

A theory of slow fluid flow through a porous thermoelastic matrix

L. W. Morland^{*} *Department of Mechanical Engineering, University of Colorado, Boulder, Colorado 80309, USA*

Received 1978 April 3; in original form 1977 August 1

Summary. Slow flow of linearly viscous fluid through a linear isotropic thermoelastic matrix is described. The interaction body force is Darcy's law, and the constitutive laws for the partial stresses assume that porosity changes of strain order occur and that porosity is a linear function of the partial pressures in matrix and fluid. A further dependence on deviatoric matrix stress allows a description of dilatancy. The laws are expressed in terms of four distinct compressibilities and a mixture shear modulus, and various strong inequalities between the compressibilities are examined. A consolidation theory for an incompressible fluid is derived, and the restrictions required to recover an uncoupled diffusion equation for the matrix compression are determined. Convection equations for large temperature differences across a horizontal layer are derived, allowing finite fluid expansion, and it is shown that the fluid flow equations uncouple from the matrix equilibrium equation in the case of steady flow, but not in the case of unsteady flow.

Introduction

The flow of fluids through permeable porous media is fundamental to hydrology, geothermal systems, oil recovery and soil consolidation. Applied theories range from rigid matrix and incompressible fluid (e.g. Bear 1972; Verruijt 1970), through elastic matrix and incompressible fluid (e.g. Biot 1941; Lambe & Whitman 1969; Schiffman, Chen & Jordan 1969), to elastic matrix and compressible fluid (e.g. Biot 1955, 1956a, b, 1973; Biot & Willis 1957; Verruijt 1969; Rice & Cleary 1976). Approximate one- and two-dimensional Terzaghi–Rendulic elastic theories of soil consolidation are described by Schiffman (1976). The significance of the elastic deformation of ground water aquifers was noted by Meinzer (1928), and a one-dimensional deformation theory was presented and applied by Jacob (1940, 1950). Biot (1941) developed the first three-dimensional theory for an isotropic elastic matrix and incompressible fluid, which is the basis for the consolidation applications by Schiffman *et al.* (1969), and recent numerical treatments by Booker & Small (1975)

^{*}Visiting professor, on leave from School of Mathematics and Physics, University of East Anglia, Norwich.

and Small, Booker & Davis (1976). Verruijt (1969) uses Biot theory for a compressible fluid with an effective stress concept to discuss flow in an elastic aquifer.

Compressibilities of soils, clays, aggregates, rocks, vary considerably, so the approximations of rigid matrix or incompressible fluid are restricted cases, and theory for a deformable matrix and compressible fluid is more widely applicable. Furthermore, with current interest in geothermal systems, thermal effects are also important. Here I describe a mixture theory for the slow flow of a compressible, linearly viscous fluid through an isotropic thermoelastic matrix. The mechanical or stress induced strains in the matrix, and pressure induced compressions in the fluid, are assumed infinitesimal, but finite thermal expansion of the fluid is allowed to include the effects of large temperature differences across geothermal layers. Infinitesimal mechanical compression in matrix and fluid are restrictions on the applied stress levels, and hence on depth in the Earth. The linear theory can be applied to infinitesimal strain fields superposed on a gravity equilibrium field involving finite strain, if any induced matrix anisotropy in the new reference configuration can be neglected, but the reference densities, at least, will vary with depth.

Mixture theory (Truesdell 1965) supposes that the length scale of interest is large compared with the size of a representative mixture element containing an abundance of each constituent. Here the length scale must be much greater than the mean pore diameter. The balance laws are expressed in terms of overlapping velocity (displacement) fields of each constituent defined at every mixture point, and the partial densities, partial stresses, partial heat fluxes, internal energies and temperatures. Partial density denotes mass of a constituent per unit mixture volume, and partial stress denotes traction carried by the constituent per unit mixture cross-section. Since a constituent velocity measures a volume flux per unit normal cross-section of mixture, it is convenient to introduce the terms partial deformation, partial strain, partial rate of strain, etc., for the quantities defined in the usual manner from this velocity (displacement) field. In contrast, material properties of a constituent are generally prescribed in terms of intrinsic stresses, strains, densities, etc., which are measures associated with unit cross-section and unit volume of the constituent.

In the isothermal, compressible fluid, isotropic elastic matrix theories of Biot (1956a, b) and Biot & Willis (1957), direct linear relations for the partial matrix stress and fluid pressure in terms of the partial matrix strain and fluid dilatation are postulated. A symmetry relation on the coefficients, based on an energy argument, leaves four independent mixture constants, one of which is a shear modulus. Various mixture tests are described to determine the four constants. Rice & Cleary (1976) start with total stress and intrinsic fluid pressure (pore pressure) as independent variables and postulate a linear relation for the matrix strain, from which, with Biot's energy result, is deduced an expression for the porosity change. Again, four mixture constants occur, and are interpreted and compared with the Biot parameters. An analogous thermodynamic treatment is presented by Schiffman (1971) on the assumption of a mixture free energy which is quadratic in the matrix strain and fluid dilatation, and reproduces the Biot symmetry relation.

An alternative approach (Morland 1972) is to incorporate the matrix and fluid material properties directly by relating partial and intrinsic stress and partial and intrinsic deformation etc. Here the term intrinsic is used in place of the earlier description, effective, since effective stress has different connotations, discussed later in the text. Porosity and porosity variation have a central role in these relations, so the partial stress laws require a constitutive law for porosity. Whenever the compressibilities of solid and fluid are different, the porosity will vary as the matrix and fluid pressures vary, and also as the matrix shear stress varies if dilatancy – the opening and closing of cracks under shear stress – is exhibited. Further, if the thermal expansion properties are different, porosity will vary as

the mixture is heated. Here we assume a common temperature T for both constituents, since flow times through a mixture element are long compared with heat conduction times. Velocity magnitudes are less than 1 cm/s even for well pumping and injection applications. The most simple assumption which includes these effects is that porosity ϕ is a frame indifferent function of the temperature and partial stresses in matrix and fluid. Thus

$$\phi = \phi(T, p^s, p^f, J_2, J_3), \quad (1)$$

where p^s, p^f are the partial pressures in matrix and fluid, and J_2, J_3 are invariants of the matrix deviatoric partial stress $\sigma^s + p^s 1$. Dependence on the viscous stress in the fluid is neglected by (1), and dilatancy is excluded if J_2, J_3 are absent. Hysteresis under shear stress cycles would be described by different dependence on J_2, J_3 during loading and unloading.

The law (1) and mixture theory (Morland 1972, 1975) is applicable to finite deformation and finite porosity variation. Here, analogous to Biot (1956a) theory, attention is restricted to infinitesimal matrix strain, fluid compression and relative porosity change, but finite thermal expansion of the fluid is allowed and dilatancy is included. With a further postulate that porosity is linear in the partial pressures, and depends only on shear stress through the quadratic invariant J_2 , (1) reduces to

$$\phi = \phi_1 [1 - ap^s + bp^f + d(J_2)]. \quad (2)$$

The parameters a and b have an order of magnitude of a matrix or fluid compressibility so that the terms ap^s, bp^f have the magnitude of a matrix or fluid compression (infinitesimal), and it is supposed that the function d has infinitesimal magnitude. For convenience all stresses are measured relative to a uniform isotropic pressure p_0 existing through the mixture at temperature T_0 , corresponding to $p^f = \phi_0 p_0, p^s = (1 - \phi_0)p_0$, where $\phi_0 = \phi_1(T_0)$ is the initial porosity and $d(0) = 0$. ϕ_1, a, b, d may be functions of temperature, and in particular $\phi_1(T) - \phi_0$ measures the porosity change through thermal expansion of the matrix at zero stress in matrix and fluid.

Use of (2) with the mixture theory (Morland 1972, 1975) determines partial stress laws incorporating directly the intrinsic moduli and thermal expansion coefficients of the matrix material and fluid. The coupling arises through the parameters a and b , and function d , which are shown to be determined by compressibility tests analogous to those of Biot & Willis (1957) together with a bi-axial test. In particular, a and b can be expressed in terms of the drained and undrained compressibilities, or drained and unjacketed compressibilities, in addition to the intrinsic matrix and fluid compressibilities and initial porosity. In the isothermal, non-dilatant case, there are five mixture constants; for example, a shear modulus and four independent compressibilities, but the Biot symmetry relation would impose a restriction on the compressibilities, to leave only four mixture constants. Measurement of the three independent drained, unjacketed and undrained compressibilities allows the symmetry to be tested directly, and further, the determination of an intrinsic solid compressibility measuring the mean solid compressibility in the matrix structure, which may be different from a grain compressibility.

The mixture theory is completed by the respective mass and momentum balance laws for the matrix and fluid, and, in view of the common temperature approximation, a single energy balance for the mixture. The interaction body force is deduced from a generalized Darcy's law for deforming matrix and fluid. Orders of magnitudes for practical applications are considered to obtain simpler approximate forms of the momentum and energy equations. In particular, the inertia terms and viscous fluid shear stress are neglected in comparison with the interaction drag, and the viscous stress working is neglected in comparison with the heat terms.

Consolidation theory for an incompressible fluid is examined, and it is shown that the incompressibility approximation alone makes no simplification in the equations, and that an uncoupled diffusion equation for the matrix compression is obtained only with further strong restrictions on the mixture compressibilities. The effective stress principle and the deduction of an effective stress expression is discussed. Finally, approximate equations for convection applications are derived. When the pressure induced compressions in matrix and fluid are negligible in comparison with the thermal expansion of the fluid, it is shown that uncoupling of the matrix and fluid momentum equations occurs for steady convection, but not in unsteady convection flows. In the steady case, the conventional reduced form of the convection equations is derived without making the unsatisfactory Boussinesq approximation, by introducing a vector potential defining a mass flux instead of a volume flux.

Mixture equations

Superscripts s and f denote quantities associated with the solid matrix and fluid respectively. $\mathbf{v}^s, \mathbf{v}^f$ are the overlapping velocity fields of the mixture theory (Truesdell 1965), and \mathbf{u}^s the mean matrix infinitesimal displacement field associated with \mathbf{v}^s , that is

$$\mathbf{v}^s = \frac{\partial \mathbf{u}^s(\mathbf{x}, t)}{\partial t} \quad (3)$$

where t denotes time and \mathbf{x} is position in a spatial Cartesian coordinate system $0x_i$ ($i = 1, 2, 3$). A partial solid strain associated with \mathbf{u}^s , representing the strain measured in the porous matrix, is defined by

$$\epsilon_{ij}^s = \frac{1}{2} \left(\frac{\partial u_i^s}{\partial x_j} + \frac{\partial u_j^s}{\partial x_i} \right), \quad \text{tr}(\boldsymbol{\epsilon}^s) = \text{div } \mathbf{u}^s, \quad (4)$$

and a partial fluid rate of strain tensor associated with \mathbf{v}^f by

$$D_{ij}^f = \frac{1}{2} \left(\frac{\partial v_i^f}{\partial x_j} + \frac{\partial v_j^f}{\partial x_i} \right), \quad \text{tr}(D^f) = \text{div } \mathbf{v}^f. \quad (5)$$

ρ^s, ρ^f are the partial densities, so the mixture density is

$$\rho = \rho^s + \rho^f. \quad (6)$$

$\boldsymbol{\sigma}^s, \boldsymbol{\sigma}^f$ are the partial stresses measured in excess of a uniform isotropic pressure p_0 supposed to exist in an undeformed rest state of the mixture at temperature $T = T_0$. E^s, E^f are the internal energies per unit mass of constituent. It is assumed that time-scales of flow are sufficiently long for a common temperature T to be attained (in mean) in both constituents in a mixture element. A subscript zero will denote quantities evaluated in the initial rest state at temperature T_0 . The mass balance laws are

$$\frac{\partial \rho^s}{\partial t} + \text{div}(\rho^s \mathbf{v}^s) = 0, \quad \frac{\partial \rho^f}{\partial t} + \text{div}(\rho^f \mathbf{v}^f) = 0. \quad (7)$$

If $\rho \boldsymbol{\beta}$ denotes the interaction body forces per unit mixture volume acting on the matrix and the external body force per unit mass is gravity \mathbf{g} , the momentum balances laws are

$$\frac{\partial \sigma_{ij}^s}{\partial x_j} + \rho^s g_i + \rho \beta_i = \rho^s \frac{D_s v_i^s}{Dt}. \quad (8)$$

$$\frac{\partial \sigma_{ij}^f}{\partial x_j} + \rho^f g_i - \rho \beta_i = \rho^f \frac{D_f v_i^f}{Dt} \quad (9)$$

where D_s/Dt and D_f/Dt are convective time derivatives associated with velocities $\mathbf{v}^s, \mathbf{v}^f$, respectively. The mixture energy balance is

$$\rho^s \frac{D_s E^s}{Dt} + \rho^f \frac{D_f E^f}{Dt} - \sigma_{ij}^s \frac{\partial v_i^s}{\partial x_j} - \sigma_{ij}^f \frac{\partial v_i^f}{\partial x_j} + \text{div}(\mathbf{q}^s + \mathbf{q}^f) = 0, \tag{10}$$

where $\mathbf{q}^s, \mathbf{q}^f$ are the partial heat fluxes within each constituent. Later, infinitesimal strains in the matrix are neglected in comparison with unity, but finite thermal expansion of the fluid is allowed. Interaction couples are excluded so that the partial stress tensors are symmetric.

A raised prefix E will denote quantities intrinsic to a constituent element. The relation between partial density ρ^f and intrinsic density ${}^E\rho^f$ in the fluid is defined simply by the volume fraction, porosity ϕ , of fluid in the mixture, and similarly the volume fraction $1 - \phi$ for the solid, thus

$$\rho^f = \phi {}^E\rho^f, \quad \rho^s = (1 - \phi) {}^E\rho^s. \tag{11}$$

It is commonly assumed that ϕ and $1 - \phi$ are also the mean fluid and solid cross-section areas respectively per unit mixture cross-section, so that

$$\boldsymbol{\sigma}^f = \phi {}^E\boldsymbol{\sigma}^f, \quad \boldsymbol{\sigma}^s = (1 - \phi) {}^E\boldsymbol{\sigma}^s. \tag{12}$$

This result can also be deduced (Morland 1975) in the absence of body couples by interpreting partial and intrinsic stresses as averages over mixture and constituent volumes in a representative mixture element. From the relation between intrinsic and partial deformation gradient proposed by Morland (1972) on the assumption that intrinsic and partial shear deformations are the same, ${}^E\mathbf{F}^f = (\phi/\phi_0)^{1/3} \mathbf{F}$, and the identity $D_f \mathbf{F}/Dt = \mathbf{L}\mathbf{F}$ where \mathbf{L} is the spatial velocity gradient with symmetric part \mathbf{D} , it follows that the intrinsic rate of strain of the fluid is given by

$${}^E\mathbf{D}^f = \mathbf{D}^f + \frac{1}{3} \left(\frac{\phi_0}{\phi} \right) \frac{D_f}{Dt} (\phi/\phi_0) \mathbf{1}. \tag{13}$$

In particular, the deviatoric parts of ${}^E\mathbf{D}^f$ and \mathbf{D}^f are the same, so for a linearly viscous compressible fluid with zero bulk viscosity and a temperature dependent shear viscosity $\mu(T)$,

$${}^E\boldsymbol{\sigma}^f + {}^E p^f \mathbf{1} = 2\mu \{ \mathbf{D}^f - \frac{1}{3} \text{div} \mathbf{v}^f \mathbf{1} \}. \tag{14}$$

Following Batchelor (1970, p. 20) and assuming $E^f = E^f({}^E\rho^f, T)$ classical thermodynamics shows that

$$\frac{D_f E^f}{Dt} - \frac{{}^E p^f}{({}^E\rho^f)^2} \frac{D_f {}^E\rho^f}{Dt} = C_p^f \frac{D_f T}{Dt} - \frac{\kappa_f}{\alpha_f} (C_p^f - C_v^f) \frac{D_f {}^E p^f}{Dt}, \tag{15}$$

where C_p^f, C_v^f are the specific heats at constant pressure and constant volume respectively, κ_f is the isothermal compressibility, and α_f the coefficient of thermal expansion. Making the approximations $\kappa_f = \text{constant}$, $\alpha_f = \alpha_f(T)$, which are appropriate to water over pressure and temperature ranges $(1 \rightarrow 200) \times 10^6 \text{ gm cm}^{-1} \text{ s}^{-2}$, $300 \rightarrow 550 \text{ K}$ (Keenan *et al.* 1969, p. 114),

$${}^E\rho^f = \rho_f \exp(\kappa_f {}^E p^f) \exp(-A_f), \tag{16}$$

where $\rho_f = {}^E\rho_0^f$ is the intrinsic density at pressure p_0 , temperature T_0 , and

$$A_f(T) = \int_{T_0}^T \alpha_f(T') dT'. \quad (17)$$

In view of the low compressibility κ_f we assume that the pressure-induced intrinsic compression $-{}^Ee^f$, at constant temperature, remains infinitesimal; that is, from (16), $|{}^Ee^f| \ll 1$ where

$$-{}^Ee^f = \kappa_f {}^E p^f = \exp(A_f) \frac{{}^E\rho^f}{\rho_f} - 1. \quad (18)$$

The relations (11), (12) with (14), (15), (18) provide constitutive laws for E^f and σ^f once a law for ϕ is prescribed. Here temperature differences $\Delta T = T - T_0$ are allowed for which A_f is not infinitesimal. For example, for water over the range 300–500 K, $A_f \sim 0.3$.

The analogous laws for a linear thermoelastic solid are (Chadwick 1960)

$${}^E\sigma^s + {}^E p^s \mathbf{1} = 2G \{ {}^E\epsilon^s - \frac{1}{3} \text{tr}({}^E\epsilon^s) \mathbf{1} \}, \quad (19)$$

$$-{}^Ee^s = \kappa_s {}^E p^s = \frac{{}^E\rho^s}{\rho_s} - 1 + \alpha_s(T - T_0) = -\text{tr}({}^E\epsilon^s) + \alpha_s(T - T_0), \quad (20)$$

where $\rho_s = {}^E\rho_0^s$ is the intrinsic density at (p_0, T_0) , and ${}^E\epsilon^s$ is the intrinsic infinitesimal strain. G , κ_s are the constant isothermal shear modulus and compressibility respectively, and α_s is the constant coefficient of thermal expansion. Here $|\alpha_s(T - T_0)| \ll 1$ so both $|{}^Ee^s|$ and $|\text{tr}({}^E\epsilon^s)| \ll 1$. From Morland (1972)

$${}^E\epsilon^s - \frac{1}{3} \text{tr}({}^E\epsilon^s) \mathbf{1} = \epsilon^s - \frac{1}{3} \text{tr}(\epsilon^s) \mathbf{1}, \quad (21)$$

$$\text{tr}({}^E\epsilon^s) = \text{tr}(\epsilon^s) - \frac{\phi - \phi_0}{1 - \phi_0}, \quad -\text{tr}(\epsilon^s) = \frac{\rho^s}{\rho_0^s} - 1. \quad (22)$$

Assuming $E^s = E^s({}^E\epsilon^s, T)$, analogous to (15)

$$\frac{D_s E^s}{Dt} - \frac{1}{{}^E\rho^s} {}^E\sigma_{ij}^s \frac{D_s E_{ij}^s}{Dt} = C_p^s \frac{D_s T}{Dt} - \frac{\kappa_s}{\alpha_s} (C_p^s - C_v^s) \frac{D_s {}^E p^s}{Dt} \quad (23)$$

where C_p^s, C_v^s are the specific heats at constant stress and constant strain respectively.

It is convenient to introduce partial mechanical dilatations e^s, e^f by

$$-e^s = -\text{tr}(\epsilon^s) + \alpha_s(T - T_0), \quad -e^f = \exp(A_f) \frac{\rho^f}{\rho_0^f} - 1, \quad (24)$$

so that

$${}^Ee^s = e^s - \frac{\phi - \phi_0}{1 - \phi_0}, \quad {}^Ee^f = \frac{\phi_0}{\phi} \left(e^f + \frac{\phi - \phi_0}{\phi_0} \right), \quad (25)$$

and

$$\rho^f = \rho_0^f \exp(-A_f) [1 - e^f]. \quad (26)$$

When $T \equiv T_0$, e^s , e^f are simply the partial dilatations, but even for large $(T - T_0)$ with finite thermal expansion of the fluid, since $|\phi - \phi_0| \ll \phi_0$,

$$|e^s|, \quad |e^f| \ll 1. \tag{27}$$

Now

$$\sigma^s = (1 - \phi) \left[2G \left(\epsilon^s - \frac{1}{3} \text{div } \mathbf{u}^s \mathbf{1} \right) + \frac{1}{\kappa_s} \left(e^s - \frac{\phi - \phi_0}{1 - \phi_0} \right) \mathbf{1} \right], \tag{28}$$

$$\sigma^f = \frac{\phi}{\kappa_f} \left[e^f + \frac{\phi - \phi_0}{\phi_0} \right] \mathbf{1} + 2\mu\phi \left[\mathbf{D}^f - \frac{1}{3} \text{div } \mathbf{v}^f \mathbf{1} \right], \tag{29}$$

showing the explicit dependence on the strain-order porosity change $(\phi - \phi_0)$.

Assuming a constant mixture thermal conductivity $\bar{\lambda}$, then the total heat flux is

$$\mathbf{q}^s + \mathbf{q}^f = -\bar{\lambda} \text{grad } T, \tag{30}$$

and

$$\text{div} (\mathbf{q}^s + \mathbf{q}^f) = -\bar{\lambda} \nabla^2 T. \tag{31}$$

In practice $\bar{\lambda}$ is not a simple weighted combination of the solid and fluid conductivities λ_s , λ_f (e.g. Bear 1972, p. 648; Ramey *et al.* 1974), and direct measure is required. With (15), (23) and (31), and making use of the relations (7), (11), (12), (14), (21), (22), the energy balance (10) becomes

$$(\rho^s C_p^s + \rho^f C_p^f) \frac{\partial T}{\partial t} + (\rho^f C_p^f v_j^f + \rho^s C_p^s v_j^s) \frac{\partial T}{\partial x_j} - \bar{\lambda} \nabla^2 T = R \tag{32}$$

where

$$R = \frac{\rho^s \kappa_s}{\alpha_s} (C_p^s - C_v^s) \frac{D_s^E p^s}{Dt} + \frac{\rho^f \kappa_f}{\alpha_f} (C_p^f - C_v^f) \frac{D_f^E p^f}{Dt} + (\sigma_{ij}^f + p^f \delta_{ij}) \frac{\partial v_i^f}{\partial x_j} + E_{p^f} \frac{D_f \phi}{Dt} - E_{p^s} \frac{D_s \phi}{Dt}. \tag{33}$$

If constant temperature is assumed the balance laws (7), (8), (9), relations (3), (4), (5) and constitutive relations (28), (29), (2) fully determine the stresses and velocities, and (10) merely calculates the necessary energy storage. However, the resulting value of R given by (33) may not be identically zero, when (32) implies there are small temperature variations in contradiction to the isothermal assumption, and that the present thermodynamic model is an approximation.

For thermal applications with a temperature difference ΔT , velocity magnitude V , and maximum mechanical compression e in matrix or fluid, across a length scale h , the left-hand terms of (32) have magnitudes

$$\rho C_p \Delta T V / h, \quad \bar{\lambda} \Delta T / h^2,$$

while R has groups with magnitudes

$$\frac{V}{h} \left\{ \frac{\rho(C_p - C_v)}{\alpha} e, \quad \mu \frac{V}{h}, \quad \frac{e^2}{\kappa} \right\},$$

evaluated for matrix and fluid. For water over the temperature range 298–548 K,

$$C_p^f \sim 4.2 \times 10^7 \text{ cm}^2 \text{ s}^{-2} \text{ K}^{-1}, \quad \lambda_f \sim 6 \times 10^4 \text{ gm cm s}^{-3} \text{ K}^{-1}, \quad \kappa_f \sim 0.5 \times 10^{-10} \text{ gm}^{-1} \text{ cm s}^2, \\ \alpha_f \sim (3.2-21) \times 10^{-4} \text{ K}^{-1}, \quad \mu \sim (8.7-0.87) \times 10^{-3} \text{ gm cm}^{-1} \text{ s}^{-1}. \quad (34)$$

Values of C_p^s , λ_s are of similar magnitude for rocks, κ_s is less and of the same magnitude for granites, but will be of greater magnitude for some rocks and aggregates, and for soils. α_s is typically of smaller magnitude for rocks. For these values the viscous stress working is trivially negligible and the porosity variation contribution can be neglected whenever $\Delta T \gg 10^3 e^2$ is trivially satisfied. The pressure-rate contributions can also be neglected when $\alpha C_p \Delta T \gg (C_p - C_v)e$. This is a good approximation for water in the above temperature range, but in view of the smaller α_s may not be acceptable for the matrix in all applications. However, for steady flow with the matrix at rest, there is no pressure-rate contribution from the matrix. In the subsequent thermal theory, (32) is adopted with the common assumption

$$R = 0. \quad (35)$$

Now consider steady uniform flow in a horizontal x_i direction under uniform, intrinsic, fluid pressure gradient, with no gravity or other body force. Assuming that the interaction body force is given by Darcy's law (Bear 1972, p. 159) with the Hubbert potential for a compressible fluid, then

$$-\frac{\partial^E p^f}{\partial x_i} = \frac{\mu}{k} Q_i \quad (36)$$

where $k(\phi)$ is the permeability of the matrix–fluid system (with dimension length squared) and Q_i is the specific discharge, namely the volume of fluid discharging per unit time across unit cross-section of a mixture surface normal to the x_i direction. By mixture theory definitions the mass flux of fluid and mass flux of solid across a surface with unit normal \mathbf{n} are respectively $\rho^f \mathbf{v}^f \cdot \mathbf{n}$ and $\rho^s \mathbf{v}^s \cdot \mathbf{n}$, corresponding to overlapping space fluxes $\mathbf{v}^f \cdot \mathbf{n}$ and $\mathbf{v}^s \cdot \mathbf{n}$ respectively, but intrinsic volume fluxes $\phi \mathbf{v}^f \cdot \mathbf{n}$ and $(1 - \phi) \mathbf{v}^s \cdot \mathbf{n}$ respectively (mass flux divided by intrinsic density). The total volume flux across a fixed surface is therefore $[\phi \mathbf{v}^f + (1 - \phi) \mathbf{v}^s] \cdot \mathbf{n}$. Thus the fluid discharge across a matrix boundary is the total volume flux less the space flux of the matrix; that is,

$$Q_i = \phi(v_i^f - v_i^s). \quad (37)$$

Applying the momentum equation (9) to this flow, and using (12)₁, gives

$$\rho \beta_i = -\frac{\partial}{\partial x_i} (\phi^E p^f) = \frac{\mu \phi^2}{k} (v_i^f - v_i^s) - \frac{p^f}{\phi} \frac{\partial \phi}{\partial x_i}, \quad (38)$$

which is the generalization of the one-dimensional result given by Garg (1971) to include effects of porosity gradient. This result is also derived by Garg *et al.* (1975) by introducing an effective velocity field for each constituent and postulating equality of convective derivatives associated with the two velocities.

Permeability of various porous rocks and aggregates has a wide range $10^{-3} \rightarrow 10^{-12} \text{ cm}^2$ (Bear 1972, p. 136), but a mean value $k = 10^{-9} \text{ cm}^2$ seems typical of reasonably permeable media. For $k \leq 10^{-7} \text{ cm}^2$ and $\phi \geq 10^{-2}$, the drag force (38) is significantly greater than the inertia terms in (8), (9) provided that $h \text{ cm} \gg V \text{ cm/s}$, a trivial restriction in practical applications, and always much greater than the viscous stress gradient determined by (14).

Also $|p^f \nabla \phi / \phi| = 0$ $|p^f a \nabla p^s, p^f b \nabla p^f| \ll |\nabla p^f, \nabla p^s|$ since a, b have the magnitudes of compressibility, and so the momentum balances (8), (9) reduce, to a good approximation, to

$$\begin{aligned} \frac{\partial \sigma_{ij}^s}{\partial x_j} + \rho^s g_i + \frac{\mu \phi^2}{k} (v_i^f - v_i^s) &= 0, \\ -\frac{\partial p^f}{\partial x_i} + \rho^f g_i - \frac{\mu \phi^2}{k} (v_i^f - v_i^s) &= 0, \end{aligned} \tag{39}$$

with $\phi = \phi_0 = \phi_1(T_0)$, $k(\phi) = k(\phi_0)$, and $\rho^s = \rho_0^s$, to first order in strain.

Partial stress laws

The laws (28), (29) determined the partial stresses σ^s, p^f required in the equilibrium balances (8), (9) once a constitutive law for ϕ is prescribed. Adopting the simple model (2) for infinitesimal strain order porosity change, functions of temperature $\phi_1, a, b, d(J_2)$ must be determined. Uniform heating of a unit mixture volume at zero stress, allowing fluid to drain freely in or out, produces a new volume $1 + tr(\epsilon^s)$ of the same matrix element. Introducing a constant coefficient of thermal expansion α_m for these conditions, so $tr(\epsilon^s) = \alpha_m(T - T_0)$, and eliminating $tr(\epsilon^s)$ between (20) and (22), shows that

$$\phi_1(T) = \phi_0 \{1 + \gamma \alpha_s (T - T_0)\}, \tag{40}$$

where

$$\gamma = \frac{(1 - \phi_0)(\alpha_m - \alpha_s)}{\phi_0 \alpha_s}. \tag{41}$$

Measurements of the expansion $tr(\epsilon^s)$ determines α_m and hence γ in the constitutive law (40). It is supposed that $\gamma \leq 0(1)$. If $\alpha_m = \alpha_s$, then $\gamma = 0$ and $\phi_1(T) \equiv \phi_0$. Note that a porosity determined from the new solid volume $(1 - \phi_0) [1 + tr(\epsilon^s)]$ and matrix volume $1 + tr(\epsilon^s)$, using (20) and the linearized relation (22), is trivially ϕ_0 , since $\gamma \alpha_s (T - T_0)$ is a strain-order term neglected in comparison with unity. This would also be the conclusion from displacement solutions to stress-free linear thermoelastic bodies containing holes. Thus α_m , and hence γ , must be determined from thermal response of the matrix, though the simple result $\alpha_m = \alpha_s$ may be exact or a good approximation.

Next consider isotropic pressure, isothermal, loading of the mixture body from this zero stress state and porosity ϕ_1 at temperature T . If ϵ^s, ϵ^f denote the changes of e^s, e^f induced by the partial pressures p^s, p^f then by (28), (29)

$$-p^s = \frac{1 - \phi_0}{\kappa_s} \left(\epsilon^s - \frac{\phi - \phi_1}{1 - \phi_0} \right), \quad -p^f = \frac{\phi_0}{\kappa_f} \left(\epsilon^f + \frac{\phi - \phi_1}{\phi_0} \right). \tag{42}$$

Now by (2), with $d = d(0) = 0$,

$$(1 - \phi_0) \epsilon^s = -(\kappa_s + \phi_1 a) p^s + \phi_1 b p^f, \tag{43}$$

$$\phi_0 \epsilon^f = \phi_1 a p^s - (\kappa_f + \phi_1 b) p^f,$$

or

$$-Dp^s = (1 - \phi_0) (\kappa_f + \phi_1 b) \epsilon^s + \phi_0 \phi_1 b \epsilon^f, \tag{44}$$

$$-Dp^f = \phi_1 (1 - \phi_0) a \epsilon^s + \phi_0 (\kappa_s + \phi_1 a) \epsilon^f,$$

where

$$D = \kappa_s \kappa_f + \phi_1 \kappa_s b + \phi_1 \kappa_f a. \quad (45)$$

Now with the neglect of the strain order term $\gamma \alpha_s (T - T_0)$, $\phi_1 = \phi_0$, and it is assumed that κ_s, κ_f, a, b , are independent of temperature to this order, so

$$\begin{aligned} -p^s &= c_{11} \epsilon^s + c_{12} \epsilon^f, \\ -p^f &= c_{21} \epsilon^s + c_{22} \epsilon^f, \end{aligned} \quad (46)$$

where the constant coefficients are given by

$$D \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} (1 - \phi_0) (\kappa_f + \phi_0 b) \phi_0^2 b \\ \phi_0 (1 - \phi_0) a \phi_0 (\kappa_s + \phi_0 a) \end{pmatrix}. \quad (47)$$

Now apply the total pressure p in a jacketed (free draining) test and an unjacketed test in turn, measuring overall compressibilities κ, δ respectively (Biot & Willis 1957), and then in an undrained test measuring an overall compressibility $\bar{\delta}$. For free drainage

$$p^f = 0, \quad p^s = p = -\epsilon^s / \kappa. \quad (48)$$

In the unjacketed test

$$p^f = \phi_0 p, \quad p^s = (1 - \phi_0) p, \quad p = -\epsilon^s / \delta. \quad (49)$$

In the undrained test there is no mass flux of fluid in or out of the mixture body, so

$$\epsilon^f = \epsilon^s, \quad p^f + p^s = p = -\epsilon^s / \bar{\delta}. \quad (50)$$

Any two of (48), (49), (50) are sufficient to determine the parameters a and b .

Combining (48) and (50) gives

$$\begin{aligned} \phi_0 a &= (1 - \phi_0) \kappa - \kappa_s, \\ \phi_0 (\bar{\delta} - \kappa_s) b &= (1 - \phi_0) (\bar{\delta} \kappa_s - \bar{\delta} \kappa_f - \bar{\delta} \kappa + \kappa \kappa_f), \\ (\bar{\delta} - \kappa_s) D &= \bar{\delta} (1 - \phi_0) (\kappa_s - \kappa) (\kappa_s - \kappa_f). \end{aligned} \quad (51)$$

The pressure sharing between matrix and fluid in the undrained test is given by

$$p^f = \eta p, \quad p^s = (1 - \eta) p, \quad \eta = \frac{\bar{\delta} - \kappa_s}{\kappa_f - \kappa_s}. \quad (52)$$

Alternatively, combining (48) and (49) gives (51)₁ and

$$\begin{aligned} \phi_0^2 b &= (1 - \phi_0) [(1 - \phi_0) \kappa - \delta], \\ \phi_0 D &= (1 - \phi_0) [\phi_0 \kappa (\kappa_f - \kappa_s) + \kappa_s (\kappa - \delta)]. \end{aligned} \quad (53)$$

In the isothermal case $T = T_0$, Biot (1956a, b), Biot & Willis (1957) and Rice & Cleary (1976) use equations analogous to (46) with a symmetry restriction $c_{12} = c_{21}$ deduced from an assumption that a mixture element has a potential energy. However, since diffusion takes place during general loading, a mixture element is not a fixed element of matter. Here $c_{12} = c_{21}$ implies

$$\phi_0 (\kappa_s - \kappa_f) (\kappa - \bar{\delta}) = (\kappa - \kappa_s) (\kappa_s - \bar{\delta}) \quad \text{and} \quad \kappa_s = \delta, \quad (54)$$

reducing the mixture parameters to three independent compressibilities and a shear modulus, analogous to the Biot theory. The symmetry restrictions (54) can be tested directly. Here four independent compressibilities are assumed, and the consequences of the restriction (54) deduced as a special case.

From (28), a simple shear test on a mixture body determines a mixture shear modulus

$$G_m = (1 - \phi_0)G_m \tag{55}$$

for all drainage conditions. Now the intrinsic solid shear modulus G and compressibility κ_s must measure mean response within a representative mixture element of all material other than the fluid in interconnected pores, and so will be affected by gas and fluid trapped in closed pores, and any other structural features. Thus grain properties may not be appropriate interpretations of G and κ_s , and direct measure of G_m and deduction of κ_s from the three independent tests (48), (59), (50), are more valid procedures. Here any shear hysteresis in loading–unloading cycles is neglected. Also the connecting pore fluid compressibility κ_f can be influenced by trapped gas.

If the matrix solid is more compressible than the fluid, the following inequalities are expected:

$$\kappa_f < \bar{\delta} < \kappa_s < \kappa, \tag{56}$$

but if the fluid is more compressible:

$$\kappa_s < \bar{\delta} < \kappa, \quad \bar{\delta} < \kappa_f, \tag{57}$$

so in both cases $D > 0$.

Choose

$$J_2 = \frac{3}{2} \text{tr} \{(\boldsymbol{\sigma}^s + p^s \mathbf{1})^2\}, \tag{58}$$

and consider a bi-axial loading test on the mixture body with free drainage at uniform temperature T , with principal total compressive stresses p_1, p_2, p_3 and additional principal compressions $-\epsilon_1^s, -\epsilon_2^s, -\epsilon_3^s$ of the matrix. Let $p_2 = p_3 = 0$ and measure the dilatation versus p_1 relations for loading and unloading, $\theta = \theta_l(p_1)$ and $\theta = \theta_u(p_1)$ respectively where $\theta = \epsilon_1^s + 2\epsilon_2^s$. Opening and closing of cracks in rocks under shear stress, dilatancy, typically exhibits hysteresis and non-linearity; that is, distinct non-linear functions θ_l and θ_u . In both loading and unloading the trace of (28) shows that

$$d(p_1^2) - \frac{\kappa_s + \phi_0 a}{3\phi_0} p_1 = \frac{1 - \phi_0}{\phi_0} \theta(p_1), \tag{59}$$

which determines the respective functions d at temperature T . Independent bi-axial loading tests will check the consistency of the restriction to a single dilatancy function $d(J_2)$.

Hence, for non-isothermal general stress loading, assuming the variation of d with temperature is negligible, inserting the full expression (2), (40) for ϕ in (28), (29), gives

$$-p^f = c_{21}e^s + c_{22}e^f + \frac{\phi_0 \kappa_s}{D} [\gamma \alpha_s (T - T_0) + d(J_2)], \tag{60}$$

$$-p^s = c_{11}e^s + c_{12}e^f - \frac{\phi_0 \kappa_f}{D} [\gamma \alpha_s (T - T_0) + d(J_2)], \tag{61}$$

and the deviatoric matrix stress

$$\boldsymbol{\sigma}^s + p^s \mathbf{1} = 2G_m (\boldsymbol{\epsilon}^s - \frac{1}{3} \text{div} \mathbf{u}^s \mathbf{1}). \tag{62}$$

J_2 is calculated from (62) and then (60), (61) give expressions for p^f, p^s with non-linear dependence on the deviatoric matrix strain due to dilatancy. Alternatively, inversion of (60), (61) gives non-linear laws for e^s, e^f in terms of stresses and temperature.

The laws (60)–(62) with $d \equiv 0$, and balance laws (40) and (32), have been used to investigate vertical oscillatory motions in a horizontal permeable layer due to Earth tides modelled as time harmonic perturbations in the gravity force g (Morland 1977a), and also to investigate two-dimensional time harmonic flows due to Earth tides, in the isothermal case, when the permeable layer is connected to a well (Morland 1977b).

Limit cases and consolidation theory

For the conventional theory of flow through a rigid matrix set $d = 0, G = \infty, \kappa_s = 0, \alpha_s = 0$, and then, consistent with (57), let $\bar{\delta} = \kappa \rightarrow 0$. Trivially $a = b = 0$ so $\phi \equiv \phi_0$, and $D = 0$, but

$$c_{11} \rightarrow \infty, \quad c_{12} \rightarrow 0, \quad c_{21} \rightarrow \frac{\phi_0(1 - \phi_0)}{\kappa_f}, \quad c_{22} \rightarrow \frac{\phi_0}{\kappa_f}, \quad (63)$$

and so p^s is indeterminate and $-\kappa_f p^f = \phi_0 \epsilon^f$, or, using (12), (18) and (24), $-\kappa_f E p^f = E \epsilon^f$ as expected.

In the incompressible fluid limit used in soil consolidation, $\kappa_f = 0, \alpha_f = 0$, so $e^f = \epsilon^f$. Let $\bar{\delta}, \kappa_s, \kappa$ be non-zero and ordered by (56), and consider the non-dilatant case $d = 0$. Now a, b, D are non-zero and

$$\begin{aligned} \kappa_s(c_{11}, c_{12}) &= (1 - \phi_0, \phi_0), \\ \bar{\delta}\kappa_s(\kappa_s - \kappa)(c_{21}, c_{22}) &= (\bar{\delta} - \kappa_s)([1 - \phi_0]\kappa - \kappa_s, \phi_0\kappa). \end{aligned} \quad (64)$$

That is, there is complete coupling between p^s, p^f and ϵ^s, ϵ^f , and the partial fluid pressure p^f is determined by (46). By (18), (11) and (24) the zero intrinsic fluid dilatation is given by

$$0 = \phi^E \epsilon^f = \phi_0(\epsilon^f - a p^s + b p^f + \gamma \alpha_s [T - T_0]), \quad (65)$$

which is identically zero with the coefficients (64), so neither is ϵ^f zero nor explicitly related to ϵ^s by the fluid incompressibility condition. In fact, by (65), $\epsilon^f \equiv 0$ if $\phi \equiv \phi_0$ which is not a reasonable conclusion for an incompressible fluid and compressible matrix. The consolidation equations used by Schiffman *et al.* (1969), Booker & Small (1975) and Small *et al.* (1976) have no partial stress dependence on a partial fluid dilatation ϵ^f , and in the latter papers the mass balance equation is consistent with the solid and fluid balances (7) only if $\epsilon^f = 0$.

Another feature of the consolidation theories referenced above is an initial condition assumption that no matrix compression occurs when a load is applied instantaneously (because fluid requires a finite time to drain). That is, the undrained compressibility $\bar{\delta} = 0$. This is compatible with the approximation $\bar{\delta} \ll \kappa$ noted by Biot (1941) from tests on completely saturated clays. With this strong inequality, (64) implies $(c_{11}, c_{12}) = 0(1/\kappa_s)$, $(c_{21}, c_{22}) = 0(\kappa/\bar{\delta}\kappa_s)$, and hence $\bar{\delta} p^f = 0(\kappa p^s)$; that is, $|p^f| \gg |p^s|$. However, inverting (60), (61) in the isothermal, non-dilatant, case gives

$$\begin{aligned} \epsilon^s &= -\kappa p^s + \frac{\bar{\delta}(\kappa - \kappa_s)}{\kappa_s - \bar{\delta}} p^f, \\ \epsilon^f &= \frac{(1 - \phi_0)\kappa - \kappa_s}{\phi_0} p^s - \frac{(1 - \phi_0)\bar{\delta}(\kappa - \kappa_s)}{\phi_0(\kappa_s - \bar{\delta})} p^f, \end{aligned} \quad (66)$$

which shows that $\epsilon^f = 0(\epsilon^s)$. With the earlier interpretation of κ_s it is possible that $\kappa_s = 0(\kappa)$, but also there is the possibility $\kappa_s \ll \kappa$, is compatible with the theories mentioned above which show no dependence on κ_s . In the latter case (66) reduces to

$$\epsilon^s = -\frac{\phi_0}{1 - \phi_0} \epsilon^f = -\kappa p^s + \frac{\bar{\delta}\kappa}{\kappa_s - \bar{\delta}} p^f, \tag{67}$$

so that p^s, p^f are not separately determined by ϵ^s, ϵ^f , but again $\epsilon^f = 0(\epsilon^s)$. In fact, for a low porosity matrix, ϵ^f dominates ϵ^s . Thus, an approximation $\epsilon^f = 0$ appears to have no validity.

In an isothermal consolidation theory for incompressible fluid with $\bar{\delta} \neq 0$ or $\bar{\delta} = 0(\kappa_s \neq 0)$ all c_{ij} are non-zero, and the general equations apply. Let the stresses, strains, etc. denote values in excess of an equilibrium solution for the gravity force and zero (atmospheric pressure) traction on the surface, which will also depend on imposed lateral constraints. The mass balance laws are

$$\epsilon^s = \text{div } \mathbf{u}^s \quad \text{or} \quad \frac{\partial \epsilon^s}{\partial t} = \text{div } \mathbf{v}^s, \quad \frac{\partial \epsilon^f}{\partial t} = \text{div } \mathbf{v}^f, \tag{68}$$

so that

$$\text{div } \mathbf{v} = \frac{\partial}{\partial t} (\epsilon^f - \epsilon^s) \quad \text{where} \quad \mathbf{v} = \mathbf{v}^f - \mathbf{v}^s. \tag{69}$$

The momentum balance equations become

$$c_{21} \nabla \epsilon^s + c_{22} \nabla \epsilon^f = \frac{\mu \phi_0^2}{k} \mathbf{v}, \tag{70}$$

$$G_m \nabla^2 \mathbf{u}^s + (c_{11} + c_{21} + \frac{1}{3} G_m) \nabla \epsilon^s + (c_{12} + c_{22}) \nabla \epsilon^f = 0, \tag{71}$$

where (71) is the mixture balance. Equations (68)₂, (70), (71) are seven equations for the seven variables $\mathbf{u}^s, \mathbf{v}^f, \epsilon^f$, since \mathbf{u}^s determines ϵ^s, \mathbf{v}^s by (68)₁. The divergence of (71) and the divergence of (70) combined with (69) give

$$(c_{11} + c_{21} + \frac{4}{3} G_m) \nabla^2 \epsilon^s + (c_{12} + c_{22}) \nabla^2 \epsilon^f = 0, \tag{72}$$

$$\frac{\partial}{\partial t} (\epsilon^f - \epsilon^s) = \frac{k}{\mu \phi_0^2} \nabla^2 (c_{21} \epsilon^s + c_{22} \epsilon^f). \tag{73}$$

By (72),

$$\epsilon^f = \xi \epsilon^s + w, \quad \nabla^2 w = 0, \tag{74}$$

where ξ is a constant and w an arbitrary harmonic function, and (73) becomes

$$\frac{\partial}{\partial t} (\epsilon^s - \epsilon^f) = \frac{k}{\mu \phi_0^2} \frac{c_{11} c_{22} - c_{21} c_{12} + \frac{4}{3} G_m c_{22}}{c_{22} + c_{12}} \nabla^2 \epsilon^s. \tag{75}$$

A result analogous to (74) for p^f instead of ϵ^f is derived by Verruijt (1969). The consolidation equation (8a) of Schiffman *et al.* (1969) relates $\partial \epsilon^s / \partial t$ to $\nabla^2 \epsilon^s$ which follows from (75) only if $w \equiv 0$ (or if $\epsilon^f \equiv 0$). No simplification in form follows for the case $\bar{\delta} \ll \kappa_s = 0(\kappa)$.

Consider the approximation

$$\kappa_f = 0, \quad \bar{\delta} < \kappa_s \ll \kappa, \tag{76}$$

for which the relations (67) hold. Eliminating ϵ^f and p^s by (67) gives the momentum and mass balance equations

$$G_m \nabla^2 \mathbf{u}^s + \left(\frac{1}{\kappa} + \frac{1}{3} G_m \right) \nabla \epsilon^s - \frac{\kappa_s}{\kappa_s - \bar{\delta}} \nabla p^f = 0, \quad (77)$$

$$\frac{\partial \epsilon^s}{\partial t} = C_M \nabla^2 \epsilon^s \quad \text{where} \quad C_M = \frac{k(1/\kappa + 4/3 G_m)(\kappa_s - \bar{\delta})}{\mu \phi_0 \kappa_s} \quad (78)$$

is the consolidation coefficient, giving an uncoupled diffusion equation for ϵ^s . (77), (78) are the equations given by Schiffman *et al.* (1969) if their permeability is k/ϕ_0 and if $\bar{\delta}$ can be neglected in comparison with κ_s . That is, under the stronger restrictions

$$\kappa_f = 0, \quad \bar{\delta} \ll \kappa_s \ll \kappa. \quad (79)$$

Only for the approximation (76) does the uncoupled law (78) arise, and less restricted compressibilities κ_f , κ_s , $\bar{\delta}$, κ require the full equations. While this restriction may be appropriate to saturated clays, it is not likely to be generally applicable to aquifers in porous rocks and aggregates. Verruijt's (1969) equations are based on an assumption that an effective matrix stress depends only on the partial matrix strain, so no explicit dependence on the partial fluid dilatation is exhibited. Rice & Cleary (1976) achieve a diffusion equation by taking \mathbf{v}^f instead of $\mathbf{v}^f - \mathbf{v}^s$ in Darcy's law.

For steady flow with the matrix necessarily at rest, $\mathbf{v}^s = 0$, $\mathbf{v} = \mathbf{v}^f$, and

$$\text{div } \mathbf{v}^f = 0, \quad \text{curl } \mathbf{v}^f = 0; \quad \mathbf{v}^f = \nabla \Omega, \quad \nabla^2 \Omega = 0, \quad (80)$$

$$p^f + \frac{\mu \phi_0^2}{k} \Omega = \text{constant}, \quad (81)$$

so harmonic Ω determines \mathbf{v}^f and p^f . Now (70) expresses $\nabla \epsilon^f$ in terms of $\nabla \epsilon^s$ and Ω and (71) becomes

$$G_m \nabla^2 \mathbf{u}^s + \frac{c_{11} c_{22} - c_{12} c_{21} + 1/3 c_{22} G_m}{c_{22}} \nabla \epsilon^s = - \frac{\mu \phi_0^2 (c_{12} + c_{22})}{k c_{22}} \nabla \Omega, \quad (82)$$

to determine \mathbf{u}^s . With the restriction to $0x_1x_2$ plane flow and plane strain or plane stress in the matrix, the matrix equilibrium is satisfied by introducing a plane biharmonic stress function ψ ,

$$\nabla^4 \psi = 0, \quad (83)$$

such that

$$\sigma_{11}^s = \frac{\partial^2 \psi}{\partial x_2^2} - \frac{\mu \phi_0^2}{k} \Omega, \quad \sigma_{12}^s = - \frac{\partial^2 \psi}{\partial x_1 \partial x_2}, \quad \sigma_{22}^s = \frac{\partial^2 \psi}{\partial x_1^2} - \frac{\mu \phi_0^2}{k} \Omega. \quad (84)$$

In the non-isothermal case the thermal expansion terms are expressed in terms of a given harmonic temperature field.

Effective stress

The term intrinsic stress has been introduced here in place of the description effective stress used by Morland (1972), to avoid confusion with the Terzaghi effective stress principle and extensions of this concept (e.g. Schiffman 1970) which define the effective (matrix) stress as the stress dependent only on the matrix (partial) strain. This is written in the form

$$\bar{\sigma} = \sigma + \alpha^E p^f \mathbf{1}, \quad (85)$$

where the intrinsic fluid pressure $E p^f$ is termed pore pressure, and α is the soil–water interaction coefficient. Schiffman (1970) proposes $\phi \leq \alpha \leq 1$ where the limit $\alpha = \phi$ gives $\bar{\sigma} = \sigma^s$, the partial matrix stress, and the limit $\alpha = 1$ is the original Terzaghi postulate. The effective stress principle is adopted by Verruijt (1969) and Rice & Cleary (1976) to formulate partial stress laws, but effective stress does not arise directly in the balance laws.

Nur & Byerlee (1971) discuss proposed expressions for α , and by constructing a solution for a mixture body under confining (total) pressure P_c and pore pressure P_p , conclude that

$$\alpha = 1 - \kappa_s / \kappa. \quad (86)$$

The load is applied in two stages. First both confining pressure and pore pressure are raised to P_p , allowing the necessary fluid diffusion, producing a matrix compression θ_1 , then an additional confining pressure $P_c - P_p$, is applied retaining the pore pressure at P_p . At the end of stage 1, the intrinsic pressure in the solid is P_p , and so by (20), (22), if ϕ_1 is the new porosity,

$$\theta_1 = \kappa_s P_p - \frac{\phi_1 - \phi_0}{1 - \phi_0}. \quad (87)$$

Stage 2 is a free drainage loading, with the fluid reservoir at pressure P_p , and so the additional matrix compression is

$$\theta_2 = \kappa (P_c - P_p). \quad (88)$$

Hence, evaluating ϕ_1 by (2), (51)₁ and (53)₁, the total compression is

$$\theta = \kappa P_c - (\kappa - \delta) P_p, \quad = \kappa \langle p \rangle \quad (89)$$

where

$$\langle p \rangle = P_c - \alpha P_p, \quad \alpha = 1 - \delta / \kappa. \quad (90)$$

Thus the result (86) is obtained only if $\phi_1 = \phi_0$, equivalent to the unjacketed compressibility $\delta = \kappa_s$, which follows if the Biot symmetry restriction holds. This was also deduced by Garg & Nur (1973) on the basis of the Morland (1972) theory in conjunction with the symmetry relation.

Nur & Byerlee point out that the Terzaghi postulate $\alpha = 1$ is justified if $\kappa_s \ll \kappa$, here replaced by $\delta \ll \kappa$, but also that $\alpha \sim 0$ is feasible in low porosity aggregates if κ_s approaches κ , here δ approaches κ .

Convection equations

In geothermal systems with large temperature differences across permeable horizontal layers, buoyancy-driven convection flows may be significant. Classical treatments use the porous media equations for a rigid matrix, so now the interaction with a deforming thermoelastic

matrix is examined. Let the stresses, strains and displacements denote values in excess of an equilibrium field under gravity and the imposed temperature increase ΔT downwards across the layer, with equilibrium quantities denoted by a raised *. Define

$$\theta = (T - T^*)/\Delta T, \quad \text{where} \quad \nabla^2 T^*(\mathbf{x}) = 0, \quad (91)$$

then it is supposed that

$$|\theta| \ll 1 \quad (92)$$

in convection flows, and terms of order θ compared with unity are neglected.

By (7) and (24) the mass balances become

$$\rho^s - \rho^{s*} = -\rho_0^s \operatorname{div} \mathbf{u}^s, \quad \frac{\partial \rho^f}{\partial t} + \operatorname{div}(\rho^f \mathbf{v}^f) = 0, \quad (93)$$

where ρ^s, ρ^f are actual partial densities and the energy balance (32) with (35) becomes

$$(\rho^s C_p^s + \rho^f C_p^f) \frac{\partial \theta}{\partial t} + (\rho^f C_p^f v_j^f + \rho^s C_p^s v_j^s) \left(\frac{\partial \theta}{\partial x_j} + \frac{\partial T^*}{\partial x_j} / \Delta T \right) = \bar{\lambda} \nabla^2 \theta. \quad (94)$$

By definition $\nabla T^*/\Delta T$ is order $(1/h)$ where h is the layer thickness, so using (92) the $\nabla \theta$ term is neglected in the linear theory. Also μ may be evaluated at T^* (expansions of $\mu(T)$ and $\alpha_f(T)$ about T^* in powers of θ for water show that leading coefficients are order unity when $300 \text{ K} \leq T \leq 555 \text{ K}$) and the matrix momentum balance (39)₁ becomes

$$\frac{\partial \sigma_{ij}^s}{\partial x_j} + \frac{\mu(T^*) \phi_0^2}{k} (v_i^f - v_i^s) = 0. \quad (95)$$

However, the fluid convection is driven by buoyancy arising from the density decrease $\rho^{f*} - \rho^f$. By (26),

$$\rho^f - \rho^{f*} = -\rho_0^f \exp \{-A_f(T^*)\} \{\alpha_f(T^*) \Delta T \theta + e^f - e^{f*}\}, \quad (96)$$

to first order in θ , neglecting e^f compared with unity, and (39)₂ gives

$$-\frac{\partial p^f}{\partial x_i} + (\rho^f - \rho^{f*}) g_i - \frac{\mu(T^*) \phi_0^2}{k} (v_i^f - v_i^s) = 0. \quad (97)$$

Now $e^f - e^{f*} = 0(\kappa_f p^f)$ and $p^f = 0\{(\rho^f - \rho^{f*}) h g\}$ by (97), so $e^f - e^{f*} = 0\{\rho_0^f g h \kappa_f \alpha_f \Delta T \theta\} = 0(e^f \alpha_f \Delta T \theta)$ which is strain order and negligible compared with $\alpha_f \Delta T \theta$. Thus (97) becomes

$$-\frac{\partial p^f}{\partial x_i} - \rho_0^f g_i \exp \{-A_f(T^*)\} \alpha_f(T^*) \Delta T \theta - \frac{\mu(T^*) \phi_0^2}{k} (v_i^f - v_i^s) = 0. \quad (98)$$

For unsteady flow with $\mathbf{v}^s \neq \mathbf{0}$, (93), (94), (95), (98) with the laws (60), (61), (62) are a coupled system for $\theta, \mathbf{u}^s, \mathbf{v}^f, \rho^s, \rho^f$, and the matrix motion influences the convection flow. However, for steady flow with $\mathbf{v}^s \equiv \mathbf{0}$, the fluid laws are

$$\operatorname{div}(\rho^f \mathbf{v}^f) = 0, \quad C_p^f \rho^f v_j^f \frac{\partial T^*}{\partial x_j} / \Delta T = \bar{\lambda} \nabla^2 \theta, \quad (99)$$

and (98) with $\mathbf{v}^s = 0$; that is, five equations for θ , ρ^f , \mathbf{v}^f , independent of ρ^s , \mathbf{u}^s except through the law for p^t . But the curl of (98) gives (assuming ρ_0^f , ϕ_0 , k are uniform)

$$\text{curl} \left[\exp \{-A_f(T^*)\} \alpha_f(T^*) \theta \mathbf{g} + \frac{\mu(T^*) \phi_0^2}{\rho_0^f \Delta T k} \mathbf{v}^f \right] = 0, \quad (100)$$

in which p^f is absent. Thus (99) and (100) are complete equations for the steady fluid convection.

Introducing a vector potential Ψ by

$$\rho^f \mathbf{v}^f = \text{curl } \Psi \quad (101)$$

satisfies (99), identically and the energy balance becomes

$$(\text{curl } \Psi) \cdot \nabla T^* = \frac{\bar{\lambda} \Delta T}{C_p^f} \nabla^2 \theta. \quad (102)$$

Ψ is the generalization of a two-dimensional stream function ψ defining a mass flux instead of volume flux in plane flow, which leads to linear equations (100), (102) for θ , Ψ without a Boussinesq approximation $\rho^f = \rho_0^f$. This formulation was used by Morland, Zebib & Kassoy (1977) to investigate the effects of temperature dependent $\mu(T)$, $\alpha_f(T)$ on the onset of steady plane convection in a horizontal layer.

Acknowledgments

This work was supported by National Science Foundation Grants AER 74-04329 A02 directed by Dr D. R. Kassoy and DES 75-20137 directed by Dr C. B. Archambeau, and an Air Force Cambridge Research Laboratory Grant F19628-76-C-0061 directed by Dr C. B. Archambeau, while the author was a visiting professor in the University of Colorado in 1976. I thank Dr R. L. Schiffman for helpful discussions on consolidation theory, and am grateful to reviewers who pointed out omissions and misinterpretations of previous work, and made many constructive comments.

References

- Batchelor, G. K., 1970. *An introduction to fluid dynamics*, 615 pages, Cambridge University Press.
- Bear, J., 1972. *Dynamics of fluids in porous media*, 764 pages, American Elsevier, New York.
- Biot, M. A., 1941. General theory of three-dimensional consolidation, *J. Appl. Phys.*, **12**, 155–164.
- Biot, M. A., 1955. Theory of elasticity and consolidation for a porous anisotropic solid, *J. Appl. Phys.*, **26**, 182–185.
- Biot, M. A., 1956a. General solutions of the equations of elasticity and consolidation for a porous material, *J. Appl. Mech.*, **78**, 91–96.
- Biot, M. A., 1956b. Theory of propagation of elastic waves in a fluid saturated porous solid, Parts I and II, *J. acoust. Soc. Am.*, **28**, 168–191.
- Biot, M. A., 1973. Non-linear and semilinear rheology of porous solids, *J. geophys. Res.*, **78**, 4924–4937.
- Biot, M. A. & Willis, D. G., 1957. The elastic coefficients of the theory of consolidation, *J. Appl. Mech.*, **79**, 594–601.
- Booker, J. R. & Small, J. C., 1975. An investigation of the stability of numerical solutions of Biot's equations of consolidation, *Int. J. solids Struct.*, **11**, 907–917.
- Chadwick, P., 1960. Thermoelasticity: the dynamical theory, in *Progress in solid mechanics*, **1**, pp. 265–328, eds Sneddon, I. N. & Hill, R., North Holland, Amsterdam.
- Garg, S. K., 1971. Wave propagation effects in a fluid-saturated porous solid, *J. geophys. Res.*, **76**, 7947–7962.

- Garg, S. K. & Nur, A., 1973. Effective stress laws for fluid-saturated porous rocks, *J. geophys. Res.*, **78**, 5911–5921.
- Garg, S. K., Brownell, D. H., Pritchett, J. W. & Herrmann, R. G., 1975. Shock-wave propagation in fluid-saturated porous media, *J. Appl. Phys.*, **46**, 702–713.
- Jacob, C. E., 1940. On the flow of water in an elastic artesian aquifer, *Trans. Am. Geophys. Un.*, **21**, 574–586.
- Jacob, C. E., 1950. *Flow of ground water in engineering hydraulics*, pp. 321–386, ed. Rouse, H., Wiley, New York.
- Keenan, J. H., Keyes, F. G., Hill, P. G. & Moore, J. G., 1969. *Steam tables*, 162 pages, Wiley, New York.
- Lambe, T. W. & Whitman, R. V., 1969. *Soil mechanics*, 533 pages, Wiley, New York.
- Meinzer, O. E., 1928. Compressibility and elasticity of artesian aquifers, *Econ. Geol.*, **23**, 263–291.
- Morland, L. W., 1972. A simple constitutive theory for a fluid-saturated porous solid, *J. geophys. Res.*, **77**, 890–900.
- Morland, L. W., 1975. Effective stress in mixture theory, *Arch. Mech.*, **27**, 883–887.
- Morland, L. W., 1977a. Oscillatory flows in a porous thermoelastic matrix due to earth tides, *Geotherm.*, **6**, 107–116.
- Morland, L. W., 1977b. Earth tide effects on flows in horizontal permeable elastic layers connected to wells, *Geophys. J. R. astr. Soc.*, **51**, 371–385.
- Morland, L. W., Zebib, A. & Kassoy, D. R., 1977. Variable property effects on the onset of convection in an elastic porous matrix, *Phys. Fluids*, **20**, 1255–1259.
- Nur, A. & Byerlee, J. D., 1971. An exact effective stress law for elastic deformation of rock with fluid, *J. geophys. Res.*, **76**, 6414–6419.
- Ramey, H. J., Brigham, W. E., Chen, H. K., Atkinson, P. G. & Arihara, N., 1974. *Thermodynamic and hydrodynamic properties of hydrothermal systems*, Stanford Geothermal Report TR-6, Stanford University, Stanford, California.
- Rice, J. R. & Cleary, M. P., 1976. Some basic stress diffusion solutions for fluid-saturated elastic porous media with compressible constituents, *Rev. Geophys. Space Phys.*, **14**, 227–241.
- Schiffman, R. L., 1970. The stress components of a porous medium, *J. geophys. Res.*, **75**, 4035–4038.
- Schiffman, R. L., 1971. *A thermoelastic theory of consolidation*, Winter annual meeting of ASME, Washington, D.C.
- Schiffman, R. L., 1976. *The theory of consolidation*, Rep. UCCC 76-13, Computing Center, University of Colorado, Boulder, Colorado.
- Schiffman, R. L., Chen, A. T. F. & Jordan, J. C., 1969. An analysis of consolidation theories, *J. Soil Mech. Found. Div., ASCE*, **95**, 285–312.
- Small, J. C., Booker, J. R. & Davis, E. H., 1976. Elasto-plastic consolidation of soil, *Int. J. Solids Struct.*, **12**, 431–448.
- Truesdell, C., 1965. The rational mechanics of materials, in *On the foundations of mechanics and energetics*, pp. 293–304, Gordon and Breach, New York.
- Verruijt, A., 1969. Elastic storage of aquifers, in *Flow through porous media*, pp. 331–376, ed. DeWiest, R. J. M., Academic Press, New York.
- Verruijt, A., 1970. *Theory of groundwater flow*, 190 pages, Macmillan, London.