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THEORY OF SEMIREGENERATIVE PHENOMENA

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Abstract: We develop a theory of semiregenerative phenomena. These may be viewed as a family of linked regenerative phenomena, for which Kingman ([6],[7]) developed a theory within the framework of quasi–Markov chains. We use a different approach and explore the correspondence between semiregenerative sets and the range of a Markov subordinator with a unit drift (or a Markov renewal process in the discrete time case). We use techniques based on results from Markov renewal theory.

Keywords: Semiregenerative phenomena and sets, linked regenerative phenomena, quasi–Markov chains, standard phenomena, stable states, lifetime, Markov renewal processes, Markov additive processes.

1. Introduction. Let the set T be either $[0,\infty)$ or $\{0,1,2,...\}$, E a countable set and $(\Omega,\mathcal{F},P)$ a probability space.

Definition 1. A semiregenerative phenomenon $Z = \{Z_{t\ell}, (t,\ell) \in T\times E\}$ on a probability space $(\Omega,\mathcal{F},P)$ is a stochastic process taking values 0 or 1 and such that for $(t_r,\ell_r) \in T\times E$ $(r \geq 1)$, with $0 = t_0 \leq t_1 \leq ... \leq t_r$, $j \in E$ we have

$$\begin{aligned} P\{Z_{t_1\ell_1} &= Z_{t_2\ell_2} = ... = Z_{t_r\ell_r} = 1 \mid Z_{0j} = 1\} \\ &= \prod_{i=1}^{r} P\{Z_{t_i-t_{i-1},\ell_i} = 1 \mid Z_{0,\ell_{i-1}} = 1\} \quad (\ell_0 = j). \end{aligned} \tag{1}$$

For each $\ell \in E$, denote $Z_\ell = \{Z_{t,\ell}, t \in T\}$. Since

$$\begin{aligned} P\{Z_{t_1\ell} &= Z_{t_2\ell} = ... = Z_{t_r\ell} = 1 \mid Z_{0j} = 1\} \\ &= P\{Z_{t_1\ell} = 1 \mid Z_{0j} = 1\} \prod_{i=2}^{r} P\{Z_{t_i-t_{i-1},\ell} = 1 \mid Z_{0\ell} = 1\}, \end{aligned} \tag{2}$$

Z_ℓ is a (possibly delayed) regenerative phenomenon in the sense of Kingman [7] in the continuous time case $T = [0,\infty)$, and a recurrent event (phenomenon) in the sense of Feller [5] in the discrete time case $T = \{0,1,2,...\}$. The family $Z' = \{Z_\ell, \ell \in E\}$ is a family of linked regenerative phenomena, for which a theory was developed by Kingman [6] in the case of finite E; later he reformulated the results in terms of quasi–Markov chains (Kingman [7]). We explain this concept below.

Example 1. Let $J = \{J_t, t \in T\}$ be a time–homogeneous Markov chain on the state space E and denote

$$Z_{t\ell} = 1_{\{J_t=\ell\}} \quad \text{for} \quad (t,\ell) \in T\times E. \tag{3}$$

The random variables $Z_{t\ell}$ satisfy the relationship (1), which is merely the Markov property. More generally, let C be a fixed subset of E and

$$Z_{t\ell} = 1_{\{J_t=\ell\}} \quad \text{for} \quad (t,\ell) \in T\times C. \tag{4}$$

These random variables also satisfy (1) and thus $Z = \{Z_{t\ell}, \ (t,\ell) \in T\times C\}$ is a semiregenerative phenomenon. In particular, suppose that C is a finite subset of E and define

$$K_t = J_t \ \text{ if } \ J_t \in C, \ \text{ and } \ = 0 \ \text{ if } \ J_t \notin C. \tag{5}$$

Then $\{K_t, t \in T\}$ is a quasi–Markov chain on the state space $C \cup \{0\}$. □

While the quasi–Markov chain does provide a good example of a semiregenerative phenomenon (especially in the case of finite E), it does not reveal the full features of these phenomena; in particular, it does not establish their connection with Markov additive processes. Thus, let

$$\zeta = \{(t,\ell) \in T\times E: Z_{t\ell} = 1\}. \tag{6}$$

We shall call ζ the semiregenerative set associated with Z. The main theme of this paper is the correspondence between the set ζ and the range of a Markov renewal process (in the discrete time case) and of a Markov subordinator with a unit drift (in the continuous time case). Kingman ([7], p. 123) has remarked that associated with a quasi–Markov chain there is a process of type F studied by Neveu [9]. The Markov subordinator we construct for our purpose is indeed a process of type F, but we concentrate on properties of the range of this process. For a detailed description of Markov additive processes see Cinlar ([2],[3]).

To complete Definition 1 we specify the initial distribution $\{a_j, j \in E\}$, where

$$P\{Z_{0j} = 1\} = a_j \tag{7}$$

with $a_j \geq 0$, $\Sigma a_j = 1$. As in the case of regenerative phenomena, it can be proved that the relation (1) determines all finite dimensional distributions of Z and that Z is strongly regenerative (that is, (1) holds for stopping times). We shall write P_j and E_j for the probability and the expectation conditional on the event $\{Z_{0j} = 1\}$.

In the discrete time case we call Z a semirecurrent phenomenon and denote

$$u_{jk}(n) = P\{Z_{nk} = 1 | Z_{0j} = 1\} \tag{8}$$

where $u_{jk}(0) = \delta_{jk}$. In the continuous time case let

$$P_{jk}(t) = P\{Z_{tk} = 1 | Z_{0j} = 1\} \tag{9}$$

where $P_{jk}(0) = \delta_{jk}$. The phenomenon is standard if

$$P_{jk}(t) \to \delta_{jk} \text{ as } t \to 0+. \tag{10}$$

In this case it is known that the limit

$$\lim_{t \to 0+} \frac{1 - P_{jj}(t)}{t} \quad (j \in E) \tag{11}$$

is known to exist (possibly infinite); if this limit is finite, then j is said to be stable.

In section 2 we consider semirecurrent phenomena and provide some examples. The main result is that the semirecurrent set ζ corresponds to the range of a Markov renewal process (MRP) and conversely, a semirecurrent set can only arise in this manner (Theorem 3). For details of the results from Markov renewal theory used in this paper see Cinlar ([4], Chapter 10). In section 3 we construct a Markov subordinator with a unit drift whose range turns out to be a semiregenerative set (Theorem 4). In the case where E is finite we prove that every semiregenerative set corresponds to the range of a Markov subordinator (Theorem 7). Our approach yields results analogous to Kingman's ([7], Chapter 5) for quasi–Markov chains. While our approach (based on Definition 1) is thus more rewarding in these respects, our techniques are simpler, being based on properties of Markov renewal processes. Bondesson [1] has investigated the distribution of occupation times of quasi–Markov processes. We shall not investigate this problem for semiregenerative phenomena.

In the literature there are extensive investigations of semiregenerative processes. These are processes imbedded in which there is an MRP (or equivalently, in view of Theorem 3, a semirecurrent phenomenon). We take the view that semiregenerative phenomena are important by themselves and therefore worthy of study.

Professor Erhan Cinlar has remarked to the author that several papers by him, J. Jacod, H. Kaspi and B. Maisonneuve on regenerative systems have a bearing on the theory developed in this paper. However, our approach is different from theirs and makes the results more accessible to applied probabilists. In particular, the theory developed in this paper provides a proper perspective to the work of Kulkarni and Prabhu [8] and Prabhu [10], to which no reference is made by the above authors; see Examples 3 and 5.

2. Semirecurrent phenomena. We write $\mathscr{N}_+ = \{0,1,2,...\}$ and $\mathscr{L} = \mathscr{N}_+ \times E$. Let ζ be the semirecurrent set defined by (6).

Definition 2. Let $T_0 = 0$ and for $r \geq 1$

$$T_r = \min\{n > T_{r-1}: (n,\ell) \in \zeta \text{ for some } \ell\}. \tag{12}$$

We shall call T_r the semirecurrence times of Z.

Let $J_r = \ell$ when $Z_{T_r,\ell} = 1$. Definition 1 shows that this ℓ is unique. We have the following.

Theorem 1. The process $\{(T_r,J_r), r \geq 0\}$ is a Markov renewal process (MRP) on the state space $\mathscr{L}$.

Proof: We have

$$\begin{aligned} P\{T_{r+1} = n, J_{r+1} = \ell | J_0,T_1,J_1,...,T_r,J_r\} \\ = P_{J_r}\{T_{r+1} - T_r = n - T_r, J_{r+1} = \ell\} \text{ a.s. } \square \end{aligned} \tag{13}$$

We shall denote by

$$q_{jk}^{(r)}(n) = P_j\{T_r = n, J_r = k\} \tag{14}$$

the semirecurrence time distribution of Z. We have $q_{jk}^{(0)}(0) = \delta_{jk}$ and $q_{jk}^{(0)}(n) = 0$ for $n \geq 1$. We shall write $q_{jk}^{(1)}(n) = q_{jk}(n)$. On account of the Markov renewal property we have for $r,s \geq 0$

$$\sum_{m=0}^{n} \sum_{\ell \in E} q_{j\ell}^{(r)}(m) q_{\ell k}^{(s)}(n-m) = q_{jk}^{(r+s)}(n). \tag{15}$$

<u>Theorem 2</u>. The probabilities $\{u_{jk}(n),\ n \in \mathscr{N}_+,\ j,k \in E\}$ defined by (8) form the unique solution of the equations

$$x_{jk}(n) = q_{jk}(n) + \sum_{m=1}^{n-1} \sum_{\ell \in E} q_{j\ell}(m) x_{\ell k}(n-m) \tag{16}$$

with $0 \leq x_{jk}(n) \leq 1$. This solution is given by

$$u_{jk}(n) = \sum_{r=1}^{\infty} q_{jk}^{(r)}(n) \tag{17}$$

(the summation being effectively finite).

Proof: From (6) and (12) it follows that for $n \geq 1$,

$$\begin{aligned} u_{jk}(n) &= P_j\{(T_r, J_r) = (n,k) \text{ for some } r \geq 1\} \\ &= \sum_{r=1}^{n} q_{jk}^{(r)}(n), \end{aligned} \tag{18}$$

the sum going only upto $r = n$ since $T_r \geq r$ a.s. We have

$$\begin{aligned} u_{jk}(n) &= q_{jk}^{(1)}(n) + \sum_{t=1}^{\infty} q_{jk}^{(t+1)}(n) \\ &= q_{jk}(n) + \sum_{t=1}^{\infty} \sum_{m=1}^{n-1} \sum_{\ell \in E} q_{j\ell}(m) q_{\ell k}^{(t)}(n-m) \end{aligned}$$

using (15). This gives

$$u_{jk}(n) = q_{jk}(n) + \sum_{m=1}^{n-1} \sum_{\ell \in E} q_{j\ell}(m) u_{\ell k}(n-m). \tag{19}$$

Thus the $u_{jk}(n)$ satisfy the equations (16). To prove uniqueness we find that if $\{x_{jk}(n)\}$ is a solution of (16), then

$$x_{jk}(n) = q_{jk}(n) + \sum_{m=1}^{n-1} \sum_{\ell \in E} q_{j\ell}(m)[q_{\ell k}(n-m) + \sum_{m'=1}^{n-m-1} \sum_{\ell' \in E} q_{\ell\ell'}(n-m-m') x_{\ell' k}(m')]$$

$$= q_{jk}^{(1)}(n) + q_{jk}^{(2)}(n) + \sum_{m'=1}^{n-1} \sum_{\ell' \in E} q_{j\ell'}^{(2)}(n-m') x_{\ell' k}(m').$$

By induction we find that

$$x_{jk}(n) = \sum_{t=1}^{r} q_{jk}^{(t)}(n) + \sum_{m=1}^{n-1} \sum_{\ell \in E} q_{j\ell}^{(r)}(m) x_{\ell k}(n-m).$$

Since $q_{jk}^{(r)}(n) = 0$ for $r > n$ we have

$$x_{jk}(n) = \sum_{t=1}^{n} q_{jk}^{(t)}(n) = u_{jk}(n),$$

which proves the uniqueness of the solution (17). □

We have thus seen that a semirecurrent phenomenon gives rise to an MRP with the sojourn times $T_r - T_{r-1}$ $(r \geq 1)$ concentrated on the set $\{1,2,\ldots\}$. The following theorem shows that this is the only way that a semi–recurrent phenomenon can occur.

Theorem 3. (i) Let $\{(T_r,J_r), r \geq 0\}$ be an MRP on the state space $\mathscr{L}$, with the sojourn time distribution concentrated on $\{1,2,\ldots\}$. Denote by

$$\mathscr{R} = \{(n,\ell) \in \mathscr{L}\colon (T_r,J_r) = (n,\ell) \text{ for some } r \geq 0\} \tag{20}$$

the range of this process and $Z'_{n\ell} = 1_{\{(n,\ell)\in\mathscr{R}\}}$. Then $Z' = \{Z'_{n\ell}, (n,\ell) \in \mathscr{L}\}$ is a semirecurrent phenomenon.

(ii) Conversely, any semirecurrent phenomenon Z is equivalent to a phenomenon Z' generated in the above manner in the sense that Z and Z' have the same $\{u_{jk}(n)\}$ sequence.

Proof: (i) We have $P\{Z'_{0j} = 1\} = P\{J_0 = j\}$. Let $v_{jk}(0) = \delta_{jk}$ and for $n \geq 1$

$$\begin{aligned} v_{jk}(n) &= P_j\{Z'_{nk} = 1\} = P_j\{(n,k) \in \mathscr{R}\} \\ &= \sum_{r=1}^{\infty} P_j\{T_r = n, J_r = k\}. \end{aligned} \tag{21}$$

For $0 = n_0 \leq n_1 \leq n_2 \leq \ldots \leq n_r$ $(r \geq 1)$ we have

$$P\{Z'_{n_1\ell_1} = Z'_{n_2\ell_2} = \dots = Z'_{n_r\ell_r} = 1 \mid Z'_{0j} = 1\}$$

$$= P_j \bigcap_{i=1}^{r} \bigcup_{t_i \geq t_{i-1}} \{(T_{t_i}, J_{t_i}) = (n_i, \ell_i)\}$$

$$= P_j \bigcup_{0=t_0 \leq t_1 \leq \dots \leq t_r} \bigcap_{i=1}^{r} \{(T_{t_i}, J_{t_i}) = (n_i, \ell_i)\}$$

$$= \sum_{0=t_0 \leq t_1 \leq \dots \leq t_r} \prod_{i=1}^{r} P\{(T_{t_i}, J_{t_i}) = (n_i, \ell_i) \mid (T_{t_{i-1}}, J_{t_{i-1}}) = (n_{i-1}, \ell_{i-1})\}$$

$$= \Sigma \prod_{i=1}^{r} P_{\ell_{i-1}}\{(T_{t_i - t_{i-1}}, J_{t_i - t_{i-1}}) = (n_i - n_{i-1}, \ell_i)\} \quad (\ell_0 = j)$$

$$= \sum_{t_1 \geq 0} P_j\{(T_{t_1}, J_{t_1}) = (n_1, \ell_1)\} \sum_{t_2 \geq t_1} P_{\ell_1}\{(T_{t_2 - t_1}, J_{t_2 - t_1}) = (n_2 - n_1, \ell_2)\}$$

$$\dots \sum_{t_r \geq t_{r-1}} P_{\ell_{r-1}}\{(T_{t_r - t_{r-1}}, J_{t_r - t_{r-1}}) = (n_r - n_{r-1}, \ell_r)\}$$

$$= v_{j\ell_1}(n_1) v_{\ell_1\ell_2}(n_2 - n_1) \dots v_{\ell_{r-1}\ell_r}(n_r - n_{r-1}).$$

This shows that Z' is a semirecurrent phenomenon. From

$$P_j\{T_{r+1} = n, J_{r+1} = k\} = \sum_{m=1}^{n-1} \sum_{\ell \in E} P_j\{T_1 = m, J_1 = \ell\} \cdot P_\ell\{T_r = n-m, J_r = k\}$$

we obtain the relation

$$v_{jk}(n) = q_{jk}(n) + \sum_{m=1}^{n-1} \sum_{\ell \in E} q_{j\ell}(m) v_{\ell k}(n-m) \quad (n \geq 1) \tag{22}$$

where $q_{jk}(n) = P_j\{T_1 = n, J_1 = k\}$.

(ii) Conversely, let Z be a semirecurrent phenomenon with the associated sequence $\{u_{jk}(n), (n,k) \in \mathscr{L}\}$. Let $\{q_{jk}(n)\}$ be the associated semirecurrence time distribution, so that by Theorem 2, equation (19) holds. Let Z' be the semirecurrent phenomenon constructed as in (i) from the sojourn time distribution $\{q_{jk}(n)\}$. Then $\{v_{jk}(n)\}$ satisfies (22). Because of the uniqueness of the solution of (16) we find that $v_{jk}(n) = u_{jk}(n)$ as required. □

The following are some illustrative examples.

Example 1 (continuation). For a quasi–Markov chain in discrete time the semirecurrence times are the hitting times of the set C. Thus if $K_0 = j \in C$, the process spends one unit of time in j and T_1-1 units outside C before returning to C. This remark is helpful in understanding the results of section 3 (continuous time). If $C = E$, then $T_r = r$ a.s. for all $r \geq 0$. □

Example 2. Let $\{(K_n,J_n), n \in \mathscr{N}_+\}$ be a time–homogeneous Markov chain on the state space $S \times E$ (with S arbitrary). Let $a \in S$ (fixed) and assume that $K_0 = a$ a.s. Define

$$Z_{n\ell} = 1_{\{K_n=a,\ J_n=\ell\}} \quad \text{for } (n,\ell) \in \mathscr{L}$$

On account of the Markov property, $Z = \{Z_{n\ell}\}$ is a semirecurrent phenomenon. The semirecurrence times are the successive hitting times of the line $K_n = a$. □

Example 3. Let $\{X_n, n \in \mathscr{N}_+\}$ be a Markov chain on the state space E, and

$$M_n = \max(X_0,X_1,X_2,\ldots,X_n),\ n \in \mathscr{N}$$

be its maximum functional. Also define

$$Z_{n\ell} = 1_{\{X_n=M_n=\ell\}} \quad \text{for } (n,\ell) \in \mathscr{L}$$

We have $P\{Z_{0j}=1\} = P\{X_0=j\}$, and for $0=n_0 \le n_1 \le \dots \le n_r$ $(r \ge 1)$, using the Markov property

$$P\{Z_{n_1\ell_1} = Z_{n_2\ell_2} = \dots = Z_{n_r\ell_r} = 1 \,|\, Z_{0j}=1\}$$

$$= P_j \bigcap_{i=1}^{r} \{X_{n_{i-1}+t} \le X_{n_i} \ (0 \le t \le n_i - n_{i-1}),\ X_{n_i} = \ell_i\}$$

$$= \prod_{i=1}^{r} P\{X_{n_{i-1}+t} \le X_{n_i} \ (0 \le t \le n_i - n_{i-1}),\ X_{n_i} = \ell_i \,|\, X_{n_{i-1}} = \ell_{i-1}\}$$

$$= \prod_{i=1}^{r} P\{X_t \le X_{n_i-n_{i-1}} \ (0 \le t \le n_i - n_{i-1}),\ X_{n_i-n_{i-1}} = \ell_i \,|\, X_0 = \ell_{i-1}\} \quad (\ell_0 = j)$$

$$= \prod_{i=1}^{r} P\{Z_{n_i-n_{i-1},\ell_i} = 1 \,|\, Z_{0\ell_{i-1}} = 1\} = \prod_{i=1}^{r} u_{\ell_{i-1},\ell_i}(n_i - n_{i-1}),$$

where

$$u_{jk}(n) = P\{X_t \le X_n \ (0 \le t \le n),\ X_n = k \,|\, X_0 = j\}.$$

This shows that $Z = \{Z_{n\ell}\}$ is a semirecurrent phenomenon. The semirecurrence times of this phenomenon are called ascending ladder epochs by Kulkarni and Prabhu [8], who study the fluctuation theory of the Markov chain. □

Example 4. Let $\{(S_n, J_n), n \in \mathcal{N}_+\}$ be a Markov random walk (discrete time version of Markov additive process) on the state space $R \times E$, and

$$M_n = \max(0, S_1, S_2, \dots, S_n),\ n \in \mathcal{N}_+$$

its maximum functional. Denote

$$Z_{n\ell} = 1_{\{M_n - S_n = 0,\ J_n = \ell\}} \quad \text{for } (n,\ell) \in \mathcal{L}$$

We have $P\{Z_{0j} = 1\} = P\{J_0 = j\}$, and for $0 = n_0 \le n_1 \le \dots \le n_r$ $(r \ge 1)$

$$P\{Z_{n_1\ell_1} = Z_{n_2\ell_2} = \dots = Z_{n_r\ell_r} = 1 \mid Z_{0j} = 1\}$$

$$= P_j \bigcap_{i=1}^{r} \{S_{n_{i-1}+t} \le S_{n_i} \quad (0 \le t \le n_i - n_{i-1}),\ J_{n_i} = \ell_i\}$$

$$= P_j \bigcap_{i=1}^{r} \{S_{n_{i-1}+t} - S_{n_{i-1}} \le S_{n_i} - S_{n_{i-1}} \quad (0 \le t \le n_i - n_{i-1}),\ J_{n_i} = \ell_i\}$$

$$= \prod_{i=1}^{r} P\{S_t \le S_{n_i - n_{i-1}} \quad (0 \le t \le n_i - n_{i-1}),\ J_{n_i - n_{i-1}} = \ell_i \mid J_0 = \ell_{i-1}\}$$

$$= \prod_{i=1}^{r} P\{Z_{n_i - n_{i-1}, \ell_i} = 1 \mid Z_{0,\ell_{i-1}} = 1\} \quad (\ell_0 = j),$$

$$= \prod_{i=1}^{r} u_{\ell_{i-1},\ell_i}(n_i - n_{i-1})$$

where

$$u_{jk}(n) = P\{S_t \le S_n \quad (0 \le t \le n),\ J_n = k \mid J_0 = j\}.$$

Thus $Z = \{Z_{n\ell}\}$ is a semirecurrent phenomenon. We obtain a second such phenomenon by considering the minimum functional. These two phenomena determine the fluctuation behavior of the random walk. □

<u>3. A continuous time semiregenerative phenomenon</u>. As before, let E be a countable set and $\mathbb{R}_+ = [0,\infty)$. Let $J = \{J(\tau),\ \tau \ge 0\}$ be a time–homogeneous Markov process on the state space E, all of whose states are stable. Let $T_0 = 0$ and T_n $(n \ge 1)$ the epochs of successive jumps in J; for convenience we denote $J_n = J(T_n)$ $(n \ge 0)$. We define a sequence of continuous time processes $\{X_n^{(1)},\ n \ge 1\}$ and a sequence of random variables $\{X_n^{(2)},\ n \ge 1\}$ as follows.

(i) On $\{T_n \leq \tau \leq T_{n+1}, J_n = j\}$, $X_{n+1}^{(1)}(\tau)$ is a subordinator with a unit drift and Lévy measure $\mu_{jj}^{(0)}$.

(ii) Given $X_m^{(1)}, X_m^{(2)}, J_m$ $(m \geq 1)$, J_0, the increment $X_{n+1}^{(1)}(\tau) - X_{n+1}^{(1)}(T_n)$ and the pair of random variables $(X_{n+1}^{(2)}, J_{n+1})$ depend only on J_n.

(iii) Given J_n, $X_{n+1}^{(1)}(\tau) - X_{n+1}^{(1)}(T_n)$ $(T_n \leq \tau \leq T_{n+1})$ and $(X_{n+1}^{(2)}, J_{n+1})$ are conditionally independent, with respective distribution measures

$$H_j\{s;A\} \text{ and } \lambda_{jk}F_{jk}\{A\} \quad (j \neq k) \tag{23}$$

for any Borel subset A of $\mathbb{R}_+$. Here the F_{jk} are concentrated on $[0,\infty)$, while H_j is concentrated on $[0,\infty]$; λ_{jk} $(j \neq k)$ are the transition rates of the process. We denote $\lambda_{jj} = \Sigma_{k \neq j} \lambda_{jk}$ $(0 < \lambda_{jj} < \infty)$. Let

$$S_0 = 0, \; S_n = \sum_{i=1}^{n} [X_i^{(1)}(T_i) - X_i^{(1)}(T_{i-1}) + X_i^{(2)}] \quad (n \geq 1). \tag{24}$$

From the above conditions it follows that $\{(S_n, J_n), n \geq 0\}$ is an MRP on the state space $\mathbb{R}_+ \times E$, whose transition distribution measure

$$P\{S_{n+1} \in A, J_{n+1} = k \mid S_n = s, J_n = j\} = Q_{jk}\{A-s\}$$

is given by

$$Q_{jk}\{A\} = \int_0^\infty e^{-\lambda_{jj}s} \lambda_{jk} H_j\{s;dx\} F_{jk}\{A-x\} ds \quad (k \neq j). \tag{25}$$

We denote by $U_{jk}\{A\}$ the Markov renewal measure associated with this process, so that

$$U_{jk}\{A\} = \sum_{n=0}^{\infty} P_j\{S_n \in A, J_n = k\}. \tag{26}$$

Let

$$L' = \sup_{n\geq 0} T_n, \quad L = \sup_{n\geq 0} S_n \quad (L' \leq L \leq \infty). \tag{27}$$

We construct a process $(Y,J) = \{(Y(\tau), J(\tau)), \tau \geq 0\}$ as follows:

$$\begin{aligned} \{Y(\tau),J(\tau)\} &= \{(S_n + X^{(1)}_{n+1}(\tau) - X^{(1)}_{n+1}(T_n), J_n)\} \text{ for } T_n \leq \tau < T_{n+1} \\ &= (L,\Delta) \text{ for } \tau \geq L', \end{aligned} \tag{28}$$

where Δ is a point of compactification of the set E. Denoting

$$N(\tau) = \max\{T_n \leq \tau\}$$

we can write

$$\{Y(\tau),J(\tau)\} = \{S_{N(\tau)} + X^{(1)}_{N(\tau)+1}(\tau) - X^{(1)}_{N(\tau)+1}(T_{N(\tau)}), J_{N(\tau)}\} \text{ on } \{\tau < L'\}. \tag{29}$$

In view of the assumptions (i)–(iii) above we find that

$$\begin{aligned} &P\{Y(\tau+\tau') \in A, J(\tau+\tau') = k \,|\, (Y(s),J(s)), (0 \leq s \leq \tau)\} \\ &\quad = P\{S_{N(\tau+\tau')} - S_{N(\tau)} + X^{(1)}_{N(\tau+\tau')+1}(\tau+\tau') - X^{(1)}_{N(\tau+\tau')+1}(T_{N(\tau+\tau')}) \\ &\qquad - X^{(1)}_{N(\tau)+1}(\tau) + X^{(1)}_{N(\tau)+1}(T_{N(\tau)}) \in A - Y(\tau), J(\tau+\tau') = k \,|\, Y(\tau),J(\tau)\} \\ &\quad = P\{S_{N(\tau')} + X^{(1)}_{N(\tau')+1}(\tau') - X^{(1)}_{N(\tau')+1}(T_{N(\tau')}) \in A - Y(0), J(\tau') = k \,|\, J(0)\}. \end{aligned}$$

This shows that (Y,J) is a Markov additive process (MAP) on the state space $\mathbb{R}_+ \times E$. Let $f(t,j)$ be a bounded function on $\mathbb{R}_+ \times E$ such that for each fixed j, f is continuous and has a bounded continuous derivative $\partial f/\partial t$. Then the infinitesimal geneator $\mathscr{A}$ of (Y,J) is found to be

$$(\mathscr{A}f)(t,j) = \frac{\partial f}{\partial t} + \int_{0-}^{\infty+} [f(t+v,j) - f(t,j)]\mu_{jj}^{(0)}\{dv\}$$
$$+ \sum_{k\neq j} \int_{0-}^{\infty} [f(t+v,k) - f(t,j)]\lambda_{jk}F_{jk}\{dv\}. \tag{30}$$

This shows that the jumps in Y are those in the additive component plus the Markov–modulated jumps with the distribution measures $\lambda_{jk}F_{jk}$ $(k \neq j)$.

Let us denote by

$$\mathscr{R} = \{(t,\ell) \in \mathbb{R}_+ \times E\colon (Y(\tau),J(\tau)) = (t,\ell) \text{ for some } \tau \geq 0\} \tag{31}$$

the range of this process, and

$$Z_{t\ell} = 1_{\{(t,\ell)\in\mathscr{R}\}},\ P_{jk}(t) = P\{Z_{tk} = 1 \mid Z_{0j} = 1\}. \tag{32}$$

An inspection of the sample path of the process shows that we can write the range $\mathscr{R}$ as

$$\mathscr{R} = \bigcup_{n=0}^{\infty} [\mathscr{R}_{X_{n+1}^{(1)}} \times \{J_n\}], \tag{33}$$

where $\mathscr{R}_{X_{n+1}^{(1)}}$ is the range of the subordinator $X_{n+1}^{(1)}$. Let

$$p_j(t) = P\{t \in \mathscr{R}_{X_{n+1}^{(1)}} \mid J_n = j\}. \tag{34}$$

We know that each $\mathscr{R}_{X_{n+1}^{(1)}}$ is a regenerative set (Kingman [7], section 4.2) with its p–function given by (34). We have

$$\int_A p_j(t)dt = E \int_0^{T_1} H_j\{s;A\}ds = \int_0^{\infty} e^{-\lambda_{jj}s} H_j\{s;A\}ds \tag{35}$$

for any Borel subset A of $\mathbb{R}_+$. We have then the following.

Theorem 4. The family $Z = \{Z_{t\ell}, (t,\ell) \in \mathbb{R}_+ \times E\}$ is a standard semiregenerative phenomenon, for which

$$P_{jk}(t) \geq \int_{0-}^{t} U_{jk}\{ds\}p_k(t-s). \tag{36}$$

The equality in (36) holds iff $L = \infty$ a.s.

Proof: We have $P\{Z_{0j} = 1\} = P\{J_0 = j\}$. The semiregenerative property (1) can be established exactly as in the proof of Theorem 3(i) using the Markov additive property of (Y,J). We have

$$\begin{aligned} P_{jk}(t) &= P_j\{(t,k) \in \mathscr{R}_{X_1^{(1)}} \times \{J_0\}\} + \sum_{\ell \in E} \int_{0-}^{t} P_j\{S_1 \in ds, J_1 = \ell\} \\ &\qquad \cdot P\{(t,k) \in \mathscr{R} | S_1 = s, J_1 = \ell\} \\ &= p_j(t)\, \delta_{jk} + \sum_{\ell \in E} \int_{0-}^{t} Q_{j\ell}\{ds\}P\{(t-s,k) \in \mathscr{R} | J_0 = \ell\} \\ &= p_j(t)\, \delta_{jk} + \sum_{\ell \in E} \int_{0-}^{t} Q_{j\ell}\{ds\}P_{\ell k}(t-s). \end{aligned}$$

Thus $P_{jk}(t)$ satisfies the integral equation

$$G_{jk}(t) = h_{jk}(t) + \sum_{\ell \in E} \int_{0-}^{t} Q_{j\ell}\{ds\}G_{\ell k}(t-s), \tag{37}$$

with $h_{jk}(t) = p_j(t)\, \delta_{jk}$. We seek a solution $G_{jk}(t)$ such that for fixed $j,k \in E$, G_{jk} is bounded over finite intervals, and for each $t \in \mathbb{R}_+$, G_{jk} is bounded. The inequality (36) follows from the fact that the minimal solution of (37) is given by

$$\sum_{\ell \in E} \int_{0-}^{t} U_{j\ell}\{ds\}h_{\ell k}(t-s) = \int_{0-}^{t} U_{jk}\{ds\}p_k(t-s).$$

This solution is unique iff $L = \infty$ (see Cinlar [4], Chapter 10, Section 3). From (36) we find, in particular, that

$$P_{jj}(t) \geq p_j(t) \to 1 \quad \text{as} \quad t \to 0+ \quad (j \in E) \tag{38}$$

This shows that Z is a standard phenomenon. □

From the definition of the (Y,J) process it is clear that the lifetime of the phenomenon Z is given by

$$\sup\{t: Z_{t\ell} = 1 \text{ for some } \ell \in E\} = L \leq \infty \text{ a.s.} \tag{39}$$

and the total duration (occupation time) of Z by

$$\sum_{\ell \in E} \int_0^L Z_{t\ell} dt \geq L' \quad \text{a.s.} \tag{40}$$

In the discrete time case, the representation (18) shows that the Green measure of the MRP $\{(T_r,J_r)\}$ has weight $u_{jk}(n)$ at its atom n. In the present situation it can be proved that

$$\int_0^\infty P\{(Y(\tau),J(\tau)) \in (A \times \{k\}) \mid J(0) = j\} d\tau = \int_A P_{jk}(t) dt, \tag{41}$$

so that $P_{jk}(t)$ is the density of the Green measure of the process (Y,J). We shall not prove (41) here; more useful to us is the result (36), which relates $P_{jk}(t)$ to an MRP. The following results follow from (36) in the case $L = \infty$. We have then

$$P_{jk}(t) = \int_{0-}^t U_{jk}\{ds\} p_k(t-s). \tag{42}$$

We first note the following. Let

$$N = \min\{n \geq 1: J_n = k\}, \quad G_{jk}\{A\} = P_j\{T_N + S_N \in A\}. \tag{43}$$

Using the relation

$$U_{jk}\{A\} = \int_0^t G_{jk}\{ds\}U_{kk}\{A-s\} \quad (j \neq k) \tag{44}$$

in (42) we can write

$$P_{jk}(t) = \int_0^t G_{jk}\{ds\}P_{kk}(t-s) \quad (j \neq k); \tag{45}$$

(cf. Kingman ([7], Theorem 5.3). The regularity properties of $P_{jk}(t)$ and its asymptotic behaviour follow from (42)–(45).

Theorem 4 shows that the range of the MAP constructed above is a semiregenerative set. The converse statement is that every semi–regenerative set corresponds to the range of a Markov subordinator with unit drift. We are able to prove this only in the case of the Markov chain J having a finite state space (Theorem 7 below). However, the following examples show that the converse is true in two important cases.

Example 1 (continuation). Let $\{J_t\}$ be the Markov chain of this example, with $T = [0,\infty)$. Assume that all of its states are stable and use the notation of this section. Let $Z_{t\ell}$ be as defined by (3). We have already observed that $Z = \{Z_{t\ell}\}$ is a semiregenerative phenomenon. An inspection of the sample path of J shows that the semiregenerative set of Z is given by

$$\zeta = \bigcup_{n=0}^{\infty} \{[T_n,T_{n+1}) \times \{J_n\}\}. \tag{46}$$

This set is the range of the Markov subordinator $(Y,J) = \{\tau,J(\tau)\}$. This means that $X_{n+1}^{(1)}(\tau) - X_{n+1}^{(1)}(T_n) = \tau - T_n$ $(T_n \le \tau \le T_{n+1})$ and $F_{jk}\{0\} = 1$ $(k \neq j)$, so that the transition distribution measure Q_{jk} has density

$$e^{-\lambda_{jj}s}\lambda_{jk} \quad (k \neq j). \tag{47}$$

Since $\mathscr{R}_{X_1^{(1)}} = [0,T_1)$ we find that

$$p_j(t) = P\{T_1 > t \mid J_0 = j\} = e^{-\lambda_{jj}t} \tag{48}$$

and

$$P_{jk}(t) \geq \int_{0-}^{t} U_{jk}\{ds\} e^{-\lambda_{kk}(t-s)}. \tag{49}$$

The equality in (49) holds iff $L' = \infty$ a.s. □

<u>Example 5</u>. This is the continuous time version of Example 3. Let $X = \{X(t),\ t \geq 0\}$ be a continuous Markov chain on E, all of whose states are stable. Let

$$M(t) = \sup_{0 \leq s \leq t} X(s) \tag{50}$$

$$L(t) = \text{Lebesgue measure of } \{t \geq 0\colon M(t)-X(t) = 0\} \cap (0,t] \tag{51}$$

$$Y(\tau) = \inf\{t \geq 0\colon L(t) > \tau\},\quad J(\tau) = M(Y(\tau)). \tag{52}$$

Prabhu [10] showed that the process $(Y,J) = \{Y(\tau),\ J(\tau)),\ \tau \geq 0\}$ is a Markov subordinator on the state space $\mathbb{R}_+ \times E$, with the infinitesimal generator $\mathscr{A}$ given by

$$(\mathscr{A}f)(t,j) = \frac{\partial f}{\partial t} + \int_{0-}^{\infty} [f(t+v,j) - f(t,j)]\mu_{jj}^{(0)}\{dv\} \tag{53}$$

$$+ \sum_{k>j} \int_{0-}^{\infty} [f(t+v,k) - f(t,j)]\mu_{jk}\{dv\}$$

where

$$\mu_{jk}\{dv\} = q_{jk}\epsilon_0\{dv\} + \sum_{\ell<j} q_{j\ell} B_{\ell k}^{j}\{dv\} \quad (k > j) \tag{54}$$

$$\mu_{jj}^{(0)}\{dv\} = \sum_{\ell<j} q_{j\ell} B_{\ell j}^{j}\{dv\} \quad (0 \leq v < \infty) \tag{55}$$

$$\mu_{jj}^{(0)}\{\infty\} = \sum_{\ell<j} q_{j\ell} P_{\ell}\{T_j = \infty\}, \tag{56}$$

q_{jk} $(k \neq j)$ are the transition rates of the process X, ϵ_0 is a distribution measure concentrated at the origin and $B^j_{\ell k}$ are given by

$$B^j_{\ell k}\{A\} = P_\ell\{T \in A, X(T) = k\} \quad (k \geq j > \ell) \tag{57}$$

$$T = \inf\{t \geq 0: X(t) \geq j\}. \tag{58}$$

We note from (53) that the subordinator $X^{(1)}_{n+1}$ is a compound Poisson process with unit drift.

Now let

$$\zeta = \{(t,\ell) \in T \times E: M(t) = X(t) = \ell\}. \tag{59}$$

Proceeding as in Example 3 we see that ζ is a semiregenerative set. An inspection of the sample path of the process X shows that ζ is the range of the Markov subordinator (Y,J). □

4. Semiregenerative phenomena with finite E. For the semiregenerative phenomenon constructed in the last section we now suppose that E is a finite set. In this case $L = \infty$ a.s. and $P_{jk}(t)$ is given by (42). For $\theta > 0$ let

$$\hat{P}_{jk}(\theta) = \int_0^\infty e^{-\theta t} P_{jk}(t)dt, \quad \hat{P} = (P_{jk}(\theta)). \tag{60}$$

We have then the following.

Theorem 5. For the semiregenerative phenomenon arising from the Markov additive process (Y,J) of section 3, with finite E, we have

$$\hat{P}^{-1} = R \tag{61}$$

where $R = (R_{jk}(\theta))$, with

$$R_{jj}(\theta) = \theta + \int_{0-}^{\infty+} (1-e^{-\theta x})\mu_{jj}\{dx\} \tag{62}$$

$$R_{jk}(\theta) = -\int_{0-}^{\infty} e^{-\theta x}\mu_{jk}\{dx\} \quad (j \neq k) \tag{63}$$

where μ_{jj} is a Lévy measure identical with $\mu_{jj}^{(0)}$ except that it has an additional weight λ_{jj} at infinity,

$$\mu_{jk}\{A\} = \lambda_{jk}F_{jk}\{A\} \tag{64}$$

and we note that

$$\sum_{k\neq j} \mu_{jk}\{\mathbb{R}_+\} \leq \mu_{jj}\{\infty\}. \tag{65}$$

Proof: We first calculate the transform $r_j(\theta)$ of $p_j(t)$. We have

$$\int_0^{\infty} e^{-\theta x}H_j\{s;dx\} = e^{-s\phi_{jj}(\theta)} \quad (\theta > 0)$$

where

$$\phi_{jj}(\theta) = \theta + \int_{0-}^{\infty+} (1 - e^{-\theta x})\mu_{jj}^{(0)}\{dx\}.$$

From (35) we find that

$$\begin{aligned} r_j(\theta) &= \int_0^{\infty} e^{-\theta t}p_j(t)dt = \int_0^{\infty} e^{-\lambda_{jj}s-s\phi_{jj}(\theta)}\,ds \\ &= [\theta + \int_{0-}^{\infty+} (1 - e^{-\theta x})\mu_{jj}\{dx\}]^{-1}. \end{aligned} \tag{66}$$

From (25) we find that

$$\hat{Q}_{jk}(\theta) = \int_{0-}^{\infty} e^{-\theta t} Q_{jk}\{dt\} = \lambda_{jk} r_j(\theta) \hat{F}_{jk}(\theta) \tag{67}$$

where $\hat{F}_{jk}$ is the transform of F_{jk}. Let us denote

$$\hat{U}_{jk}(\theta) = \int_{0-}^{\infty} e^{-\theta t} U_{jk}\{dt\}, \tag{68}$$

$\hat{Q} = (\hat{Q}_{jk}(\theta))$ and $\hat{U} = (\hat{U}_{jk}(\theta))$. From (42) we find that

$$\hat{P} = \hat{U}(\delta_{jk} r_j(\theta))$$

where it is known from Markov renewal theory that $\hat{U}^{-1} = I - \hat{Q}$. Therefore

$$R = \hat{P}^{-1} = ((r_j(\theta)^{-1})\delta_{jk})(I - \hat{Q}).$$

Using (66)–(67) it is easily verified that the elements of the matrix on the right side of this last relation are given by (62)–(64). The inequality (65) follows from the fact that $\Sigma_{k \neq j} \lambda_{jk} F_{jk}\{\mathbb{R}_+\} \leq \Sigma_{k \neq j} \lambda_{jk} = \lambda_{jj} \leq \mu_{jj}\{\infty\}$. □

We now ask whether semiregenerative phenomena with finite E can arise only in this manner (their semiregenerative sets corresponding to the range of a Markov additive process of the type described in section 3). In order to investigate this, we first prove the following result, which is essentially due to Kingman ([7], Theorem 5.2), who proved it in the setting of quasi–Markov chains. We shall only indicate the starting point of the proof.

Theorem 6. Let Z be a standard semiregenerative phenomenon with finite E and $P_{jk}(t)$ defined by (9). Denote $\hat{P}$ as in (60). Then

$$\hat{P}^{-1} = R \tag{69}$$

where $R = (R_{jk}(\theta))$, with $R_{jk}(\theta)$ given by (62)–(64), μ_{jk} $(j \neq k)$ being totally finite measures on $[0,\infty)$ and μ_{jj} a Lévy measure on $[0,\infty]$. Moreover, the inequality (65) holds.

Proof: For each $h > 0$, $Z_h = \{Z_{nh,\ell}, (n,\ell) \in \mathscr{L}\}$ is a semi–recurrent phenomenon, for which by Theorem 2,

$$P_{jk}(nh) = \sum_{r=1}^{\infty} q_{jk}^{(r)}(nh).$$

This gives

$$\sum_{n=0}^{\infty} P_{jk}(nh)z^n = \delta_{jk} + \sum_{r=1}^{\infty} \sum_{n=1}^{\infty} q_{jk}^{(r)}(nh)z^n$$

or

$$\begin{aligned}(\sum_{n=0}^{\infty} P_{jk}(nh)z^n) &= I + \sum_{r=1}^{\infty} (\sum_{n=1}^{\infty} q_{jk}(nh)z^n)^r \\ &= [I - (\sum_{n=1}^{\infty} q_{jk}(nh)z^n)]^{-1}.\end{aligned}$$ □

It turns out that the answer to the question raised above is affirmative. We prove this below.

Theorem 7. Let Z be a standard semiregenerative phenomenon with finite E. Then Z is equivalent to a phenomenon $\bar{Z}$ constructed as in Theorem 5, in the sense that Z and Z' have the same $P_{jk}(t)$.

Proof: We ignore the trivial case where all the measures μ_{jk} $(j \neq k)$ are identically zero. Then from (65) we find that $\mu_{jj}\{\infty\} > 0$. Define the probability measures F_{jk} by setting

$$\lambda_{jk} F_{jk}\{A\} = \mu_{jk}\{A\}/\mu_{jj}\{\infty\} \quad (j,k \in E,\ j \neq k).$$

We construct an MAP on the state space $\mathbb{R}_+ \times E$ as in section 3, with μ_{jj} and $\lambda_{jk}F_{jk}$. Let Z' be the semiregenerative phenomenon obtained from this MAP and $\overline{P}_{jk}(t) = P\{\overline{Z}_{tk} = 1 \mid \overline{Z}_{0j} = 1\}$. By Theorem 5 we have

$$\hat{\overline{P}}^{-1} = \overline{R}$$

where $\overline{R} = (\overline{R}_{jk}(\theta))$, with $\overline{R}_{jk}(\theta) = R_{jk}(\theta)$. Thus $\hat{P} = \hat{\overline{P}}$. Since the $P_{jk}(t)$ are continuous functions, it follows that $P_{jk}(t) = \overline{P}_{jk}(t)$, as was required to be shown. □

<u>Example 1 (continuation)</u>. For the continuous time Markov chain J with finite state space E we find from (48) that

$$r_j(\theta) = (\theta + \lambda_{jj})^{-1},$$

so that the Lévy measure μ_{jj} is concentrated at ∞ with weight $\lambda_{jj} > 0$. Also the F_{jk} are concentrated at zero with weight 1. Equality holds in (49). Theorem 5 gives

$$R_{jj}(\theta) = \theta + \lambda_{jj},\ R_{jk} = -\lambda_{jk}\ \ (k \neq j),$$

so that

$$R = \hat{P}^{-1} = \theta I - Q \tag{70}$$

where Q is the infinitesimal generator matrix of the chain. This is in agreement with the known result. By Theorem 7, the associated semiregenerative phenomenon is unique upto equivalence of $P_{jk}(t)$. □

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