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A theory of the destabilization paradox in non-conservative systems

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Summary. In the present paper, a theory is developed qualitatively and quantitatively describing the paradoxical behavior of general non-conservative systems under the action of small dissipative and gyroscopic forces. The problem is investigated by the approach based on the sensitivity analysis of multiple eigenvalues. The movement of eigenvalues of the system in the complex plane is analytically described and interpreted. Approximations of the asymptotic stability domain in the space of the system parameters are obtained. An explicit asymptotic expression for the critical load as a function of dissipation and gyroscopic parameters allowing to calculate a jump in the critical load is derived. The classical Ziegler–Herrmann–Jong pendulum considered as a mechanical application demonstrates the efficiency of the theory.

1 Introduction

In 1952, Ziegler [1] studying the stability of a double pendulum loaded by a follower force came to the unexpected conclusion that the critical load of the non-conservative system with vanishingly small dissipation is considerably lower than in the case when dissipation is completely absent. The analytical description of this phenomenon called *the destabilization paradox* was recognized as one of the main theoretical challenges in the non-conservative stability theory [2]. However, the questions provoked by the destabilization paradox have not yet been answered in the general form.

To formulate the problem, we consider a linear autonomous non-conservative mechanical system

$$\mathbf{M}\frac{d^2\mathbf{y}}{dt^2} + \mathbf{D}(\mathbf{k})\frac{d\mathbf{y}}{dt} + \mathbf{A}(q)\mathbf{y} = 0, \tag{1}$$

where $\mathbf{y}$ is the vector of generalized coordinates, and $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{A}$ are real square matrices of order m corresponding to the inertial, dissipative and gyroscopic, and non-conservative positional forces, respectively. It is assumed that the matrix $\mathbf{D}$ is a smooth function of the vector of real parameters $\mathbf{k} = (k_1, \dots, k_{n-1})$, $\mathbf{D}(0) = 0$, the matrix $\mathbf{A}$ smoothly depends on the real load parameter $q \geq 0$, and the matrix $\mathbf{M}$ is parametrically independent. The vector $\mathbf{k}$ of the parameters corresponding to the velocity-dependent forces is assumed to be small ($\|\mathbf{k}\| \ll 1$).

Seeking a solution to Eq. (1) in the form $\mathbf{y} = \mathbf{u}\exp(\lambda t)$ we get the generalized eigenvalue problem

$$\mathbf{L}\mathbf{u} = 0, \quad \mathbf{L} \equiv \lambda^2\mathbf{M} + \lambda\mathbf{D}(\mathbf{k}) + \mathbf{A}(q), \tag{2}$$

where $\mathbf{u}$ is an eigenvector and λ is an eigenvalue of the operator $\mathbf{L}(\lambda, \mathbf{k}, q)$. A non-conservative system without dissipative and gyroscopic forces ($\mathbf{k} = 0$)

$$\mathbf{M}\frac{d^2\mathbf{y}}{dt^2} + \mathbf{A}(q)\mathbf{y} = 0 \tag{3}$$

is called the *circulatory system* [2], [3]. The spectrum of the circulatory system is mirror symmetric; i.e., if λ is an eigenvalue of the linear operator $\lambda^2\mathbf{M}+\mathbf{A}(q)$, then $-\lambda$, $\overline{\lambda}$, and $-\overline{\lambda}$, where the bar stands for complex conjugation, are also eigenvalues. Hence, the circulatory system is stable in the Lyapunov sense if and only if all the eigenvalues λ lie on the imaginary axis of the complex plane and are semisimple. The latter means that the number r of linearly independent eigenvectors corresponding to an eigenvalue is equal to its algebraic multiplicity μ. If $r < \mu$ then *secular* terms proportional to $t^\alpha e^{\lambda t}$, where $\alpha \leq \mu - 1$, appear in the general solution of Eq. (3) causing flutter instability (i.e., oscillations with growing amplitude).

Let system (3) be stable for q=0. When the load parameter q increases and reaches a certain critical value q=q_0, two simple purely imaginary eigenvalues can collide with the origination of a double eigenvalue $i\omega_0$ with only one eigenvector and associated vector forming a so-called Keldysh chain [4], while other eigenvalues remain simple and purely imaginary. The further increase in the load causes splitting of the double eigenvalue into a pair of complex-conjugate eigenvalues, one of them with positive real part, Fig. 1a (flutter). Therefore, the interval $0 \leq q < q_0$ belongs to the stability domain of system (3), and the critical value q_0 defines the boundary between the stability and flutter instability domains [2], [5].

It turns out [6] that perturbation of a circulatory system by weak dissipative and gyroscopic forces ($\mathbf{k} \neq 0$) destroys the interaction of the eigenvalues: For a certain value of the load parameter q=$q_{cr}(\mathbf{k})$, one of the eigenvalues crosses the imaginary axis and then moves into the right half of the complex plane without origination of a double eigenvalue and its bifurcation, Fig.1b. Moreover, for $\mathbf{k}$=$\epsilon\widetilde{\mathbf{k}}$, where $\widetilde{\mathbf{k}}$ is the fixed vector and $\epsilon \geq 0$ is a small parameter, the following relation holds:

$$\widetilde{q}_{cr} \equiv \lim_{\epsilon \to 0} q_{cr}(\epsilon\widetilde{\mathbf{k}}) \leq q_0. \tag{4}$$

Inequality (4) shows that in non-conservative systems the critical load can decrease in a discontinuous manner when infinitesimally small dissipative and gyroscopic forces are taken into account. This is considered paradoxical because dissipation is expected to provide stability rather than instability. The destabilization paradox discovered by Ziegler [1] for a pendulum with two degrees of freedom and one dissipation parameter, loaded by a follower force, has attracted much attention in the world literature, see the book by Bolotin [2] and the review articles [7]–[12].

Bolotin and Herrmann were the first who initiated the systematic study of non-conservative systems with small dissipation. For a number of specific mechanical systems with two

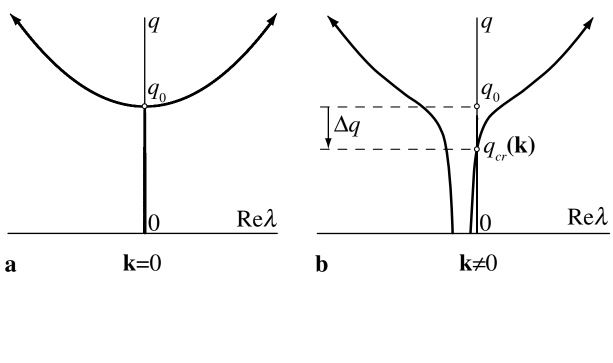

Fig. 1. Trajectories of eigenvalues illustrating the destabilization paradox

dissipation parameters k_1, k_2, Bolotin established that the limit of the critical load $\widetilde{q}_{cr}$ depends on the choice of the vector $\widetilde{\mathbf{k}}$. In particular, changing the ratio of the parameters k_1 and k_2 it is possible to avoid the jump in the critical load and therefore the destabilization effect [2],[13]. Herrmann and Jong [14],[15] studied the limit of the critical load $\widetilde{q}_{cr}$ as a function of the ratio k_1/k_2 for the Ziegler pendulum with two dissipation parameters and showed that it attains its maximum $\widetilde{q}_{cr}=q_0$ at $k_1/k_2=4+5\sqrt{2}$. Bolotin [2] and Herrmann and Jong [14] also examined the trajectories of eigenvalues of undamped and damped specific non-conservative systems on the complex plane and pointed out that small dissipation generally destroys the interaction of the eigenvalues. They established the qualitative pictures of the eigenvalue motion, shown in Fig. 1, and recognized that the change of instability mechanism is at the heart of the destabilization paradox. However, in those pioneering works the destabilizing effect of small dissipation was not placed into the framework of a theory of sufficient generality. Nor was it shown whether a system with many degrees of freedom can exhibit such behavior.

As a generalization, Nemat-Nasser and Herrmann [16] suggested to consider an m-degrees-of-freedom non-conservative system (1) with the matrix $\mathbf{D}=\epsilon\widetilde{\mathbf{D}}$, where $\widetilde{\mathbf{D}}$ is fixed and $\epsilon \geq 0$ is a small parameter. Analyzing the characteristic polynomial of the eigenvalue problem (2), they established inequality (4) and noted the strong dependence of the critical load $q_{cr}(\epsilon)$ on the structure of the matrix $\widetilde{\mathbf{D}}$, but came to the wrong conclusion that $q_{cr}(\epsilon)$ never exceeds q_0. Using the analogous approach, Bolotin and Zhinzher [6] showed that for the dissipative forces with the matrix $\mathbf{D}=\epsilon\mathbf{M}$ (external damping) or $\mathbf{D}=\epsilon\mathbf{A}$ the critical load $q_{cr}(\epsilon)\geq q_0$. These conclusions were later confirmed by Done [17] and Kounadis [18], and became widely known in the mechanical literature [8]. However, already Walker [19] using the direct Lyapunov method found a class of stabilizing configurations for the matrix $\mathbf{D}$, which includes the results of [6] as a particular case. O'Reilly et al. [20] succeeded in getting the necessary and sufficient conditions for the 2×2 matrix $\mathbf{D}=\epsilon\widetilde{\mathbf{D}}$ to be stabilizing in terms of invariants of the matrices $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{A}$, assuming that the unperturbed system has only simple purely imaginary eigenvalues. Their contribution was extended to the case of m-dimensional systems by Gallina [21]. Recently Seyranian and Kirillov [22] generalized the results of those works, establishing the necessary and sufficient conditions for the $m \times m$ matrix $\mathbf{D}=\epsilon\widetilde{\mathbf{D}}$ to make the circulatory system (3), which is stable or located on the flutter boundary, asymptotically stable.

Note that despite the progress in finding the stabilizing matrices $\mathbf{D}$, none of the afore mentioned papers has answered the question: How evaluate a jump in the critical load caused by the perturbation $\mathbf{D}=\epsilon\widetilde{\mathbf{D}}$ with a given matrix $\widetilde{\mathbf{D}}$? An important step to the solution of this problem was taken by Seyranian and Pedersen [23], and Seyranian [24], who obtained explicit asymptotic formulae describing the movement of the eigenvalues of the non-conservative system (1) with small dissipation $\mathbf{D}=\epsilon\widetilde{\mathbf{D}}$ in the complex plane. They explained the break of the coupling between the eigenvalues, but did not calculate a jump in the critical load due to small dissipation because their approximation was not accurate enough. Besides, for the Ziegler–Herrmann–Jong pendulum with two degrees of freedom and two dissipation parameters, Seyranian and Pedersen [25], and Seyranian [26] found the domain in the parameter plane where the non-conservative system perturbed by small dissipative forces is asymptotically stable and $q_{cr}(\mathbf{k})>q_0$. They established that the domain of asymptotic stability has a singularity at the point $(\mathbf{k}=0, q=q_0)$ corresponding to the unperturbed circulatory system. Seyranian [26] and Zhinzher [27] noted that the critical load of the Ziegler pendulum as a function of two dissipative parameters has no limit when dissipation goes to zero. Troger and Zeman [28] were the first who applied the singularity theory to the study of the Ziegler pendulum. Recently Mailybaev and Seyranian [29] realized that the paradoxical behavior of the Ziegler pendulum is closely connected with the typical singularity of the asymptotic stability domain known as the Whitney–Cayley umbrella [30]. Hoffmann and Gaul [40] studied the effects of damping on mode-coupling instability in friction induced oscillations.

Our paper presents a new theory of the destabilization paradox in linear non-conservative systems of general type with small dissipative and gyroscopic forces. This theory is based on the bifurcation analysis of multiple eigenvalues of a linear matrix operator with coefficients smoothly dependent on a spectral parameter and a vector of real parameters, which is done in Sect. 2 of the paper.

In Sect. 3, explicit asymptotic expressions describing the trajectories of the eigenvalues are derived for the general non-conservative system (1) with m degrees of freedom. They generalize and improve the results presented in [23]–[25] and make possible the analytical investigation of the splitting of the eigenvalue trajectories into independent curves due to a perturbation of the circulatory system (3) by small dissipative and gyroscopic forces. From the analysis of the eigenvalue trajectories, an explicit asymptotic expression for the critical load $q_{cr}(\mathbf{k})$ as a function of the vector of dissipative and gyroscopic parameters $\mathbf{k}$ is obtained.

In Sect. 4, the properties of the function $q_{cr}(\mathbf{k})$ are thoroughly investigated. It is shown that the critical load has no limit as $\mathbf{k}\to 0$, although $\lim_{\epsilon\to 0} q_{cr}(\epsilon\widetilde{\mathbf{k}})$ exists for almost all direction vectors $\widetilde{\mathbf{k}}$. An explicit formula approximating the jump in the critical load $q_0 - \lim_{\epsilon\to 0} q_{cr}(\epsilon\widetilde{\mathbf{k}})$ due to small velocity-dependent forces is found. In the case of two degrees of freedom the jump is expressed by means of the invariants of the matrices of the system. For the general non-conservative system (1) with small velocity-dependent forces, explicit approximations of the domains of asymptotic stability in the space of the parameters $k_1, \ldots, k_{n-1}, q$ are found. For a system with one load parameter and two dissipative and gyroscopic parameters it is shown that the surface $q_{cr}(k_1, k_2)$ bounding the stabilization domain has the singularity known as the Whitney–Cayley umbrella. Besides, a simple explicit relation between the parameters k_1, k_2 necessary for the stabilization of system (1) is established.

The classical Ziegler–Herrmann–Jong pendulum considered in Sect. 5 as a mechanical application shows the efficiency of the developed theory, which is in a good qualitative and quantitative agreement with the known results.

2 Bifurcation of multiple eigenvalues

Since the destabilization paradox is closely related to the splitting of the double eigenvalues, it is first necessary to study the bifurcation of multiple eigenvalues with a change of parameters. We consider a generalized eigenvalue problem for a linear matrix operator $\mathbf{L}$ whose coefficients smoothly depend on a complex spectral parameter λ and a vector of real parameters $\mathbf{p} \in \mathbb{R}^n$,

$$\mathbf{L}(\lambda, \mathbf{p})\mathbf{u} = 0. \tag{5}$$

For a fixed vector $\mathbf{p} = \mathbf{p}_0$, a value λ_0 of the spectral parameter, at which there exists a nontrivial solution $\mathbf{u}_0$ of Eq. (5), is called an eigenvalue whereas the vector $\mathbf{u}_0$ is called *an eigenvector* of the operator $\mathbf{L}$ at the eigenvalue λ_0. The eigenvalues λ are found from the characteristic equation $\det \mathbf{L}(\lambda, \mathbf{p}) = 0$.

Let λ_0 be a μ-fold eigenvalue, which possesses a Keldysh chain of vectors consisting of one eigenvector $\mathbf{u}_0$ and associated vectors $\mathbf{u}_1, \ldots, \mathbf{u}_{\mu-1}$. Denote $\mathbf{L}_0 = \mathbf{L}(\lambda_0, \mathbf{p}_0)$. The vectors of the Keldysh chain satisfy the Eqs. [4]

$$\mathbf{L}_0\mathbf{u}_0 = 0, \quad \mathbf{L}_0\mathbf{u}_s = -\sum_{r=1}^{s} \frac{1}{r!}\frac{\partial^r \mathbf{L}}{\partial \lambda^r}\mathbf{u}_{s-r}, \quad s = 1, \ldots, \mu - 1, \tag{6}$$

where partial derivatives are evaluated at $\lambda=\lambda_0$ and $\mathbf{p}=\mathbf{p}_0$. The left Keldysh chain is formed by the vectors $\mathbf{v}_0, \ldots, \mathbf{v}_{\mu-1}$ satisfying the equations

$$\mathbf{v}_0^T\mathbf{L}_0 = 0, \quad \mathbf{v}_s^T\mathbf{L}_0 = -\sum_{r=1}^{s} \mathbf{v}_{s-r}^T \frac{1}{r!}\frac{\partial^r \mathbf{L}}{\partial \lambda^r}, \quad s = 1, \ldots, \mu - 1. \tag{7.1,2}$$

Multiplying Eq. (6) by the left eigenvector $\mathbf{v}_0$ and Eq. (7) by the right eigenvector $\mathbf{u}_0$ we get the orthogonality conditions for the vectors of the right and left Keldysh chains,

$$\sum_{r=1}^{s} \frac{1}{r!}\mathbf{v}_0^T \frac{\partial^r \mathbf{L}}{\partial \lambda^r}\mathbf{u}_{s-r} = \sum_{r=1}^{s} \frac{1}{r!}\mathbf{v}_{s-r}^T \frac{\partial^r \mathbf{L}}{\partial \lambda^r}\mathbf{u}_0 = 0, \quad s = 1, \ldots, \mu - 1. \tag{8}$$

The notion of the Keldysh chain is an extension of the Jordan chain to the case of linear operators [4].

Consider a variation of the vector of parameters

$$\mathbf{p}(\epsilon) = \mathbf{p}_0 + \epsilon\dot{\mathbf{p}} + \frac{\epsilon^2}{2}\ddot{\mathbf{p}} + o(\epsilon^2), \quad \epsilon \geq 0, \tag{9}$$

where a dot indicates differentiation with respect to the small parameter ϵ and the derivatives are evaluated at $\epsilon = 0$. Then, $\mathbf{L}(\lambda, \mathbf{p}(\epsilon))$ can be represented in the form

$$\mathbf{L}(\lambda, \mathbf{p}(\epsilon)) = \sum_{r=0}^{\infty} \frac{(\lambda-\lambda_0)^r}{r!}\left(\frac{\partial^r \mathbf{L}}{\partial \lambda^r} + \epsilon\frac{\partial^r \mathbf{L}_1}{\partial \lambda^r} + \epsilon^2\frac{\partial^r \mathbf{L}_2}{\partial \lambda^r} + o(\epsilon^2)\right), \tag{10}$$

where

$$\frac{\partial^r \mathbf{L}_1}{\partial \lambda^r} = \sum_{j=1}^{n} \frac{\partial^{r+1}\mathbf{L}}{\partial \lambda^r \partial p_j}\dot{p}_j, \qquad \frac{\partial^r \mathbf{L}_2}{\partial \lambda^r} = \frac{1}{2}\sum_{j=1}^{n} \frac{\partial^{r+1}\mathbf{L}}{\partial \lambda^r \partial p_j}\ddot{p}_j + \frac{1}{2}\sum_{j,t=1}^{n} \frac{\partial^{r+2}\mathbf{L}}{\partial \lambda^r \partial p_j \partial p_t}\dot{p}_j\dot{p}_t, \tag{11}$$

and all the partial derivatives are evaluated at $\mathbf{p} = \mathbf{p}_0$, $\lambda = \lambda_0$. For r=0, formulae (11) give expressions for the operators $\mathbf{L}_1$ and $\mathbf{L}_2$.

The perturbed eigenvalue $\lambda(\epsilon)$ and eigenvector $\mathbf{u}(\epsilon)$ are expressed by means of the Newton-Puiseux series, see [31],

$$\lambda = \lambda_0 + \lambda_1\epsilon^{1/\mu} + \lambda_2\epsilon^{2/\mu} + \ldots + \lambda_{\mu-1}\epsilon^{(\mu-1)/\mu} + \lambda_\mu\epsilon + \ldots, \tag{12}$$

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{w}_1\epsilon^{1/\mu} + \mathbf{w}_2\epsilon^{2/\mu} + \ldots + \mathbf{w}_{\mu-1}\epsilon^{(\mu-1)/\mu} + \mathbf{w}_\mu\epsilon + \ldots \ . \tag{13}$$

Next, we substitute expansions (10)–(13) into the eigenvalue problem (5) and collect the terms with the same powers of the small parameter ϵ. Then, the first μ relations are

$$\mathbf{L}_0\mathbf{w}_r = -\sum_{j=0}^{r-1}\left(\sum_{s=1}^{r-j}\frac{1}{s!}\frac{\partial^s \mathbf{L}}{\partial \lambda^s}\sum_{|\alpha|_s=r-j}\lambda_{\alpha_1}\cdots\lambda_{\alpha_s}\right)\mathbf{w}_j, \ r = 1, \ldots, \mu{-}1; \tag{14}$$

$$\mathbf{L}_0\mathbf{w}_\mu = -\mathbf{L}_1\mathbf{w}_0 - \sum_{j=0}^{\mu-1}\left(\sum_{s=1}^{\mu-j}\frac{1}{s!}\frac{\partial^s \mathbf{L}}{\partial \lambda^s}\sum_{|\alpha|_s=\mu-j}\lambda_{\alpha_1}\cdots\lambda_{\alpha_s}\right)\mathbf{w}_j, \ |\alpha|_s = \alpha_1{+}\ldots{+}\alpha_s, \tag{15}$$

where $\mathbf{w}_0$ is equal to $\mathbf{u}_0$, and the indices α_1, ..., $\alpha_{\mu-1}$ are positive integers.

Comparison of Eqs. (14) with equations of the Keldysh chain (6) gives the vectors $\mathbf{w}_r$ in expansions (13):

$$\mathbf{w}_r = \sum_{j=1}^{r}\mathbf{u}_j\sum_{|\alpha|_j=r}\lambda_{\alpha_1}\cdots\lambda_{\alpha_j}, \ r = 1, \ldots, \mu - 1. \tag{16}$$

The vectors $\mathbf{w}_r$ are determined by Eq. (16) up to the linear combinations $\gamma_{r,1}\mathbf{u}_0 + \ldots + \gamma_{r,r-1}\mathbf{u}_{r-1}$, where $\gamma_{r,j}$ are arbitrary constants. However, without loss of generality we can assume that all the coefficients $\gamma_{r,j} = 0$, because otherwise all the terms with these coefficients will vanish in our further calculations due to the orthogonality conditions (8).

With the use of the vectors (16), we transform Eq. (15) into the form

$$\mathbf{L}_0\mathbf{w}_\mu = -\mathbf{L}_1\mathbf{u}_0 - \lambda_1^\mu \sum_{r=1}^{\mu} \frac{1}{r!}\frac{\partial^r \mathbf{L}}{\partial \lambda^r}\mathbf{u}_{\mu-r} + \sum_{j=1}^{\mu-1}\mathbf{L}_0\mathbf{u}_j \sum_{|\alpha|_j=\mu} \lambda_{\alpha_1}\cdots\lambda_{\alpha_j}. \tag{17}$$

Multiplying Eq. (17) by the left eigenvector $\mathbf{v}_0$ and taking into account Eq. (7.1), we get the coefficient λ_1 in expansions (12):

$$\lambda_1^\mu = -\mathbf{v}_0^T\mathbf{L}_1\mathbf{u}_0\left(\sum_{r=1}^{\mu}\frac{1}{r!}\mathbf{v}_0^T\frac{\partial^r \mathbf{L}}{\partial \lambda^r}\mathbf{u}_{\mu-r}\right)^{-1}. \tag{18}$$

Note that for $\mu = 1$ Eqs. (12) and (18) describe the increment of a simple eigenvalue λ_0 due to perturbation of the vector of parameters.

Consider in more detail the splitting of a double eigenvalue λ_0 with the Keldysh chain of length 2 due to perturbation of the vector of parameters (9). In this case the eigenvalue $\lambda(\epsilon)$ and eigenvector $\mathbf{u}(\epsilon)$ are expressed by means of the Newton-Puiseux series (12) and (13) with $\mu = 2$. Substituting these expansions along with Eqs. (10) and (11) into the eigenvalue problem (5) and collecting the terms with the same powers of ϵ, we get the equations

$$\mathbf{L}_0\mathbf{w}_1 = -\lambda_1\mathbf{L}'\mathbf{u}_0, \tag{19}$$

$$\mathbf{L}_0\mathbf{w}_2 = -\lambda_1\mathbf{L}'\mathbf{w}_1 - \lambda_2\mathbf{L}'\mathbf{u}_0 - \mathbf{L}_1\mathbf{u}_0 - \frac{\lambda_1^2}{2!}\mathbf{L}''\mathbf{u}_0, \tag{20}$$

$$\mathbf{L}_0\mathbf{w}_3 = -\lambda_1\mathbf{L}'\mathbf{w}_2 - \lambda_2\mathbf{L}'\mathbf{w}_1 - \lambda_3\mathbf{L}'\mathbf{u}_0 - \mathbf{L}_1\mathbf{w}_1 - \lambda_1\mathbf{L}_1'\mathbf{u}_0 - \frac{\lambda_1^2}{2!}\mathbf{L}''\mathbf{w}_1 - \lambda_1\lambda_2\mathbf{L}''\mathbf{u}_0 - \frac{\lambda_1^3}{3!}\mathbf{L}'''\mathbf{u}_0, \tag{21}$$

$$\begin{aligned}\mathbf{L}_0\mathbf{w}_4 = {}& -\lambda_3\mathbf{L}'\mathbf{w}_1 - \lambda_2\mathbf{L}'\mathbf{w}_2 - \mathbf{L}_1\mathbf{w}_2 - \lambda_2\mathbf{L}'_1\mathbf{u}_0 - \frac{1}{2}\lambda_2^2\mathbf{L}''\mathbf{u}_0 - \lambda_4\mathbf{L}'\mathbf{u}_0 - \mathbf{L}_2\mathbf{u}_0\\ & - \lambda_1(\mathbf{L}'\mathbf{w}_3 + \lambda_2\mathbf{L}''\mathbf{w}_1 + \mathbf{L}_1'\mathbf{w}_1 + \lambda_3\mathbf{L}''\mathbf{u}_0) - \frac{\lambda_1^2}{2!}(\mathbf{L}''\mathbf{w}_2 + \lambda_2\mathbf{L}'''\mathbf{u}_0 + \mathbf{L}_1''\mathbf{u}_0)\\ & - \frac{\lambda_1^3}{3!}\mathbf{L}'''\mathbf{w}_1 - \frac{\lambda_1^4}{4!}\mathbf{L}''''\mathbf{u}_0,\end{aligned} \tag{22}$$

where the prime $('= \partial/\partial\lambda)$ indicates the partial derivative with respect to the spectral parameter. Note that Eq. (19) follows from Eq. (14) with $r = 1$ and Eq. (20) is a particular case of Eq. (15) for $\mu = 2$. Thus, the coefficient λ_1 in expansion (12) is given by formula (18) where one should take $\mu = 2$,

$$\lambda_1^2 = -\frac{\mathbf{v}_0^T\mathbf{L}_1\mathbf{u}_0}{\mathbf{v}_0^T\mathbf{L}'\mathbf{u}_1 + \frac{1}{2}\mathbf{v}_0^T\mathbf{L}''\mathbf{u}_0}. \tag{23}$$

Formula (23) was derived earlier in [8] and [23] for a matrix polynomial $\mathbf{L}$ of degree 2.

From Eq. (19) it follows that $\mathbf{w}_1 = \lambda_1\mathbf{u}_1 + \gamma\mathbf{u}_0$, where γ is an unknown constant. To get the coefficient λ_2, one needs first to substitute the vector $\mathbf{w}_1$ in the explicit form into Eqs. (20) and (21), then multiply Eq. (20) by the left associated vector $\mathbf{v}_1$ to find the quantity $\mathbf{v}_0^T\mathbf{L}'\mathbf{w}_2$. Finally, Eq. (21) is multiplied by the left eigenvector $\mathbf{v}_0$, the term $\mathbf{v}_0^T\mathbf{L}'\mathbf{w}_2$ is substituted into the expression obtained, and taking into account Eqs. (8) and (23), the coefficient λ_2 is isolated,

$$\lambda_2 = -\frac{\mathbf{v}_1^T\mathbf{L}_1\mathbf{u}_0 + \mathbf{v}_0^T\mathbf{L}_1\mathbf{u}_1 + \mathbf{v}_0^T\mathbf{L}_1'\mathbf{u}_0 + \lambda_1^2 Q}{2\mathbf{v}_0^T\mathbf{L}'\mathbf{u}_1 + \mathbf{v}_0^T\mathbf{L}''\mathbf{u}_0}, \tag{24}$$

$$Q = \mathbf{v}_1^T\mathbf{L}'\mathbf{u}_1 + \frac{1}{2!}(\mathbf{v}_1^T\mathbf{L}''\mathbf{u}_0 + \mathbf{v}_0^T\mathbf{L}''\mathbf{u}_1) + \frac{1}{3!}\mathbf{v}_0^T\mathbf{L}'''\mathbf{u}_0. \tag{25}$$

If the vectors $\mathbf{u}_0$, $\mathbf{v}_0$, $\mathbf{u}_1$, $\mathbf{v}_1$ are chosen so that Q=0 we can rewrite Eq. (24) in the form

$$\lambda_2 = -\frac{\mathbf{v}_1^T\mathbf{L}_1\mathbf{u}_0 + \mathbf{v}_0^T\mathbf{L}_1\mathbf{u}_1 + \mathbf{v}_0^T\mathbf{L}_1'\mathbf{u}_0}{2\mathbf{v}_0^T\mathbf{L}'\mathbf{u}_1 + \mathbf{v}_0^T\mathbf{L}''\mathbf{u}_0}. \tag{26}$$

Therefore, the double eigenvalue λ_0 of the linear matrix operator $\mathbf{L}(\lambda, \mathbf{p}(\epsilon))$ splits in the case of general position according to the formula

$$\lambda(\epsilon) = \lambda_0 + \lambda_1\epsilon^{1/2} + \lambda_2\epsilon + o(\epsilon), \tag{27}$$

with the coefficients λ_1 and λ_2 from Eqs. (23)–(26).

The case when the coefficient λ_1 equals zero in Eq. (27) is referred to as degenerate [31] and should be investigated separately. Substituting λ_1=0 into Eqs. (19)–(22) we find

$$\mathbf{L}_0\mathbf{w}_1=0, \quad \mathbf{L}_0\mathbf{w}_2=-\lambda_2\mathbf{L}'\mathbf{u}_0-\mathbf{L}_1\mathbf{u}_0, \tag{28.1, 2}$$

$$\mathbf{L}_0\mathbf{w}_4=-\lambda_3\mathbf{L}'\mathbf{w}_1-\lambda_2\mathbf{L}'\mathbf{w}_2-\mathbf{L}_1\mathbf{w}_2-\lambda_2\mathbf{L}'_1\mathbf{u}_0-\frac{1}{2}\lambda_2^2\mathbf{L}''\mathbf{u}_0-\lambda_4\mathbf{L}'\mathbf{u}_0-\mathbf{L}_2\mathbf{u}_0. \tag{29}$$

Solving Eqs. (28) yields the vectors $\mathbf{w}_1$ and $\mathbf{w}_2$:

$$\mathbf{w}_1 = \beta\mathbf{u}_0, \quad \mathbf{w}_2 = \lambda_2\mathbf{u}_1 + \gamma\mathbf{u}_0 - \mathbf{S}_0(\mathbf{L}_1\mathbf{u}_0), \tag{30}$$

where β and γ are unknown constants, and $\mathbf{S}_0$ is the operator inverse to $\mathbf{L}_0$. Next we multiply Eq. (29) by the left eigenvector $\mathbf{v}_0$ and substitute the quantity $\mathbf{v}_0^T\mathbf{L}'\mathbf{w}_2$ obtained from the multiplication of Eq. (28.2) by the left associated vector $\mathbf{v}_1$ into the result. After this transformation we substitute the vectors (30) into Eq. (29). Finally, taking into account Eqs. (8) with μ=2 and the degeneration condition $\mathbf{v}_0^T\mathbf{L}_1\mathbf{u}_0$=0, we arrive at the quadratic equation serving for the determination of the coefficient λ_2 in the degenerate case,

$$\lambda_2^2+\lambda_2\frac{\mathbf{v}_1^T\mathbf{L}_1\mathbf{u}_0+\mathbf{v}_0^T\mathbf{L}_1\mathbf{u}_1+\mathbf{v}_0^T\mathbf{L}_1'\mathbf{u}_0}{\mathbf{v}_0^T\mathbf{L}'\mathbf{u}_1+\frac{1}{2}\mathbf{v}_0^T\mathbf{L}''\mathbf{u}_0}+\frac{\mathbf{v}_0^T\mathbf{L}_2\mathbf{u}_0-\mathbf{v}_0^T\mathbf{L}_1\mathbf{S}_0(\mathbf{L}_1\mathbf{u}_0)}{\mathbf{v}_0^T\mathbf{L}'\mathbf{u}_1+\frac{1}{2}\mathbf{v}_0^T\mathbf{L}''\mathbf{u}_0}=0. \tag{31}$$

Thus, the double eigenvalue splits in the degenerate case according to the formula $\lambda = \lambda_0 + \epsilon\lambda_2 + o(\epsilon)$ with the coefficient λ_2 determined from Eq. (31). Note that the obtained formulae generalize the results on bifurcation of eigenvalues derived earlier in [25], [29], and [32]–[35].

3 Effect of small velocity-dependent perturbation on the spectrum of a circulatory system

Let us now return to the general non-conservative system described by Eq. (1). In the n-dimensional space of the system parameters $k_1, \ldots, k_{n-1}, q$, consider a point $\mathbf{p}_0$=$(0, \ldots, 0, q_0)$. Assume that $\pm i\omega_0$, $\omega_0 > 0$, are double eigenvalues with the Keldysh chains of length $\mu = 2$ of the operator $\mathbf{L}(\lambda, 0, q_0) = \mathbf{A}_0 + \lambda^2\mathbf{M}$, where $\mathbf{A}_0 = \mathbf{A}(q_0)$ and the operator $\mathbf{L}(\lambda, \mathbf{k}, q)$ is defined by Eq. (2). The remaining eigenvalues $\pm i\omega_{0,s}$, $\omega_{0,s}$>0, s=1, …, m−2, are assumed to be simple. The non-conservative system corresponding to

$\mathbf{k}$=0, q=q_0 is a circulatory system described by Eq. (3) and the point $\mathbf{p}_0$ belongs to the boundary between its stability and flutter domains.

The right and left eigenvectors $\mathbf{u}_0$, $\mathbf{v}_0$ and associated vectors $\mathbf{u}_1$, $\mathbf{v}_1$ of the double eigenvalue $i\omega_0$ satisfy Eqs. (6) and (7). With the use of the relations

$$\mathbf{L}_0{=}\mathbf{A}_0{-}\omega_0^2\mathbf{M},\quad \left.\frac{\partial \mathbf{L}}{\partial \lambda}\right|_{\substack{\mathbf{p}=\mathbf{p}_0\\ \lambda=i\omega_0}}{=}2i\omega_0\mathbf{M},\quad \left.\frac{\partial^2 \mathbf{L}}{\partial \lambda^2}\right|_{\substack{\mathbf{p}=\mathbf{p}_0\\ \lambda=i\omega_0}}=2\mathbf{M},\quad \left.\frac{\partial^3 \mathbf{L}}{\partial \lambda^3}\right|_{\substack{\mathbf{p}=\mathbf{p}_0\\ \lambda=i\omega_0}}{=}0, \tag{32}$$

valid for the operator $\mathbf{L}(\lambda, \mathbf{k}, q)$ defined by Eq. (2), we transform Eqs. (6) and (7) into the following form:

$$(\mathbf{A}_0 - \omega_0^2\mathbf{M})\mathbf{u}_0 = 0, \quad (\mathbf{A}_0 - \omega_0^2\mathbf{M})\mathbf{u}_1 = -2i\omega_0\mathbf{M}\mathbf{u}_0, \tag{33}$$

$$\mathbf{v}_0^T(\mathbf{A}_0 - \omega_0^2\mathbf{M}) = 0, \quad \mathbf{v}_1^T(\mathbf{A}_0 - \omega_0^2\mathbf{M}) = -2i\omega_0\mathbf{v}_0^T\mathbf{M}. \tag{34}$$

Since the vectors $\mathbf{u}_0$, $\mathbf{v}_0$ are defined up to arbitrary multipliers and the vectors $\mathbf{u}_1$, $\mathbf{v}_1$ are defined up to the terms $\gamma_1\mathbf{u}_0$, $\gamma_2\mathbf{v}_0$, respectively, where γ_1 and γ_2 are arbitrary constants, we choose the real vectors $\mathbf{u}_0$, $\mathbf{v}_0$ and purely imaginary vectors $\mathbf{u}_1$, $\mathbf{v}_1$ satisfying the normalization and orthogonality conditions

$$2i\omega_0\mathbf{v}_0^T\mathbf{M}\mathbf{u}_1 = 1, \qquad 2i\omega_0\mathbf{v}_1^T\mathbf{M}\mathbf{u}_1 + \mathbf{v}_1^T\mathbf{M}\mathbf{u}_0 + \mathbf{v}_0^T\mathbf{M}\mathbf{u}_1 = 0. \tag{35}$$

Besides, the left and right eigenvectors satisfy the orthogonality condition $\mathbf{v}_0^T\mathbf{M}\mathbf{u}_0$=0 following from Eq. (8) for $\mu = 2$.

Let us study how the stability of system (1) depends on the *linear* perturbation of the vector of parameters $\mathbf{p}$=$(\mathbf{k}, q)$,

$$\mathbf{p}(\epsilon) = \mathbf{p}_0 + \epsilon\dot{\mathbf{p}}, \quad \epsilon \geq 0, \tag{36}$$

where the dot over a symbol indicates the derivative with respect to ϵ evaluated at $\epsilon = 0$. In the case of general position the perturbed double eigenvalue is determined by the Newton-Puiseux series (12) with $\mu = 2$,

$$\lambda = i\omega_0 + \epsilon^{1/2}\lambda_1 + \epsilon\lambda_2 + \dots\ . \tag{37}$$

Substituting the operator $\mathbf{L}$ given by Eq. (2) into Eqs. (11), (23) and (24), and taking into account the orthogonality and normalization conditions (35) we get the coefficients λ_1 and λ_2,

$$\lambda_1^2 = -i\omega_0\langle \mathbf{f}, \dot{\mathbf{k}}\rangle - \widetilde{f}\dot{q}, \quad 2\lambda_2 = -\langle \mathbf{f}{-}\omega_0\mathbf{h}, \dot{\mathbf{k}}\rangle - i\widetilde{h}\dot{q}. \tag{38.1,2}$$

In Eqs. (38) the vector $\dot{\mathbf{k}}$=$(\dot{k}_1, \dots, \dot{k}_{n-1})$, the real vectors $\mathbf{f}$, $\mathbf{h}$ have the components

$$f_r = \mathbf{v}_0^T\frac{\partial \mathbf{D}}{\partial k_r}\mathbf{u}_0, \quad ih_r = \mathbf{v}_1^T\frac{\partial \mathbf{D}}{\partial k_r}\mathbf{u}_0 + \mathbf{v}_0^T\frac{\partial \mathbf{D}}{\partial k_r}\mathbf{u}_1, \quad r = 1, \dots, n-1, \tag{39.1,2}$$

the angular brackets denote the inner product of real vectors in $\mathbb{R}^{n-1}$, and the real scalars $\widetilde{f}$, $\widetilde{h}$ are defined by the expressions

$$\widetilde{f} = \mathbf{v}_0^T\frac{\partial \mathbf{A}}{\partial q}\mathbf{u}_0, \quad i\widetilde{h} = \mathbf{v}_1^T\frac{\partial \mathbf{A}}{\partial q}\mathbf{u}_0 + \mathbf{v}_0^T\frac{\partial \mathbf{A}}{\partial q}\mathbf{u}_1. \tag{40}$$

Thus, from Eqs. (36)–(38) we obtain

$$\lambda = i\omega_0 \pm \sqrt{-i\omega_0\langle \mathbf{f}, \mathbf{k}\rangle - \widetilde{f}(q{-}q_0)} - \frac{1}{2}\Big(\langle \mathbf{f}{-}\omega_0\mathbf{h}, \mathbf{k}\rangle + i\widetilde{h}(q{-}q_0)\Big) + o(\|\mathbf{p}{-}\mathbf{p}_0\|). \tag{41}$$

Equation (41) describes splitting of the double eigenvalue $i\omega_0$ with a change of the parameters $\mathbf{k}$=$(k_1, \dots, k_{n-1})$ and q in the case when the radicand is not zero.

If $\mathbf{k} = 0$ the double eigenvalue splits into a pair of purely imaginary eigenvalues (stability) for $\widetilde{f}(q - q_0) > 0$. Let us assume that $\widetilde{f} < 0$. Then the circulatory system (3) is stable for $q < q_0$ and unstable for $q > q_0$. The case $\widetilde{f} = 0$ is degenerate and will not be considered.

For a fairly small variation of the parameters $\mathbf{k}$ and q the double eigenvalue $i\omega_0$ splits in general into two simple complex-conjugate eigenvalues, one of them with positive real part (flutter instability). However, if $\langle \mathbf{f}, \mathbf{k} \rangle = 0$ and $\langle \mathbf{h}, \mathbf{k} \rangle < 0$, then for $q < q_0$ the radicand in Eq. (41) is purely imaginary, and for a sufficiently small perturbation of parameters the double eigenvalue $i\omega_0$ (as well as $-i\omega_0$) splits into two simple eigenvalues with negative real parts.

The asymptotic stability of system (1) after perturbation (36) also depends on the behavior of the remaining $2m - 4$ simple purely imaginary eigenvalues $\pm i\omega_{0,s}$. We choose the real right $\mathbf{u}_{0,s}$ and left $\mathbf{v}_{0,s}$ eigenvectors of the eigenvalues $i\omega_{0,s}$ satisfying the normalization conditions

$$2\omega_{0,s}\mathbf{v}_{0,s}^T\mathbf{M}\mathbf{u}_{0,s} = 1. \tag{42}$$

According to Eqs. (11), (12) and (18) with $\mu=1$ the increments of the eigenvalues $\pm i\omega_{0,s}$ due to change of parameters are determined by the expansions

$$\lambda = \pm i\omega_{0,s} \mp i\widetilde{g}_s(q - q_0) - \omega_{0,s}\langle \mathbf{g}_s, \mathbf{k} \rangle + o(\|\mathbf{p} - \mathbf{p}_0\|^2), \quad s = 1, \ldots, m-2, \tag{43}$$

where the real scalar $\widetilde{g}_s$ and the components of the real vector $\mathbf{g}_s$ are

$$\widetilde{g}_s = \mathbf{v}_{0,s}^T \frac{\partial \mathbf{A}}{\partial q}\mathbf{u}_{0,s}, \quad g_{s,r} = \mathbf{v}_{0,s}^T \frac{\partial \mathbf{D}}{\partial k_r}\mathbf{u}_{0,s}, \quad r = 1, \ldots, n-1. \tag{44}$$

The sufficient condition for $\mathrm{Re}\lambda_s$ to be negative is $\langle \mathbf{g}_s, \mathbf{k} \rangle > 0$. Therefore, the system (1) is asymptotically stable for fairly small linear perturbations of the parameters $\mathbf{k}$, q, given by Eq. (36), if the following conditions are satisfied:

$$\langle \mathbf{f}, \mathbf{k} \rangle = 0, \quad q < q_0, \quad \langle \mathbf{h}, \mathbf{k} \rangle < 0, \quad \langle \mathbf{g}_s, \mathbf{k} \rangle > 0, \quad s = 1, \ldots, m-2. \tag{45}$$

The inequalities (45) show that the set of directions leading from the point $\mathbf{p}_0$ to the domain of asymptotic stability has the dimension $n-1$ in the n-dimensional space of the system parameters $k_1, \ldots, k_{n-1}, q$. However, it is known that for the multiparameter families of linear matrix operators of the general type the dimension of the asymptotic stability domain coincides with that of the parameter space [30]. Therefore, starting from the point $\mathbf{p}_0$ it is possible to reach other points of the asymptotic stability domain only following the curves which are tangential to the plane $\langle \mathbf{f}, \mathbf{k} \rangle = 0$ at $\mathbf{p}_0$.

To obtain more accurate information on the geometry of the stability domain in the vicinity of the point $\mathbf{p}_0 = (0, \ldots, 0, q_0)$ we consider a variation of the vector of parameters along a smooth curve

$$\mathbf{p}(\epsilon) = \begin{bmatrix} 0 \\ q_0 \end{bmatrix} + \epsilon \begin{bmatrix} \dot{\mathbf{k}} \\ 0 \end{bmatrix} + \frac{\epsilon^2}{2} \begin{bmatrix} \ddot{\mathbf{k}} \\ \ddot{q} \end{bmatrix} + o(\epsilon^2), \tag{46}$$

assuming that

$$\langle \mathbf{f}, \dot{\mathbf{k}} \rangle = 0. \tag{47}$$

The curve (46), (47) is orthogonal to the axis q in the space of the parameters $\mathbf{k}, q$ because $\dot{q} \equiv 0$.

The coefficient λ_1 in expansion (37) determined by Eq. (38.1) vanishes along the curve (46), (47). Thus, the double eigenvalue $i\omega_0$ in this degenerate case splits linearly with respect to ϵ:

$$\lambda = i\omega_0 + \lambda_2\epsilon + o(\epsilon). \tag{48}$$

The coefficient λ_2 is a root of the quadratic equation (31), which for the operator $\mathbf{L}$ given by Eq. (2) and eigen- and associated vectors satisfying conditions (35) takes the form

$$\lambda_2^2 - \lambda_2 \quad \omega_0\langle \mathbf{h}, \dot{\mathbf{k}}\rangle + \left(\frac{1}{2}\widetilde{f}\ddot{q} + \omega_0^2\langle \mathbf{G}\dot{\mathbf{k}}, \dot{\mathbf{k}}\rangle\right) + i\omega_0\left(\frac{1}{2}\langle \mathbf{f}, \ddot{\mathbf{k}}\rangle + \langle \mathbf{H}\dot{\mathbf{k}}, \dot{\mathbf{k}}\rangle\right) = 0. \tag{49}$$

The real vectors $\mathbf{f}$, $\mathbf{h}$ and scalars $\widetilde{f}$, $\widetilde{h}$ in the coefficients of the polynomial (49) are determined by Eqs. (39) and (40), the real matrix $\mathbf{H}$ has the components

$$H_{rs} = \frac{1}{2}\mathbf{v}_0^T \frac{\partial^2 \mathbf{D}}{\partial k_r \partial k_s}\mathbf{u}_0, \quad r, s = 1, \ldots, n-1, \tag{50}$$

and the real matrix $\mathbf{G}$ is defined by the expression

$$\langle \mathbf{G}\dot{\mathbf{k}}, \dot{\mathbf{k}}\rangle = \sum_{r=1}^{n-1} \dot{k}_r \mathbf{v}_0^T \frac{\partial \mathbf{D}}{\partial k_r}\mathbf{S}_0\left(\sum_{s=1}^{n-1}\dot{k}_s \frac{\partial \mathbf{D}}{\partial k_s}\mathbf{u}_0\right), \tag{51}$$

where $\mathbf{S}_0 = (\mathbf{A}_0 - \omega_0^2\mathbf{M} - 2i\omega_0\mathbf{v}_0\mathbf{v}_1^T\mathbf{M} - 2\mathbf{v}_0\mathbf{v}_0^T\mathbf{M})^{-1}$ is the operator inverse to $\mathbf{L}_0 = \mathbf{A}_0 - \omega_0^2\mathbf{M}$ [36].

In view of Eqs. (46) and (47), which explicitly specify the curve $\mathbf{p}(\epsilon)$, and the expansion (48), Eq. (49) is represented in the form

$$(\lambda - i\omega_0)^2 - \omega_0\langle \mathbf{h}, \mathbf{k}\rangle(\lambda - i\omega_0) + \widetilde{f}(q - q_0) + \omega_0^2\langle \mathbf{G}\mathbf{k}, \mathbf{k}\rangle + i\omega_0(\langle \mathbf{f}, \mathbf{k}\rangle + \langle \mathbf{H}\mathbf{k}, \mathbf{k}\rangle) = 0. \tag{52}$$

Equation (52) describes splitting of the double eigenvalue $i\omega_0$ due to small perturbation of the parameters $\mathbf{k}$ and q. To investigate this process in detail we substitute $\lambda = \mathrm{Re}\lambda + i\mathrm{Im}\lambda$ into Eq. (52) and separate real and imaginary parts. Transforming the real and imaginary parts of Eq. (52) we get the following relations:

$$(\mathrm{Im}\lambda - \omega_0 + \mathrm{Re}\lambda + a/2)^2 - (\mathrm{Im}\lambda - \omega_0 - \mathrm{Re}\lambda - a/2)^2 = -2d, \tag{53}$$

$$\left(\mathrm{Re}\lambda + \frac{a}{2}\right)^4 + \left(c - \frac{a^2}{4}\right)\left(\mathrm{Re}\lambda + \frac{a}{2}\right)^2 = \frac{d^2}{4}, \tag{54}$$

$$(\mathrm{Im}\lambda - \omega_0)^4 - \left(c - \frac{a^2}{4}\right)(\mathrm{Im}\lambda - \omega_0)^2 = \frac{d^2}{4}, \tag{55}$$

where

$$a = -\omega_0\langle \mathbf{h}, \mathbf{k}\rangle, \quad c = \widetilde{f}(q - q_0) + \omega_0^2\langle \mathbf{G}\mathbf{k}, \mathbf{k}\rangle, \quad d = \omega_0(\langle \mathbf{f}, \mathbf{k}\rangle + \langle \mathbf{H}\mathbf{k}, \mathbf{k}\rangle). \tag{56.1 – 3}$$

Consider first the case when the system is circulatory ($\mathbf{k}=0$). Then, according to Eq. (56) the quantities $a=0$, $c=\widetilde{f}(q - q_0)$, $d=0$, and Eqs. (54), (55) take the form

$$q \le q_0: \quad \mathrm{Re}\lambda = 0, \quad \mathrm{Im}\lambda = \omega_0 \pm \sqrt{\widetilde{f}(q - q_0)}; \tag{57}$$

$$q \ge q_0: \quad \mathrm{Re}\lambda = \pm\sqrt{-\widetilde{f}(q - q_0)}, \mathrm{Im}\lambda = \omega_0. \tag{58}$$

Equations (57) and (58) show that with an increase in the load parameter q two simple purely imaginary eigenvalues move along the imaginary axis, collide at $q=q_0$ and then diverge in the direction perpendicular to the imaginary axis with the origination of a pair of simple complex-conjugate eigenvalues (flutter instability). Such a behavior of eigenvalues is known as the strong interaction and is typical for circulatory systems [5]. The trajectories of eigenvalues of a circulatory system with a change of the parameter q are shown in Figs. 2 and 3 by the fine lines.

If $\mathbf{k}\neq 0$ and $d\neq 0$, then dissipative and gyroscopic forces destroy the strong interaction of eigenvalues shifting and splitting their trajectories as shown in Figs. 1 and 2. This qualitative

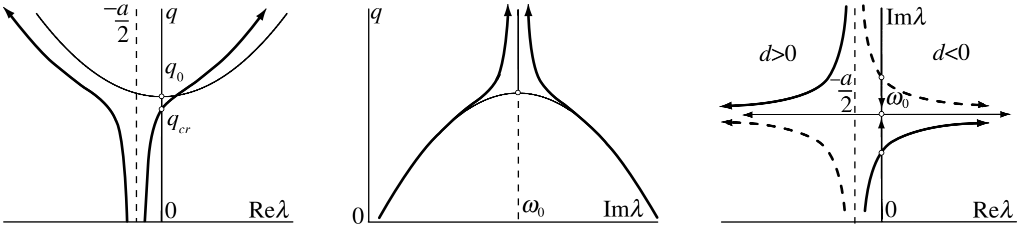

Fig. 2. Trajectories of eigenvalues of a circulatory system (fine lines) and the system with small velocity-dependent forces (bold lines) for $d \neq 0$

effect known in the literature only from the numerical analysis of the specific mechanical examples [2], [8], [10] is described here analytically by Eqs. (53)–(55).

Indeed, for a fixed $\mathbf{k} \neq 0$ with a change of the parameter q the eigenvalues move in the complex plane along the branches of the hyperbola given by Eq. (53). This hyperbola has two asymptotes Re$\lambda = -a/2$ and Im$\lambda = \omega_0$ where a is determined by Eq. (56.1). If $a > 0$, then one of the two eigenvalues is in the left-hand half of the complex plane while another one passes through the imaginary axis to the right when $q = q_{cr}(\mathbf{k})$. Thus, $a > 0$ or equivalently $\langle \mathbf{h}, \mathbf{k} \rangle < 0$ is a necessary condition for asymptotic stability.

Equations (54) and (55) describe the real and imaginary parts of eigenvalues λ as functions of the parameters q and $\mathbf{k}$. The functions Re$\lambda(q)$ and Im$\lambda(q)$ for $\mathbf{k} \neq 0$ are shown in Fig. 2 by the bold lines. The value of the parameter q at which one of the eigenvalues crosses the imaginary axis follows from Eq. (54) if we assume there Reλ=0. This yields the relation $ca^2 = d^2$, which after taking into account the explicit expressions for a, c, and d from Eqs. (56) takes the form

$$q_{cr}(\mathbf{k}) = q_0 + \frac{(\langle \mathbf{f}, \mathbf{k} \rangle + \langle \mathbf{Hk}, \mathbf{k} \rangle)^2}{\tilde{f} \langle \mathbf{h}, \mathbf{k} \rangle^2} - \frac{\omega_0^2}{\tilde{f}} \langle \mathbf{Gk}, \mathbf{k} \rangle. \tag{59}$$

Thus, both eigenvalues belong to the left-hand half of the complex plane if

$$q < q_0 + \frac{(\langle \mathbf{f}, \mathbf{k} \rangle + \langle \mathbf{Hk}, \mathbf{k} \rangle)^2}{\tilde{f} \langle \mathbf{h}, \mathbf{k} \rangle^2} - \frac{\omega_0^2}{\tilde{f}} \langle \mathbf{Gk}, \mathbf{k} \rangle, \tag{60}$$

$$\langle \mathbf{h}, \mathbf{k} \rangle < 0. \tag{61}$$

The necessary and sufficient conditions (60), (61) for all the roots of the complex polynomial (52) to have negative real parts can also be obtained with the use of the Bilharz criterion [37], which is the analogue of the Routh–Hurwitz conditions for complex polynomials.

Note that an equation similar to (53) was obtained by Seyranian and Pedersen [25], [26]. However, according to their approximation the eigenvalue trajectories due to small velocity-dependent perturbation can only split without shifting (a=0). Due to that reason Seyranian and Pedersen did not calculate the critical load $q_{cr}(\mathbf{k})$.

In the case $d \equiv \omega_0(\langle \mathbf{f}, \mathbf{k} \rangle + \langle \mathbf{Hk}, \mathbf{k} \rangle) = 0$ the strong interaction of the eigenvalues is preserved with the introduction of small velocity-dependent forces ($\mathbf{k} \neq 0$). According to formulae (54) and (55), which for this case take the form

$$q \le q_* : \quad \text{Re}\lambda = \omega_0 \frac{\langle \mathbf{h}, \mathbf{k} \rangle}{2}, \qquad \text{Im}\lambda = \omega_0 \pm \sqrt{\tilde{f}(q - q_*)}, \tag{62}$$

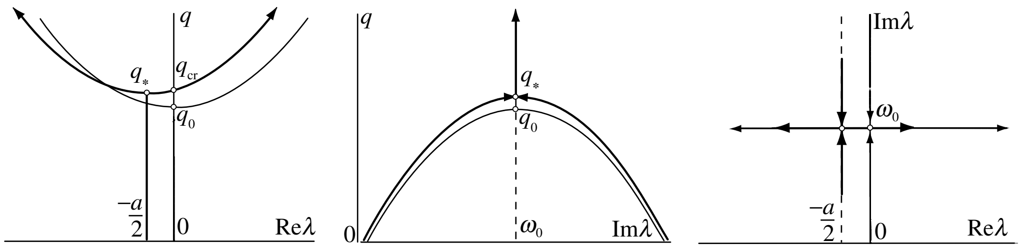

Fig. 3. Trajectories of eigenvalues of a circulatory system (fine lines) and the system with small velocity-dependent forces (bold lines) for $d = 0$

$$q \geq q_* : \quad \mathrm{Re}\lambda = \omega_0 \frac{\langle \mathbf{h}, \mathbf{k} \rangle}{2} \pm \sqrt{-\tilde{f}(q - q_*)}, \qquad \mathrm{Im}\lambda = \omega_0, \tag{63}$$

the complex eigenvalues λ with $\mathrm{Re}\lambda = -a/2$ interact strongly at $q = q_*$, where

$$q_* = q_0 + \omega_0^2 \frac{\langle \mathbf{h}, \mathbf{k} \rangle^2 - 4\langle \mathbf{Gk}, \mathbf{k} \rangle}{4\tilde{f}}. \tag{64}$$

With the further increase in the parameter q the double eigenvalue $\lambda_* = -a/2 + i\omega_0$ splits into two simple complex-conjugate eigenvalues, and one of them crosses the imaginary axis at $q = q_{cr}(\mathbf{k})$ given by Eq. (59), which takes here the form

$$q_{cr}(\mathbf{k}) = q_0 - \frac{\omega_0^2}{\tilde{f}} \langle \mathbf{Gk}, \mathbf{k} \rangle. \tag{65}$$

We can conclude that in the case $d=0$ small dissipative and gyroscopic forces just shift the picture of the strong interaction of eigenvalues off the imaginary axis, as shown in Fig. 3 for $a > 0$. As in the previous case ($d \neq 0$) both eigenvalues are in the left-hand half of the complex plane if conditions (60) and (61) are satisfied simultaneously. If, additionally, $\langle \mathbf{Gk}, \mathbf{k} \rangle > 0$, then according to Eq. (65) the critical load is increased in the presence of small velocity-dependent forces (stabilization).

4 Function of the critical load $q_{cr}(\mathbf{k})$ and its properties

Consider the function of the critical load $q_{cr}(\mathbf{k})$ in more detail. We restrict our further consideration to the case when

$$\{\mathbf{k} : \langle \mathbf{f}, \mathbf{k} \rangle = 0, \langle \mathbf{h}, \mathbf{k} \rangle < 0\} \subset \{\mathbf{k} : \langle \mathbf{g}_s, \mathbf{k} \rangle > 0, s = 1, \ldots, m-2\}, \tag{66}$$

meaning that all simple eigenvalues $\pm i\omega_{0,s}$ move into the left half of the complex plane due to small perturbation of the system parameters. Thus, stability of system (1) depends only on the splitting of the double eigenvalues $\pm i\omega_0$. Therefore, the surface $q_{cr}(k_1, \ldots, k_{n-1})$ approximated by Eq. (59) under constraint (61) is the boundary of the domain of asymptotic stability approximated by inequalities (60) and (61).

The function $q_{cr}(\mathbf{k})$ given by Eq. (59) consists of rational and polynomial terms. The rational term has squared linear forms with respect to the vector $\mathbf{k}$ both in the numerator and denominator. Thus, the function $q_{cr}(\mathbf{k})$ is singular at the point $\mathbf{k} = 0$, and the critical load as a function of $n - 1$ variables has no limit when $\mathbf{k} = (k_1, \ldots, k_{n-1})$ tends to zero. This fact was first established for the critical load of the Ziegler–Herrman–Jong pendulum in [26] and [27] but it was not known for arbitrary linear non-conservative systems.

However, the homogeneity property of the numerator and denominator of the rational term of $q_{cr}(\mathbf{k})$ guarantees the existence of $\lim_{\epsilon\to 0} q_{cr}(\epsilon\widetilde{\mathbf{k}})$ for any direction $\widetilde{\mathbf{k}}$ such that $\langle \mathbf{h}, \widetilde{\mathbf{k}}\rangle \neq 0$. Substituting $\mathbf{k} = \epsilon\widetilde{\mathbf{k}}$ into Eq. (59) we find the explicit expression describing approximately a *jump in the critical load* due to small dissipative and gyroscopic forces,

$$\Delta q \equiv q_0 - \lim_{\epsilon\to 0} q_{cr}(\epsilon\widetilde{\mathbf{k}}) = -\frac{1}{\widetilde{\widetilde{f}}} \frac{\langle \mathbf{f}, \widetilde{\mathbf{k}}\rangle^2}{\langle \mathbf{h}, \widetilde{\mathbf{k}}\rangle^2}. \tag{67}$$

If $\langle \mathbf{f}, \widetilde{\mathbf{k}}\rangle$=0, then $\lim_{\epsilon\to 0} q_{cr}(\epsilon\widetilde{\mathbf{k}}) = q_0$. For the two-dimensional vector $\mathbf{k}$=(k_1, k_2) this condition gives the ratio of the parameters k_1 and k_2 for which small velocity-dependent forces do not destabilize a circulatory system as

$$\frac{k_i}{k_j} = -\frac{f_j}{f_i}, \quad i,j = 1,2, \tag{68}$$

where f_1,f_2 are determined by Eq. (39.1). The strong influence of the ratio of the dissipation parameters on the critical load was first noted in [2] and [6].

In the classical formulation [16]–[27], when the operator of dissipative and gyroscopic forces has the form $k\mathbf{D}$, where k is a scalar parameter and the matrix $\mathbf{D}$ is constant, the equation of motion of the non-conservative system (1) has the form

$$\frac{d^2\mathbf{y}}{dt^2} + k\mathbf{D}\frac{d\mathbf{y}}{dt} + \mathbf{A}(q)\mathbf{y} = 0. \tag{69}$$

Without loss of generality it is assumed in Eq. (69) that $\mathbf{M}$ is the identity matrix. Calculating the vectors $\mathbf{f}$, $\mathbf{h}$, and the scalar $\widetilde{f}$ by Eqs. (39) and (40) we get from Eq. (67) the explicit expression for the jump of the critical load in the non-conservative system (69)

$$\Delta q = \frac{1}{\mathbf{v}_0^T\mathbf{A}_1\mathbf{u}_0}\left(\frac{\mathbf{v}_0^T\mathbf{D}\mathbf{u}_0}{\mathbf{v}_0^T\mathbf{D}\mathbf{u}_1 + \mathbf{v}_1^T\mathbf{D}\mathbf{u}_0}\right)^2, \tag{70}$$

where $\mathbf{A}_1$=$d\mathbf{A}/dq$, and the derivative is evaluated at q=q_0.

In a particular case when system (69) has 2 degrees of freedom, the eigen- and associated vectors of the double eigenvalue $i\omega_0$ in Eq. (70) can be expressed in terms of the entries a_{ij}, $i,j = 1,2$, of the matrix $\mathbf{A}_0$=$\mathbf{A}(q_0)$:

$$\mathbf{u}_0 = \frac{1}{4a_{12}\mathrm{tr}\mathbf{A}_0}\begin{bmatrix} 2a_{12} \\ a_{22}-a_{11}\end{bmatrix}, \qquad \mathbf{v}_0 = \frac{1}{2}\begin{bmatrix} a_{11}-a_{22} \\ 2a_{12}\end{bmatrix},$$

$$\mathbf{u}_1 = \frac{-i\sqrt{2\mathrm{tr}\mathbf{A}_0}}{4a_{12}(\mathrm{tr}\mathbf{A}_0)^2}\begin{bmatrix} -2a_{12} \\ 3a_{11}+a_{22}\end{bmatrix}, \qquad \mathbf{v}_1 = -i\sqrt{2\mathrm{tr}\mathbf{A}_0}\begin{bmatrix} 1 \\ 0\end{bmatrix}, \quad \mathrm{tr}\mathbf{A}_0 = 2\omega_0^2, \quad a_{12} \neq 0. \tag{71}$$

Using the vectors given by Eqs. (71) we obtain

$$\mathbf{v}_0^T\mathbf{A}_1\mathbf{u}_0 = \frac{2\mathrm{tr}\mathbf{A}_0\mathbf{A}_1 - \mathrm{tr}\mathbf{A}_0\mathrm{tr}\mathbf{A}_1}{4\mathrm{tr}\mathbf{A}_0}, \qquad \mathbf{v}_0^T\mathbf{D}\mathbf{u}_0 = \frac{2\mathrm{tr}\mathbf{A}_0\mathbf{D} - \mathrm{tr}\mathbf{A}_0\mathrm{tr}\mathbf{D}}{4\mathrm{tr}\mathbf{A}_0},$$

$$\mathbf{v}_0^T\mathbf{D}\mathbf{u}_1 + \mathbf{v}_1^T\mathbf{D}\mathbf{u}_0 = i\frac{2\mathrm{tr}\mathbf{A}_0\mathbf{D} - 3\mathrm{tr}\mathbf{A}_0\mathrm{tr}\mathbf{D}}{4(\mathrm{tr}\mathbf{A}_0)^2}\sqrt{2\mathrm{tr}\mathbf{A}_0}. \tag{72}$$

The case $a_{12} = 0$ is considered in the same way and yields the same results. Substituting the quantities (72) into Eq. (70) we find that in the non-conservative system (69) with two degrees of freedom the jump in the critical load caused by small velocity-dependent forces is approximated by the expression

$$\Delta q \equiv q_0 - \lim_{k\to 0} q_{cr}(k) = \frac{-2(\mathrm{tr}\mathbf{A}_0)^2}{2\mathrm{tr}\mathbf{A}_0\mathbf{A}_1 - \mathrm{tr}\mathbf{A}_0\mathrm{tr}\mathbf{A}_1}\left(\frac{2\mathrm{tr}\mathbf{A}_0\mathbf{D} - \mathrm{tr}\mathbf{A}_0\mathrm{tr}\mathbf{D}}{2\mathrm{tr}\mathbf{A}_0\mathbf{D} - 3\mathrm{tr}\mathbf{A}_0\mathrm{tr}\mathbf{D}}\right)^2, \quad (73)$$

where $\mathbf{A}_0 = \mathbf{A}(q_0)$, $\mathbf{A}_1 = d\mathbf{A}/dq|_{q=q_0}$. According to Eq. (73), $\Delta q = 0$ for the matrices $\mathbf{D}$ satisfying the condition $\mathrm{tr}\mathbf{A}_0\mathbf{D} = \omega_0^2 \mathrm{tr}\mathbf{D}$. This condition was obtained earlier in [39] from the study of the characteristic polynomial of system (69). Note that expressions (70) and (73) for the jump in the critical load have not been obtained in the previous works [16]–[27], where the non-conservative system (69) was studied.

The function $q_{cr}(\mathbf{k})$ determined by Eq. (59) under constraints (66) is the boundary between the asymptotic stability and flutter domains of system (1) with m degrees of freedom in the n-dimensional space of parameters $\mathbf{k}, q$. Level sets of function (59) are the stability boundaries in the space of parameters $\mathbf{k} = (k_1, \dots, k_{n-1})$. The level set $q_{cr} = q_0$, where q_0 is the critical value of the parameter q for the unperturbed circulatory system (3), is given by the expression

$$\langle \mathbf{f}, \mathbf{k}\rangle = \pm\omega_0 \langle \mathbf{h}, \mathbf{k}\rangle \sqrt{\langle \mathbf{G}\mathbf{k}, \mathbf{k}\rangle} - \langle \mathbf{H}\mathbf{k}, \mathbf{k}\rangle, \quad \langle \mathbf{G}\mathbf{k}, \mathbf{k}\rangle > 0. \quad (74)$$

Consider the case when the vector of dissipation and gyroscopic parameters consists of only two components $\mathbf{k} = (k_1, k_2)$. Then, the stability boundary described by the function $q_{cr}(k_1, k_2)$ is a surface in the 3-dimensional space of the parameters k_1, k_2, q. Since at the point $\mathbf{p}_0 = (0, 0, q_0)$ the spectrum of system (1) contains a double purely imaginary eigenvalue with the Keldysh chain of length 2, then in the case of general position this surface has at $\mathbf{p}_0$ a singularity known as the Whitney–Cayley umbrella [29], [30]. Therefore, in the vicinity of $\mathbf{p}_0$ the asymptotic stability boundary $q_{cr}(k_1, k_2)$ qualitatively looks as shown in Fig. 4.

To confirm this qualitative conclusion we find the asymptotic formulae for the level curves of the function $q_{cr}(k_1, k_2)$ in the vicinity of the origin in the plane of parameters k_1, k_2 for q_{cr} close to q_0. First we get approximations of the level curves for $q_{cr} < q_0$ assuming k_1 as a smooth function of k_2 (or vice versa). Substituting into Eq. (59) the expansion $k_i = \beta_j k_j + o(k_j)$, $j = 1, 2$, where β_j are unknown constants, and collecting the terms with the same powers of k_j, we get

$$k_i = -\frac{f_j \pm h_j \sqrt{\widetilde{f}(q_{cr} - q_0)}}{f_i \pm h_i \sqrt{\widetilde{f}(q_{cr} - q_0)}} k_j + o(k_j), \quad i,j = 1, 2. \quad (75)$$

Since $\widetilde{f} < 0$ and $q_{cr} < q_0$, the radicals in Eq. (75) are real quantities. Thus, for $q_{cr} < q_0$ the asymptotic stability domain in the plane k_1, k_2 is bounded in the first approximation by the two different straight lines intersecting at the origin as shown in Fig. 5a. Note that only the part of

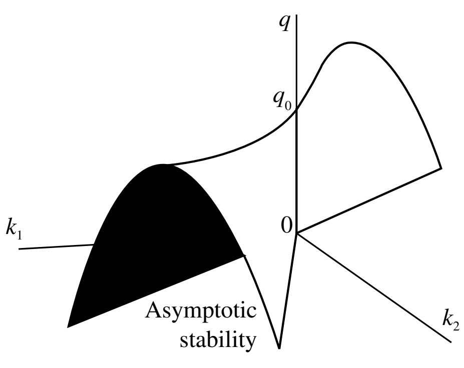

Fig. 4. The surface of the function $q_{cr}(k_1, k_2)$ given by Eq. (59) (the Whitney–Cayley umbrella)

the surface $q_{cr}(k_1, k_2)$ which belongs to the half-space $\langle \mathbf{h}, \mathbf{k}\rangle < 0$ bounds the asymptotic stability domain.

It follows from Eq. (75) that with the increase in q_{cr}, the angle between the lines bounding the asymptotic stability domain is decreased, being zero for $q_{cr} = q_0$. In this case the first approximation (75) gives only the ratio of the parameters k_1 and k_2 coinciding with Eq. (68). Substituting into Eq. (74) the expansion $k_i = -(f_j/f_i)k_j + \gamma_j k_j^2 + o(k_j^2)$ with unknown constants γ_j, and collecting the terms with the same powers of k_j, we find the second-order approximation of the level curve at $q_{cr} = q_0$

$$k_i = -\frac{f_j}{f_i}k_j - \frac{\mathbf{f}^T\mathbf{H}^\dagger\mathbf{f} \pm \omega_0(h_i f_j - h_j f_i)\sqrt{\mathbf{f}^T\mathbf{G}^\dagger\mathbf{f}}}{f_i^3}k_j^2 + o(k_j^2), \quad i,j = 1,2, \tag{76}$$

$$\mathbf{H}^\dagger = \begin{bmatrix} H_{22} & -H_{12} \\ -H_{21} & H_{11} \end{bmatrix}, \quad \mathbf{G}^\dagger = \begin{bmatrix} G_{22} & -G_{12} \\ -G_{21} & G_{11} \end{bmatrix},$$

where the components H_{rs}, G_{rs} $(r,s = 1,2)$ of the matrices $\mathbf{H}$ and $\mathbf{G}$ are determined by Eqs. (50) and (51). Equation (76) describes two curves tangential to each other at the origin of the plane of parameters k_1, k_2, forming a degenerate singularity of the asymptotic stability domain known as the cusp [26], [30].

In general the line $k_i = -(f_j/f_i)k_j$ does not always belong to the cusp. However, in the case when the matrix $\mathbf{D}(\mathbf{k})$ is *a linear* function of parameters this line is always inside of the asymptotic stability domain (Fig. 5b), because the matrix $\mathbf{H}$, consisting of the second derivatives of the matrix $\mathbf{D}(\mathbf{k})$ with respect to parameters k_1 and k_2, is identically zero.

To study the level curves for $q_{cr} > q_0$ we rewrite Eq. (59) as follows:

$$\langle \mathbf{f}, \mathbf{k}\rangle + \langle \mathbf{Hk}, \mathbf{k}\rangle = \pm\langle \mathbf{h}, \mathbf{k}\rangle\sqrt{\tilde{f}(q_{cr} - q_0) + \omega_0^2\langle \mathbf{Gk}, \mathbf{k}\rangle}. \tag{77}$$

If $\langle \mathbf{Gk}, \mathbf{k}\rangle > 0$, then the real solutions to Eq. (77) describing the level curves for $q_{cr} > q_0$ exist only if the radicand is positive or, equivalently,

$$\|\mathbf{k}\| \equiv \sqrt{\langle \mathbf{k}, \mathbf{k}\rangle} > \sqrt{\frac{-\tilde{f}(q_{cr} - q_0)}{\omega_0^2\langle \mathbf{Ge}, \mathbf{e}\rangle}} > 0, \tag{78}$$

where $\mathbf{e} = \mathbf{k}/\|\mathbf{k}\|$. Condition (78) means that the level curves $q_{cr} > q_0$ do not pass through the origin. Moreover, they are moved from the origin at the distance prescribed by the right-hand side of inequality (78), as shown in Fig. 5c.

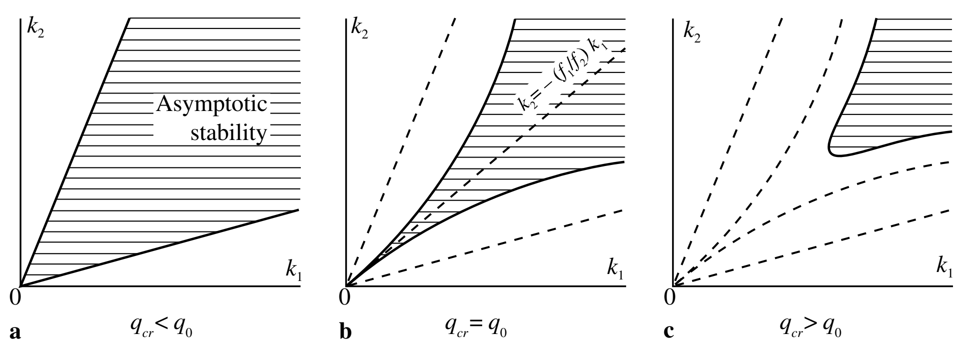

Fig. 5. Level curves of the function $q_{cr}(k_1, k_2)$ in the vicinity of the origin; the asymptotic stability domains are hatched

Therefore, analyzing the level curves of the function $q_{cr}(k_1, k_2)$ we showed that the boundary of the asymptotic stability domain described by Eq. (59) has the singularity Whitney–Cayley umbrella at the point $(0, 0, q_0)$ in the system parameters space. Note that in mechanical applications this singularity was found first on the asymptotic stability boundary of the Ziegler–Herrmann–Jong pendulum in [25], [26] and [29].

5 The Ziegler–Herrmann–Jong pendulum

As a mechanical example we consider a classical model of non-conservative system introduced first by Ziegler [1] and then extended by Herrmann and Jong [14]. This is a double pendulum composed of two rigid weightless bars of equal length l, which carry concentrated masses $m_1{=}2m$, $m_2{=}m$. The generalized coordinates φ_1 and φ_2 are assumed to be small. A load Q is applied at the free end and is always parallel to the second bar, as shown in Fig. 6a. At the hinges, the restoring moments $c\varphi_1{+}b_1 d\varphi_1/dt$ and $c(\varphi_2{-}\varphi_1){+}b_2(d\varphi_2/dt{-}d\varphi_1/dt)$ are induced. The oscillations of the pendulum near vertical equilibrium are described by the equations

$$3ml^2\frac{d^2\varphi_1}{dt^2} + (b_1+b_2)\frac{d\varphi_1}{dt} - (Ql-2c)\varphi_1 + ml^2\frac{d^2\varphi_2}{dt^2} - b_2\frac{d\varphi_2}{dt} + (Ql-c)\varphi_2 = 0,$$

$$ml^2\frac{d^2\varphi_1}{dt^2} - b_2\frac{d\varphi_1}{dt} - c\varphi_1 + ml^2\frac{d^2\varphi_2}{dt^2} + b_2\frac{d\varphi_2}{dt} + c\varphi_2 = 0, \tag{79}$$

where t indicates time, b_1 and b_2 are the damping coefficients and c characterizes the elastic properties of the hinges [14]. Introducing the dimensionless quantities

$$q = \frac{Ql}{c}, \quad k_1 = \frac{b_1}{\sqrt{cml^2}}, \quad k_2 = \frac{b_2}{\sqrt{cml^2}}, \quad \tau = t\sqrt{\frac{c}{ml^2}},$$

we get the equation of motion in the form (1) with the matrices

$$\mathbf{M}{=}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{D}{=}\begin{bmatrix} k_1{+}k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}, \quad \mathbf{A}{=}\begin{bmatrix} 2{-}q & q{-}1 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{y}{=}\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}. \tag{80}$$

Calculation of the determinant $\det(\mathbf{M}\lambda^2 + \mathbf{D}\lambda + \mathbf{A})$ results in the characteristic equation of the system

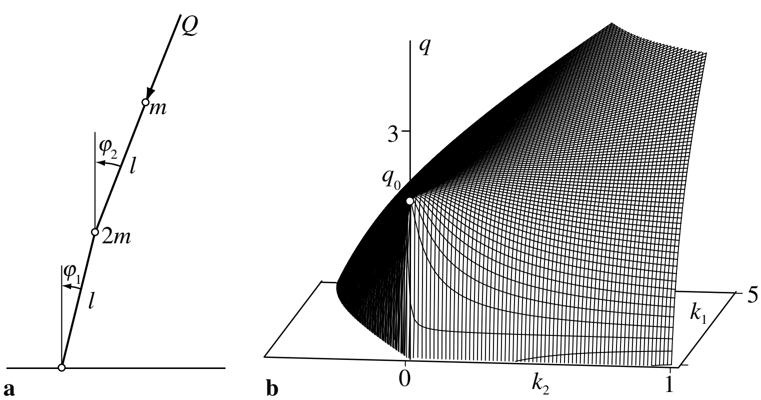

Fig. 6. The Ziegler–Herrmann–Jong pendulum and its asymptotic stability domain given by Eqs. (83) and (84)

$$2\lambda^4 + (k_1 + 6k_2)\lambda^3 + (k_1k_2 - 2q + 7)\lambda^2 + \lambda(k_1 + k_2) + 1 = 0. \tag{81}$$

If the damping is absent ($k_1{=}k_2{=}0$), then with the use of the Gallina criterion [38] we find from Eq. (81) that the system is marginally stable for $q<q_0$, where the critical load q_0 corresponds to the double eigenvalues $\pm i\omega_0$ with the Keldysh chains of length 2, and

$$\omega_0 = 2^{-1/4}, \quad q_0 = \frac{7}{2} - \sqrt{2} \simeq 2.09. \tag{82}$$

The critical value q_0 given by Eq. (82) was first obtained by Ziegler [1].

From the Routh-Hurwitz conditions applied to the characteristic polynomial (81) we conclude that the damped system is asymptotically stable iff

$$q < q_{cr}(k_1, k_2), \quad k_1 > -k_2, \tag{83}$$

where the critical load of the damped system is [14]

$$q_{cr}(k_1, k_2) = q_0 - \frac{(3-2\sqrt{2})}{2}\frac{(k_1-(4+5\sqrt{2})k_2)^2}{(k_1+k_2)(k_1+6k_2)} + \frac{1}{2}k_1k_2. \tag{84}$$

From Eqs. (83) and (84) it follows that as $q\to-\infty$ the domain of asymptotic stability is defined by the inequalities

$$k_1 > -k_2, \quad k_1 > -6k_2. \tag{85}$$

The domain of asymptotic stability of the Ziegler–Herrmann–Jong pendulum determined by inequalities (83) and Eq. (84) is shown in Fig. 6b. Note that this domain was drawn first in the works [25] and [26].

If we assume $k_1{=}k_2$ in Eq. (84) and consider a limit of $q_{cr}(k_2)$ as k_2 goes to zero, then we obtain

$$\widetilde{q}_{cr} \equiv \lim_{k_2\to 0} q_{cr}(k_2) = \frac{41}{28} \simeq 1.46 < q_0 = \frac{7}{2} - \sqrt{2} \simeq 2.09. \tag{86}$$

Inequality (86) established by Ziegler in 1952 shows that the critical load of the pendulum with the equal damping coefficients decreases in a discontinuous manner due to infinitesimally small dissipation. This inequality is known as the Ziegler paradox [2], [8].

Now we apply the theory developed in Sects. 3 and 4 to approximate the asymptotic stability domain. First, we find the right and left Keldysh chains of the double eigenvalue $\lambda_0{=}i\omega_0$ at the load q_0 given by Eqs. (82). Solution of Eqs. (33) and (34) yields

$$\mathbf{u}_0=\begin{bmatrix} 5\sqrt{2}-6 \\ 3\sqrt{2}+2 \end{bmatrix}, \quad \mathbf{u}_1=8i\omega_0\begin{bmatrix} 5\sqrt{2}-6 \\ 0 \end{bmatrix},$$

$$\mathbf{v}_0=\frac{-1}{112}\begin{bmatrix} \sqrt{2}+4 \\ -7 \end{bmatrix}, \quad \mathbf{v}_1=\frac{i\omega_0}{112}\begin{bmatrix} 10-8\sqrt{2} \\ 19\sqrt{2}-36 \end{bmatrix}. \tag{87}$$

The vectors found satisfy the normalization and orthogonality conditions (35). Substituting the eigen- and associated vectors into Eqs. (39), (40), (50) and (51) we get the real quantity $\widetilde{f}$, vectors $\mathbf{f}$ and $\mathbf{h}$, and the matrices $\mathbf{G}$ and $\mathbf{H}$:

$$\widetilde{f}=-\frac{1}{4}, \quad \mathbf{f}=\frac{1}{8}\begin{bmatrix} 1-\sqrt{2} \\ 6-\sqrt{2} \end{bmatrix}, \quad \mathbf{h}=\frac{-1}{8\omega_0}\begin{bmatrix} 1+\sqrt{2} \\ 6+\sqrt{2} \end{bmatrix}, \quad \mathbf{G}=\frac{1}{8}\begin{bmatrix} 0 & 2-\sqrt{2} \\ 2-\sqrt{2} & 14-8\sqrt{2} \end{bmatrix}, \quad \mathbf{H}\equiv 0. \tag{88}$$

Finally, with the vectors and matrices given by Eqs. (88) the approximation of the function $q_{cr}(k_1, k_2)$ (59) and inequality (61) take the form

$$q_{cr}=q_0-2\sqrt{2}(3-2\sqrt{2})^2\frac{(k_1-(4+5\sqrt{2})k_2)^2}{(k_1-(4-5\sqrt{2})k_2)^2}+2\sqrt{2}\left(\left(\frac{7}{4}-\sqrt{2}\right)k_2^2+\left(\frac{1}{2}-\frac{1}{4}\sqrt{2}\right)k_1k_2\right),$$

$$k_1>(4-5\sqrt{2})k_2. \tag{89}$$

Equation and inequality (89) approximate the asymptotic stability boundary given by Eqs. (83) and (84) in the vicinity of the point $(k_1=0, k_2=0, q=q_0)$. The level curves of this boundary on the plane of the damping parameters k_1, k_2 given by Eq. (84) and (89) are shown in Fig. 7 by the solid and dashed lines, respectively. These curves are in fact the sections of the asymptotic stability boundary, shown in Fig. 6b, by the planes $q_{cr}=const$. The left drawing of Fig. 7 shows that both the exact function $q_{cr}(k_1, k_2)$ and its approximation are symmetrical with respect to the origin of the damping parameters plane. Inequalities (83) and (89) define which half of the surface $q_{cr}(k_1, k_2)$ bounds the asymptotic stability domain. One can see from Fig. 7 that Eq. (89) approximates the exact stability boundary (84) with good accuracy even for large variations of the parameters k_1, k_2, and q.

The level curves corresponding to $q_{cr}=q_0$ can be approximated as well by formula (76). Indeed, substituting expressions (88) into Eq. (76) and taking into account that

$$\frac{f_2}{f_1}=-(4+5\sqrt{2}),\quad \omega_0(h_1f_2-h_2f_1)=-\frac{5\sqrt{2}}{32},\quad \mathbf{f}^T\mathbf{G}^{\dagger}\mathbf{f}=f_1^2\frac{5+2\sqrt{2}}{4},$$

we find

$$k_1=k_2(4+5\sqrt{2})\pm k_2^2\sqrt{50(133+94\sqrt{2})}+O(k_2^3). \tag{90}$$

Approximation (90) is shown in Fig. 7 by the dotted lines. One can see that Eq. (89) gives a more accurate approximation of the level curve for $q_{cr}=q_0$ than Eq. (90) because the latter contains only the second order terms with respect to the parameter k_2. Note that the asymptotic expression (90) was derived first in [25] and [26] from the analysis of the Routh–Hurwitz inequalities for the Ziegler–Herrmann–Jong pendulum. We obtained exactly the same result analyzing splitting of the double eigenvalue of the non-conservative system with a change of parameters.

Let us now find how the critical load of the Ziegler–Herrmann–Jong pendulum jumps due to small damping for various ratios of damping coefficients. Assume $k_1=\delta\cos\alpha$, $k_2=\delta\sin\alpha$ in Eqs. (84) and (89) and plot the critical value $q_{cr}(\delta)$ and its approximation as functions of the

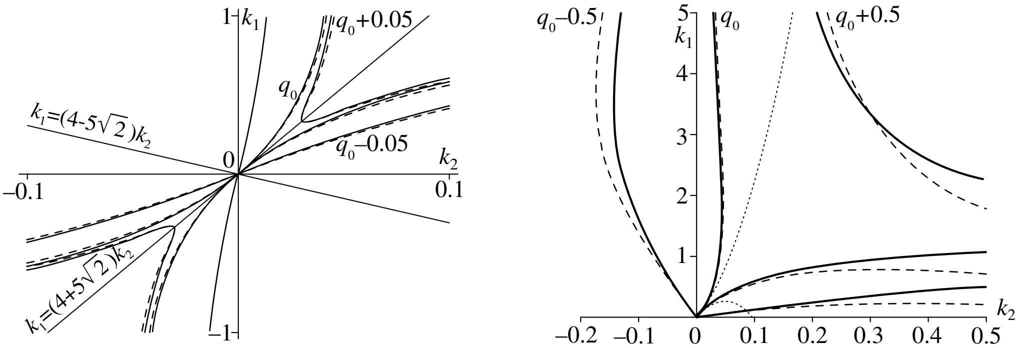

Fig. 7. The level curves of the asymptotic stability boundary $q_{cr}(k_1, k_2)$: solid lines – exact solution (84), dashed lines – approximation (89), dotted lines – Eq. (90)

parameter δ for different values of the parameter α, as shown in Fig 8a. Consider a limit of $q_{cr}(\delta\cos\alpha, \delta\sin\alpha)$ as δ goes to zero. Then,

$$\widetilde{q}_{cr}(\alpha) = q_0 - \frac{(3-2\sqrt{2})}{2}\frac{(\cos\alpha-(4+5\sqrt{2})\sin\alpha)^2}{(\cos\alpha+\sin\alpha)(\cos\alpha+6\sin\alpha)}. \tag{91}$$

Transforming Eq. (89) the same way we get

$$\widetilde{q}_{cr}(\alpha) = q_0-2\sqrt{2}(3-2\sqrt{2})^2\left(\frac{\cos\alpha-(4+5\sqrt{2})\sin\alpha}{\cos\alpha-(4-5\sqrt{2})\sin\alpha}\right)^2. \tag{92}$$

The graphs of function (91) and its approximation (92) are shown in Fig. 8b. One can see that the limit of the critical load smoothly depends on the direction in the plane of the damping parameters determined by the angle α and always does not exceed q_0: $\widetilde{q}_{cr}(\alpha) \le q_0$. Therefore, the limit of the function $q_{cr}(k_1, k_2)$ when k_1 and k_2 tend to zero is not defined as was first noted in [26], [27]. The infinitesimally small dissipation usually destabilizes the circulatory system, and $\Delta q \equiv q_0-\widetilde{q}_{cr}(\alpha)$ measures a jump in the critical load for the different directions α. Equation (92) evaluates the jump with the accuracy up to 1% for the direction angles $\pi/96\le\alpha\le\pi/28$, up to 10% for $-\pi/63\le\alpha\le 2\pi/13$, and up to 30% for the angles $-\pi/30\le\alpha\le 2\pi/5$. For the angle $\alpha=\pi/2$ the error in the evaluation of the jump is less than 40%.

However, for $k_1=(4+5\sqrt{2})k_2$ (corresponding to $\alpha \simeq 0.09$) a small dissipation stabilizes the pendulum, because in this case the exact equation (91) as well as its approximation (92) take the form

$$q_{cr} = q_0 + \frac{k_2^2}{2}(4+5\sqrt{2}). \tag{93}$$

One can see that q_{cr} is a function of one of the damping parameters and goes to q_0 as $k_2 \to 0$. Both curves given by Eqs. (91) and (92) have a maximum $\widetilde{q}_{cr}(\alpha_m) = q_0$ exactly at the same point $\alpha_m = \arctan(1/(4+5\sqrt{2}))$ as shown in Fig. 8.

Finally, we obtain the approximate expressions describing the trajectories of the eigenvalues of the Ziegler–Herrmann–Jong pendulum and compare the approximations with the numerical solution of the characteristic equation (81). Substituting the vectors and matrices given by Eqs. (88) into expressions (56), we find

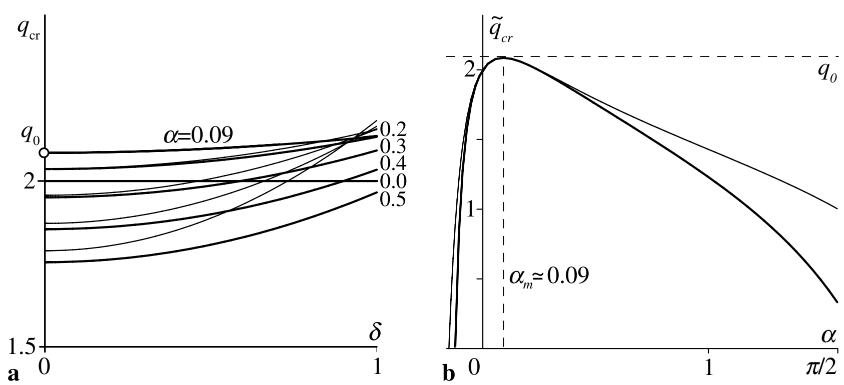

Fig. 8. The critical load $q_{cr}(\delta\cos\alpha, \delta\sin\alpha)$ and its limit as $\delta\to 0$ for the Ziegler–Herrmann–Jong pendulum: bold lines – exact solutions, fine lines – approximations

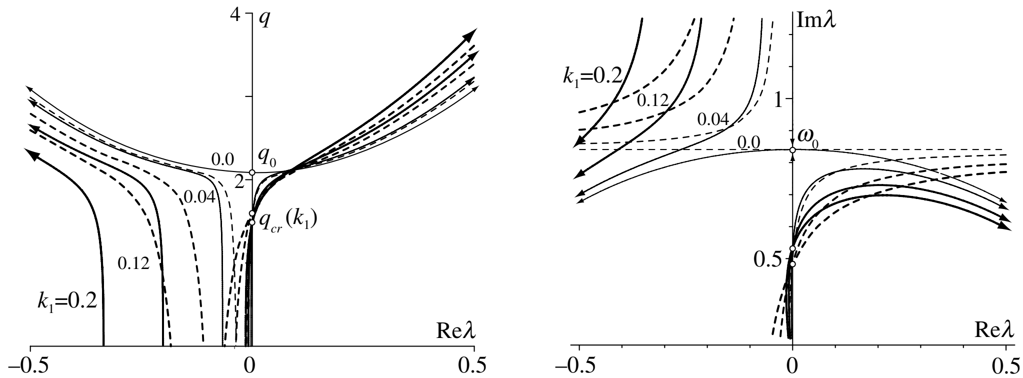

Fig. 9. The trajectories of the eigenvalues of the Herrmann–Jong pendulum given by Eq. (81) (solid lines) and their approximations by Eqs. (54), (94) and (95) (dashed lines) for $k_2/k_1 = 1$

$$a=\frac{\sqrt{2}+1}{8}(k_1-k_2(4-5\sqrt{2})),\ c=\frac{1}{4}(q_0-q)+\omega_0^2\left(\left(\frac{7}{4}-\sqrt{2}\right)k_2^2+\left(\frac{1}{2}-\frac{1}{4}\sqrt{2}\right)k_1k_2\right),$$

$$d=\omega_0\frac{1-\sqrt{2}}{8}(k_1-k_2(4+5\sqrt{2})). \tag{94}$$

The real and imaginary parts of the eigenvalues as functions of parameters are determined by Eqs. (54) and (55) with the coefficients a, c, and d given by Eqs. (94). The function $\mathrm{Re}\lambda(q)$ is shown in Fig. 9 for different values of k_1 under the assumption that $k_2=k_1$. Since $k_1\neq(4+5\sqrt{2})k_2$ the small damping destabilizes the pendulum. In this case the eigenvalues move on the complex plane along the branches of the hyperbola given by Eq. (53), which takes here the form

$$\left(\mathrm{Im}\lambda-\omega_0+\mathrm{Re}\lambda+\frac{\sqrt{2}+1}{16}(k_1-k_2(4-5\sqrt{2}))\right)^2-\left(\mathrm{Im}\lambda-\omega_0-\mathrm{Re}\lambda-\frac{\sqrt{2}+1}{16}(k_1-k_2(4-5\sqrt{2}))\right)^2$$
$$=\omega_0\frac{\sqrt{2}-1}{4}(k_1-k_2(4+5\sqrt{2})). \tag{95}$$

The approximations of the eigenvalue trajectories shown in Fig. 9 by the dashed lines give the right qualitative picture of the behavior of the eigenvalues being in a good agreement with the exact solutions of the characteristic Eq. (81) for small variations of the parameters. However, as one can see in Figs. 7–9, the critical load $q_{cr}(k_1,k_2)$ is approximated with a good accuracy even for the large deviations of the damping parameters.

6 Conclusion

A new theory describing the paradoxical behavior of general linear non-conservative systems due to small dissipative and gyroscopic forces qualitatively and quantitatively has been presented. The theory is substantially based on the sensitivity analysis of multiple eigenvalues. The behavior of eigenvalues of the system in the complex plane is described analytically. Approximations of the stabilization domain in the space of the system parameters are obtained. An explicit asymptotic expression for the critical load as a function of dissipation and gyroscopic parameters allowing to calculate a jump in the critical load is derived. The results are of general nature and give a constructive solution to the problem recognized as one of the main theoretical challenges in the non-conservative stability theory [2].

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References

[1] Ziegler, H.: Die Stabilitätskriterien der Elastomechanik. Ing.-Arch. **20**, 49–56 (1952).
[2] Bolotin, V. V.: Non-conservative problems of the theory of elastic stability. Oxford: Pergamon Press 1963.
[3] Ziegler, H.: Principles of structural stability. Waltham, Mass.: Blaisdell 1968.
[4] Gohberg I. C., Lancaster P, Rodman L.: Matrix polynomials. San Diego: Academic Press 1982.
[5] Seyranian, A. P.: Bifurcations in one-parameter circulatory systems. Izv. Ross. Acad. Nauk. MTT. **19**, 142–148 (1994).
[6] Bolotin, V. V., Zhinzher, N. I.: Effects of damping on stability of elastic systems subjected to nonconservative forces. Int. J. Solids Struct. **5**, 965–989 (1969).
[7] Herrmann, G.: Stability of equilibrium of elastic systems subjected to non-conservative forces. Appl. Mech. Revs. **20**, 103–108 (1967).
[8] Seyranian, A. P.: Destabilization paradox in stability problems of non-conservative systems. Adv. Mech. **13**, 89–124 (1990).
[9] Bloch, A. M., Krishnaprasad, P. S., Marsden, J. E., Ratiu, T. S.: Dissipation induced instabilities. Annales de l'Institut Henri Poincaré **11**, 37–90 (1994).
[10] Langthjem, M. A., Sugiyama, Y.: Dynamic stability of columns subjected to follower loads: a survey. J Sound Vibr. **238**, 809–851 (2000).
[11] Lobas, L. G.: The dynamics of finite-dimensional systems under non-conservative position forces. Int. Appl. Mech. **37**, 38–64 (2001).
[12] Agafonov, S. A.: Stability and motion stabilization of non-conservative mechanical systems. J. Math. Sci. **112**, 4419–4497 (2002).
[13] Bolotin, V. V., Grishko, A. A., Panov, M. Yu.: Effect of damping on the postcritical behavior of autonomous non-conservative systems. Int. J. Non-Linear Mech. **37**, 1163–1179 (2002).
[14] Herrmann, G., Jong, I. C.: On the destabilizing effect of damping in non-conservative elastic systems. ASME J. Appl. Mech. **32**, 592–597 (1965).
[15] Herrmann, G., Jong, I. C.: On non-conservative stability problems of elastic systems with slight damping. ASME J. Appl. Mech. **33**, 125–133 (1966).
[16] Nemat-Nasser, S., Herrmann, G.: Some general considerations concerning the destabilizing effect in non-conservative systems. Z. Angew. Math. Phys. (ZAMP) **17**, 305–313 (1966).
[17] Done, G. T. S.: Damping configurations that have a stabilizing influence on non-conservative systems. Int. J. Solids Struct. **9**, 203–215 (1973).
[18] Kounadis, A. N.: On the paradox of the destabilizing effect of damping in non-conservative systems. Int. J. Non-Linear Mechs. **27**, 597–609 (1992).
[19] Walker, J. A.: A note on stabilizing damping configurations for linear non-conservative systems. Int. J. Solids Struct. **9**, 1543–1545 (1973).
[20] O'Reilly, O. M., Malhotra, N. K., Namachchivaya, N. S.: Some aspects of destabilization in reversible dynamical systems with application to follower forces. Nonlinear Dyn. **10**, 63–87 (1996).
[21] Gallina, P.: Effect of damping on asymmetric systems. J. Vibr. Acoustics **125**, 359–364 (2003).
[22] Seyranian, A. P., Kirillov, O. N.: Effect of small dissipative and gyroscopic forces on the stability of non-conservative systems. Doklady Physics **48**, 679–684 (2003).

[23] Seyranian, A. P., Pedersen, P.: On interaction of eigenvalue branches in non-conservative multi-parameter problems. In: Dynamics and vibration of time-varying systems and structures, vol. 56., ASME, pp. 19–31, 1993.

[24] Seyranian, A. P.: Collisions of eigenvalues in linear oscillatory systems. J. Appl. Math. Mech. **58**, 805–813 (1994).

[25] Seyranian, A. P., Pedersen, P.: On two effects in fluid/structure interaction theory. In: Flow-induced vibration (Bearman, P.W., ed.), pp. 565–576. Rotterdam: Balkema 1995.

[26] Seyranian, A. P.: On stabilization of non-conservative systems by dissipative forces and uncertainty of critical load. Doklady Akademii Nauk. **348**, 323–326 (1996).

[27] Zhinzher, N. I.: Effect of dissipative forces with incomplete dissipation on the stability of elastic systems. Izv. Ross. Acad. Nauk. MTT **19**, 149–155 (1994).

[28] Troger, H., Zeman, K.: Zur korrekten Modellbildung in der Dynamik diskreter Systeme. Ing.-Arch. **51**, 31–43 (1981).

[29] Mailybaev, A. A., Seyranian, A. P.: On singularities of a boundary of the stability domain. SIAM J. Matrix Anal. Appl. **21**, 106–128 (1999).

[30] Arnold, V. I.: Geometrical methods in the theory of ordinary differential equations. New York Berlin: Springer 1983.

[31] Vishik, M. I., Lyusternik, L. A.: Solution of some perturbation problems in the case of matrices and selfadjoint or non-selfadjoint equations. Russian Math. Surveys. **15**, 1–73 (1960).

[32] Deif, A. S., Hagedorn, P.: Matrix polynomials subjected to small perturbations. Z. Angew. Math. Mech. (ZAMM) **66**, 403-412 (1986).

[33] Pedersen, P., Seyranian, A. P.: Sensitivity analysis for problems of dynamic stability. Int. J. Solids Struct. **19**, 315–335 (1983).

[34] Seyranian, A. P.: Sensitivity analysis of multiple eigenvalues. Mech. Struct. Mach. **21**, 261–284 (1993).

[35] Seyranian, A. P., Kirillov, O. N.: Bifurcation diagrams and stability boundaries of circulatory systems. Theor. Appl. Mech. **26**, 135–168 (2001).

[36] Yakubovich, V. A., Starzhinskii, V. M.: Parametric resonance in linear systems. Moscow: Nauka 1987.

[37] Bilharz, H.: Bemerkung zu einem Satz von Hurwitz. Z. Angew. Math. Mech. (ZAMM) **24**, 77–82 (1944).

[38] Gallina, P.: About the stability of non-conservative undamped systems. J. Sound Vibr. **262**, 977–988 (2003).

[39] Kirillov, O. N.: How do small velocity-dependent forces (de)stabilize a non-conservative system? Copenhagen: DCAMM Report No. 681, 2003.

[40] Hoffmann, N., Gaul, L.: Effects of damping on mode-coupling instability in friction induced oscillations: imperfect merging of modes and viscous instability. Z. Angew. Math. Mech. (ZAMM) **838**, 524–534 (2003).

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