Collins I. F. & Kelly, P. A. (2002). *Géotechnique* **52**, No. 7, 507–518

A thermomechanical analysis of a family of soil models

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Techniques of thermomechanics, based upon the use of internal variables, are used to develop a systematic procedure for deriving elastic/plastic models of soils and granular materials. Fundamental thermodynamic state variables are defined and used to formulate free energy and dissipation potentials. These are used to determine yield conditions and flow rules. It is demonstrated that it is necessary to distinguish between plastic work and plastic energy dissipation. It is suggested that the stored energy associated with the plastic deformations is due to the fact that only a proportion of the intergrain contacts are actually plastic in a plastically deforming continuum element. The stored plastic energy of the continuum model arises from locked-in elastic energy on the micro scale. Some well-known existing critical-state models are re-examined and some of their shortcomings are highlighted. New models are proposed that overcome some of these objections. These models are able to predict non-associated flow rules, contractive behaviour and pre-peak failure for 'loose' soils, and aspects of static liquefaction, and can predict the position of the failure, phase change, instability and ultimate state lines. In some extreme cases the yield surfaces are found to contain concave segments, and dilatant behaviour can occur below the critical (characteristic) state line.

KEYWORDS: constitutive relations; numerical modelling and analysis; plasticity; sands; theoretical analysis

Nous utilisons des techniques de thermomécanique, basées sur l'utilisation de variables internes, pour développer une procédure systématique permettant de dériver des modèles élastiques/plastiques de sols et de matières granulaires. Les variables fondamentales d'état thermodynamique sont définies et utilisées pour formuler des potentiels d'énergie libre et de dissipation. Ceux-ci sont utilisés pour déterminer les conditions de limite élastique et les règles d'écoulement. Nous démontrons qu'il est nécessaire de faire la distinction entre travail plastique et dissipation de l'énergie plastique. Nous suggérons que l'énergie emmagasinée associée à la déformation plastique est due au fait que seule une proportion des contacts entre grains est plastique dans un élément de continuum à déformation plastique. L'énergie plastique emmagasinée du modèle de continuum vient d'une énergie élastique contenue sur l'échelle micro. Nous examinons à nouveau certains modèles d'état critique existants et bien connus et nous montrons leurs défauts. Nous proposons de nouveaux modèles qui viennent à bout de certaines de ces objections. Ces modèles sont capables de prédire les règles d'écoulement non associées, le comportement contre-actif et la rupture de pré-crête pour des sols 'meubles' et les aspects de liquéfaction statique ; nous pouvons ainsi prévoir la position de la faille, le changement de phase, l'instabilité et les lignes d'état ultime. Dans certains cas extrêmes, nous avons trouvé que les surfaces d'écoulement contiennent des segments concaves et que le comportement dilatant peut se produire en dessous de la ligne (caractéristique) d'état critique.

INTRODUCTION

There are a variety of ways of constructing mathematical models of the mechanical behaviour of soils. One is to curve-fit experimental data and formulate the basic equations needed to describe the model in terms of these empirically derived functions. However, even in this empirical approach, it is necessary to have some underlying theoretical framework that enables the model to have the necessary predictive qualities. The well-developed theory of rate-independent elastic/plastic materials is by far the most popular such background framework currently in use, and the theory developed by Lade and co-workers is one of the best known of such empirically based models (e.g. Lade, 1975).

A second approach to modelling is to start with some specific theoretical assumptions, such as a dilatancy law, an expression for 'plastic work' or 'energy dissipation', as in the original Cam clay studies (Schofield & Wroth, 1968; Gens & Potts, 1988; Wood, 1990), or to make some other micro-mechanical assumptions, as in the recently developed models involving particle crushing by McDowell & Bolton (1998) and McDowell (2000). These Cam clay, critical-state models and the vast hierarchy of extensions are excellent examples of this approach to modelling. Because these theories are based on conceptual models of macro or micro behaviour, such models are capable of giving deep physical insights into the engineering behaviour of soils, although quite frequently at the expense of detailed numerical accuracy. The pedagogical merits of this approach are elegantly argued in Wood (2000). The choice of complexity of the model depends on the practical purpose underlying the model construction. There are now some very complex models available, involving large numbers of material parameters, but, as pointed out by Kolymbas (2000), such models are seldom 'transportable' from one research group to another.

A basic requirement of all such models is that they satisfy the basic laws of physics. For example, any equations describing the properties of isotropic materials must be expressed in terms of the invariants of stress, strain, strain increment, etc. The second law of thermodynamics is one of these basic laws that govern the dissipative behaviour of materials, but it is seldom invoked in geomechanical theories. Until recently thermodynamics and 'heat flow' were seldom seen as relevant to geotechnical problems. A noticeable exception is the overview paper by Mitchell (1991). However, this situation is now changing, particularly as a result of the development of geo-environmental engineering: see Smith (2000) for a recent review of some of the application areas. In the early days of the development of the theory of elastic/plastic materials, quasi-thermodynamic postulates were introduced in an effort to ensure that the dissipation of energy was always positive in a closed cycle of stress (Drucker's postulate) or strain (Il'iushin's postulate).

Manuscript received 21 September 2001; revised manuscript accepted 7 June 2002.

Discussion on this paper closes 1 March 2003, for further details see p. ii.

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However, it was soon realised that these postulates were actually classifications of types of material behaviour and not, in any sense, equivalent to the second law (Drucker, 1988; Lubliner, 1990).

In the last 20 years, however, there have been major developments in the theory of thermomechanics of continua, which have thrown new light on many long-standing issues in a number of application areas. In the case of isothermal deformations of rate-independent elastic/plastic solids, the new theories have shown that, under certain minimal assumptions, knowledge of the material's free energy and dissipation rate potentials is sufficient to uniquely determine the elasticity law, the yield function, the flow rule and the hardening rules (both isotropic and kinematic) of the material. The simultaneous determination of both the yield function and the plastic potential is particularly significant, since, in geomechanics, these two functions are normally chosen independently. (A notable exception to this is the modelling process used by Chandler (1985, 1988) and Chandler & Song (1990), whose approach using the theory of envelopes has several similarities to the more general thermomechanical procedure.)

The early developments of these general ideas were due to Ziegler (1983) and Ziegler & Wehrli (1987), who noted some applications of their general theory to the classical Coulomb model. A rigorous general theory was then developed by a number of researchers in France, accounts of which can be found in the books by Maugin (1992, 1999), Maugin *et al.* (2000), Besseling & van der Giessen (1994) and Lemaitre & Chaboche (1990). Applications of these ideas to soil mechanics were pioneered by Houlsby (1981, 1982, 1993). A comprehensive analysis of the isothermal thermomechanics of geomaterials was given by Collins & Houlsby (1997), who demonstrated that a *non-associated flow rule* is a *necessary* property of a '*frictional material*', in which the plastic deformations are governed by stress ratios rather than by the magnitudes of certain yield stresses. Houlsby & Puzrin (2000) generalised some aspects of this work to non-isothermal conditions, and in a series of papers have developed a family of sophisticated models, based on the use of internal functions (e.g. Puzrin & Houlsby, 2001b, 2001c). These authors have proposed the term 'hyperplasticity' to describe theories in which the irreversible plastic behaviour is determined by a dissipation potential function, by analogy with the well-established term 'hyperelasticity' used in the analogous situation in elasticity theory.

The object of this paper is to present a systematic procedure for establishing a hierarchy of models, starting from the familiar Cam clay models, which include internal friction, non-associated flow rules and a variety of biased volumetric/shear hardening laws. These models have the advantage of automatically satisfying the second law of thermodynamics. This is not as trivial an exercise as may at first be thought, since, as will be seen, the well-known original Cam clay model actually violates the second law, even though it satisfies Drucker's postulate. It will also be shown that some models with significant amounts of internal friction actually have yield surfaces parts of which are *concave,* a property that violates Drucker's postulate but not the second law.

In the next section the basic thermodynamic concepts of state and state variable are discussed in a geomechanics context. The following section is a brief outline of the procedure for constructing the yield function and flow rule of the material from a known dissipation function. A key step in this procedure is the use of the intermediate *dissipative stress space*. The well-known elementary models for isotropic compression, Coulomb friction and original and modified Cam clay are then reviewed in the light of this new procedure. This review suggests ways in which the original dissipation functions for these simple models can be modified to include internal friction and shear hardening. Some illustrative examples are given in the final section.

VOLUME FRACTIONS AND ELASTIC/PLASTIC STRAINS: OBSERVABLE STATE PARAMETERS

There are several differences between the fundamental properties of metals and soils, which require a number of modifications to be made to the standard elastic/plastic theories for metals, if they are to be extended to soils. Here we concentrate on two of these factors:

(*a*) A soil does not have a 'natural state' to which it returns when all the stresses are removed. Here we shall adopt the convention of introducing an effective pressure, p'_{R}, which is the pressure in an arbitrarily chosen reference state.

(*b*) Since a soil is a two-phase material, any description of the 'state' of a soil must include some volume fraction parameter, such as voids ratio, e, or specific volume, v. However, knowledge of v alone is not sufficient to determine the state uniquely. Another state variable is needed. This is frequently taken to be the effective pressure, so that a state is defined by knowledge of both v and p' – the coordinates of a point in the standard isotropic compression diagram.

When constructing thermomechanical descriptions of states, it is usual to start with 'observable' quantities such as volumes or strains, which are directly measurable geometric quantities, as the *fundamental, independent state variables*. Other state variables such as pressure and stress, which describe 'what is being done to the specimen', are viewed, at least initially, as the *dependent state variables*. These variables are frequently interchanged in the subsequent theory development, the general theory of which is most elegantly expressed in terms of Legendre transformations (Callen, 1960; Collins, 1996; Wilmanski, 1998). There are, however, rules governing these interchanges. Most importantly, it is impossible to have a geometric variable and its work conjugate force variable as a pair of *independent* state variables. By work conjugate variables we mean those variables whose product gives the work done. For example, the product of pressure and volume strain increment give the work done, so that pressure and volume strain are conjugate variables, as indeed are p' and v, since $p'\delta v$ is also the work increment, under the standard assumption that the solid phase of the material is incompressible. These work conjugate variables are related through a constitutive equation, or equation of state, and cannot both be chosen as the primary independent state variables.

In a thermomechanical formulation of geomaterials we cannot therefore choose v and p' as our initial independent state variables. In the standard development of isotropic compression of soils p' is frequently replaced by p_{c}, the normal consolidation pressure, and v and p_{c} are taken as the fundamental pair of state variables. The variables in this formulation are still mixed, however. A completely geometric description requires us to find a geometric parameter that defines the position of the elastic swelling line in place of p_{c}. Iwan & Chelvakumar (1988) use the voids ratio corresponding to p_{c} in their strain-space formulation of clays. Here, however, we prefer to follow Hashiguchi (1995) and introduce the *reference, plastic specific volume*, v^{p}, defined to be the specific volume of the sample, attained when it is unloaded from its current state to the reference pressure, p'_{R}. This is an example of an *internal variable*, as it is 'observable', but not 'controllable'. We hence take v

and v^{p} as the defining independent volume fraction variables. This is in line with modern thermomechanical developments of plasticity theory, where the total and plastic strains are taken as the defining state variables.

We shall use the finite logarithmic (or Hencky) measure of strain, so that the volume strain is $e_{\mathrm{v}} = \ln(V_0/V)$, and the corresponding volumetric strain increment is $\delta e_{\mathrm{v}} = -\delta V/V$. Assuming that the lid phase is incompressible, these relations are equally valid when the total volumes are replaced by specific volumes. The corresponding finite shear strain can be defined by

$$e_\gamma = \tfrac{2}{3}(e_1 - e_2) = \tfrac{2}{3}[\ln(L_0/L) - \ln(R_0/R)] = \ln(\Lambda_0/\Lambda) \tag{1}$$

where the aspect ratio of the cylindrical specimen is defined to be $\Lambda = (L/R)^{2/3}$. This definition ensures that the shear strain is work conjugate to q, as conventionally defined in the analysis of triaxial tests. An advantage of the use of logarithmic strains is that the total strain can be expressed as the sum of the elastic and plastic strains, just as for small strains. Consider the deformation, which transforms the sample from state A to state B in Fig. 1. The total strain is

$$e_{\mathrm{v}} = \ln(v_0/v_1) = \ln(v_0/v_0^{\mathrm{p}}) + \ln(v_0^{\mathrm{p}}/v_1^{\mathrm{p}}) + \ln(v_1^{\mathrm{p}}/v_1) \tag{2}$$

The first term is the elastic strain from A to X. The second term is $e_{\mathrm{v}}^{\mathrm{p}}$, the plastic strain from A to B (that is, the actual strain between the unloaded states, from X to Y), and the last term is the elastic strain from Y to B. The sum of the first and last terms is hence $e_{\mathrm{v}}^{\mathrm{e}}$, the elastic component of the total strain.

If we further assume that the material is decoupled, by which we mean that the elastic moduli are independent of the accumulated plastic strain, the elastic loading/unloading lines are now parallel curves, so that their slopes are independent of v. That is,

$$\frac{\mathrm{d}(\ln v)}{\mathrm{d}\,p'} = F(p') \tag{3}$$

The function F has the dimensions of $(\text{stress})^{-1}$, so that in a purely frictional or cohesionless material, which has no material parameter with the dimensions of stress, this function can only be of the form of $(\text{constant}/p')$, so that on the elastic line

$$\frac{\mathrm{d}(\ln v)}{\mathrm{d}\,p'} = -\frac{\kappa}{p'}$$

and hence

$$\ln v + \kappa\tau\eta \ln p' = \text{constant} \tag{4}$$

This is the slightly modified form of the standard equation, with v having been replaced by $\ln v$. However, here we have *proved* this result for *decoupled, frictional materials* by appealing to the principles of dimensional analysis. Arguments for favouring this $\ln v - \ln p'$ relation over the traditional $v - \ln p'$ relation have been advanced by Hashiguchi (1974, 1995), Butterfield (1979) and Houlsby (1981). Pestana & Whittle (1995) have presented an overview of various models, and argue the merits of a $\ln e$–$\ln p'$ relation. Of course, once one departs from the decoupled and frictional assumptions, the above proof breaks down.

Since the reference pressure is arbitrary, we can choose it to be the initial pressure, so that $v_0^{\mathrm{p}} = v_0$, in which case equation (2) can be rewritten in the more useful form

$$e_{\mathrm{v}} = \ln(v_0/v) = e_{\mathrm{v}}^{\mathrm{e}} + e_{\mathrm{v}}^{\mathrm{p}} = \ln(v^{\mathrm{p}}/v) + \ln(v_0/v^{\mathrm{p}}) \tag{5}$$

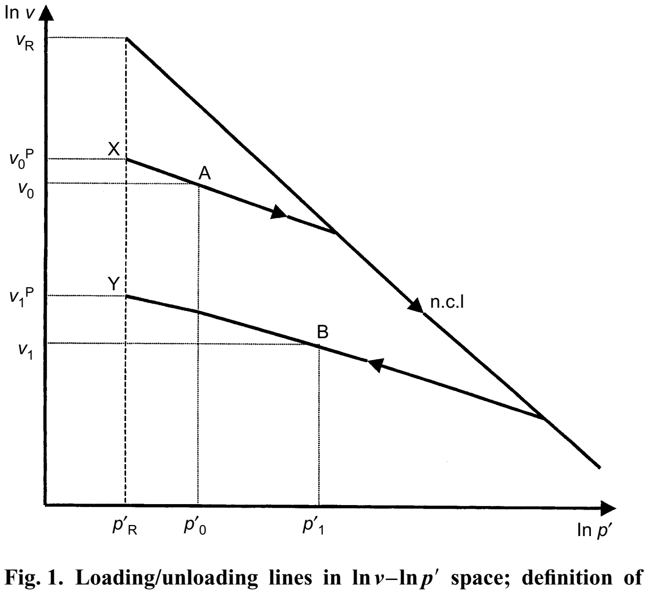


Fig. 1. Loading/unloading lines in ln$v$–lnp' space; definition of reference, plastic specific volume

THE THERMOMECHANICAL PROCEDURE FOR ESTABLISHING ELASTIC/PLASTIC CONSTITUTIVE LAWS

The general procedure for establishing constitutive laws for rate-independent materials, starting from the principles of thermodynamics, will be illustrated here. For textbook accounts, see the books cited in the introduction. The papers by Collins & Houlsby (1997), Houlsby & Puzrin (2000) and Collins (2002) contain accounts particularly relevant to geomechanics. When the deformation is isothermal, the incremental work done by the applied stresses is the sum of the increment in the free energy function, Ψ, and the increment in the dissipation function, Φ, all these functions being defined per unit volume. Here we use the notation appropriate for analysing triaxial tests. The basic energy relation equates the incremental work done by the applied stresses p' and q to the increase in free energy plus the energy that is dissipated:

$$p'\delta e_{\mathrm{v}} + q\delta e_\gamma = \delta\Psi + \delta\Phi, \quad \text{where } \delta\Phi \geqslant 0 \tag{6}$$

This inequality is the statement of the second law of thermodynamics appropriate for isothermal deformations, and is a strict 'greater than' whenever any irreversible plastic deformations occur (Callen, 1960; Dugdale, 1996; Wilmanski; 1998). (Note that Collins & Houlsby (1997) and Houlsby (2000) drew attention to the possibility of adding part of the free energy function to the dissipation function and still keeping $\delta\Phi$ non-negative. However, these modified dissipation functions do not satisfy the stronger *converse* statement of the second law, namely that $\delta\Phi$ must be strictly positive when *any* plastic deformation occurs.)

The free energy is a function of the observable kinematic state variables, e.g. the total and plastic strains e_{v}, e_γ and $e_{\mathrm{v}}^{\mathrm{p}}$, e_γ^{p} respectively. Instead of using these volumetric strains, one could use the logarithm of the total and plastic specific volumes, as described above. Here, however, we shall continue to use strains as the independent variables. Hence the increment in the free energy is

$$\begin{aligned}\delta\Psi = {} & (\partial\Psi/\partial e_{\mathrm{v}})\delta e_{\mathrm{v}} + (\partial\Psi/\partial e_\gamma)\delta e_\gamma + (\partial\Psi/\partial e_{\mathrm{v}}^{\mathrm{p}})\delta e_{\mathrm{v}}^{\mathrm{p}} \\ & + (\partial\Psi/\partial e_\gamma^{\mathrm{p}})\delta e_\gamma^{\mathrm{p}}\end{aligned} \tag{7}$$

The dissipation function is *not* a state function, as it depends on the increments in the plastic strains as well as the strains themselves. It cannot depend on the *total strain increments*, since otherwise a purely elastic deformation would produce dissipation. Hence the dissipation increment function is of the form $\delta\Phi(e^{\mathrm{p}}_{\mathrm{v}}, e^{\mathrm{p}}_{\gamma}; \delta e^{\mathrm{p}}_{\mathrm{v}}, \delta e^{\mathrm{p}}_{\gamma})$. In the case of rate-independent, elastic/plastic materials, this function is homogeneous of degree 1 in the plastic strain increments. It follows from Euler's theorem for such functions that

$$\delta\Phi = [\partial(\delta\Phi)/\partial(\delta e^{\mathrm{p}}_{\mathrm{v}})]\delta e^{\mathrm{p}}_{\mathrm{v}} + [\partial(\delta\Phi)/\partial(\delta e^{\mathrm{p}}_{\gamma})]\delta e^{\mathrm{p}}_{\gamma} \tag{8}$$

In the case of *frictional materials* it has been shown by Collins & Houlsby (1997) that $\delta\Phi$ depends, in addition, on the *effective pressure*, or equivalently on the *total* volumetric strain (Collins, 1998). This does not affect the validity of equation (8), however. Eliminating $\delta\Psi$ and $\delta\Phi$ between equations (6)–(8), and equating the four independent strain increment terms, gives the fundamental relations

$$p' = \partial\Psi/\partial e_{\mathrm{v}} \text{ and } q = \partial\Psi/\partial e_{\gamma} \tag{9}$$

and

$$\pi' = -\partial\Psi/\partial e^{\mathrm{p}}_{\mathrm{v}} = \partial(\delta\Phi)/\partial(\delta e^{\mathrm{p}}_{\mathrm{v}}) \text{ and } \tau = -\partial\Psi/\partial e^{\mathrm{p}}_{\gamma} = \partial(\delta\Phi)/\partial(\delta e^{\mathrm{p}}_{\gamma}) \tag{10}$$

Equations (9) show that the effective stresses can be deduced from the free-energy function, whereas equations (10) *define* the *dissipative* (thermodynamic or generalised) effective pressure, π', and *dissipative* shear invariant, τ, in terms of the free energy function, and show that they can also be derived from the dissipation function. The dissipative stresses are the work conjugate stresses to the plastic strains, just as the effective stresses are conjugate to the total strains. The exact conditions, which are very weak, under which the deductions of equations (9) and (10) are valid are fully discussed in the books and papers cited above.

It is important to distinguish between dissipation and plastic work. Using equations (8) and (10) the dissipation increment is given by

$$\delta\Phi = \pi'\delta e^{\mathrm{p}}_{\mathrm{v}} + \tau\delta e^{\mathrm{p}}_{\gamma} \tag{11}$$

and the plastic work increment is defined by

$$\delta W^{\mathrm{p}} = p'\delta e^{\mathrm{p}}_{\mathrm{v}} + q\delta e^{\mathrm{p}}_{\gamma} \tag{12}$$

The second law states that $\delta\Phi$ can never be negative, but the sign of the plastic work increment is not restricted (Mroz, 1973; Lubliner, 1990). Since $\delta\Phi$ is homogeneous of degree 1 in the plastic strain increments, the two derivatives of $\delta\Phi$ occurring in equations (10) are *homogeneous of degree zero*: that is, they depend just on the ratio of the plastic strain increments, namely the plastic dilation:

$$\delta^{\mathrm{p}} \equiv -\delta e^{\mathrm{p}}_{\mathrm{v}}/\delta e^{\mathrm{p}}_{\gamma} \tag{13}$$

This ratio can then be eliminated from these two equations, giving a relation between π' and τ which is the yield condition in dissipative stress space (Maugin, 1992, 1999; Collins & Houlsby, 1997; Houlsby & Puzrin, 2000). Moreover, the general theory shows that the plastic strain increments are always given by the normal flow rule in this stress space. Several examples of the construction of such yield surfaces will be given below.

The relationship between the true stresses p' and q and the dissipative stresses π' and τ depends on the form of the free-energy function. It follows from equations (9) and (10) that if the free energy function depends only on the elastic strains—that is, just on the *difference* between the total and plastic strains—then the dissipative and true stresses are identical. The yield surface and flow rule can hence be readily transferred to true stress space. However, Collins & Houlsby (1997) showed that in the special case of a *frictional material*, where the dissipation function and hence the yield function in (π', τ) space also depend explicitly on p', the normality property of the flow rule is lost when it is transferred to true stress space.

In the case of a *decoupled material* Ψ is the sum of two functions: one is dependent only on the elastic strains, and the other depends only on the plastic strains, as shown by Collins & Houlsby (1997):

$$\Psi = \Psi_1(e^{\mathrm{e}}_{\mathrm{v}}, e^{\mathrm{e}}_{\gamma}) + \Psi_2(e^{\mathrm{p}}_{\mathrm{v}}.e^{\mathrm{p}}_{\gamma}) \tag{14}$$

In this case equations (9) and (10) give the relations between the two sets of stress variables:

$$p' = \rho' + \pi' \text{ and } q = \zeta + \tau, \text{ where } \rho' = \partial\Psi_2/\partial e^{\mathrm{p}}_{\mathrm{v}} \text{ and } \zeta = \partial\Psi_2/\partial e^{\mathrm{p}}_{\gamma} \tag{15}$$

The stresses ρ' and ζ are termed the *shift stresses*. These stresses have the effect of translating the yield surfaces without change of shape. These shift (or back) stresses play an important role in linear, *anisotropic*, kinematic hardening models, where they represent the stress at the 'centre' of the shifted yield surface (Lemaitre & Chaboche, 1990; Puzrin & Houlsby, 2001a). However, it does not seem to have been fully appreciated that they also play an important role in formulating *isotropic* models of geomaterials with different strengths in tension and compression, as will be shown below. These shift stresses may be thought of as arising from recoverable (elastic) deformations, which are triggered only by the occurrence of plastic strains. They can be represented by a spring *in parallel* with the plastic slider, as described in Puzrin & Houlsby (2001a). Such models have been developed by Walton & Braun (1986), in order to describe the elastic/plastic deformations of the contacts between individual granular particles. However, perhaps a more satisfying conceptual model, which is relevant to geomaterials, and which demonstrates the natural occurrence of such shift stresses, is that described in the books by Mroz (1973) and Besseling & van der Giessen (1994). In this model it is recognised that although a macro-continuum element is assumed to be deforming plastically, some micro-elements within this continuum element will not be plastically stressed, but will still be deforming elastically. This recoverable energy, 'locked' into the macro-deformation, gives rise to the second free energy function, Ψ_2, and its associated shift stress. This energy can be released only when the plastic strains are reversed. In the case of isotropic compression, the macroscopic plastic deformations are normally assumed to be due to the plastic deformation of the interparticle contacts. However, in reality only a fraction of the particle contacts will be plastically stressed, as is graphically illustrated in the well-known, discrete element simulations, such as those of Cundall & Strack (1979). The applied continuum stresses are transmitted through 'force chains' joining only a subset of all the possible particle contacts. The remaining elastically stressed contacts can provide the locked-in elastic energy represented by the second part of the free energy function. The effect of the ratio of the number of plastic/elastic contacts on the stress–strain curves has been studied in the simulations of Thornton (2000).

Once the yield function and flow rule have been determined, the incremental form of the constitutive equations needed for modelling drained and undrained stress and strain paths can be found by the well-established procedures. The yield condition is differentiated to give the consistency equation. The flow rule is then used to determine the hardening modulus and the magnitude of the strain increments and their relation to the stress increments (Wood, 1990; Lubliner, 1990). An overview of the structure of this theory, indicating the steps needed to deduce the form of the yield function, flow rule, incremental form etc. from the free energy and dissipation functions, is given in Fig. 2.

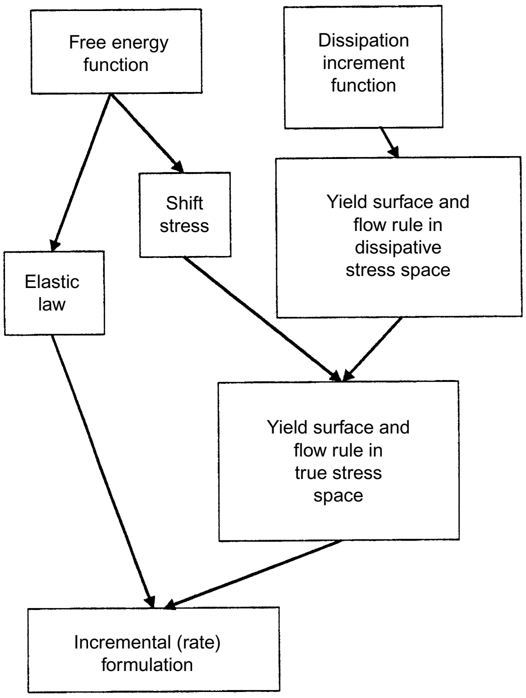


Fig. 2. Flow chart illustrating the steps in constructing the incremental form of the elastic/plastic constitutive law, starting with the free energy and dissipation functions

A REVIEW OF EXISTING MODELS

Here we examine a number of the standard one- and two-dimensional models in the light of the above thermomechanical considerations. We shall consider only cohesionless materials, so that the effective pressure is always positive.

Isotropic compression

In isotropic compression the only non-zero strains are the volumetric components, and the only effective stress is the effective pressure, p'. The standard linear elastic model, with constant bulk modulus K, is obtained by taking the first part of the free energy function to be $\Psi_1 = \frac{1}{2}Ke_{\mathrm{v}}^{\mathrm{e}^2}$, so that, using equation (9), we obtain $p' = Ke_{\mathrm{v}}^{\mathrm{e}}$, and hence $\delta p' = K\delta e_{\mathrm{v}}^{\mathrm{e}}$. If the material is frictional, however, the material does not possess a characterising modulus. Nevertheless, the free energy has the dimensions of stress, and the only stress variable available is the reference pressure, p'_{R}. A possible form for the free energy is hence

$$\Psi_1 = \kappa p'_{\mathrm{R}} \exp(e_{\mathrm{v}}^{\mathrm{e}}/\kappa) \tag{16}$$

where κ is a dimensionless constant. The corresponding effective pressure and effective pressure increment are obtained by differentiation:

$$p' = p'_{\mathrm{R}} \exp(e_{\mathrm{v}}^{\mathrm{e}}/\kappa), \text{ and } \delta p' = (p'_{\mathrm{R}}/\kappa) \exp(e_{\mathrm{v}}^{\mathrm{e}}/\kappa)\delta e_{\mathrm{v}}^{\mathrm{e}}$$
$$= -(p'/\kappa)(\delta v/v) \tag{17}$$

This last equation integrates back to give the linear swelling line (equation (4)) once more.

Since the dissipation increment function has the dimensions of stress/time and is non-negative, in the case of normal compression we can take it to be of the form

$$\delta\Phi(e_{\mathrm{v}}^{\mathrm{p}}, \delta e_{\mathrm{v}}^{\mathrm{p}}) = \pi_{\mathrm{c}}(e_{\mathrm{v}}^{\mathrm{p}})|\delta e_{\mathrm{v}}^{\mathrm{p}}| \tag{18}$$

so that from equation (10), when plastic flow occurs:

$$\pi' = \partial(\delta\Phi)/\partial(\delta e_{\mathrm{v}}^{\mathrm{p}}) = \pm\pi_{\mathrm{c}}(e_{\mathrm{v}}^{\mathrm{p}}) \tag{19}$$

It follows that the yield condition in *dissipative* stress space is $\pi' = \pm\pi_{\mathrm{c}}(e_{\mathrm{v}}^{\mathrm{p}})$, according as $\delta e_{\mathrm{v}}^{\mathrm{p}} >$ or < 0, and $\pi_{\mathrm{c}}(e_{\mathrm{v}}^{\mathrm{p}})$ can be interpreted as the volumetric hardening yield function. It is clear that, if we want to model a material with no strength in tension, we must choose the second part of the free energy function Ψ_2, so that the shift stress, ρ', is also $\pi_{\mathrm{c}}(e_{\mathrm{v}}^{\mathrm{p}})$, and the effective pressure is $p' = \pi' + \pi_{\mathrm{c}}$. Hence in *true* stress space, yield occurs when $p' = 0$ or $p_{\mathrm{c}}(e_{\mathrm{v}}^{\mathrm{p}})$, where $p_{\mathrm{c}} \equiv 2\pi_{\mathrm{c}}$ is, of course, the normal consolidation pressure. This situation is illustrated in Fig. 3(a). The shifting operation has the effect of shifting the elastic interval from $(-\pi_{\mathrm{c}}, \pi_{\mathrm{c}})$ in *dissipative* stress space, to $(0, p_{\mathrm{c}})$ in *true* stress space. As the material hardens under compression, its tensile strength remains at zero. This is a kind of 'passive' Bauschinger effect.

The presence of this shift stress in the modified Cam clay model was noted by Houlsby (1981) and Collins & Houlsby (1997). Here it has been shown to be a natural part of any model with different yield stresses in isotropic tension and compression. (It is *not*, however, a feature of the original Cam clay model, as was also noted in these two references. This apparent contradiction will be resolved below.) If the model is desired to have a non-zero tensile strength, t_{s} say, the shift stress must be chosen to be $\rho' = \pi_{\mathrm{c}} - t_{\mathrm{s}}$, in which case $p_{\mathrm{c}} = 2\pi_{\mathrm{c}} - t_{\mathrm{s}}$.

The second part of the free energy, and the incremental dissipation functions, are

$$\Psi_2 = \frac{1}{2}\int p_{\mathrm{c}}(e_{\mathrm{v}}^{\mathrm{p}})\mathrm{d}e_{\mathrm{v}}^{\mathrm{p}}, \text{ so that } \delta\Psi_2 = \frac{1}{2}p_{\mathrm{c}}(e_{\mathrm{v}}^{\mathrm{p}})\delta e_{\mathrm{v}}^{\mathrm{p}}, \text{ and}$$

$$\delta\Phi = \frac{1}{2}p_{\mathrm{c}}(e_{\mathrm{v}}^{\mathrm{p}})|\delta e_{\mathrm{v}}^{\mathrm{p}}| \tag{20}$$

The plastic work increment, which is the sum of $\delta\Psi_2$ and $\delta\Phi$, is hence $p_{\mathrm{c}}(e_{\mathrm{v}}^{\mathrm{p}})\delta e_{\mathrm{v}}^{\mathrm{p}}$ in compression, but zero in tension. The introduction of the shift stress hence explains how it is possible to induce plastic dissipation during plastic flow taking place at constant *zero* effective pressure. The stored elastic energy released is exactly equal to the plastically dissipated energy. We can hence expect that the pressure component of the shift stress will always be equal to half of the consolidation pressure, in any model in which plastic yielding is assumed to occur at constant, zero effective pressure. A generalisation of this result has been given by Collins and Hilder (2003). These remarks, of course, apply only to 'single surface' models. This loading/unloading process can be modelled much more accurately with 'multiple surface' or 'internal function' models. Houlsby (1981) also shows that this theory, involving the additional energy terms that arise in an internal variable formulation, is very similar to the early energy theory for clays developed by Palmer (1967).

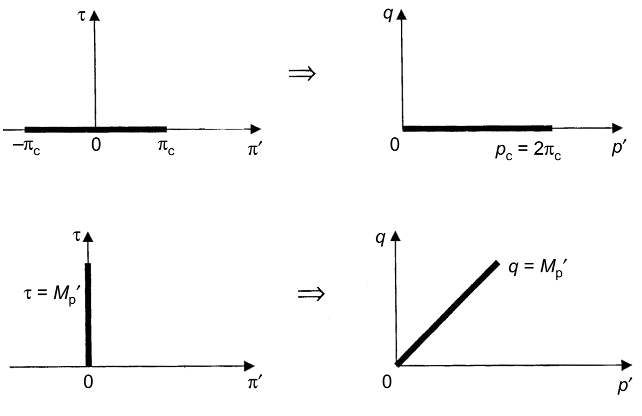


Fig. 3. Transformation from dissipative stress space to true stress space for: (a) isotropic compression; (b) pure friction

For frictionless materials, the linear isotropic consolidation line in $\ln(v) - \ln(p')$ space is generated by choosing the hardening function to be $p_c = p'_R \exp(e_v^p/\gamma)$, where the strains are measured from the state where $v = v^p = v_R$, namely the point on the normal consolidation line where the effective pressure is the reference value p'_R (Fig. 1).

The linear friction model

This familiar model has been discussed, in a thermomechanical context, before by Ziegler & Wehrli (1987), Houlsby (1981) and Collins & Houlsby (1997). We recall the basic results here, as they are needed for the development of new models. Consider a material whose dissipation function increment is proportional to the plastic shear increment and the mean effective pressure, $Mp'|\delta e_\gamma^p|$, so that from equation (11):

$$\pi' \delta e_v^p + \tau \delta e_\gamma^p = \delta \Phi = Mp'|\delta e_\gamma^p| \tag{21}$$

This is the simplest model that attempts to model Coulomb's law of dry friction. From equations (10) and (21) it follows that the shear component of the dissipative stress is given by: $|\tau| = Mp'$, and either π' is identically zero, and/or the volumetric plastic strain rates are identically zero. It is clear that this dissipation function is meaningful only if the plastic volume strains are identically zero, since otherwise, according to equation (21), a plastic volume change can occur without producing any energy dissipation. If we now transfer to true stress space, by simply equating the true and dissipative stress (no shift stress), we deduce the classical linear, Drucker–Prager version of the model, but with a non-associated, incompressible flow rule as illustrated in Fig. 3(b):

$$q = Mp' \text{ and } de_v^p = 0 \tag{22}$$

Original Cam clay

Although this model is the precursor of many important developments, it is a pathological example from a thermomechanics viewpoint. The dissipation function is the same as in equation (21). The standard argument is then to equate the plastic work-rate to this dissipation function, $p'\delta e_v^p + q\delta e_\gamma^p = Mp'|\delta e_\gamma^p|$, set up a differential equation for the plastic potential $q = q(p')$, by noting that $\mathrm{d}q/\mathrm{d}p' = -\delta e_v^p/\delta e_\gamma^p$, and integrate to derive the potential function $q = Mp' \ln(p'_c/p')$. An associated flow rule is then assumed, so that the yield surface and potential are identical. From the thermomechanics viewpoint this argument is flawed in many respects. It fails to draw the distinction between dissipation and plastic work, and it assumes a normal flow rule, whereas in fact the thermomechanical potentials uniquely determine *both* the flow rule and yield condition. Most importantly of all, however, is the fact that, if we use the dissipation function $\delta\Phi = Mp'|\delta e_\gamma^p|$, then isotropic plastic compression cannot be modelled, since these irreversible deformations would occur without producing any dissipation, which is in violation of the second law. The simple linear model above is the true model associated with this dissipation function. The original Cam clay model may satisfy Drucker's postulate, but it nevertheless violates the laws of thermodynamics. This model is of course well known to be unsatisfactory, as it predicts shear strains at the vertex on the pressure axis (Roscoe and Burland, 1968). The argument presented here is perhaps a slightly more fundamental way of making the same objection. According to Schofield (2000) the original Cam clay model was developed as a result of the experimental observation that, in the absence of volume changes, the dissipative energy increment was found to be always given by $Mp'|\delta e_\gamma^p|$. In this restricted context the model is free from these objections.

As will now be seen, the modified Cam clay model is not open to these objections. However, in this modification the frictional mechanisms of energy dissipation, as embodied in the original model, are abandoned. This was not of course the intention of its originators, Roscoe & Burland (1968), and the correct relationship between a dissipation function and the yield condition and flow rule had not been developed at that time. It will be argued that a more natural generalisation of the original Cam clay model, which preserves the frictional mechanism for energy dissipation, is the new 'alpha' model described below.

Modified Cam clay

This model stems from a dissipation function that is proportional to the isotropic consolidation, volumetric hardening function, and is the natural extension of that for normal compression used in equation (18):

$$\pi' \delta e_v^p + \tau \delta e_\gamma^p = \delta\Phi = \pi_c(e_v^p)[(\delta e_v^p)^2 + M^2(\delta e_\gamma^p)]^{1/2} \tag{23}$$

The dissipative stresses are obtained directly by differentiating equation (23) with respect to the plastic strain increments and using equation (10):

$$\pi' = \pi_c^2 \delta e_v^p/\delta\Phi \text{ and } \tau = M^2 \pi_c^2 \delta e_\gamma^p/\delta\Phi \tag{24}$$

so that, eliminating the strain increments between equations (23) and (24), we obtain the family of elliptical yield surface in dissipative stress space:

$$(\pi'/\pi_c)^2 + (\tau/M\pi_c)^2 = 1 \tag{25}$$

Introducing the same shift stress as for isotropic compression, $\rho' = \pi_c = p_c/2$, so $\pi' = p' - p_c/2$, the yield locus in true stress space is

$$(p' - p_c/2)^2/(p_c/2)^2 + q^2/M^2(p_c/2)^2 = 1$$
$$\text{or } q = M[p'(p_c - p')]^{1/2} \tag{26}$$

The familiar family of yield loci is shown in Fig. 4. The essential point being made here is that this construction is a two-stage process. The dissipative stress picture is a necessary intermediate step. Appreciation of this procedure makes it possible to generalise this model in a number of directions.

The flow rule is necessarily associated, since the dissipation function does not involve the true stresses:

$$\delta e_v^p = \delta\mu(p' - \tfrac{1}{2}p_c) \text{ and } \delta e_\gamma^p = \delta\mu q/M^2 \tag{27}$$

and the corresponding increments in plastic work and plastic dissipation are

$$\delta W^p = \delta\mu \tfrac{1}{2} p' p_c \text{ and } \delta\Phi = \delta\mu \tfrac{1}{4} p_c^2 \tag{28}$$

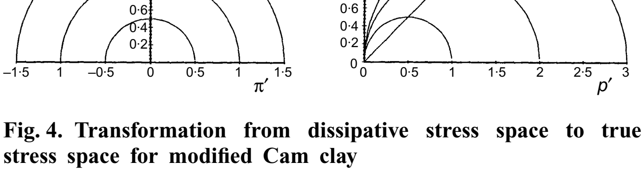


Fig. 4. Transformation from dissipative stress space to true stress space for modified Cam clay

A NON-ASSOCIATED MODEL (THE ALPHA MODEL)

A feature of the above models is that, in the dissipative stress plane, the critical-state line, defined to be the line on which the volumetric plastic strain increment is zero, is always a segment of the τ-axis (Figs 3 and 4). This is because of the symmetry of the yield loci and the fact that the flow rule is always associated in this plane. When we transform to the true stress plane, the critical-state line is now $q = Mp'$. In the case of modified Cam clay, this transformation is accomplished by the addition of the shift stress $\frac{1}{2}p_c$, which varies linearly with distance along the τ-axis. However, for the linear frictional model this same transformation is accomplished, as a result of the fact that the yield condition in (π', τ) space involves p' as a parameter. As a result the yield surface is 'sheared' when π' and τ are identified with p' and q. In addition the flow rule is no longer associated. This immediately suggests a procedure for generalising the modified Cam clay model to one involving non-associated flow rules, by superposing these two types of model.

We choose the dissipation increment function to be

$$\delta\Phi(e_v^p, p'; \delta e_v^p, \delta e_\gamma^p) = \pi_c(e_v^p)\{(\delta e_v^p)^2 + M^2[\alpha + (1-\alpha)(p'/\pi_c)]^2(\delta e_\gamma^p)^2\}^{1/2} \quad (29)$$

where α is a parameter lying between 0 and 1. This dissipation function reduces to that for modified Cam clay model when $\alpha = 1$, and to the linear friction model when $\pi_c = 0$. This model hence combines the volumetric hardening behaviour of modified Cam clay and the 'Coulomb frictional' shearing of the linear frictional and original Cam clay models. The smaller the value of α, the more dominant are the frictional effects. The corresponding dissipative stresses are hence, using the standard procedure in equation (10),

$$\pi' = \pi_c^2\delta e_v^p/\delta\Phi \text{ and } \tau = \pi_c^2 M^2\Pi^2\delta e_\gamma^p/\delta\Phi \quad (30)$$

where

$$\Pi \equiv [\alpha + (1-\alpha)(p'/\pi_c)]$$

Eliminating the strain increments between equations (29) and (30) gives the family of concentric, elliptical yield loci, with semi-axes of length π_c and $M\Pi\pi_c$, in dissipative stress space (Fig. 5):

$$\pi'^2/\pi_c^2 + \tau^2/(\pi_c^2 M^2\Pi^2) = 1 \quad (31)$$

When this is transformed to true stress space, by using the standard shift stress $\pi_c(e_v^p) = \frac{1}{2}p_c(e_v^p)$, so that $\pi' = p' - \frac{1}{2}p_c$, the resulting yield condition is

$$(p' - p_c'/2)^2/(p_c'/2)^2 + q^2/[M^2(p_c'/2)^2\Pi^2] = 1 \quad (32)$$

or

$$q = M\Pi[p'(p_c' - p)]^{1/2} \quad (33)$$

where now we can write $\Pi \equiv \alpha + (1-\alpha)(2p'/p_c)$. This expression only differs from that for the modified Cam clay yield loci by the addition of the factor Π. Representative graphs of these yield loci are shown in Fig. 6, where it is to be noted that for sufficiently small values of α the loci have concave segments.

The plastic strain increments are given by the normal flow rule in dissipative stress space:

$$\delta e_v^p = \delta\mu\pi' \text{ and } \delta e_\gamma^p = \delta\mu\tau/M^2\Pi^2 \quad (34)$$

which, when transferred to true stress space, gives

$$\delta e_v^p = \delta\mu(p' - p_c/2) \text{ and } \delta e_\gamma^p = \delta\mu q/M^2\Pi^2 \quad (35)$$

so that the plastic dilation is

$$\delta^p = -\frac{\dot{e}_v^p}{\dot{e}_\gamma^p} = -\frac{M^2\Pi^2(p' - p_c'/2)}{q} \equiv \frac{q(p' - p_c/2)}{p'(p_c - p')} \quad (36)$$

Note that when $p' = p_c/2$ then $\Pi = 1$ and $q = Mp'$ from equation (33), so that the plastic volume strain and the dilation are still zero on this critical-state line. This essential feature of the modified Cam clay model is hence preserved. However, the flow rule is no longer associated. The equations for the plastic potentials in (p', q) space are complex and not useful. The present procedure of deriving the plastic strain increments from the normal flow rule in dissipative stress space, and then transforming to the true stress space, is algebraically much more straightforward.

This model has much in common with those proposed by Nova & Wood (1979) and Chandler (1985). Two primary mechanisms of producing plastic deformations are envisaged

(*a*) compaction, dominating at low stress ratios, involving the deformation of contact bonds between the particles and eventually bond fracture and particle crushing
(*b*) particle rearrangement, which dominates at high stress ratios, resulting from the sliding and rolling of the soil particles.

In both these papers yield surfaces and flow rules were proposed for both regimes. However, these surfaces had to be joined together in an artificial manner at a certain transitional stress ratio. One advantage of the present approach is that the presence of the shift stress, which has emerged naturally from the thermomechanical formulation, gives rise to a model in which this transition is continuous. The relative importance of the two mechanisms is determined by the value of the parameter α.

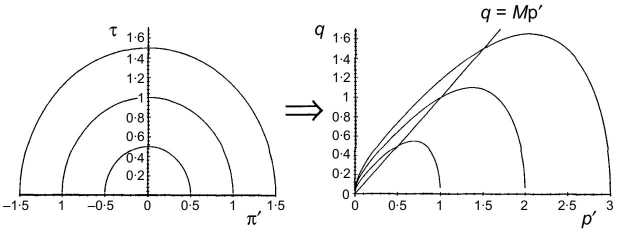


Fig. 5. Transformation from dissipative stress space to true stress space for the alpha model ($\alpha = 0{\cdot}5$)

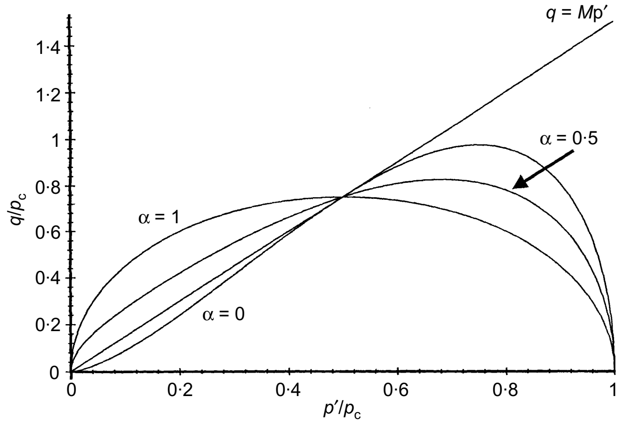


Fig. 6. Yield loci for the alpha model ($M = 1{\cdot}5$, $p_c = 1{\cdot}0$, $\alpha = 0{\cdot}0$, $0{\cdot}5$, $1{\cdot}0$)

The limiting situation where $\alpha = 0$ is of particular interest. The dissipation increment function is now

$$\delta\Phi(e_{\mathrm{v}}^{\mathrm{p}}, p'; \delta e_{\mathrm{v}}^{\mathrm{p}}, \delta e_{\gamma}^{\mathrm{p}}) = [\pi_{\mathrm{c}}^2(\delta e_{\mathrm{v}}^{\mathrm{p}})^2 + M^2 p'^2(\delta e_{\gamma}^{\mathrm{p}})^2]^{1/2} \tag{37}$$

In this case all the energy dissipation associated with the increment in *plastic shear strain*, $\delta e_{\gamma}^{\mathrm{p}}$, is due to friction and is given by $Mp'|\delta e_{\gamma}^{\mathrm{p}}|$, as in the linear frictional, or original Cam clay models. However, here, in addition, the model includes the dissipation associated with the *volumetric plastic strain* increment. The yield loci *always* lie below the critical-state line, which they touch at $p' = p_{\mathrm{c}}/2$. The yield loci are concave when $p' < p_{\mathrm{c}}/2$. It is frequently stated that yield surfaces must always be convex. This is not true, however. The positive dissipation inequality implies only that the plastic potentials must be convex surfaces (Lemaitre & Chaboche, 1990). The yield surfaces will hence also be convex when the flow rule is *associated,* but this does not have to be so when this rule is *non-associated,* as discussed by Mroz (1963) for example. It is, of course, very difficult to obtain reliable experimental results at low stress levels. Lade (2001, personal communication), however, reports that concave yield loci could possibly explain the apparent curvature of the instability lines observed at low stress levels on sands with significant fines content, as discussed in Yamamuro & Lade (1997, 1998) and Lade & Yamamuro (1997).

From the flow rule (equation (35)) we observe that, in the ‘dense region’ where $p' < p_{\mathrm{c}}/2$, the plastic volume strains are dilatant, even though the yield locus lies below the critical-state line. In this model therefore it is possible to have plastic dilation below the critical-state line. As the value of α is increased, at least part of the yield curve in this ‘dense region’, $p' \leqslant p_{\mathrm{c}}/2$, lies above the critical-state line. For values of α in the interval 0 to 0·172 there are two such arcs, necessarily convex, separated by a concave arc, which still lies below the critical-state line. For higher values of α the whole of the yield locus in the ‘dense’ region, lies above $q = Mp'$, as in the classical models.

Before discussing the predicted form of the drained and undrained stress paths, we discuss a modification to the classical volumetric hardening model, which includes the effect of shear strains.

A BIASED WORK-HARDENING MODEL (THE BETA MODEL)

The volumetric hardening model used in modified Cam clay is now generalised to include contributions from the shear strains. Specifically we use a law in which the increment in the hardening parameter p_{c} is a weighted average of the work done by the volumetric and shear strains:

$$\delta p_{\mathrm{c}} = (p'\delta e_{\mathrm{v}}^{\mathrm{p}} + \beta q \delta e)/\gamma \tag{38}$$

Similar approaches have been used by Nova (1977) and Nova & Wood (1979), who used a weighted average of the volumetric and shear *strain* increments, rather than of the associated *work* increments. In using work increments we are following the suggestion of Krenk (1996, 2000). Work would seem to be a more natural choice than strain, in a thermomechanical formulation. This model does reduce to the standard volumetric strain law under isotropic compression conditions, with $p' = p_{\mathrm{c}}$. However, when $\beta = 0$ it does *not* reduce to the standard volumetric strain-hardening model of critical-state soil mechanics, which would have $\delta p_{\mathrm{c}} = p_{\mathrm{c}}\delta e_{\mathrm{v}}^{\mathrm{p}}/\gamma$. Lade (1975, 1988) has developed models in which the *total* plastic work is taken as the hardening parameter. This corresponds to a value of $\beta = 1$ in the current model. Lade’s procedure is open to the objection that it is then possible to deduce the hardening behaviour over the full range of normal and shear stresses just from a series of isotropic, consolidation tests ($q = 0$). This seems an unlikely possibility, as the microscopic mechanisms dominating in normal compaction are very different from those at high stress ratios. In support of this viewpoint it will be seen below that some of the observed features of sand deformations can be simulated with this model only when $\beta < \frac{1}{2}$.

Here we are principally interested in the effect on the incremental form of the constitutive law of the inclusion of the plastic shear strains in the hardening law. Application of the standard procedure for constructing such a law, by using the consistency equation, flow rule and hardening rule, in the case of modified Cam clay, leads to the incremental plastic strain–stress relations:

$$\delta e_{\mathrm{v}}^{\mathrm{p}} = (\gamma\varpi/2\Omega)(\varpi\delta p'/p' + 2\bar{\eta}^2\delta q/q)$$
$$\text{and } \delta e_{\gamma}^{\mathrm{p}} = (\gamma\bar{\eta}/M\Omega)(\varpi\delta p'/p' + 2\bar{\eta}^2\delta q/q) \tag{39}$$

where

$$\bar{\eta} = q/Mp',\ \varpi = 1 - \bar{\eta}^2 \equiv 1 - (q/Mp')^2,$$
$$\text{and } \Omega = 1 - (1 - 2\beta)\bar{\eta}^2 \tag{40}$$

Drained tests

In a standard drained test, the ratio of the stress increments is kept constant. The volume strain increment is zero on the characteristic (critical) state line—that is, when $\bar{\eta} = 1$ and $\varpi = 0$, but the shear strain increment is non-zero. The peak stress is reached when Ω, which is essentially the hardening modulus, vanishes: that is, when

$$\bar{\eta} \equiv q/Mp' = [1/(1 - 2\beta)]^{1/2} \tag{41}$$

This defines the failure line (f.l.) for such stress-controlled drained tests. Note that we get such peaks only when $\beta < \frac{1}{2}$, and that the peak stress always occurs above the critical-state line. The unweighted plastic work model ($\beta = 1$) would not predict failure, at least with this elliptic yield condition.

Undrained tests

In an undrained test, the sum of the volumetric elastic and plastic strain increments is zero, so that

$$\delta e_{\mathrm{v}}^{\mathrm{p}} = -\delta e_{\mathrm{v}}^{\mathrm{e}} = -\kappa\delta p'/p' \tag{42}$$

The characteristic ‘kink’ on the undrained path occurs when $\delta p' = 0$. This is the *phase transition* or *characteristic state line* (c.s.l.), at which the shear strains start to dominate and on which the elastic, plastic and total volumetric strain increments are all zero (Ishihara, 1996; Zienkiewicz *et al.*, 1998; Krenk, 2000). This line is identical with the classical critical-state line of the classical models, if that line is regarded as the line on which the plastic volumetric strain increment vanishes, rather than the ultimate state line.

It follows from equation (39) that the stress increments defining the loading path are given by

$$\delta p'/\delta e_{\gamma}^{\mathrm{p}} = -p'M\varpi/2\kappa\bar{\eta}$$
$$\text{and } \delta q/\delta e_{\gamma}^{\mathrm{p}} = qM(\gamma\varpi^2 + \kappa\Omega)/4\kappa\gamma\bar{\eta}^3 \tag{43}$$

If the undrained effective stress path has a peak, this will occur where $\delta q = 0$: that is, where

$$\gamma\varpi^2 + \kappa\Omega = 0 \tag{44}$$

This equation defines the *instability line*. This stress path becomes asymptotic to a radial line through the origin when $\delta\eta/\eta \equiv \delta q/q - \delta p'/p' = 0$. The slope of this ‘ultimate’ or *steady-state line* is given by $\bar{\eta} = \bar{\eta}_{ss}$ say, where $\bar{\eta}_{ss}$ is the root of

$$\gamma(1 - \bar{\eta}^4) + \kappa\Omega = 0 \tag{45}$$

Some sample undrained paths for various values of β are shown in Fig 7. Since the yield loci are geometrically self-similar, the various possible types of behaviour can be illustrated by starting the undrained paths at various points on a normalised yield locus. When $\beta = 0$ the paths terminate on the c.s.l., but for higher values of β paths starting on the 'loose' side of critical cross this line and become tangential to the u.s.l., as do paths starting in the 'dense' zone.

THE ALPHA-BETA MODEL

Drained tests

Equation (41) for the failure line now becomes

$$\bar{\eta} \equiv q/Mp' = [(1 + \alpha\beta - 2\beta)/(1 - \beta)][1/(1 - 2\beta)]^{1/2} \tag{46}$$

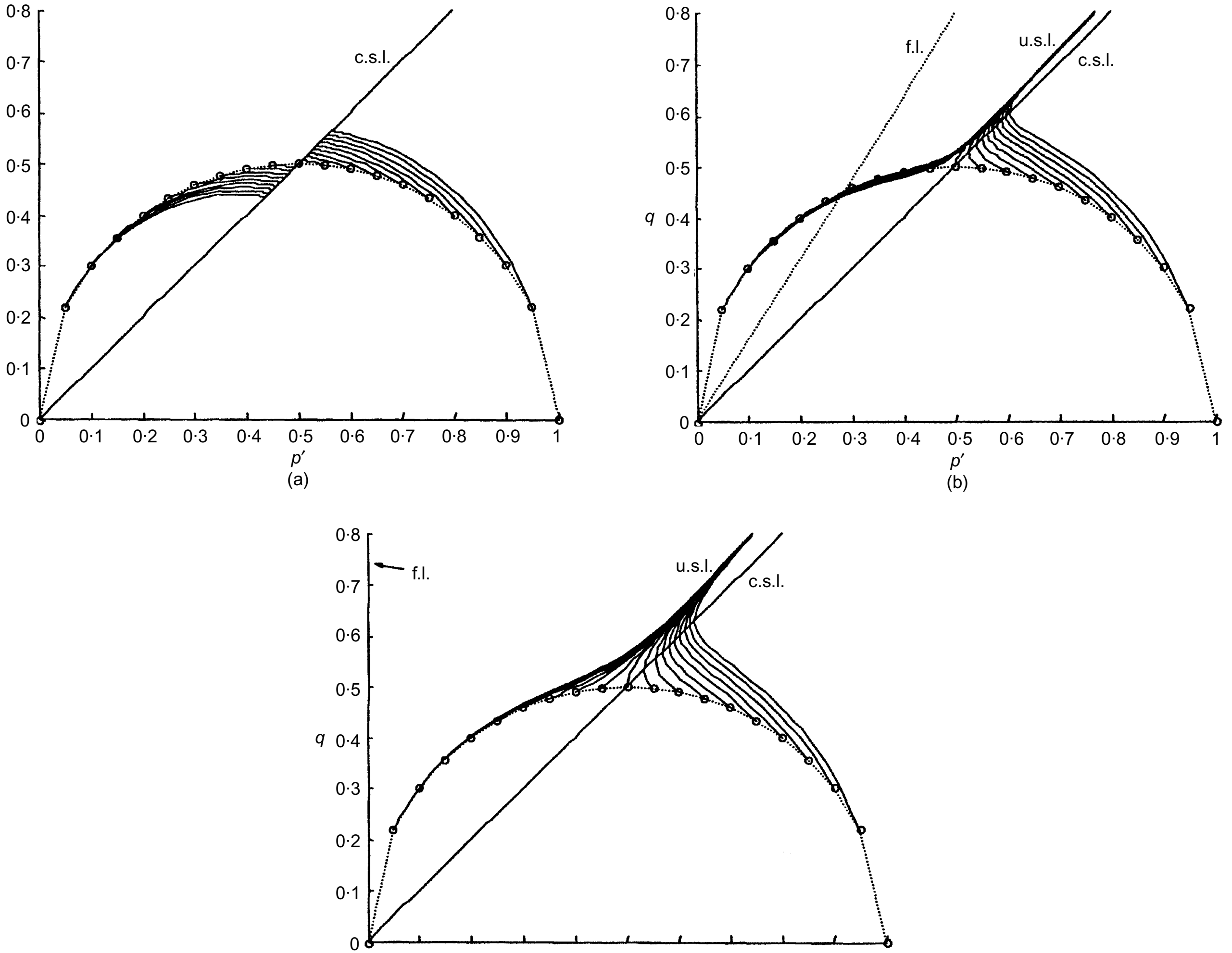
The presence of internal friction ($\alpha < 1$) reduces this peak stress ratio slightly.

Undrained tests

The undrained paths can be readily found by numerical integration. For each increment $\delta p'$ the corresponding increment in p_c is obtained from

$$\gamma\delta p_c + [1 + 2\beta(p_c - p')/(2p - p_c)]\kappa\delta p' = 0 \tag{47}$$

which follows from the flow rule (equation (36)), the hardening law (equation (38)) and the basic property of an undrained path (equation (42)). The updated value of q is then obtained from the yield condition (equation (33)). Representative paths are shown in Fig. 8. When $\alpha = 0$ the yield locus touches the c.s.l. only once, so that all paths go to this point (Fig. 8(a)). The undrained paths cannot cross this line, since there is no segment of yield surface above the c.s.l. Instead the paths continue up the c.s.l., which hence coincides with the u.s.l., irrespective of the value of β. Paths starting in the 'loose' region show a marked drop in strength before the c.s.l. is reached. This pre-failure softening behaviour is typical of loose contractive sands. This softening behaviour is less marked when the value of α is increased, as shown in Fig. 8(b), but now the paths can cross the c.s.l. and bend upwards and asymptotic to the u.s.l. In both cases paths starting in the 'dense' region follow paths going away from the origin, with both p' and q increasing. The paths lie close to the c.s.l. and eventually merge with the u.s.l. This is true even for paths starting *below* the c.s.l



(a)

(b)

(c)

Fig. 7. Undrained stress paths for modified Cam clay with biased, work-hardening law (beta model): (a) $\beta = 0$; (b) $\beta = 0{\cdot}25$; (c) $\beta = 0{\cdot}5$ ($M = 1$, $\kappa = 0{\cdot}05$, $\lambda = 0{\cdot}25$)

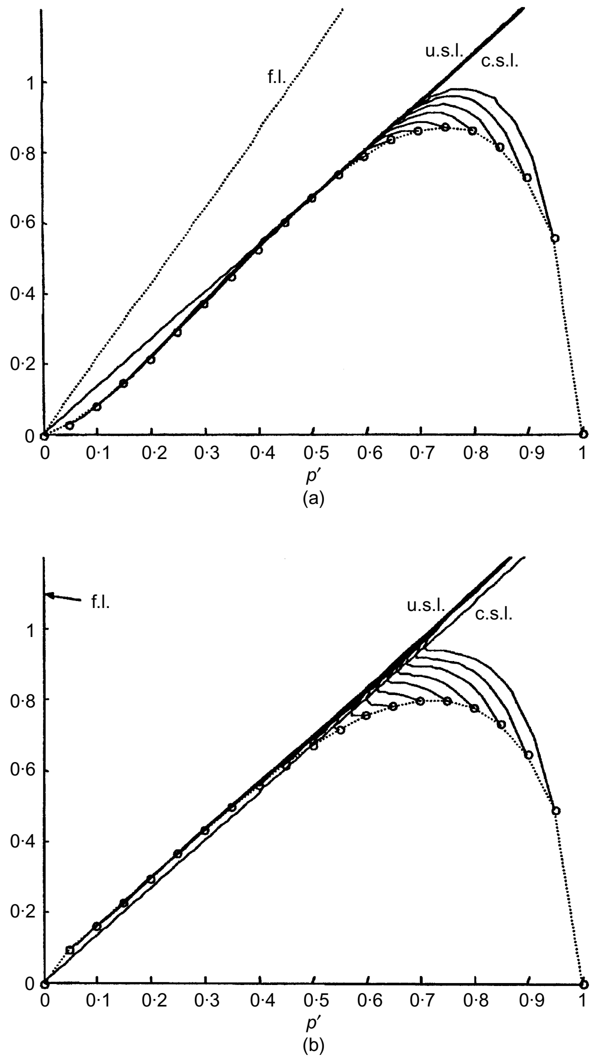


Fig. 8. Undrained stress paths for alpha model with biased, work-hardening law (alpha-beta model): (a) $\alpha = 0$, $\beta = 0{\cdot}25$; (b) $\alpha = 0{\cdot}25$, $\beta = 0{\cdot}5$ ($M = 1{\cdot}34$, $\kappa = 0{\cdot}0054$, $\lambda = 0{\cdot}181$)

in the $\alpha = 0$ model, since the plastic volume strains are still dilative.

The main object of this paper has been to demonstrate a procedure for model construction, based upon the laws and techniques of thermomechanics. The alpha and beta models have been introduced to illustrate this general procedure. It is appreciated that, to be useful, these models must be fully validated against experimental data, and the results of such comparisons will be discussed elsewhere. There is a particular difficulty in trying to use these types of model for sands, owing to the problematic nature of the normal consolidation line, as discussed in Pestana & Whittle (1995) and Jefferies & Been (2000) for example. For sands it may well be preferable to refer the thermomechanical state parameters to the critical-state line, as in the analyses of Been & Jefferies (1985).

SUMMARY AND CONCLUSIONS

Although modern developments in thermomechanics have had a large influence on many branches of mechanics, this is not yet true of geomechanics. Perhaps, in part at least, this is due to the commonly held view that non-associated flow rules are not covered by modern thermomechanics theories. However, Collins & Houlsby (1997) showed that this was not true, and that such flow rules arise naturally for frictional materials, where the plastic dissipation depends on the effective pressure. The main achievements and conclusions of the present analysis are as follows:

(*a*) A set of thermodynamically consistent state variables, which describe the state of the soil, has been developed. In particular it has proved helpful to define a reference, plastic specific volume, by analogy with the plastic strain, which is used as a key internal state variable in modern plasticity theories.

(*b*) The construction of the plasticity models from the free energy and dissipation functions is a two-stage process. The yield condition and plastic potential are first constructed in dissipative stress space, where the flow rule is always associated. The yield condition and flow rule are then constructed in true stress space, either by using a shift stress and/or by using the implicit dependence of these functions on the true stress variables, such as the effective pressure. It is important to distinguish between plastic work and plastic energy dissipation. Not all the plastic work is dissipated. The stored plastic energy gives rise to these shift stresses. These arise naturally when modelling isotropic compression with single-surface models. This stored energy can be interpreted on the micro scale as locked-in elastic energy, which can accompany the macro-level plastic deformations.

(*c*) The two-step process has been used to analyse a number of the basic extant models, including Drucker–Prager and the original and modified Cam clay models. As a result of the understanding obtained from these analyses, a family of new models has been proposed. These incorporate some of the observed features of real soils and granular materials, not predicted by the classical models, such as non-associated flow behaviour at high stress ratios, contractive behaviour at low stress levels, static liquefaction and instability. Yet these models maintain much of the familiar, simple structure of the classical critical-state theories. They also demonstrate that concave yield surfaces are possible at low stress levels.

There are a large number of models, which have extended the classical critical-state theories in a number of directions, available in the literature. These include CANA Sand (Poorooshasb, 1994), NOR Sand (Jefferies, 1993), Superior Sand (Boukpeti & Drescher, 1999), Severn–Trent Sand (Gajo & Muir Wood, 1999), CASM (Yu, 1998), and other ‘anonymous’ models due to Pender (1978), Krenk (2000) and Manzari & Dafalias (1997) etc. The special feature of the present approach is that the models are developed using ideas of modern internal variable thermomechanics, and are based on the fundamental physical concepts of work, energy and dissipation. In many ways this procedure is closest to the original ideas of critical-state soil mechanics.

For simplicity and relevance, this paper has dealt only with the formulation of these models for the analysis of triaxial tests. It is not difficult to generalise the general theory to fully general three-dimensional situations. The main limiting factor, however, is that the algebra of the relationship between the yield function and the dissipation function for specific models can become prohibitive, once one departs from the simple quadratic form of the yield locus. We have also been concerned only with single-surface models. More realistic models using bounding surfaces, multiple yield surfaces, or sub-loading surfaces can be analysed in a similar fashion. They require the introduction of multiple internal variables. The internal function models of Puzrin & Houlsby (2001b, 2001c) represent one such generalisation.

ACKNOWLEDGEMENTS

The authors are grateful to Koichi Hashiguchi, Guy Houlsby, Poul Lade, Mick Pender, Andrew Schofield and David Smith for many helpful discussions on the fundamental problems of modelling soil behaviour related to this study.

NOTATION

e	voids ratio
e_1, e_2, e_3	principal components of logarithmic, finite, strain tensor
e_v, e_v^e, e_v^p	total, elastic and plastic, volumetric, finite, logarithmic strains
$e_\gamma, e_\gamma^e, e_\gamma^p$	total, elastic and plastic, finite shear strains
L	length of test specimen
M	slope of critical-state line
N	constant determining position of normal consolidation line
p'	effective pressure
p_c	normal consolidation pressure
p'_0	initial effective pressure
p'_R	reference effective pressure
q	shear stress invariant
R	radius of test specimen
u	pore pressure
U	internal energy
V	volume of test specimen
V_s	volume of solid phase
V^p	plastic volume of test specimen
v	specific volume
v^p	reference, plastic specific volume
δW	work increment
α	parameter in non-associated model
β	parameter in weighted work-hardening model
γ	$= \lambda - \kappa$
δ^p	$= -\delta e_v^p / \delta e_\gamma^p$, the plastic dilation
η	$= q/p'$, stress ratio
$\bar{\eta}$	$= q/Mp'$
κ	slope of elastic loading/unloading line
λ	slope of normal consolidation line
Λ	$= (L/R)^{2/3}$ aspect ratio
$\delta\mu$	scale parameter in flow rule
π'	effective, dissipative pressure
π_c	normal consolidation pressure in dissipative stress plane
$\sigma'_1, \sigma'_2, \sigma'_3$	principal effective stresses
τ	shear stress invariant of dissipative stress
Φ	rate of dissipation function
Ψ	free energy function
Ω	$= 1 - (1 - 2\beta)\bar{\eta}^2$
ϖ	$= 1 - \bar{\eta}^2 \equiv 1 - (q/Mp')^2$

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