$$(5.11) \quad f(t, y_\infty(\mu)) \equiv 0$$

and  $\phi(t, \mu)[\hat{G}_0, G_-]$  decays exponentially. In this case the right hand side of (5.10)

contains the exponential factor  $\|\phi(t, \mu)[\hat{G}_0, G_-]\|^2$  and the algebraic and logarithmic

factors change. It follows from this analysis that the boundary value problem (5.1),

(5.2), (5.3) is soluble if the equation

$$(5.12) \quad b(y(1, \mu, \eta)) = 0$$

is soluble where  $b : R^n + R^{n_1+r_0+r_-}$  and  $y(t, \mu, \eta)$  denotes the continuation to  $[1, \infty]$

(if it exists). We assume that  $b \in C^1(R^n)$ .

The approximating problems have the form

$$(5.13) \quad x_T' = t^\alpha f(t, x_T), \quad 1 \leq t \leq T,$$

$$(5.14) \quad b(x_T(1)) = 0,$$

$$(5.15) \quad S(x_T(T), T) = 0$$

and  $S : R^{n+1} + R^{n-(n_1+r_0+r_-)}$ . We assume that we have obtained a solution

$y^* \equiv y(\cdot, \mu^*, \eta^*)$  fulfilling (5.10) and that this solution is isolated, i.e. the linearized problem

$$(5.16) \quad w' = t^\alpha \frac{\partial f(t, y^*(t))}{\partial y} w,$$

$$(5.17) \quad w \in C([1, \infty]),$$

$$(5.18) \quad \frac{\partial b(y^*(1))}{\partial y(1)} w(1) = 0$$

has only the trivial solution  $w \equiv 0$ . Using (5.10) we get for  $f_y$  Lipschitz continuous

$$(5.19) \quad \frac{\partial f(t, y^*(t))}{\partial y} = A(t, \mu^*) + O(t^{-(\alpha+1)\bar{r}-\frac{\epsilon(\mu^*)}{2}}).$$

From Markowich (1980b), Chapter 4 we derive that the general solution of (5.16), (5.17) is

$$(5.20) \quad w = E(\mu^*) \psi_-^0(t, \mu^*, \eta^*) \xi, \quad \xi \in C^{\tilde{r}_0 + r_-}.$$

$\psi_-^0$  is a  $n \times (\tilde{r}_0 + r_-)$  matrix.

For the following we assume that

$$(5.21) \quad \tilde{r}_0 = n_1 + \hat{r}_0$$

holds, which means that the nonlinear problem and the linearized problem have solution manifolds of the same dimension. The isolatedness of  $y^*$  now implies that the

$(\tilde{r}_0 + r_-) \times (\tilde{r}_0 + r_-)$  matrix

$$(5.22) \quad \frac{\partial b(y^*(1))}{\partial y(1)} E(\mu^*) \psi_-^0(1, \mu^*, \eta^*) \text{ is regular.}$$

Now we define:

$$(5.23) \quad (Fy)(t) = (t^{-\alpha} y' - f(t, y), b(y(1)), S(y(T), T) - S(y(T), T)),$$

$$(5.24) \quad F : (C([1, \infty]) \cap C^1([1, \infty]) \cap \{y | t^{-\alpha} y' \in C([1, \infty])\}, \|\cdot\|_{[1, \infty]}) + \\ + (C([1, \infty]) \times R^n, \|(v, a)\| = \|v\|_{[1, \infty]} + \|a\|)$$

and

$$(5.25) \quad (F_T x)(t) = (t^{-\alpha} x' - f(t, x), b(x(1)), S(x(T), T)),$$

$$(5.26) \quad F_T : (C^1([1, T]), \|x\| = \|x\|_{[1, T]} + T^{-\alpha} \|x'\|_{[1, T]}) + \\ + (C([1, T]) \times R^n, \|(w, b)\| = \|w\|_{[1, T]} + \|b\|).$$

All involved spaces are linear normed spaces and the space on which  $F_T$  is defined is a Banach space. (\*) We calculate the Frechet-derivative of  $F_T(y^*)$  where  $y^*$  is an

(isolated) solution of  $F(y^*) = 0$  assuming that  $S \in C^1(R^n)$ :

$$(5.27) \quad ((F_T'(y^*))z)(t) = (t^{-\alpha} z' - \frac{\partial f(t, y^*(t))}{\partial y} z, \frac{\partial b(y^*(1))}{\partial y(1)} z, \frac{\partial S(y^*(T), T)}{\partial y(T)} z).$$

Assuming that

$$(5.28) \quad \frac{\partial f}{\partial y}(t, \cdot) \text{ is locally Lipschitz continuous in the } \|\cdot\|_{[1, \infty]} \text{-norm} \\ \text{around } y^* \text{ uniformly in } t \in [1, \infty],$$

(\*) Norms are always assumed to be taken in the appropriate spaces.



(5.29)  $\frac{\partial b}{\partial y(1)}(y(1)), \frac{\partial S}{\partial y(T)}(y(T))$  are locally Lipschitz continuous around  $y^*(1)$  resp.  $y^*(T)$

hold, we derive

$$(5.30) \quad \|F'_T(y_1) - F'_T(y_2)\| \leq \text{const.} \|y_1 - y_2\|$$

for

$$(5.31) \quad \|y^* - y_i\|_{[1,T]} + T^{-\alpha} \|y^{*'} - y_i'\|_{[1,T]} \leq \text{const.}, \quad i = 1, 2.$$

Moreover Theorem 4.2 assures that the problem

$$(5.32) \quad z' = t^\alpha \frac{\partial f(t, y^*(t))}{\partial y} z + t^\alpha f(t),$$

$$(5.33) \quad \frac{\partial b(y^*(1))}{\partial y(1)} z(1) = \beta,$$

$$(5.34) \quad \frac{\partial S(y^*(T), T)}{\partial y(T)} z(T) = \gamma(T)$$

is for all  $f \in C([1, T])$ ,  $\beta \in \mathbb{R}^{\tilde{r}_0^+ + r_-}$ ,  $\gamma(T) \in \mathbb{R}^{n - (\tilde{r}_0^+ + r_-)}$  uniquely soluble if

$$(5.35) \quad \frac{\partial S(y^*(T), T)}{\partial y(T)} \text{ fulfills } (A_2), (B_2), (C_2)$$

where

$$(5.36) \quad E(\mu^*)\phi(t) = E(\mu^*)\phi(t, \mu^*) = E(\mu^*)P(t, \mu^*)t^{D(\mu^*)}e^{Q(t, \mu^*)}$$

is the fundamental matrix of the problem

$$(5.37) \quad \tilde{w}' = t^\alpha \frac{\partial f(t, y_\infty(\mu^*))}{\partial y} \tilde{w}.$$

It also implies that the unique solution fulfills

$$(5.38) \quad \|x\|_{[1,T]} + \frac{1}{T^\alpha} \|x'\|_{[1,T]} \leq \text{const.} T^{(\alpha+1)\bar{r}_{(\ln T)^2}} \|(f, \beta, \gamma(T))\|$$

so that  $F'_T(y^*)$  is invertible and

$$(5.39) \quad \|(F'_T(y^*))^{-1}\| \leq \text{const.} T^{(\alpha+1)\bar{r}_{(\ln T)^2}}.$$

We have used that (5.19) holds. From the nonlinear stability-consistency concept in

Spijker (1971) we conclude that

$$(5.40) \quad \|x_T - y^*\|_{[1,T]} + T^{-\alpha} \|x_T - (y^*)'\|_{[1,T]} \leq \text{const.} T^{(\alpha+1)\bar{r}_{(\ln T)^2}} \|S(y^*(T), T)\|$$

if

$$(5.41) \quad \|S(y^*(T), T)\| \leq \rho_1 T^{-2(\alpha+1)\bar{r}} (\ln T)^{-4j}$$

holds, where  $\rho_1$  is sufficiently small.  $x_T$  is a solution of  $F_T(x) = 0$  which is unique in a sphere whose center is the restriction of  $y^*$  to  $[1, T]$  and whose radius is smaller than  $\rho_2 T^{-(\alpha+1)\bar{r}} (\ln T)^{-2j}$  with  $\rho_2$  sufficiently small. This holds in the

$$(5.42) \quad \|x_T - y^*\|_{[1, T]} \leq \text{const. } T^{(\alpha+1)\bar{r}} (\ln T)^{2j} \|S(y^*(T), T)\|$$

if (5.35) and (5.41) holds.

Because of (5.10) it is sufficient to require that  $(A_2)$ ,  $(B_2)$ ,  $(C_2)$  hold for the matrix  $\frac{\partial S}{\partial y}(y_\infty(\mu^*), T)$  instead for  $\frac{\partial S}{\partial y}(y(T, \mu^*, \eta^*), T)$ . Moreover (5.41) is fulfilled if

$$(5.43) \quad S(y_\infty(\mu^*), T) \equiv 0 \text{ for } T \text{ sufficiently large}$$

and

$$(5.44) \quad \left\| \frac{\partial S}{\partial y}(y_\infty(\mu^*), T)(y(T, \mu^*, \eta^*) - y_\infty(\mu^*)) \right\| = O(T^{-2(\alpha+1)\bar{r}-\epsilon}), \quad \epsilon > 0.$$

In most cases of physical interest  $y^*(T)$  converges exponentially so that (5.44) is fulfilled automatically. Therefore, if  $(A_2)$ ,  $(B_2)$ ,  $(C_2)$  hold for  $\frac{\partial S}{\partial y}(y_\infty(\mu^*), T)$  and if (5.43) is fulfilled, convergence follows at isolated solutions and the order of convergence can be estimated by (5.42).

For the case when  $f$  is independent of  $t$  Lentini and Keller (1980) have generalized the projection condition and an example for the construction of an appropriate asymptotic boundary condition in the other case will be presented in Chapter 6.

### 6. A Case Study

The problem we analyse is a similarity equation for a combined forced and free convection flow over a horizontal plate (see Schneider (1979)). The governing equations are

$$(6.1) \quad y' = x \begin{pmatrix} \frac{y_2}{x} \\ \frac{y_3}{x} \\ -\frac{1}{2} \left(1 + \frac{y_1}{x}\right) y_2 - \frac{k}{2} y_4 \\ -\frac{1}{2} \left(1 + \frac{y_1}{x}\right) y_4 \end{pmatrix} = xf(x,y), \quad 0 < x < \infty,$$

$$(6.2) \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} y(0) = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix},$$

$$(6.3) \quad y \in C([0, \infty)).$$

From (6.1) we conclude that

$$(6.4) \quad y_\infty = y_\infty(\mu) = (\mu, 0, 0, 0)^T, \quad \mu \in \mathbb{R}$$

and

$$(6.5) \quad \frac{\partial f(x, y_\infty(\mu))}{\partial y} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{k}{2} \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}}_{A_0} + \frac{1}{x} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{\mu}{2} & 0 \\ 0 & 0 & 0 & -\frac{\mu}{2} \end{bmatrix}}_{A_1(\mu)}.$$

For this problem  $\alpha = 1$ ,  $\bar{r} = 2$  hold. We calculate:

$$(6.6) \quad J_0 = E^{-1} A_0 E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}, \quad E = \text{diag}(1, 1, 1, -\frac{2}{k}),$$

$$(6.7) \quad J_1(\mu) = E^{-1} A_1(\mu) E = A_1(\mu) .$$

Markowich (1980b) calculated an asymptotic expression for the fundamental matrix  $\phi(x, \mu)$  of the system

$$(6.8) \quad \tilde{w}' = x(J_0 + \frac{1}{x} J_1(\mu)) \tilde{w} ,$$

$$(6.9) \quad \phi(x, \mu) = \underbrace{\begin{bmatrix} 1 & 1 & 0(x^{-2}) & 0(x^{-1}) \\ 0 & x^{-1} & 0(x^{-1}) & 0(x^{-1}) \\ 0 & 0 & 1 & 1 - x^{-2} \\ 0 & 0 & 0 & x^{-2} \end{bmatrix}}_{P(x, \mu)} \underbrace{\text{diag}(1, x, 1, x^2)}_{x^D} \underbrace{\text{diag}(1, 1, e^{-\frac{x^2}{4} - \frac{\mu}{2}x}, e^{\frac{x^2}{4} - \frac{\mu}{2}x})}_{e^{Q(x, \mu)}} .$$

Markowich (1980b) showed that the problem (6.1) has solutions  $y(x, \xi_1, \xi_2, \mu)$  which fulfills

$$(6.10) \quad \|y(x, \xi_1, \xi_2, \mu) - y_\infty(\mu)\| \leq \text{const. } x^2 e^{-\frac{x^2}{4} - \frac{\mu}{2}x}, \quad x \text{ suff. large}$$

where the constant depends linearly on  $\xi_1, \xi_2$ . These solutions are in  $A_{\varepsilon, 0} + \{y_\infty(\mu)\}$

where

$$(6.11) \quad A_{\varepsilon, 0} = \{u | u(x) = x^{-4-\varepsilon} U(x), \quad U \in C_b([0, \infty))\}, \quad \varepsilon > 0 .$$

From (6.9) we conclude that

$$(6.12) \quad \tilde{G}_0 = [1, 0, 0, 0]^T, \quad \bar{G}_0 = [0, 1, 0, 0]^T .$$

The simplest choice of  $S$  is a linear function so we set

$$(6.13) \quad S(x) = (s_1(x), s_2(x), s_3(x), s_4(x)) .$$

Condition  $(B_2)$  of Theorem 4.1 applied to our problem gives

$$(6.14) \quad s_1(x) = 0(x^{-2}) .$$

We choose  $s_1(x) \equiv 0$ . Condition  $(C_2)$  gives

$$(6.15) \quad (s_2(x)x^{-1})^{-1} = 0(x^2) .$$

Therefore any matrix  $S(x)$  of the form

$$(6.16) \quad S(x) = [0, s_2(x), s_3(x), s_4(x)]$$

where

$$(6.17) \quad s_2(x) = \text{const.} \neq 0, \quad s_3(x) = 0(1), \quad s_4(x) = 0(1)$$

fulfills  $(A_2), (B_2), (C_2)$ . A natural choice is the following asymptotic boundary condition

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The problem we analyse is a similarity equation for a combined forced and free convection flow over a horizontal plate (see Schneider (1979)). The governing equations are

$$(6.1) \quad y' = x \begin{pmatrix} \frac{y_2}{x} \\ \frac{y_3}{x} \\ -\frac{1}{2} \left(1 + \frac{y_1}{x}\right) y_2 - \frac{k}{2} y_4 \\ -\frac{1}{2} \left(1 + \frac{y_1}{x}\right) y_4 \end{pmatrix} = xf(x,y), \quad 0 < x < \infty,$$

$$(6.2) \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} y(0) = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix},$$

$$(6.3) \quad y \in C([0, \infty)).$$

From (6.1) we conclude that

$$(6.4) \quad y_\infty = y_\infty(\mu) = (\mu, 0, 0, 0)^T, \quad \mu \in \mathbb{R}$$

and

$$(6.5) \quad \frac{\partial f(x, y_\infty(\mu))}{\partial y} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{k}{2} \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}}_{A_0} + \frac{1}{x} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{\mu}{2} & 0 \\ 0 & 0 & 0 & -\frac{\mu}{2} \end{bmatrix}}_{A_1(\mu)}.$$

For this problem  $\alpha = 1$ ,  $\bar{r} = 2$  hold. We calculate:

$$(6.6) \quad J_0 = E^{-1} A_0 E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}, \quad E = \text{diag}(1, 1, 1, -\frac{2}{k}),$$

$$(6.7) \quad J_1(\mu) = E^{-1} A_1(\mu) E = A_1(\mu) .$$

Markowich (1980b) calculated an asymptotic expression for the fundamental matrix  $\phi(x, \mu)$  of the system

$$(6.8) \quad \tilde{w}' = x(J_0 + \frac{1}{x} J_1(\mu)) \tilde{w} ,$$

$$(6.9) \quad \phi(x, \mu) = \underbrace{\begin{bmatrix} 1 & 1 & O(x^{-2}) & O(x^{-1}) \\ 0 & x^{-1} & O(x^{-1}) & O(x^{-1}) \\ 0 & 0 & 1 & 1 - x^{-2} \\ 0 & 0 & 0 & x^{-2} \end{bmatrix}}_{P(x, \mu)} \underbrace{\text{diag}(1, x, 1, x^2)}_{x^D} \underbrace{\text{diag}(1, 1, e^{-\frac{x^2}{4} - \frac{\mu}{2}x}, e^{\frac{x^2}{4} - \frac{\mu}{2}x})}_{e^{Q(x, \mu)}} .$$

Markowich (1980b) showed that the problem (6.1) has solutions  $y(\cdot, \xi_1, \xi_2, \mu)$  which fulfills

$$(6.10) \quad \|y(x, \xi_1, \xi_2, \mu) - y_\infty(\mu)\| \leq \text{const. } x^2 e^{-\frac{x^2}{4} - \frac{\mu}{2}x} , \quad x \text{ suff. large}$$

where the constant depends linearly on  $\xi_1, \xi_2$ . These solutions are in  $A_{\epsilon, 0} + \{y_\infty(\mu)\}$

where

$$(6.11) \quad A_{\epsilon, 0} = \{u | u(x) = x^{-4-\epsilon} U(x), \quad U \in C_b([0, \infty))\}, \quad \epsilon > 0 .$$

From (6.9) we conclude that

$$(6.12) \quad \tilde{G}_0 = [1, 0, 0, 0]^T, \quad \tilde{G}_0 = [0, 1, 0, 0]^T .$$

The simplest choice of  $S$  is a linear function so we set

$$(6.13) \quad S(x) = (s_1(x), s_2(x), s_3(x), s_4(x)) .$$

Condition (B<sub>2</sub>) of Theorem 4.1 applied to our problem gives

$$(6.14) \quad s_1(x) = O(x^{-2}) .$$

We choose  $s_1(x) \equiv 0$ . Condition (C<sub>2</sub>) gives

$$(6.15) \quad (s_2(x)x^{-1})^{-1} = O(x^2) .$$

Therefore any matrix  $S(x)$  of the form

$$(6.16) \quad S(x) = [0, s_2(x), s_3(x), s_4(x)]$$

where

$$(6.17) \quad s_2(x) = \text{const.} \neq 0, \quad s_3(x) = O(1), \quad s_4(x) = O(1)$$

fulfills (A<sub>2</sub>), (B<sub>2</sub>), (C<sub>2</sub>). A natural choice is the following asymptotic boundary condition

$$(6.18) \quad [0,1,0,0]v_X(X) = 0$$

which assures convergence of the order

$$(6.19) \quad \|v_X - y^{\dagger}\|_{[1,T]} < \text{const. } x^6 \exp\left(-\frac{x^2}{4} - \frac{\mu^*}{2} x\right) (\ln x)^2$$

where  $\mu^*$  is the parameter value of the actual solution  $y(\cdot, \mu^*, \xi_1^*, \xi_2^*)$  of (6.1), (6.2),

(6.3) which is assumed to be isolated. (6.19) holds because of

$$(6.20) \quad [0,1,0,0]y_{\infty}(\mu) \equiv 0 \quad \text{for } \mu \in \mathbb{R}$$

and because of (5.41).

Numerical calculations can be found in Schneider (1978).

#### REFERENCES

- E. Coddington and N. Levinson (1955) *Theory of Ordinary Differential Equations*, McGraw Hill, New York.
- F. R. de Hoog and Richard Weiss (1980a) On the boundary value problem for systems of ordinary differential equations with a singularity of the second kind, *SIAM J. Math. Anal.*
- F. R. de Hoog and R. Weiss (1980b) An approximation method for boundary value problems on infinite intervals, *Computing* 24, pp. 227-239.
- M. Lentini (1978) *Boundary Value Problems over Semi-finite Intervals*, Ph.D. Thesis, California Institute of Technology, Pasadena, CA 91125.
- M. Lentini and H. B. Keller (1980) The von Karman swirling flows, *SIAM J. Appl. Math.* 32, pp. 52-64.
- P. A. Markowich (1980a) *Randwertprobleme auf unendlichen intervallen*, Dissertation, Technische Universität Wien.
- P. A. Markowich (1980b) *Analysis of Boundary Value Problems on Infinite Intervals*, Mathematics Research Center Technical Summary Report #2138, University of Wisconsin-Madison, Madison, WI 53706.
- W. Schneider (1979) A similarity solution for combined forced and free convection flow over a horizontal plate, *Int. J. Heat Mass Transfer* 22, pp. 1401-1406.
- M. N. Spijker (1971) Equivalence Theorems for Nonlinear Finite Difference Methods, in 'Numerische Lösung nichtlinearer partieller Differential-und Integrodifferentialgleichungen, Springer Verlag, Lecture Notes in Mathematics, vol. 267.
- W. Wasow (1965) *Asymptotic Expansion for Ordinary Differential Equations*, Series in Pure and Applied Mathematics, vol. XIV, Wiley & Sons, New York.

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20. ABSTRACT - Cont'd.

↓ derived which defines classes of appropriate additional boundary conditions. Appropriate is to be understood in the sense that the solutions of the approximate problems converge to the actual solution of the 'infinite' problem as the length of the finite interval tends to infinity. Moreover boundary conditions which produce convergence with the largest expectable order are devised.

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