A THEORY OF MINIMAL K-TYPES FOR FLAT G-BUNDLES

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Abstract. The theory of minimal K-types for p-adic reductive groups was developed in part to classify irreducible admissible representations with wild ramification. An important observation was that minimal K-types associated to such representations correspond to fundamental strata. These latter objects are triples (x, r, β), where x is a point in the Bruhat-Tits building of the reductive group G, r is a nonnegative real number, and β is a semistable functional on the degree r associated graded piece of the Moy-Prasad filtration corresponding to x.

Recent work on the wild ramification case of the geometric Langlands conjectures suggests that fundamental strata also play a role in the geometric setting. In this paper, we develop a theory of minimal K-types for formal flat G-bundles. We show that any formal flat G-bundle contains a fundamental stratum; moreover, all such strata have the same rational depth. We thus obtain a new invariant of a flat G-bundle called the slope, generalizing the classical definition for flat vector bundles. The slope can also be realized as the minimum depth of a stratum contained in the flat G-bundle, and in the case of positive slope, all such minimal depth strata are fundamental. Finally, we show that a flat G-bundle is irregular singular if and only if it has positive slope.

1. Introduction

The theory of unrefined minimal K-types for representations of p-adic reductive groups arose as a response to two *a priori* distinct problems concerning admissible representations of GL_n. First, Bushnell and Frölich [10] introduced the notion of a *fundamental stratum* contained in a representation and showed that for representations containing a fundamental stratum, the stratum could be used to calculate the constants in the functional equation of a zeta integral. Bushnell [9] later proved that any irreducible admissible representation of GL_n contains a fundamental stratum. In another direction, Howe and Moy [27, 20], motivated by work of Vogan [35] on representations of real groups, developed a theory of (unrefined) minimal K-types for GL_n in order to better understand the parameterization of irreducible representations. In particular, they were interested in establishing a well-defined notion of the *depth* of a representation V by studying the congruence level of compact subgroups that fix a vector in V. In retrospect, one sees that fundamental strata and unrefined minimal K-types contain equivalent information about admissible representations of GL_n with positive depth. Thus, fundamental strata can be viewed as a theory of minimal K-types in this case.

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This theory plays a vital role in the classification of supercuspidal representations with wild ramification [11, 24] as well as in the proof of the local Langlands conjecture for GL_n [18, 19]. Recent work suggests that minimal K-types also play a role in the geometric setting. Somewhat unexpectedly, the applications so far have been on the "Galois" side of the correspondence. Let G be a complex reductive group with Langlands dual ${}^L G$, and let F be the field of complex Laurent series $\mathbb{C}((z))$. In the geometric Langlands program, the role of Galois representations is played by monodromy data associated to flat ${}^L G$-bundles: over a smooth complex curve X or the formal punctured disk $\Delta^\times = \mathrm{Spec}(F)$ depending on whether one is in the global or local context. Consider, for example, the case $G = {}^L G = \mathrm{GL}_n$, so that a flat ${}^L G$-bundle on X is just a rank n vector bundle endowed with a meromorphic connection. For regular singular connections, i.e., those whose connection matrix at each singular point can be chosen to have simple poles, the monodromy data is just a representation of the fundamental group. Most previous analyses of geometric Langlands have concentrated on such connections; indeed, a detailed correspondence has been formulated by Frenkel and Gaitsgory in the "tame" case, where the connections considered are regular singular with unipotent monodromy [14].

Much less is known about the wild case, where irregular singularities are allowed. Here, one must also include "wild" monodromy data. This consists of a collection of Stokes matrices at each singular point, describing the "jumps" in the asymptotic behavior of a horizontal section as it is analytically continued around each irregular singularity. Thus, the Stokes data is simply an enhancement of the usual monodromy data, which allows one to establish a Riemann-Hilbert correspondence for irregular singular flat vector bundles [26, Chapitre IV]. The deviation of an irregular singular connection from the regular singular case, or equivalently the complexity of the Stokes data, is measured by the *slope* of the connection. By analogy with the p-adic case, a geometric theory of minimal K-types ought to detect both whether a flat G-bundle is irregular and the degree of its irregularity, thereby giving a definition for general G of the discrete invariant slope (akin to the depth of a representation). It should yield information about the moduli stack of flat G-bundles. Moreover, it should illuminate certain transcendental invariants such as the irregular monodromy map, just as the classical theory was used to calculate constants in the functional equation of zeta integrals. Finally, the theory should be effectively computable, i.e., it should provide an algorithm for finding a minimal K-type associated to a flat G-bundle.

The classical approach to studying the local behavior of meromorphic differential equations in one variable, or equivalently, of meromorphic connections on $\mathbb{P}^1$, makes use of the naive "leading term" of the connection. More generally, let X be a smooth curve. Let V be a rank n vector bundle on X endowed with a meromorphic connection ∇, and assume that $y \in X$ is an irregular singular point. After choosing a trivialization for V on a formal neighborhood of y and a local parameter z, ∇ has the local description

$$\nabla = d + [\nabla] = d + (M_{-r} z^{-r} + M_{-r+1} z^{-r+1} + \dots) \frac{dz}{z}, \tag{1}$$

where $[\nabla]$ (the *matrix* of the connection) is a $\mathfrak{gl}_n(\mathbb{C}((z)))$-valued one-form, $M_j \in \mathfrak{gl}_n(\mathbb{C})$, and $r \geq 0$. When the leading term M_{-r} is well-behaved, asymptotic analysis using this expansion produces detailed information about the connection and the form of fundamental solutions at y. (See, for instance, [36].) For example, if M_{-r}

is nonnilpotent, then the slope of the connection is r. The case when M_{-r} is regular semisimple has been studied extensively in the literature [3, 21], and the contribution of the leading term to the Stokes data is well understood.

This perspective is much less useful when the formal connection has nonintegral slope because in this situation, the leading term M_{-r} is nilpotent for every formal trivialization at y. This is the case for many irregular singular connections that arise naturally in the geometric Langlands program, such as connections corresponding to cuspidal representations. For example, Frenkel and Gross have constructed a rigid flat G-bundle on $\mathbb{P}^1$ (for any reductive G) which is the de Rham analogue of the automorphic representation with the Steinberg representation at 0 and a certain "small" supercuspidal representation at ∞ [15]. Here, the leading term at the irregular singular point at ∞ is always nilpotent. More concretely, Witten considers Airy-type connections of the form

$$\nabla = d + \begin{pmatrix} 0 & z^{-s} \\ z^{-s+1} & 0 \end{pmatrix} \frac{dz}{z}, \tag{2}$$

which always have nilpotent leading terms at 0 [37]. When $s = 2$, this is the classical Airy connection; when $s = 1$, it is the GL_2 case of the Frenkel-Gross flat G-bundle with the roles of 0 and ∞ reversed.

One obtains the naive notion of the leading term of a connection by studying the connection in terms of the obvious degree filtration on $\mathfrak{gl}_n(\mathbb{C}(\!(z)\!))$. In [4], the authors introduced a more powerful notion of the "leading term" of a connection by considering more general filtrations on the loop algebra defined in terms of *lattice chains* [4, 31]. Let V be a finite-dimensional vector space over k. A lattice chain $\mathcal{L} = \{L^j\}_{j\in\mathbb{Z}}$ in $V(\!(z)\!)$ is a collection of lattices (i.e., maximal rank $\mathbb{C}[\![z]\!]$-submodules) such that $L^j \supset L^{j+1}$ and $zL^j = L^{j+e}$ with fixed period e. The stabilizer $P \subset \mathrm{GL}(V(\!(z)\!))$ of $\mathcal{L}$ is called a *parahoric subgroup*. The lattice chain is uniquely determined by P up to translation of the indexing of $\mathcal{L}$; in particular, the period $e = e_P$ is determined by P. One further obtains congruent subalgebras $\mathfrak{p}^m$, consisting of the endomorphisms that map L^j to L^{j+m} for each $j \in \mathbb{Z}$ in particular, $\mathfrak{p} = \mathfrak{p}^0$ is the Lie algebra of P. Using this data, we can associate a triple (P, r, β) called a *stratum* to the connection ∇, where r is a nonnegative integer and β is a $\mathbb{C}$-linear functional on $\mathfrak{p}^r/\mathfrak{p}^{r+1}$. This means that the matrix $[\nabla]$ lies in $\mathfrak{p}^{-r}\frac{dz}{z}$ and that the functional on $\mathfrak{gl}(V(\!(z)\!))$ given by $Y \mapsto \mathrm{Res}(\mathrm{Tr}(Y[\nabla]))$ induces β on the quotient algebra. To give a simple example, the usual leading term comes from the lattice chain $L^j = z^j V[\![z]\!]$. The stabilizer here is the maximal parahoric subgroup $\mathrm{GL}(V[\![z]\!])$ and $\mathfrak{p}^r = z^r\,\mathfrak{gl}(V[\![z]\!])$. The connection in (1) satisfies $[\nabla](L^j) \subset L^{j-r}\frac{dz}{z}$ while the functional on $z^r\,\mathfrak{gl}(V[\![z]\!])/z^{r+1}\,\mathfrak{gl}(V[\![z]\!])$ induced by $[\nabla]$ is the same as that induced by the leading term $M_{-r}\frac{dz}{z}$. In the terminology of [4], we say that the stratum

$$(\mathrm{GL}(V[\![z]\!]), r, M_{-r}z^{-r}\frac{dz}{z})$$

is contained in the induced formal connection at y.

As has been noted above, the fact that the stratum (1) is contained in a connection is primarily useful when M_{-r} is nonnilpotent. More generally, we say that the stratum (P, r, β) is *fundamental* if it satisfies an analogous nondegeneracy condition. To be precise, the functional β comes from some $Y\frac{dz}{z} \in \mathfrak{p}^{-r}\frac{dz}{z}$ as described above, and the stratum (P, r, β) is fundamental if the action of Y on the associated graded space $\bigoplus L^i/L^{i+1}$ is not nilpotent. It is fundamental strata that provide a better notion of the leading term of a connection. In [4], we showed that every formal connection contains a fundamental stratum. Moreover, if (P, r, β) is contained in $(V(\!(z)\!), \nabla)$, then $r/e_P \geq \mathrm{slope}(V(\!(z)\!), \nabla)$ with equality (in the irregular singular case) precisely when the stratum is fundamental. We remark that this geometric theory of strata was motivated by the analogous p-adic theory, which plays a crucial role in the representation theory of GL_n over p-adic fields [10, 9, 11].

A fundamental stratum contained in a connection at a given point should provide a coarse approximation to the local behavior of the connection and its solutions. This can be seen most strikingly when the fundamental stratum is *regular*, a condition introduced by the authors generalizing the classical assumption of regular semisimplicity of the naive leading term [4]. The data of a regular stratum includes a (not necessarily split) maximal torus S in $\mathrm{GL}_n(V(\!(z)\!))$. Consider, for example, the collection of meromorphic connections (V, ∇) on $\mathbb{P}^1$ with singularities at $\mathbf{y} = (y_1, \dots, y_m)$ such that the induced formal connection at each y_i contains an S_i-regular stratum of depth r_i. Let $\mathbf{r}$ and $\mathbf{S}$ denote the collection of r_i's and S_i's respectively. One can construct the moduli space of such connections as a Poisson manifold $\widetilde{\mathcal{M}}(\mathbf{y}, \mathbf{r})$. Moreover, the construction of this moduli space is "automorphic", i.e., it is realized as the Poisson reduction of products of smooth varieties describing local data at the singularities.[1] Finally, the monodromy map induces an integrable system on $\widetilde{\mathcal{M}}(\mathbf{y}, \mathbf{S}, \mathbf{r})$, and the authors have analyzed the relationship between the monodromy foliation and the corresponding regular strata in [5].

The goal of this paper is to develop a geometric theory of minimal K-types.[2] We will do so by generalizing the theory of strata and its application to formal flat G-bundles for an arbitrary reductive group G over an algebraically closed field of characteristic 0. Again, one wants to study the behavior of a connection in terms of suitable filtrations on the loop algebra $\mathfrak{g}(\!(z)\!)$. In the general setting, we will use the filtrations introduced by Moy and Prasad in their theory of minimal K-types for admissible representations of p-adic groups [28, 29]. Given any algebraic group H defined over a discrete valuation field, there is an associated complex called its Bruhat-Tits building. For any point x in this building, Moy and Prasad defined a decreasing $\mathbb{R}_{\geq 0}$-filtration $\{H_{x,r}\}$ on the parahoric subgroup $H_x = H_{x,0}$ with a discrete number of jumps [30, 28]. There is also a compatible $\mathbb{R}$-filtration on the Lie algebra $\mathfrak{h}$. An *unrefined K-type* (at least for $r > 0$) is then a triple (x, r, β) with β a character of $H_{x,r}/H_{x,r+}$ (with $H_{x,r+}$ the next step up in the filtration); it is called minimal if it satisfies a certain nondegeneracy condition. For p-adic groups, Moy and Prasad showed that every irreducible admissible representation V of H contains a minimal unrefined K-type. Moreover, if one defines the depth of V to be the smallest r appearing in a K-type in V, then (x', r', β') contained in V is minimal if and only if $r' = r$. Finally, they showed that two minimal K-types contained in V are closely related; in their terminology, the minimal K-types are *associates* of each other [28, 29].

Returning to the geometric setting, we define a $G(\!(z)\!)$-stratum to be a triple (x, r, β) where x is a point in the Bruhat-Tits building for $G(\!(z)\!)$, $r \in \mathbb{R}_{\geq 0}$, and β is a functional on the rth step $\mathfrak{g}(\!(z)\!)_{x,r}/\mathfrak{g}(\!(z)\!)_{x,r+}$ in the filtration on $\mathfrak{g}(\!(z)\!)$ determined by x. Again, we call a stratum fundamental if the functional β satisfies a certain nondegeneracy condition. We remark that if $r > 0$, the exponential map induces

[1] See [32] for some explicit examples.

[2] Further extensions and applications appear in [6, 22].

an isomorphism between $\mathfrak{g}(\!(z)\!)_{x,r}/\mathfrak{g}(\!(z)\!)_{x,r+}$ and $G(\!(z)\!)_{x,r}/G(\!(z)\!)_{x,r+}$, so there is a bijection between strata and unrefined K-types. In the case of flat vector bundles (i.e., $G = \mathrm{GL}_n$), we can recover our previous version of strata by choosing appropriate points in the building. Indeed, any parahoric subgroup P corresponds to a unique facet in the building, and this facet decomposed as the product of a simplex and $\mathbb{R}$. If x is any point lying over the barycenter of this simplex, then the jumps in the associated filtration occur at $\frac{1}{e_P}\mathbb{Z}$ and $P^j = G_{x,j/e_P}$, where e_P is the period of a lattice chain with stabilizer P.

Our main results, given in Theorems 2.14, 2.15, and 2.17, are the geometric analogue of Moy and Prasad's theorem on minimal K-types for admissible representations of p-adic groups. We show that any formal flat G-bundle $(\mathcal{G}, \nabla)$ contains a fundamental stratum (x, r, β) with r a nonnegative rational number and all such strata have the same depth. In fact, we exhibit an explicit algorithm for finding such a fundamental stratum. Moreover, any two fundamental strata contained in ∇ are associates of each other. We thus obtain a new invariant of a flat G-bundle called the slope, generalizing the classical definition for flat connections. We further show that $(\mathcal{G}, \nabla)$ is irregular singular if and only if $\mathrm{slope}(\nabla) > 0$. Finally, we prove that a stratum (x', r', β') contained in the formal G-bundle ∇ satisfies $r' \geq \mathrm{slope}(\nabla)$, and in the irregular singular case, it is fundamental if and only if $r' = r$.

We remark that there are other approaches to defining the slope. In [15], Frenkel and Gross suggest a procedure which involves pulling the flat G-bundle back to a ramified cover where the connection matrix (for a suitable trivialization) is well-behaved with respect to the usual degree filtration. Corollary 2.20 shows that their approach is well-defined and gives the same invariant as our definition. There is also a recent preprint of Chen and Kamgarpour which shows how to define slope in terms of opers [12]. These alternate definitions are discussed in more detail in Remark 2.21.

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2. Preliminaries and main results

In this section, we state the main results of the paper. In order to make the paper accessible to readers with a background in p-adic groups, who may not be familiar with the geometric setting, we begin by illustrating the theory for flat vector bundles. We also provide some background on flat G-bundles.

Throughout the paper, k will be an algebraically closed field of characteristic 0, and $F = k(\!(z)\!)$ will be the field of formal Laurent series over k with ring of integers $\mathfrak{o} = k[\![z]\!]$. We write $\Delta^\times = \mathrm{Spec}(F)$ for the formal punctured disk and $\Omega^1 = \Omega^1_{F/k}$ for the space of differential one-forms on $\Delta^\times$. We will denote the Euler vector field $z\frac{d}{dz}$ by τ. Finally, ι_τ will be the inner derivation by τ, so that a 1-form ω can be written $\omega = \iota_\tau(\omega)\frac{dz}{z}$.

2.1. **Flat vector bundles.** A flat vector bundle over $\Delta^\times$ is an F-vector space U equipped with a connection ∇, which is k-derivation $\nabla : U \to \Omega^1(U)$. Throughout, we shall assume that U is finite dimensional. The simplest example is the trivial connection on F^n. In the standard basis $\{e_i\}_{i=1}^n$, ∇ is the usual exterior derivative

d defined by $d(e_i) = 0$ and extended to F^n by k-linearity and the Leibniz rule: for $f \in F$, $d(fu) = \frac{df}{dz}u + fd(u)$. More generally, for any $n \times n$ matrix M with coefficients in Ω^1, it is easily checked that the following is a k-derivation from F^n to $\Omega^1(F^n)$:

$$\nabla(u) = d(u) + Mu.$$

We will use the shorthand $\nabla = d + M$ for this operator. Moreover, any connection on F^n is of this form. Accordingly, if (U, ∇) is a rank n connection and $\phi : U \to F^n$ is a trivialization, then there exists a matrix $[\nabla]_\phi$ of 1-forms such that the induced connection ∇_ϕ on F^n can be written as $\nabla_\phi = d + [\nabla]_\phi$. One can also write ∇ in terms of an ordinary matrix as $\nabla_\phi = d + \iota_\tau([\nabla]_\phi)\frac{dz}{z}$. The left action of $\mathrm{GL}_n(F)$ on trivializations has the following effect on the matrix of a connection, known as a gauge transformation: for $g \in \mathrm{GL}_n(F)$,

$$[\nabla]_{g\phi} = g \cdot [\nabla]_\phi = g[\nabla]_\phi g^{-1} - (dg)g^{-1}. \tag{3}$$

We remark that if $T \in \Omega^1(\mathrm{End}(U))$, the formula $d + T$ does not give a well-defined connection on U. However, if U is endowed with a fixed k-structure, say $U = U_k \otimes F$, then $d + T$ makes sense: $(d + T)(fv) = \frac{df}{dz}v + fT(v)$ for $f \in F$ and $v \in U_k$. In particular, below we will frequently consider flat vector bundles with $U = \hat{V} := V \otimes F$, where V is a k-vector space, and expressions of the form $d + T$ will be defined in terms of the natural k-structure.

A flat vector bundle (U, ∇) is called *regular singular* if there exists an $\mathfrak{o}$-lattice $L \subset U$ with the property that $\nabla(L) \subset L \otimes_\mathfrak{o} \Omega^1_{\mathfrak{o}/k}(1)$. Equivalently, (U, ∇) is regular singular if and only if there exists a lattice L for which $(\iota_\tau \circ \nabla)(L) \subset L$. This means that there is some trivialization ϕ with respect to which the matrix $[\nabla]_\phi$ has at worst a simple pole. Otherwise, ∇ is said to be *irregular singular*.

The deviation of an irregular singular connection from the regular singular case is measured by an invariant called the *slope*. We recall the precise definition. Fix a lattice $L \subset U$. If $\mathbf{e} = \{e_j\}$ is a finite collection of vectors in U, we define $v(\mathbf{e}) = m$ if m is the greatest integer such that $\mathbf{e} \subset z^m L$. Let (U, ∇) be an irregular flat vector bundle. Take $\mathbf{e}$ to be any basis for U. By a theorem of Katz [13, Theorem II.1.9], there is a unique positive rational number r such that the subset of $\mathbb{Q}$ given by

$$\{v((\iota_\tau \circ \nabla)^i \mathbf{e}) + ri \mid i > 0\} \tag{4}$$

is bounded. Here, $(\iota_\tau \circ \nabla)^i \mathbf{e} = \{(\iota_\tau \circ \nabla)^i(e_j)\}$. This number is independent of the choice of basis $\mathbf{e}$ and is called the slope of (U, ∇). For a regular singular connection, the set in (4) is never bounded for positive r, and we say that its slope is 0.

2.2. **Filtrations on F-vector spaces.** A previous paper by the authors [4] is concerned with the interaction between flat vector bundles (U, ∇) and decreasing $\mathbb{R}$-indexed filtrations on U consisting of $\mathfrak{o}$-lattices $\{U_r\}_{r \in \mathbb{R}}$. We will only consider *periodic* (more specifically, 1-periodic) filtrations, for which $zU_r = U_{r+1}$ for all r. Such filtrations are parameterized by points in a certain complex $\mathcal{B} = \mathcal{B}(\mathrm{GL}(U))$ called the Bruhat-Tits building for $\mathrm{GL}(U)$. We will not describe this correspondence in detail here; see Section 2.6 where we discuss analogous filtrations for an arbitrary reductive group. For the present, we only recall a few properties of the building.[3] First, $\mathcal{B}$ is the union of n-dimensional real affine spaces called *apartments*, which are in one-to-one correspondence with the set of split maximal tori in $\mathrm{GL}(U)$. A

[3]References for Bruhat-Tits buildings include the survey article [34] and the book [25], as well as the original papers of Bruhat and Tits [7, 8].

choice of basis for U determines a specific apartment together with an origin for the affine space. Second, $\mathcal{B}$ is endowed with a surjection onto a simplicial complex $\bar{\mathcal{B}}$ called the reduced building with fibers isomorphic to $\mathbb{R}$; furthermore, one can decompose the building as $\mathcal{B} = \bar{\mathcal{B}} \times \mathbb{R}$. If $x \in \mathcal{B}$ lies over $\bar{x} \in \bar{\mathcal{B}}$, then the filtrations for points lying above $\bar{x}$ are precisely those of the form $U'_r = U_{x,r+s}$ for some fixed $s \in \mathbb{R}$. Finally, the origin determined by a choice of basis lies over a fixed vertex in $\bar{\mathcal{B}}$.

Example 2.1 (Degree filtration). Let $U = F^n$ and $U_j = z^j \mathfrak{o}^n$ whenever $j \in \mathbb{Z}$. This $\mathbb{Z}$-filtration extends to an $\mathbb{R}$-filtration by $U_i = U_{\lceil i \rceil}$. This periodic filtration corresponds to the origin of $\mathcal{B}$ determined by the standard basis of F^n.

The most familiar class of periodic filtrations comes from *lattice chains* in U. Recall that a lattice chain $\mathcal{L} = \{L^j\}_{j\in\mathbb{Z}}$ in U is a collection of $\mathfrak{o}$-lattices in U such that $L^j \supset L^{j+1}$ and $zL^j = L^{j+e}$ with fixed period e [4, 31]. The corresponding $\mathbb{R}$-filtration is given by setting $U^{\mathcal{L}}_{j/e} = L^{\lceil j \rceil}$. Note that lattice chain filtrations are uniform in the sense that the "jumps" in the filtration are evenly spaced a distance $1/e$ apart. More formally, given any filtration $\{U_r\}$, set $U_{r+} = \cup_{s>r} U_s$. The critical numbers of the filtration are those r for which $U_r \neq U_{r+}$. A filtration is called *uniform* if the distance between consecutive critical numbers is a constant. An arbitrary uniform filtration is obtained from an appropriate $\{U^{\mathcal{L}}_r\}$ by translating the indices by a constant.

Uniform filtrations correspond to distinguished points in the building. Let $P \subset \mathrm{GL}(U)$ be the stabilizer of $\mathcal{L}$; equivalently, it is the stabilizer of any uniform filtration coming from $\mathcal{L}$ up to translation of the indices. Such subgroups are called *parahoric subgroups*, and they parameterize the facets in both $\mathcal{B}$ and $\bar{\mathcal{B}}$ in a way compatible with the natural map $\mathcal{B} \to \bar{\mathcal{B}}$. The uniform filtrations coming from $\mathcal{L}$ are precisely those $U_{x,r}$ with $x \in \mathcal{B}$ lying above the barycenter of the simplex in $\bar{\mathcal{B}}$ corresponding to P (c.f. Appendix A). We denote the Lie algebra of P by $\mathfrak{p}$; it is called a *parahoric subalgebra*.

A periodic filtration on U also induces a periodic filtration on the endomorphism ring $\mathfrak{gl}(U)$ of U. Indeed, if $x \in \mathcal{B}$ corresponds to the filtration $U_{x,r}$, then one sets $\mathfrak{gl}(U)_{x,r} = \{X \in \mathfrak{gl}(U) \mid X(U_{x,s}) \subset U_{x,s+r} \text{ for all } s\}$. In particular, if x is in the facet corresponding to the parahoric subgroup P, then $\mathfrak{gl}(U)_{x,0} = \mathfrak{p}$. These subspaces should be viewed as generalized congruence subalgebras. To see why, let $\mathcal{L}$ be a lattice chain with period e. The corresponding sequence of congruence subalgebras is given by $\mathfrak{p}^m = \{X \in \mathfrak{gl}(U) \mid X(L^i) \subset L^{i+m} \text{ for all } i\}$, with $\mathfrak{p}^0 = \mathfrak{p}$. It is now easy to check that the filtration $\{\mathfrak{gl}(U)^{\mathcal{L}}\}$ with critical numbers $\frac{1}{e}\mathbb{Z}$ and $\mathfrak{gl}(U)^{\mathcal{L}}_{m/e} = \mathfrak{p}^m$ for all m is the same as the filtration induced from $\{U^{\mathcal{L}}_r\}$.

The degree filtration is an example of a uniform filtration coming from a lattice chain of period 1. The corresponding parahoric subgroup is the maximal parahoric subgroup $\mathrm{GL}_n(\mathfrak{o})$. We now give an example corresponding to a minimal parahoric (or Iwahori) subgroup.

Example 2.2 (Iwahori filtration). Let $U = F^n$. For $m \in \mathbb{Z}$, write $m = sn + j$ with $0 \leq j < n$, and let L^m be the lattice with $\mathfrak{o}$-basis $\{z^s e_i \mid i \leq n-j\} \cup \{z^{s+1} e_i \mid i > n-j\}$. The lattice chain $\mathcal{L} = \{L^m\}$ has period n, so that the associated periodic filtration has critical numbers $\frac{1}{n}\mathbb{Z}$. The stabilizer of this lattice chain is the standard Iwahori subgroup I, consisting of the pullback of the standard upper triangular Borel subgroup under the natural map $\mathrm{GL}_n(\mathfrak{o}) \to \mathrm{GL}_n(k)$. The point

$x_I \in \mathcal{B}$ corresponding to the Iwahori filtration lies above the barycenter of the chamber (i.e., maximal facet) in $\bar{\mathcal{B}}$ corresponding to I.

We will also need to consider periodic filtrations on the space of 1-forms $\Omega^1(U)$ and on the smooth k-dual space $\mathfrak{gl}(U)^\vee$, i.e., the space of k-functionals vanishing on a nonempty bounded open subgroup. First, we define $\Omega^1(U)_{x,r}$ as the image of $U_{x,r}$ via the isomorphism $U \to \Omega^1(U)$ given by $u \mapsto u\frac{dz}{z}$. Next, observe that there is a pairing

$$\mathfrak{gl}(U) \times \mathfrak{gl}(U) \to k, \qquad (X,Y) \mapsto \operatorname{Res}\operatorname{Tr}(XY)\tfrac{dz}{z}, \tag{5}$$

which induces an isomorphism $\mathfrak{gl}(U) \to \mathfrak{gl}(U)^\vee$. We set $\mathfrak{gl}(U)^\vee_{x,r}$ to be the image of $\mathfrak{gl}(U)_{x,r}$ under this isomorphism.

2.3. **U-strata and flat vector bundles.** In order to relate filtrations to connections, we introduce the notion of a U-stratum. We will denote the k-dual of a finite-dimensional k-vector space W by $W^\vee$.

Definition 2.3. A *U-stratum* of depth r is a triple (x, r, β), where $x \in \mathcal{B}$, $r \geq 0$, and $\beta \in (\mathfrak{gl}(U)_{x,r}/\,\mathfrak{gl}(U)_{x,r+})^\vee$.

The pairing (5) induces an isomorphism $(\mathfrak{gl}(U)_{x,r}/\,\mathfrak{gl}(U)_{x,r+})^\vee \cong \mathfrak{gl}(U)^\vee_{x,-r}/\,\mathfrak{gl}(U)^\vee_{x,-r+}$, and we say that $\tilde{\beta} \in \mathfrak{gl}(U)^\vee_{x,-r}$ is a *representative* of β if $\tilde{\beta} + \mathfrak{gl}(U)^\vee_{x,-r+}$ corresponds to β. The stratum (x,r,β) is called *fundamental* if every representative is nonnilpotent. (We define nilpotent elements in $\mathfrak{gl}(U)^\vee$ via transport of structure from $\mathfrak{gl}(U)$.)

We can now show how U-strata can be used to define the leading term of a flat connection with respect to a periodic filtration. Given $\tilde{\beta} \in \mathfrak{gl}(U)^\vee$, set $X_{\tilde{\beta}} \in \Omega^1(\mathfrak{gl}(U))$ equal to the unique 1-form such that $\operatorname{Res}\tilde{\beta}(Y)\frac{dz}{z} = \operatorname{Res}\operatorname{Tr}(X_{\tilde{\beta}}Y)$.

Definition 2.4. Let (U, ∇) be a flat vector bundle. We say that (U, ∇) contains the stratum (x, r, β) if, given any (or equivalently, some) representative $\tilde{\beta}$ for β,

$$(\nabla - j\frac{dz}{z} - X_{\tilde{\beta}})(U_{x,j}) \subseteq \Omega^1(U)_{x,(j-r)+}, \quad \forall j \in \mathbb{R}. \tag{6}$$

Example 2.5. Let $o \in \mathcal{B}$ correspond to the degree filtration on F^n determined by the standard basis as in Example 2.1. The connection given in (1) makes F^n into a flat vector bundle that contains the stratum $(o, r, M_{-r}z^{-r}\frac{dz}{z})$. Here, $M_{-r}\frac{dz}{z}$ is viewed as the functional on $z^r\,\mathfrak{gl}_n(k) \cong z^r\,\mathfrak{gl}_n(\mathfrak{o})/z^{r+1}\,\mathfrak{gl}_n(\mathfrak{o})$ given by the residue of the trace form. This stratum is fundamental if and only if M_{-r} is nonnilpotent.

Example 2.6. The connection on F^2 given by (2) contains the nonfundamental stratum $(o, s, \left(\begin{smallmatrix} 0 & z^{-s} \\ 0 & 0 \end{smallmatrix}\right)\frac{dz}{z})$ based at the degree filtration. However, it contains a fundamental stratum with respect to the Iwahori filtration of Example 2.2, namely

$$(x_I, s-\frac{1}{2}, \left(\begin{smallmatrix} 0 & z^{-s} \\ z^{-s+1} & 0 \end{smallmatrix}\right)\frac{dz}{z}).$$

Setting $\mathfrak{i} = \operatorname{Lie}(I)$, the sth Iwahori congruence subalgebra is given by

$$\mathfrak{i}^s = \begin{pmatrix} z^{\lceil s/2\rceil}\mathfrak{o} & z^{\lfloor s/2\rfloor}\mathfrak{o} \\ z^{\lfloor s/2\rfloor+1}\mathfrak{o} & z^{\lceil s/2\rceil}\mathfrak{o} \end{pmatrix}.$$

The 1-form in the stratum acts by $\operatorname{Res}\operatorname{Tr}$ on $\operatorname{span}_k\{\left(\begin{smallmatrix} 0 & 0 \\ z^s & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & z^{s-1} \\ 0 & 0 \end{smallmatrix}\right)\} \cong \mathfrak{i}^{2s-1}/\mathfrak{i}^{2s}$.

Remark 2.7. The theory of U-strata described here is somewhat different from that given in [4]. For example, the definition of strata in [4, Definition 2.13] (as well as the original definition from [10] in the context of p-adic representation theory) only allows for filtrations coming from lattice chains. Also, the present definition of containment of a stratum in a connection differs from that in [4, Definition 4.1]; it is equivalent for strata of positive depth. This is discussed in the appendix; see, in particular, Proposition A.3.

2.4. **Flat G-bundles.** We now turn from flat vector bundles to flat G-bundles. We will need to fix some notation.

Let G be a connected reductive group over k with Lie algebra $\mathfrak{g}$. The category of finite-dimensional representations of G over k will be denoted by $\mathrm{Rep}(G)$. We fix a nondegenerate invariant symmetric bilinear form $\langle , \rangle$ on $\mathfrak{g}$ throughout. We set $\hat{G} = G_F$ and $\hat{\mathfrak{g}} = \mathfrak{g} \otimes_k F$; note that $\hat{G}$ represents the functor sending a k-algebra R to $G(R(\!(z)\!))$. We will use the analogous notation $\hat{H}$ and $\hat{\mathfrak{h}}$ for any algebraic group H over k.

A formal principal G-bundle $\mathcal{G}$ is a principal G-bundle over $\Delta^\times$. The G-bundle $\mathcal{G}$ induces a tensor functor from $\mathrm{Rep}(G)$ to the category of formal vector bundles via $(V, \rho) \mapsto V_{\mathcal{G}} = \mathcal{G} \times_G V$, and by Tannakian formalism, this tensor functor uniquely determines $\mathcal{G}$. Formal principal G-bundles are trivializable, so we may always choose a trivialization $\phi : \mathcal{G} \to \hat{G}$. Note that this trivialization induces an isomorphism between the groups $\mathrm{Aut}(\mathcal{G})$ and $\hat{G}$. Moreover, there is a left action of $\hat{G}$ on the set of trivializations of $\mathcal{G}$. The trivialization ϕ induces a compatible collection of maps $\phi_V : V_{\mathcal{G}} \to \hat{V} := V \otimes_k F$; we will usually omit the superscript from the notation.

A flat structure on a principal G-bundle is a formal derivation ∇ that determines a compatible family of flat connections ∇_V (which we usually write simply as ∇) on $V_{\mathcal{G}}$ for all $(V, \rho) \in \mathrm{Rep}(G)$. In practice, once one has fixed a trivialization ϕ for $\mathcal{G}$, ∇ may be expressed in terms of a one-form with coefficients in $\hat{\mathfrak{g}}$. This means that there exists an element $[\nabla]_\phi \in \Omega^1(\hat{\mathfrak{g}})$, called the *matrix* of ∇ with respect to the trivialization ϕ, for which the induced connection on $\hat{V}$ is given by $d + \rho([\nabla]_\phi)$. We will formally write $\nabla_\phi = d + [\nabla]_\phi$ for the flat structure on $\hat{G}$ induced by ϕ. To express the effect of change of trivialization (or gauge change) on the matrix, we first observe that there is a natural action of $\hat{G}$ on $\Omega^1(\hat{\mathfrak{g}})$ which we will denote by Ad^*.[4] Recalling that ι_τ is the inner derivation by the Euler vector field, we can write $[\nabla]_\phi = \iota_\tau([\nabla]_\phi)\frac{dz}{z} \in \hat{\mathfrak{g}}\frac{dz}{z}$; with this notation, $\mathrm{Ad}^*(g)([\nabla]_\phi) = \mathrm{Ad}(g)(\iota_\tau([\nabla]_\phi))\frac{dz}{z}$. The gauge transformation action of $\hat{G}$ is then given by

$$[\nabla]_{g\phi} = g \cdot [\nabla]_\phi = \mathrm{Ad}^*(g)([\nabla]_\phi) - (dg)g^{-1}. \tag{7}$$

Here, the right-invariant Maurer-Cartan form $(dg)g^{-1}$ lies in $\Omega^1(\hat{\mathfrak{g}})$ and may be calculated explicitly.

We remark that there is an equivalence of categories between flat GL_n-bundles and flat rank n vector bundles given by $(\mathcal{G}, \nabla) \mapsto (V_{\mathcal{G}}, \nabla_V)$, where V is the standard representation.

2.5. **The Bruhat-Tits building.** As for flat vector bundles, we will study flat G-bundles in terms of appropriate periodic $\mathbb{R}$-filtrations; however, here we will need compatible filtrations on $\hat{V}$ for all $V \in \mathrm{Rep}(G)$. We will consider a class of filtrations

[4]As we will see in Remark 3.5, this may be viewed as the coadjoint action on $\mathfrak{g}^\vee \otimes F$.

introduced by Moy and Prasad that are parameterized by a complex $\mathcal{B}(\hat{G})$ called the Bruhat-Tits building of $\hat{G}$ [28].

In this section, we recall some basic information about the Bruhat-Tits building. Fix a maximal torus $T \subset G$ with corresponding Cartan subalgebra $\mathfrak{t}$. Let $N = N(T)$ be the normalizer of T, so that the Weyl group W of G is isomorphic to N/T. We denote the set of roots with respect to T by Φ; if $\alpha \in \Phi$, $U_\alpha \subset G$ is the associated root subgroup and $\mathfrak{u}_\alpha \subset \mathfrak{g}$ is the weight space for $\mathfrak{t}$ corresponding to α. We will write Z for the center of G and $\mathfrak{z}$ for its Lie algebra. We also let $W_{\mathrm{aff}} = N(T(F))/T(\mathfrak{o})$ denote the affine Weyl group.

We will define two versions of the building, the reduced building $\bar{\mathcal{B}}(\hat{G})$ and the enlarged building $\mathcal{B}(\hat{G})$, which we will simply refer to as the building of $\hat{G}$. Both are defined as appropriate unions of affine spaces called apartments, which are in one-to-one correspondence with the split maximal tori in $\hat{G}$. We start by defining the apartments $\bar{\mathcal{A}}_0 = \bar{\mathcal{A}}(\hat{T})$ and $\mathcal{A}_0 = \mathcal{A}(\hat{T})$ associated to $\hat{T}$, which we call standard apartments; they are affine spaces isomorphic to $X_*(T \cap [G,G]) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\bar{\mathcal{A}}_0 \times (X_*(Z) \otimes_{\mathbb{Z}} \mathbb{R}) \cong X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ respectively. The standard apartments are endowed with a cell structure induced by the roots. Explicitly, the facets are intersections of half-spaces determined by the affine hyperplanes $\{x \in \mathcal{A}_0 \mid \alpha(x) = j\}$ for $\alpha \in \Phi$ and $j \in \mathbb{Z}$ (and similarly for $\bar{\mathcal{A}}_0$). A facet in $\bar{\mathcal{A}}_0$ is a polysimplex while the facets in $\mathcal{A}_0$ are pullbacks of those in $\bar{\mathcal{A}}_0$ under the projection map. The group $N(\hat{T})$ acts on these apartments by affine transformations which preserve the cell structure.

We next need to define the parahoric subgroup $\hat{G}_x$ associated to $x \in \mathcal{A}_0$. Given $\alpha \in \Phi$, let $\hat{U}_{\alpha,x}$ be the image of $z^{\lceil -\alpha(x) \rceil}\mathfrak{o}$ under the isomorphism $F \cong \hat{U}_\alpha$. The subgroup $\hat{G}_x$ is then generated by $T(\mathfrak{o})$ and the $\hat{U}_{\alpha,x}$'s. Its pro-unipotent radical is denoted by $\hat{G}_{x,+}$. The reduced building $\bar{\mathcal{B}}(\hat{G})$ is defined as the quotient of $\hat{G} \times \bar{\mathcal{A}}_0$ by the following equivalence relation [25, §9]:

$$(g,x) \sim (h,y) \iff \text{there exists } n \in N(\hat{T}) \text{ such that } y = n \cdot x \text{ and } g^{-1}hn \in \hat{G}_{x+}.$$

We then set $\mathcal{B}(\hat{G}) = \bar{\mathcal{B}}(\hat{G}) \times (X_*(Z) \otimes_{\mathbb{Z}} \mathbb{R})$. We will view $\bar{\mathcal{B}}(\hat{G})$ as a subset of $\mathcal{B}(\hat{G})$ via the zero section. The group $\hat{G}$ acts on the buildings, and the buildings inherit cell structures from the standard apartments via these actions. The cell structures are compatible with the projection $\mathcal{B}(\hat{G}) \to \bar{\mathcal{B}}(\hat{G})$.

Since every point is conjugate to an element in the standard apartments, the buildings are covered by translates of the standard apartments. These are the apartments of the buildings. They are parameterized by the split maximal tori in $\hat{G}$. More specifically, any split maximal torus T' is of the form $g\hat{T}g^{-1}$ for some $g \in \hat{G}$; we define the corresponding apartments via $\mathcal{A}(T') = g\mathcal{A}_0$ and $\bar{\mathcal{A}}(T') = g\bar{\mathcal{A}}_0$.

Given $x \in \mathcal{B}(\hat{G})$, we define the parahoric subgroup $\hat{G}_x$ to be the connected stabilizer of x; it is a finite index subgroup of $\mathrm{Stab}(x)$.[5] This definition agrees The corresponding Lie subalgebra is denoted $\hat{\mathfrak{g}}_x$. The parahoric subgroups are in bijective correspondence with the facets of $\mathcal{B}(\hat{G})$. Indeed, $\hat{G}_x = \hat{G}_y$ if and only if x and y are in the same facet. In particular, the parahoric subgroups only depend on the image of x in $\bar{\mathcal{B}}(\hat{G})$. The maximal parahoric subgroup $G(\mathfrak{o})$ corresponds to the origin in $X_*(T \cap [G,G]) \otimes_{\mathbb{Z}} \mathbb{R}$. See [7, 8, 34, 25] for more details.

[5] If $x \in \mathcal{A}_0$, this definition agrees with our previous definition. Indeed, this follows from the fact that for $x \in \mathcal{A}_0$, $\mathrm{Stab}(x)$ is generated by $\mathrm{Stab}_{N(\hat{T})}(x)$ and the $\hat{U}_{\alpha,x}$'s [25, §8].

We will also need to consider the Bruhat-Tits building $\mathcal{B}(\mathrm{Aut}(\mathcal{G}))$ for the F-group of automorphisms of the formal principal G-bundle $\mathcal{G}$. This may be defined in exactly the same way, starting with a fixed maximal torus of $\mathrm{Aut}(\mathcal{G})$. Alternatively, any trivialization of $\mathcal{G}$ induces an isomorphism between $\mathcal{B}(\mathrm{Aut}(\mathcal{G}))$ and $\mathcal{B}(\hat{G})$, and in fact, we see that $\mathcal{B}(\mathrm{Aut}(\mathcal{G})) = \mathcal{B}(\hat{G}) \times^{\hat{G}} \mathcal{G}$.

2.6. **Moy-Prasad filtrations.** We now define filtrations associated to points in $\mathcal{B}(\hat{G})$.

Let V be a finite-dimensional representation of G over k, and let $\hat{V}$ be the corresponding representation of $\hat{G}$. For any $x \in \mathcal{B}(\hat{G})$, the Moy-Prasad filtration associated to x is a decreasing $\mathbb{R}$-filtration $\{\hat{V}_{x,r} \mid r \in \mathbb{R}\}$ of $\hat{V}$ by $\mathfrak{o}$-lattices [17, 28]. We briefly review the construction. First, assume $x \in \mathcal{A}_0$. In this case, the filtration can be obtained from an $\mathbb{R}$-grading on $V \otimes_k k[z, z^{-1}]$ (cf. [17]). If $\chi \in X^*(T)$, let $V_\chi \subset V$ be the weight space $\{v \in V \mid sv = \chi(s)v \quad \forall s \in T\}$. The rth graded subspace is defined to be

$$\hat{V}_{x,\mathcal{A}_0}(r) = \bigoplus_{\chi(x)+m=r} V_\chi z^m \subset \hat{V}. \tag{8}$$

Note that the set $\{r \mid \hat{V}_{x,\mathcal{A}_0}(r) \neq 0\}$ is discrete and closed under translations by $\mathbb{Z}$. For any $r \in \mathbb{R}$, define

$$\hat{V}_{x,r} = \prod_{s \geq r} \hat{V}_{x,\mathcal{A}_0}(s) \subset \hat{V}; \quad \hat{V}_{x,r+} = \prod_{s > r} \hat{V}_{x,\mathcal{A}_0}(s) \subset \hat{V}.$$

If $x \in \mathcal{B}(\hat{G})$ is arbitrary, we write $x = gy$ for $g \in \hat{G}$, $y \in \mathcal{A}_0$, and set $\hat{V}_{x,r} = g\hat{V}_{y,r}$. It is shown in [28] that this definition is independent of the choice of g and y.

The collection of lattices $\{\hat{V}_{x,r}\}$ is the Moy-Prasad filtration on $\hat{V}$ associated to x. It is immediate that $\hat{V}_{x,r+1} = z\hat{V}_{x,r}$. Note that if $x \in \mathcal{A}_0$, $\hat{V}_{x,r}/\hat{V}_{x,r+} \cong \hat{V}_{x,\mathcal{A}_0}(r) \neq \{0\}$ if and only if there exists $\chi \in X^*(T)$ such that $V_\chi \neq \{0\}$ and $r - \chi(x) \in \mathbb{Z}$. We call the real numbers for which $\hat{V}_{x,r} \neq \hat{V}_{x,r+}$ the *critical numbers* of V at x, and denote the set of such points by $\mathrm{Crit}_x(V)$. We will write Crit_x for $\mathrm{Crit}_x(\mathfrak{g})$, the critical numbers of the adjoint representation. The set $\mathrm{Crit}_x(V)$ is a discrete subset of $\mathbb{R}$ closed under translation by $\mathbb{Z}$. If the set of weights in V is closed under inversion, then $\mathrm{Crit}_x(V)$ is also symmetric around 0. In particular, this is the case for representations on which $\mathfrak{z}$ acts trivially such as the adjoint and coadjoint representations $\mathfrak{g}$ and $\mathfrak{g}^\vee$. We further observe that when $\mathfrak{z}$ acts trivially, the Moy-Prasad filtrations depend only on the image $\bar{x} \in \bar{\mathcal{B}}(\hat{G})$ of x.

Moy-Prasad filtrations are canonically associated to points in the building, but this is not true for gradings. Given $T' \subset G$ a maximal torus and $x' \in \mathcal{A}' = \mathcal{A}(\hat{T}')$, one can define an analogous grading $\hat{V}_{x',\mathcal{A}'}(r)$; moreover, if $T' = gTg^{-1}$ and $gx = x'$ for some $g \in G$, then $\hat{V}_{x',\mathcal{A}'}(r) = g\hat{V}_{x,\mathcal{A}_0}(r)$. Accordingly, if $gx = x$, but $gTg^{-1} \neq T$, we need not have $\hat{V}_{x,\mathcal{A}'}(r) = \hat{V}_{x,\mathcal{A}_0}(r)$, so the grading depends on a choice of apartment containing x. In this paper, we will only consider gradings coming from $\mathcal{A}_0$, so we will usually simplify notation by writing $\hat{V}_x(r)$ instead of $\hat{V}_{x,\mathcal{A}_0}(r)$.

We will also need to consider Moy-Prasad filtrations on the space of 1-forms $\Omega^1(\hat{V})$ and on the smooth k-dual space $\hat{\mathfrak{g}}^\vee$. We define $\Omega^1(\hat{V})_{x,r}$ to be the image of $\hat{V}_{x,r}$ via the isomorphism $\hat{V} \to \Omega^1(\hat{V})$ given by $u \mapsto u\frac{dz}{z}$. Next, our fixed nondegenerate invariant symmetric bilinear form $\langle , \rangle$ on $\mathfrak{g}$ induces an isomorphism $\hat{\mathfrak{g}} \to \hat{\mathfrak{g}}^\vee$ given by $X \mapsto \mathrm{Res}\,\langle X, \cdot\rangle\,\frac{dz}{z}$. We set $\hat{\mathfrak{g}}^\vee_{x,r}$ equal to the image of $\hat{\mathfrak{g}}_{x,r}$ under this isomorphism. Note that composing the isomorphisms $\Omega^1(\hat{\mathfrak{g}}) \to \hat{\mathfrak{g}}$ and $\hat{\mathfrak{g}} \to \hat{\mathfrak{g}}^\vee$ gives an isomorphism $\Omega^1(\hat{\mathfrak{g}}) \to \hat{\mathfrak{g}}^\vee$ which preserves Moy-Prasad filtrations.

There is also a corresponding filtration $\{\hat{G}_{x,r}\}_{r\in\mathbb{R}_{\geq 0}}$ of the parahoric subgroup $\hat{G}_x = \hat{G}_{x,0}$ for $x \in \mathcal{B}(\hat{G})$ [28, Section 2.6]. To define $\hat{G}_{x,r}$, first let $\hat{T}_r$ be the subgroup of $\hat{T}$ generated by the images of the $\lceil r \rceil$th congruent subgroup in $\mathfrak{o}^\times$ under each cocharacter $\lambda \in X_*(T)$. Similarly, if $\psi \in \Phi$, let $\hat{U}_{\psi,x,r}$ be the image of $z^{\lceil r-\psi(x)\rceil}\mathfrak{o}$ under the isomorphism $F \cong \hat{U}_\psi$. The subgroup $\hat{G}_{x,r}$ is generated by the $\hat{T}_r$ and the $\hat{U}_{\psi,x,r}$'s. We set $\hat{G}_{x,r+} = \bigcup_{s>r} \hat{G}_{x,s}$. In particular, we write $\hat{G}_{x+} = \hat{G}_{x,0+}$; it is the pro-unipotent radical of $\hat{G}_x$ [28, p.397]. (We use similar notation for the Lie algebras: $\hat{\mathfrak{g}}_x = \hat{\mathfrak{g}}_{x,0}$ and $\hat{\mathfrak{g}}_{x+} = \hat{\mathfrak{g}}_{x,0+}$.) For $r > 0$, there is a natural isomorphism $\hat{G}_{x,r}/\hat{G}_{x,r+} \cong \hat{\mathfrak{g}}_{x,r}/\hat{\mathfrak{g}}_{x,r+}$ [28, p.399], so that this quotient group is unipotent. However, $\hat{G}_x/\hat{G}_{x+}$ is reductive. Indeed, in Section 3.1, we will construct an explicit isomorphism between this group and a maximal rank reductive subgroup of G. The parahoric subgroup $\hat{G}_x$ stabilizes $\hat{V}_{x,r}$, and $\hat{V}_{x,r}/\hat{V}_{x,r+}$ is a representation of $\hat{G}_x/\hat{G}_{x+}$. Note that when G is a torus, i.e., $G = T$, then the reduced building is a point. Accordingly, there is a unique Moy-Prasad filtration on $\hat{T}_0 = T(\mathfrak{o})$ and $\hat{\mathfrak{t}}$; the filtration on $T(\mathfrak{o})$ is given by the subgroups $\hat{T}_r$ above.

One can similarly define Moy-Prasad filtrations on the vector bundles $V_{\mathcal{G}}$ parameterized by points in $\mathcal{B}(\mathrm{Aut}(\mathcal{G}))$. Of course, fixing a trivialization identifies these filtrations with filtrations on the F-vector spaces $\hat{V}$. To be explicit, let ϕ be a trivialization of $\mathcal{G}$. Given $x \in \mathcal{B}(\mathrm{Aut}(\mathcal{G}))$, let $\phi(x)$ be the induced point in $\mathcal{B}(\hat{G})$. Then, the trivialization ϕ_V identifies $(V_{\mathcal{G}})_{x,r}$ and $\hat{V}_{\phi(x),r}$ for all r.

2.7. **Strata and flat G-bundles.** In this section, we generalize the geometric theory of strata for GL_n given in [4] to arbitrary reductive groups. More precisely, given a formal flat G-bundle $(\mathcal{G}, \nabla)$, its leading term with respect to a Moy-Prasad filtration is described either in terms of a $\mathcal{G}$-stratum or in terms of a trivialization and a $\hat{G}$-stratum.

Definition 2.8. Let $x \in \mathcal{B}(\hat{G})$ and let $r \geq 0$ be a real number. A *$\hat{G}$-stratum* of depth r is a triple (x, r, β) such that $\beta \in (\hat{\mathfrak{g}}_{x,r}/\hat{\mathfrak{g}}_{x,r+})^\vee$.

One can similarly define $\mathcal{G}$-strata associated to points in $\mathcal{B}(\mathrm{Aut}(\mathcal{G}))$. For a stratum (x, r, β) of this type, $x \in \mathcal{B}(\mathrm{Aut}(\mathcal{G}))$ and $\beta \in ((\mathfrak{g}_{\mathcal{G}})_{x,r}/(\mathfrak{g}_{\mathcal{G}})_{x,r+})^\vee$.

Recall from geometric invariant theory that if W is a representation of a reductive group H, then a point $w \in W$ is called unstable if 0 is in the Zariski closure of the orbit $H \cdot w$; otherwise, it is semistable. In characteristic zero, w is unstable if and only if there exists a one-parameter subgroup $\gamma : \mathbb{G}_m \to H$ such that $\lim_{t\to 0} \gamma(t) \cdot w = 0$ [23]. For example, $X \in \hat{\mathfrak{g}}$ is unstable if and only if it is nilpotent. We call a functional in $\hat{\mathfrak{g}}^\vee$ to be nilpotent if it is unstable.

Definition 2.9. We say that a stratum (x, r, β) is *fundamental* if the functional β is a semistable point of the $\hat{G}_x/\hat{G}_{x+}$-representation $(\hat{\mathfrak{g}}_{x,r}/\hat{\mathfrak{g}}_{x,r+})^\vee$.

Note that a $\mathrm{GL}(\hat{V})$-stratum is the same thing as a $\hat{V}$-stratum in the sense of Definition 2.3.

Remark 2.10. Observe that (x, r, β) can only be fundamental when $r \in \mathrm{Crit}_x$.

We will give some equivalent conditions which are easier to compute in Proposition 3.7. For example, we will see in Proposition 3.6 that $(\hat{\mathfrak{g}}_{x,r}/\hat{\mathfrak{g}}_{x,r+})^\vee$ may be identified with $\hat{\mathfrak{g}}^\vee_{x,-r}/\hat{\mathfrak{g}}^\vee_{x,-r+}$. We will call $\tilde{\beta} \in \hat{\mathfrak{g}}^\vee_{x,-r}$ a *representative* for β if β corresponds to $\tilde{\beta} + \hat{\mathfrak{g}}^\vee_{x,-r+} \in \hat{\mathfrak{g}}^\vee_{x,-r}/\hat{\mathfrak{g}}^\vee_{x,-r+}$. Then, (x, r, β) is fundamental if and only if every representative $\tilde{\beta}$ is nonnilpotent.

We now show how to associate strata to formal flat G-bundles. Recall that $\hat{\mathfrak{g}}^\vee \cong \Omega^1(\hat{\mathfrak{g}})$ and similarly $\mathfrak{g}^\vee_{\mathcal{G}} \cong \Omega^1(\mathfrak{g}_{\mathcal{G}})$. In both cases, we will let $X_{\tilde{\beta}}$ be the one-form corresponding to the functional $\tilde{\beta}$.

Definition 2.11. A flat G-bundle $(\mathcal{G}, \nabla)$ contains the $\mathcal{G}$-stratum (x, r, β) if, for any (or equivalently, some) representative $\tilde{\beta} \in {\mathfrak{g}_{\mathcal{G}}}^\vee_{x,-r}$ of β,

$$(\nabla - i\frac{dz}{z} - X_{\tilde{\beta}})((V_{\mathcal{G}})_{x,i}) \subset \Omega^1(V_{\mathcal{G}})_{x,(i-r)+}, \quad \forall i \in \mathbb{R}, V \in \mathrm{Rep}(G). \tag{9}$$

It will be convenient to make use of a similar concept involving an explicit trivialization.

Definition 2.12. A flat G-bundle $(\mathcal{G}, \nabla)$ contains the $\hat{G}$-stratum (x, r, β) with respect to the trivialization ϕ if, for any (or equivalently, some) representative $\tilde{\beta} \in \hat{\mathfrak{g}}^\vee_{x,-r}$ of β,

$$(\nabla_\phi - i\frac{dz}{z} - X_{\tilde{\beta}})(\hat{V}_{x,i}) \subset \Omega^1(\hat{V})_{x,(i-r)+}, \quad \forall i \in \mathbb{R}, V \in \mathrm{Rep}(G). \tag{10}$$

Lemma 4.2 states that if $(\mathcal{G}, \nabla)$ contains the stratum (x, r, β) with respect to ϕ, then it contains $(gx, r, g\beta)$ with respect to $g\phi$.

Since $X_{\tilde{\beta}}(\hat{V}_{x,i}) \subset \Omega^1(\hat{V})_{x,i-r}$ and $i\frac{dz}{z}(\hat{V}_{x,i}) \subset \Omega^1(\hat{V}_{x,i}) \subset \Omega^1(\hat{V})_{x,i-r}$, an immediate consequence of (10) is that $\nabla_\phi(\hat{V}_{x,i}) \subset \Omega^1(\hat{V})_{x,i-r}$ for all i. Also, note that the term $i\frac{dz}{z}$ can be omitted from (10) if $r > 0$. Similar considerations apply to Definition 2.11.

As we will see in Proposition 4.3, when $x \in \mathcal{A}_0$, stratum containment with respect to ϕ is equivalent to a concrete and easily verified condition on the matrix $[\nabla]_\phi$.

Just as for flat vector bundles, there is a notion of regular and irregular singularity for a flat G-bundle.

Definition 2.13. We say that a formal flat G-bundle $(\mathcal{G}, \nabla)$ is *regular singular* if for all representations V, the associated flat vector bundle $(V_{\mathcal{G}}, \nabla_V)$ is regular singular. Otherwise, it is called *irregular singular*.

As one would hope, a flat GL_n-bundle is regular singular if and only if the corresponding flat vector bundle is regular singular. This is indeed true; see Corollary A.6.

It is more subtle to define the *slope* of $(\mathcal{G}, \nabla)$. One of the major results of this paper is using the theory of fundamental strata to provide the appropriate generalization.

2.8. **Main theorems.** We can now state the main results of the paper. The proofs are given in Section 4.3.

In the following theorem, $\mathcal{A}_0 \subset \mathcal{B}(\hat{G})$ is the apartment corresponding to the split maximal torus $\hat{T} \subset \hat{G}$, where T is a fixed maximal torus in G. We call a point in $\mathcal{B}(\hat{G})$ (resp. $\mathcal{B}(\mathrm{Aut}(\mathcal{G}))$) rational if for every $V \in \mathrm{Rep}(G)$, the critical numbers for the filtration on $\hat{V}$ (resp. $V_{\mathcal{G}}$) are rational. We denote the set of rational points in

$\mathcal{B}(\mathrm{Aut}(\mathcal{G}))$, $\mathcal{B}(\hat{G})$, and $\mathcal{A}_0$ by $\mathcal{B}(\mathrm{Aut}(\mathcal{G}))^{\mathrm{rat}}$, $\mathcal{B}(\hat{G})^{\mathrm{rat}}$, and $\mathcal{A}_0^{\mathrm{rat}}$ respectively. We call a stratum rational if it is based at a rational point in the building. *Optimal points* are certain points in $\mathcal{A}_0$ that generalize the barycenters of simplices in the reduced building for GL_n. (See Section 4.3 for the definition.) One of their nice properties is that they lie in $\mathcal{A}_0^{\mathrm{rat}}$.

We only wish to consider nonrational strata when $\mathbb{R} \subset k$. Accordingly, throughout the paper, we adopt the following convention:

Convention. *All strata are assumed to be rational without further comment unless* $\mathbb{R} \subset k$.

Theorem 2.14. *Every flat G-bundle $(\mathcal{G}, \nabla)$ contains a fundamental $\mathcal{G}$-stratum (x, r, β) with $x \in \mathcal{B}(\mathrm{Aut}(\mathcal{G}))^{\mathrm{rat}}$; the depth r is positive if and only if $(\mathcal{G}, \nabla)$ is irregular singular. More precisely, x can be chosen to be the preimage of an optimal point in $\mathcal{A}_0$ under some trivialization ϕ. Moreover, the following statements hold.*

1. *If $(\mathcal{G}, \nabla)$ contains the stratum (y, r', β'), then $r' \geq r$.*
2. *If $(\mathcal{G}, \nabla)$ is irregular singular, a stratum (y, r', β') contained in $(\mathcal{G}, \nabla)$ is fundamental if and only if $r' = r$.*

It is an important question to understand the set of strata contained in a given flat G-bundle. As a first step in this direction, we have shown that two such strata of the same depth r are *associates* of each other. The formal definition is given in Definition 3.10.

Theorem 2.15. *Suppose that $(\mathcal{G}, \nabla)$ contains the $\hat{G}$-strata (x, r, β) and (y, r, β') with respect to the trivializations ϕ and ϕ' respectively. Then, (x, r, β) and (y, r, β') are associates of each other. In particular, all fundamental strata contained in $(\mathcal{G}, \nabla)$ are associates of each other.*

We now define the slope of a flat G-bundle.

Definition 2.16. The *slope* of the flat G-bundle $(\mathcal{G}, \nabla)$ is the depth of any fundamental stratum contained in $(\mathcal{G}, \nabla)$.

This definition makes sense by Theorem 2.14. It also follows that the slope is always an *optimal number* in the sense of [1], i.e., a critical number for the filtration on $\hat{\mathfrak{g}}$ determined by an optimal point.

Theorem 2.17. *The slope of the flat G-bundle $(\mathcal{G}, \nabla)$ is a nonnegative rational number. It is positive if and only if $(\mathcal{G}, \nabla)$ is irregular singular. The slope may also be characterized as*

1. *the minimum depth of any stratum contained in $(\mathcal{G}, \nabla)$;*
2. *the minimum depth of any stratum contained in $(\mathcal{G}, \nabla)$ and based at an optimal point;*
3. *the maximum slope of the associated flat vector bundles $(V_{\mathcal{G}}, \nabla_V)$; or*
4. *the maximum slope of the flat vector bundles associated to the adjoint representations and the characters.*

Remark 2.18. Since the category of formal flat $\mathrm{GL}_n(V)$-bundles is equivalent to the category of rank n flat vector bundle, one would expect the notions of strata, stratum containment, and slope on the two categories to correspond. This is indeed the case; see Corollaries A.2 and A.6.

If E is a degree e field extension of F, then E is generated by an element u_E with $u_E^e = z$. We let $\pi_E : \mathrm{Spec}(E) \to \mathrm{Spec}(F)$ be the associated map of spectra. The pullback of a flat G-bundle $(\mathcal{G}, \nabla)$ to $\mathrm{Spec}(E)$ will be denoted by $(\pi_E^* \mathcal{G}, \pi_E^* \nabla)$.

Lemma 2.19. *If a flat G-bundle $(\mathcal{G}, \nabla)$ over $\mathrm{Spec}(F)$ has slope r, then $(\pi_E^* \mathcal{G}, \pi_E^* \nabla)$ has slope $[E : F]r$.*

Proposition 2.20. *A flat G-bundle $(\mathcal{G}, \nabla)$ has slope r if and only if there exists a finite field extension E/F and a trivialization ϕ of the pullback of $(\mathcal{G}, \nabla)$ to $\mathrm{Spec}(E)$ such that*

$$(\pi_E^*[\nabla])_\phi = \sum_{i=-n}^{\infty} M_i u^i \frac{du}{u}, \tag{11}$$

with $M_i \in \mathfrak{g}$, M_{-n} is nonnilpotent, and $r = n/[E : F]$.

Remark 2.21. In [15, Section 5], Frenkel and Gross suggest the condition of Proposition 2.20 as a definition of the slope of a flat G-bundle. The corollary shows that this approach is independent of the choices and hence provides an alternate description of the slope. (This can also be shown directly using results from Section 9 of [2].)

We remark that there are advantages to the approach to the slope taken in this paper. For example, there is no need to pass to a ramified cover in order to put $[\nabla]$ into the appropriate form; indeed, there is an explicit algorithm appearing in the proof of Theorem 2.14 that allows one to find a trivialization of $\mathcal{G}$ with respect to which $(\mathcal{G}, \nabla)$ contains a fundamental stratum. Moreover, one is able to predict which rational numbers occur as the slope of a flat G-bundle, since these must be optimal numbers for $\hat{G}$. Finally, a fundamental stratum provides additional structural information about a formal flat G-bundle beyond the slope. This is explored further in [6].

There is also a recent preprint of Chen and Kamgarpour which shows how the slope may be characterized using *opers* [12]. More specifically, they define the slope of an oper and show that it agrees with the slope of the underlying flat G-bundle, using the formulation of [15]. Since any formal flat G-bundle has an oper structure [16], its slope can be given as the slope of any associated oper. However, unlike fundamental strata, there is no known algorithm for effectively computing the oper structure on a flat G-bundle.

3. Moy-Prasad filtrations and strata

We continue with the notation of Section 2, so G is a connected reductive group over k with Lie algebra $\mathfrak{g}$. All results about Bruhat-Tits buildings, Moy-Prasad filtrations, and strata will be stated only for $\hat{G}$, but analogous statements hold throughout for $\mathrm{Aut}(\mathcal{G})$. To simplify notation, we will write $\mathcal{B} = \mathcal{B}(\hat{G})$ and $\bar{\mathcal{B}} = \bar{\mathcal{B}}(\hat{G})$.

3.1. **Filtrations on loop group representations.**

In this section, we will discuss some further properties of Moy-Prasad filtrations.

Recall that $\mathcal{A}_0 \cong X_*(T) \otimes \mathbb{R}$ is the standard apartment of $\mathcal{B}$ determined by the split maximal torus $\hat{T}$. As long as $\mathbb{R} \subset k$, points in $\mathcal{A}_0$ may also be viewed as elements of $\mathfrak{t}$. The Lie algebra $\mathfrak{t}'$ of any split torus T' over k is canonically isomorphic to $X_*(T') \otimes_{\mathbb{Z}} k$; the map is induced by sending a cocharacter λ to $d\lambda(1)$.

If k' is a subfield of k, we let $\mathfrak{t}'_{k'}$ be the image of $X_*(T') \otimes_{\mathbb{Z}} k'$ under this map. Assuming that $\mathbb{R} \subset k$, we have $\mathcal{A}_0 \cong \mathfrak{t}_{\mathbb{R}}$, and we write $\tilde{x} \in \mathfrak{t}_{\mathbb{R}}$ for the image of $x \in \mathcal{A}_0$.

Remark 3.1. A point $x \in \mathcal{B}$ is called rational if $\mathrm{Crit}_x(V) \subset \mathbb{Q}$ for all $V \in \mathrm{Rep}(G)$. Every element of $\mathcal{B}^{\mathrm{rat}}$ is conjugate to a point in $\mathcal{A}_0^{\mathrm{rat}}$, and $\mathcal{A}_0^{\mathrm{rat}}$ consists precisely of those elements of $\mathcal{A}_0$ coming from $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. In particular, the map $x \mapsto \tilde{x}$ defined above when $\mathbb{R} \subset k$ always makes sense when restricted to $\mathcal{A}_0^{\mathrm{rat}}$; it gives an isomorphism $\mathcal{A}_0^{\mathrm{rat}} \cong \mathfrak{t}_{\mathbb{Q}}$. We adopt the convention that when the notation $\tilde{x}$ for $x \in \mathcal{A}_0$ is used, it will automatically mean that $x \in \mathcal{A}_0^{\mathrm{rat}}$ unless $\mathbb{R} \subset k$.

There is a very useful alternate description of Moy-Prasad filtrations for $x \in \mathcal{A}_0$ in terms of operators related to the Euler vector field $\tau = z\frac{d}{dz}$.

Proposition 3.2. *Let V be a finite-dimensional representation of G, and fix $x \in \mathcal{A}_0$ and $r \in \mathbb{R}$.*

1. *The space $\hat{V}_x(r)$ is the eigenspace corresponding to the eigenvalue r in $\hat{V}$ for the differential operator $\tau + \tilde{x}$.*
2. *An element $v \in \hat{V}$ lies in $\hat{V}_{x,r}$ if and only if $(\tau + \tilde{x})(v) - rv \in \hat{V}_{x,r+}$.*
3. *The set $\hat{V}_x(r)$ constitutes a full set of coset representatives for the coset space $\hat{V}_{x,r}/\hat{V}_{x,r+}$.*
4. *If $X \in \hat{\mathfrak{g}}_x(s)$, then $\mathrm{ad}(X)(\hat{V}_x(r)) \subset \hat{V}_x(r+s)$.*

Proof. Each of these statements follows immediately from the definitions of $\hat{V}_x(r)$ and $\hat{V}_{x,r}$. □

The following lemma allows one to express the action of $\hat{N} = N_F$ on $\mathcal{A}_0$ in term of the differential calculus above.

Lemma 3.3. *Suppose that $n \in \hat{N}$ is a coset representative of $w \in W_{\mathit{aff}}$. For all $x \in \mathcal{A}_0$, $\mathrm{Ad}(n)(\tilde{x}) - \tau(n)n^{-1} \in \widetilde{wx} + \hat{\mathfrak{t}}_{0+}$.*

Proof. Write $n = tn'$, with $t \in \hat{T}$ and $n' \in N$. Since $\hat{T}/\hat{T}_0 \cong X_*(T)$ (as abstract groups), we may write $t = z^y t'$ for some $y \in X_*(T)$ and $t' \in \hat{T}_0$. An application of the Leibniz rule shows that

$$\tau(n)n^{-1} = \mathrm{Ad}(t)\left[\tau(n')(n')^{-1}\right] + \tau(t)t^{-1} = \tau(z^y)z^{-y} + \tau(t')(t')^{-1} \in \tilde{y} + \hat{\mathfrak{t}}_{0+}. \quad (12)$$

Here, $\tau(n')(n')^{-1} = 0$, since $n' \in G$.

Define $a \in \mathcal{A}_0$ by $\tilde{a} = \mathrm{Ad}(n)(\tilde{x}) - \tilde{y}$. Suppose that $v \in \hat{V}_{x,r}$ for some representation V, and let $u = nv \in \hat{V}_{wx,r}$. We will show that

$$\tau(u) + \tilde{a}u \in ru + \hat{V}_{wx,r+}. \quad (13)$$

Assuming this, Proposition 3.2(2) implies that $(\tilde{a} - \widetilde{wx})(u) \in \hat{V}_{wx,r+}$. Furthermore, if this expression holds for arbitrary V, r, and u, then $\tilde{a} = \widetilde{wx}$. (Otherwise, let V be a faithful representation. Then, there exists $r \in \mathbb{R}$ and an eigenvector in $\hat{V}_{wx,r}/\hat{V}_{wx,r+}$ with nonzero eigenvalue for the action of $\tilde{a} - \widetilde{wx}$, a contradiction.)

It remains to prove (13). Calculating, we obtain

$$\begin{aligned} \tau(nv) + \tilde{a}u &= \tau(n)n^{-1}u + n(\tau(v)) + \tilde{a}u && \text{by the Leibniz rule} \\ &\in n(\tau(v)) + (\mathrm{Ad}(n)\tilde{x})(u) + \hat{\mathfrak{t}}_{0+}u && \text{by (12)} \\ &= n((\tau + \tilde{x})(v)) + \hat{\mathfrak{t}}_{0+}u && \text{by definition of } a \\ &\subset rn(v) + n\hat{V}_{x,r+} = ru + \hat{V}_{wx,r+} && \text{by Proposition 3.2(2).} \end{aligned}$$

□

Next, we discuss further the quotient group $\hat{G}_x/\hat{G}_{x+}$. This group is isomorphic to a reductive maximal rank subgroup of G. We will construct an explicit isomorphism. We will first define this isomorphism for $x \in \mathcal{A}_0$ and then will apply equivariance to define the map in the general case.

Fix $x \in \mathcal{A}_0$. Let $\mathfrak{h}_x \subset \mathfrak{g}$ be the subalgebra $\mathfrak{h}_x = \mathfrak{t} \oplus \bigoplus_{\{\alpha | d\alpha(\tilde{x}) \in \mathbb{Z}\}} \mathfrak{u}_\alpha$. It is the Lie algebra of the reductive subgroup $H_x \subset G$ generated by T and the corresponding root subgroups U_α. The group H_x is the connected centralizer of $\exp(2\pi i \tilde{x}) \in G$.[6] Note that H_x only depends on the image of x in $\bar{\mathfrak{B}}$.

Next, we define a T-equivariant homomorphism $\theta'_x : H_x \to \hat{G}_x$. Choose isomorphisms $u_\alpha : \mathbb{G}_a \to U_\alpha$ giving a realization of the root system of H_x with respect to T in the sense of Springer [33, Section 8.1]. On the generating subgroups, θ'_x is defined via $T \hookrightarrow T(\mathfrak{o})$ and $\theta'_x(u_\alpha(c)) = u_\alpha(cz^{-\alpha(\tilde{x})})$ for $c \in k$. It is easy to check that θ'_x satisfies the relations among the generators (see for example [33, Section 9.4]) and hence is a homomorphism. (If $\tilde{x} \in \mathfrak{t}_\mathbb{Q}$, then θ'_x is just conjugation by $z^{-\tilde{x}}$; here, if $\lambda_1, \dots \lambda_n$ is a basis for $X_*(T)$ and $\tilde{x} = \sum d\lambda_i(\tilde{x}_i)$, then $z^{-\tilde{x}} = \prod \lambda_i(z^{-\tilde{x}_i}) \in T(\bar{F})$.)

Let $\theta_x : H_x \to \hat{G}_x/\hat{G}_{x+}$ be the induced map. Since

$$\hat{\mathfrak{g}}_x/\hat{\mathfrak{g}}_{x+} \cong \hat{\mathfrak{g}}_x(0) = \mathfrak{t} \oplus \bigoplus_{\alpha : d\alpha(\tilde{x}) \in \mathbb{Z}} \mathfrak{u}_\alpha z^{-\alpha(\tilde{x})},$$

it is clear that $d\theta_x$ is an isomorphism. The map θ_x is thus an isogeny which restricts to an isomorphism on the maximal torus T, hence is an isomorphism. The group H_x acts on $\hat{V}_x(r)$ for any r. It is now easy to see that θ_x intertwines the representations $\hat{V}_x(r)$ and $\hat{V}_{x,r}/\hat{V}_{x,r+}$.

If $y = gx$ for some $g \in \hat{G}$, then θ_x composed with conjugation by g gives an isomorphism $\theta_{x,g} : H_x \xrightarrow{\sim} \hat{G}_y/\hat{G}_{y+}$. This map depends on the choice of g. If $y = g'x$ also, then $g' = gh$ for some $h \in \mathrm{Stab}(x)$, and $\theta_{x,g'}$ and $\theta_{x,g}$ differ by an automorphism of H. However, if $\mathrm{Stab}(x) = \hat{G}_x$, then $\theta_{x,g'}$ is just conjugation by $\theta_x^{-1}(h\hat{G}_{x+})$ composed with $\theta_{x,g}$. Thus, in this case, the isomorphism is well-defined up to an inner automorphism of H_x. In particular, this is true when x lies in a minimal facet (i.e., $\hat{G}_x$ is a maximal parahoric) or G is semisimple and simply connected [34, p.50]. We sum up this discussion in the following proposition.

Proposition 3.4. *Fix $x \in \mathcal{A}_0$.*

(1) *The map $\theta_x : H_x \to \hat{G}_x/\hat{G}_{x+}$ is an isomorphism that intertwines the representations $\hat{V}_x(r)$ and $\hat{V}_{x,r}/\hat{V}_{x,r+}$ for all r.*

---

[6] If k contains $\mathbb{R}$ (and hence $\mathbb{C}$), then $\exp(2\pi i \tilde{x})$ makes sense; otherwise, by our convention, $\tilde{x} \in \mathfrak{t}_\mathbb{Q}$.

(2) *If* $\operatorname{Stab}(x) = \hat{G}_x$, *the maps* $\theta_{x,g}$ *induce a compatible family of bijections between the sets of conjugacy classes of* $\hat{G}_y/\hat{G}_{y+}$ *for* $y \in \hat{G} \cdot x$ *and the set of conjugacy classes of* H_x.

Moy-Prasad filtrations are well-behaved with respect to the usual algebraic operations. The filtration on the trivial representation $\hat{k} = F$ is just the usual degree filtration; it is independent of $x \in \mathcal{B}$. If W is a subrepresentation of V, then $\hat{W}_{x,r} = \hat{V}_{x,r} \cap \hat{W}$. If V and W are two representations, then $(\widehat{V \oplus W})_{x,r} = \hat{V}_{x,r} \oplus \hat{W}_{x,r}$. Next, under the identification $\widehat{\operatorname{Hom}_k(V, W)} \cong \operatorname{Hom}_F(\hat{V}, \hat{W})$, we have

$$\widehat{\operatorname{Hom}_k(V, W)}_{x,r} = \{f \mid f(\hat{V}_{x,s}) \subset \hat{W}_{x,s+r} \text{ for all } s \in \mathbb{R}\}. \tag{14}$$

If we let $V^\vee = \operatorname{Hom}(V, k)$ and denote the F-linear dual of an F-vector space U by U^*, then this formula also gives the filtrations on $\widehat{(V^\vee)} = \hat{V}^*$ and on $\hat{V} \otimes_F \hat{W} \cong \widehat{V \otimes W} \cong \widehat{\operatorname{Hom}_k(V^\vee, W)}$.

We have already seen how to define Moy-Prasad filtrations on the space of one-forms $\Omega^1(\hat{V})$; we set $\Omega^1(\hat{V})_{x,r}$ equal to the image of $\hat{V}_{x,r}$ under the isomorphism $\hat{V} \to \Omega^1(\hat{V})$ given by $u \mapsto u\frac{dz}{z}$. While this map depends on the choice of uniformizer, it is easy to see that the filtrations do not.

We have also defined Moy-Prasad filtrations on the smooth k-dual of $\hat{\mathfrak{g}}$. To do this, we identified $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{g}}^\vee$ using the fixed G-invariant form $\langle , \rangle$ and defined filtrations on the dual space by transport of structure. For completeness, we will show how to define filtrations on smooth dual spaces $\hat{V}^\vee$, where $V \in \operatorname{Rep}(G)$ is not necessarily endowed with an appropriate invariant form. The new approach will of course give the same filtrations on $\hat{\mathfrak{g}}^\vee$.

If V is a representation of G, we define an isomorphism $\hat{V}^* = \widehat{(V^\vee)} \xrightarrow{\kappa} (\hat{V})^\vee$ via $\kappa(\alpha)(v) = \operatorname{Res} \alpha(v)\frac{dz}{z}$, and set $(\hat{V}^\vee)_{x,r} = \kappa(\hat{V}^*_{x,r})$. Again, although the map depends on the uniformizer, the filtrations do not. (To see this, simply observe that κ is the composition of the map $\hat{V}^* \to \Omega^1(\widehat{V^\vee})$ with the canonical isomorphism $\Omega^1(\widehat{V^\vee}) \to (\hat{V})^\vee$ sending ω to $v \mapsto \operatorname{Res} \omega(v)$.)

If one further supposes that V is endowed with a nondegenerate G-invariant symmetric bilinear form $(,)$, then the resulting isomorphism $V \to V^\vee$ induces an isomorphism $\hat{V} \to (\hat{V})^\vee$ given by $v \mapsto \operatorname{Res}(v, \cdot)\frac{dz}{z}$. We denote the corresponding k-bilinear pairing $\hat{V} \times \hat{V} \to k$ by $(,)_{\frac{dz}{z}}$.

Remark 3.5. In the case of the adjoint representation, the isomorphism $\Omega^1(\hat{\mathfrak{g}}) \cong \hat{\mathfrak{g}}^*$ intertwines the G-action defined in Section with the coadjoint action.

We now give the basic facts about the relationship between duality and the Moy-Prasad filtration. For the adjoint representation, these results appear in [28, Sections 3.5 and 3.7].

If W is a k-subspace of $\hat{V}$, we let $W^\perp = \{\phi \in \hat{V}^\vee \mid \phi(W) = 0\}$.

Proposition 3.6. *Let* V *be a finite-dimensional representation of* G. *Fix* $x \in \mathcal{B}$ *and* $r \in \mathbb{R}$.

(1) *The Moy-Prasad filtrations on* $\hat{V}^\vee$ *may be expressed in terms of the annihilators of the filtrations on* $\hat{V}$ *as* $\hat{V}^\vee_{x,-r} = \hat{V}^\perp_{x,r+}$ *and* $\hat{V}^\vee_{x,-r+} = \hat{V}^\perp_{x,r}$.

(2) *There is a natural* $\hat{G}_x$*-invariant perfect pairing*

$$\hat{V}^\vee_{x,-r}/\hat{V}^\vee_{x,-r+} \times \hat{V}_{x,r}/\hat{V}_{x,r+} \to k,$$

which induces the isomorphism $(\hat{V}_{x,r}/\hat{V}_{x,r+})^\vee \cong \hat{V}^\vee_{x,-r}/\hat{V}^\vee_{x,-r+}$.

(3) *There are* $\hat{G}_x$*-isomorphisms* $(\hat{V}_{x,r})^\vee \cong \hat{V}^\vee/\hat{V}^\vee_{x,-r+}$ *and* $(\hat{V}_{x,r+})^\vee \cong \hat{V}^\vee/\hat{V}^\vee_{x,-r}$.

(4) *Suppose that* V *is endowed with a nondegenerate* G*-invariant symmetric bilinear form* $(,)$*. Then,* $(,)_{\frac{dz}{z}}$ *induces* $\hat{G}_x$*-isomorphisms* $\hat{V}^\vee_{x,-r} \cong \hat{V}_{x,-r} \cong \Omega^1(\hat{V})_{x,-r}$ *and* $\hat{V}^\vee_{x,-r+} \cong \hat{V}_{x,-r+} \cong \Omega^1(\hat{V})_{x,-r+}$*; in particular,* $(\hat{V}_{x,r}/\hat{V}_{x,r+})^\vee \cong \hat{V}_{x,-r}/\hat{V}_{x,-r+}$.

Proof. If $x \in \mathcal{A}_0$, these statements are easily checked using the gradings. For example, to prove (1), one need only observe that the natural pairing $\hat{V}^\vee_x(s) \times \hat{V}_x(r) \to k$ is perfect if $s = -r$ and 0 otherwise. (Here, the graded components of $\hat{V}^\vee_x$ are defined using transport of structure from $\hat{V}^*$ via κ.) Parts (2) and (3) then follow from (1), and part (4) is a consequence of (2) and (3).

The general case is obtained by conjugating x into $\mathcal{A}_0$. □

In the situation of part (4), we obtain $\mathrm{Crit}_x(V^\vee) = -\mathrm{Crit}_x(V)$. If in addition $\mathfrak{z}$ acts trivially on V, $\mathrm{Crit}_x(V^\vee) = \mathrm{Crit}_x(V)$; in particular, $\mathrm{Crit}_x(\mathfrak{g}^\vee) = \mathrm{Crit}_x$.

3.2. **Fundamental Strata.** In this section, we provide more information about strata. Recall that a $\hat{G}$-stratum of depth r is a triple (x, r, β) such that $\beta \in (\hat{\mathfrak{g}}_{x,r}/\hat{\mathfrak{g}}_{x,r+})^\vee$. We denote the set of G-strata (resp, G-strata of depth r) by $\mathcal{S}^G$ (resp. $\mathcal{S}^G_r$).

By Proposition 3.6, we may identify $(\hat{\mathfrak{g}}_{x,r}/\hat{\mathfrak{g}}_{x,r+})^\vee$ with $\hat{\mathfrak{g}}^\vee_{x,-r}/\hat{\mathfrak{g}}^\vee_{x,-r+}$. We will call $\tilde{\beta} \in \hat{\mathfrak{g}}^\vee_{x,-r}$ a *representative* for β if β corresponds to $\tilde{\beta} + \hat{\mathfrak{g}}^\vee_{x,-r+} \in \hat{\mathfrak{g}}^\vee_{x,-r}/\hat{\mathfrak{g}}^\vee_{x,-r+}$. If $x \in \mathcal{A}_0$, we let $\tilde{\beta}_0$ denote the unique homogeneous representative in $\hat{\mathfrak{g}}^\vee_x(-r)$.

The $\hat{G}$-equivariance of Moy-Prasad filtrations induces a natural action of $\hat{G}$ on $\mathcal{S}^G$ and on each $\mathcal{S}^G_r$. Indeed, the coadjoint action induces a map $\bar{\mathrm{Ad}}^*(g) : \hat{\mathfrak{g}}^\vee_{x,-r}/\hat{\mathfrak{g}}^\vee_{x,-r+} \to \hat{\mathfrak{g}}^\vee_{gx,-r}/\hat{\mathfrak{g}}^\vee_{gx,-r+}$. If we let $g\beta \in (\hat{\mathfrak{g}}_{gx,r}/\hat{\mathfrak{g}}_{gx,r+})^\vee$ be the functional induced by $\bar{\mathrm{Ad}}^*(g)(\tilde{\beta})$, then the action on strata is defined by $g \cdot (x, r, \beta) = (gx, r, g\beta)$.

Definition 2.9 states that a stratum (x, r, β) is fundamental if the functional β is a semistable point of the $\hat{G}_x/\hat{G}_{x+}$-representation $(\hat{\mathfrak{g}}_{x,r}/\hat{\mathfrak{g}}_{x,r+})^\vee$. This condition may be expressed more explicitly as follows.

Proposition 3.7. *The stratum* (x, r, β) *is nonfundamental if and only if the coset* $\tilde{\beta} + \hat{\mathfrak{g}}^\vee_{x,-r+}$ *contains a nilpotent element. Moreover, if* $x \in \mathcal{A}_0$*,* (x, r, β) *is nonfundamental if and only if the graded representative* $\tilde{\beta}_0$ *is nilpotent.*

Proof. Since the stratum (x, r, β) is fundamental if and only if $g(x, r, \beta)$ is fundamental, it suffices to assume that $x \in \mathcal{A}_0$. Suppose that (x, r, β) is nonfundamental. By Proposition 3.4(1), the homogeneous representative $\tilde{\beta}_0$ is an unstable point of the H_x-representation $\hat{\mathfrak{g}}^\vee_x(-r)$. Let $\gamma : k^* \to H_x$ be a one-parameter subgroup for which $\lim_{t\to 0} \gamma(t) \cdot \tilde{\beta}_0 = 0$. Let $\hat{\gamma} : F^* \to \hat{G}$ be the one-parameter subgroup obtained from γ by extension of scalars. Since $\tilde{\beta}_0$ lies in the sum of positive weight spaces for γ, it also is in the sum of positive weight spaces for $\hat{\gamma}$. Thus, $\lim_{t\to 0} \hat{\gamma}(t) \cdot \tilde{\beta}_0 = 0$, and $\tilde{\beta}_0$ is nilpotent.

Next, if $\tilde{\beta} + \hat{\mathfrak{g}}^\vee_{x,-r+}$ contains a nilpotent element, then β is unstable by [28, Proposition 4.3]. Finally, if $\tilde{\beta}_0$ is nilpotent, then $\tilde{\beta} + \hat{\mathfrak{g}}^\vee_{x,-r+}$ of course contains a nilpotent element.

□

Remark 3.8. The proof of the proposition gives a practical way of determining whether a stratum is fundamental. One conjugates into the standard apartment (or any apartment corresponding to a maximal torus of G) and checks to see if the graded component of the functional is nonnilpotent.

We conclude this section by defining what it means for two strata to be *associates* of each other. We have seen in Theorem 2.15 that two fundamental strata contained in a flat G-bundle are associates of each other. Before giving the definition, we need a proposition.

Proposition 3.9. *Suppose that $x, y \in \mathcal{B}$. There exists a element $\delta_{x,y} \in \hat{\mathfrak{g}}^\vee_{x,0} \cap \hat{\mathfrak{g}}^\vee_{y,0}$ with the following property: for all $g \in \hat{G}$ such that $gx, gy \in \mathcal{A}_0$,*

$$\mathrm{Ad}^*(g)(\delta_{x,y}) \subset (\widetilde{gy} - \widetilde{gx})\frac{dz}{z} + \hat{\mathfrak{g}}^\vee_{gx,0+} + \hat{\mathfrak{g}}^\vee_{gy,0+}. \tag{15}$$

The proof will be deferred to Section 4.3. When $x, y \in \mathcal{A}_0$, it will follow from the proof that we may take $\delta_{x,y} = (\tilde{x} - \tilde{y})\frac{dz}{z}$. We will usually do so without comment.

Definition 3.10. Let (x, r, β) and (y, r, β') be two G-strata. Choose representatives $\tilde{\beta} \in \hat{\mathfrak{g}}^\vee_{x,-r}$ and $\tilde{\beta}' \in \hat{\mathfrak{g}}^\vee_{y,-r}$ for β and β', respectively. We say that (x, r, β) and (y, r, β') are *associates* of each other if there exists $g \in \hat{G}$ such that

$$\left(\mathrm{Ad}^*(g)(\tilde{\beta}) + \hat{\mathfrak{g}}^\vee_{gx,-r+}\right) \cap \left(\tilde{\beta}' - \delta_{gx,y} + \hat{\mathfrak{g}}^\vee_{y,-r+}\right) \neq \emptyset. \tag{16}$$

Note that when $r > 0$, the $\delta_{gx,y}$ term is unnecessary.

It is immediate that this definition is independent of the particular elements $\delta_{x,y}$ chosen to satisfy (15). It is also symmetric. We observe further that conjugate strata are associates. Indeed, if $(y, r, \beta') = (gx, r, g\beta)$, then the two cosets coincide. (One may take $\delta_{y,y} = 0$.)

4. Formal flat G-bundles and fundamental strata

Recall from Section 2.4 that a formal flat G-bundle is a principal G-bundle $\mathcal{G}$ over $\Delta^\times$ endowed with a connection ∇. Upon fixing a trivialization $\phi : \mathcal{G} \to \hat{G}$, the corresponding connection on the trivial flat G-bundle can be written as $\nabla_\phi = d + [\nabla]_\phi$ with $[\nabla]_\phi \in \Omega^1(\hat{\mathfrak{g}})$. Equivalently, $(\mathcal{G}, \nabla)$ can be viewed as a compatible family of flat vector bundles $(V_\mathcal{G}, \nabla_V)$ indexed by $\mathrm{Rep}(G)$. In terms of the trivialization ϕ, the flat vector bundle associated to (V, ρ) is $(\hat{V}, d + \rho([\nabla]_\phi)$. We remark that if V is the standard representation of GL_n, then the functor $(\mathcal{G}, \nabla) \mapsto (V_\mathcal{G}, \nabla_V)$ is an equivalence of categories between flat GL_n-bundles and flat rank n vector bundles.

As discussed before Proposition 3.6, $\Omega^1(\hat{\mathfrak{g}}) \cong \hat{\mathfrak{g}}^\vee$, where the isomorphism depends only on our fixed choice of invariant form $\langle, \rangle$. Recall that $X_{\tilde{\beta}} \in \Omega^1(\hat{\mathfrak{g}})$ denotes the one-form corresponding to $\tilde{\beta} \in \hat{\mathfrak{g}}^\vee$.

4.1. Strata contained in flat G-bundles.

In Section 2.7, we showed how to associate G-strata to flat G-bundles. In particular, a flat G-bundle $(\mathcal{G}, \nabla)$ contains the $\hat{G}$-stratum (x, r, β) with respect to the trivialization ϕ if, for any representative $\tilde{\beta} \in \hat{\mathfrak{g}}^\vee_{x,-r}$ of β, $(\nabla_\phi - i\frac{dz}{z} - X_{\tilde{\beta}})(\hat{V}_{x,i}) \subset \Omega^1(\hat{V})_{x,(i-r)+}$ for all $i \in \mathbb{R}$ and $V \in \mathrm{Rep}(G)$. More succinctly, this holds if for all (V, ρ), the induced flat vector bundle $(V_\mathcal{G}, \nabla_V)$ contains the $\hat{V}$-stratum $(\rho(x), r, \rho(\beta))$.

In this section, we will explain some properties of stratum containment. In particular, we will prove an equivariance result for containment of $\hat{G}$-strata which shows that the trivialization-free version of containment in terms of $\mathcal{G}$-strata (Definition 2.11) is well-defined. We will also show that when $x \in \mathcal{A}_0$, $\hat{G}$-stratum containment with respect to the trivialization ϕ is equivalent to a concrete and easily verified condition on the matrix $[\nabla]_\phi$.

Remark 4.1. As one would expect, the notions of containment of strata in flat GL_n-bundles and in rank n vector bundles are the same. This is shown in Corollary A.2. The proof uses the fact that $\mathrm{Rep}(\mathrm{GL}_n)$ is generated by the standard representation V in the sense that every representation of G may be obtained from V via some combination of duals, direct sums, tensor products, and subrepresentations. More generally, suppose that $\mathrm{Rep}(G)$ is generated (in the same sense) by $\{V_i\} \subset \mathrm{Rep}(G)$. We show in Proposition A.1 that a flat G-bundle contains a stratum if and only if (2.12) is satisfied for each V_i.

Stratum containment is well-behaved with respect to change of trivialization:

Lemma 4.2. *If $(\mathcal{G}, \nabla)$ contains (x, r, β) with respect to ϕ, then it contains $(gx, r, g\beta)$ with respect to $g\phi$.*

Proof. Fix a representation (V, ρ). Changing the trivialization ϕ by g changes $\nabla_\phi - i\frac{dz}{z} - X_{\tilde{\beta}}$, viewed as an element of $\mathrm{End}_k(\hat{V})\frac{dz}{z}$, to $\rho(g)(\nabla_\phi - i\frac{dz}{z} - X_{\tilde{\beta}})\rho(g)^{-1}$. (From now on, we will omit the ρ's from the notation.) However, the gauge change formula implies that $\nabla_{g\phi} = g\nabla_\phi g^{-1}$ and equivariance of Moy-Prasad filtrations gives $X_{g\tilde{\beta}} = gX_{\tilde{\beta}}g^{-1}$. Accordingly,

$$\begin{aligned} g(\nabla_\phi - i\frac{dz}{z} - X_{\tilde{\beta}})g^{-1}\hat{V}_{gx,i} &= g(\nabla_\phi - i\frac{dz}{z} - X_{\tilde{\beta}})g^{-1}g\hat{V}_{x,i} \\ &= g(\nabla_\phi - i\frac{dz}{z} - X_{\tilde{\beta}})\hat{V}_{x,i} \\ &\subset g(\Omega^1(\hat{V})_{x,(i-r)+}) = \Omega^1(\hat{V})_{gx,(i-r)+}. \end{aligned} \tag{17}$$

□

We now give a more explicit description of what is means for a stratum to be contained in a flat G-bundle when the stratum is based at a point in the standard apartment. This characterization will be useful in calculations throughout the rest of the paper.

Proposition 4.3. *Let $x \in \mathcal{A}_0$. Then, $(\mathcal{G}, \nabla)$ contains the stratum (x, r, β) with respect to the trivialization ϕ if and only if $[\nabla]_\phi - \tilde{x}\frac{dz}{z} \in \hat{\mathfrak{g}}^\perp_{x,r+}$ and the coset $\left([\nabla]_\phi - \tilde{x}\frac{dz}{z}\right) + \hat{\mathfrak{g}}^\vee_{x,-r+}$ determines the functional $\beta \in (\hat{\mathfrak{g}}_{x,r}/\hat{\mathfrak{g}}_{x,r+})^\vee$.*

Proof. Assume that $[\nabla]_\phi$ satisfies the given conditions, and take $V \in \mathrm{Rep}(G)$. By the hypothesis, we can take $X_{\tilde{\beta}} = [\nabla]_\phi - \tilde{x}\frac{dz}{z} \in \Omega^1(\hat{\mathfrak{g}})_{x,-r}$, so $\nabla_\phi - i\frac{dz}{z} - X_{\tilde{\beta}} = d + [\nabla]_\phi - i\frac{dz}{z} - ([\nabla]_\phi - \tilde{x}\frac{dz}{z}) = d + \tilde{x}\frac{dz}{z} - i\frac{dz}{z}$. Since Proposition 3.2(2) implies

$$(d + \tilde{x}\frac{dz}{z} - i\frac{dz}{z})(\hat{V}_{x,i}) \subset \Omega^1(\hat{V})_{x,i+} \subset \Omega^1(\hat{V})_{x,(i-r)+}, \tag{18}$$

the defining property (10) is satisfied.

For the converse, suppose that $(\mathcal{G}, \nabla)$ contains the stratum (x, r, β) with respect to the trivialization ϕ, and let V be a faithful representation. As usual, take

$\tilde{\beta} \in \hat{\mathfrak{g}}^\vee_{x,-r}$ a representative for β with $X_{\tilde{\beta}} \in \Omega^1(\hat{\mathfrak{g}})_{x,-r}$ the corresponding element. We then have

$$(d + [\nabla]_\phi - i\frac{dz}{z} - X_{\tilde{\beta}})\hat{V}_{x,i} \subset \Omega^1(\hat{V})_{x,(i-r)+}$$

Subtracting (18), we obtain $([\nabla]_\phi - \tilde{x}\frac{dz}{z} - X_{\tilde{\beta}})\hat{V}_{x,i} \subset \Omega^1(\hat{V})_{x,(i-r)+}$ for all $i \in \mathbb{R}$. The faithfulness of V now implies that $[\nabla]_\phi - \tilde{x}\frac{dz}{z} - X_{\tilde{\beta}} \in \Omega^1(\hat{\mathfrak{g}})_{x,-r+}$. In particular, $X_{\tilde{\beta}} \in ([\nabla]_\phi - \tilde{x}\frac{dz}{z}) + \Omega^1(\hat{\mathfrak{g}})_{x,-r+}$. Viewing $[\nabla]_\phi - \tilde{x}\frac{dz}{z}$ as a functional, we see that it lies in it $\hat{\mathfrak{g}}^\vee_{x,-r} \cong \hat{\mathfrak{g}}^\perp_{x,r+}$ and determines the same coset in $\hat{\mathfrak{g}}^\vee_{x,-r}/\hat{\mathfrak{g}}^\vee_{x,-r+}$ as $\tilde{\beta}$. This proves the result.

□

Remark 4.4. The additional $-i\frac{dz}{z}$ terms in Definition 2.12 (or equivalently, the $-\tilde{x}\frac{dz}{z}$ term in the previous proposition) are needed to make the map $\mathcal{Q} \to \mathcal{S}$ equivariant. The $-\tilde{x}\frac{dz}{z}$ term also plays an important role in our study of the isomonodromy equations for flat vector bundles (c.f. [5, Definition 2.12]).

Remark 4.5. Any trivialization ϕ and $x \in \mathcal{B}$ (or equivalently, any point in $\mathcal{B}(\mathrm{Aut}(\mathcal{G}))$ determines a stratum contained in $(\mathcal{G}, \nabla)$. By equivariance, it is enough to show this for $x \in \mathcal{A}_0$. Let r be the smallest critical number satisfying $[\nabla]_\phi - \tilde{x}\frac{dz}{z} \in \hat{\mathfrak{g}}^\perp_{x,r+}$. If $\beta \in (\hat{\mathfrak{g}}_{x,r}/\hat{\mathfrak{g}}_{x,r+})^\vee$ is the induced functional, then by Proposition 4.3, $(\mathcal{G}, \nabla)$ contains (x, r, β) with respect to ϕ.

In general, a nonfundamental stratum provides very little information about a flat G-bundle. However, we will see that all flat G-bundles contain fundamental strata, and the depth of any such stratum determines whether the flat G-bundle is regular singular, and if not, how irregular it is. Moreover, we will see that it always possible to find a fundamental stratum for which $x \in \mathcal{B}$ is an *optimal point* in the sense of [28, Section 6].

We now recall the definition of optimal points. Fix an alcove $C \subset \mathcal{A}_0$, and let Σ_C be the collection of minimal affine roots on C, that is, the set of affine roots ψ such that for all $x \in \bar{C}$, $0 \leq \psi(x) \leq 1$. For any nonempty subset $\Xi \subset \Sigma_C$, define a function f_Ξ on $\bar{C}$ by

$$f_\Xi(x) = \min\{\psi(x) \mid \psi \in \Xi\}.$$

Choose a point $x_\Xi \in \bar{C}$ at which f_Ξ attains its maximum value and $\widetilde{x_\Xi} \in \mathfrak{t}_{\mathbb{Q}}$. We can further assume that $x_\Xi \in \bar{\mathcal{B}}$. The set of optimal points in $\bar{C}$ is given by $\Psi_C = \{x_\Xi\}$. It can be shown that Ψ_C contains all the vertices of $\bar{C} \cap \bar{\mathcal{B}}$.

Example 4.6. For $\mathrm{SL}(V)$, there is no ambiguity about the optimal points in a closed alcove; they can only be taken to be the barycenters of the simplices. With our convention that optimal points are in the reduced building, the optimal points for $\mathrm{GL}(V)$ are also uniquely determined. Thus, optimal points here give rise to lattice chain filtrations.

We will need to understand the effect of change of trivialization on strata contained in $\mathcal{G}$. The following lemma establishes the necessary calculus.

Lemma 4.7.

(1) *If* $n \in \hat{N}$, $([\nabla]_{n\phi} - \widetilde{nx}\frac{dz}{z}) \in \mathrm{Ad}^*(n)([\nabla]_\phi - \tilde{x}\frac{dz}{z}) + \hat{\mathfrak{t}}_{0+}\frac{dz}{z}$.

(2) *If* $X \in \hat{\mathfrak{u}}_\alpha \cap \hat{\mathfrak{g}}_{x,\ell}$, *then*

$$[\nabla]_{\exp(X)\phi} - \tilde{x}\frac{dz}{z} \in \mathrm{Ad}^*(\exp(X))([\nabla]_\phi - \tilde{x}\frac{dz}{z}) - \ell X\frac{dz}{z} + \hat{\mathfrak{g}}^\vee_{x,\ell+}.$$

(3) *If* $p \in \hat{G}_x$, *then* $[\nabla]_{p\phi} - \tilde{x}\frac{dz}{z} \in \mathrm{Ad}^*(p)([\nabla]_\phi - \tilde{x}\frac{dz}{z}) + \hat{\mathfrak{g}}^\vee_{x,0+}$.

(4) *If* $p \in \hat{G}_{x,\ell}$ *for* $\ell > 0$, *then* $[\nabla]_{p\phi} - \tilde{x}\frac{dz}{z} \in \mathrm{Ad}^*(p)([\nabla]_\phi - \tilde{x}\frac{dz}{z}) + \hat{\mathfrak{g}}^\vee_{x,\ell}$.

Proof. By Lemma 3.3, $\widetilde{nx} \in \mathrm{Ad}(n)(\tilde{x}) - \tau(n)n^{-1} + \hat{\mathfrak{t}}_{0+}$. Dualizing, we obtain $\widetilde{nx}\frac{dz}{z} \in \mathrm{Ad}^*(n)(\tilde{x}\frac{dz}{z}) - (dn)n^{-1} + \hat{\mathfrak{t}}_{0+}\frac{dz}{z}$. Part (1) follows by applying the gauge transformation formula and substituting the above expression into $[\nabla]_{n\phi} - \widetilde{nx}\frac{dz}{z}$.

Suppose that $X \in \hat{\mathfrak{u}}_\alpha \cap \hat{\mathfrak{g}}_{x,\ell}$, and write $u = \exp(X)$. In order to prove (2), it suffices to show that

$$(du)u^{-1} \in \mathrm{Ad}^*(u)(\tilde{x}\frac{dz}{z}) - \tilde{x}\frac{dz}{z} + \ell X\frac{dz}{z} + \hat{\mathfrak{g}}^\vee_{x,\ell+}.$$

Recall that $\tau(X) + \mathrm{ad}(\tilde{x})(X) \in \ell X + \hat{\mathfrak{g}}_{x,\ell+}$ by Proposition 3.2. Thus,

$$\begin{aligned}(du)u^{-1} = \tau(X)\frac{dz}{z} &\in -\mathrm{ad}(\tilde{x})(X)\frac{dz}{z} + \ell X\frac{dz}{z} + \hat{\mathfrak{g}}^\vee_{x,\ell+}\\ &= (\mathrm{Ad}(u)(\tilde{x}) - \tilde{x})\,\frac{dz}{z} + \ell X\frac{dz}{z} + \hat{\mathfrak{g}}^\vee_{x,\ell+}.\end{aligned}$$

The right hand side of this equation proves the desired result.

Observe that when $t \in T_\ell$, $(dt)(t^{-1}) \in \mathfrak{t}_1\frac{dz}{z}$ (resp. $\mathfrak{t}_\ell\frac{dz}{z}$ when $t \in T_\ell$ and $\ell > 0$). Furthermore, $\widetilde{tx} = \tilde{x}$ for all $t \in T_\ell$, since $T_\ell \subseteq T_0$. Since $\hat{G}_{x,\ell}$ is generated by the root subgroups $\hat{U}_\alpha \cap \hat{G}_{x,\ell}$ and the congruence subgroups $T_\ell \subset T$, statements (3) and (4) now follow from (2). □

Recall that the treatment of formal flat vector bundles in [4] only involved strata coming from lattice chain filtrations. Moreover, the definition of containment there differs from Definition 2.4, and it is not equivalent for strata of depth 0. However, we will need to know that certain results from [4] remain true using the present definition of containment and allowing arbitrary periodic filtrations. The following theorem generalizes [4, Theorem 4.10]; we relegate the proof to Section A.2 of the appendix.

Proposition 4.8. *The slope of a flat vector bundle* (U, ∇) *is the minimum of the depths of the strata* (x, r, β) *contained in it. If* (x, r, β) *is fundamental, then* $r = \mathrm{slope}(U, \nabla)$*; the converse is true if* $r > 0$*. In particular,* (U, ∇) *is regular singular if and only if it contains a stratum of depth* 0*.*

4.2. **Regular and irregular singularity.** Recall that a formal flat G-bundle $(\mathcal{G}, \nabla)$ is regular singular if for all representations V, the associated flat vector bundle $(V_{\mathcal{G}}, \nabla_V)$ is regular singular. Otherwise, it is called irregular singular. As one would hope, a flat GL_n-bundle is regular singular if and only if the corresponding flat vector bundle is regular singular. This is indeed true; see Corollary A.6.

We next show that if a flat G-bundle contains a stratum of depth r, then the same is true for each associated flat vector bundle.

Proposition 4.9. *Let* $\rho : G \to \mathrm{GL}(V)$ *be a representation for* G.

(1) *For any* $x \in \mathcal{B}$, *there is a uniquely determined* $\rho_*(x) \in \mathcal{B}(\mathrm{GL}(\hat{V}))$ *that induces the same filtration on* $\hat{V}$ *as* x*. Moreover,* $\mathfrak{gl}(\hat{V})_{\rho_*(x),r} \cap \rho(\hat{\mathfrak{g}}) = \rho(\hat{\mathfrak{g}}_{x,r})$ *for all* $r \in \mathbb{R}$*.*

(2) *If* T_ρ *is a maximal torus in* $\mathrm{GL}(V)$ *containing* $\rho(T)$*, then* ρ_* *restricts to the map* $\mathcal{A}_0 \to \mathcal{A}(\widehat{T_\rho})$ *determined by* $\widetilde{\rho_*(x)} = \rho(\tilde{x})$*.*

Proof. Since x determines a periodic filtration on $\hat{V}$ and $\mathcal{B}(\mathrm{GL}(\hat{V}))$ parameterizes such filtrations, there is a unique $\rho_*(x) \in \mathcal{B}(\mathrm{GL}(\hat{V}))$ such that $\hat{V}_{\rho_*(x),r} = \hat{V}_{x,r}$ for all r. Next, choose a maximal torus $T_\rho \subset \mathrm{GL}(V)$ such that $\rho(T) \subset T_\rho$. If $x \in \mathcal{A}_0$, let $y \in \mathcal{A}(\widehat{T_\rho}) \subset \mathcal{B}(\mathrm{GL}(\hat{V}))$ be the point determined by $\tilde{y} = \rho(\tilde{x})$. To prove that $y = \rho_*(x)$, we must show that $\hat{V}_{y,r} = \hat{V}_{x,r}$ for all r. It suffices to show that $\hat{V}_y(r) = \hat{V}_x(r)$ for all r. By Proposition 3.2(1), $\hat{V}_x(r)$ is the r-eigenspace of the action of $\tau + \tilde{x}$ on $\hat{V}$. Since this operator on $\hat{V}$ coincides with $\tau + \rho(\tilde{x}) = \tau + \widetilde{\rho_*(x)}$, another application of Proposition 3.2(1) gives the desired equality of graded spaces.

To prove the remaining statement, first note that by equivariance of Moy-Prasad filtrations, we may assume that $x \in \mathcal{A}_0$. Observe that $(\tau + \mathrm{ad}(\rho(\tilde{x}))\rho(X) = \rho((\tau + \mathrm{ad}(\tilde{x}))(X))$ for all $X \in \hat{\mathfrak{g}}$. Therefore, if $X_r \in \hat{\mathfrak{g}}_x(r)$, then $\rho(X_r) \in \mathfrak{gl}(\hat{V})_{\rho_*(x)}(r)$. By continuity, there exists $s \geq r$ such that $\hat{\mathfrak{g}}_{x,s} \subset \rho^{-1}(\mathfrak{gl}(V)_{\rho_*(x),r+})$. Repeated application of Proposition 3.2(3) shows that $\sum_{r \leq j < s} \hat{\mathfrak{g}}_x(j)$ constitutes a full set of coset representatives for $\hat{\mathfrak{g}}_{x,r}/\hat{\mathfrak{g}}_{x,s}$. Since $\rho(\hat{\mathfrak{g}}_x(j)) \subset \mathfrak{gl}(V)_{\rho_*(x),r}$ by the work above, we deduce that $\rho(\hat{\mathfrak{g}}_{x,r}) \subset \mathfrak{gl}(V)_{\rho_*(x),r}$.

We now show that $X \in \mathfrak{gl}(\hat{V})_{\rho(x),r} \cap \rho(\hat{\mathfrak{g}})$ implies $X \in \rho(\hat{\mathfrak{g}}_{x,r})$. If this is false, let $r' < r$ be the largest critical number s for which $\hat{\mathfrak{g}}_{x,s}$ intersects $\rho^{-1}(X)$. Take $Y \in \hat{\mathfrak{g}}_{x,r'}$ with $\rho(Y) = X$. By Proposition 3.2(3), there exists $Y_{r'} \in \hat{\mathfrak{g}}_x(r')$ such that $Y \in Y_{r'} + \hat{\mathfrak{g}}_{x,r'+}$ The argument above shows that $\rho(Y_{r'}) \in X + \mathfrak{gl}(\hat{V})_{\rho_*(x),r'+}$, and since $r' < r$, this implies that $\rho(Y_{r'}) \in \mathfrak{gl}(\hat{V})_{\rho_*(x),r'+}$. However, we also have $\rho(Y_{r'}) \in \mathfrak{gl}(\hat{V})_{\rho_*(x)}(r')$. Since $\mathfrak{gl}(\hat{V})_{\rho_*(x),r'+} \cap \mathfrak{gl}(\hat{V})_{\rho_*(x)}(r') = \{0\}$, we deduce that $\rho(Y_{r'}) = 0$. This means that $\rho(Y - Y_{r'}) = X$, but $Y - Y_{r'} \notin \hat{\mathfrak{g}}_{x,r'}$, contradicting the definition of r'. It follows that $X \in \rho(\hat{\mathfrak{g}}_{x,r})$.

□

If β is a functional on $\hat{\mathfrak{g}}_{x,r}/\hat{\mathfrak{g}}_{x,r+}$ and (V, ρ) is a representation of G, we can use the proposition to define a functional $\rho(\beta)$ on $\mathfrak{gl}(V)_{\rho_*(x),r}/\,\mathfrak{gl}(V)_{\rho_*(x),r+}$ which is represented (with respect to the residue of the trace form) by $\rho(\tilde{\beta}) \in \mathfrak{gl}(V)^\vee_{\rho_*(x),-r}$. The proposition also shows that the functional is independent of the choice of representative $\tilde{\beta}$.

Corollary 4.10. *Let (V, ρ) be a representation of G. If $(\mathcal{G}, \nabla)$ contains the stratum (x, r, β) with respect to ϕ, then $(V_\mathcal{G}, \nabla_V)$ contains the stratum $(\rho_*(x), r, \rho(\beta))$ with respect to the induced trivialization ϕ_V.*

Proof. By Definition 2.12, $(\nabla_\phi - i\frac{dz}{z} - X_{\tilde{\beta}})(\hat{V}_{x,i}) \subset \Omega^1(\hat{V})_{x,(i-r)+}$ for all $i \in \mathbb{R}$. Since is obvious that $X_{\rho(\tilde{\beta})} = \rho(X_{\tilde{\beta}}) \in \Omega^1(\mathfrak{gl}(\hat{V}))$, the action of $(\nabla_\phi - i\frac{dz}{z} - X_{\tilde{\beta}})$ on $\hat{V}$ is the same as that given by $\rho(\nabla_\phi - i\frac{dz}{z} - X_{\tilde{\beta}}) = \nabla_{V,\phi} - i\frac{dz}{z} - X_{\rho(\tilde{\beta})}$. Applying Proposition 4.9, we obtain $(\nabla_\phi - i\frac{dz}{z} - X_{\tilde{\beta}})(\hat{V}_{\rho_*(x),i}) \subset \Omega^1(\hat{V})_{\rho_*(x),(i-r)+}$ for all i. By Definition 2.4, $(\hat{V}, \nabla_{V,\phi})$ contains the stratum $(\rho_*(x), r, \rho(\beta))$. □

Since the slope of a vector bundle is the minimum depth of a stratum that it contains, we immediately obtain an upper bound on the slope of $(V_\mathcal{G}, \nabla_V)$.

Corollary 4.11. *The slope of $(V_\mathcal{G}, \nabla_V)$ is not greater than the minimum depth of a stratum contained in $(\mathcal{G}, \nabla)$.*

Next, we show that if a flat G-bundle contains a stratum of depth 0, then it is regular singular. We will see later that the converse is also true.

Proposition 4.12. *Suppose that the flat G-bundle $(\mathcal{G}, \nabla)$ contains a stratum $(x, 0, \beta)$ of depth 0 with respect to the trivialization ϕ. The following statements hold.*

1. *The flat G-bundle $(\mathcal{G}, \nabla)$ is regular singular.*
2. *Suppose $x \in \mathcal{A}_0$, and let $\Delta \subset \mathcal{A}_0$ be the open facet containing x. Then, for any $y \in \bar{\Delta}$, $(\mathcal{G}, \nabla)$ contains the stratum $(y, 0, \beta^y)$ with respect to ϕ, with β^y induced by $[\nabla]_\phi - \tilde{y}\frac{dz}{z}$. In particular, this is true for y in a minimal facet contained in $\bar{\Delta}$.*

Proof. By equivariance, we may assume that $x \in \mathcal{A}_0$. By Proposition 4.3, $[\nabla]_\phi - \tilde{x}\frac{dz}{z} \in \hat{\mathfrak{g}}_{x,0+}^\perp$. Since the same is true for $\tilde{x}\frac{dz}{z}$, $[\nabla]_\phi \in \hat{\mathfrak{g}}_{x,0+}^\perp$ as well. Take $y \in \bar{\Delta}$. Since $\hat{\mathfrak{g}}_{y+} \subset \hat{\mathfrak{g}}_{x+}$, we have $[\nabla]_\phi \in \hat{\mathfrak{g}}_{x+}^\perp \subset \hat{\mathfrak{g}}_{y+}^\perp$. Accordingly, $[\nabla]_\phi - \tilde{y}\frac{dz}{z} \in \hat{\mathfrak{g}}_{y+}^\perp$, and $(y, 0, \beta^y)$ is contained in $(\mathcal{G}, \nabla)$.

Now, suppose that V is a finite dimensional representation for G. We have $[\nabla]_\phi = \iota_\tau([\nabla]_\phi)\frac{dz}{z}$ with $\iota_\tau([\nabla]_\phi) \in \hat{\mathfrak{g}}_{x,0}$. Since $\hat{\mathfrak{g}}_{x,0}$ preserves the lattice $\hat{V}_{x,0} \subset \hat{V}$, $\iota_\tau([\nabla]_\phi)(\hat{V}_{x,0}) \subset \hat{V}_{x,0}$, and it follows that $(V_\mathcal{G}, \nabla_V)$ is regular singular. □

If a flat G-bundle contains a fundamental stratum of positive depth, then at least one of the associated vector bundles is irregular singular. In fact, we can be more specific.

Proposition 4.13. *If $(\mathcal{G}, \nabla)$ contains a fundamental stratum (x, r, β) of depth $r > 0$, then either the flat bundle $(\mathfrak{g}_\mathcal{G}, \nabla_\mathfrak{g})$ corresponding to the adjoint representation has slope r or G has a one dimensional representation W such that $(W_\mathcal{G}, \nabla_W)$ has slope r.*

Proof. Suppose that $\operatorname{slope}(\mathfrak{g}_\mathcal{G}, \nabla_\mathfrak{g}) < r$. We will show that G has a character (W, χ) for which $(W_\mathcal{G}, \nabla_W)$ has slope r.

As usual, we can assume that $x \in \mathcal{A}_0$. Since (x, r, β) is fundamental, Proposition 3.7 implies that the coset $\iota_\tau([\nabla]_\phi) + \hat{\mathfrak{g}}_{x,-r+}$ contains no nilpotent elements. (We omit the $\tilde{x}$ term, since $r > 0$.) Let $Y \in \hat{\mathfrak{g}}_x(-r)$ be the homogeneous coset representative. By corollary 4.10, $(\mathfrak{g}_\mathcal{G}, \nabla_\mathfrak{g})$ contains the corresponding $\mathrm{GL}(\mathfrak{g})$-stratum $(\mathrm{ad}_*(x), r, \mathrm{ad}(Y)\frac{dz}{z})$. Proposition 4.8 implies that this stratum is not fundamental, so the graded representative $\mathrm{ad}(Y)$ is nilpotent. This means that $Y = Y_1 + Y_2$ with $Y_1 \in \hat{\mathfrak{z}}(-r)$ nonzero and Y_2 a nilpotent element of $\mathfrak{g}_x(-r) \cap [\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]$. Note that this already implies that $r \in \mathbb{Z}_{>0}$, since homogeneous element of the center have integral degrees.

Since the connected center $Z^0 \cong G/[G, G]$ is a torus, there exists a character (W, χ) of G, vanishing on $[G, G]$ such that $\chi(Y_1) \in z^{-r}k^*$. The corresponding stratum contained in $(W_\mathcal{G}, \nabla_W)$ is $(\chi_*(x), r, \chi(Y_1)\frac{dz}{z})$, which is evidently fundamental. □

Flat line bundles have integral slope, so we obtain the following corollary.

Corollary 4.14. *If $(\mathcal{G}, \nabla)$ contains a fundamental stratum of nonintegral depth r, then* $\operatorname{slope}(\mathfrak{g}_\mathcal{G}, \nabla_\mathfrak{g}) = r$.

4.3. **Proofs of the main theorems.** Recall that Ψ_C denotes the set of optimal points in a given closed chamber $\bar{C} \subset \mathcal{A}_0$.

Lemma 4.15. *Suppose that $(\mathcal{G}, \nabla)$ contains a stratum (x, r, β) of depth $r > 0$*

(1) *If r is not a critical number for x, then $(\mathcal{G}, \nabla)$ contains a stratum of the form (x, s, β') where $s < r$ is a critical number.*
(2) *If r is a critical number and (x, r, β) is not fundamental, then $(\mathcal{G}, \nabla)$ contains a stratum (y, s, β') with $y \in \Psi_C$ and $s < r$.*

Proof. By Lemma 4.2, we may assume without loss of generality that $x \in \bar{C}$. First, suppose that $(\mathcal{G}, \nabla)$ contains (x, r, β) with respect to the trivialization ϕ and that $r \notin \mathrm{Crit}_x$. Let s be the greatest critical number for x less than r. We claim that $[\nabla]_\phi \in \hat{\mathfrak{g}}^\vee_{x,-s}$. Write $[\nabla]_\phi = (Y_0 + \sum_{\alpha\in\Phi} Y_\alpha)\frac{dz}{z}$, where $Y_0 \in \hat{\mathfrak{t}}$ and $Y_\alpha \in \hat{\mathfrak{u}}_\alpha$. If $[\nabla]_\phi \notin \hat{\mathfrak{g}}^\vee_{x,-s}$, then $Y_\gamma \frac{dz}{z} \notin \hat{\mathfrak{g}}^\vee_{x,-s}$ for some $\gamma \in \Phi \cup \{0\}$. Since $Y_\gamma \frac{dz}{z} \in \hat{\mathfrak{g}}^\vee_{x,-r}$, we may take $s' = \min\{t \in (s, r] \mid Y_\gamma \frac{dz}{z} \in \hat{\mathfrak{g}}^\vee_{x,-t}\} > s$. However, since $Y_\gamma \frac{dz}{z}$ has nonzero image in $\hat{\mathfrak{g}}^\vee_{x,-s'}/\hat{\mathfrak{g}}^\vee_{x,-s'+}$, s' is a critical point with $s' < r$. This contradicts the assumption that s is the greatest critical point less than r. We now apply Proposition 4.3 to see that $(\mathcal{G}, \nabla)$ contains the stratum (x, s, β') with respect to ϕ, where β' is induced by $[\nabla]_\phi - \tilde{x}\frac{dz}{z}$.

We now assume that r is a critical number for x and (x, r, β) is not fundamental. By [28, Proposition 6.3], there exists $p \in \hat{G}_x$ and $y \in \Psi_C$ such that for some $s < r$, $\mathrm{Ad}^*(p)([\nabla]_\phi - \tilde{x}\frac{dz}{z}) + \hat{\mathfrak{g}}^\vee_{x,-r+} \subset \hat{\mathfrak{g}}^\vee_{y,-s}$. Lemma 4.7 (3) implies that $\mathrm{Ad}^*(p)([\nabla]_\phi - \tilde{x}\frac{dz}{z}) + \hat{\mathfrak{g}}^\vee_{x,-r+} = [\nabla]_{p\phi} - \tilde{x}\frac{dz}{z} + \hat{\mathfrak{g}}^\vee_{x,-r+}$. Since $[\nabla]_{p\phi} - \tilde{x}\frac{dz}{z} \in \hat{\mathfrak{g}}^\vee_{y,-s}$ and $(\tilde{x} - \tilde{y})\frac{dz}{z} \in \hat{\mathfrak{g}}^\vee_{y,0}$, $[\nabla]_{p\phi} - \tilde{y}\frac{dz}{z} \in \hat{\mathfrak{g}}^\vee_{y,-s}$. Letting β' be the functional induced by $[\nabla]_{p\phi} - \tilde{y}\frac{dz}{z}$, it follows that $(\mathcal{G}, \nabla)$ contains the stratum (y, s, β') with respect to the trivialization $p\phi$. □

Proposition 4.16. *Suppose that $(\mathcal{G}, \nabla)$ contains a fundamental stratum (x, r, β) of depth $r > 0$. If (y, s, β') is another stratum contained in $(\mathcal{G}, \nabla)$, then $s \geq r$. Moreover, if $s = r$, this stratum is fundamental.*

Proof. Recall that any two points in the building lie in a common apartment. Hence, by equivariance, we may assume without loss of generality that the two strata are contained in $(\mathcal{G}, \nabla)$ with respect to the same trivialization and that $x, y \in \mathcal{A}_0$.

Suppose that $s < r$. We can apply [28, Proposition 6.4] to obtain $\hat{\mathfrak{g}}^\vee_{y,-s} \cap (\tilde{\beta} + \hat{\mathfrak{g}}^\vee_{x,-r+}) = \emptyset$. By Proposition 4.3, there exists $\omega \in \hat{\mathfrak{g}}^\vee_{x,-r+}$ and $\omega' \in \hat{\mathfrak{g}}^\vee_{y,-s+}$ such that $\tilde{\beta} = [\nabla]_\phi - \tilde{x}\frac{dz}{z} + \omega$ and $\tilde{\beta}' = [\nabla]_\phi - \tilde{y}\frac{dz}{z} + \omega'$. Since $(\tilde{y} - \tilde{x})\frac{dz}{z} \in \hat{\mathfrak{g}}^\vee_{y,0}$, it follows that $\tilde{\beta}' - \omega' + (\tilde{y} - \tilde{x})\frac{dz}{z} = \tilde{\beta} - \omega \in \hat{\mathfrak{g}}^\vee_{y,-s} \cap (\tilde{\beta} + \hat{\mathfrak{g}}^\vee_{x,-r+})$. This contradiction implies that $s \geq r$.

Now, suppose that $s = r$. If (y, r, β') is not fundamental, then Lemma 4.15 states that $(\mathcal{G}, \nabla)$ contains a stratum of depth s' strictly less than r. This contradicts the conclusion of the previous paragraph.

□

Corollary 4.17. *All fundamental strata contained in $(\mathcal{G}, \nabla)$ have the same depth.*

Proof. Suppose $(\mathcal{G}, \nabla)$ contains two fundamental strata with different depths r and s, say with $r > s$. Then $r > 0$, so the proposition gives the contradiction $s \geq r$. □

Proof of Theorem 2.14. We only need to consider $\hat{G}$-strata in the proof as the statements about $\mathcal{G}$-strata are immediate consequences. We first prove that every flat G-bundle $(\mathcal{G}, \nabla)$ contains a fundamental stratum (resp. a stratum of depth 0) if it is irregular singular (resp. regular singular). Fix a chamber $C \subset \mathcal{A}_0$, and let Ψ_C be the set of optimal points in $\bar{C}$. Consider the set of positive real numbers s such that

$(\mathcal{G}, \nabla)$ contains a stratum (x, s, β) with $x \in \Psi_C$. This set is clearly nonempty, as by Remark 4.5, any choice of trivialization and optimal point determines a stratum for $(\mathcal{G}, \nabla)$. Since Ψ_C is finite, it attains its lower bound $r > 0$.

Suppose that (x, r, β) is a stratum contained in $(\mathcal{G}, \nabla)$ such that $x \in \Psi_C$ and r is the lower bound described above. We write ϕ for the associated trivialization of $\mathcal{G}$. Assume that this stratum is not fundamental. Note that Proposition 4.13 guarantees that this is the case when $(\mathcal{G}, \nabla)$ is regular singular. Lemma 4.15 implies that $(\mathcal{G}, \nabla)$ contains a stratum of the form (y, s, β'), where $y \in \Psi_C$ and $s < r$. (If r is not a critical point, one can take $y = x$.) By minimality of r, we must have $s = 0$, and it follows from Proposition 4.12 that $(\mathcal{G}, \nabla)$ is regular singular. Thus, (x, r, β) is fundamental if $(\mathcal{G}, \nabla)$ irregular singular, and $(\mathcal{G}, \nabla)$ contains a stratum of depth 0 if it is regular singular. Statements (1) and (2) now follow immediately from Proposition 4.16 and Corollary 4.17.

Next, suppose that $r = 0$. It remains to show that $(\mathcal{G}, \nabla)$ contains a fundamental stratum. We have shown that $(\mathcal{G}, \nabla)$ contains a stratum of the form $(x, 0, \beta)$ with respect to a trivialization ϕ, and by Proposition 4.12 we may assume that x is in a minimal facet $\Delta \subset \mathcal{A}_0$. If this stratum is nonfundamental, then, using the notation of Proposition 3.4, $d\theta_x^*(\beta) \in \mathfrak{h}_x^\vee$ is unstable. In particular, there exists a Borel subgroup $B_\beta \subset H_x$ such that $d\theta_x^*(\beta) \in (\mathfrak{b}_\beta)^\perp$. Choose $h \in H_x$ such that $hB_\beta h^{-1}$ is the standard Borel subgroup $B_x \subset H_x$ containing T coming from a choice of positive roots for G, and let $m \in G_x$ be a lift of $\theta_x(h)$. By part (3) of Lemma 4.7, $(\mathcal{G}, \nabla)$ contains the stratum $m \cdot (x, 0, \beta)$. Thus, we may assume without loss of generality that the stratum $(x, 0, \beta)$ satisfies $d\theta_x^*(\beta) \in (\mathfrak{b}_x)^\perp$, where $\mathfrak{b}_x = \mathrm{Lie}(B_x)$.

Let $\delta \in \mathfrak{t}$ be the element corresponding to the half sum of positive coroots of H_x. We define $x_\epsilon \in \mathcal{A}_0$ via $\tilde{x}_\epsilon = \tilde{x} + \epsilon\delta$. For small $\epsilon > 0$, we will show that $\tilde{\beta}_0 + \hat{\mathfrak{g}}_{x+}^\vee \subset \hat{\mathfrak{g}}_{x_\epsilon+}^\vee$. (Recall that $\tilde{\beta}_0$ is the homogeneous representative for β.) First, since x is in a minimal facet, $x+\epsilon\delta$ must be contained in a chamber C with $x \in \bar{C}$ for ϵ sufficiently small. (Note that $x \notin C$). Therefore, $\hat{\mathfrak{g}}_{x_\epsilon} \subset \hat{\mathfrak{g}}_x$, so taking annihilators gives $\hat{\mathfrak{g}}_{x+}^\vee \subset \hat{\mathfrak{g}}_{x_\epsilon+}^\vee$. An elementary calculation, using the fact that $\mathrm{ad}^*(\epsilon\delta)$ has strictly positive eigenvalues on $\mathfrak{b}_x^\perp$, shows that $(d\theta_x^*)^{-1}(\mathfrak{b}_x^\perp) \subset \hat{\mathfrak{g}}_{x_\epsilon+}^\vee + \hat{\mathfrak{g}}_{x+}^\vee$. We conclude that $\tilde{\beta}_0 \in \hat{\mathfrak{g}}_{x_\epsilon+}$ since $d\theta_x^*(\beta) \in (\mathfrak{b})^\perp$. This proves the assertion above.

Write $\tilde{\beta}_\epsilon = \tilde{\beta}_0 - \epsilon\delta\frac{dz}{z} \in \hat{\mathfrak{g}}_{x_\epsilon}^\vee$ and let β_ϵ be the corresponding element of $(\hat{\mathfrak{g}}_{x_\epsilon}/\hat{\mathfrak{g}}_{x_\epsilon+})^\vee$. We deduce that $(\mathcal{G}, \nabla)$ contains the stratum $(x_\epsilon, 0, \beta_\epsilon)$ with respect to the trivialization ϕ. By Proposition 4.12, $(\mathcal{G}, \nabla)$ contains the stratum $(y, 0, \beta^y)$for any $y \in \bar{C}$, where β^y is determined by $[\nabla]_{m\phi} - \tilde{y}\frac{dz}{z}$.

Finally, since $[\nabla]_\phi - \tilde{x}\frac{dz}{z} \in \tilde{\beta}_0 + \hat{\mathfrak{g}}_{x+}^\vee \subset \hat{\mathfrak{g}}_{x_\epsilon+}^\vee$, it follows that $[\nabla]_\phi - \tilde{y}\frac{dz}{z} + \hat{\mathfrak{g}}_{x_\epsilon+}^\vee \in (\tilde{x} - \tilde{y})\frac{dz}{z} + \hat{\mathfrak{g}}_{x_\epsilon+}^\vee$; as $(\tilde{x} - \tilde{y}) \in \hat{\mathfrak{g}}^\vee(0)$, the proof of Proposition 3.7 shows that this coset is semistable if $\tilde{y} \neq \tilde{x}$. Therefore, $[\nabla]_\phi - \tilde{y}\frac{dz}{z} + \hat{\mathfrak{g}}_{y,0+}^\vee$ has the same property, so $(y, 0, \beta')$ is fundamental. In particular, this is true for any optimal point in $\bar{C}$ besides x, for example, any other vertex of $\bar{C} \cap \bar{\mathcal{B}}$.

□

Remark 4.18. If $(\mathcal{G}, \nabla)$ contains a nonfundamental stratum $(x, 0, \beta)$ with x in a minimal facet, then the proof above, replacing δ with $w\delta$ for w in the Weyl group of H_x, gives a construction of a fundamental stratum contained in $(\mathcal{G}, \nabla)$ at every $y \neq x$ in the closed star of x in $\mathcal{A}_0$. Moreover, this stratum induces a fundamental stratum in $\mathfrak{g}_\mathcal{G}$ as long as y is not in the same facet as x.

We now turn to the proof of Theorem 2.15, which states that strata of the same depth contained in a flat G-bundle are associates of each other. First, we supply the proof of Proposition 3.9, which was needed for Definition 3.10.

Proof of Proposition 3.9. Choose $g \in \hat{G}$ such that $gx, gy \in \mathcal{A}_0$. (For example, let $\mathcal{A}$ be an apartment containing x and y, and take g such that $g\mathcal{A} = \mathcal{A}_0$.) We now set $\delta_{x,y} = \mathrm{Ad}^*(g^{-1})((\widetilde{gy} - \widetilde{gx})\frac{dz}{z})$. This, of course, depends on the choice of g, but the defining property will be satisfied independently of the choice.

In order to show (15), one may easily reduce to the special case of $x, y \in \mathcal{A}_0$ and $g \in \hat{G}_x \cap \hat{G}_y$. Here, we will set $\delta_{x,y} = (\tilde{x} - \tilde{y})\frac{dz}{z}$. In other words, we must show that whenever $g \in \hat{G}_x \cap \hat{G}_y$,

$$\mathrm{Ad}^*(g)(\delta_{x,y}) \in \delta_{x,y} + \hat{\mathfrak{g}}^\vee_{x,0+} + \hat{\mathfrak{g}}^\vee_{y,0+}. \tag{19}$$

Using the notation introduced after Lemma 4.2, observe that $\delta_{x,y} = d_{\mathcal{A}_0,y} - d_{\mathcal{A}_0,x}$. However, by (20), $gd_{\mathcal{A}_0,x}g^{-1} \in d_{\mathcal{A}_0,x} + \hat{\mathfrak{g}}^\vee_{x,0+}$ and $gd_{\mathcal{A}_0,y}g^{-1} \in d_{\mathcal{A}_0,y} + \hat{\mathfrak{g}}^\vee_{y,0+}$. Subtracting, we obtain (19).

□

Proof of Theorem 2.15. Since conjugate strata are associates of each other, we may assume without loss of generality that $\phi' = \phi$. Applying Lemma 4.2 and the fact that we can find $h \in \hat{G}$ such that $hx, hy \in \mathcal{A}_0$, we may further assume that $x, y \in \mathcal{A}_0$.

We now verify (16) in this situation with $g = 1$. By Proposition 4.3, $\tilde{\beta} \in [\nabla]_\phi - \tilde{x}\frac{dz}{z} + \hat{\mathfrak{g}}^\vee_{x,-r+}$ and $\tilde{\beta}' \in [\nabla]_\phi - \tilde{y}\frac{dz}{z} + \hat{\mathfrak{g}}^\vee_{y,-r+}$. Taking $\delta_{x,y} = (\tilde{x} - \tilde{y})\frac{dz}{z}$, it is immediate that $[\nabla]_\phi - \tilde{x}\frac{dz}{z}$ lies in the intersection $\left(\tilde{\beta} + \hat{\mathfrak{g}}^\vee_{x,-r+}\right) \cap \left(\tilde{\beta}' - \delta_{x,y} + \hat{\mathfrak{g}}^\vee_{y,-r+}\right)$.

□

Proof of Theorem 2.17. By Corollary 4.11, $\mathrm{slope}(V_{\mathcal{G}}, \nabla_V)$ is at most the minimum of the depths of the strata contained in $(\mathcal{G}, \nabla)$. The equivalence of the four characterizations and the positivity statement follow from Theorem 2.14 and Proposition 4.13. The slope is rational, since slopes of flat vector bundles are rational. □

Finally, we turn to the proof of Proposition 2.20. Let E is a degree e field extension of F, and fix a generator u_E satisfying $u_E^e = z$. We let $\pi_E : \mathrm{Spec}(E) \to \mathrm{Spec}(F)$ be the associated map of spectra. We let $\mathcal{B}(E)$ be the building for $G(E)$ and denote the apartment corresponding to $T(E)$ by $\mathcal{A}_0(E)$. The pullback of the flat G-bundle $(\mathcal{G}, \nabla)$ to $\mathrm{Spec}(E)$ will be denoted by $(\pi_E^*\mathcal{G}, \pi_E^*\nabla)$. We will suppress the subscripts when the field E is clear from context.

Proof of Lemma 2.19. Set $e = [E : F]$. Take $x \in \mathcal{A}_0$, and let $ex \in \mathcal{A}_0(E)$ be the point corresponding to $e\tilde{x}$. If V is any representation of G, it is easily checked that $\hat{V}_x(r) = V(E)_{ex}(er) \cap \hat{V}$ and $\hat{V}_{x,r} = V(E)_{ex,er} \cap \hat{V}$. Indeed, it suffices to check the statement about gradings. Setting $\tau_E = u\frac{d}{du} = e\tau$, Proposition 3.2(1) implies that $V(E)_{ex}(er) \cap \hat{V}$ consists of those elements $v \in \hat{V}$ such that $erv = (\tau_E + \widetilde{ex})(v) = e(\tau + \tilde{x})(v)$, which is precisely $\hat{V}_x(r)$. We can now define a pullback map on strata based at points in $\mathcal{A}_0$: $\pi^*(x, r, \beta) = (ex, er, \beta')$; here, β' is the functional on $\mathfrak{g}(E)_{ex,er}/\mathfrak{g}(E)_{ex,er+}$ determined by the representative $e\tilde{\beta}$ coset in $\hat{\mathfrak{g}}^\vee_{x,-r} \subset \mathfrak{g}(E)^\vee_{ex,-er}$. Moreover, since β and β' have graded representatives differing by a factor of e, Remark 3.8 implies that $\pi^*(x, r, \beta)$ is fundamental if and only if (x, r, β) is.

Since $(\mathcal{G}, \nabla)$ has slope r, there exists a fundamental stratum (x, r, β) contained in $(\mathcal{G}, \nabla)$ with respect to a trivialization ϕ. By equivariance, we may assume that $x \in \mathcal{A}_0$. Applying Proposition 4.3, the functional β is determined by $[\nabla]_\phi - \tilde{x}\frac{dz}{z} \in \hat{\mathfrak{g}}^\vee_{x,-r} \subset \mathfrak{g}(E)^\vee_{ex,-er}$. However, $e[\pi^*\nabla]_\phi - \widetilde{ex}\frac{du}{u} = [\nabla]_\phi - \tilde{x}\frac{dz}{z}$, so the same proposition shows that $(\pi^*\mathcal{G}, \pi^*\nabla)$ contains the fundamental stratum (ex, er, β'). It follows that the pullback bundle has slope er. □

Proof of Proposition 2.20. Set $e = [E : F]$. The condition involving (11) is equivalent to the statement that $(\pi^*\mathcal{G}, \pi^*\nabla)$ contains a fundamental stratum (o, n, β) based at the origin of $\mathcal{A}_0(E) \subset \mathcal{B}(E)$. It must then have slope $n = re$. By Lemma 2.19, its slope is also equal to $e\,\mathrm{slope}(\mathcal{G}, \nabla)$, so the original flat G-bundle has slope r. On the other hand, by [2, Theorem 9.5], there exists an algebraic field extension E/F and a trivialization ϕ for $\pi^*\mathcal{G}$ such that $(\pi^*[\nabla])_\phi$ has nonnilpotent leading term with respect to powers of u, say in degree $-n$. Again, this means that the pullback bundle has slope n. If the original bundle has slope r, then the same lemma implies that $r = n/e$ as desired. □

5. Examples

In this section, we provide some examples to illustrate the theory. In each example, we write down the matrix for the flat G-bundle $(\mathcal{G}, \nabla)$ with respect to a fixed trivialization ϕ, which will be omitted from the notation.

Example 5.1. Let m be a nonnegative integer, and set $[\nabla] = (\sum_{i \geq -m} X_i z^i)\frac{dz}{z}$, where $X_i \in \mathfrak{g}$ and $X_{-m} \neq 0$. Then, $(\mathcal{G}, \nabla)$ contains the stratum (o, m, β^o), where $o \in \mathcal{A}_0$ is the origin and β^o is induced by $X_m \frac{dz}{z}$, so $\mathrm{slope}(\nabla) \leq m$. If $m > 0$, then this stratum is fundamental at the origin in $\mathcal{A}_0$ if and only if X_{-m} is not nilpotent, in which case, ∇ has slope m. If we assume that X_{-m} is contained in a Borel subalgebra $\mathfrak{b} \supset \mathfrak{t}$ (which we can accomplish by a constant change of gauge), then (x, m, β^x) is contained in $(\mathcal{G}, \nabla)$ for all $x \in \mathcal{A}_0$ if and only if $X_{-m} \in \mathfrak{t}$. If $m = 0$, $\mathrm{slope}(\nabla) = 0$ for any X_0 while the stratum at o is fundamental if and only if X_0 is nonnilpotent. If we assume that $X_0 \in \mathfrak{b}$, then (x, m, β^x) is fundamental for all $x \in \mathcal{A}_0$ precisely when $X_0 \in \mathfrak{t} - \mathfrak{t}_\mathbb{R}$.

Example 5.2. Suppose that $\mathfrak{g}$ has connected Dynkin diagram. Let $\alpha_1, \ldots, \alpha_n$ be a set of simple roots, and let α_0 be the highest root. Let $y_{-i} \in \mathfrak{u}_{-\alpha_i}$, $y_{\alpha_0} \in \mathfrak{u}_{\alpha_0}$ be a collection of nonzero root vectors, and set $X = (z^{-1}y_{\alpha_0} + \sum_{i=1}^n y_{-i})$. Fix $m \in \mathbb{Z}_{\geq 0}$, and let $[\nabla] = Xz^{-m}\frac{dz}{z}$. As explained in the previous example, this flat G-bundle contains the stratum $(o, m+1, \beta^o)$, but the stratum is not fundamental, since it is induced by the nilpotent element $z^{-m-1}y_{\alpha_0}$. However, one readily checks that $X \in \hat{\mathfrak{g}}_x(-1/h)$, where h is the Coxeter number and x is any $\mathfrak{z}_\mathbb{R}$-translate of the barycenter corresponding to the standard Iwahori subgroup. Since X is regular semisimple, it follows that $(\mathcal{G}, \nabla)$ contains a fundamental stratum based at x of depth $m + \frac{1}{h}$, which is accordingly the slope. Note that when $m = 0$, this is the rigid flat G-bundle described by Katz for GL_n and Frenkel-Gross in general [15].

In fact, no other points in $\mathcal{A}_0$ support a fundamental stratum contained in ∇ with respect to ϕ. To see this, note that $X \in \hat{\mathfrak{g}}_{y,-1/h}$ implies that $\alpha_i(y) \leq \frac{1}{h}$ for $1 \leq i \leq n$ and $\alpha_0(y) \geq \frac{h-1}{h}$. Using the fact that α_0 has height $h-1$ and adding the first inequalities appropriately, we get $\alpha_0(y) \leq \frac{h-1}{h}$, so $\alpha_0(y) = \frac{h-1}{h}$.

This immediately gives $\alpha_i(y) = \frac{1}{h}$, so y lies over the vertex in the reduced building corresponding to the standard Iwahori subgroup.

A variation on this construction gives flat G-bundles with slope $m + \frac{h-1}{h}$: define $[\nabla'] = X' z^{-m}\frac{dz}{z}$, where $X' = (z^{-1}y_{-\alpha_0} + \sum_{i=1}^n y_i)$ and $y_i \in \mathfrak{u}_{\alpha_i}$, $y_{-\alpha_0} \in \mathfrak{u}_{-\alpha_0}$ are nonzero root vectors.

For SL_n, the previous example gives flat SL_n-bundles of slope $m + 1/n$ and $m + (n-1)/n$. The next example constructs flat SL_n-bundles with slopes that are integer translates of $1/(n-1)$. Here, ∇ supports fundamental strata on a line in $\mathcal{A}_0$. For clarity, we take $n = 3$.

Example 5.3. Let $G = \mathrm{SL}_3(k)$. Set $X = z^{-1}e_{12} + e_{21} \in \hat{\mathfrak{g}}$, and consider the flat G-bundle $[\nabla] = Xz^{-m}\frac{dz}{z}$. We show that $\mathrm{slope}(\nabla) = m + \frac{1}{2}$. It suffices to find $x \in \mathcal{A}_0$ for which the regular semisimple matrix X lies in $\hat{\mathfrak{g}}_x(-\frac{1}{2})$. Writing $\tilde{x} = \mathrm{diag}(x_1, x_2, x_3)$ with $x_1 + x_2 + x_3 = 0$, this is equivalent to $x_1 - x_2 = 1/2$. This is the line connecting the barycenters of the faces of the fundamental alcove where the last simple root α_2 (with respect to the usual order) vanishes and the highest root α_0 equals 1. It is easy to check that x supports a stratum of depth $m + \frac{1}{2}$ (with respect to ϕ) precisely for x on this line. Note that this line does not contain a barycenter of an alcove; if it did, the slope of ∇ would have to be a multiple of $1/3$.

A more intuitive explanation is obtained by looking at lattice chains. For any $s \in \mathbb{Z}$, write $s = 3q + j$ with $0 \leq j < 3$. Now, define L^s be the lattice with $\mathfrak{o}$-basis $\{z^q e_i \mid i \leq 3 - j\} \cup \{z^{q+1}e_i \mid i > 3 - j\}$. The lattice chain $\mathcal{L} = (L^s)$ corresponds to the fundamental alcove while the period 2 lattice chains $\mathcal{L}_j = (L^s \mid s \not\equiv j \mod 3)$ correspond to the faces. In this terminology, $\mathcal{L}_1$ and $\mathcal{L}_0$ correspond to the faces $\alpha_2 = 0$ and $\alpha_0 = 1$ respectively. Note that $X(L^s) \subset L^{s-1} - L^s$ unless $s \equiv 1$, in which case $X(L^s) \subset L^{s-2} - L^{s-1}$. Accordingly, the depth of the stratum contained in ∇ induced by the lattice filtrations for $\mathcal{L}$ is $m + 2/3$ while for $\mathcal{L}_j$, it is $m + 1/2$ if $j = 0, 1$ and $m + 1$ if $j = 2$.

A similar analysis shows that the flat $\mathrm{SL}_n(k)$-bundle $[\nabla] = Xz^{-m}\frac{dz}{z}$ with $X = z^{-1}e_{1(n-1)} + \sum_{i=1}^{n-2} e_{(i+1)i}$ has slope $m + \frac{1}{n-1}$.

The following example shows that one can have a flat G-bundle that supports a fundamental stratum only at a single point in $\bar{\mathcal{B}}$, which, unlike Example 5.2, is not in an alcove.

Example 5.4. Let $G = \mathrm{Sp}_4(k)$, with the form defined by $\langle e_i, e_{j+2}\rangle = \delta_{ij} = -\langle e_{i+2}, e_j\rangle$ and $\langle e_i, e_j\rangle = 0$ for $1 \leq i, j \leq 2$. Set $Y = z^{-1}(e_{13} - e_{24}) + e_{31} + e_{42}$, and let ∇ be the flat $\mathrm{Sp}_4(k)$-bundle defined by $[\nabla] = Yz^{-m}\frac{dz}{z}$. Here, ∇ is a connection of slope $m + \frac{1}{2}$, and there is a unique point in the standard apartment supporting a fundamental stratum for ϕ with this depth. Indeed, setting $\tilde{x} = (x_1, x_2, -x_1, -x_2)$, the 4 inequalities $-2x_i \geq -1/2$ and $2x_i - 1 \geq -1/2$ immediately give $x_1 = x_2 = 1/4$. This point is the barycenter of the edge of the fundamental alcove where the short simple root α_1 vanishes. Since Y is regular semisimple and is graded with respect to this filtration, the corresponding stratum at x is fundamental.

One can give a lattice-theoretic interpretation here as well. Parahoric subgroups of $\mathrm{Sp}_{2n}(k)$ are stabilizers of *symplectic* lattice chains: lattice chains which are closed under homothety and duality with respect to the symplectic form [31]. For $n = 2$, one period of the lattice chain $\mathcal{L}$ stabilized by the standard Iwahori subgroup is

$$L^0 = \mathfrak{o}^4 \supset L^1 = \operatorname{span}\{e_1, e_2, ze_3, e_4\} \supset \\ L^2 = \operatorname{span}\{e_1, e_2, ze_3, ze_4\} \supset L^3 = \operatorname{span}\{e_1, ze_2, ze_3, ze_4\}.$$

The edges are obtained from $\mathcal{L}_j$ for $0 \leq j \leq 2$, the symplectic lattice chain generated by L_i, $0 \leq i \leq 2$, $i \neq j$. In particular, $\alpha_1 = 0$, $\alpha_2 = 0$, and $\alpha_0 = 1$ (with α_2 the long simple root and α_0 the highest root) correspond to j equal to 1, 2, and 0 respectively. We have $L^{4q+3} \subset L^{4q} - L^{4q+1}$ and $Y(L^j) \subset L^{j-2} - L^{j-1}$ otherwise. Thus, Y shifts $\mathcal{L}^1$ by -1, $\mathcal{L}^0$ and $\mathcal{L}^2$ by -2, and $\mathcal{L}$ by -3. The depth of the corresponding strata are $m + 1/2$, $m + 1$, and $m + 3/4$. (Note that for partial symplectic lattice chains, one does not obtain the depth by dividing the magnitude of the shift by the period.)

The previous three examples have the special property that they contain fundamental strata whose graded representative in $\hat{\mathfrak{g}}^\vee$ has a maximal *nonsplit* torus in $\hat{G}$ as its connected stabilizer under the coadjoint action. In the case of GL_n, these *regular strata* have been the key ingredient in constructing smooth symplectic and Poisson moduli spaces of connections and have allowed the realization of the isomonodromy equations as an integrable system [4, 5]. Regular strata and flat G-bundles containing them for reductive G are studied in detail in [6].

Appendix A. Complements on flat vector bundles

The authors have studied flat vector bundles using GL_n-strata in previous work [4], and these results are cited frequently in this paper. However, there are certain differences between the set-up in [4] and our present approach to flat vector bundles. For example, only lattice chain filtrations were considered in [4], and the notation for strata was given in terms of parahoric subgroups instead of points in the building. Furthermore, the definition of containment of a stratum in a flat vector bundle ([4, Definition 4.1]) is different from Definition 2.4 and is only equivalent for strata of depth greater than 0. Here, we provide the necessary explanations. We also show that our theory gives the same results for the equivalent concepts of flat rank n vector bundles and flat GL_n-bundles.

A.1. **Definitions of containment.** We begin with a general proposition about stratum containment.

Proposition A.1. *The subset of* $\operatorname{Rep}(G)$ *satisfying* (10) *for the flat G-bundle* $(\mathcal{G}, \nabla)$ *and the stratum* (x, r, β) *is closed under taking subrepresentations, duals, direct sums, tensor products, and homomorphism spaces. Moreover, it always contains the trivial representation.*

Proof. The fact that this set is closed under subrepresentations and direct sums is trivial. Let U and W be two representations satisfying (10). Using the fact that $U^\vee = \operatorname{Hom}(U, k)$ and $U \otimes W \cong \operatorname{Hom}(U^\vee, W)$, it suffices to check (10) for the trivial representation k and for $\operatorname{Hom}(U, W)$. For $\hat{k} = F$, each Moy-Prasad filtration is the usual one, so we need only check that the operator $\tau - i$ strictly increases the valuation for Laurent series of valuation at least i. This follows since, for any $a \in \mathfrak{o}$ and $k \geq i$, $(\tau - i)(az^k) = (k - i)az^k + \frac{da}{dz} z^{k+1}$ has valuation at least $i + 1$.

Next, take $f \in \widehat{\mathrm{Hom}(U, W)}_{x,i} = \mathrm{Hom}(\hat{U}, \hat{W})_{x,i}$, so $f(U_{x,s}) \subset W_{x,s+i}$ for all s. Recalling that ∇_ϕ acts on f via $\nabla_\phi \circ f - f \circ \nabla_\phi$ and similarly for $X_{\tilde{\beta}}$, we compute:

$$\begin{aligned}
&[(\nabla_\phi - i\frac{dz}{z} - X_{\tilde{\beta}})(f)](U_{x,s}) \\
&\qquad = (\nabla_\phi - (i+s)\frac{dz}{z} - X_{\tilde{\beta}})(f(U_{x,s})) - f((\nabla_\phi - s\frac{dz}{z} - X_{\tilde{\beta}})(U_{x,s})) \\
&\qquad \subset (\nabla_\phi - (i+s)\frac{dz}{z} - X_{\tilde{\beta}})(W_{x,s+i}) - f(U_{x,(s-r)+})\frac{dz}{z} \\
&\qquad \subset \Omega^1(\hat{W})_{x,(s+i-r)+}.
\end{aligned}$$

Hence, $(\nabla_\phi - i\frac{dz}{z} - X_{\tilde{\beta}})(\widehat{\mathrm{Hom}(U, W)}_{x,i}) \subset \Omega^1(\widehat{\mathrm{Hom}(U, W)})_{x,(i-r)+}$ as desired. □

Corollary A.2. *The* GL_n*-stratum* (x, r, β) *is contained in the flat* GL_n*-bundle* $(\mathcal{G}, \nabla)$ *if and only if it is contained in the associated vector bundle for the standard representation.*

Proof. Recall that (x, r, β) is contained in $(\mathcal{G}, \nabla)$ if and only if (10) holds for $W \in \mathrm{Rep}(\mathrm{GL}(V))$ while it is contained in $(\mathcal{G}_V, \nabla_V)$ if the same equation holds for the standard representation V. It thus suffices to show that if (10) holds for the standard representation V, then it holds for any representation. Since any representation of $\mathrm{GL}(V)$ is obtained from V via some combination of the operations in the previous proposition, the result follows. □

In [4], a U-stratum was defined as a triple (P, s, β) with $P \subset \mathrm{GL}(U)$ a parahoric subgroup, s a nonnegative integers, and β a functional on the quotient of consecutive congruent subalgebras $\beta \in (\mathfrak{p}^s/\mathfrak{p}^{s+1})^\vee$. The congruent subalgebras are defined in terms of a lattice chain $\mathcal{L}$ (say of period e) satisfying $\mathrm{Stab}(\mathcal{L}) = P$. (This lattice chain is unique up to translation of the indices by an integer.) As explained in Section 2.2, $\mathcal{L}$ gives rise to uniform $\mathbb{R}$-filtrations $\{U_r^{\mathcal{L}}\}$ and $\{\mathfrak{gl}(U)_r^{\mathcal{L}}\}$ with critical numbers at $\frac{1}{e}\mathbb{Z}$. The latter filtration is the Moy-Prasad filtration at x, where $x \in \mathcal{B}$ is any point in the building lying above the barycenter of the simplex in $\bar{\mathcal{B}}$ corresponding to P; the filtration on U comes from a unique such x which we denote by $x_{\mathcal{L}}$.

To see this, fix a k-structure for U, i.e., a k-subspace V such that $U = V \otimes F$. Recall that any parahoric subgroup in $\mathrm{GL}(U)$ is conjugate to a *standard parahoric subgroup* with respect to V, i.e., the pullback under $\mathrm{GL}(V \otimes \mathfrak{o}) \to \mathrm{GL}(V)$ of a standard parabolic subgroup $Q \subset \mathrm{GL}(V)$ (so $Q \supset B$). We may thus assume without loss of generality that P is a standard parahoric subgroup of this form. Any lattice chain with stabilizer contains $V \otimes \mathfrak{o}$, so we may assume that this lattice is L^0. If $e_1, \dots e_n$ is an ordered basis of V compatible with the flag of Q, then there exist indices $s_0 = n+1 > s_1 > s_2 > \cdots > s_e = 1$ such that for $q \in \mathbb{Z}$ and $0 \leq j < e$, L^{qe+j} has $\mathfrak{o}$-basis $\{z^q e_i \mid i < s_j\} \cup \{z^{q+1} e_i \mid i \geq s_j\}$. Let $\tilde{x} = \sum a_i e_{ii}^*$ where $a_i = \frac{e-j}{e}$ for $s_j \leq i < s_{j-1}$, so that $x \in \mathcal{A}_0$ is the barycenter of the simplex in $\bar{\mathcal{B}}$ corresponding to P. A direct calculation now shows that $\mathrm{Crit}_x = \frac{1}{e}\mathbb{Z}$ and for all $m \in \mathbb{Z}$, $U_{x,m/e} = L^m$ and $\mathfrak{gl}(U)_{x,m/e} = \mathfrak{p}^m$.

In [4], the association of U-strata with a formal flat vector bundle (U, ∇) was slightly different than that given here. We will show that for strata of positive depth, the two formulations agree. For (U, ∇) to contain a stratum (P, s, β) in the sense of [4, Definition 4.1] (with P the stabilizer of a period e lattice chain

$\mathcal{L}$), one first needs $\mathcal{L}$ to be "compatible" with ∇. One can then consider the endomorphism of the associated graded space $\mathrm{gr}(\mathcal{L}) = \bigoplus L^i/L^{i+1}$ induced by $\iota_\tau \circ \nabla$. Note that β also induces an endomorphism $\mathrm{gr}(\mathcal{L})$; indeed, this endomorphism is given by multiplication by an element $Y \in \mathfrak{p}^{-r}$ corresponding to a representative $\tilde{\beta}$. Containment now means that these two endomorphism coincide on all sufficiently large graded subspaces. (If $r > 0$, the endomorphisms will actually be the same.)

By choosing a trivialization of U compatible with $\mathcal{L}$ (as in [4, Remark 2.9]) and an appropriate k-rational structure V for U, one may assume without loss of generality that P is a standard parahoric subgroup in $\mathrm{GL}_n(\mathfrak{o})$ associated to the optimal point $x_{\mathcal{L}} \in \mathcal{A}_0$. For $s > 0$, it is now immediate that the criterion described above is the same as that given in Proposition 4.3 for containment of $(x_{\mathcal{L}}, s/e, \beta)$. Here, we are using the fact that the normalization term $-\tilde{x}\frac{dz}{z}$, which does not appear in [4], makes no contribution in this case. We are also using Corollary A.2, which allows us to apply Proposition 4.3 instead of Definition 2.4.

When $s = 0$, the definitions are not equivalent due to the normalization term. However, to apply results from [4], we only need to know that (U, ∇) contains a stratum $(x_{\mathcal{L}}, 0, \beta')$ if and only if it contains a stratum $(P, 0, \beta)$. In the latter case, after choosing a trivialization identifying $L^0 \in \mathcal{L}$ with $\mathfrak{o}^n$, we see that $\iota_\tau([\nabla]_\phi) \in \mathfrak{gl}_n(\mathfrak{o})$. Accordingly, ∇ contains the depth 0 stratum supported at the origin of $\mathcal{A}_0$ determined by $[\nabla]_\phi$. Conversely, suppose that ∇ contains $(x, 0, \beta)$ for x an optimal point corresponding to the lattice chain $\mathcal{L}$. Since $[\nabla]_\phi - \tilde{x}\frac{dz}{z}$ and $\tilde{x}\frac{dz}{z}$ are both in $\mathfrak{gl}_n(F)^\perp_{x+}$, the same is true for $[\nabla]_\phi$. It follows that $\iota_\tau([\nabla]_\phi)$ preserves every lattice in $\mathcal{L}$, so by [13, Lemme 6.21], ∇ is regular singular. Lemma 4.8 of [4] now implies that ∇ contains a depth 0 stratum $(P, 0, \beta')$.

Summing up the proceeding discussion, we see:

Proposition A.3. *Let (U, ∇) be a flat vector bundle. Let $P \subset \mathrm{GL}(U)$ be a parahoric subgroup corresponding to a lattice chain $\mathcal{L}$ of period e, and let $x \in \mathcal{B}$ be any point which has the same image in $\bar{\mathcal{B}}$ as $x_{\mathcal{L}}$.*

1. *If $s > 0$, (U, ∇) contains (P, s, β) in the sense of* [4, Definition 4.1] *if and only it contains $(x, s/e, \beta)$ in the sense of Definition 2.4.*
2. *The flat vector bundle (U, ∇) contains a depth 0 stratum $(P, 0, \beta)$ if and only if it contains a depth 0 stratum $(x, 0, \beta')$.*

A.2. **Slopes of vector bundles via the theory of strata.** In [4, Theorem 4.10], we showed that the slope of a flat vector bundle is determined by the strata contained in it associated to lattice chain filtrations. We are now ready to prove Proposition 4.8, which generalizes this result to allow arbitrary Moy-Prasad filtrations.

Lemma A.4. *If $(\hat{V}, \nabla)$ contains a fundamental stratum (x, r, β) with respect to the trivialization ϕ with $x \in \bar{C} \subset \mathcal{A}_0$ and $r > 0$, then it also contains a fundamental stratum (x', r, β') with respect to ϕ with the same depth and $x' \in \bar{C}$ an optimal point.*

Proof. Let $y \in \mathcal{A}_0$ be a vertex adjacent to the alcove containing x. Since $G = \mathrm{GL}_n$, there exists $n \in \hat{N}$ such that ny is the origin and nx is in the fundamental alcove. By Lemma 4.7 part (1), we see that $[\nabla]_{n\phi} - \widetilde{nx}\frac{dz}{z} \in \mathrm{Ad}^*(n)([\nabla]_\phi - \tilde{x}\frac{dz}{z}) + \hat{\mathfrak{g}}^\vee_{nx,0+}$. It follows that the stratum (nx, r, β') determined by $[\nabla]_{n\phi}$ is contained in $(\hat{V}, \nabla)$. Moreover, this stratum is fundamental if and only if (x, r, β) is.

Thus, without loss of generality, we can assume that C is the fundamental alcove and x is further in the open star of the origin. In particular, this implies that the corresponding filtration on $\hat{V}$ gives rise to a lattice chain $\hat{V}_{x,b_j+qe}$ with one term given by $V \otimes \mathfrak{o}$, say $\hat{V}_{x,b_0}$; here $b_0 < b_1 < \cdots < b_{e-1} < b_0 + 1$, and the critical numbers of the filtration are precisely at the translates of the b_i's. Up to indexing, this is just the lattice chain described above for the optimal point in the same open facet. Accordingly, the graded pieces of the filtration are given by $\hat{V}_x(b_j + qe) = \operatorname{span}\{z^{q+1}e_j \mid s_j \leq j < s_{j-1}\}$. We claim that $(\hat{V}, \nabla)$ contains a fundamental stratum at an optimal point coming from an appropriate sub-lattice chain.

Let $X \in \mathfrak{gl}(\hat{V})_x(-r)$ be the graded representative corresponding to the functional $\tilde{\beta}_0$ via the trace form as in Proposition 3.6. Since (x, r, β) is fundamental, X is not nilpotent. Hence, for some j, X^m is a nonzero map sending $\hat{V}_x(b_j)$ into $\hat{V}_x(b_j - mr)$ for all $m > 0$, and so each $b_j - mr$ is an integral translate of some b_i. Since there are only a finite number of distinct b_i's, r must be a rational number with denominator at most e, say $r = a/f$ with $(a, f) = 1$ and $1 \leq f \leq e$. The set of lattices $L^m \stackrel{\text{def}}{=} \hat{V}_x(b_j + m/f)$ for $m \in \mathbb{Z}$ is thus a sub-lattice chain of period f. Note that $\iota_\tau(([\nabla]_\phi)(L^m)) \subset L^{m-a}$. Thus, if x' is the optimal point corresponding to this lattice chain, $(\hat{V}, \nabla)$ contains the stratum (x', r, β'), with β' induced by ∇. Finally, we observe that $(\iota_\tau \circ [\nabla]_\phi)^m \notin \mathfrak{gl}(\hat{V})_{x',-mr+1/f}$ for any $m > 0$. Indeed, if this were true, then $(\iota_\tau \circ [\nabla]_\phi)^m(\hat{V}_{x,b_j}) \subset \hat{V}_{x,(b_j-mr)+}$; since $r > 0$ implies $(\iota_\tau \circ [\nabla]_\phi) - X \in \mathfrak{gl}(\hat{V})_{x,-r+}$, $X^m(\hat{V}_{x,b_j}) \subset \hat{V}_{x,(b_j-mr)+}$ as well. This contradicts the fact that (x, r, β) is fundamental. Applying [9, Lemma 2.1], we conclude that (x', r, β') is fundamental.

□

Remark A.5. It is not true that x' can be taken to be an optimal point in the same open facet as x. The rank 3 connection defined in Example 5.3 contains fundamental strata at x in the fundamental alcove $\bar{C}$ for SL_3 precisely for x lying on the line connecting the midpoints of two sides of this triangle.

Proof of Proposition 4.8. By [4, Theorem 4.10], these statements hold if one only allows x to range over uniform filtrations. By Proposition 4.12 (for $G = \mathrm{GL}_n$), if (U, ∇) contains a stratum $(x, 0, \beta)$, then it also contains a stratum $(x', 0, b')$ with x' in a minimal facet. Since x' thus corresponds to a uniform filtration, $\operatorname{slope}(U, \nabla) = 0$ and ∇ is regular singular. We can thus assume that ∇ is irregular singular, i.e., $\operatorname{slope}(\nabla) > 0$. It now follows from the previous lemma that all fundamental strata contained in (U, ∇) have depth $\operatorname{slope}(\nabla)$. It remains to show that one cannot have a nonfundamental (x, r, β) contained in the connection with $0 < r \leq \operatorname{slope}(\nabla)$. However, this follows from Corollary A.2 and the GL_n case of Proposition 4.16.

□

We may now combine this proposition with Corollary A.2 and Theorem 2.14 to obtain the following Corollary.

Corollary A.6. *The slope of a flat* $\mathrm{GL}(V)$*-bundle* $(\mathcal{G}, \nabla)$ *is the same as the slope of the flat vector bundle* $(V_{\mathcal{G}}, \nabla_V)$*. In particular,* $(\mathcal{G}, \nabla)$ *is regular singular if and only if* $(V_{\mathcal{G}}, \nabla_V)$ *is regular singular.*

Appendix B. Equivariance of stratum containment

We have seen in Lemma 4.2 that stratum containment is well-behaved with respect to change of trivialization. Here, we show that it also satisfies a stronger equivariance property. In order to state this, it will be useful to introduce some notation. Suppose that $x \in \mathcal{A}_0$. There exists a unique flat structure $d_{\mathcal{A}_0,x}$ on the trivial G bundle $\hat{G}$ that satisfies the following properties:

(1) for every $V \in \mathrm{Rep}(G)$, $(d_{\mathcal{A}_0,x} - i\frac{dz}{z})\hat{V}_x(i) = \{0\}$; and
(2) for all $g \in \hat{G}$,

$$(gd_{\mathcal{A}_0,x}g^{-1} - i\frac{dz}{z})\hat{V}_{gx,i} \subset \Omega^1(\hat{V})_{gx,i+}. \tag{20}$$

In fact, we will show that $d_{\mathcal{A}_0,x} = d + \tilde{x}\frac{dz}{z}$. It follows from Proposition 3.2(1) that $d + \tilde{x}\frac{dz}{z}$ satisfies the first property. This immediately shows that the second property holds when g is the identity. One obtains the general case of (20) from the observations that $i\frac{dz}{z} = gi\frac{dz}{z}g^{-1}$ and $\hat{V}_{gx,i} = g\hat{V}_{x,i}$.

To see that $d_{\mathcal{A}_0,x}$ is unique, note that if d' satisfies the first condition above, then $d' - d_{\mathcal{A}_0,x}$ is the zero map on all $\hat{V}_x(i)$. Since connections are continuous (with respect to the z-adic topology), $d' = d_{\mathcal{A}_0,x}$.

Consider the space $\mathcal{Q}^G$ of quintuples $(\mathcal{G}, \nabla, x, r, \phi)$, where $(\mathcal{G}, \nabla)$ is a formal flat G-bundle with trivialization ϕ, such that $\nabla_\phi(\hat{V}_{x,i}) \subset \Omega^1(\hat{V})_{x,i-r}$ for all $i \in \mathbb{R}$ and $V \in \mathrm{Rep}(G)$. We denote the subset of $\mathcal{Q}^G$ with fixed r by $\mathcal{Q}^G_r$. The group $\hat{G}$ acts on $\mathcal{Q}^G$ (and each $\mathcal{Q}^G_r$) via $g(\mathcal{G}, \nabla, x, r, \phi) = (\mathcal{G}, \nabla, gx, r, g\phi)$.

Lemma B.1. *Given $(\mathcal{G}, \nabla, x, r, \phi) \in \mathcal{Q}^G$, there is a unique stratum (x, r, β) contained in $(\mathcal{G}, \nabla)$ with respect to ϕ.*

Proof. First, we show uniqueness. Suppose that $(\mathcal{G}, \nabla)$ contains both (x, r, β) and (x, r, β') with respect to ϕ. It follows that $(X_{\tilde{\beta}} - X_{\tilde{\beta}'})\hat{V}_{x,i} \subset \Omega^1(\hat{V})_{x,i-r+}$ for all i and $V \in \mathrm{Rep}(G)$. In particular, $(X_{\tilde{\beta}} - X_{\tilde{\beta}'}) \in \Omega^1(\hat{\mathfrak{g}})_{x,-r+}$. Therefore, $\tilde{\beta} - \tilde{\beta}' \in \hat{\mathfrak{g}}^\vee_{x,-r+}$ and $\beta = \beta'$.

To prove existence, suppose that $x = gy$ for some $g \in \hat{G}$ and $y \in \mathcal{A}_0$. Write $d' = gd_{\mathcal{A}_0,y}g^{-1}$ and $X = \nabla_\phi - d'$. We see that X is F-linear; moreover, by (20) and the assumption on ∇_ϕ, $X\hat{V}_{x,i} \subset \Omega^1(\hat{V})_{x,i-r}$ for all $V \in \mathrm{Rep}(G)$. Therefore, $X \in \Omega^1(\hat{\mathfrak{g}})_{x,-r}$. Since (20) implies that $(d' - i\frac{dz}{z})\hat{V}_{x,i} \subset \hat{V}_{x,i+}$, (10) holds with $X_{\tilde{\beta}}$ replaced with X. There is a corresponding element $B \in \hat{\mathfrak{g}}^\vee_{x,-r}$, and we may take β to be the induced functional on $(\hat{\mathfrak{g}}_{x,r}/\hat{\mathfrak{g}}_{x,r+})$. □

Containment thus induces a map $\mathcal{Q}^G \to \mathcal{S}^G$ which restricts to maps $\mathcal{Q}^G_r \to \mathcal{S}^G_r$. The image of $\mathcal{Q}^G$ (resp. $\mathcal{Q}^G_r$) is the set of strata (resp. strata of depth r) contained in some flat G-bundle $(\mathcal{G}, \nabla)$ with respect to some trivialization.

Proposition B.2. *The map $\mathcal{Q}^G \to \mathcal{S}^G$ and the maps $\mathcal{Q}^G_r \to \mathcal{S}^G_r$ are $\hat{G}$-equivariant.*

Proof. Suppose that $(\mathcal{G}, \nabla, x, r, \phi)$ maps to (x, r, β). Lemma 4.2 implies that $(gx, r, g\beta)$ is contained in $(\mathcal{G}, \nabla)$ with respect to $g\phi$, and Lemma B.1 implies that $(gx, r, g\beta)$ is the image of $(\mathcal{G}, \nabla, gx, r, g\phi)$. □

References

[1] J. D Adler and S. DeBacker, "Some applications of Bruhat-Tits theory to harmonic analysis on the Lie algebra of a reductive p-adic group," Michigan Math. J. **50** (2002), 263–286.

[2] D. G. Babbitt and V. S. Varadarajan, " Formal reduction theory of meromorphic differential equations: a group theoretic view," Pacific J. Math. **109** (1983), 1–80.

[3] P. Boalch, "Symplectic manifolds and isomonodromic deformations," Adv. Math. **163** (2001), 137–205.

[4] C. Bremer and D. S. Sage, "Moduli spaces of irregular singular connections," Int. Math. Res. Not. IMRN **2013** (2013), 1800–1872.

[5] C. Bremer and D. S. Sage, "Isomonodromic deformation of connections with singularities of parahoric formal type," Comm. Math. Phys., **313** (2012), 175–208.

[6] C. Bremer and D. S. Sage, "Flat G-bundles and regular strata for reductive groups," arXiv:1309.6060[math.AG], 2014.

[7] F. Bruhat and J. Tits, "Groupes réductifs sur un corps local I," Publ. Math. I. H. E. S. **41**, 1972.

[8] F. Bruhat and J. Tits, "Groupes réductifs sur un corps local II," Publ. Math. I. H. E. S. **60**, 1984.

[9] C. J. Bushnell, "Hereditary orders, Gauss sums, and supercuspidal representations of GL_N," J. Reine Angew. Math. **375/376** (1987), 184–210.

[10] C. J. Bushnell and A. Frölich, "Nonabelian congruence Gauss sums and p-adic simple algebras," Proc. London Math. Soc. **50** (1985), 207–264.

[11] C. J. Bushnell and P. Kutzko, *The admissible dual of* GL_N *via compact open subgroups,* Annals of Mathematics Studies, **129**, Princeton University Press, Princeton, NJ, 1993.

[12] T.-H. Chen and M. Kamgarpour, "Preservation of depth in local geometric Langlands correspondence," arXiv:1404.60598[math.RT], 2014.

[13] P. Deligne, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, Vol. 163, Springer-Verlag, New York, 1970.

[14] E. Frenkel and D. Gaitsgory, "Local geometric Langlands correspondence and affine Kac-Moody algebras," in *Algebraic geometry and number theory*, Progr. Math **253**, Birkhäuser Boston, Boston, MA, 2006, pp. 69–260.

[15] E. Frenkel and B. Gross, "A rigid irregular connection on the projective line," Ann. of Math. (2) **170** (2009), 1469–1512.

[16] E. Frenkel and X. Zhu, "Any flat bundle on a punctured disc has an oper structure," Math. Res. Lett. **17** (2010), 27–37.

[17] M. Goresky, R. Kottwitz, and R. Macpherson, "Purity of equivalued affine Springer fibers," Represent. Theory **10** (2006), 130–146.

[18] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, **151**, Princeton University Press, Princeton, NJ, 2001.

[19] G. Henniart, "Une preuve simple des conjectures de Langlands pour $\mathrm{GL}(n)$ sur un corps p-adique," Invent. Math. **139** (2000), 439–455.

[20] R. Howe and A. Moy, " Minimal K-types for GL_n over a p-adic field," Astérisque **171/172** (1989), 257–273.

[21] M. Jimbo, T. Miwa, and K. Ueno, "Monodromy preserving deformations of linear ordinary differential equations with rational coefficients I," Physica D **2** (1981), 306–352.

[22] M. Kamgarpour and D. S. Sage, "A geometric analogue of a conjecture of Gross and Reeder," arXiv:1606.00943 [math.RT], 2016.

[23] G. Kempf, "Instability in invariant theory," Ann. Math. **108** (1978), 299–316.

[24] P. C. Kutzko, "Towards a classification of the supercuspidal representations of GL_n," J. London Math. Soc. (2) **37** (1988), 265–274.

[25] E. Landvogt, *A Compactification of the Bruhat-Tits Building*, Lecture Notes in Mathematics, Vol. 1619, Springer-Verlag, Berlin, 1996.

[26] B. Malgrange, *Équations Différentielles à Coefficients Polynomiaux*, Progress in Mathematics, Vol. 96, Birkhäuser Boston, Inc., Boston, MA, 1991.

[27] A. Moy, "A conjecture on minimal K-types for GL_n over a p-adic field," in *Representation theory and number theory in connection with the local Langlands conjecture: Proceedings of a conference held December 8-14, 1985*, Contemp. Math. **86** (1989), 249–154.

[28] A. Moy and G. Prasad, "Unrefined minimal K-types for p-adic groups," Invent. Math. **116** (1994), 393–408.
[29] A. Moy and G. Prasad, "Jacquet functors and unrefined minimal K-types," Comment. Math. Helv. **71** (1996), no. 1, 98–121.
[30] G. Prasad and M. S. Raghunathan, "Topological central extensions of semi-simple groups over local fields," Ann. Math. **119** (1984), 143–268.
[31] D. S. Sage, "The geometry of fixed point varieties on affine flag manifolds," Trans. Amer. Math. Soc. **352** (2000), 2087–2119.
[32] D. S. Sage, "Regular strata and moduli spaces of irregular singular connections," Proceedings of the 2015 Isaac Congress, to appear.
[33] T. A. Springer, *Linear Algebraic Groups*, second edition, Progr. Math **9**, Birkhäuser Boston, Boston, MA, 1998.
[34] J. Tits, "Reductive Groups over Local Fields," Proc. Symp. Pure Math. **33** (1979), 29–69.
[35] D. Vogan, "The algebraic structure of the representations of semisimple Lie groups," Ann. Math. **119** (1979), 1–60.
[36] W. Wasow, *Asymptotic expansions for ordinary differential equations*, Wiley Interscience, New York, 1976.
[37] E. Witten, "Gauge theory and wild ramification," Anal. Appl. (Singap.) **6** (2008), 429–501.

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