A Theory of Bond Portfolios

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Abstract

We introduce a bond portfolio management theory based on foundations similar to that of stock portfolio management. A general continuous time zero coupon market is considered. The problem of optimal portfolios of zero coupon bonds is solved for general utility functions, under a condition of no-arbitrage in the zero coupon market. A mutual fund theorem is proved, in the case of deterministic volatilities. Explicit expressions are given for the optimal solutions for several utility functions.

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1 Introduction

This paper is a first step towards a unified theory of portfolio management, including both stocks and bonds. There is a gap between the traditional approaches to manage bond portfolios and stock portfolios. Managing bond portfolios relies on concepts like duration, sensibility and convexity, while managing stock portfolios relies on optimization of expected utility. We give two results towards bridging this gap. First we set up and solve the problem of managing a bond portfolio by optimizing (over all self-financing trading

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strategies for a given initial capital) the expected utility of the final wealth. Second we express the solution of this problem as portfolios of self-financing trading strategies which include naturally stocks and bonds.

The well-established theory of portfolio management, initiated in the seminal papers [12], [19], [13], [14] and further developed by many, cf. [16], [11] and references therein, does not apply as it stands to bond portfolios. The difficulty here is that stocks and bonds differ in many ways, the most important of which is the fact that bonds mature at a prescribed date (time of maturity) after which they disappear from the market, whereas the characteristics of a stock do not change, except in reaction to business news or management decisions. Another difference is that in an unconstrained market, the time of maturity can take an infinity of values, so there is an infinity of different bonds. As a first consequence, the price of a stock depends only on the risks it carries (market risk, idiosyncratic risk), whereas the price of a bond depends both on the risks it carries (interest rate risk, credit risk) and on time to maturity. Mathematically, this is expressed by the fact that the stochastic differential equations used to model stock prices are usually autonomous (meaning that the coefficient are time-independent functions of the prices, as in geometric Brownian motion or mean-reverting processes), whereas any model for bond prices must incorporate the fact that the volatility goes to zero when time to maturity goes to zero. So the mathematical analysis of a portfolio including stocks and bonds is complicated by the fact that the prices for each type of assets evolve according to different rules, even in the most elementary case. An added difficulty is that certain strategies which are possible for stocks are no longer allowable for bonds: a simple buy-and-hold strategy, for instance, results in converting bonds to cash on maturity.

Our suggestion is to work in a "moving frame", that is, to consider time to maturity, instead of maturity, as the basic variable on which the zero-coupon depends at each time. At time t, there will be a curve $T \to p_t(T)$, $T \geq 0$, where $p_t(T)$ is the price of a standard zero-coupon maturing at time $t + T$. Here T is time to maturity and $S = t + T$ time of maturity. Such a parametrization was introduced in [15]. When t changes, so does the curve p_t, and a bond portfolio then is simply a linear functional operating on the space of such curves. Now from the financial point of view, this can be seen in different perspectives: 1) The static point of view, say, is to consider the portfolio at time t simply as a linear combination (possibly infinite) of standard zero-coupons, each of which has a fixed times of maturity $S \geq t$. Such a portfolio has to be rebalanced each time a zero-coupon in the portfolio comes to maturity. 2) The dynamic point of view, is to consider the portfolio at time t as a linear combination of self-financing instruments each one with a fixed time to maturity $T \geq 0$. We coin such an instrument a Roll-Over and it is simply a certain t-dependent multiple of a zero-coupon with time to maturity T, independent of t (see Remark 2.7). Its price has a simple expression, given by formula (2.31). Such instruments were introduced earlier in [18] under the name "rolling-horizon bond". Roll-Overs behave like stocks, in the sense that their time to maturity is constant through time, so that their price depends only on the risk they carry. One can then envision a program where portfolios are expressed as combinations of stocks and Roll-Overs, which are treated in a uniform fashion.

However, it is well-known that this program entails mathematical difficulties. The first one is that rewriting the equations for bond prices in the moving frame introduces the operator $\frac{\partial}{\partial T}$, which has to make sense as an unbounded operator in the space H of curves p_t chosen to describe zero-coupon prices. The second one is that this space H has to be contained in the space of all continuous functions on $\mathbb{R}^+$, so that its dual H^* contains the Dirac masses δ_{S-t}, corresponding at time t to one zero-coupon of maturity S, but should not be too small, otherwise H^* will contain many more objects which cannot easily be interpreted as bond portfolios. In this paper we choose H to be a standard Sobolev space, which in particular is a Hilbert space. Bond portfolios are then simply elements of the Hilbert space H^*. Reference [1] introduced portfolios being signed finite Borel measures. They also are elements of H^*. The analysis is in our case simplified by the fact that H and H^* are Hilbert spaces. In a different context, Hilbert spaces of forward rates were considered in [3], [6] and [7]. The image of these spaces, under the nonlinear map of forward rates to zero-coupons prices, is locally included in H.

We believe that this abstract, Hilbertian approach opens up many possibilities. In this paper, as mentioned above we explore one, namely portfolio management. We give existence theorems for very general utility functions and for H-valued price processes driven by a cylindrical Wiener process, i.e. in our case by a countable number of independent Brownian motions. We give explicit solutions, taking advantage of the Hilbertian setting. These solutions are expressed in terms of (non-unique) combinations of classical zero-coupon bonds (i.e. financial interpretation (1) above), but the optimal strategy can readily be translated in terms of Roll-Overs, which may not be marketed, although they are self-financing (i.e. financial interpretation (2) above). If the price of bonds depends on a d-dimensional Brownian motion, then the optimal strategy can be expressed as a linear combination of d bonds and in certain cases these can be any d marketed bonds, with time of maturity exceeding the time horizon of the optimal portfolio problem.

We note that, after the introduction of Roll-Overs, which transforms our problem into a problem similar to the case of stocks and making abstraction from the fact that there is an infinite number of different Roll-Overs, the slightly different conditions in our set-up compared with those of [10] and [11] for stock portfolios, leads to slightly different conclusions, concerning existence and uniqueness of optimal portfolios.

The portfolio theory developed here can be adapted to the management of other instruments depending on maturity dates, like portfolios of European type of derivatives. The case of maturity dependent reinsurance contracts, in discrete time, were considered in [20].

The outline of the paper is as follows. We begin by setting up the appropriate framework in §2, where bond portfolios are defined as elements of a certain Hilbert space H^*. Bond dynamics are prescribed in formula (2.12) according the the HJM methodology [8] and a self-financing portfolio is defined (cf. [1]) by formulas (2.25) and (2.26). An arbitrage free market is prescribed according to Condition A and we introduce certain self-financing trading strategies with fixed time to maturity, which we call Roll-Overs (Remark 2.7). The optimal portfolio problem is set up in §3, and solved in two special cases, the first being when the underlying Brownian motion is finite-dimensional (Theorem 3.5), the second being when it is infinite-dimensional, but the market price of risk is a deterministic function of time (Theorem 3.7). Examples of closed-form solutions are then given in §4.

Mathematical proofs are provided in §5 and Appendix A. We note that the appropriate mathematical framework for the study of infinite-dimensional (cylindrical) processes is the theory of Hilbert-Schmidt operators, to which we appeal in the proofs, although we have avoided it in the statement of the results.

Several remarks of a mathematical nature are made in §5. Remark 5.1 justifies our market condition (Condition A), in Remark 5.4 it is shown that our results apply to certain incomplete markets and in Remark 5.5 a Hamilton-Jacobi-Bellman approach is considered. We note, in Remark 5.6, that our existence result stands in apparent contrast with the earlier result of [10] and [11] for stock portfolios. Indeed, we prove existence of an optimal portfolio even in some cases where the asymptotic elasticity of the utility function equals 1, even though Theorem 2.2 (and comments) of [10] states that a necessary condition for existence is that this asymptotic utility should be strictly smaller than 1. This is because we have used a narrower definition (Condition A) of arbitrage-free prices for bonds.

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2 The bond market

We consider a continuous time bond market and without restriction we can assume that only zero-coupons are available. The time horizon in our model is some finite date $\bar{T} > 0$. At any date $t \in \mathbb{T} = [0, \bar{T}]$, one can trade zero-coupon bonds with maturity $s \in [t, \infty[$. Bonds with maturity $s = t$ at time t will be assimilated to money in a current account (see (ii) of Example 2.6 and cf. [2]).

Uncertainty is modelled by a filtered probability space $(\Omega, P, \mathcal{F}, \mathcal{A})$; here $\mathcal{A} = \{\mathcal{F}_t \mid 0 \leq t \leq \bar{T}\}$, is a filtration of the σ-algebra $\mathcal{F}$. The random sources are given by independent Brownian motions W^i, $i \in \mathbb{I}$. The index set $\mathbb{I}$ can be finite, $\mathbb{I} = \{1, \ldots, \bar{m}\}$, or infinite, $\mathbb{I} = \mathbb{N}^* = \mathbb{N} - \{0\}$. The filtration $\mathcal{A}$ is generated by the W^i, $i \in \mathbb{I}$ (one could also introduce jump processes as in [1]).

2.1 Zero-coupons and state space

As usual, we denote by $B(t, s)$ the price at time t of a zero-coupon yielding one unit of account at time s, $0 \leq t < s$, so that $B(t, t) = 1$. It is a $\mathcal{F}_t$-measurable random variable. Throughout the paper, we shall assume that, almost surely, the function $s \mapsto B(t, s)$ is strictly positive and C^1. We denote by $r(t)$ the spot interest rate at t :

$$r(t) = -\frac{1}{B(t,s)} \frac{\partial B}{\partial s}(t,s)\Big|_{s=t}, \tag{2.1}$$

which is allowed to be negative, and by $B^*(t, s)$ the price discounted to time 0 :

$$B^*(t,s) = B(t,s) \exp(-\int_0^t r(\tau) d\tau). \tag{2.2}$$

It will be convenient to characterize zero-coupons by their time to maturity. For this reason we introduce the $\mathcal{A}$-adapted C^1 ($[0, \infty[$)-valued processes p and p^* defined by

$$p_t(T) = B(t, t+T) \quad \text{and} \quad p^*_t(T) = B^*(t, t+T), \tag{2.3}$$

where $t \in \mathbb{T}$ and $T \geq 0$. This parameterization was introduced in [15]. One should here take care that T is the time to maturity and not the maturity itself. Note that $p_t(0) = 1$. We shall call p_t and (resp. p^*_t) the zero-coupon (resp. discounted zero-coupon) state at time t. For simplicity we will also use zero-coupon state or just state for both cases. The state at time t can thus be thought of as the curve: zero-coupons price at the instant t as function of time to maturity. Obviously

$$B(t,s) = p_t(s-t) \quad \text{and} \quad B^*(t,s) = p_t^*(s-t), \tag{2.4}$$

where $t \in \mathbb{T}$ and $s - t \geq 0$.

We will assume the processes p and p^* to take values in a certain Sobolev space H, the zero-coupon state space. Our choice of H is motivated by the fact that the state space of portfolios at each time, which is the dual of the zero-coupon state space, shall contain measures but not in general derivatives of measures. We now define H and recall certain elementary facts concerning Sobolev spaces.

For $s \in \mathbb{R}$, let H^s (c.f. §7.9 of [9]) be the usual Sobolev space of real tempered distributions f on $\mathbb{R}$ such that the function $x \mapsto (1+|x|^2)^{s/2}\hat{f}(x)$ is an element of $L^2(\mathbb{R})$, where $\hat{f}$ is the Fourier transform[1] of f, endowed with the norm:

$$\|f\|_{H^s} = \left(\int (1+|x|^2)^s \left|\hat{f}(x)\right|^2 dx \right)^{1/2}.$$

All the H^s are Hilbert spaces. Clearly, $H^0 = L^2$ and $H^s \subset H^{s'}$ for $s \geq s'$ and in particular $H^s \subset L^2 \subset H^{-s}$, for $s \geq 0$. If f is C^n, $n \in \mathbb{N}$ and if f together with its n first derivatives belong to L^2, then $f \in H^n$. For every s, the space $C_0^\infty(\mathbb{R})$ of C^∞ functions with compact support is dense in H^s. For every $s > 1/2$, by the Sobolev embedding theorems, we have $H^s \subset C^0 \cap L^\infty$. In addition H^s is a Banach algebra for $s > 1/2$: if $f \in H^s$ and $g \in H^s$ then $fg \in H^s$ and the multiplication is continuous. Also, if $s > 1/2$, $f \in H^s$ and $g \in H^{-s}$ then $fg \in H^{-s}$ and the multiplication is continuous also here.

We define, for $s \in \mathbb{R}$, a continuous bilinear form on $H^{-s} \times H^s$ by:

$$< f\,,\, g >= \int \overline{\left(\hat{f}(x)\right)}\, \hat{g}(x) dx, \tag{2.5}$$

where $\overline{z}$ is the complex conjugate of z. Any continuous linear form $f \to u(f)$ on H^s is of the form $u(f) =< g\,,\, f >$ for some $g \in H^{-s}$, with $\|g\|_{H^{-s}} = \|u\|_{(H^s)^*}$, so that henceforth we shall identify the dual $(H^s)^*$ of H^s with H^{-s}.

Fix some s in the interval $]1/2, 1[$. Since $s > 1/2$, we have $H^s \subset C^0 \cap L^\infty$, so that H^{-s} contains all bounded Radon measures on $\mathbb{R}$. In H^s, consider the set H^s_- of functions with support in $]-\infty, 0]$, so that $f \in H^s_-$ if and only

[1] In $\mathbb{R}^n$ we denote $x \cdot y = \sum_{1 \leq i \leq n} x_i y_i$, $x, y \in \mathbb{R}^n$ and we define the Fourier transform $\hat{f}$ of f by $\hat{f}(y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp(-iy \cdot x) f(x) dx$.

if $f(t) = 0$ for all $t > 0$. It is a closed subspace of H^s, so that the quotient space H^s/H^s_- is a Hilbert space as well. This is the space we want:

$$H = H^s/H^s_- \tag{2.6}$$

Recall that a real-valued function f on $[0, \infty[$ belongs to H if and only if it is the restriction to $[0, \infty]$ of some function in H^s, that is, if there is some function $\tilde{f} \in H^s$ (and hence defined on the whole real line) such that $\tilde{f}(t) = f(t)$ for all $t \geq 0$. The norm on H is given by

$$\|f\|_H = \inf \left\{ \|\tilde{f}\|_{H^s} \mid \tilde{f} \in H^s,\ \tilde{f}(t) = f(t)\ \forall t \geq 0 \right\}$$

and the dual space H^* by

$$H^* = \left\{ g \in H^{-s} \mid < \tilde{f}, g >= 0\ \ \forall \tilde{f} \in H^s_- \right\}$$

It follows that H^* is the set of all distributions in H^{-s} with support in $[0, \infty[$ and in particular, it contains all bounded Radon measures with support in $[0, \infty[$. H inherits the property of being a Banach algebra from H^s.

2.2 Bond dynamics

From now on, it will be assumed that the states p_t and p^*_t belong to H, so that the processes p and p^* are $\mathcal{A}$-adapted and H-valued.

We shall denote by $\mathcal{L} : [0, \infty[\times H \to H$ the semigroup of left translations in H :

$$(\mathcal{L}_a f)(s) = f(a + s) \tag{2.7}$$

where $a \geq 0$, $s \geq 0$ and $f \in H$. This is well-defined since both H^s and H^s_- in (2.6) are invariant under left translations. One readily verifies that $\mathcal{L}$ is a strongly continuous contraction semigroup in H. Therefore, cf. §3, Ch IX of [21], it has an infinitesimal generator which we shall denote by ∂, with dense and invariant domain[2], denoted by $\mathcal{D}(\partial)$. $\mathcal{D}(\partial)$is a Hilbert space with norm

$$\|f\|_{\mathcal{D}(\partial)} = (\|f\|^2_H + \|\partial f\|^2_H)^{1/2}. \tag{2.8}$$

Volatilities are assumed to take values in the Hilbert space $\tilde{H}_0$ of all real valued functions F on $[0, \infty[$ such that $F = a + f$, for some $a \in \mathbb{R}$ and $f \in H$. The norm is given by

$$\|F\|_{\tilde{H}_0} = (a^2 + \|f\|^2_H)^{1/2}, \tag{2.9}$$

[2] The domain consists of all $f \in H$ such that $\lim_{\epsilon \downarrow 0} \epsilon^{-1}(\mathcal{L}_\epsilon f - f)$ exists in H and for such f the limit is equal to ∂f.

wich is well-defined since the decomposition of $F = a+f$, $a \in \mathbb{R}$ and $f \in H$ is unique. $\tilde{H}_0$ is a subset of continuous multiplication operators on H. In fact, since H is a Banach algebra it follows that $\|Fh\|_H = C\|F\|_{\tilde{H}_0}\|h\|_H$, where $C > 0$ is independent of $F \in \tilde{H}_0$ and $h \in H$. We also introduce a Hilbert space $\tilde{H}_1$ of continuous multiplication operators on $\mathcal{D}(\partial)$. $\tilde{H}_1$ is the subspace of elements $F \in \tilde{H}_0$ with finite norm

$$\|F\|_{\tilde{H}_1} = (a^2 + \|f\|^2_{\mathcal{D}(\partial)})^{1/2}, \tag{2.10}$$

where $F = a+f$, $a \in \mathbb{R}$ and $f \in \mathcal{D}(\partial)$. Finally le us define the left translation in $\tilde{H}_0$ by

$$(\tilde{\mathcal{L}}_a F)(s) = F(a+s), \tag{2.11}$$

where $F \in \tilde{H}_0$, $a \geq 0$, $s \geq 0$. $\tilde{H}_1$ is the domain of the generator of $\tilde{\mathcal{L}}$, which we also denote ∂.

We shall assume that the bond dynamics are given by an equation of the following type:

$$p^*_t = \mathcal{L}_t p^*_0 + \int_0^t \mathcal{L}_{t-s} p^*_s m_s ds + \int_0^t \mathcal{L}_{t-s} p^*_s \sum_{i \in \mathbb{I}} \sigma^i_s dW^i_s, \tag{2.12}$$

for $t \in \mathbb{T}$, where σ^i_t, $i \in \mathbb{I}$, and m_t are $\mathcal{A}$-adapted $\tilde{H}_0$-valued processes and the W^i, $i \in \mathbb{I}$, are the already introduced standard Brownian motions. One must also take into account the boundary condition $B(t,t) = 1$, which in this context becomes

$$p^*_t(0) = \exp(-\int_0^t r(s)ds). \tag{2.13}$$

This can only be satisfied in general if

$$\sigma^i_t(0) = 0 \text{ for } i \in \mathbb{I} \tag{2.14}$$

and

$$m_t(0) = 0. \tag{2.15}$$

When $\mathbb{I}$ is finite, then (2.12) gives the usual HJM equation (equation (9) of [8]) for B.

In this paper, the process p^* is given. So formula (2.12), which then defines σ and m, can be considered as the decomposition of the real-valued semimartingale $t \mapsto p^*_t(S-t) = B^*(t,S)$, describing the value of the zero-coupon with maturity S, for each fixed value of S. Alternatively, one may want to take σ^i_t and m_t as the parameters in the model, and derive p^* as the solution of a stochastic differential equation in H. The aim of the following theorem is to ensure consistency in our model between the properties of p^*

and those of σ and m. Before stating it we note that p^* satisfying (2.12) is a H-valued semimartingale, however the second term on the right hand side is not in general the local martingale part, cf. equation (5.6).

Theorem 2.1 *If σ^i, $i \in \mathbb{I}$, and m are given $\mathcal{A}$-adapted $\tilde{H}_1$-valued processes, such that (2.14) and (2.15) are satisfied and such that*

$$\int_0^{\bar{T}} \sum_{i \in \mathbb{I}} \|\sigma_t^i\|_{\tilde{H}_1}^2 dt < \infty \ a.s. \tag{2.16}$$

and

$$\int_0^{\bar{T}} \|m_t\|_{\tilde{H}_1} dt < \infty \ a.s. \tag{2.17}$$

and if $p_0^* \in H$ is given and satisfies[3]

$$p_0^* \in \mathcal{D}(\partial),\ p_0^*(0) = 1,\ p_0^* > 0, \tag{2.18}$$

then equation (2.12) has, in the set of continuous $\mathcal{A}$-adapted H-valued semimartingales, a unique solution p^*. This solution has the following properties: p^* is strictly positive (i.e. $\forall t \in \mathbb{T}$, $p_t^* > 0$), $p_t^* \in \mathcal{D}(\partial)$ for each $t \in \mathbb{T}$, $t \mapsto \partial p_t^* \in H$ is continuous a.s., the boundary condition

$$p_t^*(0) = \exp(\int_0^t \frac{\partial p_s^*(0)}{p_s^*(0)} ds). \tag{2.19}$$

is satisfied for each $t \in \mathbb{T}$ and an explicit expression of the solution is given by

$$p_t^* = \exp\left(\int_0^t \tilde{\mathcal{L}}_{t-s}((m_s - \frac{1}{2}\sum_{i \in \mathbb{I}}(\sigma_s^i)^2)ds + \sum_{i \in \mathbb{I}} \sigma_s^i dW_s^i)\right)\mathcal{L}_t p_0^*. \tag{2.20}$$

In particular $p_t^* \in C^1([0, \infty[)$ a.s.

So, given appropriate σ^i, $i \in \mathbb{I}$, and m, the mixed initial value and boundary value problem (2.12), (2.13) has a unique solution for any initial curve of zero-coupon prices satisfying (2.18). The proof of Theorem 2.1 is given in §5.

Under additional conditions on σ^i, $i \in \mathbb{I}$, and m, we are able to prove L^p-estimates of p^*.

[3] We use obvious functional notations such as $f > 0$ for $f \in H$, meaning $\forall s > 0\ f(s) > 0$.

Theorem 2.2 *If σ^i, $i \in \mathbb{I}$, and m in Theorem 2.1 satisfy the following supplementary conditions: for each $a \in [1, \infty[$,*

$$E((\int_0^{\bar{T}} \sum_{i\in\mathbb{I}} \|\sigma_t^i\|_{\tilde{H}_1}^2 dt)^a + \exp(a \int_0^{\bar{T}} \sum_{i\in\mathbb{I}} \|\sigma_t^i\|_{\tilde{H}_0}^2 dt)) < \infty \tag{2.21}$$

and

$$E((\int_0^{\bar{T}} \|m_t\|_{\tilde{H}_1} dt)^a + \exp(a \int_0^{\bar{T}} \|m_t\|_{\tilde{H}_0} dt)) < \infty, \tag{2.22}$$

then the solution p^* in Theorem 2.1 has the following property: If $u \in [1, \infty[$, $q(t) = p_t/\mathcal{L}_t p_0$ and $q^*(t) = p_t^*/\mathcal{L}_t p_0^*$ then $p, p^* \in L^u(\Omega, P, L^\infty(\mathbb{T}, \mathcal{D}(\partial)))$ and $q, q^*, 1/q, 1/q^* \in L^u(\Omega, P, L^\infty(\mathbb{T}, \tilde{H}_1))$.

We remind that, under the hypotheses of Theorem 2.1, $p_t^*(0)$ satisfies (2.19), so it is the discount factor (2.13). Theorem 2.1 has the

Corollary 2.3 *Under the hypotheses of Theorem 2.2, if $\alpha \in \mathbb{R}$, then the discount factor satisfies*

$$E(\sup_{t\in\mathbb{T}}(p_t^*(0))^\alpha) < \infty.$$

2.3 Portfolios

The linear functionals in H^* will be interpreted as bond portfolios. More precisely, a portfolio is an H^*-valued $\mathcal{A}$-adapted process θ defined on $\mathbb{T}$. Its value at time t is:

$$V(t, \theta) =< \theta_t \, , \, p_t > \tag{2.23}$$

and its discounted value

$$V^*(t, \theta) =< \theta_t \, , \, p_t^* > . \tag{2.24}$$

Example 2.4
i) A portfolio consisting of one single zero-coupon with a fixed *time of maturity* S, $S \geq \bar{T}$. It is represented by θ, where $\theta_t = \delta_{S-t} \in H^*$, the Dirac mass with support at $S - t$, where $t \in \mathbb{R}$. Note that when t increases its support moves to the left towards the origin, which also can be expressed by $\theta_t(s) = \theta_0(s + t)$, for $s \geq 0$. Its value at time t is $p_t(S - t)$.
ii) A portfolio θ consisting of one single zero-coupon with a fixed *time of maturity* S, $0 \leq S < \bar{T}$. Then $\theta_t = \delta_{S-t} \in H^*$, for $t \leq S$ and $\theta_t = 0$, for $S < t \leq \bar{T}$. Its value at time $t \leq S$ is $p_t(S - t)$ and its value at time $t > S$ is zero.

iii) θ given by $\theta_t = \delta_T \in H^*$, the Dirac mass with fixed support at T, represents a portfolio which consists at any time of a single zero-coupon with *time to maturity* T; note that it has to be constantly readjusted to keep the time to maturity constant, and that its value at time t is $p_t(T)$.

As usual, a portfolio will be called *self-financing* if at any time, the change in its value is due to changes in market prices, and not to any redistribution of the portfolio, i.e.

$$V^*(t, \theta) = V^*(0, \theta) + G^*(t, \theta), \tag{2.25}$$

where $G^*(t, \theta)$ represents the discounted gains in the time interval $[0, t[$. We shall find the expression of $G^*(t, \theta)$. We remind that the subspace of elements f of H^* with support not containing 0 is dense in H^*. Suppose that the portfolio is already defined up to time t and that θ_t contains no zero-coupons of time to maturity smaller than some $A > 0$, i.e. θ_t has no support in $[0, A[$. At t let the portfolio evolve itself without any trading until $t + \epsilon$, where $0 < \epsilon < A$. Then $\theta_{t+\epsilon}$ is given by $\theta_{t+\epsilon}(s) = \theta_t(s + \epsilon)$, for $s \geq 0$. At $t + \epsilon$, the discounted value of the portfolio is $V^*(t + \epsilon, \theta) =< \theta_{t+\epsilon} \, , \, p^*_{t+\epsilon} >= \int_A^\infty \theta_t(s) p^*_{t+\epsilon}(s - \epsilon) ds$. We can now differentiate in ϵ. Using formulas (2.12) and (2.25) and taking the limits $\epsilon \to 0$ and then $A \to 0$ we obtain:

$$dG^*(t, \theta) =< \theta_t \, , \, p^*_t m_t > dt + \sum_{i \in \mathbb{I}} < \theta_t \, , \, p^*_t \sigma^i_t > dW^i_t. \tag{2.26}$$

We now take $G^*(0, \theta) = 0$ and this expression, in case it makes sens, as the definition of the discounted gains for an arbitrary portfolio θ.

To formalize this idea, we need to define appropriately the space of admissible portfolios. Given the process $p^*_\cdot$ an *admissible portfolio* is a H^*-valued $\mathcal{A}$-adapted process θ such that:

$$\|\theta\|^2_{\mathsf{P}} = E\left(\int_0^{\bar{T}} (\|\theta_t\|^2_{H^*} + \|\sigma^*_t \theta_t p^*_t\|^2_{H^*}) dt + (\int_0^{\bar{T}} | < \theta_t \, , \, p^*_t m_t > | dt)^2 \right) < \infty, \tag{2.27}$$

where we have used the *notation*

$$\|\sigma^*_t \theta_t p^*_t\|^2_{H^*} = \sum_{i \in \mathbb{I}} (< \theta_t \, , \, p^*_t \sigma^i_t >)^2. \tag{2.28}$$

For the mathematically minded reader this notation will be given a meaning in §5. The set of all admissible portfolios is Banach space P and the subset of all admissible self-financing portfolios is denoted by P_{sf}. The discounted gain process for a portfolio in P is a continuous square integrable processes:

Proposition 2.5 *Assume that p_0^*, m and σ are as in Theorem 2.1. If $\theta \in \mathsf{P}$, then $G^*(\cdot, \theta)$ is continuous a.s. and $E(\sup_{t\in\mathbb{T}}(G^*(t,\theta))^2) < \infty$.*

Example 2.6
i) The portfolio of Example 2.4 (i) is self-financing and the portfolios of Example 2.4 (ii) and (iii) are not self-financing.
ii) We define a self-financing portfolio θ of zero-coupons with constant *time to maturity T*; Let θ be given by $\theta_t = x(t)\delta_T$, where

$$x(t) = x(0)\exp\left(\int_0^t f_s(T)ds\right) \tag{2.29}$$

and $f_t(T) = -(\partial p_t^*)(T)/p_t^*(T)$ is the instantaneous forward rate contracted at $t \in \mathbb{T}$ for time to maturity T. That θ is self-financing is readily established by observing that in this case $x(t)p_t^*(T) = V^*(t,\theta)$, $V^*(t,\theta) = V^*(0,\theta) + \int_0^t V^*(s,\theta)(m_s(T)ds + \sum_{i\in\mathbb{I}}\sigma_s^i(T)dW_s^i)$ and by applying Itô's lemma to $x(t) = V^*(t,\theta)/p_t^*(T)$, cf. [18].

We note that $x(t) = V(t,\theta)/p_t(T)$ is the wealth at time t expressed in units of zero-coupons of time to maturity T. According to (2.29), the self-financing portfolio θ is then given by the initial number $x(0)$ of bonds and by the growth rate $f(T)$ of x. So this is as a money account, except that here we count in zero-coupons of time to maturity T.

In particular, if $T = 0$, then the equality $x(t) = V(t,\theta)$, the definition (2.1) of r and the definition (2.29) show that θ can be assimilated to money at a usual bank account with spot rate r, c.f. [2].

Remark 2.7 ***(Roll-Overs)***
i) Let $T \geq 0$, $x(0) = 1$ and the portfolio θ be as in (ii) of Example 2.6. Of course $X = V(\bar{T}, \theta)$ is then an attainable interest rate derivative, for which θ is a replicating portfolio. We name this derivative a Roll-Over or more precisely a T-Roll-Over to specify the time to maturity of the underlying zero-coupon. Let $\tilde{p}_t(T)$ be the discounted price of a T-Roll-Over at time t. Then $\tilde{p}_0(T) = p_0(T)$ by definition and the price dynamics of roll-overs is simply given by

$$\tilde{p}_t = p_0 + \int_0^t \tilde{p}_s m_s ds + \int_0^t \tilde{p}_s \sum_{i\in\mathbb{I}} \sigma_s^i dW_s^i, \tag{2.30}$$

$t \in \mathbb{T}$, *which solution is given by*

$$\tilde{p}_t = p_t^* \exp\left(\int_0^t f_s(T)ds\right). \tag{2.31}$$

ii) Zero-coupons do not in general permit self-financing buy-and-hold portfolios, i.e. constant portfolios. However Roll-Overs do, since a constant portfolio of roll-overs is always self-financing. Mathematically, this can be thought of as changing from a fixed frame to a moving frame for expressing a self-financed discounted wealth process in terms of coordinates, i.e. the portfolio. To be more precise let us consider a technically simple case. Let σ be non-degenerated in the sense that the linear span of the set $\{\sigma_t^i \mid i \in \mathbb{I}\}$ is dense a.s. in $\tilde{H}_0$ for every $t \in \mathbb{T}$. Let the initial price satisfy $\sup_{t\in\mathbb{T}} \sup_{s\geq 0} p_0(s)/p_0(t+s) < \infty$ and let the hypotheses of Theorem 2.2 be satisfied. Then a self-financing portfolio $\theta \in \mathsf{P}_{sf}$ is the unique replicating portfolio in $\theta \in \mathsf{P}_{sf}$ of $V(\bar{T}, \theta)$. Moreover, there is a unique $\eta \in \mathsf{P}$ such that $< \theta_t \, , \, p_t^* > = < \eta_t \, , \, \tilde{p}_t >$, for all $t \in \mathbb{T}$. The coordinates of the self-financed discounted wealth process $V(\cdot, \theta)$, with respect to the moving frame is η. In particular, a T-Roll-Over is given by the constant portfolio η, where $\eta_t = \delta_T$.

We next set up an arbitrage-free market by postulating a market-price of risk relation between m and σ. See Remark 5.1 for the motivation.

Condition A

There exists a family $\{\Gamma^i \mid i \in \mathbb{I}\}$ of real-valued $\mathcal{A}$-adapted processes such that

$$m_t + \sum_{i\in\mathbb{I}} \Gamma_t^i \sigma_t^i = 0, \tag{2.32}$$

and

$$E(\exp(a \int_0^{\bar{T}} \sum_{i\in\mathbb{I}} |\Gamma_t^i|^2 dt)) < \infty, \quad \forall a \geq 0. \tag{2.33}$$

If $\mathbb{I}$ is finite, Condition A is similar to a standard no-arbitrage condition. When Condition A is satisfied, formula (2.26) for the discounted gains of a portfolio θ becomes

$$dG^*(t, \theta) = \sum_{i\in\mathbb{I}} < \theta_t \, , \, p_t^* \sigma_t^i > (-\Gamma_t^i dt + dW_t^i). \tag{2.34}$$

The following result shows how to obtain a martingale measure in the general case of Condition A. Introduce the notation

$$\xi_t = \exp\left(-\frac{1}{2}\int_0^t \sum_{i\in\mathbb{I}} (\Gamma_s^i)^2 ds + \int_0^t \sum_{i\in\mathbb{I}} \Gamma_s^i dW_s^i)\right), \tag{2.35}$$

where $t \in \mathbb{T}$.

Theorem 2.8 *If (2.33) is satisfied, then* ξ *is a martingale with respect to* $(P, \mathcal{A})$ *and* $\sup_{t \in \mathbb{T}} \xi_t^\alpha \in L^1(\Omega, P)$ *for each* $\alpha \in \mathbb{R}$*. The measure* Q*, defined by*

$$dQ = \xi_{\bar{T}} dP,$$

is equivalent to P *on* $\mathcal{F}_{\bar{T}}$ *and* $t \mapsto \bar{W}_t^i = W_t^i - \int_0^t \Gamma_s^i ds$, $t \in \mathbb{T}$, $i \in \mathbb{I}$ *are independent Wiener process with respect to* $(Q, \mathcal{A})$*. (The Girsanov formula holds).*

The expected value of a random variable X with respect to Q is denoted $E_Q(X)$ and $E_Q(X) = E(\xi_{\bar{T}} X)$.

Proposition 2.5 and Theorem 2.8 have the

Corollary 2.9 *Assume that* p_0^* *and* σ *are as in Theorem 2.1 and assume that Condition A is satisfied. Then all conditions of Theorem 2.1 are satisfied and if* $\theta \in \mathsf{P}$*, then* $G^*(\cdot, \theta)$ *is continuous a.s.,* $E(\sup_{t \in \mathbb{T}} (G^*(t, \theta))^2) < \infty$ *and* $G^*(\cdot, \theta)$ *is a* $(Q, \mathcal{A})$*-martingale.*

By an arbitrage free market, we mean as usually, that there does not exist a self financing dynamical portfolio $\theta \in \mathsf{P}_{sf}$ such that $V(0, \theta) = 0$, $V(\bar{T}, \theta) \geq 0$ and $P(V(\bar{T}, \theta) > 0) > 0$. The following result shows that the market is arbitrage free:

Corollary 2.10 *Assume that* p_0^* *and* σ *are as in Theorem 2.1 and assume that Condition A is satisfied. If* $\theta \in \mathsf{P}_{sf}$*, its discounted price* $V^*(\cdot, \theta)$ *is a* $(Q, \mathcal{A})$*-martingale and* $E(\sup_{t \in \mathbb{T}} (V^*(t, \theta))^2) < \infty$*. In particular the market is arbitrage free.*

3 The optimal portfolio problem

The investor is characterized by his utility $u(w_{\bar{T}}^*)$, where $w_{\bar{T}}^*$ is terminal wealth, discounted to $t = 0$. Given the initial wealth x, denote by $\mathcal{C}(x)$ the set of all admissible self-financing portfolios starting from x :

$$\mathcal{C}(x) = \{\theta \in \mathsf{P}_{sf} \mid V^*(0, \theta_0) = x\}.$$

The investor's optimization problem is, for a given initial wealth K_0, find a solution $\hat{\theta} \in \mathcal{C}(K_0)$ of

$$E(u(V^*(\bar{T}, \hat{\theta}))) = \sup_{\theta \in \mathcal{C}(K_0)} E(u(V^*(\bar{T}, \theta))). \tag{3.1}$$

In the following, the utility density function is allowed to take the value $\{-\infty\}$, so $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$. Throughout this section, we make the following Inada-type assumptions:

Condition B
a) $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is strictly concave, upper semi-continuous and finite on an interval $]\underline{x}, \infty[$, with $\underline{x} \leq 0$ (the value $\underline{x} = -\infty$ is allowed).

b) u is C^1 on $]\underline{x}, \infty[$ and $u'(x) \to \infty$ when $x \to \underline{x}$ in $]\underline{x}, \infty[$.

c) there exists some $q > 0$ such that

$$\liminf_{x \downarrow \underline{x}} (1 + |x|)^{-q} u'(x) > 0 \tag{3.2}$$

and such that, if $u' > 0$ on $]\underline{x}, \infty[$ then

$$\limsup_{x \to \infty} x^q u'(x) < \infty \tag{3.3}$$

and if u' takes the value zero then

$$\limsup_{x \to \infty} x^{-q} u'(x) < 0. \tag{3.4}$$

Remark 3.1 *If u satisfies Condition B, then v obtained by an affine transformation, $v(x) = \alpha u(ax + b) - \beta$, $\alpha, a > 0$, $\beta \in \mathbb{R}$, $b \geq \underline{x}$, also satisfies Condition B. Usual utility functions, such as exponential $u(x) = -e^x$, quadratic $u(x) = -x^2/2$, power $u(x) = x^a/a$, $x > 0$, $a < 1$ and $a \neq 0$ and logarithmic $u(x) = \ln x$, $x > 0$ satisfy Condition B. Other, like HARA are obtained by affine transformations.*

It follows that u' restricted to $]\underline{x}, \infty[$ has a strictly decreasing continuous inverse φ i.e. a map such that $(\varphi \circ u')(x) = x$ for $x \in]\underline{x}, \infty[$. The domain of φ is $I = u'(\,]\underline{x}, \infty[\,)$. Condition B has an equivalent formulation in terms of φ :

Lemma 3.2 *If u satisfies Condition B then:*
i) If $u' > 0$ on $]\underline{x}, \infty[$, then $I =]0, \infty[$ and for some $C, p > 0$,

$$|\varphi(x)| \leq C(x^p + x^{-p}), \tag{3.5}$$

for all $x > 0$,
ii) If u' takes the value zero in $]\underline{x}, \infty[$, then $I = \mathbb{R}$ and for some $C, p > 0$,

$$|\varphi(x)| \leq C(1 + |x|)^p, \tag{3.6}$$

for all $x \in \mathbb{R}$.
Conversely, if $I =]0, \infty[$ (resp. $I = \mathbb{R}$), $\underline{x} \in [\infty, 0]$, and $\varphi : I \to]\underline{x}, \infty[$ satisfying (3.5) (resp. (3.6)) is a strictly decreasing continuous surjection with inverse g, $a \in \mathbb{R}$, $x_0 \in]\underline{x}, \infty[$ and if $u(x) = a + \int_{x_0}^x g(y)dy$, for $x \in]\underline{x}, \infty[$, $u(\underline{x}) = \lim_{x \downarrow \underline{x}} u(x)$, $u(x) = -\infty$, for $x < \underline{x}$, then u satisfies Condition B.

We shall next give existence results of optimal portfolios. In order to construct solutions of the optimization problem (3.1), we first solve a related problem of optimal terminal discounted wealth at time $\bar{T}$, which gives candidates of optimal terminal discounted wealths and secondly we construct, for certain of these candidates, a hedging portfolio, which then is a solution of the optimal portfolio problem 3.1. The construction of terminal discounted wealths is general and only requires that Condition A and Condition B are satisfied. For the construction of hedging portfolios, we separate the case of a finite number of random sources, i.e. $\mathbb{I} = \{1, \ldots, \bar{m}\}$ (Theorem 3.5) and the case of infinitely many random sources $\mathbb{I} = \mathbb{N}^*$ (Theorem 3.7). In the case of $\mathbb{I}$ finite, general stochastic volatilities being non-degenerated according to a certain condition are considered. In the case of $\mathbb{I}$ infinite, we only give results for deterministic σ, but which can be degenerated.

If X is the terminal discounted wealth for a self-financing strategy in $\mathcal{C}(K_0)$, then due to Corollary 2.9 $K_0 = E(\xi_{\bar{T}} X)$. We shall employ dual techniques to find candidates of the optimal X, c.f. [16].

Theorem 3.3 *Let u satisfy Condition B and let Γ satisfy condition (2.33). If $K_0 \in]\underline{x}, \infty[$ and $\mathcal{C}'(K_0) = \{X \in L^2(\Omega, P, \mathcal{F}_{\bar{T}}) \,|\, K_0 = E(\xi_{\bar{T}} X)\}$, then there exists a unique $\hat{X} \in \mathcal{C}'(K_0)$ such that*

$$E(u(\hat{X})) = \sup_{X \in \mathcal{C}'(K_0)} E(u(X)). \tag{3.7}$$

Moreover $\hat{X} \in L^p(\Omega, P)$ for each $p \in [1, \infty[$ and there is a unique $\lambda \in I$ such that $\hat{X} = \varphi(\lambda \xi_{\bar{T}})$.

Now, if $\hat{\theta}$ hedges $\hat{X}$, then $\hat{\theta}$ is a optimal portfolio. More precisely we have

Corollary 3.4 *Let m and σ satisfy the hypotheses of Theorem 2.1 and also be such that there exists a Γ with the following properties: Γ satisfies Condition A and $\hat{X}$, given by Theorem 3.3 satisfies $\hat{X} = V^*(\bar{T}, \hat{\theta})$ for some $\hat{\theta} \in \mathsf{P}_{sf}$. Then $\hat{\theta}$ is a solution of optimal portfolio problem (3.1).*

3.1 The case $\mathbb{I} = \{1, \ldots, \bar{m}\}$

Here we assume that a.s. the set of volatilities $\{\sigma_t^1, \ldots, \sigma_t^{\bar{m}}\}$ is linearly independent in $\tilde{H}_0$ for each $t \in \mathbb{T}$. Since $p_0 > 0$ and $p_0 \in H$, this is equivalent to the a.s. linear independence of $\{\sigma_t^1 \mathcal{L}_t p_0, \ldots, \sigma_t^{\bar{m}} \mathcal{L}_t p_0\}$ in H, for each t. Consider the $\bar{m} \times \bar{m}$, matrix $A(t)$ with elements

$$A(t)_{ij} = (\sigma_t^i \mathcal{L}_t p_0, \sigma_t^j \mathcal{L}_t p_0)_H,$$

(beware that we are using the scalar product in H and not in L^2):

Theorem 3.5 *Let $p_0 \in \mathcal{D}(\partial)$, $p_0(0) = 1$ and $p_0 > 0$, let $\sigma \neq 0$ satisfy conditions (2.14) and (2.21), and let Condition A and Condition B be satisfied. Assume that there exists an adapted process $k > 0$, such that for each $q \geq 1$ we have $\sup_{t \in \mathbb{T}} E(k_t^q) < \infty$ and, for each $x \in \mathbb{R}^{\bar{m}}$ and $t \in \mathbb{T}$:*

$$(x, A(t)x)_{\mathbb{R}^{\bar{m}}} k_t \geq \left(\sum_{i \in \mathbb{I}} \|\sigma_t^i \mathcal{L}_t p_0\|_H^2 \right)^{1/2} \|x\|_{\mathbb{R}^{\bar{m}}}^2, \text{ a.s.}. \tag{3.8}$$

If $K_0 \in]\underline{x}, \infty[$, then problem (3.1) has a solution $\hat{\theta}$.

We note that condition (3.8) only involves prices at time 0 and the volatilities. We also note that the optimal portfolio is never unique since one can always add a non-trivial portfolio θ' such that the linear span of the set $\{\sigma_t^1 p_t^*, \ldots, \sigma_t^{\bar{m}} p_t^*\}$ is in the kernel of θ'_t.

Remark 3.6 *Due to the non-uniqueness of the optimal portfolio $\hat{\theta}$ in Theorem 3.5, it can be realized using different numbers of bonds:*
1) One can always choose an optimal portfolio $\hat{\theta}$ such that $\hat{\theta}_t$ consists of at most $1+\bar{m}$ zero-coupons at every time t. This can be seen by a heuristic argument. Since, for every $t \geq 0$, the set of continuous functions $\{\sigma_t^1 p_t^*, \ldots, \sigma_t^{\bar{m}} p_t^*\}$ is linearly independent a.s., there exists positive $\mathcal{F}_t$-measurable finite random variables T_t^j such that $0 < T_t^1 < \ldots < T_t^{\bar{m}}$ and such that the vectors $v_t^j = (\sigma_t^1(T_t^j) p_t^*(T_t^j), \ldots, \sigma_t^{\bar{m}}(T_t^j) p_t^*(T_t^j))$, $1 \leq j \leq \bar{m}$ are linearly independent a.s. Let $\theta_t = \sum_{1 \leq j \leq \bar{m}} a_t^j \delta_{T_t^j}$, where a_t^j are real $\mathcal{F}_t$-measurable random variables. The equations $< \theta_t\, ,\, p_t^* \sigma_t^i >= y_i(t)$, $1 \leq i \leq \bar{m}$, where $y_i(t)$ is given by equation (5.33), then have a unique solution a_t. So at time t it is enough to use bonds with time to maturity $0 = T_t^0 < T_t^1 < \ldots < T_t^{\bar{m}}$ to realize an optimal portfolio $\hat{\theta}$. The number of bonds with time to maturity $0 = T_t^0$ is adjusted as to obtain a self-financing portfolio.
2) Alternatively to zero-coupon bonds, one can also use $\bar{m} + 1$ coupon bonds or Roll-Overs to realize an optimal portfolio.
3) For certain volatility structures, one can even use any $\bar{m}$ given Roll-Overs or $\bar{m}$ given marketed coupon bonds (supposed to have distinct times of maturity, each exceeding $\bar{T}$) to realize an optimal portfolio. In particular this is the case if the above vectors v_t^j, $1 \leq j \leq \bar{m}$ are linearly independent for every sequence $0 < T_t^1 < \ldots < T_t^{\bar{m}}$.

3.2 The case of deterministic σ and Γ

Condition (3.8) cannot hold in the infinite case, $\mathbb{I} = \mathbb{N}^*$. In fact this is a consequence of that $\int_0^{\bar{T}} \sum_{i \in \mathbb{I}} \|\sigma_t^i\|_{\tilde{H}_0}^2 dt < \infty$ a.s., as explained in Remark 5.3.

When σ and Γ are deterministic, we can give another, weaker, condition, which only involves σ, Γ and the zero coupon prices at time zero p_0^*. This will give us a result which will hold for the infinite case as well. Properties of the inverse φ of the derivative of the utility density function u, satisfying Condition B, were given in Lemma 3.2. For simplicity we shall need one more property, which we impose directly as a condition on φ. We keep in mind that $\varphi' < 0$, since u is strictly concave.

Condition C *Let Condition B be satisfied, assume that u is C^2 on $]\underline{x}, \infty[$ and assume that it exists $C, p > 0$ such that*
a) If $u' > 0$ on $]\underline{x}, \infty[$, then

$$|x\varphi'(x)| \leq C(x^p + x^{-p}), \tag{3.9}$$

for all $x > 0$
b) If u' takes the value zero in $]\underline{x}, \infty[$, then

$$|x\varphi'(x)| \leq C(1 + |x|)^p, \tag{3.10}$$

for all $x \in \mathbb{R}$.

We note that Condition B implies Condition C if u' is homogeneous. Condition C is satisfied for the utility functions in Remark 3.1.

Theorem 3.7 *Let σ and m be deterministic, while $\mathbb{I}$ is finite or infinite. Let $p_0 \in \mathcal{D}(\partial)$, $p_0(0) = 1$ and $p_0 > 0$, let σ satisfy conditions (2.14) and (2.16) and let Condition A and Condition C be satisfied. Assume that it exists a (deterministic) H^*-valued function $\gamma \in L^2(\mathbb{T}, H^*)$ such that*

$$< \gamma_t \, , \, \sigma_t^i \mathcal{L}_t p_0 >= \Gamma_t^i, \tag{3.11}$$

for each $i \in \mathbb{I}$ and $t \in \mathbb{T}$. If $K_0 \in]\underline{x}, \infty[$, then problem (3.1) has a solution $\hat{\theta}$.

As explained in Remark 5.4, condition (3.11) can be satisfied in highly incomplete markets. In the situation of Theorem 3.7, we can derive an explicit expression of an optimal portfolio.

Corollary 3.8 *Under the hypotheses of Theorem 3.7, an optimal portfolio is given by $\hat{\theta} = \theta^0 + \theta^1$, where $\theta^0, \theta^1 \in \mathsf{P}$, $\theta^0 = a_t \delta_0$, $\theta_t^1 = (p_t^*)^{-1} b_t \gamma_t \mathcal{L}_t p_0$. The coefficients a and b are real-valued $\mathcal{A}$-adapted processes given by*

$$b_t = E_Q(\lambda \xi_{\bar{T}} \varphi'(\lambda \xi_{\bar{T}}) \,|\, \mathcal{F}_t) \tag{3.12}$$

and

$$a_t = (p_t^*(0))^{-1}(Y(t) - b_t < \gamma_t \, , \, \mathcal{L}_t p_0 >), \tag{3.13}$$

$t \in \mathbb{T}$, where $Y(t) = E_Q(\varphi(\lambda\xi_{\bar{T}}) \,|\, \mathcal{F}_t)$ and $\lambda \in I$ is unique. The discounted price of the portfolio $\hat{\theta}$ is given by $V^*(t, \hat{\theta}) = Y(t)$, $t \in \mathbb{T}$. Moreover $< \theta^0 \,,\, \sigma^i_t p^*_t >= 0$ and

$$< \theta^1 \,,\, \sigma^i_t p^*_t >= E_Q(\lambda\xi_{\bar{T}}\varphi'(\lambda\xi_{\bar{T}}) \,|\, \mathcal{F}_t)\Gamma^i_t, \tag{3.14}$$

$i \in \mathbb{I}$ and $t \in \mathbb{T}$.

The proofs of Theorem 3.7 and Corollary 3.8 are based on a Clark-Ocone like representation of the optimal terminal discounted wealth (see Lemma A.5). Alternatively, the explicit expressions in Corollary 3.8 can be obtained by a Hamilton-Jacobi-Bellman approach (see Remark 5.5). This corollary has an important consequence, since it leads directly to mutual fund theorems. We shall state a version only involving self-financing portfolios

Theorem 3.9 *Under the hypotheses of Theorem 3.7, there exists a self-financing portfolio $\Theta \in \mathsf{P}_{sf}$, with the following properties:*

i) The initial value of Θ is 1 euro, i.e. $< \Theta_0 \,,\, p^*_0 >= 1$ and the value at each time $t \in \mathbb{T}$ is strictly positive, i.e. $< \Theta_t \,,\, p^*_t >> 0$.

ii) For each given utility density u satisfying Condition C and each initial wealth $K_0 \in \,]\underline{x}, \infty[\,$, there exist two real valued processes x and y such that if $\hat{\theta}_t = x_t\delta_0 + y_t\Theta_t$, then $\hat{\theta}$ is an optimal self financing portfolio for u, i.e. a solution of problem (3.1).

4 Examples of closed form solutions

In this section we shall give, in the situation of Corollary 3.8, examples of solutions of problem (3.1), for certain utility densities u. In particular, Condition (3.11) is satisfied, so σ and Γ are deterministic.

According to Corollary 3.8, the final optimal wealth is $Y(\bar{T}) = \varphi(\lambda\xi_{\bar{T}})$ and the optimal discounted wealth process Y is given by $Y(t) = E_Q(\varphi(\lambda\xi_{\bar{T}}) \,|\, \mathcal{F}_t)$. The initial wealth $Y(0) = K_0$. We introduce the optimal utility U as a function of discounted wealth w at time $t \in \mathbb{T}$,

$$U(t, w) = E(u(Y(\bar{T})) \mid Y(t) = w). \tag{4.1}$$

Example 4.1 *Quadratic utility;*
The utility density is

$$u(x) = \mu x - x^2/2, \tag{4.2}$$

where $\mu \in \mathbb{R}$ is given. Determination of λ gives

$$\lambda = (\mu - K_0)\exp(-\frac{3}{2}\int_0^{\bar{T}} \sum_{i\in\mathbb{I}} (\Gamma_s^i)^2 ds). \tag{4.3}$$

The optimal discounted wealth process Y, for initial wealth $K_0 \in \mathbb{R}$ at $t = 0$, is given by

$$Y(t) = \mu + (K_0 - \mu)\exp\left(\int_0^t \sum_{i\in\mathbb{I}} (\Gamma_s^i d\bar{W}_s^i - \frac{1}{2}(\Gamma_s^i)^2 ds)\right), \tag{4.4}$$

where we have used Theorem 2.8. For given $\mu, w \in \mathbb{R}$ and $t \in \mathbb{T}$, the optimal utility, is given by

$$U(t, w) = (-\frac{1}{2}w^2 + \mu w)\exp\left(-\int_t^{\bar{T}} \sum_{i\in\mathbb{I}} (\Gamma_s^i)^2 ds\right) + \frac{1}{2}\mu^2 \int_t^{\bar{T}} \sum_{i\in\mathbb{I}} (\Gamma_s^i)^2 ds. \tag{4.5}$$

One finds first $b_t = Y(t) - \mu$ and then

$$a_t = (p_t^*(0))^{-1}(Y(t) - (Y(t) - \mu) < \gamma_t\, , \mathcal{L}_t p_0 >). \tag{4.6}$$

An optimal portfolio is given by $\hat{\theta} = \theta^0 + \theta^1$, where $\theta_t^0 = a_t \delta_0$ and

$$\theta_t^1 = (Y(t) - \mu)\gamma_t (p_t^*)^{-1} \mathcal{L}_t p_0. \tag{4.7}$$

We see that the discounted wealth invested in θ^0 is $Y(t) - (Y(t) - \mu) < \gamma_t\, , \mathcal{L}_t p_0 >$ and in θ^1 is $(Y(t) - \mu) < \gamma_t\, , \mathcal{L}_t p_0 >$. If we want a certain expected return over the period $\mathbb{T}$, then this will of course fix μ in formula (4.4).

Example 4.2 *Exponential utility;*
The utility density is

$$u(x) = -\exp(-\mu x), \tag{4.8}$$

where $\mu > 0$ is given and $x \in \mathbb{R}$. Determination of λ gives

$$-\frac{1}{\mu}\ln\frac{\lambda}{\mu} = K_0 + \frac{1}{2\mu}\int_0^{\bar{T}} \sum_{i\in\mathbb{I}} (\Gamma_s^i)^2 ds. \tag{4.9}$$

The optimal discounted wealth process Y, for initial wealth $K_0 \in \mathbb{R}$, is given by

$$Y(t) = K_0 - \frac{1}{\mu}\int_0^t \sum_{i\in\mathbb{I}} \Gamma_s^i d\bar{W}_s^i. \tag{4.10}$$

The optimal utility is given by

$$U(t,w) = -\exp\left(-\mu w - \frac{1}{2}\int_t^{\bar{T}} \sum_{i\in\mathbb{I}} (\Gamma_s^i)^2 ds\right), \tag{4.11}$$

where $w \in \mathbb{R}$ and $t \in \mathbb{T}$. For an optimal portfolio we get $b_t = -1/\mu$,

$$a_t = (p_t^*(0))^{-1}(Y(t) + \frac{1}{\mu} < \gamma_t \,,\, \mathcal{L}_t p_0 >) \tag{4.12}$$

and $\hat{\theta} = \theta^0 + \theta^1$, where $\theta_t^0 = a_t\delta_0$ and

$$\theta_t^1 = -\frac{1}{\mu}(p_t^*)^{-1}\gamma_t \mathcal{L}_t p_0. \tag{4.13}$$

So in this case the portfolio θ_t^1, of risky zero coupons, is deterministic. However the portfolio θ_t^0, i.e. the number a_t of zero coupons of time to maturity 0 is random through its dependence on the wealth $Y(t)$. The discounted wealth invested in θ^0 is $Y(t) + \frac{1}{\mu} < \gamma_t \,,\, \mathcal{L}_t p_0 >$ and in θ^1 is $-\frac{1}{\mu} < \gamma_t \,,\, \mathcal{L}_t p_0 >$. Expressed in Roll-Overs the portfolio becomes $\hat{\eta} = \eta^0 + \eta^1$,

$$\eta_t^0(T) = Y(t) + \frac{1}{\mu} < \gamma_t \,,\, \mathcal{L}_t p_0 > \delta(T), \tag{4.14}$$

$$\eta_t^1(T) = -\frac{1}{\mu}\exp\left(\int_0^t (r_s - f_s(T))ds\right)\frac{p_0(t+T)}{p_t(T)}\gamma_t(T), \tag{4.15}$$

where $T \geq 0$ and $\delta = \delta_0$.

Example 4.3 *Homogeneous utility;*
The utility density is

$$u(x) = x^\mu, \tag{4.16}$$

where $0 < \mu < 1$ is given and $x > 0$. Determination of λ gives

$$(\frac{\mu}{\lambda})^{\frac{1}{1-\mu}} = K_0 \exp\left(-\frac{\mu}{2(1-\mu)^2}\int_0^{\bar{T}} \sum_{i\in\mathbb{I}} (\Gamma_s^i)^2 ds\right). \tag{4.17}$$

The optimal discounted wealth process Y, for initial wealth $K_0 > 0$, is given by

$$Y(t) = K_0 \exp\left(\int_0^t \sum_{i\in\mathbb{I}} (-\frac{1}{1-\mu}\Gamma_s^i d\bar{W}_s^i - \frac{1}{2}(\frac{1}{1-\mu}\Gamma_s^i)^2 ds)\right). \tag{4.18}$$

The optimal utility is given by

$$U(t,w) = w^{\mu} \exp\left(\frac{\mu}{2(1-\mu)} \int_t^{\bar{T}} \sum_{i \in \mathbb{I}} (\Gamma_s^i)^2 ds \right), \; w > 0. \tag{4.19}$$

The optimal portfolio $\hat{\theta}$ is given by

$$b_t = -\frac{1}{1-\mu} Y(t) \tag{4.20}$$

and

$$a_t = (p_t^*(0))^{-1}(1 + \frac{1}{1-\mu} < \gamma_t \,,\, \mathcal{L}_t p_0 >) Y(t), \tag{4.21}$$

so both θ^0 and θ^1 are proportional to the wealth.

5 Mathematical complements and Proofs

In the sequel it will be convenient to use a more compact mathematical formalism, which we now introduce. The dual of $\tilde{H}_0$ is identified with $\tilde{H}_0^* = \mathbb{R} \oplus H^*$ by extending the bi-linear form, defined in (2.5), to $\tilde{H}_0^* \times \tilde{H}_0^*$:

$$< F \,,\, G >= ab + < f \,,\, g >, \tag{5.1}$$

where $F = a + f \in \tilde{H}_0^*$, $G = b + g \in \tilde{H}_0$, $a, b \in \mathbb{R}$, $f \in H^*$ and $g \in H$. $\{e_i\}_{i \in \mathbb{N}^*}$ is a orthonormal basis in $\tilde{H}_0$. For $i \in \mathbb{N}^*$, the element $e'_i \in \tilde{H}_0^*$ is given by $< e'_i \,,\, f >= (e_i, f)_{\tilde{H}_0}$, for every $f \in \tilde{H}_0$. The map[4] $\mathcal{S} \in L(\tilde{H}_0, \tilde{H}_0^*)$ is defined by $\mathcal{S}f = \sum_{i \geq 1} < e'_i \,,\, f > e'_i$. The adjoint $\mathcal{S}^* \in L(\tilde{H}_0^*, \tilde{H}_0)$ is given by $\mathcal{S}^* f = \sum_{i \geq 1} < e_i \,,\, f > e_i$. Moreover, $(f, g)_{\tilde{H}_0} =< \mathcal{S}f \,,\, g >$ for $f, g \in \tilde{H}_0$, $(f, g)_{\tilde{H}_0^*} =< f \,,\, \mathcal{S}^* g >$ for $f, g \in \tilde{H}_0^*$ and $\mathcal{S}$ is unitary. For a given orthonormal basis $\{e'_i\}_{i \in \mathbb{I}}$ in $\tilde{H}_0^*$ we define the $L(\tilde{H}_0)$-valued process $\{\sigma_t\}_{t \in \mathbb{T}}$, by

$$\sigma_t f = \sum_{i \in \mathbb{I}} < e'_i \,,\, f > \sigma_t^i, \tag{5.2}$$

for $f \in \tilde{H}_0$. We note that if $\sum_{i \geq 1} \|\sigma_t^i\|^2_{\tilde{H}_0} < \infty$ a.s. then σ_t is a.s. a Hilbert-Schmidt operator valued process, with Schmidt norm

$$\|\sigma_t\|_{H-S} = (\sum_{i \geq 1} \|\sigma_t^i\|^2_{\tilde{H}_0})^{1/2}. \tag{5.3}$$

[4] $L(E, F)$ denotes the space of linear continuous mappings from E into F, $L(E) = L(E, E)$.

The adjoint is given by

$$\sigma_t^* f = \sum_{i \in \mathbb{I}} < f \, , \, \sigma_t^i > e_i', \tag{5.4}$$

for $f \in \tilde{H}_0^*$.

We define a cylindrical Wiener process W on $\tilde{H}_0$, c.f. §4.3.1 of reference [4]: $W_t = \sum_{i \in \mathbb{I}} W_t^i e_i$. We also define $\Gamma_t = \sum_{i=1}^{\infty} \Gamma_t^i e_i$, which is an element of $\tilde{H}_0$ a.s. if $\sum_{i=1}^{\infty} (\Gamma_t^i)^2 < \infty$ a.s. Equation (2.12) now reads

$$p_t^* = \mathcal{L}_t p_0^* + \int_0^t \mathcal{L}_{t-s} p_s^* m_s ds + \int_0^t \mathcal{L}_{t-s} p_s^* \sigma_s dW_s, \tag{5.5}$$

its differential

$$dp_t^* = (m_t p_t^* + \partial p_t^*) dt + p_t^* \sigma_t dW_t, \tag{5.6}$$

equation (2.26)

$$dG^*(t, \theta) =< \theta_t \, , \, p_t^* m_t > dt + < \sigma_t^* p_t^* \theta_t \, , \, dW_t >, \tag{5.7}$$

relation (2.32)

$$m_t + \sigma_t \Gamma_t = 0 \tag{5.8}$$

and equation (2.34)

$$dG^*(t, \theta) = - < \sigma_t^* p_t^* \theta_t \, , \, \Gamma_t > dt + < \sigma_t^* p_t^* \theta_t \, , \, dW_t >, \tag{5.9}$$

where $t \in \mathbb{T}$.

The quadratic variation for a process M is, when defined, denoted $\ll M \, , \, M \gg$.

Remark 5.1 *In order to justify condition (5.8), we note (omitting the a.s.) that if θ' is a self financing strategy such that $\theta'_t \in H^*$ is in the annihilator $\{p_t^* \sigma_t^i \,|\, i \in \mathbb{I}\}^{\perp} \subset H^*$ of the set $\{p_t^* \sigma_t^i \,|\, i \in \mathbb{I}\} \subset H$, then (2.26) gives $dG^*(t, \theta') =< \theta'_t \, , \, m_t p_t^* > dt$. θ' is therefore a risk-less self financing strategy. Since the interest rate of the discounted bank account is zero, in an arbitrage free market we must have $< \theta'_t \, , \, m_t p_t^* >= 0$. This shows that $m_t p_t^* \in (\{p_t^* \sigma_t^i \,|\, i \in \mathbb{I}\}^{\perp})^{\perp}$, i.e. $m_t p_t^*$ is an element of the closed linear span of $\{p_t^* \sigma_t^i \,|\, i \in \mathbb{I}\}$. Since $p_t^* > 0$, we choose m_t to be an element of the closed linear span F of $\{\sigma_t^i \,|\, i \in \mathbb{I}\}$ in $\tilde{H}_0$.*

When (also omitting the a.s.) the linear span of $\{\sigma_t^i \,|\, i \in \mathbb{I}\}$ has infinite dimension then condition (5.8) is slightly stronger than $m \in F$, since σ_t must be a compact operator in $\tilde{H}_0$. This phenomena, which is purely due to the infinite dimension of the state space H, is not present in the case of a market with a finite number of assets.

Remark 5.2 *The conditions involving* m *are redundant when equality (5.8) is satisfied. For example conditions (2.21) and (2.33) imply condition (2.22). In fact,* $\|m_t\|_{\tilde{H}_0} \le (\sum_{i\in\mathbb{I}} \|\sigma_t^i\|^2_{\tilde{H}_0})^{1/2}(\sum_{i\in\mathbb{I}} |\Gamma_t^i|^2)^{1/2} \le 1/2(\sum_{i\in\mathbb{I}} \|\sigma_t^i\|^2_{\tilde{H}_0} + \sum_{i\in\mathbb{I}} |\Gamma_t^i|^2)$. *By Schwarz inequality* $E(\exp(a\int_0^{\bar{T}} \sum_{i\in\mathbb{I}} \|m_t\|_{\tilde{H}_0} dt))$
$\le (E(\exp(2a\int_0^{\bar{T}} \sum_{i\in\mathbb{I}} \|\sigma_t^i\|^2_{\tilde{H}_0} dt)))^{1/2}(E(\exp(2a\int_0^{\bar{T}} \sum_{i\in\mathbb{I}} |\Gamma_t^i|^2 dt)))^{1/2}$.

Remark 5.3 *When the number of random sources is infinite, i.e.* $\mathbb{I} = \mathbb{N}^*$, *then the straight forward generalization of condition (3.8) from* $x \in \mathbb{R}^{\bar{m}}$ *to* $x \in l^2$ *can not be satisfied, since* σ_t *is a.s. a compact operator in* $\tilde{H}_0$. *In fact, in this case the left hand side of (3.8) reads* $(x, A(t)x)_{l^2}$. *Let* $l_t = \mathcal{L}_t p_0$. *By the definition of* $A(t)$ *and by the canonical isomorphism between* l^2 *and* $\tilde{H}_0$ *we obtain* $(x, A(t)x)_{l^2} = \| \sum_{i\in\mathbb{I}} x_i \sigma_t^i l_t\|^2_{\tilde{H}_0} = \|l_t\sigma_t f\|^2_{\tilde{H}_0}$, *where* $x_i = (e_i, f)_{\tilde{H}_0}$. *Condition (3.8) then reads* $\|l_t\sigma_t f\|^2_{\tilde{H}_0} k_t \ge \|l_t\sigma_t\|_{H-S}\|f\|^2_{\tilde{H}_0}$. *Since* σ_t *is a.s. compact, which then also is the case for* $l_t\sigma_t$, *it follows that* $\inf_{\|f\|_{\tilde{H}_0}=1} \|l_t\sigma_t f\|_{\tilde{H}_0} = 0$. *This is in contradiction with* k_t *finite a.s. and* $l_t\sigma_t \ne 0$ *a.s.*

Remark 5.4 *Concerning condition (3.11):*
i) Γ *is unique or more precisely: Given* σ *and* m *such that the hypotheses of Theorem 3.7 are satisfied. Then there is a unique* Γ *satisfying Condition A and satisfying condition (3.11) for some* γ. *To establish this fact let* σ_t' *be the usual adjoint operator in* $\tilde{H}_0$, *with respect to the scalar product in* $\tilde{H}_0$, *of the operator* σ_t. *Condition (3.11) can then be written* $\sigma_t'\delta_t = \Gamma_t$, *where* $\delta_t = \mathcal{S}^* l_t\gamma_t$ *and* $l_t = \mathcal{L}_t p_0$. $\delta_t \in \tilde{H}_0$, *since* $\|\delta_t\|_{\tilde{H}_0} = \|l_t\gamma_t\|_{\tilde{H}_0^*} \le C\|l_t\|_{\tilde{H}_0}\|\gamma_t\|_{\tilde{H}_0^*} < \infty$. *This shows that* Γ_t *is in the orthogonal complement, with respect to the scalar product in* $\tilde{H}_0$, *of* $Ker\,\sigma_t$. *There can not be more than one solution* Γ_t *of (5.8) with this property.*
ii) Condition (3.11) can be satisfied for arbitrary (included degenerated) volatilities σ, *resulting in incomplete markets. An example is obtained by, for given* σ, *choosing a* γ *and then defining* Γ *and* m *by (3.11) and (5.8) respectively.*

Remark 5.5 *When* m_t *and* σ_t *are given functions of* p_t^*, *for every* $t \in \mathbb{T}$, *then the optimal portfolio problem (3.1) can be solved by a Hamilton-Jacobi-Bellman approach. We illustrate this in the simplest case, when* m_t *and* σ_t *are deterministic. The optimal value function* U *then only depends of time* $t \in \mathbb{T}$ *and of the value of the discounted wealth* w *at time* t:

$$U(t,w) = \sup\{E(u(V^*(\bar{T},\theta)) \mid V^*(t,\theta) = w) \mid \theta \in \mathsf{P}_{sf}\}, \tag{5.10}$$

(here $E(Y \mid X = x)$ *is the conditional expectation of* Y *under the condition*

that $X = x$). One is then led to the HJB equation

$$\frac{\partial U}{\partial t}(t,w) + \sup_{f \in H^*}\{- < \sigma_t^* f \, , \Gamma_t > \frac{\partial U}{\partial w}(t,w) + \frac{1}{2}\|\sigma_t^* f\|^2_{H^*}\frac{\partial^2 U}{\partial w^2}(t,w)\} = 0 \tag{5.11}$$

with boundary condition

$$U(\bar{T}, w) = u(w). \tag{5.12}$$

Equation (5.11) gives

$$\frac{\partial U}{\partial t}\frac{\partial^2 U}{\partial w^2} = \frac{1}{2}\|\Gamma_t\|^2_H(\frac{\partial U}{\partial w})^2. \tag{5.13}$$

Each self financing zero coupon strategy $\hat{\theta} \in \mathsf{P}_{sf}$, such that

$$< \hat{\theta}_t \, , p_t^* \sigma_t^i >= \frac{\Gamma_t^i \frac{\partial U}{\partial w}}{\frac{\partial^2 U}{\partial w^2}}, \; i \in \mathbb{I}, \tag{5.14}$$

if $\hat{V}^*(t) = w$, is then a solution of problem (3.1). In particular, the solution of Corollary 3.8 satisfies equation (5.14). When m_t and σ_t are functions of the price p^*, then HJB equation containers supplementary terms involving the Frechét derivative with respect to p^*.

Remark 5.6 *It is interesting to look at our results in the light of asymptotic elasticity for utility functions and the results of [10] concerning markets of one money account and a fixed finite number of stocks, for which the price process is a general semi-martingale. By definition (see Definition 2.2 of [10]),*

$$e_\infty(u) = \limsup_{x \to \infty} xu'(x)/u(x)$$

is the asymptotic elasticity of a utility function u.

i) Let $\mathcal{U}$ be the set of utility functions u satisfying Condition B, $\underline{x} > -\infty$ and $u' > 0$. Let $\mathcal{U}_-$ be the set of utility functions u satisfying only (a) and (b) of Condition B, $\underline{x} > -\infty$, $u' > 0$, $\lim_{x \to \infty} u'(x) = 0$ and $e_\infty(u) < 1$. We have the following result, proved in §5.15:

$$\mathcal{U}_- \subset \mathcal{U} \tag{5.15}$$

and

$$\exists u \in \mathcal{U} \quad \text{such that} \quad e_\infty(u) = 1. \tag{5.16}$$

Of course, since only the values of u for large x are important in (5.16), there is such a u satisfying Condition B, for each $\underline{x} \in]-\infty, 0]$. Moreover u can be *chosen to be C^∞ on $]\underline{x},\infty[$.*

ii) In the situation of Theorem 3.5 and Theorem 3.7 there exists optimal portfolios in P_{sf} and the terminal discounted wealth $\hat{X}$ is unique, according to Theorem 3.3. In particular this is true for $u \in \mathcal{U}$ with $e_\infty(u) = 1$ and for all initial capitals $K_0 \in]\underline{x},\infty[$. This is in contrast to the situation considered in [10], where for such u, with $e_\infty(u) = 1$ and K_0 sufficiently large, there is for certain complete financial markets no solution $\hat{X}$ (see Proposition 5.2 of [10]).

5.1 Proof of Theorem 2.1

Proof of Theorem 2.1: Existence, uniqueness and continuity of p^* and ∂p^* follows from Lemma A.1. It then follows from formula (A.21) of Lemma A.2, with $Y(t) = \mathcal{L}_t p_0^*$ that the solution is given by (2.20). This shows that it is positive.

Finally we prove that condition (2.19) is satisfied. Formula (5.6) and conditions (2.14) and (2.15), give $d(p_t^*(0)) = (\partial p_t^*)(0)dt$. Since $p_0^*(0) = 1$, (2.19) follows by integration. End of proof.

5.2 Proof of Theorem 2.2

Proof of Theorem 2.2: It follows from the explicit expression (2.20) of p^* and formula (A.19) that $q^* = \tilde{\mathcal{E}}(L)$, where $L(t) = \int_0^t (m_s ds + \sum_{i\in\mathbb{I}} \sigma_s^i dW_s^i)$. Let $\alpha = 1$ or $\alpha = -1$ and let $J_\alpha = \int_0^t ((\alpha m_s + \alpha(\alpha-1)/2 \sum_{i\in\mathbb{I}} (\sigma_s^i)^2) ds + \sum_{i\in\mathbb{I}} \alpha\sigma_s^i dW_s^i)$. Then $(q^*)^\alpha = \tilde{\mathcal{E}}(J_\alpha)$. According to conditions (2.21), (2.22), hypotheses (i)–(iv) of Lemma A.4 (with J_α instead of L) are satisfied. We now apply estimate (A.39) of Lemma A.4 to $X = (q^*)^\alpha$, which proves that $(q^*)^\alpha \in L^u(\Omega, P, L^\infty(\mathbb{T}, \tilde{H}_1))$, for $\alpha = \pm 1$. Since $p_t^* = q^*(t)\mathcal{L}_t p_0$, $\mathcal{L}$ is a contraction semigroup and $\tilde{H}_0$ is a Banach algebra, we have $\|p_t^*\|_H^2 + \|\partial p_t^*\|_H^2 \leq C(\|p_0^*\|_H^2 + \|\partial p_0^*\|_H^2)\|q^*(t)\|_{\tilde{H}_1}^2$, for some constant C given by H. This proves the statement of the lemma in the case p^*.

To prove the case of q we note that $q^*(t) = q(t)p_t^*(0)$. Using that the case of $(q^*)^\alpha$ is already proved and Hölders inequality, it is enough to prove that $g \in L^u(\Omega, P, L^\infty(\mathbb{T}, \mathbb{R}))$, where $g(t) = (p_t^*(0))^{-\alpha}$. Since $p_t^*(0) = (\mathcal{L}_t p_0^*)(0)(q^*(t))(0) = p_0^*(t)(q^*(t))(0)$, it follows that $0 \leq g(t) = (p_0^*(t))^{-\alpha}((q^*(t))(0))^{-\alpha}$. By Sobolev embedding, p_0^* is a continuous real valued function on $[0,\infty[$ and it is also strictly positive, so $(p_0^*)^{-\alpha}$ is bounded on $\mathbb{T}$. Once more by Sobolev embedding, $((q^*(t))(0))^{-\alpha} \leq C\|(q^*(t))^{-\alpha}\|_{\tilde{H}_0}$. The result now follows, since we have already proved the case of $(q^*)^\alpha$. The case of p is so similar to the previous cases that we omit it. End of proof.

5.3 Proof of Corollary 2.3

Proof of Corollary 2.3: The second part of the proof of Theorem 2.2 gives the result. End of proof.

5.4 Proof of Proposition 2.5

Proof of Proposition 2.5: Let $\theta \in \mathsf{P}$ and introduce $X = \sup_{t\in\mathbb{T}}(G^*(t,\theta))$, $Y(t) = \int_0^t < \theta_s \, , \, p_s^* m_s > ds$ and $Z(t) = \int_0^t < \sigma_s^* p_s^* \theta_s \, , \, dW_s >$. $G^*(t,\theta) = Y(t) + Z(t)$, according to formula (5.7). Let p^* be given by Theorem 2.1, of which the hypotheses are satisfied.

We shall give estimates for Y and Z. By the definition (2.27) of P :

$$E((\sup_{t\in\mathbb{T}}(Y(t))^2) \leq E((\int_0^{\bar{T}} | < \theta_s \, , \, p_s^* m_s > | ds)^2) \leq \|\theta\|_{\mathsf{P}}^2. \tag{5.17}$$

By isometry we obtain

$$\begin{aligned} E(Z(t)^2) &= E(\int_0^t < \theta_s \, , \, p_s^* \sum_{i\in\mathbb{I}} \sigma_s^i dW_s^i >)^2 = E(\int_0^t \sum_{i\in\mathbb{I}} (< \theta_s \, , \, p_s^* \sigma_s^i >)^2 ds) \\ &\leq E(\int_0^{\bar{T}} \|\sigma_s^* \theta_s p_s^*\|_{H^*}^2 ds) \leq \|\theta\|_{\mathsf{P}}^2. \end{aligned} \tag{5.18}$$

Doob's L^2 inequality and inequality (5.18) give $E(\sup_{t\in\mathbb{T}} Z(t)^2) \leq 4\|\theta\|_{\mathsf{P}}^2$. Inequality (5.17) then gives $E(X^2) \leq 10\|\theta\|_{\mathsf{P}}^2$, which proves the proposition. End of proof.

5.5 Proof of Theorem 2.8

As we will see, the strong condition (2.33) on Γ introduced in formula (2.32) assures the existence of a martingale measures Q equivalent to P, with Radon-Nikodym derivative in $L^u(\Omega, P)$, for each $u \in [1, \infty[$.

Lemma 5.7 *If (2.33) is satisfied, then $(\xi_t)_{t\in\mathbb{T}}$ is a $(P, \mathcal{A})$-martingale and $\sup_{t\in\mathbb{T}}(\xi_t)^\alpha \in L^1(\Omega, P)$ for each $\alpha \in \mathbb{R}$.*

Proof: Let $M(t) = \int_0^t \sum_{i\in\mathbb{I}} \Gamma_s^i dW_s^i$. Then $\ll M \, , \, M \gg(t) = \int_0^t \sum_{i\in\mathbb{I}} (\Gamma_s^i)^2 ds$ and according to condition (2.33) $E(\exp(a \ll M \, , \, M \gg(\bar{T})) < \infty$, for each $a \geq 0$. By chosing $a = 1/2$ Novikov's criteria, c.f. [17], Ch. VIII, Prop.

1.15, shows that ξ is a martingale. Let $b \geq 0$. It then follows from the same reference, by choosing $a = 2b^2$, that $E(\exp(b \sup_{t\in\mathbb{T}} |M(t)|)) < \infty$.

Let $\alpha \in \mathbb{R}$ and let $c(t) = \int_0^t \sum_{i\in\mathbb{I}} (\Gamma_s^i)^2 ds$. Then $\xi_t^\alpha = \exp(\alpha M(t) - \alpha/2 c(t))$, so

$$\sup_{t\in\mathbb{T}} (\xi_t)^\alpha \leq \sup_{t\in\mathbb{T}} \exp(|\alpha| M(t) + c(\bar{T})|\alpha|/2) \leq \exp(\alpha \sup_{t\in\mathbb{T}} |M(t)| + c(\bar{T})|\alpha|/2).$$

This and Schwarz inequality show that

$$(E(\sup_{t\in\mathbb{T}} \xi_t^\alpha))^2 \leq E(\exp(2|\alpha| \sup_{t\in\mathbb{T}} |M(t)|)) E(\exp(|\alpha| c(\bar{T}))).$$

The first factor on the right hand side of the this inequality is finite as is seen by choosing $b = 2|\alpha|$, and the second is finite due to condition (2.33). End of proof.

Next corollary is a direct application of Girsanov's theorem.

Corollary 5.8 *Let (2.33) be satisfied. The measure Q, defined by $dQ = \xi_{\bar{T}} dP$, is equivalent to P on $\mathcal{F}_{\bar{T}}$ and $t \mapsto \bar{W}_t = W_t - \int_0^t \Gamma_s ds$, $t \in \mathbb{T}$ is a cylindrical H-Wiener process with respect to $(Q, \mathcal{A})$.*

Proof: According to Lemma 5.7, ξ is a martingale with respect to $(P, \mathcal{A})$. Theorem 10.14 of reference [4] then gives the result. End of proof.

Corollary 5.8 and formulas (2.20) and (5.9) give

$$p_t^* = \exp\left(\int_0^t \tilde{\mathcal{L}}_{t-s}(\sum_{i\in\mathbb{I}} \sigma_s^i d\bar{W}_s^i - \frac{1}{2}\sum_{i\in\mathbb{I}} (\sigma_s^i)^2 ds)\right) \mathcal{L}_t p_0^* \qquad (5.19)$$

and

$$dG^*(t,\theta) =< \sigma_t^* p_t^* \theta_t \, , \, d\bar{W}_t > . \qquad (5.20)$$

Proof of Theorem 2.8: The first part is a restatement of Lemma 5.7 and the second of Corollary 5.8. End of proof.

5.6 Proof of Corollary 2.9

Proof of Corollary 2.9: Let $X = \sup_{t\in\mathbb{T}} (G^*(t,\theta)$. That conditions (2.15) and (2.17) are satisfied follows as in Remark 5.2. The hypotheses of Theorem 2.1 are therefore satisfied and p^* given by Theorem 2.1 is well-defined. The square integrability property follows from Proposition 2.5. Finally we have to prove the martingale property. According to hypotheses, (2.33) is satisfied, so Lemma 5.7, Proposition 2.5 and Schwarz inequality give $(E_Q(X))^2 \leq E(\xi^2_{\bar{T}})E(X^2) < \infty$. This shows that $X \in L^1(\Omega, Q)$ and since $G^*(\cdot, \theta)$ is a local Q-martingale according to (5.20) it follows that it is a Q-martingale (c.f. comment after Theorem 4.1 of [17]). End of proof.

5.7 Proof of Corollary 2.10

Proof of Corollary 2.10: $V^*(t, \theta)$ is given by formula (2.25), since $\theta \in \mathsf{P}_{sf}$. According to Corollary 2.9, $G^*(\cdot, \theta)$ is a Q-martingale, so this is also the case for $V^*(\cdot, \theta)$. The estimate also follows from Corollary 2.9. We note that if $V^*(\bar{T}, \theta) \geq 0$ and $E_Q(V^*(\bar{T}, \theta)) > 0$, then the martingale property give $V^*(0, \theta) > 0$, so the market is arbitrage free. End of proof.

5.8 Proof of Lemma 3.2

Proof of Lemma 3.2
First suppose that u satisfies Condition B. According to condition (3.2) it exists a sufficiently small $x' \in]\underline{x}, \infty[$, $C > 0$ and $q > 0$ such that for each $x \in]\underline{x}, x'[$

$$u'(x) \geq C(1 + |x|)^q. \tag{5.21}$$

With $x = \varphi(y)$ we then have for some $C' > 0$ for and each $y = u'(x) > 0$

$$|\varphi(y)| \leq C' y^{1/q}. \tag{5.22}$$

Consider the case (i). According to condition (3.3) it exists $C > 0$ and $x'' > 0$, such that for each $x \in]x'', \infty[$

$$u'(x) \leq C x^{-q}. \tag{5.23}$$

Then $u'(]\underline{x}, \infty[) =]0, \infty[$, since $u' > 0$ and according to Condition B (b). With $x = \varphi(y)$, for some $C' > 0$ and for each $y \in]0, u'(x'')[$

$$|\varphi(y)| \leq C' y^{-1/q}. \tag{5.24}$$

The continuity of u' and inequalities (5.22) and (5.24) then prove statement (i) with $p = 1/q$.

Consider the case (ii). It exists a unique $x_0 \in]\underline{x}, \infty[$ such that $u'(x_0) = 0$. Then $u'(x) < 0$ on $]x_0, \infty[$. According to condition (3.4) $\lim_{x \to \infty} u'(x) = -\infty$, so using Condition B (b) we get $u'(]\underline{x}, \infty[) = \mathbb{R}$. Also by (3.4), for some $C > 0$ and $q > 0$, for each $x \in]x_0, \infty[\cap]0, \infty[$, $-u'(x) \geq C x^q$. We then obtain $0 \leq \varphi(y) \leq C'|y|^{1/q}$, for some C' and for $y < 0$. This inequality, the continuity of u' and inequality (5.22) then prove statement (ii).

Secondly, the proof of the converse statement is so similar to the first part of the proof, that we omit it.We only note that the definition of $u(\underline{x})$ guarantees that u is u.s.c. End of proof.

5.9 Proof of Theorem 3.3

We recall that $I =]0,\infty[$ if $u' > 0$ on $]\underline{x},\infty[$ and $I = \mathbb{R}$ if u' takes the value zero in $]\underline{x},\infty[$, according to Lemma 3.2. We first prove the following lemma:

Lemma 5.9 *Let u satisfy condition Condition B and let Γ satisfy condition (2.33). Then $\varphi(\lambda\xi_{\bar{T}}) \in L^p(\Omega,P)$ for each $p \in [1,\infty[$, $\lambda \in I$ and $\lambda \mapsto E(\xi_{\bar{T}}\varphi(\lambda\xi_{\bar{T}}))$ defines a strictly decreasing homeomorphism from I on-to $]\underline{x},\infty[$. In particular, if $K_0 \in]\underline{x},\infty[$ then there exists a unique $x \in I$ such that $K_0 = E(\xi_{\bar{T}}\varphi(x\xi_{\bar{T}}))$ and x is continuous and strictly decreasing as a function of K_0.*

Proof: Let $\lambda \in I$ and $g_\lambda = \xi_{\bar{T}}\varphi(\lambda\xi_{\bar{T}})$. Lemma 5.7, inequalities (3.5) and (3.6) of Lemma 3.2 and Hölder's inequality show that $\varphi(\lambda\xi_{\bar{T}}) \in L^p(\Omega,P)$ for each $p \in [1,\infty[$. This result and Hölder's inequality give $g_\lambda \in L^1(\Omega,P)$. It follows that $f(\lambda) = E(g_\lambda)$ is well-defined.

We show that f is continuous. Let $\{\lambda_n\}_{n\in N^*}$ be a sequence in I converging to λ. It exists $\bar{\lambda} \in I$ such that $\bar{\lambda} \leq \lambda$ and $\bar{\lambda} \leq \lambda_n$, for $n \geq 1$. Since φ is decreasing and continuous according to Lemma 3.2, we have $|g_{\lambda_n} - g_\lambda| \leq 2g_{\bar{\lambda}}$ and $g_{\lambda_n} - g_\lambda \to 0$, a.e. as $n \to \infty$. $g_{\bar{\lambda}} \in L^1(\Omega,P)$, so by Lebesgue's dominated convergence $f(\lambda_n) - f(\lambda) = E(g_{\lambda_n} - g_\lambda) \to 0$, as $n \to \infty$, which proves the continuity.

The function f is decreasing, since φ is decreasing. If $\lambda_1, \lambda_2 \in I$ are such that $f(\lambda_1) = f(\lambda_2)$, then $g_{\lambda_1} = g_{\lambda_2}$ a.e. since $\xi_{\bar{T}} > 0$ a.e. φ is strictly decreasing, so it follows that $\lambda_1\xi_{\bar{T}} = \lambda_2\xi_{\bar{T}}$. This gives $\lambda_1 = \lambda_2$, which proves that f is strictly decreasing.

The function $\varphi : I \to]\underline{x},\infty[$ is a strictly decreasing bijection, so if $y \to \inf I$ in I, then $\varphi(y) \to \infty$ and if $y \to \infty$, then $\varphi(y) \to \underline{x}$. By Fatou's lemma it follows that

$$\liminf_{n\to\infty} f(\lambda_n) \geq E(\liminf_{n\to\infty} g_{\lambda_n}) = \infty, \tag{5.25}$$

if $\lambda_n \to \inf I$ in I. Let $\lambda_n \to \infty$ in I. Choose $\bar{\lambda} \in I$ such that $\bar{\lambda} \leq \inf\{\lambda_n \,|\, n \geq 1\}$. Then $g_{\bar{\lambda}} - g_{\lambda_n} \geq 0$, since φ is decreasing. Application of Fatou's lemma to $g_{\bar{\lambda}} - g_{\lambda_n}$ gives

$$E(\limsup_{n\to\infty} g_{\lambda_n}) \geq \limsup_{n\to\infty} E(g_{\lambda_n}). \tag{5.26}$$

If $\underline{x}$ is finite, then inequality (5.26) and, according to Lemma 5.7, $E(\xi_{\bar{T}}) = 1$ give $\underline{x} \geq \limsup_{n\to\infty} E(g_{\lambda_n})$. Since $g_{\lambda_n} \geq \xi_{\bar{T}}\underline{x}$ it follows that

$$\underline{x} = \limsup_{n\to\infty} E(g_{\lambda_n}), \tag{5.27}$$

if $\underline{x}$ is finite. Inequality (5.26) gives

$$-\infty = \limsup_{n\to\infty} E(g_{\lambda_n}). \tag{5.28}$$

if $\underline{x} = -\infty$. Since f is decreasing it follows from (5.25), (5.27) and (5.28) that f is on-to $]\underline{x}, \infty[$ and therefore a homeomorphism of I to $]\underline{x}, \infty[$. This completes the proof.

Proof of Theorem 3.3: Let

$$v(x) = \sup_{y\in]\underline{x},\infty[} (xy + u(y)), \tag{5.29}$$

$x \in \mathbb{R}$. Here v is the Legendre-Fenchel transform of $-u$. It follows from condition Condition B that $v : \mathbb{R} \to]-\infty, \infty]$ is l.s.c. and strictly convex, c.f. [5]. Let $I =]0, \infty[$ if $u' > 0$ on $]\underline{x}, \infty[$ and $I = \mathbb{R}$ if u' takes the value zero on $]\underline{x}, \infty[$. Since $-u$ is C^1 and strictly convex

$$v(x) = x\varphi(-x) + u(\varphi(-x)), \tag{5.30}$$

for $-x \in I$, which are the elements of the interior of the domain of v.

If $\mu \in I$, then $\varphi(\mu\xi_{\bar{T}}) \in L^p(\Omega, P)$ for each $p \in [1, \infty[$, according to Lemma 5.9. Let λ be the unique element in I, according to Lemma 5.9, such that $\varphi(\lambda\xi_{\bar{T}}) \in \mathcal{C}'(K_0)$. Let $Y = \varphi(\lambda\xi_{\bar{T}})$. Given $X \in \mathcal{C}'(K_0)$. By definition $E(u(X)) = E(u(X)) - \mu(E(\xi_{\bar{T}}X) - K_0)$. It then follows from (5.29) that

$$E(u(X)) = E(u(X) - \lambda\xi_{\bar{T}}X)) + \lambda K_0 \leq E(v(-\lambda\xi_{\bar{T}})) + \lambda K_0. \tag{5.31}$$

Formula (5.30) gives that $E(v(-\lambda\xi_{\bar{T}})) = E(u(Y)) - \lambda E(\xi_{\bar{T}}Y)$. Since $Y \in \mathcal{C}'(K_0)$, it follows from formula (5.31) that

$$E(u(X)) \leq E(u(Y)). \tag{5.32}$$

Therefore $\hat{X} = Y$ is a solution of problem (3.7). This solution is unique since u is strictly concave, which completes the proof.

5.10 Proof of Corollary 3.4

Proof of Corollary 3.4: It follows from Corollary 2.10 that $\{V^*(\bar{T},\theta) \mid \theta \in \mathcal{C}(K_0)\} \subset \mathcal{C}'(K_0)$, where $\mathcal{C}'(K_0)$ is given by Theorem 3.3. According to Corollary 2.10, $V^*(\cdot,\hat{\theta})$ is a Q-martingale, so Theorem 3.3 shows that $V^*(0,\hat{\theta}) = K_0$ and therefore $\hat{\theta} \in \mathcal{C}(K_0)$. This and Theorem 3.3 give

$$\sup_{\theta \in \mathcal{C}(K_0)} E(u(V^*(\bar{T},\theta))) \leq \sup_{X \in \mathcal{C}'(K_0)} E(u(X)) = E(u(\hat{X})) = E(u(V^*(\bar{T},\hat{\theta}))),$$

which proves that $\hat{\theta}$ is a solution of problem (3.1). End of proof.

5.11 Proof of Theorem 3.5

Proof of Theorem 3.5: Here $\mathbb{I}$ is a finite set and $K_0 \in]\underline{x},\infty[$. We shall construct a portfolio $\hat{\theta} \in \mathcal{C}(K_0)$ such that $V^*(\bar{T},\hat{\theta}) = \hat{X}$, where $\hat{X}$ is given by Theorem 3.3.

Since $\xi_{\bar{T}}, \hat{X} \in L^p(\Omega,P)$ for each $p \in [1,\infty[$, according to Lemma 5.7 and Theorem 3.3, it follows by Hölder's inequality that $\xi_{\bar{T}}\hat{X} \in L^p(\Omega,P)$, i.e. $\hat{X} \in L^p(\Omega,Q)$, for each $p \in [1,\infty[$. In particular $\hat{X} \in L^2(\Omega,Q)$, so by Corollary 5.8 and by the representation of a square integrable random variable as a stochastic integral, there exists adapted real valued processes y_i, $i \in I$, such that $E_Q(\int_0^{\bar{T}} \sum_{i\in\mathbb{I}} y_i(t)^2 dt) < \infty$ and such that $\hat{X} = Y(\bar{T})$, where

$$Y(t) = K_0 + \sum_{i\in\mathbb{I}} \int_0^t y_i(s) d\bar{W}^i_s, \tag{5.33}$$

for $t \in \mathbb{T}$. We define $y(t) = \sum_{i\in\mathbb{I}} y_i(t)e'_i$. Then $y(t) \in H^*$ a.s. since

$$\|y(t)\|^2_{H^*} = \sum_{i\in\mathbb{I}} y_i(t)^2. \tag{5.34}$$

Let $Z = \sup_{t\in\mathbb{T}} |Y(t)|$ and let $p \geq 2$. By Doob's inequality, $E_Q(Z^p) \leq (\frac{p}{1-p})^p \sup_{t\in\mathbb{T}} E_Q(|Y(t)|^p)$. Now $|Y|^p$, is a Q-submartingale, so $E_Q(|Y(t)|^p) \leq E_Q(|Y(\bar{T})|^p) = E_Q(|\hat{X}|^p)$, which gives

$$E_Q(Z^p) < \infty. \tag{5.35}$$

By the BDG inequalities, by equality (5.34) and inequality (5.35) one obtains

$$E_Q((\int_0^{\bar{T}} \|y(t)\|^2_{H^*} dt)^{p/2}) < \infty, \tag{5.36}$$

for $p \geq 2$. Since $E(\,\cdot) = E_Q(\xi_{\bar{T}}^{-1}\,\cdot)$, it follows from this inequality and from Lemma 5.7 that

$$E((\int_0^{\bar{T}} \|y(t)\|_{H^*}^2 dt)^{p/2}) < \infty, \tag{5.37}$$

for $p \geq 2$. We also note that similarly

$$E(Z^p) < \infty. \tag{5.38}$$

$p \geq 2$.

According to inequality (3.8), $A(t)$ is invertible a.s., we set $A(t)^{-1} = 0$, when $A(t)$ is not invertible and $A(t)_{ij}^{-1}$ are the matrix elements of $A(t)^{-1}$. We then obtain

$$\|l(t)\sigma_t\|_{H-S}\|A(t)^{-1}\|_{L(\mathbb{R}^{\bar{m}}} \leq Ck(t), \tag{5.39}$$

where $l(t) = \mathcal{L}_t p_0$. Condition (3.8), Schwarz inequality and inequality (5.39) give

$$E((\sup_{t\in\mathbb{T}} \|l(t)\sigma_t\|_{H-S}\|A(t)^{-1}\|_{L(\mathbb{R}^{\bar{m}})})^p) < \infty, \tag{5.40}$$

for $p \in [1, \infty[$.

We define

$$\eta(t) = \sum_{i,j=1}^{\bar{m}} A(t)_{ij}^{-1} l(t)\sigma_t^i y_j(t). \tag{5.41}$$

It follows from formula (5.34) that

$$\|\eta(t)\|_H \leq \|A(t)^{-1}\|_{L(\mathbb{R}^{\bar{m}})}\|y(t)\|_{H^*}\|l(t)\sigma_t\|_{H-S}, \tag{5.42}$$

for $t \in \mathbb{T}$. This inequality and inequalities (5.37) and (5.40) give

$$E((\int_0^{\bar{T}} \|\eta(t)\|_H^2)dt)^{p/2}) < \infty, \tag{5.43}$$

for $p \geq 2$. By construction, $\eta(t)$ satisfies

$$(\eta(t), l(t)\sigma_t^i)_H = y_i(t), \tag{5.44}$$

for $t \in \mathbb{T}$ and $i \in \mathbb{I}$. Defining $\tilde{\theta}_t^1 = \mathcal{S}\eta(t)$ we obtain a solution of the equation

$$\sigma_t^* \tilde{\theta}_t^1 l(t) = y(t), \tag{5.45}$$

for $t \in \mathbb{T}$. Let $q^*(t) = p_t^*/l(t)$ and $\theta_t^1 = (q^*(t))^{-1}\tilde{\theta}_t^1$. We obtain $\|\theta_t^1\|_{H^*} \leq C\|(q^*(t))^{-1}\|_{\tilde{H}_0}\|\eta(t)\|_H$, where we have used that $\|\tilde{\theta}_t^1\|_{H^*} = \|\eta(t)\|_H$. Theorem 2.2 and inequality (5.43) then give

$$E((\int_0^{\bar{T}} \|\theta_t^1\|_{H^*}^2)dt)^{p/2}) < \infty, \tag{5.46}$$

for $p \geq 2$. Equation (5.45) shows that θ^1 satisfies the equation

$$\sigma_t^* \theta_t^1 p_t^* = y(t), \tag{5.47}$$

for $t \in \mathbb{T}$. This equality, expression (5.20) of the discounted gain and the martingale representation (5.33), show that

$$Y(t) = K_0 + G^*(t, \theta^1), \tag{5.48}$$

for $t \in \mathbb{T}$.

We next prove that $\theta^1 \in \mathsf{P}$. By the hypotheses of the theorem it follows that $E(\int_0^{\bar{T}} \sum_{i \in \mathbb{I}} |\Gamma_t^i|^2 dt)^{p/2}) < \infty$, for $p \geq 2$. This inequality, definition (2.27) of the portfolio norm, inequality (5.46) and Schwarz inequality give $\|\theta^1\|_{\mathsf{P}} < \infty$, which proves the statement.

Finally we shall construct the announced self-financing strategy $\hat{\theta}$. Let us define the portfolio $\hat{\theta}$ by $\hat{\theta} = \theta^0 + \theta^1$, where $\theta_t^0 = a(t)\delta_0$ and

$$a(t) = (p_t^*)(0))^{-1}(Y(t) - < \theta_t^1 \, , \, p_t^* >) \tag{5.49}$$

for $0 \leq t \leq \bar{T}$.

We have to prove that $\hat{\theta} \in \mathcal{C}(K_0)$. To this end it is enough to prove that $\theta^0 \in \mathsf{P}$, since $\theta^1 \in \mathsf{P}$. By definition, we have

$$\|\theta_t^0\|_{H^*} = \sup_{\|f\|_H \leq 1} | < \theta_t^0 \, , \, f > | \leq \sup_{\|f\|_H \leq 1} (|a(t)| \, |f(0)|) \leq C|a(t)|,$$

where the constant is given by Sobolev embedding. Let $b(t) = a(t) p_t^*(0)$. By the definition of Z and Schwarz inequality it follows that

$$(E(\int_0^{\bar{T}} |b(t)|^2 dt))^{p/2}))^{1/p} \leq \bar{T}(E(Z^p))^{1/p} + (E((\int_0^{\bar{T}} \|\theta_t^1\|_{H^*}^2 dt)^{p/2} (\sup_{t \in \mathbb{T}} \|p_t^*\|_H^p)))^{1/p},$$

$p \geq 1$. The first term on the right hand side of this inequality is finite due to inequality (5.38) and the second term is finite due to Theorem 2.2, inequality (5.46) and Schwarz inequality. Using Corollary 2.3, we obtain now

$$E(\int_0^{\bar{T}} |a(t)|^2 dt))^{p/2}) < \infty,$$

$p \geq 1$. This proves in particular that

$$E(\int_0^{\bar{T}} \|\theta_t^0\|_{H^*}^2 dt) < \infty. \tag{5.50}$$

Since $(\sigma_t)(0) = 0$ according to (2.14), $m_t(0) = 0$ according to (2.15) and by the definition of the norm in H^*, we have that $\|\sigma_t^* \theta_t^0 p_t^*\|_{H^*} = 0$ and $< \theta_t^0 \, , \, p_t^* m_t >= 0$. This proves, together with inequality (5.50) and the definition (2.27) of the portfolio norm, that $\theta^0 \in \mathsf{P}$.

We note that by the definition of $\hat{\theta}$, it follows that $V^*(t, \hat{\theta}) =< \hat{\theta}(t) \, , \, p_t^* >= Y(t)$, for $t \in \mathbb{T}$. Moreover, since $(\sigma_t)(0) = 0$, it follows from formula (2.26) that $G^*(t, \theta_0) = 0$, for each $t \in \mathbb{T}$. So by (5.48), $Y(\cdot) = K_0 + G^*(\cdot, \hat{\theta})$, which proves that $\hat{\theta}$ is self financing with initial value K_0. End of proof.

5.12 Proof of Theorem 3.7

Proof of Theorem 3.7: This proof is, with some exceptions, so similar to the proof of Theorem 3.5 that we only develop the points which are different. Here $\mathbb{I} = \mathbb{N}^*$ or $\mathbb{I} = \{1, \ldots, \bar{m}\}$.

According to Theorem 3.3, there is a unique $\lambda \in I$ such that $\hat{X} = \varphi(\lambda \xi_{\bar{T}})$. Let $M(t) = \int_0^t \sum_{i \in \mathbb{I}} \Gamma_s^i d\bar{W}_s^i$, $t \in \mathbb{T}$. Then $\ll M \, , \, M \gg$ is deterministic and according to (2.35) and Corollary 5.8, $\xi_t = \exp(M(t) + \frac{1}{2} \ll M \, , \, M \gg(t))$. Let

$$F(x) = \varphi(\lambda \exp(x + \frac{1}{2} \ll M \, , \, M \gg(\bar{T}))),$$

$x \in \mathbb{R}$. Then $F(M(\bar{T})) = \hat{X}$. We now apply Lemma A.5 to F. This gives an integral representation, as in (5.33), with

$$y_i(t) = E_Q(\lambda \xi_{\bar{T}} \varphi'(\lambda \xi_{\bar{T}}) \,|\, \mathcal{F}_t) \Gamma_t^i, \tag{5.51}$$

$i \in \mathbb{I}$ and $t \in \mathbb{T}$.

Using that φ' satisfies conditions (3.9) and (3.10) we obtain also here inequalities (5.35) to (5.38).

Let $z(t) = E_Q(\lambda \xi_{\bar{T}} \varphi'(\lambda \xi_{\bar{T}}) \,|\, \mathcal{F}_t)$ and let γ be given by (3.11). We define $\tilde{\theta}^1 = z\gamma$. By condition (3.11), equation (5.45) is satisfied.

The remaining part of the proof is the same as for Theorem 3.5. For later reference we observe that $\theta^1 = (l/p^*) z\gamma$. End of proof.

5.13 Proof of Corollary 3.8

The observation in the end of the proof of Theorem 3.7 and expression (5.49) give the stated explicit expression of the optimal portfolio.

5.14 Proof of Theorem 3.9

Proof of Theorem 3.9: We first choose a utility function u satifying Condition C, $u' > 0$ and $\underline{x} = 0$. This is possible as seen by chosing $u(x) = x^{1/2}$, for example. We define $\Theta \in \mathsf{P}_{sf}$, to be the optimal portfolio given by Corollary 3.8 for $K_0 = 1$. Let $\Theta_t = a^1_t \delta_0 + b^1_t \gamma_t (p^*_t)^{-1} \mathcal{L}_t p_0$, where a^1 and b^1 are the coefficients defined by formulas (3.12) and (3.13) respectively. Since $u' > 0$, it follows from Theorem 3.3 and Corollary 3.4 that $\lambda > 0$. It follows from $\lambda \neq 0$, $\varphi' < 0$ and formula (3.12) that $b^1_t \neq 0$, after a possible redefinition on a set of measure zero.

Since $\underline{x} = 0$, it follows by the definition of φ that $\varphi > 0$ and then by Theorem 3.3 and Corollary 3.4 that $V^*(t, \Theta) = E_Q(V^*(\bar{T}, \Theta) \,|\, \mathcal{F}_t) > 0$. This shows that statement (i) is satisfied.

Let us now consider a general u satifying Condition C. The solution $\hat{\theta}$ given by Corollary 3.8, for a general $K_0 \in]\underline{x}, \infty[$ can now be written

$$\hat{\theta}_t = (a_t - a^1_t b_t / b^1_t)\delta_0 + (b_t / b^1_t)\Theta_t,$$

which defines x and y in statement (ii) of the theorem. End of proof.

5.15 Proof statement in Remark 5.6

We first prove the inclusion (5.15). Let $u \in \mathcal{U}_-$. If u is bounded then it follows from the mean value theorem and the monotonicity of u' that it exists $C > 0$ such that $xu'(x) \leq C$, for all $x \geq 1$. If $u(x) \to \infty$ when $x \to \infty$, then $e_\infty(u) < 1$ shows that it exists $c < 1$ and $x_0 \geq 1$ such that

$$u'(x)/u(x) \leq c/x, \tag{5.52}$$

for all $x \geq x_0$. Integration gives $u(x) \leq Ax^c$, for some $A > 0$. This inequality and inequality (5.52) give $x^{1-c}u'(x) \leq cA$, for all $x \geq x_0$. This proves (3.3) of Condition B, since $1 - c > 0$. Inequality (3.2) is trivially satisfied. This shows that u satisfies Condition B, which proves (5.15).

To prove (5.16), we shall construct $u \in \mathcal{U}$ such that $e_\infty(u) = 1$. Let $0 < q < 1$ be given. We construct inductively an increasing sequence $(x_n)_{n \in \mathbb{N}}$ satisfying

$$x_0 = 1 \quad \text{and} \quad x_{n+1} \geq x_n + 1 \tag{5.53}$$

and a function $v \in C^\infty([1, \infty[)$ satisfying

$$v(1) = 1,\ \forall x \in [1, \infty[\ 0 < v(x) \leq \frac{1}{x^q},\ v' < 0, \tag{5.54}$$

$$\forall n \in \mathbb{N}\ v(x) = \frac{1}{x^q} \text{ in a neighbourhood of } x_n \tag{5.55}$$

and

$$G_n \equiv \frac{x_n^{1-q}}{1 + \int_1^{x_{n-1}} v(y)dy + \epsilon_n + x_n^{1-q}} \geq 1 - \frac{1}{n+1}, \tag{5.56}$$

for all $n \geq 1$, where

$$\epsilon_n = \int_{x_{n-1}}^{x_n} v(y)dy - (x_n - x_{n-1})v(x_n). \tag{5.57}$$

For the moment, suppose that $(x_n)_{n\in\mathbb{N}}$, v satisfying (5.53)–(5.57) and $\underline{x} \in]-\infty, 0]$ are given. We define

$$u'(x) = v(x) \quad \text{and} \quad u(x) = 1 + \int_1^x u'(y)dy, \tag{5.58}$$

for all $x \geq 1$. Inequality (3.3) is then satisfied, due to (5.54). Also due to (5.54), u can be extended to a C^∞ strictly concave and strictly increasing function on $]\underline{x}, -\infty[$ such that $\lim_{x\downarrow\underline{x}} u'(x) = \infty$. We define $u(\underline{x}) = \lim_{x\downarrow\underline{x}} u(x)$ and $u(x) = -\infty$, for all $x < \underline{x}$. Then u satisfies Condition B with $u' > 0$ and $\underline{x} > -\infty$. We shall prove that

$$e_\infty(u) = 1. \tag{5.59}$$

Let $F(x) = xu'(x)/u(x)$. Then $F(x_n) = x_n v(x_n)/(u(x_{n-1}) + \int_{x_{n-1}}^{x_n} v(y)dy)$, for $n \geq 1$. Property (5.55) and formula (5.57) give

$$F(x_n) = \frac{x_n^{1-q}}{1 + \int_1^{x_{n-1}} v(y)dy + \epsilon_n + (x_n - x_{n-1})x_n^{-q}} \tag{5.60}$$

Property (5.53) and inequality (5.56) then give

$$F(x_n) \geq G_n \geq 1 - \frac{1}{n+1}, \tag{5.61}$$

$n \geq 1$. Since u' is decreasing it follows from the mean value theorem that $F(x_n) = u'(x_n)/(u(x_n)/x_n) < 1$. It then follows from inequality (5.61) that (5.59) is true.

Finally we construct the sequence $(x_n)_{n\in\mathbb{N}}$ and the function v. For $n \geq 1$, let S_n be the set of C^∞ functions on $[x_{n-1}, x_n]$ such that $f(x) = 1/x^q$ in a neighbourhood of x_{n-1} and in a neighbourhood of x_n, such that $f(x) \leq 1/x^q$, for $x \in [x_{n-1}, x_n]$ and such that $f' < 0$. Given $\epsilon_n > 0$ sufficiently small, we can choose $v_n \in S_n$ such that

$$\int_{x_{n-1}}^{x_n} v_n(y)dy = (x_n - x_{n-1})v_n(x_n) + \epsilon_n. \tag{5.62}$$

We choose $x_1 \in [x_0 + 1, \infty]$ such that

$$\frac{x_1^{1-q}}{2 + x_1^{1-q}} \geq \frac{1}{2}, \tag{5.63}$$

and we choose $v_1 \in S_1$ such that $0 < \epsilon_1 \leq 1$. Let $v(x) = v_1(x)$, for $x \in [x_0, x_1]$. Inequality (5.63) gives $G_1 = x_1^{1-q}/(1 + \epsilon_1 + x_1^{1-q}) > 1/2$, so (5.56) is satisfied for $n = 1$.

Now, suppose that $x_1, \ldots, x_n$ are constructed as well as v on $[x_0, x_n]$. We shall construct x_{n+1} and extend v to $[x_0, x_{n+1}]$. We first choose $x_{n+1} \in [x_n + 1, \infty[$ such that

$$\frac{x_{n+1}^{1-q}}{2 + \int_1^{x_n} v(y)dy + x_{n+1}^{1-q}} \geq 1 - \frac{1}{n+2}, \tag{5.64}$$

then we choose $\epsilon_{n+1} \in]0, 1]$ and finally we choose $v_{n+1} \in S_{n+1}$ such that (5.62) is satisfied, with n replaced by $n + 1$. We define $v(x) = v_{n+1}(x)$ for $x \in]x_n, x_{n+1}]$. Then

$$G_{n+1} \geq \frac{x_{n+1}^{1-q}}{2 + \int_1^{x_n} v(y)dy + \epsilon_{n+1} + x_{n+1}^{1-q}} \geq 1 - \frac{1}{n+2}, \tag{5.65}$$

which shows that (5.56) is satisfied. By recursion we obtain $(x_n)_{n\in\mathbb{N}}$ and a function v defined on $[1, \infty[$. Properties (5.53) and (5.54) are satisfied by construction and we have proved (5.56). Since, for some $a \in]0, 1[$, $v_n(x) = x^{-q}$ for all $x \in]x_n - a, x_n]$ and $v_{n+1}(x) = x^{-q}$ for all $x \in [x_n, x_n + a[$ it follows that $v(x) = x^{-q}$ in a neighbourhood of x_n. This proves that also (5.55) is true. End of proof.

A Appendix: SDE's and L^p estimates

In this appendix, we state and prove results, used in the article, concerning existence of solutions of some SDE's and L^p estimates of these solutions. Through the appendix m and σ^i, $i \in \mathbb{I}$, are $\mathcal{A}$ progressively measurable $\tilde{H}_0$-valued processes satisfying

$$\int_0^{\bar{T}} (\|m_t\|_{\tilde{H}_0} + \sum_{i\in\mathbb{I}} \|\sigma_t^i\|^2_{\tilde{H}_0})dt < \infty, a.s. \tag{A.1}$$

The $\tilde{H}_0$-valued semi-martingale L is given by

$$L(t) = \int_0^t (m_s ds + \sum_{i\in\mathbb{I}} \sigma_s^i dW_s^i), \quad \text{if } 0 \leq t \leq \bar{T} \tag{A.2}$$

and by $L(t) = L(\bar{T})$, if $t > \bar{T}$. We introduce, for $t \geq 0$, the random variable

$$\mu(t) = t + \int_0^t (\|m_s\|_{\tilde{H}_0} + \sum_{i\in\mathbb{I}} \|\sigma_s^i\|^2_{\tilde{H}_0})ds, \quad \text{if } 0 \leq t \leq \bar{T} \tag{A.3}$$

and $\mu(t) = t - \bar{T} + \mu(\bar{T})$ if $t > \bar{T}$. μ is a.s. strictly increasing, absolutely continuous and on-to $[0, \infty[$. The inverse τ of μ also have these properties and $\tau(t) \leq t$. For a continuous $\tilde{H}_0$-valued processes Y on $[0, \bar{T}]$ we introduce

$$\rho_t(Y) = (E(\sup_{s \in [0,t]} \|Y(\tau(s))\|^2_{\tilde{H}_0}))^{1/2}, \tag{A.4}$$

for $t \in [0, \infty[$, where we have defined $Y(t)$ for $t > \bar{T}$ by $Y(t) = Y(\bar{T})$. We note that $\rho_t(Y) \leq (E(\sup_{s \in [0,t]} \|Y(s)\|^2_{\tilde{H}_0}))^{1/2}$, since $\tau(t) \leq t$.

We will use certain supplementary properties of the Sobolev spaces H^s (c.f. §7.9 of [9]) and the space H. Let $s \in]0, 1[$. There is a norm $\mathcal{N}_s$ equivalent to $\| \ \|_{H^s}$, satisfying $\|f\|^2_{H^s} \leq (\mathcal{N}_s(f))^2 \leq 2\|f\|^2_{H^s}$ given by

$$(\mathcal{N}_s(f))^2 = \int_{\mathbb{R}} |f(x)|^2 \, dx + c_s \int_{\mathbb{R}^2} \frac{|f(x) - f(y)|^2}{|x - y|^{1+2s}} \, dx \, dy, \tag{A.5}$$

for a certain $c_s > 0$. Let s be the given number $s \in]1/2, 1[$ in formula (2.6) defining H. For $f \in H^s$, let $(\kappa f) = f(x)$, $x \geq 0$. The map $\kappa : H^s \to H$ is continuous and surjective. If $g \in H$, then there is an even function in the equivalence class of functions in H^s defining g. In fact if $f(x) = g(x)$, for $x \geq 0$, and $f(x) = g(-x)$, for $x < 0$, then it follows using (A.5) that $\mathcal{N}_s(f) \leq C_s \|g\|_H$. So $f \in H^s$ and $\kappa f = g$. Therefore we define the linear map $\iota : H \to H^s$, which is continuous and injective, by $(\iota g)(x) = g(x)$, if $x \geq 0$, and $(\iota g)(x) = g(-x)$, if $x < 0$. Let $\mathbb{R} \ni t \mapsto \mathcal{L}'_t$ be the C^0 unitary group of left translations in H^s, i.e. $(\mathcal{L}'_t f)(x) = f(x + t)$, for $f \in H^s$ and $t, x \in \mathbb{R}$.

The map κ is extended to $\kappa : \mathbb{R} \oplus H^s \to \tilde{H}_0$ by $\kappa(a + f) = a + \kappa f$, where $a \in \mathbb{R}$ and $f \in \tilde{H}_0$. The map ι is extended to $\iota : \tilde{H}_0 \to \mathbb{R} \oplus H^s$ by $\iota(a + f) = a + \iota f$, where $a \in \mathbb{R}$ and $f \in H$. $\mathcal{L}'$ is extended to a C^0 unitary group in $\mathbb{R} \oplus H^s$ by $\mathcal{L}'_t(a + f) = a + \mathcal{L}'_t f$, where $t \in \mathbb{R}$, $a \in \mathbb{R}$ and $f \in H$. One easily establish that with this extended $\mathcal{L}'$

$$\tilde{\mathcal{L}}_t \kappa = \kappa \mathcal{L}'_t, \tag{A.6}$$

for all $t \geq 0$.

Lemma A.1 *If condition (A.1) is satisfied and if Y is an $\mathcal{A}$-adapted $\tilde{H}_0$-valued continuous process on $[0, \bar{T}]$, satisfying $\rho_t(Y) < \infty$, for all $t \geq 0$, then the equation*

$$X(t) = Y(t) + \int_0^t \tilde{\mathcal{L}}_{t-s} X(s)(m_s ds + \sum_{i \in \mathbb{I}} \sigma^i_s dW^i_s), \tag{A.7}$$

$t \in [0, \bar{T}]$, has a unique solution X, being an $\mathcal{A}$-adapted $\tilde{H}_0$-valued continuous process. Moreover this solution satisfies:

i) If $\int_0^{\bar{T}}(\|m_t\|_{\tilde{H}_1} + \sum_{i\in\mathbb{I}} \|\sigma_t^i\|^2_{\tilde{H}_1})dt < \infty$ *and* Y *is a continuous* $\tilde{H}_1$*-valued process with* $\rho_t(\partial Y) < \infty$*, for all* $t \geq 0$*, then* X *is a continuous* $\tilde{H}_1$*-valued process.*
ii) If (i) is satisfied and if Y *is a semi-martingale, then* X *is a semi-martingale.*
iii) If Y *is* H*-valued, then* X *is* H*-valued.*

Proof: The given continuous process Y is extended to $t > \bar{T}$ by $Y(t) = Y(\bar{T})$. For a continuous $\mathcal{A}$-adapted $\tilde{H}_0$ valued process X on $[0, \infty[$, let

$$(AX)(t) = \int_0^t \tilde{\mathcal{L}}_{t-s} X(s) dL(s), \tag{A.8}$$

$0 \leq t$. A is a linear operator from the space of continuous $\tilde{H}_0$ valued processes into itself, since $\tilde{H}_0$ is a Banach algebra, $t \mapsto \mathcal{L}_t$ a C^0 semi-group in $\tilde{H}_0$ and $\mu(\bar{T}) < \infty$ a.s. We note that $(AX)(t)$ is constant for $t \geq \bar{T}$.

It is enough to prove that the equation

$$X = Y + AX \tag{A.9}$$

has a unique solution X being a continuous process. Its restriction to $[0, \bar{T}]$ is then the unique solution of the lemma.

In order to introduce the time transformed equation of (A.9) with respect to τ let $X'(t) = X(\tau(t))$, $Y'(t) = Y(\tau(t))$, $(A'X')(t) = (AX)(\tau(t))$ and $\rho'_t(X') = (E(\sup_{s\in[0,t]} \|X'(s)\|^2_{\tilde{H}_0}))^{1/2}$. Let also $\mathcal{A}' = (\mathcal{F}_{\tau(t)})_{t\geq 0}$ be the time transformed filtration. Equation (A.9) has a continuous solution if and only if the time transformed equation

$$X' = Y' + A'X' \tag{A.10}$$

has a continuous solution X'.

For given $T > 0$ let F be the Banach space of $\mathcal{A}'$-adapted $\tilde{H}_0$-valued continuous a.s. processes Z on $[0, T]$, with finite norm $\|Z\|_F = \rho'_T(Z)$.

We denote, for $0 \leq t \leq T$, $K_1(t) = \int_0^{\tau(t)\wedge\bar{T}} \tilde{\mathcal{L}}_{\tau(t)-s} X'(\mu(s)) m_s ds$ and $K_2(t) = \int_0^{\tau(t)\wedge\bar{T}} \tilde{\mathcal{L}}_{\tau(t)-s} X'(\mu(s)) \sum_{i\in\mathbb{I}} \sigma_s^i dW_s^i$, where $a \wedge b = \min\{a, b\}$. Since $\tilde{H}_0$ is an algebra, $\tilde{\mathcal{L}}$ a contraction C^0 semi-group, $\|m_t\|_{\tilde{H}_0} \leq d\mu(t)/dt$ and τ

the inverse of μ it follows from Schwarz inequality that

$$\begin{aligned}(\rho'_t(K_1))^2 &\leq CE((\int_0^{\tau(t)} \|X'(\mu(s))\|_{\tilde{H}_0} \|m_s\|_{\tilde{H}_0} ds)^2) \leq CE((\int_0^{\tau(t)} \|X'(\mu(s))\|_{\tilde{H}_0} d\mu(s))^2) \\ &\leq CE((\int_0^{\tau(t)} d\mu(s))(\int_0^{\tau(t)} \|X'(\mu(s))\|^2_{\tilde{H}_0} d\mu(s))) \leq CtE(\int_0^t \|X'(s))\|^2_{\tilde{H}_0} ds) \\ &\leq CtE(\int_0^t \sup_{s'\in[0,s]} \|X'(s')\|^2_{\tilde{H}_0} ds) \leq Ct \int_0^t (\rho'_s(X'))^2 ds,\end{aligned} \tag{A.11}$$

for some $C \geq 0$.

To establish an estimate of K_2, we shall use the the property (A.6) of the left translation. Since κ and ι are continuous linear operators and $\kappa\iota$ is the identity operator on $\tilde{H}_0$, it follows from (A.6) that

$$K_2(t) = \kappa \mathcal{L}'_{\tau(t)} \int_0^{\tau(t)\wedge\bar{T}} \mathcal{L}'_{-s} \iota X'(\mu(s)) \sum_{i\in\mathbb{I}} \sigma^i_s dW^i_s, \tag{A.12}$$

for all $t \geq 0$. Let $K''_2(t) = \int_0^{\tau(t)\wedge\bar{T}} \mathcal{L}'_{-s} \iota X'(\mu(s)) \sum_{i\in\mathbb{I}} \sigma^i_s dW^i_s$, for $t \geq 0$. Then K''_2 is a $\mathbb{R} \oplus H^s$-valued square integrable martingale, with respect to the time transformed filtration $\mathcal{A}'$. In fact, we obtain by isometry, the unitarity of $\mathcal{L}'$ and as in the case of K_1, that

$$\begin{aligned}E(\|K''_2(t)\|^2_{\mathbb{R}\oplus H^s}) &\leq E(\int_0^{\tau(t)} \|\mathcal{L}'_{-u} \iota X(u) \sum_{i\in\mathbb{I}} \sigma^i_u\|^2_{\mathbb{R}\oplus H^s} du) \\ &\leq CE(\int_0^{\tau(t)} \|X'(\mu(u))\|^2_{\tilde{H}_0} \sum_{i\in\mathbb{I}} \|\sigma^i_u\|^2_{\tilde{H}_0} du) \\ &\leq CE(\int_0^t \sup_{u'\in[0,u]} \|X'(\mu(u'))\|^2_{\tilde{H}_0} d\mu(u)) \leq C \int_0^t (\rho'_u(X'))^2 du,\end{aligned} \tag{A.13}$$

for some $C > 0$ and for all $t \geq 0$. Since $\mathcal{L}'_t$ is unitary and an κ is continuous with norm equal to 1, it follows from (A.12) that $(\rho'_t(K_2))^2 \leq E(\sup_{u\in[0,t]} \|K''_2(u)\|^2_{\mathbb{R}\oplus H^s})$. By Doob's inequality (c.f. theorem 3.8 of [4]) we have $E(\sup_{u\in[0,t]} \|K''_2(\tau(u))\|^2_{\mathbb{R}\oplus H^s}) \leq 4 \sup_{u\in[0,t]} E(\|K''_2(\tau(u))\|^2_{\mathbb{R}\oplus H^s})$. This gives, together with inequality (A.13) that

$$(\rho'_t(K_2))^2 \leq C \int_0^t (\rho'_s(X'))^2 ds, \tag{A.14}$$

for $t \geq 0$, where C chosen sufficiently big is independent of t. Formula (A.8) and inequalities (A.11) and (A.14) show that for $t \in [0, T]$,

$$(\rho'_t(A'X'))^2 \leq C'^2(1+t) \int_0^t (\rho'_s(X'))^2 ds, \tag{A.15}$$

where C' is a constant independent of T. In particular

$$\rho'_t(A'X') \leq C'(1+t)^{1/2} t^{1/2} \rho'_t(X'), \tag{A.16}$$

$t \in [0, T]$.

If $T > 0$ is sufficiently small, then (A.16) gives that $\|A'X'\|_F \leq a\|X'\|_F$, where $0 \leq a < 1$. Therefore $A' \in L(F)$ and $I + A'$ has bounded inverse. Equation (A.10) has then a unique solution $X' \in F$. Let $h(t) = \int_0^t (\rho'_s(X'))^2 ds$ and $a(t) = \int_0^t (\rho'_s(Y'))^2 ds$. Equation (A.10) and inequality (A.15) show that a solution $X' \in F$ satisfies

$$h(t) \leq 2a(t) + 2C'^2 \int_0^t (1+s) h(s) ds, \tag{A.17}$$

for $t \in [0, T]$. Grönwall's inequality gives $h(t) \leq 2a(t) \exp(C't(2+t))$. Equation (A.10) and inequality (A.15) then show that there exists a finite constant C''_T for every $T > 0$ independent of X', such that $\|X'\|_F^2 = (\rho'_T(X'))^2 \leq C''_T(\rho'_T(Y'))^2$. It follows that the solution can be extended to all $T > 0$ and this extended solution is unique. This proves the statement of the lemma concerning the existence and uniqueness of a $\tilde{H}_0$-valued continuous solution of equation (A.7).

We next prove the supplementary statements (i), (ii) and (iii) :
(i) We have just to replace, in the above proof, the space $\tilde{H}_0$ by $\tilde{H}_1$ and redefine appropriately the maps ι and κ.
(ii) If Y is a semi-martingale, then Itô's lemma and the fact that ∂X is a continuous process gives

$$dX(t) = dY(t) + \partial(X(t) - Y(t))dt + X(t)dL(t). \tag{A.18}$$

This shows that X is a semi-martingale.
(iii) H is a closed subspace of $\tilde{H}_0$ and if X is H-valued, then AX is also H-valued. This shows that the unique solution of equation (A.9) is H-valued. End of proof.

The solution of equation (A.7) can be given explicitly, which we shall use to derive estimates of the solution. Let

$$(\tilde{\mathcal{E}}(L))(t) = \exp(\int_0^t \tilde{\mathcal{L}}_{t-s}((m_s - \frac{1}{2}\sum_{i \in \mathbb{I}} (\sigma_s^i)^2) ds + \sum_{i \in \mathbb{I}} \sigma_s^i dW_s^i)), \tag{A.19}$$

for $t \in \mathbb{T}$.

Lemma A.2 *Let condition (A.1) be satisfied. Then $\tilde{\mathcal{E}}(L)$ is the unique $\tilde{H}_0$-valued continuous a.s. solution of*

$$(\tilde{\mathcal{E}}(L))(t) = 1 + \int_0^t \tilde{\mathcal{L}}_{t-s}(\tilde{\mathcal{E}}(L))(s)dL(s), \tag{A.20}$$

for $t \in \mathbb{T}$. Let also $L'(t) = \int_0^t \sum_{i\in\mathbb{I}}(\sigma_s^i)^2 ds - L(t)$. Then the unique solution X of equation (A.7) in Lemma A.1 is given by

$$X(t) = Y(t) - (\tilde{\mathcal{E}}(L))(t) \int_0^t \tilde{\mathcal{L}}_{t-s} Y(s)(\tilde{\mathcal{E}}(L'))(s)dL'(s), \tag{A.21}$$

for $t \in \mathbb{T}$.

Proof: Let $l_T(t) = \int_0^t \tilde{\mathcal{L}}_{T-s}(m_s + \sum_{i\in\mathbb{I}} \sigma_s^i dW_s^i)$, let $n_T(t) = \int_0^t \tilde{\mathcal{L}}_{T-s}((m_s - \frac{1}{2}\sum_{i\in\mathbb{I}}(\sigma_s^i)^2)ds + \sum_{i\in\mathbb{I}} \sigma_s^i dW_s^i)$, for $0 \leq t \leq T \leq \bar{T}$ and let $N(t) = n_t(t)$, for $t \in \mathbb{T}$. Then N is a $\tilde{H}_0$-valued continuous process, according to the hypothesis on m and σ and since $\tilde{H}_0$ is a Banach algebra. This is then also the case of $\tilde{\mathcal{E}}(L)$, since $\|(\tilde{\mathcal{E}}(L))(t)\|_{\tilde{H}_0} \leq \exp(\|N(t)\|_{\tilde{H}_0})$. We note that $dl_T(t) = \tilde{\mathcal{L}}_{T-t}dL(t)$ and that $\tilde{\mathcal{L}}_{T-t}(\tilde{\mathcal{E}}(L))(t) = \exp(n_T(t))$. Integration gives

$$\begin{aligned}\int_0^t \tilde{\mathcal{L}}_{t-s}(\tilde{\mathcal{E}}(L))(s)dL(s) &= \int_0^t \exp(n_t(s))dl_t(s) \\ &= \exp(n_t(t)) - 1 = \tilde{\mathcal{E}}(L)(t) - 1.\end{aligned} \tag{A.22}$$

This proves that $\tilde{\mathcal{E}}(L)$ is a solution of (A.20). The uniqueness follows from Lemma A.1.

To prove formula (A.21), let $l'_T(t) = \int_0^t \tilde{\mathcal{L}}_{T-s}\sum_{i\in\mathbb{I}}(\sigma_s^i)^2 ds - l_T(t)$, for $0 \leq t \leq T \leq \bar{T}$. Then

$$d\exp(n_T(t)) = \exp(n_T(t))dl_T(t) \text{ and } d\exp(-n_T(t)) = \exp(-n_T(t))dl'_T(t). \tag{A.23}$$

Let also $y_T(t) = \tilde{\mathcal{L}}_{T-s}Y(t)$ and $z_T(t) = \tilde{\mathcal{L}}_{T-s}(X(t) - Y(t))/(\tilde{\mathcal{E}}(L))(t)$, for $0 \leq t \leq T \leq \bar{T}$. Let X be the unique solution given by Lemma A.1. Equation (A.7) then reads, $X(t) = Y(t) + \int_0^t \tilde{\mathcal{L}}_{t-s}X(s)dL(s))$, for $t \in \mathbb{T}$. Applying $\tilde{\mathcal{L}}_{T-t}$, with $0 \leq t \leq T \leq \bar{T}$, on both sides we obtain

$$z_T(t) = \exp(-n_T(t)) \int_0^t (y_T(s) + z_T(s)\exp(n_T(s)))dl_T(s) \tag{A.24}$$

Ito's lemma and formulas (A.23), (A.24), $z_T(0) = 0$ give

$$\begin{aligned}z_T(t) = &\int_0^t \exp(-n_T(s))(y_T(s) + z_T(s)\exp(n_T(s)))dl_T(s) \\ &+ \int_0^t z_T(s)dl'_T(s) - \int_0^t \tilde{\mathcal{L}}_{T-s}\sum_{i\in\mathbb{I}}(\sigma_s^i)^2(\exp(-n_T(s))y_T(s) + z_T(s))ds,\end{aligned}$$

for $0 \le t \le T \le \bar{T}$. Rewriting this formula we obtain

$$\begin{aligned} z_T(t) = &\int_0^t z_T(s)(dl_T(s) + dl'_T(s) - \tilde{\mathcal{L}}_{T-s} \sum_{i \in \mathbb{I}} (\sigma_s^i)^2 ds) \\ &+ \int_0^t y_T(s) \exp(-n_T(s))(dl_T(s) - \tilde{\mathcal{L}}_{T-s} \sum_{i \in \mathbb{I}} (\sigma_s^i)^2 ds). \end{aligned}$$

The definitions of l and l' then give

$$z_T(t) = -\int_0^t y_T(s) \exp(-n_T(s)) d' l_T(s), \tag{A.25}$$

for $0 \le t \le T \le \bar{T}$. Choosing $T = t$, we now obtain formula (A.21), since $y_T(s) \exp(-n_T(s)) d' l_T(s) = \tilde{\mathcal{L}}_{T-s} Y(s)(\tilde{\mathcal{E}}(L'))(s) dL'(s)$, for $0 \le s \le T \le \bar{T}$. End of proof.

Next technical lemma collects estimates, of norms of certain $\tilde{H}_0$-valued processes, that we need later.

Lemma A.3 *Let*

$$\|(m,\sigma)\|_j = \int_0^{\bar{T}} \sum_{0 \le k \le j} \|\partial^k m_t\|_{\tilde{H}_0} dt + (\int_0^{\bar{T}} \sum_{0 \le k \le j} \|\partial^k \sigma_t\|^2_{H-S} dt)^{1/2},$$

$j \in \mathbb{N}$ *and let* $Z(t) = \int_0^t \tilde{\mathcal{L}}_{t-s} dL(s)$, $t \in \mathbb{T}$. *Let* $F : [0,\infty[\to [0,\infty[$ *be a function, which is continuous together with its first two derivatives and which has* $F' \ge 0$.

i) If $\|(m,\sigma)\|_0 < \infty$, *then*

$$F(\|Z(t)\|^2_{\tilde{H}_0}) \le F(0) + \int_0^t (a(s)ds + \sum_{i \in \mathbb{I}} b_i(s) dW_s^i), \tag{A.26}$$

where a *and* b_i, $i \in \mathbb{I}$, *are progressively measurable processes satisfying*

$$\begin{aligned} |a(t)| \le &F'(\|Z(t)\|^2_{\tilde{H}_0})(2\|Z(t)\|_{\tilde{H}_0}\|m_t\|_{\tilde{H}_0} + \|\sigma_t\|^2_{H-S}) \\ &+ 2|F''(\|Z(t)\|^2_{\tilde{H}_0})|\, \|Z(t)\|^2_{\tilde{H}_0} \|\sigma_t\|^2_{H-S} \end{aligned} \tag{A.27}$$

and

$$b_i(t) = 2F'(\|Z(t)\|^2_{\tilde{H}_0})(Z(t), \sigma_t^i)_{\tilde{H}_0}, \tag{A.28}$$

$t \in \mathbb{T}$.

ii) Moreover, if $\|(m,\sigma)\|_1 < \infty$, *then*

$$F(\|Z(t)\|^2_{\tilde{H}_0}) = F(0) + \int_0^t ((-v(s)F'(\|Z(s)\|^2_{\tilde{H}_0}) + a(s))ds + \sum_{i\in\mathbb{I}} b_i(s)dW^i_s), \tag{A.29}$$

where

$$v(t) = -2(Z(s), \partial Z(s))_{\tilde{H}_0} \geq 0, \tag{A.30}$$

with $t \in \mathbb{T}$.

Proof: Suppose first that $\|(m,\sigma)\|_j < \infty$, for each $j \in \mathbb{N}$. We remember that the set $\mathcal{D}_\infty$ of all $f \in \tilde{H}_0$, such that $\partial^j f \in \tilde{H}_0$ for each $j \in \mathbb{N}$, is dense in $\tilde{H}_0$. Then $Z(t) \in \mathcal{D}_\infty$ a.s. Ito's lemma gives

$$\|Z(t)\|^2_{\tilde{H}_0} = \int_0^t ((2(Z(s), \partial Z(s))_{\tilde{H}_0} + a^{(1)}(s))ds + \sum_{i\in\mathbb{I}} b^{(1)}_i(s)dW^i_s), \tag{A.31}$$

where

$$a^{(1)}(t) = 2(Z(t), m_t)_{\tilde{H}_0} + \|\sigma_t\|^2_{H-S} \tag{A.32}$$

and

$$b^{(1)}_i(t) = 2(Z(t), \sigma^i_t)_{\tilde{H}_0}. \tag{A.33}$$

We note that $|a^{(1)}(t)| \leq 2\|Z(t)\|_{\tilde{H}_0}\|m_t\|_{\tilde{H}_0} + \|\sigma_t\|^2_{H-S}$ and that $|b^{(1)}_i(t)| \leq 2\|Z(t)\|_{\tilde{H}_0}\|\sigma^i_t\|_{\tilde{H}_0}$. Once more, by Ito's lemma we obtain

$$F(\|Z(t)\|^2_{\tilde{H}_0}) = F(0) + \int_0^t ((2(Z(s), \partial Z(s))_{\tilde{H}_0} F'(\|Z(s)\|^2_{\tilde{H}_0}) + a(s))ds + \sum_{i\in\mathbb{I}} b_i(s)dW^i_s), \tag{A.34}$$

where

$$a(t) = F'(\|Z(t)\|^2_{\tilde{H}_0})a^{(1)}(t) + \frac{1}{2}F''(\|Z(t)\|^2_{\tilde{H}_0}) \sum_{i\in\mathbb{I}} (b^{(1)}_i(t))^2. \tag{A.35}$$

and

$$b_i(t) = F'(\|Z(t)\|^2_{\tilde{H}_0}) b^{(1)}_i(t). \tag{A.36}$$

Inequality (A.27) of the lemma follows from the noted estimates for $a^{(1)}(t)$ and $b^{(1)}_i(t)$, from formula (A.35) and from $F' \geq 0$. Formula (A.28) of the lemma follows from formulas (A.33) and (A.36). Inequality (A.26) of the lemma follows from equality (A.29) of the lemma. Equality (A.29) follows from equality (A.34) and the definition of v in (A.30). $Z(t) \in \mathcal{D}_\infty$ and ∂ is the generator of a C^0 contraction semi-group, in a real Hilbert space, which give the inequality in (A.30).

We have now proved all statements of the lemma under the supplementary hypothesis that $\|(m, \sigma)\|_j < \infty$, for each $j \in \mathbb{N}$. The general case is now obtained by continuity. End of proof.

In the next lemma we establish that the solution of equation (A.7) is in L^p, $p \in [0, \infty[$.

Lemma A.4 *Let condition (A.1) be satisfied and let* (i) $E(\exp(p \int_0^{\bar{T}} (\|m_t\|_{\tilde{H}_0} + \sum_{i \in \mathbb{I}} \|\sigma_t^i\|^2_{\tilde{H}_0}) dt)) < \infty$, *for each* $p \in [1, \infty[$. *Suppose that* Y *in Lemma A.1 satisfies* (ii) $E(\sup_{t \in \mathbb{T}} \|Y(t)\|^p_{\tilde{H}_0}) < \infty$, *for each* $p \in [1, \infty[$. *Then the unique solution* X *of equation (A.7) in Lemma A.1 satisfies*

$$E(\sup_{t \in \mathbb{T}} \|X(t)\|^p_{\tilde{H}_0}) < \infty, \tag{A.37}$$

for each $p \in [1, \infty[$. *In particular, if* $\tilde{\mathcal{E}}(L)$ *is as in Lemma A.2, then*

$$E(\sup_{t \in \mathbb{T}} \|\tilde{\mathcal{E}}(L)\|^p_{\tilde{H}_0}) < \infty, \tag{A.38}$$

for each $p \in [1, \infty[$. *Moreover if* (iii) $E((\int_0^{\bar{T}} (\|m_t\|_{\tilde{H}_1} + \sum_{i \in \mathbb{I}} \|\sigma_t^i\|^2_{\tilde{H}_1}) dt)^p) < \infty$ *and* (iv) $E(\sup_{t \in \mathbb{T}} \|Y(t)\|^p_{\tilde{H}_1}) < \infty$, *for each* $p \in [1, \infty[$, *then also*

$$E(\sup_{t \in \mathbb{T}} \|X(t)\|^p_{\tilde{H}_1}) < \infty, \tag{A.39}$$

for each $p \in [1, \infty[$.

Proof: Suppose that conditions (i) and (ii) are satisfied.

We first prove inequality (A.38). Let $N(t) = \int_0^t \tilde{\mathcal{L}}_{t-s}((m_s - \frac{1}{2} \sum_{i \in \mathbb{I}} (\sigma_s^i)^2) ds + \sum_{i \in \mathbb{I}} \sigma_s^i dW_s^i)$, for $t \in \mathbb{T}$. Then $\tilde{\mathcal{E}}(L) = \exp(N(t))$ according to (A.19). Since $\tilde{H}_0$ is a Banach algebra it follows that

$$\|(\tilde{\mathcal{E}}(L))(t)\|_{\tilde{H}_0} \leq \exp(C \|N(t)\|_{\tilde{H}_0}) \leq \exp(C(1 + \|N(t)\|^2_{\tilde{H}_0})^{1/2}), \tag{A.40}$$

for a constant C given by $\tilde{H}_0$.

We use Lemma A.3 to find a bound of the right hand side of (A.40). Let a and b_i be given by Lemma A.3, with $F(x) = (1+x)^{1/2}$, let $A(t) = \int_0^t |a(s)| ds$ and let $M(t) = \int_0^t \sum_{i \in \mathbb{I}} b_i(s) dW_s^i$. Then inequality (A.26) give

$$(1 + \|N(t)\|^2_{\tilde{H}_0})^{1/2} \leq 1 + A(t) + M(t), \tag{A.41}$$

inequality (A.27) give

$$|a(t)| \leq \|m_t\|_{\tilde{H}_0} + \frac{3}{2}C\|\sigma_t\|^2_{H-S} \tag{A.42}$$

and (A.28) give

$$b_i(t) = (1 + \|N(t)\|^2_{\tilde{H}_0})^{-1/2}(N(t), \sigma^i_t)_{\tilde{H}_0}, \tag{A.43}$$

where $i \in \mathbb{I}$ and $t \in \mathbb{T}$. Obviously $|b_i(t)| \leq \|\sigma^i_t\|_{\tilde{H}_0}$ and the quadratic variation $\ll M \,,\, M \gg(t) \leq \int_0^t \|\sigma_s\|^2_{H-S} ds$.

By the hypothesis of the lemma and (A.42) it follows that

$$E(\exp(pA(\bar{T}) + p \ll M \,,\, M \gg(\bar{T}))) < \infty \tag{A.44}$$

for each $p \in [1, \infty[$. Novikov's criteria (c.f. [17], Ch. VIII, Prop. 1.15) and inequality (A.44) give

$$E(\exp(p \sup_{t \in \mathbb{T}} |M(t)|)) < \infty \tag{A.45}$$

for each $p \in [1, \infty[$. Inequality (A.41) gives

$$E(\exp(q(1 + \|N(t)\|^2_{\tilde{H}_0})^{1/2})) \leq E(\exp(q(1 + A(\bar{T}) + \sup_{t \in \mathbb{T}} |M(t)|)) \tag{A.46}$$

for each $q \in [0, \infty[$. It follows from Schwarz inequality and inequalities (A.44), (A.45) and (A.46) that

$$E(\exp(q(1 + \|N(t)\|^2_{\tilde{H}_0})^{1/2})) < \infty \tag{A.47}$$

for each $q \in [0, \infty[$. Statement (A.38) now follows from inequalities (A.40) and (A.47) by choosing $q = p\,C$.

We use the explicit expression (A.21) for X to prove (A.37). Let $Z(t) = \int_0^t \tilde{\mathcal{L}}_{t-s} dV(s)$, where $V(t) = \int_0^t Y(s)(\tilde{\mathcal{E}}(L'))(s) dL'(s)$ and L' is as in Lemma A.2. Explicitly

$$V(t) = \int_0^t (\alpha(s)ds + \sum_{i \in \mathbb{I}} \beta_i(s) dW^i_s),$$

where $\alpha(t) = Y(t)(\tilde{\mathcal{E}}(L'))(t)((\sum_{i \in \mathbb{I}} \sigma^i_t)^2 - m_t)$ and $\beta_i(t) = -Y(t)(\tilde{\mathcal{E}}(L'))(t)\sigma^i_t$. Since we have proved (A.38), by Schwarz inequality it is enough to prove

$$E(\sup_{t \in \mathbb{T}} \|Z(t)\|^p_{\tilde{H}_0}) < \infty, \tag{A.48}$$

for each $p \in [1,\infty[$, to establish (A.37). We proceed similarly as we did earlier in this proof to obtain (A.41). We now obtain using Lemma A.3

$$(1+\|Z(t)\|^2_{\tilde{H}_0})^{1/2} \leq 1 + A_1(t) + M_1(t), \tag{A.49}$$

where $A_1(t) = \int_0^t |a_1(s)|ds$, $M_1(t) = \int_0^t \sum_{i\in\mathbb{I}} b_{1i}(s)dW^i_s$,

$$\begin{aligned} |a_1(t)| \leq\ & C'\|Y(t)\|_{\tilde{H}_0}\|(\tilde{\mathcal{E}}(L'))(t)\|_{\tilde{H}_0} \\ & (\|m_t\|_{\tilde{H}_0} + (1+\|Y(t)\|_{\tilde{H}_0}\|(\tilde{\mathcal{E}}(L'))(t)\|_{\tilde{H}_0})\|\sigma_t\|^2_{H-S}), \end{aligned} \tag{A.50}$$

with C' given by H and

$$b_{1i}(t) = -(1+\|Z(t)\|^2_{\tilde{H}_0})^{-1/2}(Z(t), Y(t)(\tilde{\mathcal{E}}(L'))(t)\sigma^i_t)_{\tilde{H}_0}. \tag{A.51}$$

Choosing the constant C' sufficiently big, (A.51) give

$$\ll M_1\,,\, M_1 \gg(t) \leq C' \int_0^t \|Y(s)\|^2_{\tilde{H}_0}\|(\tilde{\mathcal{E}}(L'))(s)\|^2_{\tilde{H}_0}\|\sigma_s\|^2_{H-S}ds. \tag{A.52}$$

Hölders inequality, inequalities (A.41) and (A.50) and the hypotheses of the lemma give

$$E(\sup_{t\in\mathbb{T}}((A_1(t))^p) < \infty, \tag{A.53}$$

for each $p \in [1,\infty[$. Similarly, using $\int_0^t \|Y(s)\|^2_{\tilde{H}_0}\|(\tilde{\mathcal{E}}(L'))(s)\|^2_{\tilde{H}_0}\|\sigma_s\|^2_{H-S}ds$ $\leq (\sup_{s\in\mathbb{T}}\|Y(s)\|^2_{\tilde{H}_0})(\sup_{s\in\mathbb{T}}\|(\tilde{\mathcal{E}}(L'))(s)\|^2_{\tilde{H}_0}) \int_0^t \|\sigma_s\|^2_{H-S}ds$, (A.52) give

$$E((\ll M_1\,,\, M_1 \gg(\bar{T}))^{p/2}) < \infty, \tag{A.54}$$

for each $p \in [1,\infty[$. The B-D-G inequality then give

$$E(\sup_{t\in\mathbb{T}}((M_1(t))^p) < \infty, \tag{A.55}$$

for each $p \in [1,\infty[$. Now inequalities (A.49), (A.53) and (A.55) prove (A.48).

Finally to prove inequality (A.39), we suppose that also conditions (iii) and (iv) are satisfied.

The solution X of equation (A.7) is, according to Lemma A.1 in the domain of ∂, i.e. $X(t) \in \tilde{H}_1$. Since ∂ is continuous from $\tilde{H}_1$ to $\tilde{H}_0$, we have $\partial \int_0^t \tilde{\mathcal{L}}_{t-s}X(s)dL(s) = \int_0^t \tilde{\mathcal{L}}_{t-s}\partial X(s)dL(s)$. Application of ∂ on both sides of equation (A.7) then give

$$X_1(t) = Y_1(t) + \int_0^t \tilde{\mathcal{L}}_{t-s}X_1(s)dL(s) \tag{A.56}$$

where $X_1(t) = \partial X(t)$, $Y_1(t) = \partial Y(t) + \int_0^t \tilde{\mathcal{L}}_{t-s} X(s) dL_1(s)$, with $L_1(t) = \int_0^t (\partial m_s ds + \sum_{i \in \mathbb{I}} \partial \sigma_s^i dW_s^i)$. We can now use inequality (A.37) for X_1, since in the context of equation (A.56) hypotheses (i) and (ii) are satisfied. This proves inequality (A.39). End of proof.

For completeness we prove, for the case of an infinite number of random sources, a representation result. The measure Q and the cylindrical Wiener process $\bar{W}$ are as in Corollary 5.8.

Lemma A.5 *Let* Γ *be deterministic and satisfy condition*

$$\int_0^{\bar{T}} \sum_{i \in \mathbb{I}} |\Gamma_t^i|^2 dt < \infty \ a.s. \tag{A.57}$$

and let $M(t) = \int_0^t \sum_{i \in \mathbb{I}} \Gamma_s^i d\bar{W}_s^i$, $t \in \mathbb{T}$. *If* $F \in C(\mathbb{R})$ *is absolutely continuous, with derivative* F', *and* $E_Q(F(M(\bar{T}))^2 + F'(M(\bar{T}))^2) < \infty$, *then*

$$F(M(\bar{T})) = E_Q(F(M(\bar{T}))) + \int_0^{\bar{T}} E_Q(F'(M(\bar{T})) \,|\, \mathcal{F}_t) dM(t). \tag{A.58}$$

for each $t \in \mathbb{T}$.

Proof: We have $\ll M\,,\, M \gg(t) = \int_0^t \sum_{i \in \mathbb{I}} (\Gamma_s^i)^2 ds < \infty$ according to condition (A.57) and the quadratic variation $\ll M\,,\, M \gg$ is deterministic. Let $n_{\mu,t}(x) = \exp(ix\mu + \frac{\mu^2}{2} \ll M\,,\, M \gg(t))$, $\mu \in \mathbb{R}$ and let $n'_{\mu,t}(x)$ be the derivative with respect to x of $n_{\mu,t}(x)$. Then $\mathbb{T} \ni t \mapsto n_{\mu,t}(M(t))$ is a complex Q-martingale and $n_{\mu,\bar{T}}(M(\bar{T})) = 1 + \int_0^{\bar{T}} n'_{\mu,t}(M(t)) dM(t)$. Since also $t \mapsto n'_{\mu,t}(M(t))$ is a Q-martingale it follows that

$$n_{\mu,\bar{T}}(M(\bar{T})) = 1 + \int_0^{\bar{T}} E_Q(n'_{\mu,\bar{T}}(M(\bar{T})) \,|\, \mathcal{F}_t) dM(t). \tag{A.59}$$

Let $g \in C_0^\infty(\mathbb{R})$ be real-valued with Fourier transform $\hat{g}$. Multiplication of both sides of equality (A.59) with the complex number

$$c(\mu) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2} \ll M\,,\, M \gg (\bar{T}))} \hat{g}(\mu)$$

gives

$$c(\mu) n_{\mu,\bar{T}}(M(\bar{T})) = c(\mu) + \int_0^{\bar{T}} E_Q(c(\mu) n'_{\mu,\bar{T}}(M(\bar{T})) \,|\, \mathcal{F}_t) dM(t).$$

Integration in μ and the stochastic Fubini theorem then gives

$$g(M(\bar{T})) = \int_{\mathbb{R}} c(\mu)d\mu + \int_0^{\bar{T}} E_Q(g'(M(\bar{T})) \,|\, \mathcal{F}_t)dM(t). \tag{A.60}$$

Since $(E_Q(g'(M(\bar{T})) \,|\, \mathcal{F}_\cdot))^2$ is a submartingale it follows that

$$E_Q(\int_0^{\bar{T}} (E_Q(g'(M(\bar{T})) \,|\, \mathcal{F}_t))^2 d \ll M\,,\, M \gg(t)) \\ \leq E_Q((g'(M(\bar{T})))^2) \ll M\,,\, M \gg(\bar{T}),$$

which is finite. Therefore $\int_{\mathbb{R}} c(\mu)d\mu = E(g(M(\bar{T})))$, so formula (A.60) proves the representation formula (A.58) for $F \in C_0^\infty(\mathbb{R})$. The general case now follows by dominated convergence since F in the lemma is the limit, in the topology defined by the norm $G \mapsto (E_Q(F(M(\bar{T}))^2 + F'(M(\bar{T}))^2))^{1/2}$, of a sequence in $C_0^\infty(\mathbb{R})$. End of proof.

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