

A THIRD ORDER BOUNDARY VALUE PROBLEM ARISING IN AEROELASTIC WING THEORY*

By GEORGE SEIFERT** (*Cornell University*)

The differential equation arising in the problems of chordwise divergence or swept-forward wing bending divergence [1] is of the type

$$\frac{d^2}{dx^2} EI(x) \frac{dz}{dx} - \lambda c(x)z = 0 \quad (1a)$$

where λ is a parameter. The boundary conditions are $z(0) = z'(l) = (EI(x)z'(x))'_{x=l} = 0$. Since this represents a non self-adjoint boundary value problem, the ordinary Rayleigh-Ritz variational method for approximating its characteristic values is not applicable. Formal extensions of this variational approach have been suggested by Flax [1] and used by Cheng [2] but no mathematical evidence for their validity is at present available. In this connection, two questions have been raised [1].

(A) Under what conditions on $EI(x)$ and $c(x)$ are all the characteristic values of (1a) real?

(B) Under what conditions on $EI(x)$, $c(x)$ and $f(x)$ can the solution $z_\lambda(x)$ of

$$[EI(x)z'(x)]'' - \lambda c(x)z(x) = f(x), \quad z(0) = z'(l) = (EIz')'_l = 0$$

be expanded in terms of the biorthogonal system of functions arising from the system (1a) and its adjoint?

Although this paper concerns itself chiefly with (A), it may be pointed out that the investigations of expansions in terms of characteristic functions of (1a) for $EI \equiv 1$, $c(x) \equiv -1$, $l = \pi$ carried out by L. E. Ward [3] indicate difficulties inherent in such irregular boundary-value expansion problems as (B).

Part I of this paper deals with the characteristic values of the system

$$u'''(x) + p(x)u'(x) + [q(x) + \lambda]u(x) = 0, \quad u(0) = u'(0) = u'(1) = 0, \quad (1)$$

where p and q are real-valued functions analytic on $0 \leq x \leq 1$. It is found that:

(a) this system has an infinite number of real characteristic values (Theorem 1);

(b) if the upper bounds of $|p(t)|$ and $|q(t)|$, and the positive numbers $|p(1)|$, $\int_0^1 |p(t)| dt$, $\int_0^1 |r(t)| dt$, where $r(t) = q(t) - p'(t)$, are small enough, then all characteristic values of (1) are real (Theorem 3).

Just how small these positive quantities in (b) must be, in order that all the characteristic values of (1) be real, is not explicitly considered. For specific functions p and q , however, the details of the proofs of (a) and (b) enable one to answer this question by means of simple numerical computations.

Much of the method of proof and notation is similar to that used by Ward [4] in his study of the system:

$$u'''(x) + [\rho^3 + r(x)]u(x) = 0, \quad u(0) = u'(0) = u(\pi) = 0.$$

Here $r(x)$ is of a special form which permits Ward to obtain an expansion theorem. It

*Received Oct. 17, 1950.

**Now at the University of Nebraska.

is hoped that Ward's approach will also suggest an expansion theorem associated with (1) and perhaps also provide an answer to (B).

Part II indicates a pair of transformations which takes the system

$$[f(t)y''(t)]' + \lambda g(t)y(t) = 0, \quad y(0) = y'(0) = y''(1) = 0, \quad (2)$$

where $f(t) > 0$, $g(t) > 0$, are real and analytic on $0 \leq t \leq 1$, over into a system of the form of (1). Note that the adjoint of (2) is of the form of (1a), and consequently the characteristic values of (2), being identical with those of its adjoint, are of interest in aeroelastic wing theory.

PART I

We consider the system (1). Define

$$\begin{aligned} \text{(a)} \quad \delta_1(t) &= e^{\omega_1 t} + e^{\omega_2 t} + e^{\omega_3 t} \\ \delta_2(t) &= e^{\omega_1 t} - \omega_3 e^{\omega_2 t} - \omega_2 e^{\omega_3 t} \\ \delta_3(t) &= e^{\omega_1 t} - \omega_2 e^{\omega_2 t} - \omega_3 e^{\omega_3 t} \end{aligned}$$

where

$$\omega_1 = -1, \quad \omega_2 = e^{(\pi i)/3}, \quad \omega_3 = e^{-(\pi i)/3};$$

(b) the complex number ρ by $\rho^3 = \lambda$, $|\arg \rho| \leq \pi/3$;

(c) the regions S_1 and S_2 of the ρ plane by $0 \leq \arg \rho \leq \pi/3$ and $-\pi/3 \leq \arg \rho \leq 0$ respectively.

Lemma 1. *A necessary and sufficient condition that $u(x, \rho)$ satisfy the equation in (1) and $u(0, \rho) = u'(0, \rho) = 0$ is that*

$$u(x, \rho) = \delta_3(\rho x) - (3\rho^2)^{-1} \int_0^x \{ \delta_3[\rho(x-t)]r(t) - \rho \delta_2[\rho(x-t)]p(t) \} u(t, \rho) dt \quad (1.1)$$

where $r(t) = q(t) - p'(t)$.

The proof of this lemma is completely analogous to that of Theorem 1 of Ward's paper² and is omitted here.

Lemma 2. *A solution $u(x, \rho)$ of the equation in (1) such that $u(0, \rho) = u'(0, \rho) = 0$ is given by*

$$u(x, \rho) = e^{\omega_3 \rho x} [-\omega_3 - \omega_2 e^{(\omega_2 - \omega_3) \rho x} + z(x, \rho)] \quad (1.2)$$

where $|z(x, \rho)| \leq m$, m being independent of x and ρ , provided $\rho \in S_1$ and $|\rho|$ sufficiently large.

¹Unless otherwise indicated, the prime will always denote differentiation with respect to the first variable.

²L. E. Ward, [4], p. 418.

Proof. Substituting (1.2) into (1.1) one obtains

$$e^{\omega_3 \rho x} [A(x, \rho) + z(x, \rho)] \\ = \delta_3(\rho x) - (3\rho^2)^{-1} \int_0^x \{ \delta_3[\rho(x-t)]r(t) - \rho \delta_2[\rho(x-t)]p(t) \} [A(t, \rho) + z(t, \rho)] e^{\omega_3 \rho t} dt,$$

where $A(t, \rho) = -\omega_3 - \omega_2 \exp [(\omega_2 - \omega_3)\rho t]$. Hence

$$z(x, \rho) = e^{(\omega_1 - \omega_3)\rho x} \\ - (3\rho^2)^{-1} \int_0^x \{ \delta_3[\rho(x-t)]r(t) - \rho \delta_2[\rho(x-t)]p(t) \} [A(t, \rho) + z(t, \rho)] e^{-\omega_3 \rho(x-t)} dt. \quad (1.3)$$

Now for fixed $\rho \in S_1$ $|z(x, \rho)|$ attains its upper bound $m(\rho)$ on $0 \leq x \leq 1$; hence by (1.3)

$$m(\rho) \leq 1 + (3|\rho|^2)^{-1} \int_0^1 | \delta_3[\rho(x-t)]A(t, \rho) e^{-\omega_3 \rho(x-t)} r(t) | dt \\ + (3|\rho|^2)^{-1} \int_0^1 | \delta_3[\rho(x-t)]z(t, \rho) e^{-\omega_3 \rho(x-t)} r(t) | dt \\ + (3|\rho|)^{-1} \int_0^1 | \delta_2[\rho(x-t)]A(t, \rho) e^{-\omega_3 \rho(x-t)} p(t) | dt \\ + (3|\rho|)^{-1} \int_0^1 | \delta_2[\rho(x-t)]z(t, \rho) e^{-\omega_3 \rho(x-t)} p(t) | dt \\ \leq 1 + m(\rho) \left(\frac{r}{|\rho|^2} + \frac{p}{|\rho|} \right) + 2 \left(\frac{r}{|\rho|^2} + \frac{p}{|\rho|} \right)$$

where

$$r = \int_0^1 |r(t)| dt, \quad p = \int_0^1 |p(t)| dt.$$

Hence

$$m(\rho) \leq \frac{1 + 2(r/|\rho|^2 + p/|\rho|)}{1 - (r/|\rho|^2 + p/|\rho|)}$$

for $|\rho|$ sufficiently large, from which the lemma follows.

Theorem 1. *There exist an infinite number of real characteristic values λ_n for the system (1); more precisely, there exists a real λ_k such that all λ_n with $R(\lambda_n) > \lambda_k$ are necessarily real.³*

Proof. From (1.1) by differentiation,

$$u''(x, \rho) = \rho^2 \delta_1(\rho x) - \frac{1}{3} \int_0^x \{ \delta_1[\rho(x-t)]r(t) - \rho \delta_3[\rho(x-t)]p(t) \} u(t, \rho) dt \\ - p(x)u(x, \rho),$$

³If $z = x + iy$, then $R(z) = x$ defines the notation $R(z)$ which will be used throughout.

and the characteristic equation, $u''(1, \rho) = 0$, becomes

$$\rho^2 \delta_1(\rho) - \frac{1}{3} \int_0^1 \{ \delta_1[\rho(1-t)]r(t) - \rho \delta_3[\rho(1-t)]p(t) \} [A(t, \rho) + z(t, \rho)] e^{\omega \rho t} dt - p(1)[A(1, \rho) + z(1, \rho)] e^{\omega \rho} = 0.$$

Define

$$E(\rho) = (3\rho^2)^{-1} \int_0^1 \{ \delta_1[\rho(1-t)]r(t) - \rho \delta_3[\rho(1-t)]p(t) \} [A(t, \rho) + z(t, \rho)] e^{-\omega \rho(1-t)} dt + p(1)[A(1, \rho) + z(1, \rho)] \rho^{-2}.$$

It is easily verified that the zeros of $\delta_1(\rho)$ are the zeros of $\exp(-3\rho/2) + 2 \cos[3^{1/2}\rho/2]$ and are all real for $\rho \in S_1$.

It will now be shown that $|\rho^2 \delta_1(\rho)| > |\rho^2 e^{\omega \rho} E(\rho)|$ for ρ on any one of a set of contours C_n in the ρ plane which are trapezoids whose bases are the segments

$$R(\rho) = 2(n+1) \frac{\pi}{3^{1/2}}, \quad |\arg \rho| \leq \frac{\pi}{3};$$

$$R(\rho) = 2(n+2) \frac{\pi}{3^{1/2}}, \quad |\arg \rho| \leq \frac{\pi}{3};$$

$n = k, k+1, \dots, k$ sufficiently large. It is sufficient to show this inequality for the upper half of the contour C_n ; namely $C'_n = C_n \cap S_1$. This is true because $u(x, \rho)$ is real for real ρ and hence $u(x, \rho^*) = [u(x, \rho)]^*$, the star denoting the complex conjugate. Hence $u''(x, \rho^*) = [u''(x, \rho)]^*$, and consequently $\rho^2 [\delta_1(\rho) - e^{\omega \rho} E(\rho)]$ takes on values on $C''_n = C_n \cap S_2$, the lower half of C_n , which are the complex conjugates of those taken on on C'_n , the upper half. Set $F(\rho) = \rho^2 \delta_1(\rho)$, $G(\rho) = -\rho^2 e^{\omega \rho} E(\rho)$; then since $F(\rho^*) + G(\rho^*) = [F(\rho)]^* + [G(\rho)]^*$ and $F(\rho^*) = [F(\rho)]^*$, we have $G(\rho^*) = [G(\rho)]^*$, and hence $\rho^2 e^{\omega \rho} E(\rho)$ assumes values on C''_n which are the complex conjugates of those it takes on on C'_n . It clearly suffices, then, to show that

$$|e^{-\omega \rho} \delta_1(\rho)| > |E(\rho)| \quad \text{for } \rho \text{ on } C'_n. \tag{1.4}$$

On $\rho = \alpha(1 + 3^{1/2}i)$ we have

$$|e^{-\omega \rho} \delta_1(\rho)| = |2e^{-3\alpha} \cos(3^{1/2}\alpha) + 1|;$$

$$|I_1| \leq \frac{1}{12\alpha^2} 3(2+m) \int_0^1 |r(t)| dt = \frac{1}{4\alpha^2} (2+m)r,$$

where

$$I_1 = \frac{1}{3\rho^2} \int_0^1 \{ \exp[(\omega_1 - \omega_3)\rho(1-t)] + \exp[(\omega_2 - \omega_3)\rho(1-t)] + 1 \} [A(t, \rho) + z(t, \rho)] r(t) dt;$$

$$|I_2| \leq \frac{1}{6\alpha} 3(2+m) \int_0^1 |p(t)| dt = \frac{1}{2\alpha} (2+m)p,$$

where

$$I_2 = \frac{1}{3\rho} \int_0^1 \{ \exp [(\omega_1 - \omega_3)\rho(1 - t)] - \omega_2 \exp [(\omega_2 - \omega_3)\rho(1 - t)] - \omega_3 \} [A(t, \rho) + z(t, \rho)] p(t) dt;$$

$$| I_3 | \leq \frac{|p(1)|}{4\alpha^2} (1 + e^{-3\alpha} + m),$$

where

$$I_3 = p(1)[A(1, \rho) + z(1, \rho)]\rho^{-2}.$$

On $\rho = 2(n + 1)\pi/3^{1/2} + i\beta, \beta \geq 0$, we have

$$| e^{-\omega_3\rho} \delta_1(\rho) | = \left| \exp \left\{ -\frac{3}{2} \left[2(n + 1) \frac{\pi}{3^{1/2}} + i\beta \right] + \frac{3^{1/2}i}{2} \left[2(n + 1) \frac{\pi}{3} + i\beta \right] \right\} + \exp \left\{ 3^{1/2}i \left[2(n + 1) \frac{\pi}{3^{1/2}} + i\beta \right] \right\} + 1 \right|$$

$$= \left| \exp [-3^{1/2}(n + 1)\pi] \exp \left(-\frac{3}{2} i\beta \right) (-1)^n \exp \left(-\frac{3^{1/2}}{2} \beta \right) + \exp (-3^{1/2}\beta) + 1 \right|,$$

hence $| e^{-\omega_3\rho} \delta_1(\rho) | \geq 1 - \exp (3^{1/2}\pi)$; also

$$| I_1 | \leq \frac{9}{4} \frac{(2 + m)r}{[2(n + 1)\pi]^2 + 3\beta^2},$$

$$| I_2 | \leq \frac{3^{1/2}}{2} \frac{(2 + m)p}{\{ [2(n + 1)\pi]^2 + 3\beta^2 \}^{1/2}},$$

$$| I_3 | \leq \frac{3 |p(1)| (1 + \exp (-3^{1/2}\beta) + m)}{[2(n + 1)\pi]^2 + 3\beta^2}.$$

Now $| E(\rho) | \leq | I_1 | + | I_2 | + | I_3 |$, and clearly for $|\rho|$ large enough there exists k such that for $n = k, k + 1, \dots$, (1.4) holds. We have then $|\rho^2 \delta_1(\rho)| > |\rho^2 e^{\omega_3\rho} E(\rho)|$ on C_n . Now it is easily verified that there is just one zero of $\delta_1(\rho)$ inside C_n . Hence, by Rouché's theorem, there is just one root of the characteristic equation inside C_n , and since these roots must occur in complex conjugate pairs, this root is real. This proves the theorem.

By a suitable modification of the choice of contours C_n in the proof of the preceding theorem, the following result may be obtained.

Theorem 2. *Let the zeros of $\delta_1(\rho)$ be denoted by $\rho_n^{(0)}$ such that $\rho_0^{(0)} < \rho_1^{(0)} < \dots$. Let $\delta > 0$ be given, and define R_n to be the circle $|\rho - \rho_n^{(0)}| \leq \delta$. Then for k large enough, each root ρ_n of the characteristic equation of (1) such that $R(\rho_n) > 2k\pi/3^{1/2}$ lies inside one of the circles $R_n, n = k, k + 1, \dots$, there being exactly one ρ_n , necessarily real, in each such R_n .*

The proof will be omitted.⁴

From the details of the proof of Theorem 1, it may be seen that for r, p , and $|p(1)|$ small enough, k may be chosen equal zero. The question of whether or not in this case all characteristic values of (1) are real, then, reduces to the question of whether or not all roots ρ_n of the characteristic equation for which $R(\rho_n) < 2\pi/3^{1/2}$ are real. By means of the following lemma, we obtain a lower bound on $|\lambda_n|$; e.g., $|\rho_n|$; and then by means of the subsequent theorem, apply the arguments of Theorem 1 to a supplementary contour which borders on C_0 and extends to the region for which no ρ_n can occur.

Lemma. *Let λ be a characteristic value of (1). Then $|\lambda|^2 \geq 6 - (m_1 + m_2)$ where $|p(t)|^2 \leq m_1, |q(t)|^2 \leq m_2$.*

Proof. Let $v(x) = u'(x)$. Then by (1)

$$v''(x) = -[p(x)u'(x) + q(x)u(x) + \lambda u(x)], \quad v(0) = v(1) = 0.$$

Hence

$$v(x) = \sum_n \frac{c_n + b_n + \lambda a_n}{\mu_n} \phi_n(x) \tag{1.5}$$

where $\phi_n(x)$ and μ_n are the characteristic functions and numbers respectively of the system

$$\phi''(x) + \mu\phi(x) = 0, \quad \phi(0) = \phi(1) = 0$$

and

$$a_n = \int_0^1 u(t)\phi_n(t) dt, \quad b_n = \int_0^1 u(t)q(t)\phi_n(t) dt, \quad c_n = \int_0^1 u'(t)p(t)\phi_n(t) dt.$$

Hence by (1.5) and integration from 0 to x , one obtains

$$u(x) = \sum_n \frac{c_n + b_n + \lambda a_n}{\mu_n} z_n(x) \quad \text{where} \quad z_n(x) = \int_0^x \phi_n(t) dt.$$

Multiplying by $\phi_k(x)$ and integrating from 0 to 1 one obtains

$$a_k = \int_0^1 \phi_k(x)u(x) dx = \sum_n \frac{c_n + b_n + \lambda a_n}{\mu_n} \int_0^1 z_n(x)\phi_k(x) dx.$$

Hence, using Schwarz's inequality,

$$|a_k|^2 \leq M \sum_n (|c_n|^2 + |b_n|^2 + |\lambda|^2 |a_n|^2) \left| \int_0^1 z_n(x)\phi_k(x) dx \right|^2$$

where

$$M = \sum_n \frac{1}{\mu_n^2} = \frac{1}{6}.$$

Summing over k and using Parseval's theorem:

$$\begin{aligned} \sum_k |a_k|^2 &\leq M \sum_n (|c_n|^2 + |b_n|^2 + |\lambda|^2 |a_n|^2) \sum_k \left| \int_0^1 z_n(x)\phi_k(x) dx \right|^2 \\ &= M \sum_n (|c_n|^2 + |b_n|^2 + |\lambda|^2 |a_n|^2) \int_0^1 |z_n(x)|^2 dx. \end{aligned} \tag{1.6}$$

⁴See L. E. Ward, [4] pp. 419-420.

However, since

$$\int_0^1 |z_n(x)|^2 dx \leq \int_0^1 \left\{ \int_0^1 |\phi_n(t)|^2 dt \right\} dx = 1,$$

and since we may assume

$$\int_0^1 |u(x)|^2 dx = 1,$$

another application of Parseval's theorem to (1.6) gives

$$1 \leq M \left\{ \int_0^1 |p(t)u'(t)|^2 dt + \int_0^1 |q(t)u(t)|^2 dt + |\lambda|^2 \right\}. \quad (1.7)$$

Let $|p(t)|^2 \leq m_1$, $|q(t)|^2 \leq m_2$; then from (1.7) one has

$$1 \leq M \left\{ m_1 \int_0^1 |u'(t)|^2 dt + m_2 + |\lambda|^2 \right\}. \quad (1.8)$$

Multiplying (1.5) by $\phi_k(x)$ and integrating from 0 to 1, one obtains

$$\left| \int_0^1 \phi_k(x)u'(x) dx \right|^2 = \left| \frac{c_k + b_k + \lambda a_k}{\mu_k} \right|^2 \leq \frac{|c_k|^2 + |b_k|^2 + |\lambda|^2 |a_k|^2}{\mu_k^2}$$

and summing over k and applying Parseval's theorem again:

$$\int_0^1 |u'(x)|^2 dx \leq M \left(m_1 \int_0^1 |u'(x)|^2 dx + m_2 + |\lambda|^2 \right),$$

from which it follows that

$$(1 - Mm_1) \int_0^1 |u'(x)|^2 dx \leq M(m_2 + |\lambda|^2). \quad (1.9)$$

Hence, by (1.8) and (1.9):

$$(1 - Mm_1) \leq M[m_1M(m_2 + |\lambda|^2) + (m_2 + |\lambda|^2)(1 - Mm_1)] = M(m_2 + |\lambda|^2).$$

From this one obtains

$$\frac{1 - M(m_1 + m_2)}{M} \leq |\lambda|^2.$$

Since $M = 1/6$, the proof of the lemma is complete.

Theorem 3. *Let p , r , m_1 , m_2 be defined as in previous theorems and lemmas. Then if these constants and $|p(1)|$ are sufficiently small, all characteristic values of the system (1) are real.*

Proof. It has already been noted that for sufficiently small values of p , r , and $|p(1)|$, k may be chosen equal zero. We now consider the supplementary contour C_{-1} , a trapezoid whose bases are the segments:

$$R(\rho) = \frac{2\pi}{3^{1/2}}, \quad |\arg \rho| \leq \frac{\pi}{3};$$

$$R(\rho) = \frac{R}{2}, \quad |\arg \rho| \leq \frac{\pi}{3};$$

where $R = [6 - (m_1 + m_2)]^{1/6}$. By Theorem 1 and the lemma all non-real characteristic values $\lambda_n = \rho_n^3$ must be such that ρ_n is in or on C_{-1} . However, for sufficiently small p , r , m_1 , m_2 , and $|p(1)|$, the argument used to prove Theorem 1 is applicable to C_{-1} and we conclude that there is just one root ρ_{-1} , necessarily real, inside C_{-1} . This proves the theorem.

PART II

Consider the system

$$[f(t)y''(t)]' + \lambda g(t)y(t) = 0, \quad y(0) = y'(0) = y''(1) = 0, \quad (2)$$

f and g real and analytic on $0 \leq t \leq 1$, $f(t) > 0$, $g(t) > 0$. The transformation

$$x = \frac{1}{d} \int_0^t h(s) ds \quad \text{where} \quad h(s) = \left[\frac{f(s)}{g(s)} \right]^{1/3}, \quad d = \int_0^1 h(s) ds,$$

has an inverse $t = t(x)$. This inverse transforms (2) into

$$\frac{d^3}{dx^3} y[t(x)] + p_1(x) \frac{d^2}{dx^2} y[t(x)] + p_2(x) \frac{d}{dx} y[t(x)] + y[t(x)] = 0, \quad (2.1)$$

$$y(0) = y'(0) = y''(1) = 0,$$

where

$$p_1(x) = \frac{1}{d} \left\{ 3h^4[t(x)]h'[t(x)] + \frac{f'[t(x)]}{g[t(x)]} h^2[t(x)] \right\}$$

$$p_2(x) = \frac{1}{d} \left\{ h^3[t(x)]h''[t(x)] + \frac{f'[t(x)]}{f[t(x)]} h'[t(x)] \right\},$$

the notation $f'[t(x)]$ meaning that the substitution $t = t(x)$ has been made in the function of $t : f'(t)$.

Next, the substitution

$$y[t(x)] = u(x) \exp \left[-\frac{1}{3} \int_0^x p_1(s) ds \right]$$

and subsequent multiplication by

$$\exp \left[\frac{1}{3} \int_0^x p_1(s) ds \right]$$

reduces (2.1) to $u'''(x) + p(x)u'(x) + [q(x) + \lambda]u(x) = 0$, $u(0) = u'(0) = u''(1) = 0$, where

$$p(x) = p_2(x) - \frac{p_1^2(x)}{3} - p_1'(x),$$

$$q(x) = \frac{2}{27} p_1^3(x) - \frac{p_1'(x)}{3} - \frac{p_1(x)p_2(x)}{3}.$$

Hence the results of Part I apply to the system (2).

Noting the structure of p and q ; e.g., p_1 and p_2 , in terms of f and g , we state an obvious corollary of Theorem 3.

Corollary. Let $|f^{(n)}(t)| < \epsilon$, $|g^{(n)}(t)| < \epsilon$ for $n = 1, 2, 3, 4$. Then for ϵ sufficiently small, all the characteristic values of (2) are real.

It is also clear from the structure of p and q that less restrictive, although perhaps more complicated, conditions on f and g than those in the hypothesis of the above corollary will yield the same conclusion.

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ON AN EQUATION OCCURRING IN THE HARMONIC ANALYSIS OF VISCOUS FLUID FLOW*

By RICHARD BELLMAN (*Stanford University*)

1. **Introduction.** It was shown by J. Kampé de Fériet¹ that the Fourier transform

$$z(w_1, w_2, t) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \zeta(x, y) \exp[-i(w_1x + w_2y)] dx dy \quad (1)$$

of the vorticity, $\zeta(x, y)$, associated with the two-dimensional flow of an incompressible fluid extending over the entire (x, y) -plane, under mild conditions, satisfies the non-linear integro-differential equation

$$\frac{\partial}{\partial t} z(w_1, w_2, t) = -v(w_1^2 + w_2^2)z(w_1, w_2, t) \quad (2a)$$

$$+ 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{\theta_1 w_2 - \theta_2 w_1}{\theta_1^2 + \theta_2^2} \right) z(\theta_1, \theta_2, t) \bar{z}(\theta_1 + w_1, \theta_2 + w_2, t) d\theta_1 d\theta_2,$$

and the boundary condition

$$z(w_1, w_2, 0) = \phi(w_1, w_2). \quad (2b)$$

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¹J. Kampé de Fériet, *Harmonic analysis of the two-dimensional flow of an incompressible viscous fluid*, Q. Appl. Math. 6, 1-13 (1948).