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A THEORY OF NON-DEVELOPABLE GENERALIZED RULED SURFACES IN THE ELLIPTIC SPACE E^m

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1. INTRODUCTION

We assume throughout that all manifolds, maps, vector fields, etc. ... are differentiable of class C^∞. We work always in the projective model of the m-dimensional elliptic space E^m of constant curvature $+1$, that is, the points of E^m are the points of the real m-dimensional projective space $\mathscr{P}^m$, there is an absolute totally imaginary hyperquadric Γ and the totally geodesic subspaces of E^m are the linear subspaces of $\mathscr{P}^m$.

Assume that M is an $(n + 1)$-dimensional submanifold of E^m, which contains an n-dimensional submanifold (hypersurface) N, which is totally geodesic in E^m $(m > n + 1 > 2)$.

The Riemannian connections of E^m, M and N are respectively denoted by $\overline{\overline{D}}$, $\overline{D}$ and D, while $V(,)$ is the vector-valued second fundamental form of M in E^m. Suppose that X and Y are vector fields of N and that ξ is the unit normal vector field on N in M. Since N is totally geodesic in E^m, we have $V(X, Y) = 0$. Moreover, $\overline{D}_X\xi$ is orthogonal with ξ and with N, because, if $\langle , \rangle$ denotes the metric tensor of E^m (and also the induced metrics on M and on N),

$$0 = X\langle \xi, \xi \rangle = 2\langle \overline{\overline{D}}_X \xi, \xi \rangle$$

and

$$0 = X\langle \xi, Y \rangle = \langle \overline{\overline{D}}_X \xi, Y \rangle + \langle \xi, \overline{\overline{D}}_X Y \rangle ,$$

while

$$\overline{\overline{D}}_X Y = D_X Y \quad \text{and thus} \quad \langle \xi, \overline{\overline{D}}_X Y \rangle = 0 .$$

Because of all this we get $\overline{D}_X\xi = 0$ or $\overline{\overline{D}}_X\xi = V(X, \xi)$.

The Riemannian curvatures $K(X, \xi)$ of M at the points of N in the so-called normal plane directions (X, ξ) on N in M, are given by

$$K(X, \xi) = +1 - \frac{\langle V(X, \xi), V(X, \xi) \rangle}{\langle X, X \rangle} . \tag{1.1}$$

Definitions. $X_p \in N_p$ determines a *principal direction at* $p \in N$ if $K(X_p, \xi_p)$ is an extremal value of the Riemann curvatures of M in the normal plane directions on N_p in M_p. A vector field X of N is called *principal* if it gives a principal direction at each point of his domain. A *line of sectional curvature* on N is a curve on N such that the tangent vector field is principal. Because of (1.1) and since $\langle V(X, \xi), V(Y, \xi)\rangle$ determines a symmetric two-covariant tensor field on N, we have at each point of N n mutually orthogonal principal directions. The extremal values of $(K(X, \xi) - 1)$ at a point p of N are denoted by $K_i(p)$ $i = 1, \ldots, n$. The product of these "principal curvatures" is denoted by: $\mathscr{K}(p) = \prod_{i=1}^{n} K_i(p)$.

From now on we suppose that the Riemann curvature of M in any normal plane direction on N in M is never equal to $+1$, i.e. we assume that $V(X_p, \xi_p) \neq 0$ for each vector $X_p \neq 0$ at each point of N. As a corollary we have now that necessarily $m \geqq 2n + 1$.

Next, if we put for each vectors X_p and Y_p at each point p of N (supposing again that ξ is the unit normal vector field on N in M):

$$g(X_p, Y_p) = \langle \overline{\overline{D}}_{X_p}\xi, \overline{\overline{D}}_{Y_p}\xi\rangle = \langle V(X_p, \xi_p), V(Y_p, \xi_p)\rangle,$$

then, because of (1.1), $g(X_p, X_p) = \langle X_p, X_p\rangle (1 - K(X_p, \xi_p)) > 0$ if $X_p \neq 0$ and g is symmetric two-covariant positive definite. Thus g determines a metric tensor on N and N endowed with this new metric becomes a Riemannian manifold denoted by N'.

We construct on N with respect to M two Gauss maps. The first is just the natural bijection $i: N \to N'$; $p \to p$. The second is set up as follows: on the complete geodesic of E^m which is at any point p of N tangent to ξ_p, there is a unique point p' at elliptic distance $\pi/2$ and $p \to p'$ is a mapping f which sends N to the so-called dual image $f(N)$ of N with respect to M. Notice that $f(N)$ is contained in the $(m - n - 1)$-dimensional dual (with respect to the absolute hyperquadric Γ) totally geodesic subspace of N in E^m and, because of our assumptions, it is not difficult to proof that $f(N)$ is an n-dimensional submanifold which is locally isometric with N'.

For the (easy) proofs of the following results, we refer to [7]:

Theorem. 1. *The lines of sectional curvature of N are the n families of curves which are mutually orthogonal in N and in N'.*

2. *If $p \in N$, $X_p \in N_p$ and $\sigma: [a, b] \to N$; $s \to \sigma(s)$ is a curve on N with N-arc length s and N'-arc length s', such that*

$$\sigma(s_0) = p \quad \textit{and} \quad T_{\sigma(s_0)} = X_p/\langle X_p, X_p\rangle^{1/2},$$

then

$$K(X_p, \xi_p) = 1 - \left(\frac{ds'}{ds}\right)^2_{s=s_0}. \tag{1.2}$$

3. *Suppose that* ω (resp. ω') *is a volume element at the point* p *of* N (resp. N'), *then*

$$\omega' = \sqrt{[(-1)^n \mathscr{K}(p)]}\, \omega . \tag{1.3}$$

Remark. The map which assigns to each point p of M the totally geodesic $(n+1)$-dimensional subspace of E^m tangent to M_p at p is called the *generalized Gauss map* $G: M \to Q$, where Q is the set of all the $(n+1)$-dimensional totally geodesic subspaces of E^m. There is a standard Riemannian metric $d\Sigma^2$ on Q with respect to which Q is a symmetric Riemannian space. The quadratic differential form $G^*(d\Sigma^2)$ induced on M by this Gauss map is the third fundamental form on M. In [2] Obata obtained a (since then wellknown) relation among this third fundamental form on M, the Ricci form $\mathrm{Ric}(M)$ on M and the second fundamental form $\langle H, V\rangle$ on M in the direction of the mean curvature vector H of M in E^m:

$$G^*(d\Sigma^2) = (n+1)\langle H, V\rangle - \mathrm{Ric}(M) + n\langle\, ,\rangle .$$

If X, Y are vector fields of N and $e_1, \ldots, e_n, \xi$ is an orthonormal base field of M at the points of N, then, if R is the curvature tensor of M, we get because of the Gauss equation, since $V(X, Y) = 0$ and $V(e_i, e_i) = 0$, $i = 1, \ldots, n$:

$$\mathrm{Ric}(M)(X, Y) = \sum_{j=1}^{n} \langle R(e_i, X)\, Y, e_i\rangle + \langle R(\xi, X)\, Y, \xi\rangle =$$

$$= (n+1)\langle X, Y\rangle - \langle V(X, \xi), V(Y, \xi)\rangle = (n+1)\langle X, Y\rangle - g(X, Y) .$$

Thus, on N we have the following relation among the metric tensors $\langle\, ,\rangle$, g and the third fundamental form $G^*(d\Sigma^2)$:

$$g = \langle\, ,\rangle + G^*(d\Sigma^2) .$$

2. NON-DEVELOPABLE GENERALIZED RULED SURFACES (G.R.S.) IN E^m

A $(n+1)$-dimensional G.R.S. in E^m, i.e. a submanifold which admits a codimension one foliation such that each leave is a complete totally geodesic subspace (i.e. a E^n) in E^m, is a G.R.S. in $\mathscr{P}^m$ and it is non-developable iff in $\mathscr{P}^m$ for each generating space N the map: (point p) $\to$ (tangent space at p, considered as a linear subspace of $\mathscr{P}^m$) is a non-singular projectivity ([4]). Assume that N is a fixed n-dimensional generating space of the G.R.S. The tangent spaces of the G.R.S. at the points of N generate a $(2n+1)$-dimensional subspace of $\mathscr{P}^m$, i.e. a totally geodesic E^{2n+1} of E^m, and, the dual image $f(N)$ is the n-dimensional dual totally geodesic subspace of N in this E^{2n+1}. Moreover $f: N \to f(N)$ regarded as a map between the n-dimensional projective spaces N and $f(N)$ is a non-singular projectivity and $f: N' \to f(N)$ is an isometry.

The dual images $f(N)$ of the generating spaces of the G.R.S. generate the so-called dual G.R.S. It is not difficult to see that the dual image of the generating space $f(N)$ in this dual G.R.S. is again N and that this $(n+1)$-dimensional dual G.R.S. is also non-developable. Finally remark that because of the foregoing, N' is an n-dimensional elliptic space of curvature $+1$ in the elliptic space N, such that N' has an absolute imaginary hyperquadric Γ' in N (remark that $f(\Gamma') = f(N) \cap \Gamma$) and that N' and N have the same geodesic lines and totally geodesic subspaces. The absolute hyperquadric of the elliptic space N is of course $\Gamma \cap N$. We suppose throughout that we are in the "general case" that is, that Γ' is in general position with respect to $\Gamma \cap N$.

Next consider a complete geodesic line (= straight line) L of N (and thus also of N'): on L there are in the general case just two points l_1 and l_2 at distance $\pi/2$ from each other in N and in N'; i.e. l_1 and l_2 are conjugate with respect to $\Gamma \cap N$ and with respect to Γ' (thus the distance between $f(l_1)$ and $f(l_2)$ is also $\pi/2$). Call these points the *points of striction of L*. Assume that we have in E^m a projective coordinate system such that the points $l_1, l_2, f(l_1), f(l_2)$ have resp. coordinates $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, $(0, \ldots, 0, 1, 0)$, $(0, \ldots, 0, 1)$ and that the absolute hyperquadric Γ has the equation $x_0^2 + \ldots + x_m^2 = 0$. The restriction of f to L is a projectivity $f|_L: L \to f(L)$; $(1, t, 0, \ldots, 0) \to (0, \ldots, 0, 1, t')$, which has now a representation of the form $t' = t/d$, where d is a real non-zero constant. We find, if we put for a general point p of L: s = distance (l_1, p) in N and s' = distance (l_1, p) in N' = distance $(f(l_1), f(p))$ in $f(N)$,

$$e^{-2is} = (i, -i, 0, t) = \frac{1-t^2}{1+t^2} - 2i\frac{t}{1+t^2}$$

and thus

$$\cos(-2s) = \frac{1-t^2}{1+t^2}, \quad \sin(-2s) = \frac{-2t}{1+t^2}$$

or

$$\cos^2 s = \frac{1}{1+t^2}, \quad \sin s \cos s = \frac{t}{1+t^2} \quad \text{and finally} \quad \operatorname{tg} s = t\,.$$

In the same way we have $\operatorname{tg} s' = t'$ and thus there is a constant d associated with L such that (we always assume that $0 \leqq s, s' \leqq \pi/2$ and thus $d > 0$)

$$\operatorname{tg} s = d \operatorname{tg} s'\,. \tag{2.1}$$

We call d the *parameter of distribution of the line L with respect to the point of striction* l_1. It is obvious that the parameter of distribution of L with respect to l_2 is equal to $1/d$. Remark that in (2.1) s' is also the angle between the tangent space of the G.R.S. at l_1 and at the variable point p of L.

Next, in order to obtain informations about the Riemann curvature of the G.R.S. we combine (2.1) with (1.2): from (2.1) we obtain after differentiation $\mathrm{d}s/\cos^2 s' = d(\mathrm{d}s'/\cos^2 s')$ and because of (1.2) we find immediately the following:

Suppose that Y_p is a unit vector of the G.R.S. tangent to L at p and that ξ_p is the unit normal vector at p on N in the G.R.S., then the Riemann curvature $K(Y_p, \xi_p)$ of the G.R.S. is given by

$$(2.2) \qquad K(Y_p, \xi_p) = 1 - \frac{\cos^4 s'}{d^2 \cos^4 s} = 1 - \frac{d^2}{(\sin^2 s + d^2 \cos^2 s)^2} = $$
$$= 1 - \frac{(\cos^2 s' + d^2 \sin^2 s')^2}{d^2} .$$

At the point of striction l_2 of L we have $K(Y_{l_2}, \xi_{l_2}) = 1 - d^2$ and at l_1 we find $K(Y_{l_1}, \xi_{l_1}) = (d^2 - 1)/d^2$.

Remark. Suppose that a two-dimensional direction of the tangent space of the G.R.S. at p is given by the unit vector $Y_p \in N_p$ and an orthogonal unit vector $Z_p = \cos \theta \xi_p + \sin \theta e_p$, with $e_p \in N_p$, then we proved in [6] that the Riemannian curvature $K(Y_p, Z_p)$ of the G.R.S. is given by

$$K(Y_p, Z_p) = \sin^2 \theta + K(Y_p, \xi_p) \cos^2 \theta .$$

So, we find here, because of (2.2):

$$K(Y_p, Z_p) = 1 - \frac{d^2 \cos^2 \theta}{(\sin^2 s + d^2 \cos^2 s)^2} .$$

Next, there is in the general case just one polar simplex $s_0, \ldots, s_n$ in N (i.e. a simplex such that the distances in N between s_i and $s_0, \ldots, \hat{s}_i, \ldots, s_n$ are $\pi/2$, $i = 0, \ldots, n$) such that $f(s_0), \ldots, f(s_n)$ is a polar simplex in $f(N)$. The vertices $s_0, \ldots, s_n$ are called the *points of striction of* N. For each complete geodesic L of N through a point of striction s_i, s_i is a point of striction of L, while the other point of striction of L is the intersection of L with the $(n-1)$-dimensional complete totally geodesic subspace of N (or of E^m) through $s_0, \ldots, \hat{s}_i, \ldots, s_n$. In particular for the sides $S_{ij} = s_i s_j$, $i \neq j$, $i, j = 0, \ldots, n$ of the simplex, s_i and s_j are the points of striction of S_{ij} and we denote the parameter of distribution of S_{ij} with respect to s_i by d_{ij}. These d_{ij}, $i, j = 0, \ldots, n$, $i \neq j$ are called the *principal parameters of distribution of the generating space* N and the sides S_{ij} are called the *principal axes in* N.

Next, assume that we have in E^m a projective coordinate system such that $s_0, \ldots, s_n$ are the first $n + 1$ base points and that Γ has again the equation $x_0^2 + \ldots + x_m^2 = 0$. Working in the n-dimensional space N, we write only the first $n + 1$ coordinates of the points (all the others are zero). So we have $s_0(1, 0, \ldots, 0)$, $s_1(0, 1, 0, \ldots, 0), \ldots$ $\ldots, s_n(0, \ldots, 0, 1)$ and the absolute hyperquadric $\Gamma \cap N$ in N has the equation $x_0^2 + \ldots + x_n^2 = 0$. The absolute hyperquadric Γ' of N' has an equation of the form $\sum_{i=0}^{n} a_i^2 x_i^2 = 0$, $a_i > 0$, $i = 0, \ldots, n$. If we consider on the principal ax $S_{01} = s_0 s_1$ a variable point $p(1, t, 0, \ldots, 0)$, a straightforward calculation (such as we have done before) shows that if s is the distance between s_0 and p in N and s' is the distance between s_0 and p in N', then $\operatorname{tg} s = t$ and $\operatorname{tg} s' = (a_1/a_0)\, t$. Moreover we know that $\operatorname{tg} s = d_{01} \operatorname{tg} s'$ and thus $d_{01} = a_0/a_1$. In the same way we find $d_{ij} = a_i/a_j$ $i, j = 0, \ldots, n$, $i \neq j$ and from this we see that the equation of Γ' with respect to this projective coordinate system of N is for instance given by $x_0^2 + d_{10}^2 x_1^2 + \ldots + d_{n0}^2 x_n^2 = 0$ or $d_{01}^2 x_0^2 + x_1^2 + d_{21}^2 x_2^2 + \ldots + d_{n1}^2 x_n^2 = 0$ and so on .... Moreover, in the general case, the principal parameters of distribution of the generating space N are related by n^2 independant relations, namely

$$d_{ij} = 1/d_{ji}\,, \quad i, j = 0, \ldots, n\,, \quad i < j \quad \text{and (for instance)} \quad d_{0r} d_{rh} d_{h0} = 1\,,$$
$$r, h = 1, \ldots, n\,, \quad r < h\,.$$

Next, we have the following relation between the scalar curvatures $r(s_i)$, $i = 0, \ldots$, of the G.R.S. at the points of striction $s_0, \ldots, s_n$ of the generating space N:

$$\sum_{i=0}^{n} \frac{2}{n^2 + n + 2 - r(s_i)} = 1\,. \tag{2.3}$$

Proof. Because of (2.2), a straightforward calculation shows that

$$r(s_i) = n(n+1) - 2 \sum_{\substack{j=0 \\ j \neq i}}^{n} d_{ji}^2\,.$$

Put $x^i = \frac{1}{2}(n(n+1) - r(s_i))$, and eliminate the $n(n+1)$ parameters d_{rh}, $r, h = 0, \ldots, n$, $r \neq h$ out of the following system of equations

$$x^i = \sum_{\substack{j=0 \\ j \neq i}}^{n} d_{ji}^2\,, \quad i = 0, \ldots, n\,,$$
$$d_{0k}^2 d_{kf}^2 d_{f0}^2 = 1\,, \quad k, f = 1, \ldots, n\,, \quad k < f\,,$$
$$d_{rh}^2 = 1/d_{hr}^2\,, \qquad h, r = 0, \ldots, n\,, \quad h < r\,.$$

We find

$$x^0 = \sum_{k=1}^{n} \frac{x^0 + 1}{x^k + 1} \quad \text{or} \quad \sum_{h=0}^{n} \frac{1}{x^h + 1} = 1\,,$$

which completes the proof.

Remarks 1. Since

$$n^2 + n + 2 - r(s_i) = 2 \sum_{\substack{j=0 \\ j \neq i}}^{n} d_{ji}^2 + 2\,,$$

none of the denominators in (2.3) can be zero.

2. From the foregoing we see now when we have the general case: Γ' is in general position with respect to $\Gamma \cap N$ iff the principal parameters of distributions are

mutually different strict positive numbers which are moreover all different from $+1$. In order to have this, it is sufficient because of the relations connecting the principal parameters of distribution, to assume that for instance $d_{01}, \ldots, d_{0n}$ are mutually different and all different from $+1$.

3. If we are not in the general case, then for instance, we can have more than $n + 1$ points of striction in N. Consider the case where $d_{ij} = 1$, $i, j = 0, \ldots, n$, $i \neq j$, then $f: N \to f(N)$ is an isometry and each point of N can be considered as a point of striction of N. In this case it is not difficult, because of (2.2), to see that the scalar curvature of the G.R.S. is equal to $n(n - 1)$ at each point of N and thus formula (2.3) is still correct (for any $n + 1$ mutually different points of N).

4. For a non-developable ruled surface in E^m, thus for $n = 1$, the foregoing is also correct: we have now in general two points of striction s_0, s_1 on the generator N and along N the Riemannian curvature of the ruled surface is given by (2.2). Formula (2.3) becomes now, if $K(s_0)$ and $K(s_1)$ are the Riemannian curvatures of the ruled surface at s_0 and s_1: $1/(2 - K(s_0)) + 1/(2 - K(s_1)) = 1$. This is correct, because if $d_{01} = d$ is the parameter of distribution of N with respect to s_0, then $K(s_1) = 1 - d^2$ and $K(s_0) = (d^2 - 1)/d^2$ because of (2.2).

Next, consider a geodesic S of N through $s_0(1, 0, \ldots, 0)$ and assume that the point of intersection of S with the totally geodesic subspace of N through $s_1, \ldots, s_n$ has coordinates $(0, b_1, \ldots, b_n)$. Then again an analogous calculation shows that the parameter of distribution d of S with respect to s_0 is given by $d^2 = (\sum_{i=1}^{n} b_i^2)/(\sum_{i=1}^{n} d_{i0}^2 b_i^2)$. Thus, if we take any point of striction s_i of N, the geodesics of N through s_i for which the parameter of distribution with respect to s_i are extremal, are the principal axes S_{ij}, $j = 0, \ldots, \hat{\imath}, \ldots, n$, through s_i. Moreover, because of (2.2), these S_{ij} determine the principal directions of N through s_i. In connection with the lines of sectional curvature of N we have the following:

Suppose that the points of striction $s_0, \ldots, s_n$ of N are again the base points of a projective coordinate system in N such that $\Gamma \cap N$ has the equation $x_0^2 + \ldots + x_n^2 = 0$ and that Γ' has the equation $x_0^2 + d_{10}^2 x_1^2 + \ldots + d_{n0}^2 x_n^2 = 0$. Consider the class of hyperquadrics of N given by

$$\frac{x_0^2}{1 + k} + \frac{x_1^2}{d_{01}^2 + k} + \ldots + \frac{x_n^2}{d_{0n}^2 + k} = 0\,, \quad k \in R\,.$$

Through each real point p of N we have n real hyperquadrics of this kind and the lines of sectional curvature of N through p are the intersection lines of each time $n - 1$ of these hyperquadrics.

Proof. Suppose that $u_0, \ldots, u_n$ are tangential projective coordinates in N. The tangential equation of $\sum_{i=0}^{n} x_i^2 = 0$ (resp. $x_0^2 + d_{10}^2 x_1^2 + \ldots + d_{n0}^2 x_n^2 = 0$) is $\sum_{i=0}^{n} u_i^2 = 0$

(resp. $u_0^2 + (u_1^2/d_{10}^2) + \dots + (u_n^2/d_{n0}^2) = 0$ or $u_0^2 + d_{01}^2 u_1^2 + \dots + d_{0n}^2 u_n^2 = 0$). The tangential bundle determined by these two tangential hyperquadrics is given by $u_0^2(1 + k) + u_1^2(d_{01}^2 + k) + \dots + u_n^2(d_{0n}^2 + k) = 0$, $k \in R$. The punctual equation of this bundle is:

$$\frac{x_0^2}{1 + k} + \frac{x_1^2}{d_{01}^2 + k} + \dots + \frac{x_n^2}{d_{0n}^2 + k} = 0 , \quad k \in R .$$

Through a general point p $(p_0, \dots, p_n)$ of N, we have n hyperquadrics $\Sigma_1, \dots, \Sigma_n$ of this bundle, respectively corresponding with mutually different values $k_1, \dots, k_n$ of k. Thus we have

$$F_j(p) = \frac{p_0^2}{1 + k_j} + \frac{p_1^2}{d_{01}^2 + k_j} + \dots + \frac{p_n^2}{d_{0n}^2 + k_j} = 0 , \quad j = 1, \dots, n .$$

Suppose that $1 \leqq i_1 < i_2 \leqq n$, then

$$F_{i_1}(p) - F_{i_2}(p) = \left(\frac{p_0^2}{(1 + k_{i_1})(1 + k_{i_2})} + \frac{p_1^2}{(d_{01}^2 + k_{i_1})(d_{01}^2 + k_{i_2})} + \dots \right.$$
$$\left. \dots + \frac{p_n^2}{(d_{0n}^2 + k_{i_1})(d_{0n}^2 + k_{i_2})} \right) (k_{i_2} - k_{i_1}) = 0 .$$

Since $k_{i_1} \neq k_{i_2}$, this means that the tangent spaces of Σ_{i_1} and Σ_{i_2} at p are conjugate with respect to Γ. Next we have

$$k_{i_1} F_{i_1}(p) - k_{i_2} F_{i_2}(p) = \left(\frac{p_0^2}{(1 + k_{i_1})(1 + k_{i_2})} + \frac{p_1^2 d_{01}^2}{(d_{01}^2 + k_{i_1})(d_{01}^2 + k_{i_2})} + \dots \right.$$
$$\left. \dots + \frac{p_n^2 d_{0n}^2}{(d_{0n}^2 + k_{i_1})(d_{0n}^2 + k_{i_2})} \right) (k_{i_1} - k_{i_2}) = 0 ,$$

which means that the tangent spaces of Σ_{i_1} and Σ_{i_2} at p are also conjugate with respect to Γ'. So, we see that the tangents at p of the n intersection curves σ_i of $\Sigma_1, \dots, \hat{\Sigma}_i, \dots$ $\dots, \Sigma_n$ through p, $i = 1, \dots, n$, are mutually orthogonal in N and in N'. This completes the proof.

Remark that the lines of sectional curvature through a point of striction of N are the principal axes of N through that point.

Next, consider a point p of N and suppose that the unit vectors $T_p^1, \dots, T_p^n$ determine the principal directions of N at p. If ξ_p is the unit normal vector on N in the G.R.S. at p, then we have

$$\mathscr{K}(p) = \prod_{i=1}^{n} K^i(p) = \prod_{i=1}^{n} (K(T_p^i, \xi_p) - 1) ,$$

and because of (1.3), we get the geometrical signification:

$$\mathscr{K}(p) = (-1)^n \left(\frac{\omega'}{\omega} \right)^2 .$$

But we also have the following: suppose that s, resp. s', is the distance between p and the point of striction s_n in N, resp. in N', and that $\mathscr{D}_n = \prod_{j=0}^{n-1} d_{nj}$, then, if $s \neq \pi/2$ (and thus also $s' \neq \pi/2$):

$$\mathscr{K}(p) = (-1)^n \frac{\cos^{2n+2} s'}{\mathscr{D}_n^2 \cos^{2n+2} s} . (2.4) . \tag{2.4}$$

Proof. Consider an Euclidean n-space $\bar{N}$ with an orthonormal coordinate system with origin 0 and use homogeneous coordinates $(x_0, \ldots, x_n)$ with respect to this coordinate system, such that the hyperplane at infinity has the equation $x_n = 0$. Suppose that we have in $\bar{N}$ a Cayley model of an elliptic geometry N' of curvature $+1$, with absolute hyperquadric given by $x_0^2/d_{n0}^2 + x_1^2/d_{n1}^2 + \ldots + x_{n-1}^2/d_{nn-1}^2 + x_n^2 = 0$, then we proved in [4] that if $\bar{\omega}$, resp. ω', is a volume element of $\bar{N}$, resp. N', at a point p of $\bar{N}$ and if s' is the (elliptic) distance in N' between p and 0, that $(\omega'/\bar{\omega})^2 = \cos^{2n+2} s'/\mathscr{D}_n^2$ (2.5). If we have in $\bar{N}$ an other Cayley model of an elliptic geometry N of constant curvature $+1$, with absolute hyperquadric given by $\sum_{i=0}^{n} x_i^2 = 0$, then we have in the same way, if ω is a volume element at p of N and if s is the (elliptic) distance in N between p and 0, that $(\omega/\bar{\omega})^2 = \cos^{2n+2} s$ (2.6).

Since $\mathscr{K}(p) = (-1)^n (\omega'/\omega)^2$, since for a finite point p of $\bar{N}$ $s \neq \pi/2$ and $s' \neq \pi/2$ and since 0 has coordinates $(0, \ldots, 0, 1)$ formula (2.4)) follows from (2.5) and (2.6).

Remark that in (2.4), s' is the angle in E^m between the tangent spaces of the G.R.S. at p and at s_n.

An analogous formula for $\mathscr{K}(p)$ can be obtained using any point of striction of N. In particular, if $\mathscr{D}_i = \prod_{j=0, j\neq i}^{n} d_{ij}$, we have at s_i:

$$\mathscr{K}(s_i) = (-1)^n/\mathscr{D}_i^2 , \quad i = 1, \ldots, n .$$

As a corollary we get:

$$\prod_{i=0}^{n} \mathscr{K}(s_i) = +1 .$$

Next, because of (1.3) we find here also, such as in the "Euclidean case", that $\int_N^{(n)} (\sqrt{(-1)^n \mathscr{K}})\, \omega$ is equal to the volume (= n-dimensional area) of an n-dimensional half unit sphere. Thus, if $n = 2f$ $(f > 0)$, then

$$\int_N^{(n)} (\sqrt{(-1)^n \mathscr{K}})\, \omega = 2^{2f-1} \pi^f \frac{(f-1)!}{(2f-1)!}$$

and, if $n = 2f + 1$ $(f \geqq 0)$, then

$$\int_N^{(n)} (\sqrt{(-1)^n \mathscr{K}})\, \omega = \frac{\pi^{f+1}}{f!} .$$

Finally, remark that we also have immediately the analogous properties of the dual G.R.S. (D.G.R.S.). We give some examples: if L is any geodesic of N with points of striction l_1 and l_2, then $f(l_1)$ and $f(l_2)$ are the points of striction of the geodesic $f(L)$ of the generating space $f(N)$ of the D.G.R.S. If d is the parameter of distribution of L with respect to l_1, then $1/d$ is the parameter of distribution of $f(L)$ with respect to $f(l_1)$. If $p \in L$, if Y_p (resp. $\bar{Y}_{f(p)}$) and ξ_p (resp. $\bar{\xi}_{f(p)}$) is a unit vector at p tangent to L (resp. at $f(p)$ tangent to $f(L)$) and the unit normal vector at p on N in the G.R.S. (resp. at $f(p)$ on $f(N)$ in the D.G.R.S.), then the Riemann curvatures $K(Y_p, \xi_p)$ of the G.R.S. and $K(\bar{Y}_{f(p)}, \bar{\xi}_{f(p)})$ of the D.G.R.S. are related by (if both are not zero.) $1/K(Y_p, \xi_p) + 1/K(\bar{Y}_{f(p)}, \bar{\xi}_{f(p)}) = 1$. Moreover we have $K(Y_p, \xi_p) = 0 \Leftrightarrow K(\bar{Y}_{f(p)}, \bar{\xi}_{f(p)}) = 0$.

If $s_0, \ldots, s_n$ are the points of striction of N and d_{ij} the principal parameters of distribution, then $f(s_0), \ldots, f(s_n)$ are the points of striction of $f(N)$ and the principal parameters of distribution $\bar{d}_{ij}$ of $f(N)$ are given by $\bar{d}_{ij} = d_{ji}$.

At corresponding points we have $\mathscr{K}(p) = 1/\mathscr{K}(f(p))$, etc. ...

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