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A Theory of Discretization for Nonlinear Evolution Inequalities Applied to Parabolic Signorini Problems (*).

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Abstract. – *We present a discretization theory for a class of nonlinear evolution inequalities that encompasses time dependent monotone operator equations and parabolic variational inequalities. This discretization theory combines a backward Euler scheme for time discretization and the Galerkin method for space discretization. We include set convergence of convex subsets in the sense of Glowinski-Mosco-Stummel to allow a nonconforming approximation of unilateral constraints. As an application we treat parabolic Signorini problems involving the p-Laplacian, where we use standard piecewise polynomial finite elements for space discretization. Without imposing any regularity assumption for the solution we establish various norm convergence results for piecewise linear as well piecewise quadratic trial functions, which in the latter case leads to a nonconforming approximation scheme.*

1. – Introduction.

The standard approach to discretization of parabolic variational inequalities employs finite elements for space discretization and finite differences for time discretization. However, only linear interpolation of nonnegative (or nonpositive) data preserves inequalities. Thus, whenever the unilateral constraint, e.g. given by an obstacle, is inhomogeneous or whenever the finite element trial functions consist of piecewise polynomials of higher order than linear, a fully discrete approximation scheme has to include a nonconforming approximation of the unilateral constraint, which can be treated by means of Glowinski-Mosco-Stummel set convergence for the associated convex subsets of the discrete variational inequalities. This is elucidated in the monographs of Glowinski-Lions-Trémolières [15, chapter 6] and of Glowinski [14, chapter 6].

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In this paper we extend the convergence analysis of Glowinski-Lions-Trémolières [15] (dealing with variational inequalities given by bilinear forms) to nonlinear evolution problems where the underlying elliptic differential operator gives rise to a monotone operator in the sense of Browder and Minty. Furthermore we also treat the semicoercive case, that is, the case where the nonlinear form associated to the elliptic differential operator is only coercive with respect to a seminorm, what occurs when only Neumann and Signorini boundary conditions are present in the initial boundary value problem. Thus we also extend convergence results of Baiocchi [4], Gröger [16], Kačur [23, chapter 8.2], Ženišek [39], see also [40], to more general evolution problems including nonconforming approximation of unilateral constraints. Moreover by our approach via monotone-convex functions following Oettli [32] we subsume a class of differential inclusions. Thus in this respect, our analysis also complements the work of Magenes [28] on time discretization of abstract differential inclusions of parabolic type.

We emphasize that our aim is to establish norm convergence of the discretization method under weak assumptions, that is under conditions as close as possible to those of the existence theory. Therefore, error estimates of the type given in [22], [26] and [27] in the case of conforming approximation using piecewise linear trial functions are outside of the scope of the present paper which instead continues earlier work [17] on steady state problems.

As the most simple application which shows all issues of the generality of our discretization theory, namely unilateral boundary conditions and monotone nonlinearity of the associated elliptic operator, we study an initial boundary obstacle problem involving the p-Laplacian. We treat both the coercive case of a mixed Dirichlet-Signorini boundary condition and the semicoercive case of a Signorini boundary condition. For its finite element approximation we do not only consider piecewise linear trial functions, but also investigate piecewise quadratic trial functions that lead to a nonconforming approximation of the unilateral constraint. Under assumptions close to the existence theory [7,30] of monotone parabolic problems, in particular dispensing with any regularity hypotheses on the solution, we establish various (strong) convergence results for fully discrete approximations.

Let us point out that the p-Laplacian and related degenerate parabolic equations are a recently much studied subject in analysis, see e.g. [6], not only by their intrinsic mathematical interest, but also by their role as a mathematical model for a diversity of nonlinear problems in mechanics and physics. Already Ladyzenskaja [24] suggested equations of this kind as a model of motion of non-newtonian fluids. Further for the modeling and study of nonlinear diffusion and of power-law materials we refer to e.g. [1,18]. Quite naturally in addition to linear boundary conditions, here unilateral boundary conditions like Signorini condition occur that model contact in quasistationary solid mechanics or semi-permeable walls in fluid mechanics, see [12]; such degenerate parabolic equations with a Signorini boundary condition are particularly encountered in porous media [9,21].

The outline of this paper is as follows. In the next section we formulate the evolution inequality problem under study where we distinguish different variational and relaxed versions. In Section 3 we present our theory of full space time discretization for the evolution problem. Finally in Section 4, we apply our discretization theory to p-harmonic Signorini initial boundary value problems.

2. – The problem: variational and relaxed versions.

In this section we formulate the evolution inequality problem under study. As our numerical functional analysis of finite element discretization will show, it is worth distinguishing different versions of the problem. Here we state these different variational and relaxed versions and also give their relations. Finally in this section, we comment on more general time dependent problems. In particular when not only the right hand side, but also in the application to parabolic obstacle problems, the coefficients of the associated elliptic differential operator depend on time, it is shown how this more general problem can easily be reduced to the problem studied.

Let us first fix some notations. Let J denote the open interval $(0, T)$ with $T > 0$ given. Let X be a real reflexive separable Banach space with norm $\|\cdot\|_X$ and dual space X^*. In addition, let H be a real separable Hilbert space endowed with the scalar product $\langle \cdot, \cdot \rangle$ and identified with its dual such that $X \subseteq H \subseteq X^*$ continously and densely.

Throughout the paper, we fix the parameter $p \in [2, \infty)$ and consider the space $L^p(J, X)$ of (classes of) measurable functions u for the Lebesgue measure on J with values in X such that

$$\|u\| := \|u\|_{L^p(J,X)} := \left[\int_0^T \|u(t)\|_X^p \, dt \right]^{1/p} < +\infty .$$

$L^p(J, X)$ is a subspace of $\mathcal{D}'(J, X) = \mathcal{L}(\mathcal{D}(J), X)$, the latter being the space of vector-valued distributions on J (see e.g. [11], chapter 18, section 1.1). Moreover $\partial_t u$ (or interchangeably u') denotes the derivative of u with respect to time t in the sense of vector-valued distribution (see e.g. [11], chapter 18, section 1.1, Definition 3). In virtue of $X \subseteq \subseteq X^*$ it makes sense to define the space

$$W^p(J; X, X^*) := \{u \in L^p(J, X) \colon \partial_t u \in L^{p'}(J, X^*)\},$$

where $1/p + 1/p' = 1$. This Banach space, equipped with the norm

$$\|u\|_{W^p(J;X,X^*)} := \|u\|_{L^p(J,X)} + \|\partial_t u\|_{L^{p'}(J,X^*)},$$

is of fundamental importance in the analysis of evolution problems. It is known (see [11], chapter 18, section 1.2, Theorem 1 for the case $p = 2$; [13], Satz 1.17 and [37], Proposition 23.23 for the general case) that every $u \in W^p(J; X, X^*)$ is a.e. (almost everywhere) equal to a continuous function on $\bar{J} = [0, T]$ in H and that even $W^p(J; X, X^*) \subseteq C[\bar{J}, H]$ continuously, where $C[\bar{J}, H]$ denotes the space of H-valued continuous functions on $\bar{J}$ and is endowed with the topology of uniform convergence.

Next let us introduce the data of the problem. The right hand side f is assumed to be in $L^{p'}(J, X^*) \cap L^1(J, H)$. To define the unilateral constraint, let $K \subseteq X$ be a given nonvoid closed convex set. For consistency of the problem, let the initial datum u_0 belong to K. Instead of a nonlinear elliptic operator, we prefer to introduce a function $\varphi \colon X \times \times X \to \mathbb{R}$ in order to subsume partial differential inclusions by this approach and follow the terminology of Oettli [32].

DEFINITION 2.1. – A function $\varphi: X \times X \to \mathbb{R}$ is called *monotone-convex*, if for any $x \in X$, $\varphi(x, \cdot)$ is convex and lower semicontinuous, satisfies $\varphi(x, x) = 0$, and further, φ is monotone, that is

$$\varphi(x, y) + \varphi(y, x) \leqslant 0 \qquad \text{for all } x, y \in X .$$

EXAMPLE 2.2. – Let $F: X \rightharpoonup X^*$ be a set valued monotone operator in the sense of Browder and Minty [38] with convex closed bounded values $F(x) \subset X^*(x \in X)$, then

$$\varphi(x, y) := \max \{ \langle \xi, y - x \rangle_{X^* \times X}: \xi \in F(x) \}$$

defines a monotone-convex function φ. Thus by this approach we treat a class of evolution inclusions and partial differential inclusions.

In all what follows we require that φ is monotone-convex. Since we are involved with time dependent problems, we require for the nonlinear Nemytskii operator $(v(\cdot), w(\cdot)) \mapsto \varphi(v(\cdot), w(\cdot))$ that $\varphi(v(\cdot), w(\cdot)) \in L^1(J)$ for arbitrary $v, w \in L^p(J, X)$. This is guaranteed by the growth condition (with a constant $c_0 > 0$)

$$|\varphi(x, y)| \leqslant c_0 [\|x\|_X^p + \|y - x\|_X^p] \qquad \text{for all } x, y \in X . \tag{2.1}$$

Then we may define the real valued function $\Phi: L^p(J, X) \times L^p(J, X) \to \mathbb{R}$ by

$$\Phi(v, w) := \int_0^T \varphi(v(t), w(t))\, dt \qquad (v, w \in L^p(J, X)) .$$

For brevity, we introduce the following sets

$$\begin{aligned} L^p(J, K) &:= \{ v \in L^p(J, X): v(t) \in K \text{ a.e. in } J \}, \\ W^p(J, K) &:= \{ v \in W^p(J; X, X^*): v(t) \in K \text{ for all } t \in \bar{J} \}; \\ W_0^p(J, K) &:= \{ v \in W^p(J, K): v(0) = u_0 \} . \end{aligned}$$

Now we can state our evolution inequality problem in pointwise variational form:

(P) Find $u \in W_0^p(J, K)$ such that for almost all t in J

$$\langle \partial_t u - f(t), y - u(t) \rangle_{X^* \times X} + \varphi(u(t), y) \geqslant 0 , \qquad \forall y \in K . \tag{2.2}$$

This means that the preceding inequality holds pointwise on J except a fixed null set (only dependent on the solution u) for any $y \in K$. In integrated variational form our evolution inequality reads:

(P1) Find $u \in W_0^p(J, K)$ such that

$$\int_0^T \langle \partial_t u - f, v - u \rangle_{X^* \times X} dt + \Phi(u, v) \geqslant 0 , \qquad \forall v \in L^p(J, K). \tag{2.3}$$

Similar to Brèzis [7] and Naumann [30] we introduce the relaxed form:

(P2) Find $u \in L^p(J, K) \cap C[\bar{J}, H]$ with $u(0) = u_0$ such that

$$\int_0^T \langle \partial_t v - f, u - v\rangle_{X^* \times X} dt + \Phi(v, u) \leq \frac{1}{2}\|v(0) - u_0\|_H^2, \qquad \forall v \in W^p(J, K). \tag{2.4}$$

These problems are related. In the following we show the equivalence of (P), (P1), and (P2) under reasonable assumptions.

LEMMA 2.3. – *The problems* (P) *and* (P1) *are equivalent.*

PROOF. – Let $u \in W_0^p(J, K)$ solve (P) and let $v \in L^p(J, K)$ arbitrarily. Then a.e. on J we can plug in $y = v(t) \in K$ and integrate a nonnegative $L^1(J)$ function to obtain that u also solves (P1). — Vice versa, for any $y \in K$, $t_0 \in J$, $\varepsilon > 0$ with $(t_0 - \varepsilon, t_0 + \varepsilon) \subset J$, set

$$v(t) = \begin{cases} y & \text{if } t \in (t_0 - \varepsilon, t_0 + \varepsilon); \\ u(t) & \text{otherwise}. \end{cases}$$

Then $v \in L^p(J, K)$ and for a solution u of (P1), we have

$$\int_{t_0 - \varepsilon}^{t_0 + \varepsilon} [\langle u'(t) - f(t), y - u(t)\rangle_{X^* \times X} + \varphi(u(t), y)]\, dt \geq 0 .$$

Division by 2ε and letting $\varepsilon \to 0$ shows that u satisfies (2.2) except some null set, since by assumption the integrand is a $L^1(J)$ function and thus the Lebesgue differentiation theorem (see e.g. [20], (18.4)) applies. Using the separability of the graph $\{(y, s): y \in K, s = \varphi(u(t), y)\}$ as a subspace of the separable metric space $X \times \mathbb{R}$ we can get rid (similarly as in [7], Appendice I) of the dependence of the null set on $y \in K$. Thus we conclude that u solves (P). ■

In addition, we need a mild continuity assumption on φ with respect to the first variable.

DEFINITION 2.2. – A function $\varphi: X \times X \to \mathbb{R}$ is called *hemicontinuous*, if for any $x, y, z \in X$ there holds

$$\limsup_{\sigma \to 0+} \varphi(x + \sigma z, y) \leq \varphi(x, y).$$

Thus hemicontinuity means upper semicontinuity of φ with respect to the first variable on one dimensional subspaces only.

REMARK. – By the condition (2.1), the integral function Φ inherits hemicontinuity from φ. Indeed, let $u, v, w \in W^p(J; X, X^*)$. Then $w_\sigma := u + \sigma w \to u$ for $\sigma \to 0+$.

Therefore it suffices to estimate

$$\liminf_{\sigma\to 0+}\{c_0[\|w_\sigma\|^p + \|v-w_\sigma\|^p] - \Phi(w_\sigma, v)\} =$$

$$= \liminf_{\sigma\to 0+}\int_0^T \{c_0[\|w_\sigma(t)\|_X^p + \|(v-w_\sigma)(t)\|_X^p] - \varphi(u(t)+\sigma w(t), v(t))\}\,dt$$

in virtue of Fatou's lemma, since the integrand $\{\dots\}$ is nonnegative and is in $L^1(J)$, by

$$\geq \int_0^T \liminf_{\sigma\to 0+}\{c_0[\|w_\sigma(t)\|_X^p + \|(v-w_\sigma)(t)\|_X^p] - \varphi(u(t)+\sigma w(t), v(t))\}\,dt$$

$$\geq c_0[\|u\|^p + \|v-u\|^p] - \Phi(u, v).$$

Thus we obtain hemicontinuity of Φ. ■

Comparing (P1) and (P2) we see that in (P1), (weak) differentiability in time is required for the solution function u, whereas in (P2), (weak) differentiability in time is only needed for the test function v. This leads to different solution spaces and test spaces. Nevertheless, we have the following result.

LEMMA 2.2.

a) *If u solves* (P1), *then u solves* (P2), *too.*

b) *Suppose that Φ is hemicontinuous. Let u be a solution of* (P2) *such that the weak derivative $\partial_t u \in L^{p'}(J, X^*)$. Then $u \in W_0^p(J, K)$ and there holds*

$$\int_0^T \langle \partial_t u - f, v-u\rangle_{X^*\times X}\,dt + \Phi(u, v) \geq 0\,, \qquad \forall v \in W_0^p(J, K).$$

PROOF. – To verify part a) let u be a solution of (P1) and let $v \in W^p(J, K)$ arbitrarily. By monotonicity, $\Phi(u, v) + \Phi(v, u) \leq 0$. Hence by (2.3)

$$\Phi(v, u) \leq \int_0^T \langle u' - f, v-u\rangle_{X^*\times X}\,dt = \int_0^T \langle u' - v', v-u\rangle\,dt + \int_0^T \langle v' - f, v-u\rangle\,dt.$$

Thus it is enough to estimate $\int_0^T \langle e', e\rangle\,dt$ below for $e := v-u$. On the other hand, it follows from integration by parts (see [11], Chapter 18, Section 1, Theorem 2 for the case

$p=2$; [13], Satz 1.17 for the general case) that

$$\int_0^T \langle e', e\rangle_{X^*\times X} dt = \frac{1}{2}\{\|e(T)\|_H^2 - \|e(0)\|_H^2\} \geq -\frac{1}{2}\|e(0)\|_H^2,$$

where $e(0)=v(0)-u_0$. This proves part a).

To verify part b) let u be a solution of (P2) with $u' \in L^{p'}(J, X^*)$. Thus immediately $u \in W_0^p(J, K)$. Fix arbitrarily $v \in W_0^p(J, K)$ and $\sigma \in (0, 1]$. Then $w_\sigma := \sigma v + (1-\sigma)u \in$ $\in W_0^p(J, K)$, and by (2.4),

$$\Phi(w_\sigma, u) \leq \int_0^T \langle w_\sigma' - f, w_\sigma - u\rangle_{X^*\times X} dt = \sigma \int_0^T \langle u' + \sigma(v' - u') - f, v - u\rangle_{X^*\times X} dt .$$

Since Φ inherits convexity in its second argument from φ, we have

$$0 = \Phi(w_\sigma, w_\sigma) \leq \sigma\Phi(w_\sigma, v) + (1-\sigma)\,\Phi(w_\sigma, u),$$

hence

$$-\Phi(w_\sigma, v) \leq \frac{1-\sigma}{\sigma}\Phi(w_\sigma, u).$$

Therefore we conclude that for $\sigma \to 0+$

$$\liminf_{\sigma\to 0+} -\Phi(w_\sigma, v) \leq \int_0^T \langle u' - f, v - u\rangle_{X^*\times X} dt .$$

Since by assumption Φ is hemicontinuous, we arrive at

$$-\Phi(u, v) \leq \int_0^T \langle u' - f, v - u\rangle_{X^*\times X} dt ,$$

what concludes the proof of part b). ■

To return from (P2) to (P1) completely, we have to be more precise about the function φ. We decompose φ as follows:

$$(*) \qquad \varphi(y, z) = \psi(z) - \psi(y) + \widehat{\varphi}(y, z) \qquad \text{for all } y, z \in X .$$

where $\psi: X \to \mathbb{R}$ is convex and lower semicontinuous (can be more generally a proper extended real function) and $\widehat{\varphi}$ has the same properties as φ as described above. By the appropriate growth condition (2.1), time integration leads to the associated functions

$$\Psi(v) = \int_0^T \psi(v(t))\, dt\,,$$

$$\widehat{\Phi}(v, w) = \int_0^T \widehat{\varphi}(v(t), w(t))\, dt\,.$$

In addition to the requirements on φ given above, being in force for $\widehat{\varphi}$ now, we need continuity of $\widehat{\Phi}$ with respect to the second variable, what follows from the continuity of $\widehat{\varphi}(y, \cdot)$ for all $y \in X$ and from the appropriate growth assumption (2.1).

Without any restriction of generality, we can assume that $\psi \geq 0$, hence $\Psi \geq 0$. Indeed, by the strong separation theorem (consider e.g. $(u_0, \psi(u_0) - 1)$ which does not belong to the closed convex epigraph set of ψ) there exist $x_0^* \in X^*$, $r \in \mathbb{R}$ such that

$$\tilde{\psi}(y) := \psi(y) - \langle x_0^*, y \rangle - r \geq 0\,, \qquad \forall y \in X\,.$$

Hence we can decompose

$$\varphi(y, z) = \tilde{\psi}(y) - \tilde{\psi}(z) + [\widehat{\varphi}(y, z) + \langle x_0^*, y - z \rangle]\,,$$

with $\tilde{\psi}$ nonnegative as claimed.

Now we can state

LEMMA 2.6. – *Let u be a solution of* (P2) *such that $\partial_t u \in L^{p'}(J, X^*)$. Suppose that in the decomposition* $(*)$, *$\widehat{\Phi}(u, \cdot)$ is continuous on $L^p(J, X)$. Then u solves* (P1).

PROOF. – We adopt a construction due to Naumann ([30] pp. 36-37, 39). For any $v \in L^p(J, K)$ consider

$$v_\varepsilon(t) = \exp[-t/\varepsilon]\, u_0 + \frac{1}{\varepsilon} \int_0^t \exp[(s-t)/\varepsilon]\, v(s)\, ds\,, \qquad t \in J,\ \varepsilon > 0\,.$$

Because of $X \subset H \subset X^*$ continuously and $p \geq 2$, $v \in L^{p'}(J, X^*)$, and hence $v_\varepsilon \in W^p(J; X, X^*)$ with $v_\varepsilon(0) = u_0 \in K$. We split for $t \in J$

$$v_\varepsilon(t) = \exp[-t/\varepsilon]\, u_0 + \{1 - \exp[-t/\varepsilon]\}\, \tilde{v}_\varepsilon(t)\,,$$

$$\tilde{v}_\varepsilon(t) = \frac{1}{\varepsilon} \frac{1}{\exp[t/\varepsilon] - 1} \int_0^t \exp[s/\varepsilon]\, v(s)\, ds\,.$$

By an indirect argument using the separation theorem, $\tilde{v}_\varepsilon(t) \in K$ for any $t \in J$. Thus

$v_\varepsilon \in W_0^p(J, K)$. Therefore by Lemma 2.5, we have

$$\Phi(u, v_\varepsilon) + \int_0^T \langle u' - f, v_\varepsilon - u \rangle_{X^* \times X} dt \geq 0, \qquad (\forall \varepsilon > 0), \tag{2.5}$$

where

$$\Phi(u, v_\varepsilon) = \Psi(v_\varepsilon) - \Psi(u) + \widehat{\Phi}(u, v_\varepsilon).$$

Since $v_\varepsilon \to v$ strongly in $L^p(J, X)$ for $\varepsilon \to 0$ (see Footnote 11 [30], p. 36) we only have to investigate the limit behaviour of $\Psi(v_\varepsilon)$ in the following. By convexity of ψ, we have for fixed $t \in J$

$$\psi(v_\varepsilon(t)) \leq \exp[-t/\varepsilon]\, \psi(u_0) + \{1 - \exp[-t/\varepsilon]\}\, \psi(\tilde{v}_\varepsilon(t)).$$

To estimate the latter term substitute for $0 \leq s \leq t$

$$r = r(s) := \frac{1}{1 - \exp[-t/\varepsilon]} \exp[(s-t)/\varepsilon] = \frac{1}{\exp[t/\varepsilon] - 1} \exp[s/\varepsilon]$$

and vice versa

$$s = s(r) = \varepsilon \log[r(\exp[t/\varepsilon] - 1)]$$

and write with $r_0 := r(0)$, $r_1 := r(t)$

$$\tilde{v}_\varepsilon(t) = \int_{r_0}^{r_1} v(s(r))\, dr.$$

Note that $r_1 - r_0 = 1$ and that the set $\{[v(s(r)), \psi(v(s(r)))]: r_0 \leq r \leq r_1\}$ is contained in $\operatorname{epi}\psi = \{[x, \alpha] \in X \times \mathbb{R}: \psi(x) \leq \alpha\}$, the epigraph of ψ which is closed and convex. Hence by Jensen's inequality

$$\psi\left(\int_{r_0}^{r_1} v(s(r))\, dr\right) \leq \int_{r_0}^{r_1} \psi(v(s(r)))\, dr.$$

Thus we conclude

$$\psi(v_\varepsilon(t)) \leq \exp[-t/\varepsilon]\, \psi(u_0) + \frac{1}{\varepsilon}\int_0^t \exp[(s-t)/\varepsilon]\, \psi(v(s))\, ds.$$

Since $\psi \geq 0$, time integration yields

$$\Psi(v_\varepsilon) \leq \varepsilon\psi(u_0) + \frac{1}{\varepsilon}\int_0^T \int_s^T \exp[(s-t)/\varepsilon]\, \psi(v(s))\, dt\, ds \leq \varepsilon\psi(u_0) + \Psi(v).$$

This estimate and (2.5) entail in the limit $\varepsilon \to 0$ that finally u satisfies (2.3). Hence u solves (P1) as claimed. ■

Our next result concerns uniqueness.

LEMMA 2.7. – *There exists at most one solution to* (P1).

PROOF. – Let u_1, u_2 be two solutions to (P1). Using the index j modulus two we obtain from the respective inequality in (P1) by plugging in $v(t) = u_{j+1}(t)$ on $[0, t]$, $v(t) = u_j(t)$ on $(t, T]$

$$0 \leq \int_0^t \langle u_j' - f, u_{j+1} - u_j \rangle_{X^* \times X} dt + \int_0^t \varphi(u_j(t), u_{j+1}(t))\, dt .$$

Adding up these inequalities for $j = 1$ and $j = 2$ leads to

$$0 \leq \int_0^t \langle u_1' - u_2', u_2 - u_1 \rangle_{X^* \times X} dt + \int_0^t [\varphi(u_1(t), u_2(t)) + \varphi(u_2(t), u_1(t))]\, dt$$

$$\leq \int_0^t \langle e', -e \rangle_{X^* \times X} dt$$

by monotonicity, where $e := u_1 - u_2$ on J. Thus by integration by parts,

$$\frac{1}{2} \|e(t)\|_H^2 - \frac{1}{2} \|e(0)\|_H^2 \leq 0 .$$

Since $e(0) = 0$, we arrive at $e = 0$. ■

CONSEQUENCE. – By the equivalence of the problems (P), (P1) and (P2) under the stated assumptions in Lemma 2.5 and in Lemma 2.6, also uniqueness for (P) and (P2) holds.

To conclude this section we admit the nonlinear functional φ to depend on time, too. We show how this more general evolution problem can be reduced to the problem given above by increasing the dimension of the problem.

Let $\varphi: J \times X \times X \to \mathbb{R}$ be given (under analogous assumptions as above). Extending the problem (P) we now seek $u \in W_0^p(J, K)$ that satisfies for almost all $t \in J$ and for all $y \in K$

$$\langle u'(t) - f(t), y - u(t) \rangle_{X^* \times X} + \varphi(t, u(t), y) \geq 0 .$$

The classical transformation (see e.g. [3] section 6.3, p. 300)

$$\begin{pmatrix} t \\ u \end{pmatrix} = \begin{pmatrix} t \\ u \end{pmatrix}(\tau), \qquad \dot{t} = \frac{dt}{d\tau} = 1$$

leads to the equivalent problem: Find $(t, u) \in W^p(J, \mathbb{R} \times K)$ with $(t(0), u(0)) = (0, u_0)$

such that for almost all $\tau \in J$ and for all $(s, y) \in \mathbb{R} \times K$

$$\langle \dot{u}(\tau) - f(\tau), y - u(\tau) \rangle_{X^* \times X} + (\dot{t}(\tau) - 1, s - t(\tau)) + \phi((t, u)(\tau), (s, y)) \geq 0 ,$$

where $\phi((t, x), (s, y)) := \varphi(t, x, y)$. In this way we arrive at a problem of the form (P), now in the product space $\mathbb{R} \times X$. Note that ϕ inherits structural properties from φ, in particular $\phi((t, x), (t, x)) = \varphi(t, x, x) = 0$, $\phi((t, x), \cdot)$ is convex and so on.

However, by this trick, problems with time dependent obstacles, that is $K = K(t)$, cannot be simplified.

3. - The discretization theory.

In this section we present the theory of full space time discretization for the evolution problem described above. Here we investigate in detail a backward Euler scheme for time discretization. For space discretization we employ the Galerkin method and admit nonconforming approximation of unilateral constraints using Glowinski convergence of convex sets.

3.1. *Time discretization.* - In this preliminary subsection we concentrate on time discretization using a backward Euler scheme. In particular, we prove useful properties of the finite difference operator and other approximation operators.

Let us introduce a sequence $\{\pi_n\}_{n \in \mathbb{N}}$ of (not necessarily equidistant) partitions of the time interval $\bar{J} = [0, T]$ such that $\pi_n = (t_n^0, \ldots, t_n^{N_n})$, where $0 = t_n^0 < t_n^1 < \ldots < t_n^{N_n} = T$. We consider a *regular* sequence of partitions, in the sense that

$$k_n := \max \{ t_n^j - t_n^{j-1} \mid j = 1, \ldots, N_n \}$$

approaches zero for $n \to \infty$ and that there exists a constant $\tau_0 > 0$ independent of n such that

$$\min \{ t_n^j - t_n^{j-1} \mid j = 1, \ldots, N_n \} \geq \tau_0 k_n .$$

To describe the different approximations used, let V be an arbitrary reflexive Banach space (or a closed convex subset thereof), where later on, as the case may be, $V = X$, $V = H$, $V = X^*$ or $V = K$. The partitions of $[0, T]$ give rise to the space of V-valued piecewise constant functions subordinate to π_n, defined by

$$\mathcal{P}_n^0(J, V) := \{ v \in L^\infty(J, V) \mid v_{|(t_n^{j-1}, t_n^j]} = v_j \in V \ (j = 1, \ldots, N_n) \} ,$$

where for any $v \in \mathcal{P}_n^0(J, V)$, v_j denotes the constant value of v on the part interval $(t_n^{j-1}, t_n^j]$. Thus, for any fixed $a \in V$, the finite difference operator $\delta_n^a : \mathcal{P}_n^0(J, V) \to \mathcal{P}_n^0(J, V)$ is defined in a linear space V by

$$(\delta_n^a v)_j := \delta_n^a v_{|(t_n^{j-1}, t_n^j]} := \frac{v_j - v_{j-1}}{t_n^j - t_n^{j-1}} \qquad (j = 1, \ldots, N_n) ,$$

where $v_0 = v(0) := a$. Moreover, for any V-valued function v defined throughout $\bar{J}$, e.g. $v \in \mathcal{C}[\bar{J}, V]$, we construct a piecewise constant interpolation $\lambda_n^0 v \in \mathcal{P}_n^0(J, V)$ by

$$\lambda_n^0 v_{|(t_n^{j-1}, t_n^j]} := v(t_n^j) \quad (j = 1, \ldots, N_n)$$

and the piecewise linear interpolation $\lambda_n^1 v \in L^\infty(J, V)$ by

$$\lambda_n^1 v_{|(t_n^{j-1}, t_n^j]}(s) := \frac{s - t_n^{j-1}}{t_n^j - t_n^{j-1}} v(t_n^j) + \frac{t_n^j - s}{t_n^j - t_n^{j-1}} v(t_n^{j-1}) \quad (j = 1, \ldots, N_n),$$

what gives rise to the class $\mathcal{P}_n^1[J, V]$ of V-valued piecewise linear continuous functions on $\bar{J}$. Finally we introduce the mean value operator $\mu_n: L^1(J, V) \to \mathcal{P}_n^0(J, V)$ by

$$\mu_n v_{|(t_n^{j-1}, t_n^j]} := \frac{1}{t_n^j - t_n^{j-1}} \int_{t_n^{j-1}}^{t_n^j} v(s)\, ds \quad (j = 1, \ldots, N_n).$$

For these approximation operators we have the following results.

LEMMA 3.1. – *Suppose $(V, \langle \cdot, \cdot \rangle, |.|)$ is a Hilbert space. Then the mapping $\delta_n^a: \mathcal{P}_n^0(J, V) \to \mathcal{P}_n^0(J, V)$ is uniformly monotone and Lipschitz continuous.*

PROOF. – Let $u, v \in \mathcal{P}_n^0(J, V)$ and $w := u - v$. Then for any $j = 1, \ldots, N_n$ we have

$$\int_0^{t_n^j} \langle \delta_n^a u - \delta_n^a v, u - v \rangle\, dt = \sum_{i=1}^{j} \langle w_i - w_{i-1}, w_i \rangle = \sum_{i=1}^{j} \{ \langle w_i, w_i \rangle - \langle w_{i-1}, w_i \rangle \}$$

$$\geq \sum_{i=1}^{j} \{ |w_i|^2 - |w_i|\,|w_{i-1}| \},$$

and since $AB \leq 1/2A^2 + 1/2B^2$ for any reals A, B and by $w_0 = 0$

$$\geq \frac{1}{2} \sum_{i=1}^{j} \{ |w_i|^2 - |w_{i-1}|^2 \} = \frac{1}{2} |u_j - v_j|^2.$$

Thus in particular

$$\int_0^T \langle \delta_n^a u - \delta_n^a v, u - v \rangle\, dt \geq \frac{1}{2} |u_{N_n} - v_{N_n}|^2. \tag{3.1}$$

This proves uniform monotonicity.

To verify Lipschitz continuity, let u, v, w be as above. Then

$$\int_0^T |\delta_n^a u - \delta_n^a v|^2 dt = \int_0^T |\delta_n^0 w|^2 dt$$

$$= \sum_{j=1}^{N_n} \frac{1}{t_n^j - t_n^{j-1}} |w_j - w_{j-1}|^2,$$

by the parallelogram rule,

$$\leqslant \frac{2}{\tau_0 k_n} \sum_{j=1}^{N_n} \{ |w_j|^2 + |w_{j-1}|^2 \}$$

$$\leqslant \frac{4}{\tau_0 k_n} \sum_{j=1}^{N_n} |w_j|^2$$

$$\leqslant \frac{4}{(\tau_0 k_n)^2} \int_0^T |w|^2 dt .$$

Hence

$$\|\delta_n^a u - \delta_n^a v\|_{L^2(J,V)} \leqslant \frac{2}{\tau_0 k_n} \|u - v\|_{L^2(J,V)},$$

where the Lipschitz constant $\dfrac{2}{\tau_0 k_n}$ depends on n. ∎

LEMMA 3.2.

a) *Let $v \in \mathcal{C}[\bar{J}, V]$. Then for any $1 \leqslant r \leqslant \infty$ and for any $i = 0, 1$ we have $\lambda_n^i v \to v$ in $L^r(J, V)$ as $n \to \infty$.*

b) *Suppose, $v \in \mathcal{C}[\bar{J}, V]$ is absolutely continuous such that $v' \in L^r(J, V)$ for some $1 \leqslant r < \infty$. Then with $a = v(0)$, $\delta_n^a v \to v'$ in $L^r(J, V)$ as $n \to \infty$.*

PROOF. – To prove part a) note that by assumption, v is continuous, hence on $\bar{J} = [0, T]$ uniformly continuous. Therefore by construction, $\|v - \lambda_n^i v\|_{L^\infty(J,V)} \to 0$ for $n \to \infty$. This proves $\lambda_n^i v \to v$ in $L^r(J, V)$ $(n \to \infty)$ for any $1 \leqslant r \leqslant \infty$.

Part b) follows from the subsequent Lemma 3.3 applied to the Bochner integrable v'. ∎

LEMMA 3.3. – *Let $1 \leqslant r < \infty$. Then the linear operators $\mu_n: L^r(J, V) \to L^r(J, V)(n \in \mathbb{N})$ are uniformly bounded with norm 1 and $\mu_n v \to v$ in $L^r(J, V)$ as $n \to \infty$ for any $v \in L^r(J, V)$.*

PROOF. – 1) To prove the uniform boundedness of the operators μ_n we estimate

$$\int_0^T \|\mu_n v\|^r dt = \sum_{j=1}^{N_n} (t_n^j - t_n^{j-1}) \left\| \frac{1}{t_n^j - t_n^{j-1}} \int_{t_n^{j-1}}^{t_n^j} v(s)\, ds \right\|^r$$

from above in the following way:

$$\sum_{j=1}^{N_n} \frac{1}{(t_n^j - t_n^{j-1})^{r-1}} \left\| \int_{t_n^{j-1}}^{t_n^j} v(s)\, ds \right\|^r \leq \sum_{j=1}^{N_n} \frac{1}{(t_n^j - t_n^{j-1})^{r-1}} \left[\int_{t_n^{j-1}}^{t_n^j} \|v(s)\|\, ds \right]^r,$$

which in the case $r = 1$ equals $\|v\|_{L^1(J, V)}$, whereas for $r > 1$ we continue to estimate by the classical Hölder inequality $(1/r + 1/r' = 1)$,

$$\leq \sum_{j=1}^{N_n} \frac{1}{(t_n^j - t_n^{j-1})^{r-1}} [\|v\|_{L^r((t_n^{j-1}, t_n^j), V)} (t_n^j - t_n^{j-1})^{1/r'}]^r$$

$$= \|v\|^r_{L^r(J, V)}.$$

Since $\mu_n v = v$ for any constant function v, $\|\mu_n\| = 1$ follows.

2) In the case $r = 1$ we directly prove the claimed convergence statement for any step function v, what suffices in view of density and of the uniform boundedness of the operators μ_n. Thus let $v = \sum_{l=1}^{m} v_l \chi_{I_l}$ be a given left-continuous step function, where $m \in \mathbb{N}$ is fixed and χ_{I_l} are the characteristic functions of the intervals I_l for $l = 1, \ldots, m$ that give rise to a fixed partitioning of $\bar{J}$. Since $k_n \to 0$, for large enough $n \in \mathbb{N}$ any subinterval $(t_n^{(j-1)}, t_n^j]$ of π_n contains at most one jump of v. This means that for any $l \in \{1, \ldots, m-1\}$ there exist unique $j \in \{1, \ldots, N_n\}$, $\alpha \in (0, 1]$ such that for $t \in (t_n^{j-1}, t_n^j]$

$$v(t) = \begin{cases} v_l & \text{if } t - t_n^{j-1} \leq \alpha(t_n^j - t_n^{j-1}); \\ v_{l+1} & \text{if } t - t_n^{j-1} > \alpha(t_n^j - t_n^{j-1}). \end{cases}$$

Hence $(\mu_n v)(t) = \alpha v_l + (1-\alpha)\, v_{l+1}$ for $t \in (t_n^{j-1}, t_n^j]$ and thus

$$\int_{t_n^{j-1}}^{t_n^j} \|\mu_n v - v\|\, ds = 2\alpha(1-\alpha)(t_n^j - t_n^{j-1}) \|v_{l+1} - v_l\|$$

$$\leq \frac{1}{2}(t_n^j - t_n^{j-1}) \|v_{l+1} - v_l\|.$$

This gives

$$\int_J \|\mu_n v - v\|\,ds \leq \frac{1}{2} k_n \sum_{l=1}^{m-1} \|v_{l+1} - v_l\| \to 0 \quad (n \to \infty).$$

3) In contrast, the remaining case $1 < r < \infty$ needs several steps.

i) Let $V = \mathbb{R}$, $v \geq 0$. Then $\mu_n v \geq 0$ and the conclusion follows from density and the above approximation argument, which easily modifies to a nonnegative step function in $L^r(J, \mathbb{R})$, or from the reasoning in [19, exercise (7), p. 112].

ii) Let $V = \mathbb{R}$, but no sign restriction on v. Then decompose $v = v^+ - v^-$, $\mu_n v = \mu_n v^+ - \mu_n v^-$ and apply the preceding step.

iii) Let V be an arbitrary reflexive Banach space. Fix some $v^* \in V^*$ and some subinterval I of J. Then for $w = v^* \chi_I$ we have that

$$\langle \mu_n v, w \rangle = \int_I \langle (\mu_n v)(t), v^* \rangle_{V \times V^*} dt = \int_I \mu_n(\langle v, v^* \rangle_{V \times V^*})(t)\, dt$$

$$\to \int_I \langle v(t), v^* \rangle_{V \times V^*} dt = \langle v, w \rangle \ (n \to \infty),$$

since by the preceding step $\mu_n(\langle v, v^* \rangle) \to \langle v, v^* \rangle$ $(n \to \infty)$ in $L^1(J) \subset L^r(J)$. Hence $\langle \mu_n v, w \rangle \to \langle v, w \rangle$ $(n \to \infty)$ for any step function w. In view of the uniform boundedness of μ_n on $L^r(J, V)$ and the density of the step functions in $L^{r'}(J, V^*)(1/r + 1/r' = 1)$ we conclude that $\langle \mu_n v, w \rangle \to \langle v, w \rangle$ for any $w \in L^{r'}(J, V^*)$. Thus weak convergence in $L^r(J, V)$ follows.

iv) In view of weak convergence and weak lower semicontinuity of the norm, the estimate in part 1) of the proof provides convergence of the $L^r(J, V)$ norm of $\mu_n v$ towards the $L^r(J, V)$ norm of v for $n \to \infty$, what together with weak convergence yields the claimed strong convergence. ■

3.2. *Space time discretization.* – In this section, we employ the Galerkin procedure to discretize with respect to the space variable x, in addition, and thus to arrive at finite dimensional, hence computable approximate problems. In particular, we admit nonconforming approximations of the subset K.

To describe the Galerkin procedure, let a sequence $\{X_\nu\}_{\nu \in \mathbb{N}}$ of finite dimensional subspaces of X be given such that $X = \overline{\bigcup_{\nu \in \mathbb{N}} X_\nu}$ with respect to $\|.\|_X$ holds. Likewise we have a sequence $\{H_\nu\}_{\nu \in \mathbb{N}}$ of finite dimensional subspaces of H with $X_\nu \subset H_\nu$ (not necessarily $H_\nu = X_\nu$) such that $H = \overline{\bigcup_{\nu \in \mathbb{N}} H_\nu}$ with respect to $\|.\|_H$ holds. Furthermore we have a sequence $\{K_\nu\}_{\nu \in \mathbb{N}}$ of closed convex nonempty subsets of X_ν. We do not require that K_ν is contained in K (nonconforming approximation). Instead we only assume that $\bigcap_\nu K_\nu \cap K \neq \emptyset$ and follow the approximation procedure of Glowinski [14, p. 9] using the following set convergence concept.

DEFINITION 3.4. – We say that K_ν *G-converges* to $K(\nu \to \infty)$ (shortly $K_\nu \underset{G}{\to} K$), if the following two conditions are satisfied.

(G1) If for some subsequence $\{\nu_i\}_{i \in \mathbb{N}}$, $\nu_i \uparrow \infty$ (as $i \to \infty$), $x_{\nu_i} \in K_{\nu_i}$, $x_{\nu_i} \rightharpoonup x$ (weak convergence in X) as $i \to \infty$, then $x \in K$.

(G2) There exists a subset M dense in K and for all $\nu \in \mathbb{N}$, mappings $r_\nu \colon M \to X_\nu$ with the property that, for each $z \in M$, $r_\nu z \to z$ (strong convergence in X) as $\nu \to \infty$ and $r_\nu z \in K_\nu$ for all $\nu \geq \nu_0(z)$ for some $\nu_0(z) \in \mathbb{N}$.

REMARKS. – If M coincides with K, then G-convergence defined above reduces to *Mosco-convergence*:

$$w - \lim_{\nu \to \infty} \sup K_\nu \subseteq K \subseteq s - \lim_{\nu \to \infty} \inf K_\nu ,$$

introduced by Mosco [29] and investigated independently by Stummel [34]. Actually, both notions are equivalent, as Stummel [34, p. 11/12] already showed; see also the subsequent Lemma 3.7. There is a great variety in concepts of set convergence and recent research in this subject, see the paper of Sonntag and Zalinescu [33] for a survey and classification. Here we stick to G-convergence since this concept provides easily verifiable conditions in finite difference and finite element approximations using well known interpolation results in Sobolev spaces.

Concerning the initial datum $u_0 \in K$ of our problem, we let approximations $u_{0,\nu} \in K_\nu$ be given such that $u_{0,\nu} \to u_0$ in $H(\nu \to \infty)$. In view of the subsequent Lemma 3.7, such approximations $u_{0,\nu} \in K_\nu$ exist. For the right hand side f, we let approximations $f_{n,\nu} \in \mathcal{P}^0_n(J, H_\nu)$ be given, such that $f_{n,\nu} \to f$ in $L^1(J, H) \cap L^{p'}(J, X^*)(n, \nu \to \infty)$; such approximations exist for any f in $L^1(J, H) \cap L^{p'}(J, X^*)$ as shown later in Section 3.3. Similary as in Section 3.1, $f^j_{n,\nu} \in H_\nu \subset X^*$ will denote the constant value of $f_{n,\nu}$ on the part interval $(t^{j-1}_n, t^j_n]$ $(j = 1, \dots, N_n)$.

Thus we are led to the subsequent approximate problem for any $n, \nu \in \mathbb{N}$:

$(\mathrm{P}_{n,\nu})$ Find $q_{n,\nu} = \{q^j_{n,\nu} \colon j = 1, \dots, N_n\}$, $q_j := q^j_{n,\nu} \in K_\nu$ such that for all $y \in K_\nu$

$$\varphi(q_j, y) + \langle (\delta^{u_{0,\nu}}_n q_{n,\nu})_j - f^j_{n,\nu}, y - q_j \rangle_{X^* \times X} \geq 0 .$$

Let us first show the solvability of the approximate problem $(\mathrm{P}_{n,\nu})$.

LEMMA 3.5. – *For any $n, \nu \in \mathbb{N}$, $(\mathrm{P}_{n,\nu})$ admits a unique solution.*

PROOF. – By definition of the finite difference operator δ_n, the variational inequality in (P_n, ν) writes, for any step $j = 1, \dots, N_n$,

$$(3.2) \qquad (t^j_n - t^{j-1}_n)\, \varphi(q_j, y) + \langle q_j, y - q_j \rangle \geq \langle (t^j_n - t^{j-1}_n)\, f^j_{n,\nu} + q^{j-1}, y - q_j \rangle \quad (\forall y \in K_\nu).$$

While the right hand side is (within the step j) a fixed linear functional applied to the argument $y - q_j$, the left hand side defines a uniformly monotone-convex function. Thus by well-known monotonicity arguments, which in the general case of monotone-convex functions can be found in [32], solutions $q_j \in K_\nu$ exist for all steps $j = 1, \dots, N_n$.

To prove uniqueness, let p_j be another solution in step j and consider $e_j := p_j - q_j$. We proceed by induction on j, plug in $y = p_j$, respectively $y = q_j$ in the variational inequalities for q_j, p_j respectively, add up, and by monotonicity of φ obtain

$$-\|e_j\|_H^2 = -\langle e_j, e_j\rangle_{X^* \times X} \geq 0$$

to conclude $e_j = 0$.

Starting from the discrete values $q_{n,\nu}^j = q_j$ with the initial value $q_{n,\nu}^0 = u_{0,\nu}$, we can construct the piecewise constant approximation («brick» function)

$$\bar{q}_{n,\nu} = \lambda_n^0 \{q_{n,\nu}^j\}_{j=1}^{N_n} \in \mathcal{P}_n^0(J, K_\nu) \qquad \text{where } \bar{q}_{n,\nu}(0) = q_{n,\nu}^0$$

and the piecewise linear approximation («hat» function)

$$\hat{q}_{n,\nu} = \lambda_n^1 \{q_{n,\nu}^j\}_{j=0}^{N_n} \in \mathcal{P}_n^1[J, K_\nu].$$

Note that contrary to [15], we have

$$\bar{q}_{n,\nu} = \sum_{j=1}^{N_n} q_{n,\nu}^j \chi_n^j,$$

where χ_n^j denotes the characteristic function of the interval $]t_n^{j-1}, t_n^j]$ $(j = 1, \ldots, N_n)$. Although the approximations $\bar{q}_{n,\nu}$ and $\hat{q}_{n,\nu}$ being different, they shall give rise to the same limit; more precisely, we shall show in the subsequent section that

$$\lim_{n,\nu \to \infty} \|\bar{q}_{n,\nu} - \hat{q}_{n,\nu}\|_{L^2(J,H)} = 0\,.$$

It is obvious from integration that the discrete variational inequality in $(\mathrm{P}_{n,\nu})$ is equivalent to

$$\Phi(\bar{q}_{n,\nu}, v) + \int_0^T \langle \delta_n^{u_{0,\nu}} \bar{q}_{n,\nu} - f, v - \bar{q}_{n,\nu}\rangle\, dt \geq 0 \quad (\forall v \in \mathcal{P}_n^0(J, K_\nu))\,,$$

which is the counterpart in discretization to the variational inequality in (P1) in the continuous case.

Similarly to the relaxed problem (P2) in the continuous case, we formulate a relaxation of $(\mathrm{P}_{n,\nu})$ that is needed in the convergence analysis to follow.

LEMMA 3.6. – *Let $\bar{q}_{n,\nu}$ be the solution of $(\mathrm{P}_{n,\nu})$. Then for any $v \in \mathcal{P}_n^0(J, K_\nu)$ with $v_0 = v(0)$, there hold the inequalities*

$$\Phi(\bar{q}_{n,\nu}, v) + \int_0^T \langle \delta_n^{v_0} v - f, v - \bar{q}_{n,\nu}\rangle\, dt \geq -\frac{1}{2}\|v_0 - u_{0,\nu}\|_H^2, \tag{3.3}$$

$$\Phi(v, \bar{q}_{n,\nu}) + \int_0^T \langle \delta_n^{v_0} v - f, \bar{q}_{n,\nu} - v\rangle\, dt \leq \frac{1}{2}\|v_0 - u_{0,\nu}\|_H^2. \tag{3.4}$$

PROOF. – Similar to [15, p. 423/424] we use the simple relation (omitting the suffix H in the Hilbert space norm $\|.\|_H$)

$$\langle a-b, a\rangle = \frac{1}{2}\|a\|^2 - \frac{1}{2}\|b\|^2 + \frac{1}{2}\|a-b\|^2.$$

Hence we have the identity for $v = \sum_{j=1}^{N_n} v_j \chi_n^j$, $q_j := q_{n,\nu}^j$ $(j = 1, \dots, N_n)$

$$\langle v_j - v_{j-1}, v_j - q_j\rangle - \langle q_j - q_{j-1}, v_j - q_j\rangle =$$

$$= \frac{1}{2}\|v_j - q_j\|^2 - \frac{1}{2}\|v_{j-1} - q_{j-1}\|^2 + \frac{1}{2}\|(v_j - v_{j-1}) - (q_j - q_{j-1})\|^2.$$

By division by $t_n^j - t_n^{j-1}$, we obtain for the solution q_j of $(P_{n,\nu})$, for any $j = 1, \dots, N_n$ with $f_j := f_{n,\nu}^j$

$$\varphi(q_j, v_j) + \langle (\delta_n^{v_0} v)_j - f_j, v_j - q_j\rangle \geq$$

$$\geq \frac{1}{2(t_n^j - t_n^{j-1})}\{\|v_j - q_j\|^2 - \|v_{j-1} - q_{j-1}\|^2 + \|(v_j - v_{j-1}) - (q_j - q_{j-1})\|^2\}.$$

Integration over $]t_n^{j-1}, t_n^j]$ and relaxation lead to

$$\int_{t_n^{j-1}}^{t_n^j} \{\varphi(q, v) + \langle \delta_n^{v_0} v - f, v - q\rangle\}\, dt$$

$$\geq \frac{1}{2}\|v_j - q_j\|^2 - \frac{1}{2}\|v_{j-1} - q_{j-1}\|^2.$$

Hence by summation on $j = 1, \dots, N_n$, we obtain (3.3). Finally we use monotonicity to arrive at (3.4). ■

3.3. *A basic density Lemma.* – In this subsection we provide the basic tool in our convergence analysis. This density lemma is analogous to Lemma 3.2 in [15, p. 418], which covers only the case $p = 2$ and is stated there without proof. Our result also extends the analogous Lemma 1.2 in [16]; however, our proof is different from that in [16], here we use an approximation argument due to Stummel [34].

LEMMA 3.7. – *For any given* $v \in W^p(J, K)$, *there exist a subsequence* $\{\pi_{n_\nu}\}_{\nu \in \mathbb{N}}$ *of partitions, sequences* $\{a_\nu\}_{\nu \in \mathbb{N}} \subset X$ *and* $\{v_\nu\}_{\nu \in \mathbb{N}} \subset L^\infty(J, X)$ *such that*

$$a_\nu \in K_\nu (\nu \in \mathbb{N}), \qquad a_\nu \to a := v(0) \text{ in } H (\nu \to \infty);$$

$$v_\nu \in \mathcal{P}_{n_\nu}^0(J, K_\nu)(\nu \in \mathbb{N}), \qquad v_\nu \to v \text{ in } L^p(J, X)\ (\nu \to \infty);$$

$$\delta_{n_\nu}^{a_\nu} v_\nu \to v' \text{ in } L^{p'}(J, X^*)\ (\nu \to \infty).$$

PROOF. – Let $v \in W^p(J, K)$ be given, let $a := v(0)$. In virtue of Lemma 3.2, both $\lambda_n^0 v \in \mathcal{P}_n^0(J, K)$ and $\lambda_n^1 v \in \mathcal{P}_n^1[J, K]$ converge strongly to v in $L^p(J, X)$, moreover $\delta_n^a v = \delta_n^a \lambda_n^0 v = \delta_n^a \lambda_n^1 v = \partial_t \lambda_n^1 v = \mu_n \partial_t v \to \partial_t v$ (strongly) in $L^{p'}(J, X^*)$. In the following we construct different approximations to the values $v(t_n^j)$ $(j = 1, \ldots, N_n)$ leading simultaneously to a piecewise constant approximation (as needed by the claim of the lemma) and a piecewise linear approximation to v, where differentiating the latter provides an approximation to $\partial_t v$. The special initial value $v(0)$, moreover, will be approximated in H.

To begin with, fix $l_0 \in \mathbb{N}$ such that $\max\{T, T^{1/p'}\} < 2^{l_0 - 3}$. By the convergence properties stated above, for any $l \in \mathbb{N}$ we find a partition π_{n_l} of $\bar{J} = [0, T]$ such that $\lambda_{n_l}^0 v \in \mathcal{P}_{n_l}^0(J, K)$ and $\lambda_{n_l}^1 v \in \mathcal{P}_{n_l}^1[J, K]$ satisfy

$$\max\{\|v - \lambda_{n_l}^0 v\|_{L^p(J, X)}, \|v - \lambda_{n_l}^1 v\|_{L^p(J, X)}, \|\partial_t v - \partial_t \lambda_{n_l}^1 v\|_{L^{p'}(J, X^*)}\} \leq \frac{1}{2^{l+3}} .$$

Since by (G2) M is a dense subset of $K \subset X$, for the values $v(t_{n_l}^j) \in K$ $(j = 0, 1, \ldots, N_{n_l})$, we find $z_{n_l}^j \in M$ $(j = 0, 1, \ldots, N_{n_l})$ such that

$$\|v(t_{n_l}^j) - z_{n_l}^j\|_X \leq \frac{1}{2^{l+l_0}} ,$$

and in view of $X \subset H \subset X^*$ and recalling $k_n \to 0$ such that also

$$\|v(0) - z_{n_l}^0\|_H \leq \frac{1}{2^{l+2}} , \qquad \|v(t_{n_l}^j) - z_{n_l}^j\|_{X^*} \leq \frac{\tau k_{n_l}}{2^{l+l_0+1}} .$$

Thus for $w_{n_l} := \lambda_{n_l}^0 \{z_{n_l}^j\}_{j=1}^{N_{n_l}} \in \mathcal{P}_{n_l}^0(J, M)$ and for $\widehat{w}_{n_l} := \lambda_{n_l}^1 \{z_{n_l}^j\}_{j=0}^{N_{n_l}} \in \mathcal{P}_{n_l}^1(J, \operatorname{co} M)$, where co M denotes the convex hull of M, we obtain

$$\max\{\|\lambda_{n_l}^0 v - w_{n_l}\|_{L^\infty(J, X)}, \|\lambda_{n_l}^1 v - \widehat{w}_{n_l}\|_{L^\infty(J, X)}\} \leq \frac{1}{2^{l+l_0}} ,$$

hence

$$\max\{\|\lambda_{n_l}^0 v - w_{n_l}\|_{L^p(J, X)}, \|\lambda_{n_l}^1 v - \widehat{w}_{n_l}\|_{L^p(J, X)}\} \leq \frac{T}{2^{l+l_0}} < \frac{1}{2^{l+3}} .$$

Furthermore, we estimate

$$\|\partial_t \widehat{w}_{n_l} - \partial_t \lambda_{n_l}^1 v\|_{L^{p'}(J, X^*)}^{p'} = \sum_{j=1}^{N_{n_l}} \frac{\|(z_{n_l}^j - z_{n_l}^{j-1}) - (v(t_{n_l}^j) - v(t_{n_l}^{j-1}))\|_{X^*}^{p'}}{(t_{n_l}^j - t_{n_l}^{j-1})^{p'}} \cdot (t_{n_l}^j - t_{n_l}^{j-1})$$

using $(t_{n_l}^j - t_{n_l}^{j-1}) \geq \tau k_{n_l}$ from above by $(1/2^{l+l_0})^{p'} T$, hence

$$\|\partial_t \widehat{w}_{n_l} - \partial_t \lambda_{n_l}^1 v\|_{L^{p'}(J, X^*)} \leq \frac{T^{1/p'}}{2^{l+l_0}} < \frac{1}{2^{l+3}} .$$

Altogether we obtain approximations $w_{n_l} \in \mathcal{P}^0_{n_l}(J, M)$ that satisfy with $a_{n_l} := \widehat{w}_{n_l}(0) = z^0_{n_l}$, $\delta^{a_{n_l}}_{n_l} w_{n_l} = \partial_t \widehat{w}_{n_l}$

$$\max\{\|v - w_{n_l}\|_{L^p(J,X)}, \|v(0) - a_{n_l}\|_H, \|\partial_t v - \partial_t \widehat{w}_{n_l}\|_{L^{p'}(J,X^*)}\} < \frac{1}{2^{l+2}}.$$

In the next approximation step, we use $M \subset s - \lim\inf_{\nu\to\infty} K_\nu$. Namely by (G2), for any $l \in \mathbb{N}$ fixed, for every $z^j_{n_l}$ ($j = 0, 1, \ldots, N_{n_l}$) there exist $r_\nu z^j_{n_l} \in K_\nu$ for all $\nu \geq \nu_0(l) \in \mathbb{N}$ such that $r_\nu z^j_{n_l} \to z^j_{n_l}$ in X as $\nu \to \infty$. Then similar as above, we construct

$$v^{(\nu)}_{n_l} = \lambda^0_{n_l}\{r_\nu z^j_{n_l}\}^{N_{n_l}}_{j=1} \in \mathcal{P}^0_{n_l}(J, K_\nu) \quad \text{and} \quad \widehat{v}^{(\nu)}_{n_l} = \lambda^1_{n_l}\{r_\nu z^j_{n_l}\}^{N_{n_l}}_{j=0} \in \mathcal{P}^1_{n_l}[J, K_\nu].$$

Again, we have for $\nu \to \infty$

$$\max\{\|v^{(\nu)}_{n_l} - w_{n_l}\|_{L^p(J,X)}, \|\widehat{v}^{(\nu)}_{n_l} - \widehat{w}_{n_l}\|_{L^p(J,X)}\} \to 0,$$

$$\|\widehat{v}^{(\nu)}_{n_l}(0) - \widehat{w}_{n_l}(0)\|_H \to 0,$$

$$\|\partial_t \widehat{v}^{(\nu)}_{n_l} - \partial_t \widehat{w}_{n_l}\|_{L^{p'}(J,X^*)} \to 0.$$

Now in the final step, we adopt a construction due to Stummel [34, p. 12], where in the space $W^p(J, X) \subset C[J, H] \cap L^{p'}(J, X^*)$ we use the equivalent norm

$$|||v||| := \max\{\|v\|_{L^p(J,X)}, \|v(0)\|_H, \|\partial_t v\|_{L^{p'}(J,X^*)}\}$$

and also abbreviate terms like

$$\max\{\|w_{n_l} - v\|_{L^p(J,X)}, \|\widehat{w}_{n_l}(0) - v(0)\|_H, \|\partial_t \widehat{w}_{n_l} - \partial_t v\|_{L^{p'}(J,X^*)}\}$$

simply by $|||w_{n_l} - v|||$.

Let $\kappa_0 = 1$ and for each $l \in \mathbb{N}$ let $\kappa_l \in \mathbb{N}$ be an index with the property

$$\kappa_l > \kappa_{l-1}, \max\{|||v^{(\nu)}_{n_l} - w_{n_l}|||, |||v^{(\nu)}_{n_{l+1}} - w_{n_{l+1}}|||\} \leq \frac{1}{2^{l+2}} \quad \text{for all } \nu \geq \kappa_l.$$

Then for these $\nu \geq \kappa_l$

$$|||v^{(\nu)}_{n_{l+1}} - v^{(\nu)}_{n_l}||| \leq |||v^{(\nu)}_{n_{l+1}} - w_{n_{l+1}}||| + |||w_{n_{l+1}} - v||| + |||v - w_{n_l}||| + |||w_{n_l} - v^{(\nu)}_{n_l}||| <$$

$$< \frac{1}{2^{l+2}} + \frac{1}{2^{l+3}} + \frac{1}{2^{l+2}} + \frac{1}{2^{l+2}} < \frac{1}{2^l}.$$

For each $\nu \in \mathbb{N}$, obviously, there exists a uniquely determined number $i_\nu := i$ such that $\kappa_{i-1} \leq \nu < \kappa_i$. So $u^{(\nu)} := v^{(\nu)}_{n_{i_\nu}} \in \mathcal{P}^0_{n_{i_\nu}}(J, K_\nu)$ and $\widehat{u}^{(\nu)} := \widehat{v}^{(\nu)}_{n_{i_\nu}} \in \mathcal{P}^1_{n_{i_\nu}}[J, K_\nu]$ are well-defined for all $\nu \in \mathbb{N}$.

Now let ε be an arbitrary positive number and choose $l \in \mathbb{N}$ such that $2^{2-l} < \varepsilon$. For every $\nu \geqslant \kappa_l$, we then have for $i := i_\nu$, $\kappa_{i-1} \leqslant \nu < \kappa_i$, hence $l < i$, and

$$|||u^{(\nu)} - v_{n_l}^{(\nu)}||| = |||v_{n_i}^{(\nu)} - v_{n_l}^{(\nu)}||| \leqslant |||v_{n_i}^{(\nu)} - v_{n_{i-1}}^{(\nu)}||| + \ldots + |||v_{n_{l+1}}^{(\nu)} - v_{n_l}^{(\nu)}||| \leqslant$$

$$< \frac{1}{2^{i-1}} + \ldots + \frac{1}{2^l} < \frac{1}{2^{l-1}} < \frac{\varepsilon}{2}.$$

Finally for all $\nu \geqslant \kappa_l$,

$$|||v - u^{(\nu)}||| \leqslant |||v - w_{n_l}||| + |||w_{n_l} - v_{n_l}^{(\nu)}||| + |||v_{n_l}^{(\nu)} - u^{(\nu)}|||$$

$$< \frac{1}{2^{l+2}} + \frac{1}{2^{l+2}} + \frac{\varepsilon}{2} < \varepsilon .$$

This proves that $u^{(\nu)} \in \mathcal{P}_{n_{i_\nu}}^0(J, K_\nu)$, $a_\nu := \widehat{u}^{(\nu)}(0) \in K_\nu$ and $\delta_{n_{i_\nu}}^{a_\nu} u^{(\nu)} = \partial_t \widehat{u}^{(\nu)}$ satisfy the desired approximation properties. ■

REMARK. – The proof shows that for any given strongly monotone number sequences $\{n_k\}_{k\in\mathbb{N}}$, $\{\nu_k\}_{k\in\mathbb{N}}$ with $n_k \to \infty$, $\nu_k \to \infty$ as $k \to \infty$, there exist n_{k_ι} for $\iota = \nu_k$, $a^{(k)} \in K_{\nu_k}$, and $v^{(\nu_k)} \in \mathcal{P}_{n_{k_\iota}}^0(J, K_{\nu_k})$ that have the claimed convergence properties.

Since in Theorem 3.15, the central result of our discretization theory, we require for the approximations $f_{n,\nu}$ that $f_{n,\nu} \to f$ in $L^{p'}(J, X^*)$, also in $L^1(J, H)$, respectively in $L^2(J, H)$, the subsequent addendum is of interest.

ADDENDUM TO LEMMA 3.7. – *Let $f \in L^r(J, H) \cap L^{p'}(J, X^*)$ for some $r \in [1, \infty)$. Then there exists a sequence $\{f_\nu\}_{\nu\in\mathbb{N}} \subset L^\infty(J, H)$ and the subsequence $\{\pi_{n_\nu}\}_{\nu\in\mathbb{N}}$ of Lemma 3.7 can be constructed such that in addition,*

$$f_\nu \in \mathcal{P}_{n_\nu}^0(J, H_\nu)(\nu \in \mathbb{N}), \qquad f_\nu \to f \text{ in } L^r(J, H) \cap L^{p'}(J, X^*) \quad (\nu \to \infty).$$

PROOF. – According to Lemma 3.3, $\mu_n f \to f$ in $L^r(J, H) \cap L^{p'}(J, X^*)$ as $n \to \infty$. Further by construction of the subspaces H_ν, $(\nu \in \mathbb{N})$, for any $z \in H$ there exist $z_\nu \in H_\nu$ such that $z_\nu \to z$ in H. Since the embedding $H \subset X^*$ is continuous, it follows that $z_\nu \to z$ also in X^*. Now the proof of Lemma 3.7 modifies to construct a subsequence of $\{\mu_n f\}$ that satisfies the claimed convergence properties in addition. ■

3.4. *Stability*. – To obtain a priori estimates and thus to prove the desired convergence properties, we need additional assumptions on the nonlinear function φ and on the right hand side f.

Let us recall that there is some $z_0 \in K$ such that $z_0 \in K_\nu$ for all $\nu \in \mathbb{N}$. In view of our applications, either simply $z_0 = 0$ in the case of homogeneous obstacles (constraints) or we can choose z_0 as a large (respectively small) enough constant function, if the given obstacle is not homogeneous. Since the function φ represents the variational form associated to an elliptic differential operator, it is no restriction of generality to assume that $\varphi(z_0, \cdot) \equiv 0$ and hence by monotonicity, $\varphi(\cdot, z_0) \leqslant 0$.

LEMMA 3.8. – *Suppose that* $f_{n,\nu} \to f$ *in* $L^1(J, H)$ $(n, \nu \to \infty)$. *Then* $\bar{q}_{n,\nu}$ *remains in a bounded set of* $L^\infty(J, H)$, *and moreover there is a constant* $\tilde{c}$ *such that for all* $n, \nu \in \mathbb{N}$

$$\sum_{j=1}^{N_n} \|q^j_{n,\nu} - q^{j-1}_{n,\nu}\|^2_H \leq \tilde{c}.$$

PROOF. – The choice $y = z_0$ in the variational inequality of $(\mathrm{P}_{n,\nu})$ and $\varphi(q^j_{n,\nu}, z_0) \leq 0$ (as discussed above) entail for $j = 1, \ldots, N_n$, $\underline{q}^j := q^j_{n,\nu} - z_0$

$$\langle (\underline{q}^j - \underline{q}^{j-1}) - (t^j_n - t^{j-1}_n) f^j_{n,\nu}, \underline{q}^j \rangle \leq 0 .$$

Using the simple identity (suppressing the suffix H in the Hilbert norm $\|.\|_H$)

$$\langle a - b, a \rangle = \frac{1}{2}\{\|a\|^2 - \|b\|^2 + \|a - b\|^2\}$$

provides

$$\frac{1}{2}\{\|\underline{q}^j\|^2 - \|\underline{q}^{j-1}\|^2 + \|\underline{q}^j - \underline{q}^{j-1}\|^2\} \leq (t^j_n - t^{j-1}_n)\langle f^j_{n,\nu}, \underline{q}^j \rangle .$$

Summation from $j = 1$ to $j = i$ leads to

$$\|\underline{q}^i\|^2 + \sum_{j=1}^{i} \|\underline{q}^j - \underline{q}^{j-1}\|^2 \leq \|u_{0,\nu} - z_0\|^2 + 2 \max_{k=1,\ldots,N_n} \|\underline{q}^k\| \sum_{j=1}^{N_n} (t^j_n - t^{j-1}_n)\|f^j_{n,\nu}\| .$$

Since by assumption, $u_{0,\nu} \to u_0$ in H and $f_{n,\nu} \to f$ in $L^1(J, H)$, we conclude from this estimate that $\max_i \|\underline{q}^i\|$, hence $\|\bar{q}_{n,\nu}\|_{L^\infty(J,H)}$ and also $\sum_j \|q^j_{n,\nu} - q^{j-1}_{n,\nu}\|^2$ remain bounded for $n, \nu \to \infty$, as claimed. ■

An easy consequence of the boundedness of $\sum_j \|q^j_{n,\nu} - q^{j-1}_{n,\nu}\|^2_H$ is the following convergence estimate for $\bar{q}_{n,\nu} - \hat{q}_{n,\nu}$:

$$\|\bar{q}_{n,\nu} - \hat{q}_{n,\nu}\|^2_{L^2(J,H)} = \sum_{j=1}^{N_n} \|\bar{q}_{n,\nu} - \hat{q}_{n,\nu}\|^2_{L^2(]t^{j-1}_n, t^j_n[, H)} = \frac{1}{3}\sum_{j=1}^{N_n} (t^{j-1}_n - t^j_n)^2 \|q^{j-1}_{n,\nu} - q_{n,\nu}{}^j\|^2_H \leq$$

$$\leq \frac{k_n^2}{3} \sum_{j=1}^{N_n} \|q^{j-1}_{n,\nu} - q^j_{n,\nu}\|^2_H \leq \frac{\tilde{c}}{3} k_n^2 \to 0 .$$

Further to obtain norm boundedness of the approximations $\bar{q}_{n,\nu}$ in the «energy» space $L^p(J, X)$, we can use the coercivity that results from the uniform monotonicity of the functional φ. Thus we have the following result.

LEMMA 3.9. – *Suppose in addition, that φ is uniformly monotone, that is, there exists a constant c_m such that*

$$c_m \|y - z\|_X^p \leq -\{\varphi(y, z) + \varphi(z, y)\}, \qquad \forall y, z \in X. \tag{3.5}$$

Then the approximations $\bar{q}_{n,\nu}$ remain bounded in $L^p(J, X)$.

PROOF. – From the variational inequality in $(\mathrm{P}_{n,\nu})$, by $\varphi(z_0, \cdot) \equiv 0$ and by (3.5), we have for $j = 1, \ldots, N_n$; $\underline{q}_{n,\nu}^j := q_{n,\nu}^j - z_0$

$$c_m(t_n^j - t_n^{j-1})\|q_{n,\nu}^j - z_0\|^p \leq -(t_n^j - t_n^{j-1})\,\varphi(q_{n,\nu}^j, z_0) \leq$$
$$\leq \langle (t_n^j - t_n^{j-1})\, f_{n,\nu}^j - (\underline{q}_{n,\nu}^j - \underline{q}_{n,\nu}^{j-1}),\, \underline{q}_{n,\nu}^j \rangle.$$

Hence by summation,

$$c_m \|\bar{q}_{n,\nu} - z_0\|_{L^p(J,X)}^p \leq \|\bar{q}_{n,\nu} - z_0\|_{L^\infty(J,H)} \|f_{n,\nu}\|_{L^1(J,H)} + \frac{1}{2}\|u_{0,\nu} - z_0\|_H^2,$$

from which by Lemma 3.8 the conclusion follows. ∎

In some applications (see the concluding section on the p-harmonic Signorini problem) the hypothesis of uniform monotonicity with respect to the norm $\|.\|_X$ is too strong. Therefore it is useful to obtain stability also in the *semicoercive* case as follows.

LEMMA 3.10. – *Suppose in addition that in X there is given a seminorm $[\cdot]$ such that*

$$\exists \beta > 0, \qquad \gamma > 0 \text{ such that } [x] + \gamma\|x\|_H \geq \beta\|x\|_X, \qquad \forall x \in X, \tag{3.6}$$

and that φ is uniformly monotone with respect to the seminorm $[.]$, that is,

$$\exists \alpha > 0 \text{ such that } \varphi(x, y) + \varphi(x, y) \leq -\alpha[x - y]^p, \qquad \forall x, y \in X. \tag{3.7}$$

Moreover let $f_{n,\nu} \to f$ in $L^1(J, H) \cap L^{p'}(J, X^*)$. Then the approximates $\bar{q}_{n,\nu}(n, \nu \in \mathbb{N})$ are bounded in $L^p(J, X)$.

PROOF. – The proof of Lemma 3.9 has to be modified. Here from the variational inequality in $(\mathrm{P}_{n,\nu})$, we have by (3.7) for $j = 1, \ldots, N_n$

$$\alpha(t_n^j - t_n^{j-1})[\underline{q}_{n,\nu}^j]^p \leq (t_n^j - t_n^{j-1})\langle f_{n,\nu}^j, \underline{q}_{n,\nu}^j\rangle_{X^* \times X} - \langle \underline{q}_{n,\nu}^j - \underline{q}_{n,\nu}^{j-1}, \underline{q}_{n,\nu}^j\rangle_{H \times H}.$$

Hence by summation till t^i, $i \leq N_n$ we obtain as in the proof of Lemma 3.9

$$\alpha \int_0^{t^i} [\underline{q}_{n,\nu}]^p\, dt + \frac{1}{2}\|\underline{q}_{n,\nu}^i\|_H^2 \leq \frac{1}{2}\|u_{0,\nu} - z_0\|_H^2 + \int_0^{t^i} \langle f_{n,\nu}, \underline{q}_{n,\nu}\rangle_{X^* \times X}\, dt, \tag{3.8}$$

where $\underline{q}_{n,\nu} = \bar{q}_{n,\nu} - z_0 \in \mathcal{P}_n^0(J, X)$. The latter integral can be estimated similarly as in

[25, Chap. 2.1.4., p. 163] as follows. We have

$$\int_0^{t^i} \langle f_{n,\nu}, \underline{q}_{n,\nu} \rangle_{X^* \times X} dt \leq \left(\int_0^{t^i} \| f_{n,\nu} \|_{X^*}^{p'} dt \right)^{1/p'} \left(\int_0^{t^i} \| \underline{q}_{n,\nu} \|_X^p dt \right)^{1/p},$$

using (3.6) and the boundedness of $f_{n,\nu}$

$$\leq c_1 \left\{ \left(\int_0^{t^i} [\underline{q}_{n,\nu}]^p dt \right)^{1/p} + \left(\int_0^{t^i} \| \underline{q}_{n,\nu} \|_H^p dt \right)^{1/p} \right\}$$

$$\leq \frac{\alpha}{2} \int_0^{t^i} [\underline{q}_{n,\nu}]^p dt + c_2(\alpha, p, c_1) + \frac{c_1}{2} \left\{ 1 + \left(\int_0^{t^i} \| \underline{q}_{n,\nu} \|_H^p dt \right)^{2/p} \right\}$$

by Young's inequality and its special form $2A \leq 1 + A^2$. Therefore (3.8) implies

$$\int_0^{t^i} [\underline{q}_{n,\nu}]^p dt + \| \underline{q}_{n,\nu}^i \|_H^2 \leq c_3 + c_3 \left(\int_0^{t^i} \| \underline{q}_{n,\nu} \|_H^p dt \right)^{2/p}. \tag{3.9}$$

This gives for any $t \in [0, T]$

$$\| \underline{q}_{n,\nu}(t) \|_H^2 \leq c_4 \max \left(1, \int_0^t \| \underline{q}_{n,\nu} \|_H^p ds \right)^{2/p},$$

hence

$$\| \underline{q}_{n,\nu}(t) \|_H^p \leq c_5 + c_5 \int_0^t \| \underline{q}_{n,\nu} \|_H^p ds .$$

Therefore by Gronwall's inequality the boundedness of $\| \underline{q}_{n,\nu} \|_H$ in $L^\infty(J)$ follows, what by (3.9) implies the boundedness of $[\underline{q}_{n,\nu}]$ in $L^p(J)$. Altogether by (3.6) we obtain the stability result. ■

Finally in this section, we want to refine the boundedness result of Lemma 3.8 and obtain the boundedness of the difference quotients. To achieve this goal, we need further assumptions. One way is by imposing higher regularity in the initial data u_0 – an argument of this kind is given by Wei [35] in his study of the p-harmonic parabolic problem in the simpler case of *linear* time-independent boundary conditions. Instead, similar to the work of Liu and Barrett (see e.g. [27]), we exploit the variational structure in our application to the p-harmonic Signorini parabolic problem. Therefore we take here for granted that our function φ arises from the subgradient operator F of a continuous convex function $\mathcal{F}: X \to \mathbb{R}$ (see Example 2.2); in other words, we assume the existence of a continuous convex potential $\mathcal{F}$. Then convex analysis tells us the simple inequality

$$\varphi(x, y) + \mathcal{F}(x) \leq \mathcal{F}(y) \quad \text{for all } x, y \in X. \tag{3.10}$$

There is no restriction of generality to assume in addition that $\mathcal{F}$ is nonnegative. Thus we arrive at the following stability result.

LEMMA 3.11. – *Suppose* $u_{0,\nu} \to u_0 (\nu \to \infty)$ *in* X *and* $f_{n,\nu} \to f$ *in* $L^2(J, H)(n, \nu \to \infty)$. *Moreover suppose that there exists a convex continuous potential* $\mathcal{F}\colon X \to \mathbb{R}_+$ *for the function* φ. *Then* $\partial_t \widehat{q}_{n,\nu}$ *and* $\mathcal{F}(q^j_{n,\nu})$ *are bounded in* $L^2(J, H)$, *respectively in* $\mathbb{R}_+$.

PROOF. – We test the j-th discrete variational inequality in $(\mathrm{P}_{n,\nu})$ by $q^{j-1}_{n,\nu} \in K_\nu$ for $j = 1, \ldots, N_n$, sum from $j = 1$ to $j = m (m = 1, \ldots, N_n)$, insert $q^0_{n,\nu} = u_{0,\nu}$, and derive from (3.10) the telescoping sum estimate

$$\sum_{j=1}^{m} \varphi(q^j_{n,\nu}, q^{j-1}_{n,\nu}) \leq \sum_{j=1}^{m} \{\mathcal{F}(q^{j-1}_{n,\nu}) - \mathcal{F}(q^j_{n,\nu})\} = \mathcal{F}(u_{0,\nu}) - \mathcal{F}(q^m_{n,\nu}).$$

Thus when writing

$$(\partial_t \widehat{q}_{n,\nu})_j := \partial_t \widehat{q}_{n,\nu} \,|(t^{j-1}_n, t^j_n] = \delta^{u_{0,\nu}}_n q_{n,\nu|(t^{j-1}_n, t^j_n)} = \frac{q^j_{n,\nu} - q^{j-1}_{n,\nu}}{t^j_n - t^{j-1}_n} \qquad (j = 1, \ldots, N_n),$$

we obtain for any $m = 1, \ldots, N_n$

$$\mathcal{F}(q^m_{n,\nu}) + \sum_{j=1}^{m} (t^j_n - t^{j-1}_n) \|(\partial_t \widehat{q}_{n,\nu})_j\|^2_H \leq \mathcal{F}(u_{0,\nu}) + \sum_{j=1}^{m} (t^j_n - t^{j-1}_n) \langle f^j_{n,\nu}, (\partial_t \widehat{q}_{n,\nu})_j \rangle,$$

hence

$$\|\partial_t \widehat{q}_{n,\nu}\|^2_{L^2(J,H)} \leq c(\mathcal{F}(u_0), \|f\|^2_{L^2(J,H)}), \quad \sup_{j=0,\ldots,N_n} \mathcal{F}(q^j_{n,\nu}) \leq c(\mathcal{F}(u_0), \|f\|^2_{L^2(J,H)}). \qquad \blacksquare$$

From Lemma 3.8 and Lemma 3.11 we conclude the following compactness result.

COROLLARY 3.12. – *Under the assumptions of Lemma* 3.11 *the sequence* $\{\widehat{q}_{n,\nu}\}_{n,\nu \in \mathbb{N}}$ *is relatively compact in* $C[\bar{J}, H]$.

PROOF. – Again we suppress the suffix H in the Hilbert space norm $\|.\|_H$. By continuity of the Bochner integral and by Hölder's inequality we obtain for $j = 1, \ldots, N_n$ and for any $t_1, t_2 \in (t^{j-1}_n, t^j_n]$,

$$\|\widehat{q}_{n,\nu}(t_2) - \widehat{q}_{n,\nu}(t_1)\| = \left\| \int_{t_1}^{t_2} \partial_t \widehat{q}_{n,\nu} \, dt \right\| \leq \left| \int_{t_1}^{t_2} \|\partial_t \widehat{q}_{n,\nu}\| dt \right| \leq |t_2 - t_1|^{1/2} \|\partial_t \widehat{q}_{n,\nu}\|_{L^2(J,H)}.$$

Hence by Lemma 3.11, $\{\widehat{q}_{n,\nu}\}$ is equicontinuous in $C[\bar{J}, H]$. Further by Lemma 3.8, $\{\widehat{q}_{n,\nu}\}$ is bounded in $C[\bar{J}, H]$. Thus in virtue of the general Arzela-Ascoli-Bourbaki theorem, $\{\widehat{q}_{n,\nu}\}$ is relatively compact. $\blacksquare$

3.5. *Feasibility.* – In this subsection we deal with another feature of the unilateral constraint in our evolution problem and its full space time discretization. We investigate whether the approximation process $\{q_{n,\nu}\}_{n,\nu\in\mathbb{N}}$ admits limit points that are feasible, i.e. that belong to K almost everywhere on J. We endeavour to obtain such a feasibility result already with weak convergence in $L^p(J, X)$ to include the semicoercive case. Therefore we have to be somewhat more precise about the unilateral constraint.

Let us suppose that $K = g + C$, $K_\nu = g_\nu + C_\nu (\nu \in \mathbb{N})$, where $X_\nu \ni g_\nu \to g$ in $X(\nu \to \infty)$ and C, $C_\nu (\nu \in \mathbb{N})$ are convex closed *cones* with their vertices at zero in X, respectively in X_ν. Then $K_\nu \underset{G}{\to} K$, if and only if C_ν Mosco-converges to C. Further we apply the equivalence of Mosco-convergence of convex lower semicontinuous functions to Mosco-convergence of their Fenchel transforms (see Theorem 3.18 in the monograph of Attouch [2]). Here we consider the indicator function $I_C: X \to \mathbb{R} \cup \{+\infty\}$ of the cone C, given by

$$I_C(x) = \begin{cases} 0 & \text{if } x \in C\,; \\ +\infty & \text{elsewhere}\,, \end{cases}$$

respectively the indicator functions I_{C_ν} of the cones C_ν. We find that the Fenchel transform I_C^* ist simply given by

$$I_C^*(\xi) = \sup\{\langle \xi, x\rangle_{X^* \times X} : x \in C\} = I_{C^-}(\xi)\,,$$

where

$$C^- = \{\xi \in X^* : \langle \xi, x\rangle \leq 0 \ \text{ for all } x \in C\}$$

denotes the *dual cone* of C. Thus we conclude that C_ν Mosco-converges to C, if and only if in the dual space, C_ν^- Mosco-converges to $C^-(\nu \to \infty)$. Based on this equivalence we obtain the following feasibility result.

LEMMA 3.13. – *Suppose that* $K_\nu \underset{G}{\to} K(\nu \to \infty)$, *where* $K = g + C$, $K_\nu = g_\nu + C_\nu (\nu \in \mathbb{N})$; $g_\nu \to g$ *in* X; C, C_ν *are convex closed cones in* X, *respectively in* $X_\nu (\nu \in \mathbb{N})$. *Then any limit point* q *of* $\{\overline{q}_{n,\nu}\}_{n,\nu\in\mathbb{N}}$ (*or of* $\{\hat{q}_{n,\nu}\}_{n,\nu\in\mathbb{N}}$) *with respect to weak convergence in* $L^p(J, X)$ *belongs to* $L^p(J, K)$.

PROOF. – First observe that since X is reflexive and separable, X^*, hence C^- is separable, too. Let $\{\zeta_\iota\}_{\iota\in\mathbb{N}}$ be dense in C^-. Since C_ν^- Mosco-converges to C^-, there exist for any ζ_ι sequences $\{\zeta_{\iota,\nu}\}_{\nu\in\mathbb{N}}$ such that $\zeta_{\iota,\nu} \in C_\nu^- (\nu \in \mathbb{N})$ and $\zeta_{\iota,\nu} \to \zeta_\iota$ in $X^*(\nu \to \infty)$. Hence for any measurable subset A of J, $\zeta_{\iota,\nu}\chi_A \to \zeta_\iota \chi_A$ in $L^{p'}(J, X^*)$.

Now let q be a limit point of $\{\overline{q}_{n,\nu}\}$ with respect to weak convergence in $L^p(J, X)$, say $q = w - \lim_{(n,\nu)\in N} \overline{q}_{n,\nu}$ where $N \subset \mathbb{N} \times \mathbb{N}$ appropriately. Then $q - g = w - \lim_{(n,\nu)\in N} (\overline{q}_{n,\nu} - g_\nu)$ in $L^p(J, X)$. Altogether we have that the reals $\langle \overline{q}_{n,\nu} - g_\nu, \zeta_{\iota,\nu}\chi_A\rangle$ converge to $\langle q - g, \zeta_\iota \chi_A\rangle$ for $(n, \nu) \in N$. Since all these reals are nonpositive by construction, we have that the limits are nonpositive, too, for any measurable subset A of J. Hence there exist null sets E_ι (possibly depending on ζ_ι) such that the $L^1(J)$ functions $\langle q(\cdot) - g, \zeta_\iota\rangle_{X\times X^*} (\iota \in \mathbb{N})$ are nonpositive on $J \backslash E_\iota$. Taking $E = \bigcup_{\iota\in\mathbb{N}} E_\iota$, a null set, and using the density of $\{\zeta_\iota\}_{\iota \in \mathbb{N}}$ in C^- we obtain that $\langle q(\cdot) - g, \zeta \rangle \leq 0$ for all $\zeta \in C^-$ on $J \backslash E$. In other words, $q(\cdot) - g \in C^{--} = C$ by the bipolar theorem or $q(\cdot) \in K$ almost everywhere on J.

The feasibility proof for a weak limit point of $\{\hat{q}_{n,\nu}\}$ is verbatim the same. ■

3.6. *The convergence result.* – In addition to hemicontinuity of the function Φ which suffices in existence theory we need a stronger hypothesis of lower semicontinuity already used in [17] to make the Galerkin procedure work.

DEFINITION 3.14. – The function Φ *satisfies the* (*LSC*) *condition*, if for any sequences $\{v_\nu\}_\nu$ and $\{w_\nu\}_\nu$ such that $v_\nu \to v$ in $L^p(J, X)$ and $w_\nu \rightharpoonup w$ (weak convergence) in $L^p(J, X)$ there holds

$$\Phi(v, w) \leq \lim_{\nu \to \infty} \inf \Phi(v_\nu, w_\nu).$$

Finally we can conclude our convergence analysis by the following result.

THEOREM 3.15. – *Suppose that Φ is hemicontinuous and satisfies the* (*LSC*) *condition; further suppose that φ is either uniformly monotone in the sense of* (3.5) *or uniformly monotone with respect to the seminorm* $[\cdot]$ *in the sense of* (3.6), (3.7). *Suppose that $f_{n,\nu} \to f$ in $L^1(J, H) \cap L^{p'}(J, X^*)(n, \nu \to \infty)$ and that $K_\nu \underset{G}{\to} K(\nu \to \infty)$, where $K = g + C$, $K_\nu = g_\nu + C_\nu (\nu \in \mathbb{N})$; $g_\nu \to g$ in X; C, C_ν are convex closed cones in X, respectively in $X_\nu (\nu \in \mathbb{N})$.*

*A) Then the sequence $\{\bar{q}_{n,\nu}\}_{n,\nu \in \mathbb{N}}$ possesses limit points with respect to weak convergence in $L^p(J, X)$ and with respect to weak*convergence in $L^\infty(J, H)$. Any such limit point belongs to $L^p(J, K)$ and satisfies* (2.4), *the variational inequality of* (P2).

B) Suppose in addition that that the function Φ is continuous with respect to its second argument on $L^p(J, X)$ and admits a convex continuous potential $\mathcal{F}: X \to \mathbb{R}_+$. Moreover suppose that $u_{0,\nu} \to u_0 (\nu \to \infty)$ in X and $f_{n,\nu} \to f$ in $L^2(J, H)(n, \nu \to \infty)$. Then the sequence $\{\hat{q}_{n,\nu}\}_{n,\nu \in \mathbb{N}}$ converges to the unique solution u of (P1) *strongly in $\mathcal{C}[\bar{J}, H]$, and the sequence $\{\bar{q}_{n,\nu}\}_{n,\nu \in \mathbb{N}}$ converges to u strongly in $L^2(J, H)$. Moreover there holds: $\hat{q}_{n,\nu} \rightharpoonup u$ in $L^p(J, X)$ and $\partial_t \hat{q}_{n,\nu} \rightharpoonup \partial_t u$ in $L^2(J, H)$.*

C) If furthermore φ is uniformly monotone in the sense of (3.5), *then $\bar{q}_{n,\nu}$ converges strongly to u as $n, \nu \to \infty$ in $L^p(J, X)$, too.*

PROOF. – Part A) - The claimed existence of limit points is a direct consequence of the stability Lemmata 3.8, 3.9, respectively 3.10. Let q be such a cluster point, that is, $\bar{q}_{n,\nu} \rightharpoonup q$ in $L^p(J, X)$ and $\bar{q}_{n,\nu} \rightharpoonup^* q$ in $L^\infty(J, H)$ for some appropriate subsequence, say $(n, \nu) \in N \subset \mathbb{N} \times \mathbb{N}$. Then by Lemma 3.13, $q \in L^p(J, K)$.

Now let $v \in W^p(J, K)$ arbitrary. Then by Lemma 3.7 we have for some appropriate subsequence $\{n_\nu\}_\nu$, thus $(n, \nu) \in N' \subset N \subset \mathbb{N} \times \mathbb{N}$, $b_\nu := v_{0,\nu} \to v(0)$ in H, $v_{n,\nu} \to v$ in $L^p(J, X)$ and $\delta_n^{b_\nu} v_{n,\nu} \to \partial_t v$ in $L^{p'}(J, X^*)$. We apply Lemma 3.6, insert $v := v_{n,\nu}$ in the relaxed variational inequality (3.4) for $\overline{q}_{n,\nu}$, and obtain

$$\Phi(v_{n,\nu}, \overline{q}_{n,\nu}) + \int_0^T \langle \delta_n^{b_\nu} v_{n,\nu} - f_{n,\nu}, \overline{q}_{n,\nu} - v_{n,\nu}\rangle_{X^* \times X} dt \leq \frac{1}{2} \|v_{0,\nu} - u_{0,\nu}\|_H^2 .$$

Here we use that $u_{0,\nu} \to u_0$, $b_\nu \to v(0)$ both in H; $f_{n,\nu} \to f$ in $L^{p'}(J, X^*)$ and $\overline{q}_{n,\nu} - v_{n,\nu} \rightharpoonup q - v$ in $L^p(J, X)$. Further by the (LSC) condition for Φ, we conclude that q satisfies (2.4).

Part B) - In virtue of Lemma 3.11 the difference quotients $\partial_t \widehat{q}_{n,\nu}$ are bounded in $L^2(J, H)$ and by Corollary 3.12 the sequence $\{\widehat{q}_{n,\nu}\}_{n,\nu \in \mathbb{N}}$ is relatively compact in $C[\overline{J}, H]$. Therefore we obtain the claimed convergence properties for some appropriate subsequence. On the other hand, for any limit point q we have $q \in L^p(J, K) \cap C[\overline{J}, H]$ and $\partial_t q \in L^2(J, H) \subset L^{p'}(J, X^*)$. Hence by Lemma 2.6, any such limit point q is a solution of (P1), and by uniqueness due to Lemma 2.7 the conclusion follows.

Part C) - We apply Lemma 3.7 to obtain approximations $u_{n,\nu} \in \mathcal{P}_n^0(J, K_\nu)$ such that $a_\nu := u_{n,\nu}^0 = u_{n,\nu}(0) \to u_0$ in H, $u_{n,\nu} \to u$ in $L^p(J, X)$ and $\delta_n^{a_\nu} u_{n,\nu} \to \partial_t u$ in $L^{p'}(J, X^*)$. We apply Lemma 3.6 and insert these $u_{n,\nu}$ in the relaxed variational inequality (3.3) for $\overline{q}_{n,\nu}$. By the uniform monotonicity of φ in the sense of (3.5), we obtain

$$0 \leq c_m \|u_{n,\nu} - \overline{q}_{n,\nu}\|^p \leq -\{\Phi(u_{n,\nu}, \overline{q}_{n,\nu}) + \Phi(\overline{q}_{n,\nu}, u_{n,\nu})\} \leq$$

$$\leq -\Phi(u_{n,\nu}, \overline{q}_{n,\nu}) + \int_0^T \langle \delta_n^{a_\nu} u_{n,\nu} - f_{n,\nu}, u_{n,\nu} - \overline{q}_{n,\nu}\rangle_{X^* \times X} dt + \frac{1}{2} \|u_{n,\nu}^0 - u_{0,\nu}\|_H^2 .$$

Taking the lim sup and using the (LSC) condition, we conclude $\|u_{n,\nu} - \overline{q}_{n,\nu}\| \to 0$, hence $\overline{q}_{n,\nu} \to u$ in $L^p(J, X)$. ■

4. - An application to p-harmonic Signorini initial boundary value problems.

Let us address the following nonlinear parabolic initial boundary value problem:

$$\partial_t u - \operatorname{div}(\|\nabla u\|^{p-2} \nabla u) = f \quad \text{in } (0, T) \times \Omega ,$$

$$u = u_0 \quad \text{in } \{0\} \times \Omega ,$$

$$u = g \quad \text{on } (0, T) \times \Gamma_D ,$$

$$u \geq g, \quad \|\nabla u\|^{p-2} \frac{\partial u}{\partial n} \geq 0, \quad (u - g)\|\nabla u\|^{p-2} \frac{\partial u}{\partial n} = 0 \quad \text{on } (0, T) \times \Gamma_S ,$$

where $p > 2$, $T > 0$, $\Omega \subset \mathbb{R}^2$ is a polygonal domain, $\overline{\Gamma}_S \cup \overline{\Gamma}_D = \partial\Omega$, $\Gamma_S \cap \Gamma_D = \emptyset$, $\|\nabla u\|^2 = (\partial_1 u)^2 + (\partial_2 u)^2$ and u_0, f, g are given data. Before we can apply Theorem 3.15 to establish the convergence in the appropriate spaces for full time space discretization including the finite element discretization, we have to discuss several more theoretical issues.

4.1. *Preliminaries.* – By invoking an appropriate Green's formula (see e.g. [5, Chapter 18]) we can see that the variational formulation of the initial boundary value problem considered is the variational inequality (P), where for all $y, z \in X = W^{1,p}(\Omega)$ we define

$$\varphi(y, z) = a(y, z-y), \qquad a(y, z) = \int_\Omega \|\nabla y\|^{p-2} \nabla y \cdot \nabla z \, d\omega ,$$

$$\mathcal{F}(z) = \frac{1}{p} \int_\Omega \|\nabla z\|^p \, d\omega ,$$

$$K = g + C, \qquad C = \{z \in W^{1,p}(\Omega) \mid z \geq 0 \text{ on } \Gamma_S\}.$$

Note that Ω is a Lipschitz domain ($\partial\Omega \in C^{0,1}$) and hence by Sobolev's embedding [31, Theorem 3.8, p. 72] $W^{1,p}(\Omega) \subset \mathcal{C}(\overline{\Omega})$. Therefore the restriction of z on Γ_S is well defined, and with g given in $W^{1,p}(\Omega)$, K is convex closed and C is a convex closed cone.

As a continuous seminorm on X we have here

$$[z] = |z|_{1,p} = \left\{ \int_\Omega \|\nabla z(\omega)\|^p \, d\omega \right\}^{1/p},$$

which is equivalent to the $\|z\|_{1,p}$ norm on X in the case of meas $\Gamma_D > 0$. We fix $H = L^2(\Omega)$ and thus (3.6) holds.

Evidently the defined φ is monotone-convex. Referring to the proof of [10, (5.3.20), Theorem 5.3.3] we have for some positive constant c_1 for all $y, z \in X$

$$-\{\varphi(y, z) + \varphi(z, y)\} = a(z, z-y) - a(y, z-y) \geq c_1 |y-z|_{1,p}^p,$$

and (3.7) holds. Since moreover by Hölder's inequality

$$|\varphi(x, y)| = |a(x, y-x)| \leq c_2 |x|_{1,p}^{p-1} |y-x|_{1,p}, \tag{4.1}$$

Young's inequality implies the growth condition (2.1). Moreover (4.1) shows the continuity of $\Phi(u, \cdot)$ on $L^p(0, T; X)$. Also the hemicontinuity of φ, hence the hemicontinuity of Φ, and also the (*LSC*) condition follow easily.

4.2. *Additional space discretization by finite elements.* – As a finite element discretization of the given bounded polygonal domain $\Omega \subset \mathbb{R}^2$ we choose a triangulation $\mathcal{T}_h$ of Ω, i.e. $\mathcal{T}_h$ is a finite set of triangles T such that

$$T \subset \overline{\Omega} \; (\forall T \in \mathcal{T}_h), \qquad \bigcup_{T \in \mathcal{T}_h} T = \overline{\Omega}$$

$$\mathring{T}_1 \cap \mathring{T}_2 = \emptyset \quad \text{if } T_1 \neq T_2 \;\; (\forall T_1, T_2 \in \mathcal{T}_h).$$

Moreover for all $T_1, T_2 \in \mathcal{T}_h$ with $T_1 \neq T_2$ exactly one of the following statements must hold:

(i) $T_1 \cap T_2 = \emptyset$,

(ii) $T_1 \cap T_2 = \{P\}$ (one common node),

(iii) $T_1 \cap T_2 = \{\overline{PP'}\}$ (one common edge).

As usual $h > 0$ denotes the length of the largest edge of the triangles in the triangulation $\mathcal{T}_h$. In the subsequent convergence analysis we consider a family of triangulations $\{\mathcal{T}_h\}_{h>0}$ with $h \to 0$, which is assumed to be *quasiuniform*, i.e. all the inner angles of all triangles of the triangulation family $\{\mathcal{T}_h\}_h$ are uniformly bounded from below by some $\theta_0 > 0$ as $h \to 0$.

Here we want to study both piecewise linear and piecewise quadratic finite element approximation of X and K. To this end we introduce the space Π_κ of polynomials in two real variables of degree less than or equal to κ ($\kappa = 1, 2$), and the following finite node sets:

$$\mathcal{N}_h = \{P \in \overline{\Omega} \mid P \text{ is a node of } T \in \mathcal{T}_h\},$$

$$N_h = \{P \in \mathcal{N}_h \mid P \in \partial\Omega\},$$

$$\mathcal{N}'_h = \{P \in \overline{\Omega} \mid P \text{ is the midpoint of an edge of } T \in \mathcal{T}_h\},$$

$$N'_h = \{P \in \mathcal{N}'_h \mid P \in \partial\Omega\},$$

$$N_h^\kappa = \begin{cases} N_h & \text{if } \kappa = 1, \\ N_h \cup N'_h & \text{if } \kappa = 2. \end{cases}$$

$$N_{h,S}^\kappa = N_h^\kappa \cap \Gamma_S, \qquad N_{h,D}^\kappa = N_h^\kappa \cap \Gamma_D,$$

where we assume that $\overline{\Gamma}_S \cap \overline{\Gamma}_D \subset N_h$ for all triangulations $\mathcal{T}_h$. Then for $\kappa = 1, 2$, the space X may be approximated by

$$X_h^\kappa = \{z_h \in \mathcal{C}^0(\overline{\Omega}) \mid z_h|_T \in \Pi_\kappa, (\forall T \in \mathcal{T}_h)\}$$

and with g_h^κ, $u_{0,h}^\kappa \in X_h^\kappa$ constructed by interpolation of g, respectively of u_0, K may be approximated by

$$K_h^\kappa = \{z_h \in X_h^\kappa \mid z_h(P) \geq g_h^\kappa(P)\ (\forall P \in N_{h,S}^\kappa),\ z_h(P) = g_h^\kappa(P)\ (\forall P \in N_{h,D}^\kappa)\}.$$

Since the gradient ∇z_h for $z_h \in X_h^\kappa$ exists a.e. on Ω and is bounded, X_h^κ is a finite dimensional subspace of $X = W^{1,p}(\Omega)$. Moreover, the sets $C_h^\kappa = K_h^\kappa - g_h^\kappa$ are closed convex nonempty cones for $\kappa = 1, 2$ and all $h > 0$.

According to Lemma 3.5, there exist for $\kappa = 1, 2$ and for all $h > 0$ unique solutions to the following approximate problems:

($\mathrm{P}_{n,h}^\kappa$) Find $q_{n,h}^\kappa = \{q_{n,h}^{j,\kappa}: j = 1, \ldots, N_n\}$, $q^{j,\kappa} := q_{n,h}^{j,\kappa} \in K_h^\kappa$ such that for all $y \in K_h^\kappa$

$$\varphi(q^{j,\kappa}, y) + \langle (\delta_n^{u_{0,h}^\kappa} q_{n,h}^\kappa)_j - f_{n,h}^j, y - q^{j,\kappa} \rangle_{X^* \times X} \geq 0\,.$$

Since it is known (see [14], [17]) that $K_h^\kappa \underset{G}{\to} K$ for $\kappa = 1, 2$ as $h \to 0$, we arrive at the following result.

COROLLARY 4.1. – *Suppose that* $u_0, g \in W^{1,p}(\Omega)$ *and* $f \in L^2(J, L^2(\Omega)) \cap \cap L^{p'}(J, (W^{1,p}(\Omega))^*)$. *Then for* $\kappa = 1, 2$, *the families* $\{\hat{q}^{\kappa}_{n,h}\}_{n \in \mathbb{N}, h > 0}$ *converge to the unique solution* u *of* (P) *strongly in* $\mathcal{C}[\bar{J}, L^2(\Omega)]$, *and the families* $\{\bar{q}^{\kappa}_{n,h}\}_{n \in \mathbb{N}, h > 0}$ *converge to* u *strongly in* $L^2(J, L^2(\Omega))$. *Moreover there holds:* $\hat{q}^{\kappa}_{n,h} \rightharpoonup u$ *in* $L^p(J, W^{1,p}(\Omega))$ *and* $\partial_t \hat{q}^{\kappa}_{n,h} \rightharpoonup \partial_t u$ *in* $L^2(J, L^2(\Omega))$.

If furthermore meas $\Gamma_D > 0$ *holds, then for* $\kappa = 1, 2$, *the families* $\bar{q}^{\kappa}_{n,h}$ *converge strongly to* u *as* $n \to \infty$, $h \to 0$ *in* $L^p(J, W^{1,p}(\Omega))$, *too.*

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