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A Theory of Nets for Polyhedra and Polytopes Related to Incidence Geometries

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Dedicated to Hanfried Lenz on the occasion of his 80th birthday.

Abstract. Our purpose is to elaborate a theory of planar nets or unfoldings for polyhedra, its generalization and extension to polytopes and to combinatorial polytopes, in terms of morphisms of geometries and the adjacency graph of facets.

Keywords: net, polytope, incidence geometry, prepolytope, category

1. Introduction

Planar nets for various polyhedra are familiar both for practical matters and for mathematical education about age 11–14. They are systematised in two directions. The most developed of these consists in the drawing of some net for each polyhedron in a given collection (see for instance [20], [21]). A less developed theme is to enumerate all nets for a given polyhedron, up to some natural equivalence. Little has apparently been done for polytopes in dimensions $d \geq 4$ (see however [18], [3]).

We have found few traces of this subject in the literature and we would be grateful for more references. An interesting reference is [1] where some formalisation takes place in a topological language. A deeper study can be found in [2]. Let us point out that the subject is related to the much studied theory of planar graphs (see for instance [11]).

Two papers by Bouzette and Vandamme [4] and Bouzette [5] are closely related to the present one.

We shall now briefly explain our approach of some theory. As a matter of fact, our starting point is physical experience dealing with planar drawings for cubes and other familiar polyhedra. Let **P** be such a polyhedron. We can think of **P** in terms of various structures. Here we want to emphasize two different structures. First comes the *combinatorial structure* of **P**. It consists of three sets of objects called respectively vertices, edges and faces, ordered by inclusion. Next comes the *metric structure* of **P**, completing the preceding one by the data of the "envelope" of **P** and the metric induced on it by the surrounding space. Actually, it suffices to deal with the metric on each face of **P**.

An *unfolding* or *net* or *development* of **P** appears as some other geometry $\mathbf{P}_1$ together with some kind of morphism from $\mathbf{P}_1$ onto **P**. In elementary geometry this goes obviously along with metrical requirements. One of our main observations is that a great deal of this study does only require the combinatorial structure. This has the usual benefits: simplification, clarity and generalization (both for polytopes and for combinatorial polytopes). The metrical viewpoint is not overlooked and it leads to interesting open problems.

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2. A Combinatorial Approach

2.1. Combinatorial Pre-Polytopes

2.1.1. Definitions and Notation

We could make use of the setting for incidence geometries developed for instance in [6], [8]. For the convenience of the reader we shall concentrate on a less general viewpoint close to the position taken in [10]; moreover we shall make all definitions explicit. As a matter of fact, our formalism could also be expressed in other contexts like lattice theory, simplicial complexes, etc. Let $d \geq 1$ be an integer and let I be the set $\{0, 1, \ldots, d-1\}$. We consider a *graded partially ordered set of rank d* namely $\Gamma = (X, \leq, t)$ where X is a set, $\leq$ is a partial order on X and t is a mapping of X onto I such that $a < b$ implies $t(a) < t(b)$ and such that for every maximal totally ordered subset Y of X, we have $t(Y) = I$. For $a \in X$, the *type* of a is the element $t(a) \in I$. We call two elements a and b of X *incident* and we write $a * b$ if either $a < b$ or $b \leq a$. An *interval* of Γ is any of the following.

(0) The set $]-\infty, \infty[$ of all x in X.

(1) For any $a < b$ in X, the set $]a, b[$ of all x in X such that $a < x < b$.

This is a graded partially ordered set of rank $t(b) - t(a) - 1$ in which the grade function maps x on $t(x) - t(a) - 1$.

(2) For any $a \in X$, the set $]a, \infty[$ of all x in X such that $a < x$ (the *upper residue* of a) and likewise the set $]-\infty, a[$ of all x in X such that $x < a$ (the *lower residue* of a). These are graded partially ordered sets of respective ranks d, $d - t(a) - 1$ and $t(a)$, in which the grade functions map x respectively on $t(x)$, $t(x) - t(a) - 1$ and $t(x)$. We call Γ *residually connected* if each of its intervals of rank ≥ 2 is a connected graph for the incidence relation $*$. We call Γ *thin* if every interval of rank 1 of Γ contains exactly two elements.

An element x of Γ with $t(x) = 0$ (resp. 1, $d-1$, $d-2$) is called *a vertex* (resp. *edge*, *facet*, *ridge*). For any $x \in X$, the *0-shadow* of x is the set $\sigma(x)$ consisting of all vertices

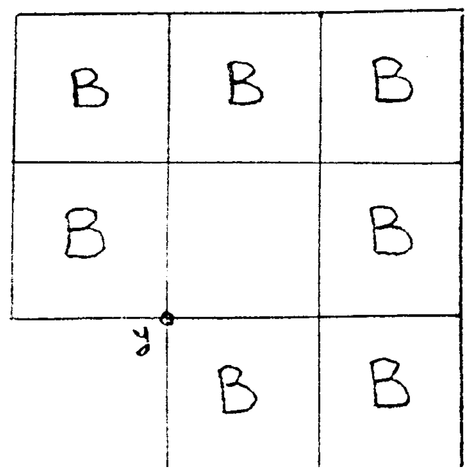

Figure 1.

y such that $y \leq x$. For a facet F, we often identify F with its lower residue $]-\infty, F[$ in order to simplify notation. We may also write Γ_F in order to denote this residue.

2.1.2. The *dual* of a graded partially ordered set of rank d say $\Gamma = (X, \leq, t)$ is the graded partially ordered set of rank d, $\Gamma^* = (X, \geq, t^*)$ where $a \geq b$ if and only if $b \leq a$ and $t^*(a) = d - t(a) - 1$ for all $a \in X$. Clearly, $\Gamma^{**} = \Gamma$. If Γ is residually connected (resp. thin) then Γ^* is also residually connected (resp. thin).

2.1.3. A *combinatorial polytope of rank d for $d \geq 1$* is a residually connected and thin graded partially ordered set of rank d. A combinatorial polytope $\mathbf{P}$ has a *dual* $\mathbf{P}^*$ which is also a combinatorial polytope.

A *combinatorial pre-polytope* is a graded partially ordered set of some rank $d \geq 2$ in which the residues of all facets are combinatorial polytopes of rank $(d-1)$ and all ridges are incident to either 1 or 2 facets. In this situation, a ridge is of *degree* 1 or 2 according to it being incident with 1 or 2 facets. Observe that a combinatorial pre-polytope does not need to be residually connected. Also, every combinatorial polytope is of course a pre-polytope.

In the rest of this paper, we deal exclusively with combinatorial pre-polytopes of rank ≥ 2. Let $\mathbf{P}$ be a combinatorial pre-polytope. The *facet graph* of $\mathbf{P}$, say $\mathcal{F}(\mathbf{P})$, consists of the set of facets of $\mathbf{P}$ and those pairs of facets that are incident to a given ridge of $\mathbf{P}$. For a combinatorial polytope $\mathbf{P}$ the facet graph is the 1-skeleton of $\mathbf{P}^*$. We call $\mathbf{P}$ *connected* if $\mathcal{F}(\mathbf{P})$ is a connected graph. This is obviously inspired by familiar unfoldings of polyhedra. Actually we need a stronger property that can also be observed on the examples. We call $\mathbf{P}$ *highly connected* if for each element y of type $t \leq d-3$, the incidence graph of facets and ridges incident with y, is connected.

The following example (figure 1) shows that $\mathbf{P}$ may be connected and not be highly connected. In that example, $d = 3$, the facets are seven squares with a label B (for black) and the vertex y has a non-connected upper residue.

2.1.4. LEMMA *In a combinatorial pre-polytope of rank d let a be an element of type i with $i \neq 0$ (reps. $i \neq d$). Then the lower residue $]-\infty, a[$ (resp. upper residue $]a, \infty[$) is a combinatorial polytope (resp. pre-polytope) of rank i (resp. $d-i-1$).*

Proof. Straightforward. ∎

2.1.5. THEOREM *A combinatorial pre-polytope* **P** *is residually connected if and only if it is highly connected and connected.*

Proof. If **P** is residually connected then it is clearly highly connected. Let **P** be highly connected. For $d = 2$, there is nothing to prove. We can assume $d \geq 3$. First we show that the incidence graph of **P** is connected. Let x, y be elements of **P**. There exist facets F, F' of **P** such that $x * F$ and $F' * y$. Moreover, since **P** is connected there exists a path of ridges and facets from F to F'. Therefore, there exists a path from x to y in the incidence graph of **P**. Let Φ be an interval of **P** of rank ≥ 2. First, take $\Phi =]a, b[$ with $a < b$. There is a facet F such that $b \leq F$. Then Φ is in an interval of the lower residue of F and since F is a combinatorial polytope, Φ is connected. This argument works also if $\Phi =]-\infty, a[$ for some element a. Hence, assume that $\Phi =]a, +\infty[$. Then there is a vertex p with $p \leq a$, Φ is an interval of the upper residue of p and the latter is a combinatorial pre-polytope of rank $d-1$ (Lemma 2.1.4). Now $t(p) = 0$ hence, by the hypothesis that **P** is highly connected, the graph of facets and ridges incident with p is connected. This means that the pre-polytope $]p, \infty[$ is connected. It is also obviously highly connected. Therefore induction on d may be applied. ∎

Let us observe further that a connected rank 2 combinatorial pre-polytope is either isomorphic to a usual n-gon ($2 \leq n \leq \infty$) or to a *string* namely a connected graph in which all vertices but two have degree 2, the two exceptions having degree one.

2.1.6. *Property.* (0). Let **P** be a combinatorial pre-polytope. We say that **P** has property (0) if any two distinct elements x, y of **P** have distinct 0-shadows $\sigma(x)$ and $\sigma(y)$ where $\sigma(x)$ is the set of vertices of **P** incident to x. This property is always assumed in Section 3. Then the data of **P** is equivalent with the graded poset of sets $\sigma(x)$ on the set of vertices.

2.2. Combinatorial Unfolding of a Combinatorial Pre-Polytope

Let **P** be a combinatorial pre-polytope of rank d over $I = \{0, 1 \ldots, d-1\}$.

A *combinatorial unfolding* of **P** is a pair $(\mathbf{Q}, \alpha)$ where **Q** is a combinatorial pre-polytope over I and α is a mapping of the set Y of elements of **Q** onto the set X of elements of **P** such that:

(1) α is type-preserving, namely $t(\alpha(y)) = t(y)$ for all $y \in Y$;

(2) α is one-to-one from the set of facets of **Q** to the set of facets of **P**;

(3) for any facet F of **Q**, α restricted to F is an isomorphism onto $\alpha(F)$.

In view of (3), α is automatically incidence-preserving. We call the unfolding $(\mathbf{Q}, \alpha)$ *connected* (resp. highly connected) if $\mathbf{Q}$ is connected (resp. *highly connected*).

These definitions are rather categorical. For instance, if $(\mathbf{Q}, \alpha)$ is an unfolding of $\mathbf{P}$ and $(\mathbf{R}, \beta)$ is an unfolding of $\mathbf{Q}$, then $(\mathbf{R}, \alpha\beta)$ is an unfolding of $\mathbf{P}$.

In the same spirit, two combinatorial unfoldings $(\mathbf{Q}, \alpha)$ and $(\mathbf{Q}', \alpha')$ of $\mathbf{P}$ are called *equivalent* if there is an isomorphism β of $\mathbf{Q}$ onto $\mathbf{Q}'$ such that $\alpha'\beta = \alpha$.

We get universal objects as well, in an obvious way. Given $\mathbf{P}$ as earlier, a *universal unfolding* $(\mathbf{Q}, \alpha)$ of $\mathbf{P}$ is an unfolding equivalent to all of its own unfoldings. It corresponds to the physical situation of a polyhedron all of whose faces are separated.

Physical unfoldings push us to look for residually connected combinatorial unfoldings. If $\mathbf{P}$ is finite and residually connected, then $\mathbf{P}$ does necessarily admit a *universal residually connected combinatorial unfolding* $(\mathbf{Q}, \alpha)$ which means that $(\mathbf{Q}, \alpha)$ is equivalent to all of its own residually connected unfoldings. As an example, if $\mathbf{P}$ is a cube, $\mathbf{P}$ has 11 equivalence classes of universal residually connected combinatorial unfoldings each one represented by one of the standard list of nets for $\mathbf{P}$.

Remarks. 1. Classical objects of recreational mathematics such as polyominos provide examples of pre-polytopes. These can help us realise that a pre-polytope is not necessarily an unfolding of some polytope.

2. An unfolding of $\mathbf{P}$ induces an unfolding on the upper residue P_v, for any vertex v of $\mathbf{P}$.

3. If $(\mathbf{Q}, \alpha)$ is an unfolding of a residually connected pre-polytope and if $\mathbf{Q}$ is connected does there follow that $\mathbf{Q}$ is residually connected? The answer is negative. For $\mathbf{Q}$, we can take two regular tetrahedra with a common vertex and for $\mathbf{P}$, two regular tetrahedra with a common face.

2.3. Combinatorial Unfolding and the Cut Set

2.3.1. Let $\mathbf{P}$ be a combinatorial pre-polytope and let $(\mathbf{Q}, \alpha)$ be a combinatorial unfolding of $\mathbf{P}$. We are interested in the ridges of $\mathbf{Q}$ of degree one whose image under α is of degree 2. These ridges of $\mathbf{Q}$ are called *cut*. Let $\mathcal{C}(\mathbf{Q}, \alpha)$ denote the *cut set* of $\mathbf{P}$ namely the set of ridges of $\mathbf{P}$ that are images under α of cut ridges of $\mathbf{Q}$.

At first view, physical experience may suggest that $(\mathbf{Q}, \alpha)$ must be determined uniquely by its cut set in $\mathbf{P}$. However, this is not quite true as we can see on an easy example.

Example. Let $\mathbf{P}$ be the union of two triangles with a common edge C and put $\mathcal{C} = \{C\}$. Let $\mathbf{Q}_1$ be the disjoint union of two triangles and let $\mathbf{Q}_2$ be the union of two triangles with one common vertex. Clearly, we get unfoldings $(\mathbf{Q}_1, \alpha_1)$ and $(\mathbf{Q}_2, \alpha_2)$ of $\mathbf{P}$ and $C(\mathbf{Q}_1, \alpha_1) = \mathcal{C} = C(\mathbf{Q}_2, \alpha_2)$.

We shall nevertheless develop a satisfactory general theory for all $(\mathbf{Q}, \alpha)$ such that $C(\mathbf{Q}, \alpha) = \mathcal{C}$ for any given set $\mathcal{C}$ of ridges of degree two in $\mathbf{P}$.

2.3.2. ***C*-unfolding.** Let $\mathbf{P}$ be a combinatorial pre-polytope over $I = \{0, \ldots, d-1\}$ with $d \geq 2$ and let $\mathcal{C}$ be a set of ridges of degree two in $\mathbf{P}$. We construct a combinatorial pre-polytope $\mathbf{Q} = Q(\mathbf{P}, \mathcal{C})$ over I, depending on $\mathbf{P}$ and $\mathcal{C}$ and we call it the *$\mathcal{C}$-unfolding* of $\mathbf{P}$.

1) The facets of $\mathbf{Q}$ are the facets of $\mathbf{P}$.

2) Each element x of $\mathbf{P}$, of type $i \leq d-2$, incident with some ridge not in $\mathcal{C}$ is also an element of $\mathbf{Q}$ and incidence among these elements and the facets is transfered from $\mathbf{P}$ to $\mathbf{Q}$.

3) For each element y of $\mathbf{P}$, of type $i \leq d-2$, such that each ridge incident with y is in $\mathcal{C}$, and for each facet F of $\mathbf{P}$, with $y * F$, we create an element (y, F) in $\mathbf{Q}$, of type i. If $x * y$, with x as in 2), we decide that $x * (y, F)$. We also decide that $(y, F) * (y', F')$ if and only if $y * y'$ and $F = F'$. Finally, $(y, F) * F'$ if and only if $F = F'$.

4) There is an obvious mapping α of $\mathbf{Q}$ onto $\mathbf{P}$, fixing each element common to $\mathbf{Q}$ and $\mathbf{P}$ and mapping (y, F) onto y for each y as in 3).

2.3.3. THEOREM *For $\mathbf{P}, \mathcal{C}, \mathbf{Q} = Q(\mathbf{P}, \mathcal{C})$ and α as above we have:*
(Q, α) is a combinatorial unfolding of $\mathbf{P}$ with $\mathcal{C}$ as cut set.

Proof. 1) We show that $\mathbf{Q}$ is a combinatorial pre-polytope. It is obviously a graded poset of rank d. Moreover the residue of a facet is a combinatorial polytope and ridges of $\mathbf{Q}$ are incident to either one or two facets.

2) We show that $(\mathbf{Q}, \alpha)$ is a combinatorial unfolding of $\mathbf{P}$. To this end, we observe that α is indeed a mapping and that it is type-preserving. Moreover, α is one-to-one from the set of facets of $\mathbf{Q}$ to the set of facets of $\mathbf{P}$ and for each facet F of $\mathbf{Q}$, the restriction of α to $\mathbf{Q}_F$ is indeed an isomorphism onto $\mathbf{P}_{\alpha(F)}$.

3) It is straightforward that $C(\mathbf{Q}, \alpha) = \mathcal{C}$ as required. ∎

2.3.4. LEMMA *Let $\mathbf{P}, \mathcal{C}$ be as above and let $(\mathbf{Q}, \alpha)$ be the $\mathcal{C}$-unfolding of $\mathbf{P}$ with $\mathbf{Q} = Q(P, \mathcal{C})$.*

Let $\mathcal{D}$ be a set of ridges of $\mathbf{P}$ of degree two, and let $(\mathbf{R}, \beta)$ be the $\mathcal{D}$-unfolding of $\mathbf{P}$ with $\mathbf{R} = Q(\mathbf{P}, \mathcal{D})$. Then the two following properties are equivalent.

(i) $\mathcal{C} \subseteq \mathcal{D}$

(ii) *there exists a unique unfolding $(\mathbf{R}, \gamma)$ of $\mathbf{Q}$ such that $\beta = \alpha\gamma$.*

Moreover, $\mathbf{R}$ is isomorphic to $Q(\mathbf{Q}, \mathcal{D} \backslash \mathcal{C})$.

Proof. (i) $\Rightarrow$ (ii). If x is an element of $\mathbf{R}$ as in 1) or 2) of 2.3.2, x is an element of $\mathbf{P}$ and it is an element of $\mathbf{Q}$ as in 1) or 2). Hence, $\gamma(x) = x$ is the only possibility for γ in that case.

If x is an element of $\mathbf{R}$ as in 3) of 2.3.2, then $x = (y, F)$ where y is in $\mathbf{P}$, F is a facet of $\mathbf{P}$ with $y * F$ and every ridge incident with y is in $\mathcal{D}$. Moreover, $\beta(x) = y$. If there is a ridge on y, not in $\mathcal{C}$, y is also an element of $\mathbf{Q}$, by 2.3.2 and then we have necessarily $\gamma(x) = y$. If each ridge on y is in $\mathcal{C}$, then (y, F) is an element of $\mathbf{Q}$, by 2.3.2 and so $\gamma(x)$ is necessarily this element.

This provides the uniqueness of γ, its construction as a mapping from $\mathbf{R}$ into $\mathbf{Q}$ and

$\beta = \alpha\gamma$. Hence γ is onto. Clearly, γ is type-preserving and one-to-one on the set of facets. Since $\beta = \alpha\gamma$, the restriction of γ to every facet is an isomorphism.

(ii) $\Rightarrow$ (i). Let $C \in \mathcal{C}$. Then C is of degree 2 in **P** and it is incident with facets F_1, F_2 of **P**. By 2.3.2, $\alpha^{-1}(C)$ consists of two ridges, namely (C, F_1) and (C, F_2), each of degree one, whose inverse images by γ are also necessarily of degree one.

Therefore $\beta^{-1}(C)$ consists of two ridges of **R** of degree one and so $C \in \mathcal{D}$, hence (i) holds.

The last statement is rather obvious. ∎

2.3.5. THEOREM *Let* **P** *be a highly connected combinatorial pre-polytope and let* $\mathcal{C}$ *be a set of ridges of degree two in* **P**. *Let* $(\mathbf{Q}, \alpha)$ *be the* $\mathcal{C}$*-unfolding of* **P**. *Assume that* **Q** *is highly connected. Then the following hold.*

(i) *for any combinatorial unfolding* $(\mathbf{R}, \beta)$ *of* **P** *with* $\mathcal{C} = C(\mathbf{R}, \beta)$ *there exists a unique combinatorial unfolding* $(\mathbf{Q}, \gamma)$ *of* **R** *such that* $\beta\gamma = \alpha$;

(ii) *if* $(\mathbf{R}, \beta)$ *is as in (i) and* **R** *is highly connected, then* $(\mathbf{R}, \beta)$ *is equivalent to* $(\mathbf{Q}, \alpha)$.

Proof. (i) We shall show that the relation $\gamma = \beta^{-1}\alpha$ is a mapping. Let x be an element of **Q** and assume that y_1, y_2 are distinct elements of $\beta^{-1}\alpha(x)$. Then there are facets F_1, F_2 of **P** such that $F_1 * \alpha(x) * F_2$, $y_1 * \beta^{-1}(F_1)$ and $y_2 * \beta^{-1}(F_2)$. Now, y_1 is not incident with $\beta^{-1}(F_2)$ because β restricted to a facet is an isomorphism and so $F_1 \neq F_2$. If x is a ridge, $\alpha(x)$ is of degree two and not in $\mathcal{C}$, y_1 is of degree one and since $\beta(y_1) = \alpha(x)$, this contradicts $\mathcal{C} = C(\mathbf{R}, \beta)$. Thus x is of type $\leq d-3$ an as **Q** is highly connected, there is a chain $\alpha^{-1}(F_1) * C_1 * E_2 * C_2 * E_3 * \cdots * \alpha^{-1}(F_2)$ where the C_i (resp. E_i) are distinct ridges (resp. facets) of **Q** incident with x. Hence, $\alpha(C_i)$ is not in $\mathcal{C}$ and β^{-1} is a mapping on the image of the above chain under α. This gives $\beta^{-1}(F_1) * \beta^{-1}\alpha(C_1) * \beta^{-1}\alpha(E_2) * \cdots * \beta^{-1}(F_2)$. Here, $y_1 * \beta^{-1}\alpha(C_1)$ because of the isomorphic action of β^{-1} on the residue of F_1. Then $y_1 * \beta^{-1}\alpha(E_2)$. Repeating the last argument, we get inductively that $y_1 * \beta^{-1}(F_2)$, a contradiction. Therefore, γ is a mapping and it is uniquely defined. It is type-preserving. It is one-to-one from the set of facets of **Q** to the set of facets of **R**. If F is a facet of **Q** it is clear that γ restricted to F is an isomorphism. Therefore, γ is a combinatorial unfolding.

(ii) Let γ be as in (i). We need only show that γ is an isomorphism. If γ is one-to-one then γ^{-1} is an isomorphism because its restriction to any facet is the restriction of $\alpha^{-1}\beta$ which is a product of isomorphisms. If γ is not one-to-one, let x, x' be elements of **Q** with $\gamma(x) = \gamma(x')$. Here x is not a ridge since otherwise $\alpha(x) \in \mathcal{C}$ contradicting $\mathcal{C} = C(\mathbf{R}, \beta)$. Therefore x is of type $\leq d-3$.

Then $\alpha(x) = \alpha(x')$ in **P**, hence $\alpha(x)$ is as in 3) of 2.3.2 and $x = (\alpha(x), F)$, $x' = (\alpha(x), F')$ where F, F' are facets of **P** incident with $\alpha(x)$. Since **R** is highly connected, there is a chain $\beta^{-1}(F) * C_1 * E_2 * C_2 * E_3 * \cdots * \beta^{-1}(F')$ where the ridges C_i and facets E_i are incident with $\gamma(x)$. All C_i may be assumed to be of degree 2 and $\alpha^{-1}\beta$ is a unique chain

$\alpha^{-1}(F) * \alpha^{-1}\beta(C_1) * \alpha^{-1}\beta(E_2) * \cdots * \alpha^{-1}(F')$. We get as in the corresponding argument of (i) that x is incident with each member of that chain, hence $x * F'$ and so $x = x'$. ■

2.3.6. The dual cut set. Let **P** be a combinatorial pre-polytope and $\mathcal{C}$ a set of ridges of degree two of **P**. Then we define the *dual* $\mathcal{C}^*$ of $\mathcal{C}$ as the set of ridges of degree two of **P**, not in $\mathcal{C}$. If $(\mathbf{Q}, \alpha)$ is a combinatorial unfolding of **P**, the *dual cut set* $C^*(\mathbf{Q}, \alpha)$ is the dual of $C(\mathbf{Q}, \alpha)$.

2.3.7. THEOREM *Let* **P** *be a highly connected and connected combinatorial pre-polytope. Let* $(\mathbf{Q}, \alpha)$ *be a highly connected combinatorial unfolding of* **P** *and let* $\mathcal{C}^*$ *be the dual cut set of* $(\mathbf{Q}, \alpha)$. *Then* $(\mathbf{Q}, \alpha)$ *is universally connected if and only if* $\mathcal{C}^*$ *is a spanning tree on the set of facets of* P.

Proof. Let $(\mathbf{Q}, \alpha)$ be universally connected. Then the graph of facets with edges in $\mathcal{C}^*$ is connected. Assume by way of contradiction that it is not a tree. Then there is a set $\mathcal{D}$ of ridges of degree two, such that $\mathcal{D}^* \subsetneq \mathcal{C}^*$ and $\mathcal{C} \subsetneq \mathcal{D}$ with $Q(\mathbf{P}, \mathcal{D})$ connected and by Lemma 2.3.4 there exists an unfolding $(Q(\mathbf{P}, \mathcal{D}), \gamma)$ of **Q**. Here, $Q(\mathbf{P}, \mathcal{D})$ cannot be equivalent to **Q**, hence we contradict the fact that $(\mathbf{Q}, \alpha)$ is universally connected.

Next, let us assume that $\mathcal{C}^*$ is a spanning tree. Then **Q** is connected. Let $(\mathbf{R}, \beta)$ be a connected combinatorial unfolding of **Q**. Then $(\mathbf{R}, \alpha\beta)$ is a connected combinatorial unfolding of **P** and if $\mathcal{D} = C(\mathbf{R}, \alpha\beta)$ we see that $\mathcal{D}^*$ gives a connected graph on the set of facets while $\mathcal{C} \subseteq \mathcal{D}$, hence $\mathcal{D}^* \subseteq \mathcal{C}^*$. Therefore, $\mathcal{D}^* = \mathcal{C}^*$. Now $\mathbf{Q}' = Q(\mathbf{P}, \mathcal{C})$ is connected and it provides an unfolding of **R**, hence of **Q**, by Theorem 2.3.5. Since **Q** is highly connected, Theorem 2.3.5 shows that $\mathbf{Q}'$, **R** and **Q** are equivalent. ■

Remark. The example given in 2.1.3 allows to show easily that the condition "**Q** highly connected" cannot be dispensed with.

2.3.8. COROLLARY *Let* **P** *be a highly connected and connected combinatorial pre-polytope. Then the equivalence classes of universal connected highly connected combinatorial unfoldings of* **P** *are in one-to-one correspondence with the spanning trees of the facet graph of* **P**.

Proof. It suffices to observe that equivalent unfoldings have the same cut set. ■

This result is a generalisation at the combinatorial level of known facts about the metrical unfolding of familiar polyhedra.

2.3.9. Let **P** be a highly connected combinatorial pre-polytope. Then the automorphism group $G = Aut\mathbf{P}$ acts on the spanning trees of the facet graph and on the universal connected combinatorial unfoldings of **P** as well. The orbits of G allow to define a slightly different equivalence on the unfoldings, namely inclusion in the same G-orbit for two equivalence classes of unfoldings.

3. A Metrical Approach

3.1. Metrical Polytopes and Pre-Polytopes

3.1.1. Consider the n-dimensional euclidean space E^n and some combinatorial pre-polytope $\mathbf{P}$ over $I = \{0, \ldots, d-1\}$. We want to express that $\mathbf{P}$ "lives" in E^n. Therefore, distinct vertices of $\mathbf{P}$ must be distinct points of E^n. Refusing that distinct vertices be represented by the same point of E^n will raise various problems afterwards but we feel that it is a necessity. We shall refer to this situation by the expression "*vertex overlapping*."

Our next concern is for edges. If E is an edge of $\mathbf{P}$ incident to the points a, b we want to identify E with the closed segment $[a, b]$, which is legitimate provided no two edges have the same vertices, in particular whenever property (0) (see 2.1.6) holds in $\mathbf{P}$. From here on we shall carefully distinguish *the points of* E namely the elements of $[a, b]$ and *the vertices of* E namely a and b.

Observe that we do not extend the incidence relation to the points of E. A point other than a vertex will not be called incident to E.

We find it convenient to identify any element x of $\mathbf{P}$, of type $i \geq 2$ with the pointset of E^n which is the union of all edges and vertices incident with x.

We shall now formalize these matters.

3.1.2. Let $\mathbf{P}$ be a combinatorial pre-polytope (resp. polytope) with property (0). We call $\mathbf{P}$ a *euclidean pre-polytope* (resp. *polytope*) of the euclidean space E^n if $\mathbf{P}$ satisfies the following:

(i) every vertex of $\mathbf{P}$ is a point of E^n and distinct vertices are distinct points;

(ii) if E is an edge of $\mathbf{P}$ incident with the vertices a, b then E is the segment $[a, b]$;

(iii) if x is an element of $\mathbf{P}$ of type $i \geq 2$, then x is the union of all edges incident with x.

Observe that on this basis, a projection of a cube on the plane E^2 is a euclidean polytope provided the projection causes no vertex-overlapping.

A euclidean pre-polytope $\mathbf{P}$ of E^d having rank d is called *straight* if each facet of $\mathbf{P}$ spans a hyperplane of E^d and each element of type i spans an i-dimensional subspace of E^d. We call it $(d-1)$-*dimensional* if it is straight and all facets of $\mathbf{P}$ span the same hyperplane.

3.1.3. THEOREM *Every finite combinatorial pre-polytope* $\mathbf{P}$ *with property (0) is isomorphic to some euclidean pre-polytope.*

Proof. Let $\mathbf{P}$ have n vertices. Then we can identify the vertices of $\mathbf{P}$ with the basis vectors of $\mathbb{R}^n$ and everything becomes obvious. ■

Observe that this representation of $\mathbf{P}$ lives in an affine hyperplane of $\mathbb{R}^n$, hence in E^{n-1}.

Notice that convenient parallel projections allow to push $\mathbf{P}$ to lower dimensions, namely to E^d with $1 \leq d \leq n-1$.

If $\mathbf{P}$ is as in the Theorem and of rank d, $\mathbf{P}$ is not always isomorphic to some straight euclidean pre-polytope of E^d. It suffices to consider the vertices, edges and Petrie polygons of a tetrahedron ($d = 3$).

3.1.4. In order to define a metrical pre-polytope we start with a euclidean pre-polytope **P** of E^n and define a metric on it, in a somewhat unusual way, inspired by the fact that familiar unfoldings preserve distances of points incident to a given facet but not necessarily any other distance. The *metric structure* of **P** attaches to any points a, b in some facet of **P** their distance $d(a, b)$ in E^n.

We define a *metrical pre-polytope* as a euclidean pre-polytope together with its metric structure. It would of course be possible to formalize this structure without any embedding in E^n.

3.2. Metrical Unfolding of a Metrical Pre-Polytope

3.2.1. Let **P** be a metrical pre-polytope in E^n. A *metrical unfolding* of **P** is a combinatorial unfolding $(\mathbf{Q}, \alpha)$ of **P** such that:

(1) **Q** is a metrical pre-polytope in E^n;

(2) the restriction of α to each facet F of **Q** is an isometry namely $d(a, b) = d(\alpha(a), \alpha(b))$ for any two points a, b of F.

Metrical unfoldings $(\mathbf{Q}, \alpha)$ and $(\mathbf{Q}', \alpha')$ of **P** are called *isometric* if there is an isomorphism β of **Q** onto $\mathbf{Q}'$ preserving distances and such that $\alpha'\beta = \alpha$. Then β does not necessarily extend to an isometry of E^n mapping **Q** onto $\mathbf{Q}'$. It suffices to think of the case where **P** has exactly two facets with a common ridge. If β can be extended to an isometry of E^n we call $(\mathbf{Q}, \alpha)$ and $(\mathbf{Q}', \alpha')$ *totally isometric*.

3.2.2. We define a *universal metrical unfolding* $(\mathbf{Q}, \alpha)$ of **P** as a metrical unfolding which is isometric to each of its own metrical unfoldings. A straight pre-polytope of E^d has an obvious universal metrical unfolding in some hyperplane E^{d-1}, just as a cube can be put to (six) pieces on a plane.

3.3. Metrical Realizability of a Combinatorial Unfolding

3.3.1. Let **P** be a finite, straight, metrical pre-polytope in E^d and let $(\mathbf{Q}, \alpha)$ be a combinatorial unfolding of **P**. We call $(\mathbf{Q}, \alpha)$ *metrically realizable* if there is a metrical unfolding $(\mathbf{R}, \beta)$ of **P** in E^d such that $(\mathbf{Q}, \alpha)$ is equivalent to $(\mathbf{R}, \beta)$. Not every $(\mathbf{Q}, \alpha)$ is metrically realizable. We produce an easy counterexample.

3.3.2. *Example.* Let **P** be a cube in E^3 and let E be an edge of **P**. Consider the pre-polytope $\mathbf{Q} = Q(\mathbf{P}, E)$ constructed in 2.3.2 which amounts to cut E on **P**. Then **Q** is not metrically realizable.

3.3.3. *Problem 1.* The preceding observation inspires two different questions.

(Q_1). Given **P** as in 3.3.1, characterize those combinatorial unfoldings of **P** that are metrically realizable.

(Q_2). In 3.3.2, we are tempted to say that the best approximation to a metrical realization of **Q** is **P** itself. Can we slightly generalize metrical realizability in order to avoid any counterexamples?

3.3.4. Towards a characterization of metrical realizability. Let us discuss question (Q_1) of 3.3.3. It obviously suffices to deal with connected combinatorial unfoldings of **P**.

Thus, let **P** be a finite, straight, connected pre-polytope in E^d and let $(\mathbf{Q}, \alpha)$ be a connected combinatorial unfolding of **P**. As in 2.3.1, let $\mathcal{C} = C(\mathbf{Q}, \alpha)$ denote the cut set of **P** namely the set of ridges of degree 2 of **P** that are images under α of ridges of degree 1 in **Q**. Let $\mathcal{C}^* = C^*(\mathbf{Q}, \alpha)$ be the set of ridges of **P** of degree 2, not in $\mathcal{C}$. Let $\mathcal{F}$ be the set of facets of **P**. We shall have to deal with the graph $(\mathcal{F}, \mathcal{C} \cup \mathcal{C}^*)$ and with its subgraph $(\mathcal{F}, C^*)$.

Physical unfoldings tell us that we ought to find edges $E_1, \dots, E_n$ in $(\mathcal{F}, \mathcal{C}^*)$ such that for every $C \in \mathcal{C}$ with facets F_1, F_2 and $F_1 * C_1 * F_2$, there is an edge E_i contained in each path from F_1 to F_2 in $(\mathcal{F}, C^*)$.

This tells us that the E_i are *bridges* in $(\mathcal{F}, \mathcal{C}^*)$, a bridge of a graph being an edge whose removal increases the number of connected components of the graph (see [11], in particular Theorem 3.2). If B is a bridge of the connected graph $(\mathcal{F}, \mathcal{C}^*)$ then the removal of B leaves two connected components and we say that B *separates* two vertices if these belong to distinct components.

In the graph $(\mathcal{F}, \mathcal{C} \cup \mathcal{C}^*)$ we define a *bridging* as a set of bridges $B_1, \dots, B_k$ of $(\mathcal{F}, \mathcal{C}^*)$ such that for any $C \in \mathcal{C}$, some B_i separates the facets incident with C. We now get a partial answer to question (Q_1), that suffices to deal with the universally connected unfoldings of polytopes.

3.3.5. THEOREM *Let* **P** *be a finite, connected, straight, metrical pre-polytope in* E^d *and let* $(\mathbf{Q}, \alpha)$ *be a combinatorial unfolding of* **P**. *Assume that* $(\mathcal{F}, \mathcal{C} \cup \mathcal{C}^*)$ *has a bridging consisting of bridges* $B_1, \dots, B_k$ *in* $(\mathcal{F}, \mathcal{C}^*)$. *Then* $(\mathbf{Q}, \alpha)$ *is metrically realizable.*

Proof. Consider B_i, for each i. There are facets F, F' of **P** incident with B_i and connected components U, U' of $(\mathcal{F}, \mathcal{C}^*)$ after the removal of B_i.

In E^d we consider a rotation ρ_i fixing all points of the ridge B_i. We can choose ρ_i in such a way that $\rho_i(U') \neq U'$ and that $\rho_i(U')$ has no vertex in common with U. Then the union of all facets in U and in $\rho_i(U')$ is a metrical unfolding of **P** that separates facets F_1, F_2 on $C \in \mathcal{C}$, whenever one of F_1, F_2 is in U and the other in U'. Applying this for all i gives a metrical realization for **Q**. ■

COROLLARY *If* **P** *is a finite, straight, connected, metrical pre-polytope in* E^d *and if* $(\mathbf{Q}, \alpha)$ *is a combinatorial unfolding of* **P** *such that* $(\mathcal{F}, \mathcal{C}^*)$ *is a tree, then* $(\mathbf{Q}, \alpha)$ *is metrically realizable.*

Proof. In a tree, every edge is a bridge and so $(\mathcal{F}, \mathcal{C} \cup \mathcal{C}^*)$ has an obvious bridging. ■

Problem 2. We might expect a converse to the theorem, namely that $(\mathbf{Q}, \alpha)$ being metrically realizable forces a bridging in $(\mathcal{F}, \mathcal{C} \cup \mathcal{C}^*)$.

Problem 3. Let $\mathbf{P}$ be a combinatorial pre-polytope and assume that $(\mathbf{Q}, \alpha)$ is a combinatorial unfolding of $\mathbf{P}$. Assume furthermore that $\mathbf{Q}$ is a metrical pre-polytope in E^d. Does there follow that $\mathbf{P}$ is metrically realizable in E^d? If this was always the case it would prove the converse mentioned in Problem 2, thanks to the Corollary.

3.3.6. Weak metrical realizability. In order to answer question (Q_2) we start with $\mathbf{P}$ and $(\mathbf{Q}, \alpha)$ as in 3.3.1.

A *weak metrical realization* of $(\mathbf{Q}, \alpha)$ is a metrical unfolding $(\mathbf{R}, \beta)$ of $\mathbf{P}$ such that there exists a combinatorial unfolding $(\mathbf{Q}, \gamma)$ of $\mathbf{R}$ with $\beta\gamma = \alpha$ and such that $(\mathbf{R}, \beta)$ is *universal* for the preceding conditions namely, if $(\mathbf{R}', \beta')$ is as $(\mathbf{R}, \beta)$ above then $(\mathbf{R}, \beta)$ and $(\mathbf{R}', \beta')$ are isometric.

3.3.7. THEOREM *Let* $\mathbf{P}$ *be a finite, straight, metrical pre-polytope in* E^d *and let* $(\mathbf{Q}, \alpha)$ *be a combinatorial unfolding of* $\mathbf{P}$. *Then the unfolding* $(\mathbf{Q}, \alpha)$ *has a weak metrical realization in* E^d.

Proof. We may assume without loss of generality that $(\mathcal{F}, \mathcal{C}^*)$ is connected. Consider the nonempty family of sets $\mathcal{D}^*$ such that $\mathcal{C}^* \subseteq \mathcal{D}^* \subseteq \mathcal{C} \cup \mathcal{C}^*$ and such that $\mathbf{Q}' = Q(\mathbf{P}, C \cup \mathcal{C}^* \backslash \mathcal{D}^*)$ is metrically realizable. Let $\mathcal{D}^*$ be a minimal member of the family, with respect to inclusion. Then $\mathbf{Q}'$ provides a weak metrical realization of $(\mathbf{Q}, \alpha)$ thanks to Lemma 2.3.4. ■

Problem 4. Can we show in the above proof that a minimal $\mathcal{D}^*$ is unique? Also, if $\mathcal{C}^* \subseteq \mathcal{D}_1^* \subseteq \mathcal{C} \cup \mathcal{C}^*$, $\mathcal{C}^* \subseteq \mathcal{D}_2^* \subseteq \mathcal{C} \cup \mathcal{C}^*$ and if $Q(\mathbf{P}, \mathcal{C} \cup \mathcal{C}^* \backslash \mathcal{D}_1^*)$ and $Q(\mathbf{P}, \mathcal{C} \cup \mathcal{C}^* \backslash \mathcal{D}_2^*)$ are metrically realizable does there follow that $Q(\mathbf{P}, C \cup C^* \backslash \mathcal{D}_1^* \cup \mathcal{D}_2^*)$ is metrically realizable? This would imply that there is a unique minimal $\mathcal{D}^*$.

4. The $(d-1)$-Unfoldings of Straight Metrical Pre-Polytopes of E^d

In this section, $\mathbf{P}$ is always a finite, connected, straight metrical pre-polytope in E^d.

4.1. $(d-1)$-Realizable Unfoldings

Consider a metrical unfolding $(\mathbf{Q}, \alpha)$ of $\mathbf{P}$ in E^d. We are now interested in the existence of a $(d-1)$-dimensional $(\mathbf{Q}', \alpha')$ which is isometric to $(\mathbf{Q}, \alpha)$. If $(\mathbf{Q}', \alpha')$ exists we call it for short, a $(d-1)$-*unfolding* of $\mathbf{P}$ and we call $(\mathbf{Q}, \alpha)$ a $(d-1)$-*realizable unfolding*. A cube, one of whose faces is detached on three of its sides, provides an example where $(\mathbf{Q}, \alpha)$ is not $(d-1)$-realizable.

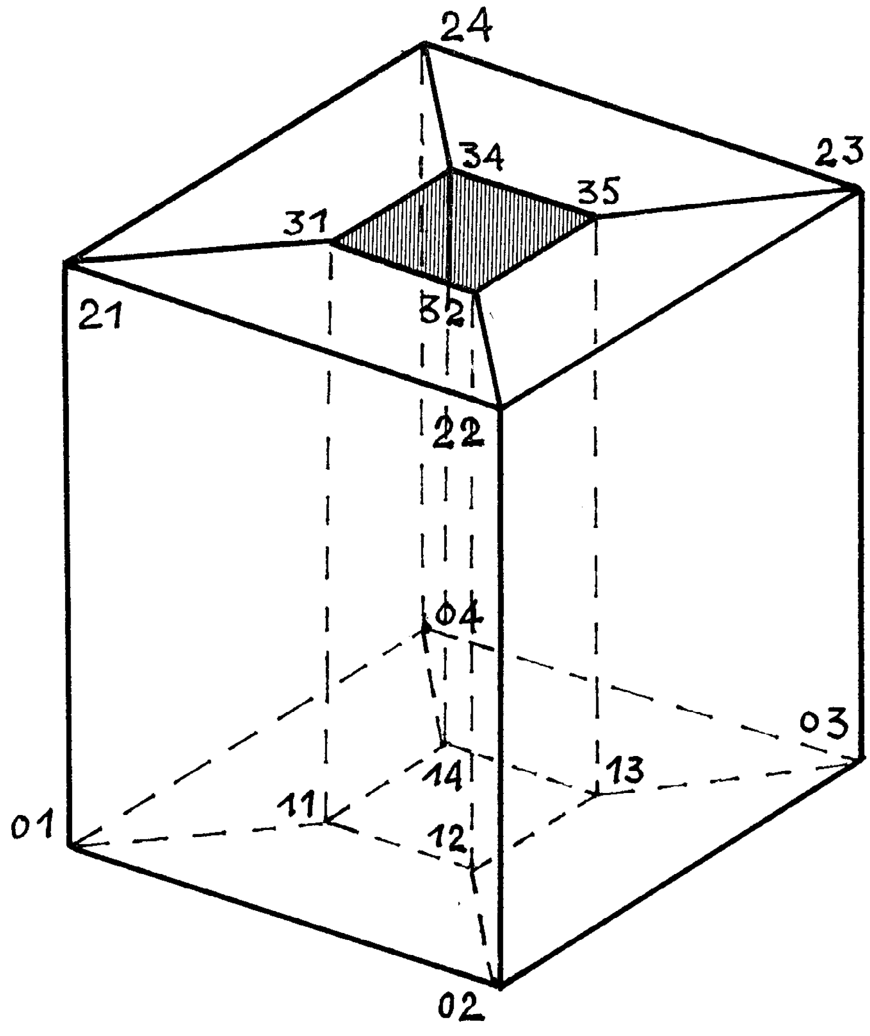

Figure 2.

4.2. $(d-1)$-*Realizability and Universal Connectedness*

In section 2, for instance 2.3.8, we got the idea that a fair combinatorial counterpart to physical $(d-1)$-dimensional nets may be the concept of universally connected combinatorial unfolding, especially because of its relationship with trees.

Therefore, we ask whether a $(d-1)$-realizable unfolding $(\mathbf{Q}, \alpha)$ of $\mathbf{P}$ is necessarily a universally connected combinatorial unfolding. We produce a counterexample.

Example. We display a picture of $\mathbf{P}$ in E^3 (figure 2). It has 16 vertices and all of its faces are convex quadrangles. The four upper (resp. lower) faces are contained in the same plane.

We display also a picture of a 2-unfolding of $\mathbf{P}$ (figure 3) which is not universally connected because it could still be cut along an edge like E.

Problem 5. Characterize $(d-1)$-realizable unfoldings in combinatorial terms.

This example also shows that a universally connected combinatorial unfolding of $\mathbf{P}$ need not be $(d-1)$-realizable. It suffices to remove the edge E. The four central quadrangles of the picture cannot be separated at E because all other configurations would lead to vertex-overlapping.

4.3. *Unambiguous Ridges*

The physical unfolding of some polyhedra suggests a role for rotations of some facets around one of their ridges. To be more specific, if F, F' are facets sharing the ridge C, we are inclined to rotate F' around C to bring it in the hyperplane $\langle F\rangle$ generated by F. Of course, this can be done in two ways but for the more regular polyhedra, there is only one way avoiding the overlapping of vertices. We formalize these ideas.

Let C be a ridge of degree 2 of $\mathbf{P}$, incident with the facets F and F'. Assume that F, F' span distinct hyperplanes $\langle F\rangle$, $\langle F'\rangle$. There are exactly two rotations ρ_1, ρ_2 of E^d (with a determinant equal to 1) fixing C pointwise and mapping $\langle F'\rangle$ onto $\langle F\rangle$. They map F' on polytopes $\rho_1(F')$ and $\rho_2(F')$ admitting C as a facet. We call C *unambiguous* if one and only one of $\rho_1(F')$ and $\rho_2(F')$ has no vertex in common with F, except those vertices in C. We call $\mathbf{P}$ *unambiguous* if each of its ridges of degree two is unambiguous. Typical examples are the convex polytopes all of whose facets are regular.

4.3.1. PROPOSITION *Assume that* $(\mathbf{Q}, \alpha)$ *is a connected metrical unfolding of* $\mathbf{P}$ *and that* $\mathbf{Q}$ *is unambiguous. Then* $(\mathbf{Q}, \alpha)$ *has at most one* $(d-1)$*-realization up to isometry.*

Proof. Straightforward. ∎

4.3.2. If $(\mathbf{Q}, \alpha)$ is as in the proposition and it is universally connected it does not necessarily have a $(d-1)$-realization. To get an example, take $\mathbf{P}$ as in the example described in 2.1.3. Replace the vertex y by two vertices y_1, y_2 where y_1 is incident to the upper facet containing y and y_2 to the lower one. This gives us a pre-polytope $\mathbf{Q}$ which is universally connected,

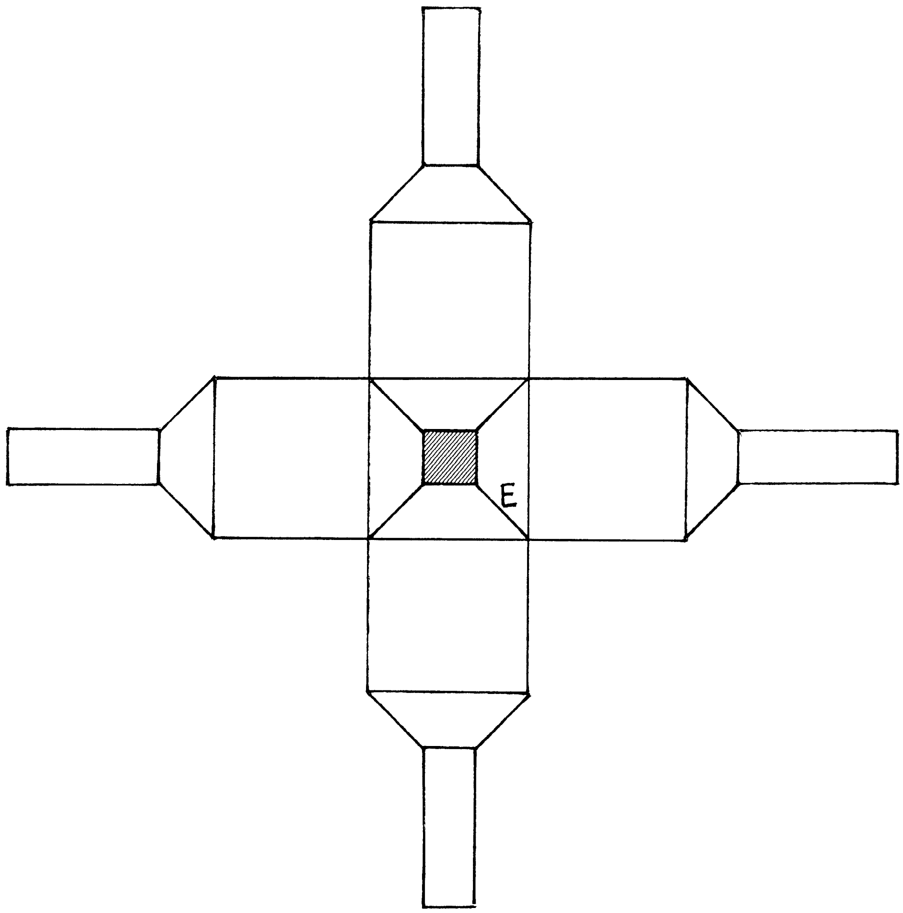

Figure 3.

unambiguous and not $(d-1)$-realizable because a realization would force y_1 and y_2 to coincide.

4.3.3. *Problem 6.* Characterize the universally connected unambiguous metrical pre-polytopes that are $(d-1)$-realizable.

4.3.4. We call **P** *totally realizable* if each of its universally connected metrical unfoldings is unambiguous and $(d-1)$-realizable.

4.3.5. THEOREM *Let* **P** *be totally realizable and assume that* Aut**P** *is a group of isometries leaving* **P** *invariant. Then the following numbers are equal:*

A) *the number of orbits of equivalence classes of universally connected combinatorial unfoldings of* **P** *under* Aut**P***;*

B) *the number of isometry classes of universally connected metrical unfoldings of* **P***;*

C) *the number of total isometry classes of* $(d-1)$*-universally connected metrical unfoldings of* **P***;*

D) *the number of orbits of spanning trees in the graph of facets and ridges of* **P** *under* Aut**P***.*

Proof. We get $A = D$ by Corollary 2.3.8 and 2.3.9. Equality $A = B$ follows from the Corollary in 3.3.5 and Theorem 2.3.7. Finally, $B = C$ because **P** is totally realizable. ■

Using pictures we can easily check that the regular tetrahedron, cube and octahedron are totally realizable.

Problem 7. Is every convex regular polytope and every deltahedron totally realizable? Are there any other totally realizable polytopes?

Let us mention here that for abstract polytopes with a flag-transitive group action namely regular polytopes, a quite detailed study of realizations can be found in [14] and [15].

4.3.6. Enumeration The number of unfoldings of the regular convex polytopes in dimension ≤ 4 is studied and determined by Buekenhout and Parker [9]. Earlier results on this theme are due to Jeger [13], Hippenmeyer [12], Tougne [18], [5] and Bouzette and Vandamme [4] (see also Reggini [16]). Let us mention here that the stellated regular $\{\frac{5}{2}, 3\}$ whose faces are pentagrams is combinatorially isomorphic to the regular dodecahedron and that it admits therefore the same number of unfoldings as the latter.

4.3.7. BOUZETTE'S THEOREM The result obtained by Bouzette [5] goes as follows.

Let P be a convex polyhedron in E^3 and let P^* be a dual of P. Each edge of P is identified with an edge of P^*. Assume that a set of edges of P say T is a spanning tree of the 1-sleleton of P. Then the set T^* of edges not in T is a spanning tree of the 1-skeleton of P^*. As a consequence there is a natural one-to-one correspondence between the unfoldings of P and those of P^*.

4.4. Non-Overlapping

If $(\mathbf{Q}, \alpha)$ is a $(d-1)$-realization of a metrical unfolding of **P**, we call **Q** *non-overlapping* if any two facets of **Q** whose residues are combinatorially disjoint are disjoint as sets of points in E^{d-1} and if their union has no knots other those of the facets themselves. A splendid example due to G. Valette [19] shows a 2-realization of a convex polyhedron (figure 4) which is not non-overlapping since $30^\circ < 36^\circ$.

We also produce a simpler example (figure 5) provided by a referee, having the combinatorial type of the cube and giving more insight. It uses an angle $\alpha < 36^\circ$ and a length $s > \frac{\cos\alpha - \cos 3\alpha}{\sin 3\alpha - \sin 2\alpha}$.

Problem 8. Does every convex polytope have some non-overlapping $(d-1)$-realization? Even the case $d = 3$ is still completely open.

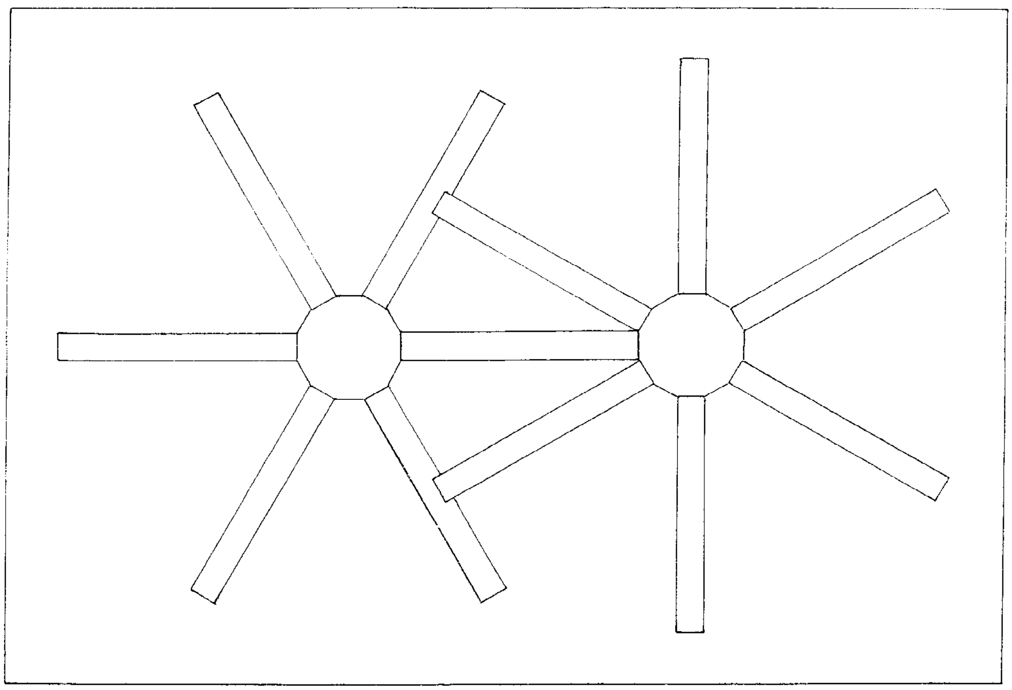

Figure 4.

4.5. Perfect Pre-Polytopes

We call **P** *perfect* if it is totally realizable and if each of its universally connected unfoldings has a non-overlapping $(d-1)$-realization.

Some experimentation shows that the regular tetrahedron, the cube and the regular octahedron are perfect.

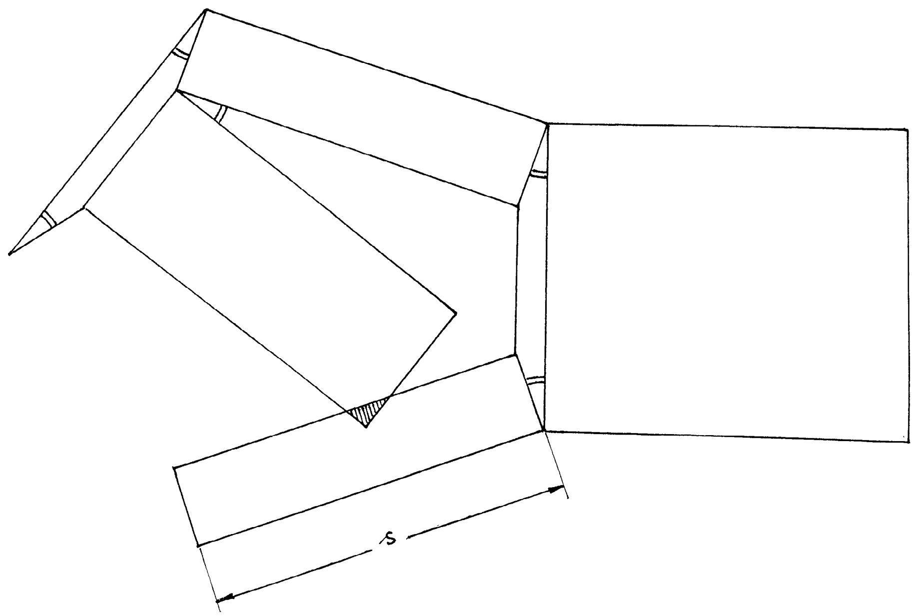

Figure 5.

Problem 10. Is every convex regular polytope perfect?

4.6. More Examples

1) We display a polyhedron in E^3 (figure 6) with 12 vertices and 12 faces which is topologically a Klein bottle. We also give a non-overlapping 2-unfolding for it.

2) We also display (figure 7) a well known toroidal polyhedron.

5. Appendix

The concept of unfolding is meaningful for geometries that are not polytopes. We shall illustrate this with an easy example based on the projective plane of order 2. The picture shows an unfolding and the corresponding morphism that identifies 1 and $1'$, 2 and $2'$. The interested reader may like to compare these morphisms with those discussed in [7].

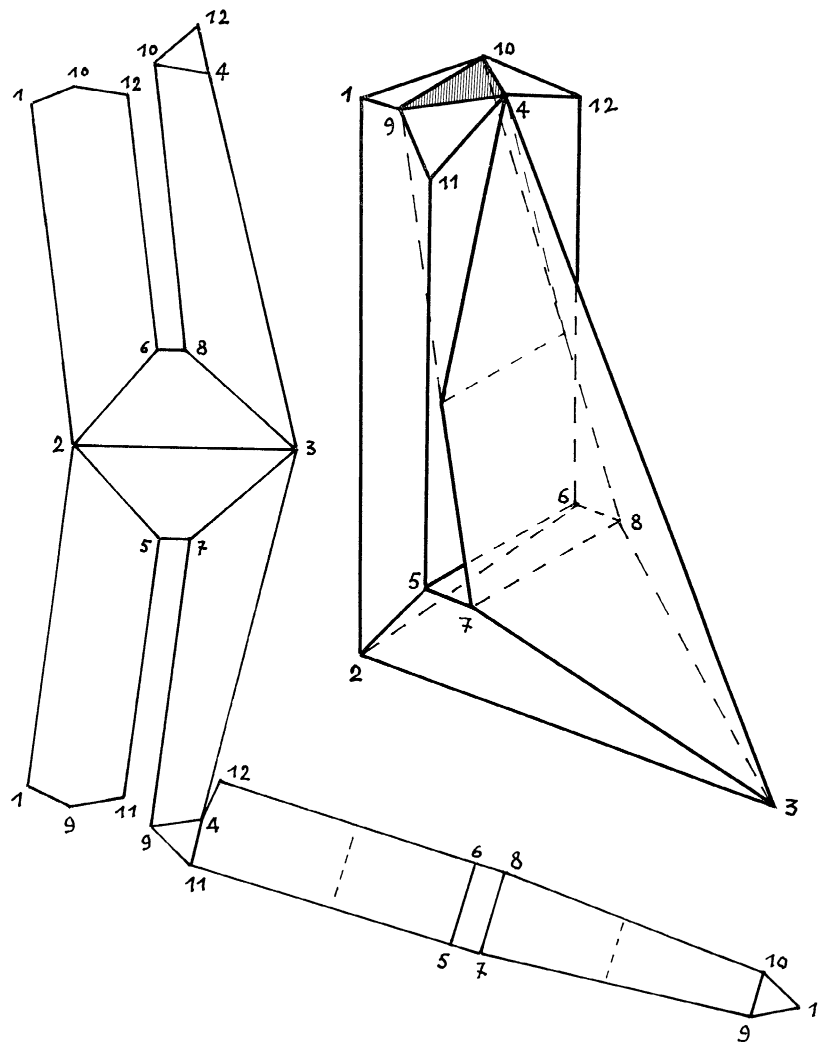

Figure 6.

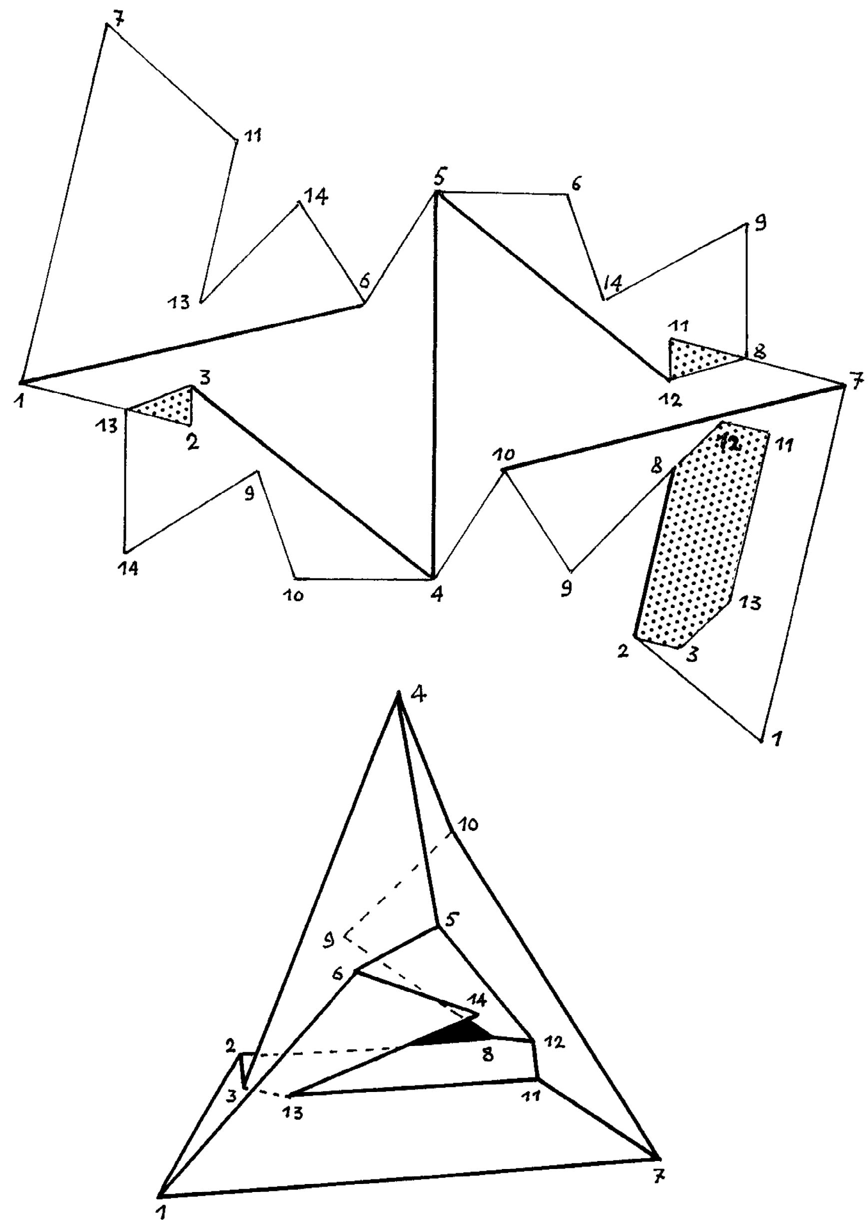
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Figure 7.

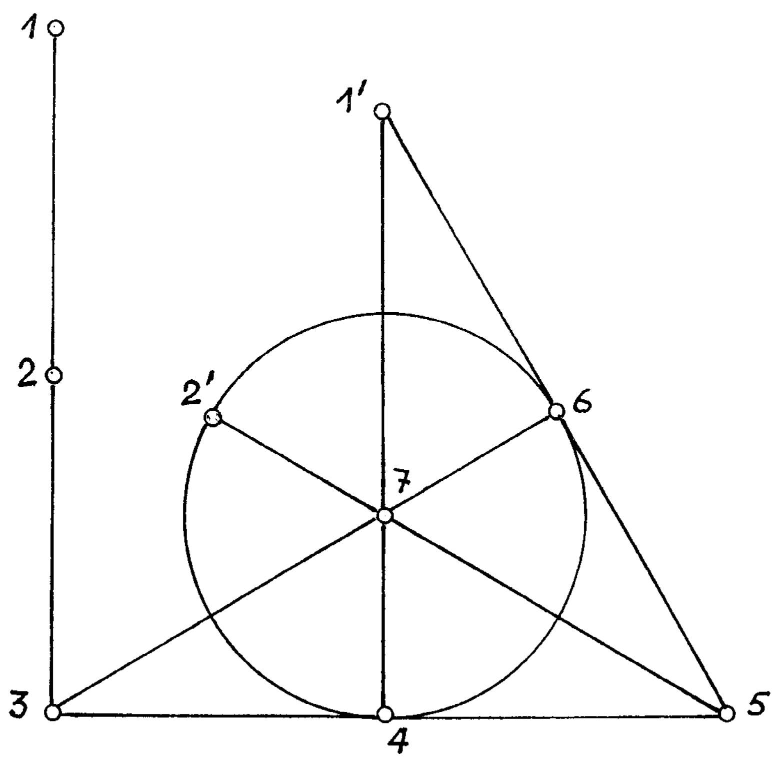

Figure 8.

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