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A theory of general solutions of 3D problems in 1D hexagonal quasicrystals

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Abstract
A theory of general solutions of three-dimensional (3D) problems is developed for the coupled equilibrium equations in 1D hexagonal quasicrystals (QCs), and two new general solutions, which are called generalized Lekhnitskii–Hu–Nowacki (LHN) and Elliott–Lodge (E–L) solutions, respectively, are presented based on three theorems. As a special case, the generalized LHN solution is obtained from our previous general solution by introducing three high-order displacement functions. For further simplification, considering three cases in which three characteristic roots are distinct or possibly equal to each other, the generalized E–L solution shall take different forms, and be expressed in terms of four quasi-harmonic functions which are very simple and useful. It is proved that the general solution presented by Peng and Fan is consistent with one case of the generalized E–L solution, while does not include the other two cases. It is important to note that generalized LHN and E–L solutions are complete in *z*-convex domains, while incomplete in the usual non-*z*-convex domains.

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1. Introduction

Since the icosahedral quasicrystal (QC) structure was observed in Al–Mn alloys by Shechtman *et al* [1], the electronic structure and the optic, magnetic, thermal and mechanical properties of the material have been extensively investigated in experimental and theoretical analyses [2–5], which show their complex structure and unusual properties. In particular, the field of linear elasticity theory of QCs has been investigated for many years [6–8]. Recently, a generalized Hooke's law of one-dimensional (1D) QCs has been derived [9], which provides us with a fundamental theory based on the notion of a continuum model to describe the elastic behavior of 1D QCs.

Due to the introduction of the phason field, the elastic equations in elasticity of QCs are much more complicated than those in classical elasticity. General solutions of elasticity of QCs are a very effective and convenient tool which helps us to obtain analytic solutions. In recent years, many efforts have been made to seek general solutions of elasticity of QCs. Most of the authors obtained only general solutions of elastic plane problems for QCs [10–12]. For 3D problems of 1D hexagonal QCs, a general solution was given first in terms of four quasi-harmonic functions [13, 14]. Wang [15] presented a general solution for 3D dynamic and static problems through introduction of two displacement functions, which satisfy a quasi-harmonic equation and a sixth-order partial differential equation, respectively. Recently, based on a differential operator matrix method, we constructively obtained a general solution by four functions [16], which satisfy a set of second-order partial differential equations as characterized by a 4×4 differential operator matrix. However, the general solutions [15, 16] are difficult to obtain rigorous analytic solutions and not applicable in most cases, since they satisfy some higher-order equations.

It is the purpose of this paper to continue our previous work [16], and develop a theory of general solutions of 3D elastic problems in 1D hexagonal QCs, which are similar in form to Lekhnitskii–Hu–Nowacki (LHN) [17–19] and Elliott–Lodge (E–L) [20, 21] solutions of transversely isotropic elasticity, and which will be called generalized LHN and E–L solutions of 1D hexagonal QCs, respectively. As a special case, the generalized LHN solution is obtained from our general solution [16] in terms of two theorems. For further simplification, considering three cases in which three characteristic roots are distinct or possibly equal to each other, the generalized E–L solution possesses different forms, and can be expressed in terms of four quasi-harmonic functions.

2. Basic equations and the general solution of 1D hexagonal QCs

3D elasticity of QCs is very complicated, even for 1D QCs, analytic solutions of the problem are difficult to obtain. To solve the boundary value problems of 3D elasticity of QCs, we need to make certain simplifications, so general solutions are studied to simplify complicated equations into a few partial differential equations of lower order. Since general solutions satisfy basic equations of elasticity, such as the deformation geometry equations, the constitutive equations and the equilibrium equations, the boundary value problems of 3D elasticity of QCs are transformed into finding a solution which satisfies the boundary conditions in general solutions.

Assume 1D hexagonal QCs are periodic in the x–y-plane and quasi-periodic in the z-direction in a Cartesian coordinate system (x, y, z). In absence of body force, the static equilibrium equations for 1D hexagonal QCs can be expressed by the following form of matrix equation,

$$\boldsymbol{AU} = 0, \tag{1}$$

in which the vector $\boldsymbol{U} = [u_x, u_y, u_z, w_z]^{\mathrm{T}}$ (the superscript 'T' denotes the transpose) is phonon and phason displacement vector, and $\boldsymbol{A}$ is a 4×4 differential operator matrix, such that

$$\boldsymbol{A} = \begin{bmatrix} \Lambda + \alpha_1\partial_x^2 + \alpha_2\partial_z^2 & \alpha_1\partial_x\partial_y & \alpha_3\partial_x\partial_z & \beta_1\partial_x\partial_z \\ \alpha_1\partial_x\partial_y & \Lambda + \alpha_1\partial_y^2 + \alpha_2\partial_z^2 & \alpha_3\partial_y\partial_z & \beta_1\partial_y\partial_z \\ \alpha_3\partial_x\partial_z & \alpha_3\partial_y\partial_z & \alpha_2\Lambda + \alpha_4\partial_z^2 & \beta_2\Lambda + \beta_3\partial_z^2 \\ \beta_1\partial_x\partial_z & \beta_1\partial_y\partial_z & \beta_2\Lambda + \beta_3\partial_z^2 & \gamma_1\Lambda + \gamma_2\partial_z^2 \end{bmatrix}, \tag{2}$$

where the subscripts $i = 1, 2, 3,\ j = 1, 2,\ k = 1, 2, 3, 4$ and $l = 0, 1, 2, 3$ will be used throughout this paper; $c_{11}, c_{12}, c_{13}, c_{33}, c_{44}, c_{66}$ are elastic constants in the phonon field and $c_{66} = (c_{11} - c_{12})/2$; K_j are elastic constants in the phason field; R_i are phonon–phason coupling elastic constants. The parameters α_k, β_i and γ_j are defined by these elastic constants, i.e.

$$\alpha_1 = \frac{c_{66} + c_{12}}{c_{66}}, \quad \alpha_2 = \frac{c_{44}}{c_{66}}, \quad \alpha_3 = \frac{c_{13} + c_{44}}{c_{66}},$$
$$\alpha_4 = \frac{c_{33}}{c_{66}}, \quad \beta_1 = \frac{R_1 + R_3}{c_{66}}, \quad \beta_2 = \frac{R_3}{c_{66}}, \tag{3}$$
$$\beta_3 = \frac{R_2}{c_{66}}, \quad \gamma_1 = \frac{K_2}{c_{66}}, \quad \gamma_2 = \frac{K_1}{c_{66}},$$

and $\Lambda = \partial_x^2 + \partial_y^2$ is the planar Laplacian.

It is clear that the governing field equations of 1D hexagonal QCs are analogous to those of transversely isotropic piezoelectric materials, with slight difference in their coefficients. Therefore, the methods developed for piezoelectric materials [22–24] were directly applied to establish general solutions of 1D hexagonal QCs [14]. As a classical elastic problem, the fundamental solutions of point phonon and phason forces applied in an infinite QC body are derived based on the general solution.

In the study of general solutions of piezoelectric media, Wang and Zheng [22] first derived the 3D general solutions of piezoelectric materials, but the solutions are restricted to the case of $s_1^2 \neq s_2^2 \neq s_3^2$. For the same problem, Ding *et al* [23] obtained three groups of general solutions, in which the solutions of [22] were included as a special case. For the cases of $s_1^2 \neq s_2^2 = s_3^2$ and $s_1^2 = s_2^2 = s_3^2$, utilizing one group of general solution [23], Ding *et al* also discussed the expressions of the general solutions [24].

By adopting Wang and Wang's technique [25] into elasticity of QCs, we have derived a new kind of general solution of 1D hexagonal QCs by using the theory of differential operator matrix [16]. In the present work, two general solutions of 1D hexagonal QCs will be established in virtue of the analysis technique of QCs [16]. The general solution is obtained by four displacement functions as follows [16]:

$$\begin{aligned}
u_x &= a\nabla_1^2\nabla_2^2\nabla_3^2\varphi_1 - \tilde{a}\tilde{\nabla}_1^2\tilde{\nabla}_2^2\partial_x\left(\partial_x\varphi_1 + \partial_y\varphi_2\right) \\
&\quad + \nabla_0^2\partial_x\partial_z\left[(\beta_1\beta_2 - \alpha_3\gamma_1)\nabla_a^2\varphi_3 + (\alpha_3\beta_2 - \alpha_2\beta_1)\nabla_b^2\varphi_4\right], \\
u_y &= a\nabla_1^2\nabla_2^2\nabla_3^2\varphi_2 - \tilde{a}\tilde{\nabla}_1^2\tilde{\nabla}_2^2\partial_y\left(\partial_x\varphi_1 + \partial_y\varphi_2\right) \\
&\quad + \nabla_0^2\partial_y\partial_z\left[(\beta_1\beta_2 - \alpha_3\gamma_1)\nabla_a^2\varphi_3 + (\alpha_3\beta_2 - \alpha_2\beta_1)\nabla_b^2\varphi_4\right], \\
u_z &= (\beta_1\beta_2 - \alpha_3\gamma_1)\nabla_0^2\nabla_a^2\partial_z\left(\partial_x\varphi_1 + \partial_y\varphi_2\right) + (1 + \alpha_1)\nabla_0^2\nabla_c^2 \\
&\quad \times \left(\gamma_1\nabla_d^2\varphi_3 - \beta_2\nabla_e^2\varphi_4\right) - \beta_1\nabla_0^2\Lambda\partial_z^2\left(\beta_1\varphi_3 - \alpha_3\varphi_4\right), \\
w_z &= (\alpha_3\beta_2 - \alpha_2\beta_1)\nabla_0^2\nabla_b^2\partial_z\left(\partial_x\varphi_1 + \partial_y\varphi_2\right) \\
&\quad + (1 + \alpha_1)\nabla_0^2\nabla_c^2\left(\alpha_2\nabla_f^2\varphi_4 - \beta_2\nabla_e^2\varphi_3\right) \\
&\quad + \alpha_3\nabla_0^2\Lambda\partial_z^2\left(\beta_1\varphi_3 - \alpha_3\varphi_4\right),
\end{aligned} \tag{4}$$

then displacement functions φ_k satisfy the following differential equations of higher order

$$\nabla_0^2\nabla_1^2\nabla_2^2\nabla_3^2\varphi_k = 0, \tag{5}$$

where the quasi-harmonic operators are expressed as

$$\nabla_0^2 = \Lambda + \frac{1}{s_0^2}\partial_z^2, \quad \nabla_i^2 = \Lambda + \frac{1}{s_i^2}\partial_z^2, \quad \tilde{\nabla}_j^2 = \Lambda + \frac{1}{\tilde{s}_j^2}\partial_z^2,$$
$$\nabla_a^2 = \Lambda + \frac{\beta_1\beta_3 - \alpha_3\gamma_2}{\beta_1\beta_2 - \alpha_3\gamma_1}\partial_z^2, \quad \nabla_b^2 = \Lambda + \frac{\alpha_3\beta_3 - \alpha_4\beta_1}{\alpha_3\beta_2 - \alpha_2\beta_1}\partial_z^2, \tag{6}$$
$$\nabla_c^2 = \Lambda + \frac{\alpha_2}{1 + \alpha_1}\partial_z^2, \quad \nabla_d^2 = \Lambda + \frac{\gamma_2}{\gamma_1}\partial_z^2,$$
$$\nabla_e^2 = \Lambda + \frac{\beta_3}{\beta_2}\partial_z^2, \quad \nabla_f^2 = \Lambda + \frac{\alpha_4}{\alpha_2}\partial_z^2.$$

The constant s_0^2 satisfies $s_0^2 = 1/\alpha_2$ and s_i^2 are three characteristic roots (or eigenvalues) of the following cubic algebra equation of s^2

$$as^6 - bs^4 + cs^2 + d = 0, \tag{7}$$

$\tilde{s}_j^2$ are two characteristic roots of the following quadratic algebra equation of $\tilde{s}^2$

$$\tilde{a}\tilde{s}^4 - \tilde{b}\tilde{s}^2 + \tilde{c} = 0. \tag{8}$$

The constants in the preceding equations are

$$\begin{aligned}
a &= \alpha_2\left(\alpha_4\gamma_2 - \beta_3^2\right), \\
b &= \alpha_2\alpha_4\gamma_1 + \left(\alpha_2^2 - \alpha_3^2 + \alpha_1\alpha_4 - \alpha_4\right)\gamma_2 - (1+\alpha_1)\,\beta_3^2 \\
&\quad - 2\alpha_2\beta_2\beta_3 + 2\alpha_3\beta_1\beta_3 - \alpha_4\beta_1^2, \\
c &= \left(\alpha_2^2 - \alpha_3^2 + \alpha_1\alpha_4 + \alpha_4\right) + (1+\alpha_1)\left(\alpha_2\gamma_2 - 2\beta_2\beta_3\right) \\
&\quad - \alpha_2\left(\beta_1^2 + \beta_2^2\right) + 2\alpha_3\beta_1\beta_2, \\
d &= (1+\alpha_1)\left(\alpha_2\gamma_1 - \beta_2^2\right), \\
\tilde{a} &= \left(\alpha_1\alpha_4 - \alpha_3^2\right)\gamma_2 - \alpha_1\beta_3^2 + 2\alpha_3\beta_1\beta_3 - \alpha_4\beta_1^2, \\
\tilde{b} &= \left(\alpha_1\alpha_4 - \alpha_3^2\right)\gamma_1 - \alpha_1(\alpha_2\gamma_2 - 2\beta_2\beta_3) + \alpha_2\beta_1^2 - 2\alpha_3\beta_1\beta_2, \\
\tilde{c} &= \alpha_1\left(\alpha_2\beta_4 - \beta_2^2\right).
\end{aligned} \tag{9}$$

With the introduction and proof of several lemmas and theorems, it can be proved that the above-mentioned general solution is complete in any limited domains in 3D Euclidean space E^3 with no loss of generality [16].

3. Generalized LHN solution of 1D hexagonal QCs

Next, in terms of our general solutions (4) and (5), we will derive a new general solution of 1D hexagonal QCs, which is similar in form to the LHN solution of transversely isotropic elasticity. Based on a differential operator matrix method, we have proved the following theorem [16].

Theorem 1. *If the solution* $\boldsymbol{U}$ *of equation (1) is represented in the same form as equations (4) and (5),* $\boldsymbol{\varphi} = [\varphi_1, \varphi_2, \varphi_3, \varphi_4]^{\mathrm{T}}$ *may be changed into*

$$\hat{\boldsymbol{\varphi}} = \boldsymbol{\varphi} + \boldsymbol{A}\boldsymbol{h}, \tag{10}$$

where $\boldsymbol{h} = [h_1, h_2, h_3, h_4]^{\mathrm{T}}$ *satisfies*

$$\nabla_0^2\nabla_1^2\nabla_2^2\nabla_3^2\boldsymbol{h} = 0, \tag{11}$$

then equation (4) still comes into existence.

From theorem 1, it is easy to verify that φ_k in equations (4) and (5) can be substituted by $\hat{\varphi}_k$ as in equation (10). Specially, taking $\boldsymbol{h} = [0, 0, h_3, 0]^{\mathrm{T}}$ in equation (10), one obtains

$$\begin{aligned}
\hat{\varphi}_1 &= \varphi_1 + \alpha_3\partial_x\partial_z h_3, \\
\hat{\varphi}_2 &= \varphi_2 + \alpha_3\partial_y\partial_z h_3, \\
\hat{\varphi}_3 &= \varphi_3 + \left(\alpha_2\Lambda + \alpha_4\partial_z^2\right)h_3, \\
\hat{\varphi}_4 &= \varphi_4 + \left(\beta_2\Lambda + \beta_3\partial_z^2\right)h_3,
\end{aligned} \tag{12}$$

where

$$\nabla_0^2\nabla_1^2\nabla_2^2\nabla_3^2 h_3 = 0. \tag{13}$$

By utilizing Almansi's theorem [26], the following theorem has been proved strictly [25, 27].

Theorem 2. *Let* Ω *be any limited domain in 3D Euclidean space* E^3*. If the domain* Ω *is z-convex (a z-convex domain is a domain which intersects each line, parallel to the z-axis at an open interval or does not intersect the line), then there always exists* h_3 *such that*

$$\text{Case a: } \hat{\varphi}_j = \hat{\varphi}_j^{(0)} + \hat{\varphi}_j^{(1)} + \hat{\varphi}_j^{(2)} + \hat{\varphi}_j^{(3)}, \quad \text{when } s_0^2 \neq s_1^2 \neq s_2^2 \neq s_3^2; \tag{14a}$$

$$\text{Case b: } \hat{\varphi}_j = \hat{\varphi}_j^{(0)} + z\hat{\varphi}_j^{(1)} + \hat{\varphi}_j^{(2)} + \hat{\varphi}_j^{(3)}, \quad \text{when } s_0^2 = s_1^2 \neq s_2^2 \neq s_3^2; \tag{14b}$$

$$\text{Case c: } \hat{\varphi}_j = \hat{\varphi}_j^{(0)} + z\hat{\varphi}_j^{(1)} + z^2\hat{\varphi}_j^{(2)} + \hat{\varphi}_j^{(3)}, \quad \text{when } s_0^2 = s_1^2 = s_2^2 \neq s_3^2; \tag{14c}$$

$$\text{Case d: } \hat{\varphi}_j = \hat{\varphi}_j^{(0)} + z\hat{\varphi}_j^{(1)} + z^2\hat{\varphi}_j^{(2)} + z^3\hat{\varphi}_j^{(3)}, \quad \text{when } s_0^2 = s_1^2 = s_2^2 = s_3^2. \tag{14d}$$

where $\varphi_j^{(l)}$ *satisfy the following equations*

$$\partial_x\hat{\varphi}_1^{(l)} + \partial_y\hat{\varphi}_2^{(l)} = 0, \qquad \nabla_l^2\varphi_j^{(l)} = 0. \tag{15}$$

(Note: repeated indices do not imply summation in this paper.)

Let

$$\begin{aligned}
A^{(l)} &= \int_{r_0}^{r} \hat{\varphi}_2^{(l)}\mathrm{d}x - \hat{\varphi}_1^{(l)}\mathrm{d}y + B^{(l)}\mathrm{d}z, \\
B^{(l)} &= \int_{r_0}^{r} \partial_z\hat{\varphi}_2^{(l)}\mathrm{d}x - \partial_z\hat{\varphi}_1^{(l)}\mathrm{d}y + s_l^2\left(-\partial_x\hat{\varphi}_2^{(l)} + \partial_y\hat{\varphi}_1^{(l)}\right)\mathrm{d}z,
\end{aligned} \tag{16}$$

where $\boldsymbol{r}_0$ is some point of the region Ω and $\boldsymbol{r}$ is any point of the region Ω. Because of conditions (15), linear integrals of equation (16) are independent of routes.

With the use of theorem 2, we have

$$\begin{aligned}
&\partial_x\nabla_l^2 A^{(l)} = \nabla_l^2\hat{\varphi}_2^{(l)} = 0, \quad \partial_y\nabla_l^2 A^{(l)} = -\nabla_l^2\hat{\varphi}_1^{(l)} = 0, \\
&\partial_z\nabla_l^2 A^{(l)} = \nabla_l^2 B^{(l)} = 0.
\end{aligned} \tag{17}$$

This shows

$$\nabla_l^2 A^{(l)} = 0. \tag{18}$$

In view of equation (16), one yields

$$\hat{\varphi}_1^{(l)} = -\partial_y A^{(l)}, \qquad \hat{\varphi}_2^{(l)} = \partial_x A^{(l)}. \tag{19}$$

Set

$$A = A^{(0)} + A^{(1)} + A^{(2)} + A^{(3)} \text{ in case a;} \tag{20a}$$

$$A = A^{(0)} + zA^{(1)} + A^{(2)} + A^{(3)} \text{ in case b;} \tag{20b}$$

$$A = A^{(0)} + zA^{(1)} + z^2A^{(2)} + A^{(3)} \text{ in case c;} \tag{20c}$$

$$A = A^{(0)} + zA^{(1)} + z^2A^{(2)} + z^3A^{(3)} \text{ in case d.} \tag{20d}$$

Equations (18) and (20) lead to

$$\nabla_0^2\nabla_1^2\nabla_2^2\nabla_3^2 A = 0. \tag{21}$$

From equations (14), (19) and (20), it follows that

$$\hat{\varphi}_1 = -\partial_y A, \quad \hat{\varphi}_2 = \partial_x A. \tag{22}$$

Since it is shown previously that in equations (4) and (5), φ_k can be replaced by $\hat{\varphi}_k$ in equation (12), substitution of equation (22) into equation (4) represented by $\hat{\varphi}_k$ leads to

$$\begin{aligned}
u_x &= -a\nabla_1^2\nabla_2^2\nabla_3^2\partial_y A + \nabla_0^2\partial_x\partial_z \\
&\quad\times\left[(\beta_1\beta_2-\alpha_3\gamma_1)\,\nabla_a^2\hat{\varphi}_3 + (\alpha_3\beta_2-\alpha_2\beta_1)\,\nabla_b^2\hat{\varphi}_4\right],\\
u_y &= a\nabla_1^2\nabla_2^2\nabla_3^2\partial_x A + \nabla_0^2\partial_y\partial_z \\
&\quad\times\left[(\beta_1\beta_2-\alpha_3\gamma_1)\,\nabla_a^2\hat{\varphi}_3 + (\alpha_3\beta_2-\alpha_2\beta_1)\,\nabla_b^2\hat{\varphi}_4\right],\\
u_z &= (1+\alpha_1)\,\nabla_0^2\nabla_c^2\left(\gamma_1\nabla_d^2\hat{\varphi}_3 - \beta_2\nabla_e^2\hat{\varphi}_4\right)\\
&\quad-\beta_1\nabla_0^2\Lambda\partial_z^2\left(\beta_1\hat{\varphi}_3-\alpha_3\hat{\varphi}_4\right),\\
w_z &= (1+\alpha_1)\,\nabla_0^2\nabla_c^2\left(\alpha_2\nabla_f^2\hat{\varphi}_4 - \beta_2\nabla_e^2\hat{\varphi}_3\right)\\
&\quad+\alpha_3\nabla_0^2\Lambda\partial_z^2\left(\beta_1\hat{\varphi}_3-\alpha_3\hat{\varphi}_4\right).
\end{aligned} \tag{23}$$

Equation (23) can be further simplified on setting

$$a\nabla_1^2\nabla_2^2\nabla_3^2 A = \psi_0, \quad \nabla_0^2\hat{\varphi}_3 = F_1, \quad \nabla_0^2\hat{\varphi}_4 = F_2. \tag{24}$$

Utilizing equation (24), the following general solution is derived

$$\begin{aligned}
u_x &= -\partial_y\psi_0 + \partial_x\partial_z[(\beta_1\beta_2-\alpha_3\gamma_1)\,\nabla_a^2 F_1\\
&\quad+(\alpha_3\beta_2-\alpha_2\beta_1)\,\nabla_b^2 F_2],\\
u_y &= \partial_x\psi_0 + \partial_y\partial_z[(\beta_1\beta_2-\alpha_3\gamma_1)\,\nabla_a^2 F_1\\
&\quad+(\alpha_3\beta_2-\alpha_2\beta_1)\,\nabla_b^2 F_2],\\
u_z &= (1+\alpha_1)\,\nabla_c^2\left(\gamma_1\nabla_d^2 F_1 - \beta_2\nabla_e^2 F_2\right)\\
&\quad-\beta_1\nabla_0^2\Lambda\partial_z^2\left(\beta_1 F_1-\alpha_3 F_2\right),\\
w_z &= (1+\alpha_1)\,\nabla_c^2\left(-\beta_2\nabla_e^2 F_1 + \alpha_2\nabla_f^2 F_2\right)\\
&\quad+\alpha_3\nabla_0^2\Lambda\partial_z^2\left(\beta_1 F_1-\alpha_3 F_2\right).
\end{aligned} \tag{25}$$

From equations (5), (21) and (24), ψ_0 and F_j follow that

$$\nabla_0^2\psi_0 = 0, \quad \nabla_1^2\nabla_2^2\nabla_3^2 F_j = 0. \tag{26}$$

4. Generalized E–L solution of 1D hexagonal QCs

Since F_j satisfy a sixth-order partial differential equation, it is difficult to obtain rigorous analytic solutions and not applicable in most cases. To circumvent the difficulties mentioned above, we may take a decomposition and superposition procedure to simplify the high-order partial differential equation in equation (26). Below we introduce a theorem to replace the high-order equation with several quasi-harmonic equations.

Theorem 3. Assume that the domain Ω is z-convex and F_j satisfy

$$\nabla_1^2\nabla_2^2\nabla_3^2 F_j = 0 \quad \text{in}\,\Omega, \tag{27}$$

there exist $F_j^{(i)}$ such that

$$\text{Case 1:}\ F_j = F_j^{(1)} + F_j^{(2)} + F_j^{(3)}, \quad \text{when } s_1^2 \neq s_2^2 \neq s_3^2; \tag{28a}$$

$$\text{Case 2:}\ F_j = F_j^{(1)} + zF_j^{(2)} + F_j^{(3)}, \quad \text{when } s_1^2 = s_2^2 \neq s_3^2; \tag{28b}$$

$$\text{Case 3:}\ F_j = F_j^{(1)} + zF_j^{(2)} + z^2F_j^{(3)}, \quad \text{when } s_1^2 = s_2^2 = s_3^2. \tag{28c}$$

where $F_j^{(i)}$ satisfy the following equations

$$\nabla_i^2 F_j^{(i)} = 0. \tag{29}$$

The preceding theorem has been proven by Wang and Wang [25] and Wang and Shi [27], so it is omitted. Next, we will deduce three different forms of generalized E–L solution, respectively, when $s_1^2 \neq s_2^2 \neq s_3^2$, $s_1^2 = s_2^2 \neq s_3^2$ and $s_1^2 = s_2^2 = s_3^2$.

4.1. Case 1: $s_1^2 \neq s_2^2 \neq s_3^2$

By utilizing this theorem, every solution of the sixth-order partial differential equation (26) can be represented by the solution of equation (29) which is only second-order. Let

$$\begin{aligned}
G_1^{(i)} &= \left(\frac{\alpha_3\gamma_1-\beta_1\beta_2}{s_i^2} + \beta_1\beta_3 - \alpha_1\gamma_2\right)\partial_z^3 F_1^{(i)},\\
G_2^{(i)} &= \left(\frac{\alpha_2\beta_1-\alpha_3\beta_2}{s_i^2} + \alpha_3\beta_3 - \alpha_4\beta_1\right)\partial_z^3 F_2^{(i)}.
\end{aligned} \tag{30}$$

By virtue of equations (7), (29) and (30), the generalized LHN solution (25) can be rewritten as

$$\begin{aligned}
u_x &= -\partial_y\psi_0 + \partial_x\sum_{i=1}^{3}\left(G_1^{(i)}+G_2^{(i)}\right),\\
u_y &= \partial_x\psi_0 + \partial_y\sum_{i=1}^{3}\left(G_1^{(i)}+G_2^{(i)}\right),\\
u_z &= \partial_z\sum_{i=1}^{3}k_{1i}\left(G_1^{(i)}+G_2^{(i)}\right),\\
w_z &= \partial_z\sum_{i=1}^{3}k_{2i}\left(G_1^{(i)}+G_2^{(i)}\right),
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
k_{1i} &= \frac{(1+\alpha_1)\,\gamma_1 - \left[(1+\alpha_1)\,\gamma_2 + \alpha_2\gamma_1 - \beta_1^2\right]s_i^2 + \alpha_2\gamma_2 s_i^4}{(\alpha_3\gamma_1-\beta_1\beta_2)\,s_i^2 + (\beta_1\beta_3-\alpha_3\gamma_2)\,s_i^4},\\
k_{2i} &= -\frac{(1+\alpha_1)\,\beta_2 - [(1+\alpha_1)\,\beta_3 + \alpha_2\beta_2 - \alpha_3\beta_1]\,s_i^2 + \alpha_2\beta_3 s_i^4}{(\alpha_3\gamma_1-\beta_1\beta_2)\,s_i^2 + (\beta_1\beta_3-\alpha_3\gamma_2)\,s_i^4},
\end{aligned} \tag{32}$$

and $G_j^{(i)}$ satisfy

$$\nabla_i^2 G_j^{(i)} = 0. \tag{33}$$

For further simplification, by assuming

$$G_1^{(i)} + G_2^{(i)} = \psi_i. \tag{34}$$

Equation (31) becomes

$$u_x = -\partial_y \psi_0 + \partial_x \sum_{i=1}^{3} \psi_i, \quad u_y = \partial_x \psi_0 + \partial_y \sum_{i=1}^{3} \psi_i,$$
$$u_z = \partial_z \sum_{i=1}^{3} k_{1i} \psi_i, \quad w_z = \partial_z \sum_{i=1}^{3} k_{2i} \psi_i. \tag{35}$$

In view of equations (26), (33) and (34), it can be seen that ψ_0 and ψ_i satisfy the following quasi-harmonic equations

$$\nabla_0^2 \psi_0 = 0, \qquad \nabla_i^2 \psi_i = 0. \tag{36}$$

4.2. Case 2: $s_1^2 = s_2^2 \neq s_3^2$

After the same manipulation as case 1, we obtain

$$\begin{aligned} u_x &= -\partial_y \psi_0 + \partial_x \psi_1 + z \partial_x \psi_2 + \partial_x \psi_3, \\ u_y &= \partial_x \psi_0 + \partial_y \psi_1 + z \partial_y \psi_2 + \partial_y \psi_3, \\ u_z &= k_{11} (\partial_z \psi_1 + z \partial_z \psi_2) + k_{13} \partial_z \psi_3 + k_{14} \psi_2, \\ w_z &= k_{21} (\partial_z \psi_1 + z \partial_z \psi_2) + k_{23} \partial_z \psi_3 + k_{24} \psi_2, \end{aligned} \tag{37}$$

where

$$k_{14} = \frac{\begin{bmatrix} -\left[\alpha_3\gamma_1 - \beta_1\beta_2 + 3(\beta_1\beta_3 - \alpha_3\gamma_2) s_1^2\right] k_{11} \\ -2\left[(1+\alpha_1)\gamma_2 + \alpha_2\gamma_1 - \beta_1^2 - 2\alpha_2\gamma_2 s_1^2\right] \end{bmatrix}}{(\alpha_3\gamma_1 - \beta_1\beta_2) s_1^2 + (\beta_1\beta_3 - \alpha_3\gamma_2) s_1^4},$$
$$k_{24} = \frac{\begin{bmatrix} -\left[\alpha_3\gamma_1 - \beta_1\beta_2 + 3(\beta_1\beta_3 - \alpha_3\gamma_2) s_1^2\right] k_{21} \\ +2\left[(1+\alpha_1)\beta_3 + \alpha_2\beta_2 - \alpha_3\beta_1 - 2\alpha_2\beta_3 s_1^2\right] \end{bmatrix}}{(\alpha_3\gamma_1 - \beta_1\beta_2) s_1^2 + (\beta_1\beta_3 - \alpha_3\gamma_2) s_1^4}. \tag{38}$$

It is also seen that ψ_0 and ψ_i satisfy equation (36).

4.3. Case 3: $s_1^2 = s_2^2 = s_3^2$

After performing similar derivations as in the above-mentioned two cases, one obtains

$$\begin{aligned} u_x &= -\partial_y \psi_0 + \partial_x \psi_1 + z \partial_x \psi_2 + z^2 \partial_x \partial_z \psi_3, \\ u_y &= \partial_x \psi_0 + \partial_y \psi_1 + z \partial_y \psi_2 + z^2 \partial_y \partial_z \psi_3, \\ u_z &= k_{11}\left(\partial_z \psi_1 + z \partial_z \psi_2 + z^2 \partial_z^2 \psi_3\right) + k_{14}(\psi_2 + 2z \partial_z \psi_3) + k_{15} \psi_3, \\ w_z &= k_{21}\left(\partial_z \psi_1 + z \partial_z \psi_2 + z^2 \partial_z^2 \psi_3\right) + k_{24}(\psi_2 + 2z \partial_z \psi_3) + k_{25} \psi_3, \end{aligned} \tag{39}$$

where

$$k_{15} = \frac{\begin{bmatrix} 6(\alpha_3\gamma_2 - \beta_1\beta_3)(k_{11} + k_{14}) s_1^2 - 2(\alpha_3\gamma_1 - \beta_1\beta_2) k_{14} \\ -2\left[(1+\alpha_1)\gamma_2 + \alpha_2\gamma_1 - \beta_1^2 - 6\alpha_2\gamma_2 s_1^2\right] \end{bmatrix}}{(\alpha_3\gamma_1 - \beta_1\beta_2) s_1^2 + (\beta_1\beta_3 - \alpha_3\gamma_2) s_1^4},$$
$$k_{25} = \frac{\begin{bmatrix} 6(\alpha_3\gamma_2 - \beta_1\beta_3)(k_{21} + k_{24}) s_1^2 - 2(\alpha_3\gamma_1 - \beta_1\beta_2) k_{24} \\ +2\left[(1+\alpha_1)\beta_3 + \alpha_2\beta_2 - \alpha_3\beta_1 - 6\alpha_2\beta_3 s_1^2\right] \end{bmatrix}}{(\alpha_3\gamma_1 - \beta_1\beta_2) s_1^2 + (\beta_1\beta_3 - \alpha_3\gamma_2) s_1^4}. \tag{40}$$

It is again seen that ψ_0 and ψ_i satisfy equation (36).

Noticeably, the generalized E–L solution in three cases is very similar to the E–L solution of transversely isotropic elasticity. When three characteristic roots are distinct, Peng and Fan [13] obtained a general solution of 1D hexagonal QCs, which is consistent with the generalized E–L solution in case 1. However, they did not consider the other two cases in which three characteristic roots are possibly equal to each other.

Up to here, two general solutions in 1D hexagonal QCs, which are called the generalized LHN and E–L solutions, respectively, are presented from our previous general solution [16]. More importantly, the generalized LHN solution is obtained for the first time, also for transversely isotropic piezoelectric materials. With the aid of a theorem concerning a decomposition and superposition procedure, the generalized LHN solution is further simplified, and is transformed into the generalized E–L solution after the reformulation. It is obvious that the generalized E–L solution is similar to the general solution obtained by Ding *et al* [23, 24] except that some notations are replaced, but the analysis techniques are distinct. The generalized E–L solution generalizes the operator method brought forward by [25, 27], while the general solution [23, 24] uses the method developed by [18].

Completeness is very important character of general solutions of elasticity of QCs. For LHN and E–L solutions of transversely isotropic elasticity, completeness for *z*-convex domains was proved, incompleteness of the usual non-*z*-convex domains was pointed out and completeness of a class of very special *z*-convex domains was also proved [25, 27]. When the phonon–phason field coupling effect is absent, namely, $R_i = 0$, LHN and E–L solutions can be obtained directly from the corresponding generalized LHN and E–L solutions, respectively. Being similar to the derivation on completeness of LHN and E–L solutions [25, 27], we can draw our conclusion that since theorems 2 and 3 presented previously come into existence when the domain Ω is *z*-convex, the generalized LHN and E–L solutions obtained here are complete in *z*-convex domains, while incomplete in the usual non-*z*-convex domains. Therefore, the applications of generalized LHN and E–L solutions are restricted. It is interesting to note that the general solution [16] is complete in any limited domains in E^3 without loss in generality. Thus the general solution [16] is considered reliable as a basis for more general applications.

5. Conclusions

Based on theorems 1 and 2, the generalized LHN solution of 1D hexagonal QCs is obtained from our general solution [16] by introducing three displacement functions ψ_0 and F_j, where ψ_0 satisfies a quasi-harmonic equation and F_j a sixth-order partial differential equation, respectively.

Owing to complexity of the higher-order equation, it is difficult to obtain rigorous analytic solutions and not applicable in most cases. Based on theorem 3, a decomposition and superposition procedure is taken, and the generalized LHN solution is simplified in terms of four quasi-harmonic functions ψ_0 and ψ_i. In consideration of the possibilities that the characteristic roots s_i^2 might be distinct or equal to each other, the generalized E–L solution of 1D hexagonal QCs possesses different forms, but all are in simple forms that are convenient to use. It is proved that the general solution [13, 14] is consistent with one case of the generalized E–L solution, while does not include the other two cases in which three characteristic roots are possibly equal to each other.

Therefore, generalized LHN and E–L solutions obtained here are simplified forms of the general solution [16] when the domain Ω is z-convex. Furthermore, it is important to note that generalized LHN and E–L solutions are complete in z-convex domains, while incomplete in the usual non-z-convex domains. Fortunately, the general solution [16] is complete in any limited domains in E^3 without loss in generality. Therefore, the applications of generalized LHN and E–L solutions are restricted, while the general solution [16] is considered reliable as a basis for more general applications.

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References

[1] Shechtman D *et al* 1984 *Phys. Rev. Lett.* **53** 1951
[2] Ronchetti M 1987 *Phil. Mag.* **56** 237
[3] Ovid'ko I A 1992 *Mater. Sci. Eng.* A **154** 29
[4] Wollgarten M *et al* 1993 *Phys. Rev. Lett.* **71** 549
[5] Fan T Y and Mai Y W 2004 *Appl. Mech. Rev.* **57** 325
[6] Ding D H *et al* 1993 *Phys. Rev.* B **48** 7003
[7] Yang W G *et al* 1993 *Phys. Rev.* B **48** 6999
[8] Hu C Z *et al* 1996 *Acta Crystallogr.* **52** 251
[9] Wang R H *et al* 1997 *J. Phys.: Condens. Matter* **9** 2411
[10] Zhou W M and Fan T Y 2000 *Chin. Phys.* **9** 295
[11] Liu G T, Fan T Y and Guo R P 2003 *Mech. Res. Commun.* **30** 335
[12] Liu G T, Fan T Y and Guo R P 2004 *Int. J. Solids Struct.* **41** 3949
[13] Peng Y Z and Fan T Y 2000 *J. Phys.: Condens. Matter* **12** 9381
[14] Chen W Q, Ma Y L and Ding H J 2004 *Mech. Res. Commun.* **31** 633
[15] Wang X 2006 *Mech. Res. Commun.* **33** 576
[16] Gao Y and Zhao B S 2006 *Phys. Status Solidi b* **243** 4007
[17] Lekhnitskii S G 1981 *Theory of Elasticity of an Anisotropic Body* (Moscow: Mir)
[18] Hu H C 1953 *Sci. Sin.* **2** 145
[19] Nowacki W 1954 *Bull. Acad. Pol. Sci. CI.* **42** 21
[20] Elliott H A 1948 *Proc. Camb. Phil. Soc.* **44** 522
[21] Lodge A S 1955 *Q. J. Mech. Appl. Math.* **8** 211
[22] Wang Z K and Zheng B L 1995 *Int. J. Solids Struct.* **32** 105
[23] Ding H J, Chen B and Liang J 1996 *Int. J. Solids Struct.* **33** 2283
[24] Ding H J, Chen B and Liang J 1997 *Int. J. Solids Struct.* **34** 3041
[25] Wang M Z and Wang W 1995 *Int. J. Solids Struct.* **32** 501
[26] Eubanks R A and Sternberg E 1954 *J. Rat. Mech. Anal.* **3** 89
[27] Wang W and Shi M X 1998 *Int. J. Solids Struct.* **35** 3283