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A Theory of the Second Order Phase Transitions in Spin Systems. III

—— *Critical Behavior of Correlation Functions in Ising Ferromagnets* ——

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Expressions for general correlation functions in Ising ferromagnets are derived in terms of the Lee-Yang theorem. By the use of these expressions, the asymptotic forms of correlation functions in the presence of magnetic field are analyzed, assuming those in the absence of the external field. In particular, the critical indices of spin-spin, energy-density-energy-density, energy-density-spin correlation functions are related with the critical indices of thermal susceptibility $\varkappa_T{}^0$, specific heat, susceptibility $\varkappa_H{}^0$, and magnetization. These results agree in form with those predicted by the scaling law approach.

§ 1. Introduction

It is our purpose in the present paper to study the relation among the critical indices of thermodynamic quantities and various spin-correlation functions. In one of the previous papers, we have discussed the functional form of the free energy in spin systems semi-phenomenologically.[1] In the other papers by the present author,[2),3)] the asymptotic form of the free energy and magnetization in the Ising model near the transition point has been derived in terms of the distribution function $g(\theta, t)$ of zeros in the fugacity plane. Rewriting the results given by Lee and Yang,[4)] the free energy is expressed in terms of the distribution function of zeros in the following:[2),3)]

$$-F/kT=\int_0^\pi g(\theta,\,t)\log 2(\cosh h-\cos\theta)\,d\theta\,, \tag{1·1}$$

and consequently the magnetization (or magnetic equation of state) is given by

$$M=m\langle s\rangle=2m\sinh h\int_0^\pi \frac{g(\theta,\,t)}{\cosh h-\cos\theta}\,d\theta\,, \tag{1·2}$$

where

$$h=2mH/kT\,,\qquad \text{and}\ \ t=(T-T_c)/T_c\,. \tag{1·3}$$

Assuming an asymptotic form of the susceptibility such as $\chi_0\sim t^{-\gamma}$, the distribution function of zeros for small θ and t has been derived in the form

$$g(\theta, t) = t^{\beta} f(\theta t^{-\Delta'}), \tag{1·4}$$

with $\Delta' = \beta + \gamma$, (see Eqs. (2·28) and (2·30) of II). As usual, α, β and γ denote the indices of specific heat, spontaneous magnetization, and susceptibility, respectively. The magnetic equation of state is given by

$$M = t^{\beta} \varphi_m (h t^{-\Delta'}) \tag{1·5}$$

in terms of Eqs. (1·2) and (1·4).

In § 2, we derive similar basic expressions for general spin-correlation functions as the magnetic equation of state (1·2), by the use of the Lee-Yang theorem. In § 3, we discuss the asymptotic form of even-even spin-correlation functions in the presence of magnetic field, such as the energy-density-energy-density correlation function, which is related with the anomaly of the specific heat. In § 4, odd-odd spin-correlation functions are analyzed. In this case a point different from that of even-even spin correlation function is that the odd-odd spin-correlation functions in the absence of the magnetic field vanish above the transition point as the distance between the two clusters becomes infinite. In particular, the critical behavior of the pair-correlation function is related with those of the magnetization and susceptibility. In § 5, we discuss odd-even spin-correlation functions such as the energy-density-spin correlation function, which is related with the susceptibility χ_H^0 or spontaneous magnetization. Finally, these results are compared with those predicted by the scaling law.

§ 2. Basic equations

As the ferromagnetic Ising model is represented by the Hamiltonian

$$\mathcal{H} = -\sum J_{ij} s_i s_j - mH \sum s_i \,;\, J_{ij} > 0, \quad \text{and } s_i = \pm 1\,, \tag{2·1}$$

correlation functions of *even number* of spins can be written as

$$\langle s_i \cdots s_j \rangle = \frac{\mathrm{Tr}\, s_i \cdots s_j \exp(-\beta\mathcal{H})}{\mathrm{Tr} \exp(-\beta\mathcal{H})} = \frac{z^N + \cdots + b_k z^k + \cdots + 1}{z^N + \cdots + a_k z^k + \cdots + 1}\,, \tag{2·2}$$

where the fugacity $z = e^{-h}$, $h = 2mH/kT$, $a_{N-k} = a_k$, $b_{N-k} = b_k$, and N is the number of spins. The theorem of Lee and Yang[4] states that the zeros of the denominator of Eq. (2·2) lie on a unit circle in the complex z-plane. Therefore, we easily obtain the following expression in the limit of N infinite in Eq. (2·2), (Appendix A):

$$\langle s_i \cdots s_j \rangle = 1 + \int_0^{\pi} \frac{\rho_e(\theta; i \cdots j; t)}{2(\cosh h - \cos\theta)} g(\theta, t) d\theta\,, \tag{2·3}$$

where $t = (T - T_c)/T_c$, $g(\theta, t)$ is the distribution function of zeros, and $\rho_e(\theta;$

$i\cdots j; t)$ "the spectral intensity" of the correlation function $\langle s_i\cdots s_j\rangle$.

In the same way, correlation functions of *odd number* of spins can be written as

$$\langle s_is_j\cdots s_m\rangle=\frac{1+\cdots+b_k'z^k+\cdots-z^N}{1+\cdots+a_kz^k+\cdots+z^N}, \tag{2·4}$$

where the coefficients of the numerator in Eq. (2·4) satisfy the reciprocal relation

$$b'_{N-k}=-b_k'. \tag{2·5}$$

Therefore, the following expression is obtained (Appendix B):

$$\langle s_is_j\cdots s_m\rangle=\int_0^\pi\frac{(\sinh h)\rho_o(\theta;ij\cdots m)}{\cosh h-\cos\theta}g(\theta,t)d\theta. \tag{2·6}$$

In the limit of h infinite in Eq. (2·6), the following normalization condition is derived:

$$\int_0^\pi\rho_o(\theta,ij\cdots m)g(\theta,t)d\theta=1, \tag{2·7}$$

where we have used the property

$$\lim_{h\to\infty}\langle s_is_j\cdots s_m\rangle=1, \tag{2·8}$$

which is obvious from Eq. (2·4). In particular, the spectral intensity $\rho_o(\theta,i,t)$ for single spin $\langle s_i\rangle$ is given by

$$\rho_o(\theta,i,t)=2 \tag{2·9}$$

from Eq. (1·2).

As was discussed in L and II, the integral for small θ is important near T_c and for small h, so that the expressions (2·3) and (2·6) can be simplified as

$$\langle s_i\cdots s_j\rangle\simeq1+\int_0^\pi\frac{\rho_e(\theta;i\cdots j,t)}{h^2+\theta^2}g(\theta,t)d\theta \tag{2·10}$$

and

$$\langle s_is_j\cdots s_m\rangle\simeq\int_0^\pi\frac{2h\rho_o(\theta;ij\cdots m,t)}{h^2+\theta^2}g(\theta,t)d\theta, \tag{2·11}$$

respectively.

When even number of spins s_i, s_j, $\cdots s_k$ are separated into two groups A_0 and B_R (R is the distance between the two groups), it is convenient to define the correlation function as

$$f^e_{AB}(t, h, R) = \langle A_0 B_R \rangle - \langle A \rangle \langle B \rangle$$

$$= \int_0^\pi \frac{\hat{\rho}_e(\theta, t; R)}{2(\cosh h - \cos\theta)} g(\theta, t) d\theta \tag{2·12}$$

$$\simeq \int_0^\pi \frac{\hat{\rho}_e(\theta, t; R)}{h^2 + \theta^2} g(\theta, t) d\theta , \tag{2·13}$$

where

$$\hat{\rho}_e(\theta, t; R) = \rho_e(\theta, t, R) - \rho_e(\theta, t, \infty) \tag{2·14}$$

in terms of the property

$$\lim_{R \to \infty} \langle A_0 B_R \rangle = \langle A \rangle \langle B \rangle . \tag{2·15}$$

In a similar way, the correlation function of odd number of spins can be expressed in the following form:

$$f^o_{AB}(t, h, R) = \langle A_0 B_R \rangle - \langle A \rangle \langle B \rangle$$

$$= \sinh h \int_0^\pi \frac{\hat{\rho}_o(\theta, t; R)}{\cosh h - \cos\theta} g(\theta, t) d\theta \tag{2·16}$$

$$\simeq \int_0^\pi \frac{2h\hat{\rho}_o(\theta, t; R)}{h^2 + \theta^2} g(\theta, t) d\theta . \tag{2·17}$$

These expressions will be used in the following sections in order to investigate the relation between the asymptotic forms of the correlation functions in the absence of the magnetic field and those in the presence of the external field.

§ 3. Even-even spin-correlation functions

(i) From the basic expressions in the foregoing section, we should classify our treatment into three cases. In this section, let us consider the case in which spin variables are separated into two clusters of *even number* of spins such as

$$A_0 = s_i \cdots s_k \quad \text{(even number)}$$

and

$$B_R = s_j \cdots s_m \quad \text{(even number)},$$

where R is a dimensionless parameter representing the distance between the clusters A and B. We start from the expression for the correlation function (2·12) or (2·13):

$$f_{AB}^{e,e}(t, h, R) = \langle A_0 B_R \rangle - \langle A \rangle \langle B \rangle$$

$$= \int_0^\pi \frac{\hat{\rho}_{e,e}(\theta, t, R)}{2(\cosh h - \cos \theta)} g(\theta, t) d\theta \tag{3·1}$$

$$\simeq \int_0^\pi \frac{\hat{\rho}_{e,e}(\theta, t, R)}{h^2 + \theta^2} g(\theta, t) d\theta \tag{3·2}$$

for small t and h. From Eqs. (2·28) and (2·30) of II, the distribution function for small θ and t takes the form

$$g(\theta, t) = t^\beta f(\theta t^{-\Delta'}); \qquad \Delta' = \beta + \gamma . \tag{3·3}$$

Substituting Eq. (3·3) into Eq. (3·2), we obtain

$$f_{AB}^{e,e}(t, h, R) = \int_0^\pi \frac{\hat{\rho}_{e,e}(\theta, t, R) t^\beta f(\theta t^{-\Delta'})}{h^2 + \theta^2} d\theta . \tag{3·4}$$

Making the change of variable $\theta = t^{\Delta'} y$, we get

$$f_{AB}^{e,e}(t, h, R) = \int_0^{\pi t^{-\Delta'}} \frac{\hat{\rho}_{e,e}(t^{\Delta'} y, t; R) t^{-\gamma} f(y)}{(h t^{-\Delta'})^2 + y^2} dy . \tag{3·5}$$

Near the Curie point, we may replace the upper limit of the integral by infinity, so that the correlation function can be expressed as

$$f_{AB}^{e,e}(t, h, R) = \int_0^\infty \frac{\hat{\rho}_{e,e}(t^{\Delta'} y, t; R) t^{-\gamma} f(y)}{(h t^{-\Delta'})^2 + y^2} dy . \tag{3·6}$$

As was discussed in L and II, there is also a case in which there exists a critical angle $\theta_c(t)$ such that $g(\theta, t) = 0$ for $0 \le |\theta| \le \theta_c(t)$. From Eq. (2·27) of II, the critical angle near T_c is of the form

$$\theta_c(t) \sim t^{\Delta'} \qquad \text{for } t > 0 , \tag{3·7}$$

so that in this case, the lower limit of the integral (3·6) is simply replaced by a finite constant, and the discussion below is quite the same.

(ii) Now, if the correlation function in the absence of a magnetic field assumes the form

$$f_{AB}^{e,e}(t, 0; R) = R^{-\lambda_1} \varphi_1(R t^{\nu_1}), \tag{3·8}$$

which will be discussed in Appendix C under a certain condition, then, in terms of Eq. (3·6), we obtain

$$\int_0^\infty \hat{\rho}_{e,e}(t^{\Delta'} y, t; R) R^{\lambda_1} t^{-\gamma} f(y) y^{-2} dy = \varphi_1(R t^{\nu_1}). \tag{3·9}$$

As a sufficient condition, the integrand of Eq. (3·9) may be a function of $R t^{\nu_1}$

and y: namely

$$\hat{\rho}_{c,c}(t^{\Delta'}y,\ t;\ R)R^{\lambda_1}t^{-\gamma}=F_1(Rt^{\nu_1},\ y), \tag{3·10}$$

where

$$\int_0^\infty F_1(x,\ y)f(y)y^{-2}dy=\varphi_1(x). \tag{3·11}$$

From Eq. (3·10), the spectral intensity takes the following asymptotic form:

$$\begin{aligned}\hat{\rho}_{c,c}(0,\ t;\ R)&=R^{-\lambda_1}t^{\gamma}F_1(Rt^{\nu_1},\ 0t^{-\Delta'}),\\ &=R^{-\lambda_1}t^{\gamma}\hat{F}_1(R^{\Delta'/\nu_1}0,\ 0t^{-\Delta'}).\end{aligned} \tag{3·12}$$

Therefore, the correlation function in the presence of the magnetic field in terms of Eqs. (3·6) and (3·12) is given in the form

$$f_{AB}^{c,c}(t,\ h,\ R)=R^{-\lambda_1}\Psi_1(Rt^{\nu_1},\ ht^{-\Delta'}), \tag{3·13}$$

where

$$\Psi_1(x,\ y)=\int_0^\infty \frac{F_1(x,\ u)f(u)}{y^2+u^2}du. \tag{3·14}$$

This result agrees with that given by the scaling law.[5)-7)]

(iii) What we are particularly interested in is an energy-density-energy-density correlation function[7)] defined by

$$f_{EE}(t,\ h,\ R)=\langle E_0E_R\rangle-\langle E_0\rangle\langle E_R\rangle, \tag{3·15}$$

where

$$E_R=\tfrac{1}{2}\sum_{\delta}J_{R,R+\delta}s_Rs_{R+\delta}.$$

The specific heat of the system is given by

$$C\simeq\frac{1}{kT^2}\int f_{EE}(t,\ h,\ R)R^{d-1}dR, \tag{3·16}$$

where d is the dimensionality of the system, and the correlation function f_{EE} in terms of Eq. (3·13) is given in the form

$$f_{EE}(t,\ h,\ R)=R^{-\lambda_1'}\Psi_{EE}(Rt^{\nu_1'},\ ht^{-\Delta'}). \tag{3·17}$$

Then, we obtain the following singularity of the specific heat:

$$\begin{aligned}C&\simeq\frac{1}{kT^2}\int_0^\infty\Psi_{EE}(Rt^{\nu_1'},\ ht^{-\Delta'})R^{d-\lambda_1'-1}dR\\ &=t^{-\nu_1'(d-\lambda_1')}\Phi_{EE}(h/t^{\Delta'}),\end{aligned} \tag{3·18}$$

where

$$\Phi_{EE}(x)=\int_0^\infty \Psi_{EE}(y,\,x)\,y^{d-\lambda_1'-1}dy\,.$$

Consequently, the index α of the specific heat is related with the parameters λ_1' and ν_1' by the equation

$$\alpha=\nu_1'(d-\lambda_1')\,. \tag{3·19}$$

In § 5, it will be discussed that the range of correlation defined by means of the energy-density-energy-density correlation function is naturally expected to be equal to that by means of the pair-correlation function. That is, we may put

$$\nu_1'=\nu\,, \tag{3·20}$$

where ν is the index for the range of correlation defined by means of the pair-correlation function. Then, from Eq. (3·19), the parameter λ_1' is given by

$$\lambda_1'=d-\alpha/\nu=2d(1-\alpha)/(2-\alpha)\,, \tag{3·21}$$

where the second equation is obtained in terms of Eqs. (5·23) and (5·25).

In the case of the two-dimensional Ising model, Hecht[7] has found the exact energy-density-energy-density correlation function in the absence of the magnetic field. His solution is

$$f_{EE}(\hat{t},\,0,\,R)=\left(\frac{2J}{\pi}\right)^2\hat{t}^2[K_1^2(\hat{t}R)-K_0^2(\hat{t}R)]\,, \tag{3·22}$$

where $\hat{t}=(4J/kT)\,|(T-T_c)/T_c|$, and K_0 and K_1 are modified Bessel functions of the second kind:

$$K_n(x)=\int_0^\infty \exp(-x\cosh y)\cosh ny\,dy\,. \tag{3·23}$$

For very large R, i.e. $\hat{t}R\gg1$,

$$f_{EE}\simeq C\,\frac{e^{-2\hat{t}R}}{R^2}\,, \tag{3·24}$$

whereas for very small $\hat{t}$, i.e. $\hat{t}R\ll1$,

$$f_{EE}\simeq C'\,\frac{1}{R^2}\,. \tag{3·25}$$

This is a nice example for the assumption (3·8) with $\lambda_1=2$ and $\nu_1=1$.

§ 4. Odd-odd spin-correlation functions

(i) Here, we discuss the case in which spin variables are separated into two groups of *odd number* of spins such as

$$A_0=s_i\cdots s_k \text{ (odd number)}$$

and

$$B_R = s_j \cdots s_m \quad \text{(odd number)}.$$

From Eq. (2·3), the correlation function can be rewritten as

$$\langle A_0 B_R \rangle = \int_0^\pi \frac{\rho_{o,o}(\theta, t, R) + 4(\cosh h - \cos\theta)}{2(\cosh h - \cos\theta)} g(\theta, t)\, d\theta\,,$$

for small h

$$\simeq \int_0^\pi \frac{\rho_{o,o}(\theta, t, R) + 2(\theta^2 + h^2)}{h^2 + \theta^2} g(\theta, t)\, d\theta$$

and neglecting h in the numerator,

$$\simeq \int_0^\pi \frac{\tilde{\rho}_{o,o}(\theta, t, R)}{h^2 + \theta^2} g(\theta, t)\, d\theta\,, \tag{4·1}$$

where

$$\tilde{\rho}_{o,o}(\theta, t, R) = \rho_{o,o}(\theta, t, R) + 2\theta^2\,, \tag{4·2}$$

and we have used the normalization

$$\int_0^\pi g(\theta, t)\, d\theta = \tfrac{1}{2}\,. \tag{4·3}$$

(ii) Noting that above T_c and in the absence of the magnetic field,

$$\lim_{R\to\infty} \langle A_0 B_R \rangle_0 = \langle A \rangle_0 \langle B \rangle_0 = 0\,, \tag{4·4}$$

which is a situation different from that in § 3, let us assume that the correlation function in the absence of the magnetic field takes the form

$$\langle A_0 B_R \rangle = R^{-\lambda_2} \varphi_2(R t^{\nu_2})\,. \tag{4·5}$$

In the same way as in § 3, the spectral intensity in terms of Eqs. (4·1) and (4·5) becomes

$$\begin{aligned}\tilde{\rho}_{o,o}(\theta, t, R) &= R^{-\lambda_2} t^{\gamma} F_2(R t^{\nu_2}, \theta t^{-\Delta'}) \\ &= R^{-\lambda_2} t^{\gamma} \hat{F}_2(R^{\Delta'/\nu_2}\theta, \theta t^{-\Delta'})\,,\end{aligned} \tag{4·6}$$

as a sufficient condition. Then, the correlation function in the presence of the magnetic field is given in the form

$$\langle A_0 B_R \rangle = R^{-\lambda_2} \Psi_2(R t^{\nu_2}, h t^{-\Delta'}) = t^{\nu_2\lambda_2} \hat{\Psi}_2(R t^{\nu_2}, h t^{-\Delta'})\,. \tag{4·7}$$

Equation (4·7) yields the expression for the product $\langle A \rangle \langle B \rangle$ in the presence of the magnetic field as

$$\langle A \rangle \langle B \rangle = \lim_{R\to\infty} \langle A_0 B_R \rangle = t^{\nu_2\lambda_2} \hat{\Psi}_2(h t^{-\Delta'})\,, \tag{4·8}$$

where

$$\hat{\Psi}_2(y)=\lim_{x\to\infty} x^{-\lambda_2}\Psi_2(x,\, y)=\hat{\Psi}_2(\infty,\, y). \tag{4·9}$$

In particular, in the case $A=B$, we obtain

$$\langle A\rangle^2=t^{\nu_2\lambda_2}\hat{\Psi}_2(ht^{-\Delta'}). \tag{4·10}$$

It is also useful to study the singularity of the quantity

$$\chi(AB)\equiv\sum_R [\langle A_0B_R\rangle-\langle A\rangle\langle B\rangle] \tag{4·11}$$

which is calculated in terms of Eq. (4·7) as

$$\chi(AB)\simeq t^{\nu_2\lambda_2}\int_0^\infty \{\hat{\Psi}_2(Rt^{\nu_2},\, ht^{-\Delta'})-\hat{\Psi}_2(ht^{-\Delta'})\}R^{d-1}dR$$

$$=t^{-\nu_2(d-\lambda_2)}\Phi_2(ht^{-\Delta'}), \tag{4·12}$$

where

$$\Phi_2(x)=\int_0^\infty \{\hat{\Psi}_2(y,\, x)-\hat{\Psi}_2(y)\}y^{d-1}dy\,. \tag{4·13}$$

(iii) The pair-correlation function is given in the form

$$\langle s_0s_R\rangle=R^{-\lambda}\Psi_{S,S}(Rt^\nu,\, ht^{-\Delta'}), \tag{4·14}$$

putting $A_0=s_0$ and $B_R=s_R$ in Eq. (4·7). From Eq. (4·10), the magnetization is

$$\langle s\rangle=t^{\nu\lambda/2}\{\hat{\Psi}_{S,S}(ht^{-\Delta'})\}^{1/2}\,. \tag{4·15}$$

Comparing this expression with Eq. (3·13) in II (or Eq. (1·5)), we obtain

$$\nu\lambda=2\beta\,. \tag{4·16}$$

Furthermore, in terms of Eq. (4·12), the susceptibility takes the form

$$\chi=t^{-\nu(d-\lambda)}\Phi_{S,S}(ht^{-\Delta'}). \tag{4·17}$$

In particular, the initial susceptibility has the following singularity:[8)]

$$\chi_0\sim t^{-\nu(d-\lambda)}\,. \tag{4·18}$$

Consequently, the index γ of the susceptibility is given by

$$\gamma=\nu(d-\lambda). \tag{4·19}$$

Therefore, the parameters λ and ν in terms of Eqs. (4·16) and (4·19) are expressed as

$$\lambda=2d\beta/(2\beta+\gamma)$$

and

$$\nu=(2\beta+\gamma)/d\,. \tag{4·20}$$

From Eq. (4·14), the temperature dependence of correlation length ξ near T_c is

$$\xi\propto t^{-\nu}\,. \tag{4·21}$$

§ 5. Odd-even spin-correlation functions

(i) As regards a correlation function of *odd number* of spins such as

$$A_0=s_i\cdots s_k \text{ (odd number)}$$

and

$$B_R=s_j\cdots s_m \text{ (even number)},$$

it is convenient to define the "semi-normalized" correlation function

$$f_{AB}^{o,e}(t,h;R)=\langle A_0B_R\rangle/\langle A_0\rangle\,. \tag{5·1}$$

In terms of Eq. (2·6), this correlation function is expressed as

$$f_{AB}^{o,e}(t,h;R)=\int_0^\pi \frac{\rho_{AB}(\theta,t,R)}{\cosh h-\cos\theta}g(\theta,t)d\theta\bigg/\int_0^\pi\frac{\rho_A(\theta,t)}{\cosh h-\cos\theta}g(\theta,t)d\theta \tag{5·2}$$

and for small h

$$\simeq\int_0^\pi\frac{\rho_{AB}(\theta,t,R)}{h^2+\theta^2}g(\theta,t)d\theta\bigg/\int_0^\pi\frac{\rho_A(\theta,t)}{h^2+\theta^2}g(\theta,t)d\theta\,. \tag{5·3}$$

(ii) If the correlation function in the absence of the magnetic field assumes the form

$$f_{AB}^{o,e}(t,0;R)=R^{-\lambda_3}\varphi_3(Rt^{\nu_3}), \tag{5·4}$$

then from Eqs. (5·3) and (5·4) we obtain

$$\int_0^\pi \rho_{AB}(\theta,t,R)g(\theta,t)\theta^{-2}d\theta\simeq t^{-\gamma_A}R^{-\lambda_3}\varphi_3(Rt^{\nu_3}), \tag{5·5}$$

where we have used the asymptotic form

$$\int_0^\pi\rho_A(\theta,t)g(\theta,t)\theta^{-2}d\theta\sim t^{-\gamma_A}\,. \tag{5·6}$$

In the same way as in the previous sections, the spectral intensity takes the form

$$\begin{aligned}\rho_{AB}(\theta,t;R)g(\theta,t)&=R^{-\lambda_3}t^{\Delta'-\gamma_A}F_3(Rt^{\nu_3},\theta t^{-\Delta'})\\&=R^{-\lambda_3}t^{\Delta'-\gamma_A}\widehat{F}_3(R^{\Delta'/\nu_3}\theta,\theta t^{-\Delta'})\,.\end{aligned} \tag{5·7}$$

Therefore, the correlation function in the presence of the magnetic field in terms of Eqs. (2·11) and (5·7) is given by

$$\langle A_0 B_R\rangle = hR^{-\lambda_3}t^{-\gamma_A}\Psi_3(Rt^{\nu_3}, ht^{-\Delta'}) = ht^{\nu_3\lambda_3-\gamma_A}\widehat{\Psi}_3(Rt^{\nu_3}, ht^{-\Delta'}). \tag{5·8}$$

From the expression (5·8), we obtain

$$\langle A\rangle\langle B\rangle = \lim_{R\to\infty}\langle A_0 B_R\rangle = ht^{-\gamma_A}t^{\lambda_3\nu_3}\widehat{\Psi}_3(ht^{-\Delta'}), \tag{5·9}$$

where

$$\widehat{\Psi}_3(y) = \lim_{x\to\infty} x^{-\lambda_3}\Psi_3(x, y) = \widehat{\Psi}_3(\infty, y). \tag{5·10}$$

Furthermore, the summation of the correlation function $\langle A_0 B_R\rangle$ has the following singularity:

$$\chi(AB) = \sum_R [\langle A_0 B_R\rangle - \langle A\rangle\langle B\rangle] \simeq ht^{-\gamma_A-\nu_3(d-\lambda_3)}\Phi_3(ht^{-\Delta'}), \tag{5·11}$$

where

$$\Phi_3(x) = \int_0^\infty \{\widehat{\Psi}_3(y, x) - \widehat{\Psi}_3(y)\} y^{d-1} dy .$$

(iii) The energy-density-spin correlation function in terms of Eq. (5·8) is

$$f_{ES}(t, h, R) = \langle s_0 E_R\rangle = \tfrac{1}{2}\sum_\delta{}' J_{R, R+\delta}\langle s_0 s_R s_{R+\delta}\rangle$$

$$= hR^{-\lambda_3'}t^{-\gamma}\Psi_{ES}(Rt^{\nu_3'}, ht^{-\Delta'}) \tag{5·12}$$

$$= R^{-\lambda_3'}t^{\beta}\Psi'_{ES}(Rt^{\nu_3'}, ht^{-\Delta}). \tag{5·12'}$$

From Eqs. (5·9) and (5·12), the following relation between the energy-density and spin-density is derived:

$$\langle s\rangle_h\langle E\rangle_h = \lim_{R\to\infty}\langle s_0 E_R\rangle_h = ht^{-\gamma+\nu_3'\lambda_3'}\widehat{\Psi}_{ES}(ht^{-\Delta'}), \tag{5·13}$$

where

$$\widehat{\Psi}_{ES}(y) = \lim_{x\to\infty} x^{-\lambda_3'}\Psi_{ES}(x, y). \tag{5·14}$$

In terms of Eq. (1·5), the magnetization can be written in the form

$$\langle s\rangle_h = t^{\beta}\varphi_m(ht^{-\Delta'}) = ht^{-\gamma}\Psi_S(ht^{-\Delta'}). \tag{5·15}$$

Consequently, the energy-density in terms of Eqs. (5·13) and (5·15), is given in the form

$$\langle E\rangle_h = t^{\nu_3'\lambda_3'}\Psi_E(ht^{-\Delta'}), \tag{5·16}$$

where

$$\Psi_E(x) = \widehat{\Psi}_{ES}(x)/\Psi_S(x).$$

Therefore, the specific heat has the singularity

$$C_0 \sim t^{-(1-\nu_3'\lambda_3')} . \tag{5·17}$$

Namely, the index of the specific heat is

$$\alpha = 1 - \nu_3'\lambda_3' . \tag{5·18}$$

On the other hand, below the Curie point, the following susceptibility can be considered:[9)]

$$\chi_H{}^0 = \lim_{H\to 0+} \left(\frac{\partial M}{\partial T}\right)_H = \frac{d}{dT} M_s \sim |t|^{\beta-1} . \tag{5·19}$$

It can also be easily shown that the susceptibility (5·19) in terms of the energy-density-spin correlation function is expressed by

$$\chi_H{}^0 \simeq \sum_R [\langle s_0 E_R \rangle - \langle s \rangle \langle E \rangle]. \tag{5·20}$$

In terms of Eqs. (5·12) and (5·13), the susceptibility $\chi_H{}^0$ has the singularity

$$\chi_H{}^0 \sim t^{\beta-\nu_3'(d-\lambda_3')} . \tag{5·21}$$

Comparing this result (5·21) with Eq. (5·19), we obtain another relation

$$\nu_3'(d-\lambda_3') = 1 . \tag{5·22}$$

From the relations (5·18) and (5·22), the parameters ν_3' and λ_3' are determined as

$$\nu_3' = (2-\alpha)/d , \tag{5·23}$$

$$\lambda_3' = d(1-\alpha)/(2-\alpha). \tag{5·24}$$

The relation[1),2),3),10),11)] $\alpha + 2\beta + \gamma = 2$ together with Eqs. (4·20) and (5·23) gives the equality

$$\nu_3' = \nu . \tag{5·25}$$

This means that the range of correlation defined by means of the pair correlation function and that by means of the energy-density-spin correlation function agree with each other. This is quite natural. Thus the index ν_1' for the range of the energy-density-energy-denstiy correlation may also be equal to that of the pair correlation:

$$\nu_1' = \nu . \tag{5·26}$$

In the case of the two-dimensional Ising model, Hecht[7)] has also found the exact solution of the energy-density-spin correlation function in the absence of the magnetic field:

$$f_{ES}(\hat{t}, 0, R) = \frac{2J}{\pi} \hat{t} \langle s \rangle \int_{2\hat{t}R}^{\infty} x^2 e^{-x} dx . \tag{5·27}$$

This is also a nice example for our assumption (5·4). In this case, the parameters λ_3' are

$$\lambda_3' = \nu_3' = 1\ . \tag{5·28}$$

§ 6. Concluding remarks

Expressions for general correlation functions in Ising ferromagnets have been derived in terms of the Lee-Yang theorem. By the use of these expressions, we have obtained the asymptotic forms of the correlation functions in the presence of the magnetic field, assuming those in the absence of the external field.

A similar discussion has been presented by Abe[12] in the case of the pair-correlation function. However, his starting point is incorrect, which should be corrected as is shown in our theory.

To summarize our results, the asymptotic forms of the energy-density-energy-density, spin-spin and energy-density-spin correlation functions are given by

$$f_{EE} = t^{2(1-\alpha)} f_1(Rt^\nu,\ ht^{-\Delta'}), \tag{6·1}$$

$$f_{SS} = t^{2\beta} f_2(Rt^\nu,\ ht^{-\Delta'}), \tag{6·2}$$

$$f_{ES} = t^{1-\alpha+\beta} f_3(Rt^\nu,\ ht^{-\Delta'}), \tag{6·3}$$

with $\nu = (2-\alpha)/d$ and $\Delta' = \beta + \gamma$. These results agree with those predicted by the scaling-law approach in Hecht's paper.[7]

It is to be hoped that various correlation functions of the three-dimensional Ising model, at least, in the absence of magnetic field should be investigated as in the two-dimensional case.

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Appendix A

Without loss of generality, we assume that $N = 2n$ (even number). Then, Eq. (2·2) can be rewritten as

$$\langle s_i \cdots s_j \rangle = \frac{(z^n + z^{-n}) + \sum b_k (z^k + z^{-k})}{(z^n + z^{-n}) + \sum a_k (z^k + z^{-k})} \tag{A·1}$$

$$= \frac{w^n + \cdots b_k' w^k + \cdots + b_n'}{w^n + \cdots a_k' w^k + \cdots + a_n'}\ , \tag{A·2}$$

where

$$w=z+1/z\,. \tag{A·3}$$

From the theorem of Lee and Yang, the zeros of the denominator of Eq. (A·1) are given in the form

$$z_k=\exp(i\theta_k)\ (k=1,\ 2,\ \cdots N)\,. \tag{A·4}$$

It means that the zeros of Eq. (A·2) are given by

$$w_k=2\cos\theta_k(0\leq\theta_k\leq\pi)\,. \tag{A·5}$$

Consequently, Eq. (A·2) can be decomposed into the following partial fractions:

$$\langle s_i\cdots s_j\rangle=1+\sum_{k=1}^{n}\frac{\rho_e(\theta_k;\ i\cdots j;\ t)}{w-w_k}\,. \tag{A·6}$$

In the limit of N infinite, the summation may be replaced by the integral

$$\langle s_i\cdots s_j\rangle=1+\int_0^\pi\frac{\rho_e(\theta;\ i\cdots j;\ t)}{2(\cosh h-\cos\theta)}g(\theta,\ t)d\theta\,. \tag{A·7}$$

Appendix B

In the same way as in Appendix A, without loss of generality, we assume that $N=2n$ (even number). Then, Eq. (2·4) can be rewritten as

$$\begin{aligned}\langle s_is_j\cdots s_m\rangle&=\frac{-\sum b_k'(z^k-z^{-k})}{\sum a_k(z^k+z^{-k})}=\left(\frac{1}{z}-z\right)\times\frac{\sum b_k''(z^k+z^{-k})}{\sum a_k(z^k+z^{-k})}\\&=\left(\frac{1}{z}-z\right)\times\frac{w^{n-1}+\cdots b_k'''w^k+\cdots b_n'''}{w^n+\cdots a_k'w^k+\cdots a_n'}\\&=2\sinh h\cdot\sum_k\frac{\rho_o(\theta_k,\ ij\cdots m;\ t)}{w-w_k}\,.\end{aligned}$$

In the limit of N infinite, we obtain the following expression:

$$\langle s_is_j\cdots s_m\rangle=\sinh h\int_0^\pi\frac{\rho_o(\theta,\ ij\cdots m;\ t)}{(\cosh h-\cos\theta)}g(\theta,\ t)d\theta\,. \tag{B·1}$$

Appendix C

First, in the case of an even-even spin-correlation function, we start from Eq. (3·2):

$$f_{AB}^{e,e}(t,\ h,\ R)=\int_0^\pi\frac{\hat{\rho}_{e,e}(\theta,\ t,\ R)}{h^2+\theta^2}g(\theta,\ t)d\theta\,. \tag{C·1}$$

Here, assuming that the spectral intensity $\rho_{AB}(\theta,\ t,\ R)$ is regular with respect to variable t near $t=0$, we neglect the t-dependence of ρ_{AB}. Then, Eq. (C·1) can be written as

$$f_{AB}^{e,e}(t,\,0,\,R)=\int_0^{\pi}\frac{\hat{\rho}_{e,e}(\theta,\,R)}{\theta^2}\,g(\theta,\,t)\,d\theta\,. \tag{C·2}$$

On the other hand, the correlation function at the Curie point may be given as

$$f_{AB}^{e,e}(0,\,0,\,R)\sim R^{-\lambda_1}\,. \tag{C·3}$$

As the distribution function at T_c takes the form

$$g(\theta,\,0)\sim\theta^{\beta/(\beta+\gamma)}\,, \tag{C·4}$$

we obtain

$$\int_0^{\pi}\hat{\rho}_{AB}(\theta,\,R)\,\theta^{\beta/(\beta+\gamma)-2}d\theta\sim R^{-\lambda_1}\,. \tag{C·5}$$

If we change a variable

$$\theta=R^{-\mu}y,\ (\mu>0;\ \text{a proper positive constant}),$$

Eq. (C·5) becomes

$$\int_0^{\pi R^{\mu}}\hat{\rho}_{AB}(R^{-\mu}y,\,R)\,R^{\lambda_1+\mu\gamma/(\beta+\gamma)}y^{\beta/(\beta+\gamma)-2}dy\sim\text{constant}\,. \tag{C·6}$$

As the upper limit of the integral may be replaced by infinity for large R, we get

$$\int_0^{\infty}\hat{\rho}_{AB}(R^{-\mu}y,\,R)\,R^{\lambda_1+\mu\gamma/(\beta+\gamma)}y^{\beta/(\beta+\gamma)-2}dy\sim\text{constant}. \tag{C·7}$$

As a sufficient condition, the integrand of the above equation (C·7) may be of the form

$$\hat{\rho}_{AB}(R^{-\mu}y,\,R)\,R^{\lambda_1+\mu\gamma/(\beta+\gamma)}=\Phi(y)\,, \tag{C·8}$$

where $\Phi(y)$ is an arbitrary function of y. Then, the spectral intensity becomes

$$\begin{aligned}\hat{\rho}_{AB}(\theta,\,R)&=R^{-\lambda_1-\mu\gamma/(\beta+\gamma)}\Phi(R^{\mu}\theta)\\&=R^{-\lambda_1}\theta^{\gamma/(\beta+\gamma)}\hat{\Phi}(R^{\mu}\theta)\,.\end{aligned} \tag{C·9}$$

Therefore, the correlation function is calculated as

$$\begin{aligned}f_{AB}(t,\,h,\,R)&=\langle A_0B_R\rangle-\langle A\rangle\langle B\rangle\\&\simeq\int_0^{\pi}\frac{1}{h^2+\theta^2}\,R^{-\lambda_1}\theta^{\gamma/(\beta+\gamma)}\hat{\Phi}(R^{\mu}\theta)\,t^{\beta}f(\theta/t^{\beta+\gamma})\,d\theta\\&\simeq R^{-\lambda_1}F(Rt^{\nu_1},\,h/t^{\beta+\gamma}),\end{aligned} \tag{C·10}$$

where

$$F(x, y)=\int_0^\infty \frac{1}{y^2+u^2}\,\hat{\Phi}(x^\mu u)\,u^{-\beta/(\beta+\gamma)}f(u)\,du \tag{C·11}$$

and

$$\nu_1=(\beta+\gamma)/\mu\,. \tag{C·12}$$

In particular, putting $h=0$ in Eq. (C·10), the correlation function in the absence of the field becomes of the form

$$\begin{aligned} f_{AB}(t,\,0,\,R) &= R^{-\lambda_1}F(Rt^{\nu_1},\,0) \\ &= R^{-\lambda_1}\varphi_1(Rt^{\nu_1})\,. \end{aligned} \tag{C·13}$$

Thus, if we assume that the spectral intensity should be regular with respect to variable t near $t=0$, we obtain the asymptotic correlation function of the form (C·10). However, the above assumption seems to be severe, so that in the text we have safely accepted the form of the correlation function (3·8) in the absence of the magnetic field, and we have discussed the relation between the correlation function in the absence of the magnetic field and that in the presence of the field.

Similar discussions are possible for odd-odd and odd-even spin-correlation functions.

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