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A THREE-DIMENSIONAL FINITE ELEMENT FORMULATION FOR THERMOVISCOELASTIC ORTHOTROPIC MEDIA

M. A. ZOCHER AND S. E. GROVES

Lawrence Livermore National Laboratory, P.O. Box 808; L-342, Livermore, CA 94551, U.S.A.

D. H. ALLEN

Center for Mechanics of Composites, Texas A&M University, College Station, TX 77843-3141, U.S.A.

SUMMARY

This paper is concerned with the development of a numerical algorithm for the solution of the uncoupled, quasistatic initial/boundary value problem involving orthotropic linear viscoelastic media undergoing thermal and/or mechanical deformation. The constitutive equations, expressed in integral form involving the relaxation moduli, are transformed into an incremental algebraic form prior to development of the finite element formulation. This incrementalization is accomplished in closed form and results in a recursive relationship which leads to the need of solving a simple set of linear algebraic equations only for the extraction of the finite element solution. Use is made of a Dirichlet–Prony series representation of the relaxation moduli in order to derive the recursive relationship and thereby eliminate the storage problem that arises when dealing with materials possessing memory. Three illustrative example problems are included to demonstrate the method.

KEY WORDS: finite element method; viscoelasticity; incremental constitutive law

1. INTRODUCTION

Applications of the FEM to the solution of initial/boundary value problems involving materials exhibiting viscoelastic behaviour have evolved over a period of time that now spans approximately thirty years. Only a brief discussion of a small sampling of the literature from this period will be provided here. This sampling will be taken from the open literature and will overlook possible sources of information such as industry or government reports. Documentation regarding the large commercial codes such as NASTRAN or ABAQUS will also be neglected in the following. For a much more in-depth review, the reader is referred to Zocher.[1]

The criterion used in the selection of papers for discussion herein was that the focus be on finite element methods as opposed to finite element analysis (i.e., code development as opposed to code use). Before launching into this discussion, however, we would be remiss if we overlooked the works of Lee and Rogers[2] and of Hopkins and Hamming.[3] It was here that the direct solution of the Volterra type integrals that arise in viscoelastic stress analysis was first accomplished. This was achieved through the application of a step-by-step finite difference (FD) integration with respect to time. Although the FEM was not used in either of these papers, they are important to the current discussion because the FD approach that was employed in them was adopted by many of the early FEM developers. It is noted that the methods of Lee and Rogers[2] and of Hopkins and Hamming[3] included no recursive relations. Consequently, the results from all previous time steps would have to be kept in memory in order to find a solution at the current time step. This requirement obviously

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limited the method to the solution of relatively simple problems. Another non-FEM paper that is important to the discussion is that of Zak.[4] In this work, a thermoviscoelastic problem is solved using the FD method only. The method involves a stepwise integration through time similar to that of Lee and Rogers[2] except for one important difference—elimination of the storage problem. The key to accomplishing this was the use of a Dirichlet–Prony series (in this case a Kelvin model) to represent the kernel of the Volterra integral equation. Zak's method made the solution of 'large' problems possible.

King[5] developed the first viscoelastic finite element code that the authors are aware of. This program was applicable to plane stress and plane strain problems. The theory was developed using integral constitutive equations in terms of creep compliance. Central to the method was the assumption that the creep compliance could be separated into an 'elastic part' and a 'creep part' and that the strain could be considered to remain constant across a time step. Taylor and Chang[6] developed a FE procedure that was limited to axisymmetric problems involving isotropic thermorheologically simple materials. The constitutive equations were expressed in integral form in terms of reduced time. The kernels in these equations were relaxation moduli. Chang[7] extended this work to include the analysis of two-dimensional problems. More importantly, Chang[7] discussed two conditions under which the requirement of storing all previous solutions could be avoided. Taylor *et al.*[8] can be thought of as a completion of the work begun in the two previous references. In this paper, the authors devise an efficient recursive relation, thereby enabling the solution of 'large' problems (this method is distinct from Zak's).

Zienkiewicz and Watson[9] developed a two-dimensional linear isotropic finite element code with which the analyst could account for both thermal and ageing effects. This theory is developed in terms of creep compliance. As King[5] had done, the creep compliance is, at the outset, separated into two parts: an 'elastic part' and a 'creep part'. The creep part is eventually expressed in terms of a Kelvin model. Problem solution is accomplished in a step-by-step manner using small time intervals during which the stresses are taken to remain constant (King[5] had taken the strains to remain constant). Rashid and Rockenhauser[10] developed a finite element program for the analysis of prestressed concrete pressure vessels. The method is developed using a single integral constitutive formula involving the relaxation modulus. In this method, the Volterra integral equations that arise in the formulation are solved using the FD procedure of Lee and Rogers.[2] White[11] included Zak's recursive relations as an option in the finite element program that he developed. If this option is not selected, however, the time integration scheme employed by White proceeds in the manner of Lee and Rogers.[2]

All of the FE papers discussed to this point have been based upon constitutive relationships of the single integral form (hereditary integrals). Zienkiewicz *et al.*[12] developed a general two-dimensional FE code in which the constitutive equations are expressed in differential form and are assumed to be modelled by Kelvin analogues. The use of Kelvin models enabled the authors to overcome the storage problem. This work employs an initial strain approach to the FD approximations. Greenbaum and Rubinstein[13] also developed a finite element program based on the use of differential constitutive equations and the initial strain method.

The paper presented by Webber[14] is somewhat unique among the early FE developments. In this work, he does not use a direct step-by-step integration in time. Instead, he combines a finite element formulation with the standard viscoelastic correspondence principle to bypass the solution of a Volterra integral in time. The key to the method is the use of simplex elements (e.g. the CST is one such element).

Lynch[15] developed a finite element procedure for the analysis of viscoelastic forming processes. In this work, an example problem involving viscoelastic sheet rolling is presented. The first step in the development of the method is to cast the constitutive equations (given in single integral form involving relaxation moduli) into a numerical form. This is accomplished using a FD method. Then a finite element procedure is pursued resulting in a global set of algebraic equations which are solved using Gauss–Seidel iteration. The methods used by Lynch[15] have been extended by Batra *et al.*[16] Batra,[17] and Purushothaman *et al.*[18]

As discussed above, Zienkiewicz *et al.*[12] and Greenbaum and Rubinstein[13] have used an initial strain method to adapt the finite element method to problems of viscoelasticity. While this method possesses the very attractive feature that the stiffness matrix is time independent and therefore need not be regenerated on each time step, it must be recognized that the method is very sensitive to time-step size. This is a direct result of the assumption, inherent in the method, that the stress is constant during a time step. To be consistent with this assumption, very small time steps may be required. Cyr and Teter,[19] Kim and Kuhlemeyer,[20] and Krishnaswamy *et al.*[21, 22] represent modifications of the basic initial strain method which employ variable stiffness. These methods are much more stable but the stiffness matrix changes with time. It is also worth noting that methods such as those employed by Zienkiewicz *et al.*[12] and Greenbaum and Rubinstein[13] which are based on a differential form of constitutive relationship result in the requirement of solving a set of ordinary differential equations simultaneously. In each of these papers, as in others, a first-order numerical procedure (the Euler method) is employed to accomplish this task. Bažant[23] and Carpenter[24] have developed methods employing higher-order numerical procedures, namely the Runge–Kutta methods.

Viscoelastic finite element analysis of bonded joints has been conducted by Nagaraja and Alwar,[25] Yadagiri *et al.*[26] and Roy and Reddy.[27, 28] Applications of the finite element method in the field of viscoelastic fracture mechanics have been presented by Krishnaswamy et al.,[21] Moran and Knauss,[29] and Warby *et al.*[30] Brinson and Knauss[31] have used the finite element method to conduct an investigation of viscoelastic micromechanics. The problem of coupled thermoviscoelasticity has been addressed in the work of Oden,[32] Oden and Armstrong,[33] Batra *et al.*[16] and Batra.[17] Non-linear thermoviscoelastic analyses were presented by Bažant,[23] Henriksen,[34] Krishnaswamy *et al.*,[22] Moran and Knauss,[29] Oden,[32] Oden and Armstrong,[33] and Roy and Reddy.[27, 28] Srinatha and Lewis[35, 36] have addressed the problem of material incompressibility.

The references cited to this point have dealt primarily with isotropic viscoelasticity. We turn our attention now to a discussion of codes that have been developed for orthotropic viscoelasticity. Lin and Hwang[37, 38] were perhaps the first to produce a FE code with the capability of predicting the time-dependent response of orthotropic viscoelastic materials. The method of Lin and Hwang assumes a plane stress constitutive relationship expressed in terms of relaxation moduli that correspond to the transformed reduced stiffness matrix of classical lamination theory.[39]. It is assumed in this formulation that Q_{11} is time independent. Time-dependent expressions of $Q_{12}(t)$, $Q_{22}(t)$ and $Q_{66}(t)$ are obtained by multiplying the elastic Q_{ij} by a given function of time (this function being in the form of a Wiechert model). The same time-dependent function is used for each of these relaxation moduli. Lin and Hwang[37, 38] employed a numerical scheme similar to Taylor *et al.*[8] to solve the resultant set of integral equations. The method is extended in Lin and Yi[40] to account for free edge effects.

Hilton and Yi[41] have also developed a two-dimensional FE code for the plane stress analysis of laminated viscoelastic composites. In this work, the form of the constitutive equations and the kernels therein are precisely the same as in Lin and Hwang.[37, 38] Also as in Lin and Hwang, the minimization of a variational statement produces a set of integral equations which must be solved for the unknown displacements. The approach taken by Hilton and Yi in solving this set of equations is very different, however, from that used by Lin and Hwang.[37, 38] Instead of solving by direct integration, Hilton and Yi chose to use the Laplace transform and solve this set of equations in Laplace space as opposed to the space of reduced time. This work has been extended in Yi,[42] wherein an ability to predict delamination onset has been added.

Kennedy and Wang[43] have developed a fully three-dimensional orthotropic viscoelastic FE code. This code uses a 20-node isoparametric solid element. The constitutive equations upon which this code is based are expressed in integral form in terms of creep compliances using a non-linear viscoelastic model proposed by Lou and Schapery.[44] The kernels are expressed in terms of Kelvin analogs.

The objective of the current work is to present a new three-dimensional finite element formulation that is suitable for the analysis of orthotropic linear viscoelastic media. This formulation has been incorporated into the three-dimensional FE program ORTHO3D. This code is a general purpose tool capable of predicting the response of a structure to complex loading/thermal histories. Phenomena such as creep, relaxation, and creep-and-recovery can all be predicted using this program. The code also includes automated mesh generators which enable convenient grid generation for problems involving internal boundaries such as matrix cracks or delaminations (see Zocher *et al*.[45] for a demonstration of this capability).

The authors benefited greatly in the development of this new formulation from the work that has preceded it. Some of the similarities to and distinctions from the papers discussed above will be mentioned here. Since this formulation is based on constitutive equations of the single integral form, it is more closely related to those papers which also assume an integral form of the stress–strain relationship. All of the papers discussed above, with the exception of References 12 and 19–24, assume such a relationship. As Lynch[15] has done, the constitutive equations in the current work are converted from integral to numerical form prior to the development of a finite element formulation. This approach has been taken by few others. As was done in many of the cited references, use is made here of a Dirichlet–Prony series in order to derive a recursive relationship, thereby obviating the need to store the results from all previous time steps. Many of the papers discussed above assume either constant stress or constant strain across a time step. Those which assume constant stress predict creep-like behaviour adequately but are ill-suited for the prediction of relaxation. Those which assume constant strain do a fine job with relaxation but are not well suited for creep. The current work assumes constant strain rate across a time step and as such is equally well suited for the prediction of creep or relaxation phenomena. Since the current work has been developed with the analysis of orthotropic media in mind, it is most closely aligned with the work presented in References 37, 38 and 40–43. Of these, only Kennedy and Wang[43] is three-dimensional. The current work then compliments the work of Kennedy and Wang for three-dimensional thermoviscoelastic analysis of laminated composites. A major difference between the present work and that of Kennedy and Wang is that they developed their method upon constitutive equations expressed in terms of creep compliances, whereas relaxation moduli are used here.

In the following, a formal statement of the uncoupled thermoviscoelastic initial/boundary value problem is provided. This is followed by a discussion of the conversion through incrementalization (essentially a FD procedure) of the thermoviscoelastic constitutive equations into a form suitable for implementation in a finite element formulation. Next the finite element formulation which is based on these incrementalized constitutive equations is presented. Solutions to three example problems for which accepted analytical solutions are available are presented for the purpose of code verification.

2. PROBLEM STATEMENT

The problem to be solved in this research, or more precisely, the class of problems for which a method of solution is presented, may be referred to as the linear three-dimensional quasistatic orthotropic uncoupled thermoviscoelastic initial/boundary value problem, hereafter referred to as

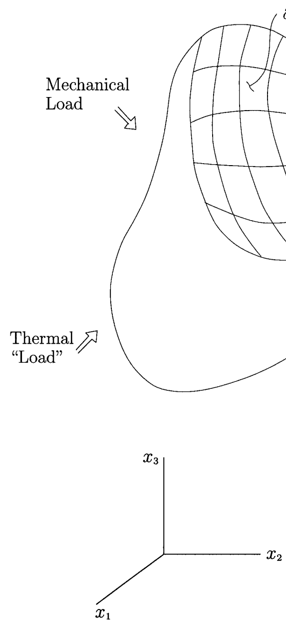

Figure 1. General three-dimensional IBVP

the IBVP. A concise statement of this problem, formulated in the context of continuum mechanics is provided below.

Consider a general three-dimensional domain Ω, bounded by surface $\partial\Omega$, and subjected to thermal and/or mechanical loading, as depicted in Figure 1. Essential boundary conditions are imposed over $\partial\Omega_1$; natural boundary conditions over $\partial\Omega_2$. Any division of $\partial\Omega$ into essential and natural parts is permissible provided the following relationship is not violated:

$$\partial\Omega_1 \cup \partial\Omega_2 = \partial\Omega \quad \text{and} \quad \partial\Omega_1 \cap \partial\Omega_2 = \emptyset \tag{1}$$

The domain may be simply or multiply connected. In addition to spatial variation, material properties may be dependent upon time and temperature. Material properties may be isotropic, transversely isotropic, or orthotropic. Our goal, to be accomplished through the solution of the IBVP, is to accurately predict the response of the body to the applied loading. The variables of state which are used to assess this response are the displacement vector $u_i(x_k, t)$, the stress tensor $\sigma_{ij}(x_k, t)$, and the strain tensor $\varepsilon_{ij}(x_k, t)$.

The governing equations which enable us to solve the IBVP are equilibrium,

$$\sigma_{ji,j} + \rho f_i = 0 \tag{2}$$

strain displacement,

$$\varepsilon_{ij} = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{3}$$

and constitution,

$$\sigma_{ij}(x_k, \xi) = \int_0^{\xi} C_{ijkl}\left(x_k, \xi - \xi'\right) \frac{\partial \varepsilon_{kl}(x_k, \xi')}{\partial \xi'}\,\mathrm{d}\xi' - \int_0^{\xi} \beta_{ij}\left(x_k, \xi - \xi'\right) \frac{\partial \Theta(x_k, \xi')}{\partial \xi'}\,\mathrm{d}\xi' \tag{4}$$

with constraints imposed on the solution by the following boundary and initial conditions:

$$\begin{aligned} u_i &= \hat{u}_i && \text{on } \partial\Omega_1 \\ T_i &= \sigma_{ji} n_j = \hat{T}_i && \text{on } \partial\Omega_2 \end{aligned} \tag{5}$$

$$\left.\begin{aligned} \Theta(x_k, t) &= 0 \\ u_i(x_k, t) &= 0 \\ \sigma_{ij}(x_k, t) &= 0 \end{aligned}\right. \quad \text{for } t < 0 \tag{6}$$

In the above, f_i is the body force, T_i is the surface traction, n_j is the unit outer normal on $\partial\Omega$, and ρ is the mass density. The terms C_{ijkl} and β_{ij} represent the fourth-order tensor of orthotropic relaxation moduli relating stress to mechanical strain, and the second-order tensor of relaxation moduli relating stress to thermal strain, respectively. The symbol Θ is used to represent the difference between the current temperature and a stress-free reference temperature. The reader will recognize from the form of the constitutive relationship that we have assumed the material to be possibly non-homogeneous, non-ageing, orthotropic, and thermorheologically simple. The symbol ξ in (4) is referred to as the reduced time and is defined as:

$$\xi = \xi(t) \equiv \int_0^t \frac{1}{a_T}\,\mathrm{d}\tau, \quad \xi' = \xi(t') \equiv \int_0^{t'} \frac{1}{a_T}\,\mathrm{d}\tau \tag{7}$$

where

$$a_T = a_T(T(\tau)) \quad \text{or equivalently} \quad a_T = a_T(\Theta(\tau)) \tag{8}$$

The term a_T is the shift factor of the time-temperature superposition principle. The shift factor is essentially a material property; it will often be expressed in terms of an Arrhenius relation or the familiar WLF formula. The symbol $\equiv$ is used herein to mean 'is defined as'.

3. INCREMENTALIZATION OF THE CONSTITUTIVE EQUATIONS

We have stated that the constitutive relationship for the class of materials considered in this work is given by (4). While it is possible to work with such a relationship in the development of a finite element method (doing so leads to the requirement of solving a set of Volterra integrals in order to extract the FE solution), a different approach is taken here. Rather than incorporating (4) directly into a finite element formulation, we shall develop a numerical incrementalization of the constitutive equations which will prove to be quite amenable to implementation in a finite element program. Use of this numerical approximation will lead to the requirement of solving a simple set of algebraic equations in order to extract the FE solution. A similar approach has been taken by Ghazlan *et al.*[46]

Let the time line (reduced time) be subdivided into discrete intervals such that $\xi_{n+1} = \xi_n + \Delta\xi$ and let us assume that the state of stress is known at reduced time ξ_n. We seek a means of expressing the state of stress at reduced time ξ_{n+1} that will be amenable to implementation in a finite element program. The state of stress at reduced time ξ_{n+1}, according to (4), is given by

$$\begin{aligned} &\sigma_{ij}(x_k, \xi_{n+1}) \\ &\quad = \int_0^{\xi_{n+1}} C_{ijkl}\left(x_k, \xi_{n+1} - \xi'\right) \frac{\partial \varepsilon_{kl}(x_k, \xi')}{\partial \xi'}\,\mathrm{d}\xi' - \int_0^{\xi_{n+1}} \beta_{ij}\left(x_k, \xi_{n+1} - \xi'\right) \frac{\partial \Theta(x_k, \xi')}{\partial \xi'}\,\mathrm{d}\xi' \end{aligned} \tag{9}$$

This may also be written as:

$$\sigma_{ij}(x_k,\xi_{n+1})$$
$$= \int_0^{\xi_n} C_{ijkl}\left(x_k,\xi_{n+1}-\xi'\right)\frac{\partial\varepsilon_{kl}(x_k,\xi')}{\partial\xi'}\,\mathrm{d}\xi' + \int_{\xi_n}^{\xi_{n+1}} C_{ijkl}\left(x_k,\xi_{n+1}-\xi'\right)\frac{\partial\varepsilon_{kl}(x_k,\xi')}{\partial\xi'}\,\mathrm{d}\xi'$$
$$- \int_0^{\xi_n} \beta_{ij}\left(x_k,\xi_{n+1}-\xi'\right)\frac{\partial\Theta(x_k,\xi')}{\partial\xi'}\,\mathrm{d}\xi' - \int_{\xi_n}^{\xi_{n+1}} \beta_{ij}\left(x_k,\xi_{n+1}-\xi'\right)\frac{\partial\Theta(x_k,\xi')}{\partial\xi'}\,\mathrm{d}\xi' \quad (10)$$

Let us define ΔC_{ijkl} and $\Delta\beta_{ij}$ as follows:

$$\Delta C_{ijkl} \equiv C_{ijkl}\left(x_k,\xi_{n+1}-\xi'\right) - C_{ijkl}\left(x_k,\xi_n-\xi'\right) \quad (11)$$

$$\Delta\beta_{ij} \equiv \beta_{ij}\left(x_k,\xi_{n+1}-\xi'\right) - \beta_{ij}\left(x_k,\xi_n-\xi'\right) \quad (12)$$

Substituting (11) and (12) into (10) yields

$$\Delta\sigma_{ij} = \int_{\xi_n}^{\xi_{n+1}} C_{ijkl}\left(x_k,\xi_{n+1}-\xi'\right)\frac{\partial\varepsilon_{kl}(x_k,\xi')}{\partial\xi'}\,\mathrm{d}\xi'$$
$$- \int_{\xi_n}^{\xi_{n+1}} \beta_{ij}\left(x_k,\xi_{n+1}-\xi'\right)\frac{\partial\Theta(x_k,\xi')}{\partial\xi'}\,\mathrm{d}\xi' + \Delta\sigma_{ij}^{\mathrm{R}} \quad (13)$$

where

$$\Delta\sigma_{ij}^{\mathrm{R}} = \int_0^{\xi_n} \Delta C_{ijkl}\frac{\partial\varepsilon_{kl}(x_k,\xi')}{\partial\xi'}\,\mathrm{d}\xi' - \int_0^{\xi_n} \Delta\beta_{ij}\frac{\partial\Theta(x_k,\xi')}{\partial\xi'}\,\mathrm{d}\xi' \quad (14)$$

and $\Delta\sigma_{ij}$ is defined as

$$\Delta\sigma_{ij} \equiv \sigma_{ij}(x_k,\xi_{n+1}) - \sigma_{ij}(x_k,\xi_n) \quad (15)$$

Let us now suppose that each member of $C_{ijkl}(x_k,\xi_{n+1}-\xi')$ and $\beta_{ij}(x_k,\xi_{n+1}-\xi')$ can be fit with a Wiechert model, i.e.

$$C_{ijkl}\left(x_k,\xi_{n+1}-\xi'\right) = C_{ijkl_\infty} + \sum_{m=1}^{M_{ijkl}} C_{ijkl_m}\mathrm{e}^{-(\xi_{n+1}-\xi')/\rho_{ijkl_m}} \quad (\text{no sum on } i,j,k,l) \quad (16)$$

$$\beta_{ij}\left(x_k,\xi_{n+1}-\xi'\right) = \beta_{ij_\infty} + \sum_{p=1}^{P_{ij}} \beta_{ij_p}\mathrm{e}^{-(\xi_{n+1}-\xi')/\rho_{ij_p}} \quad (\text{no sum on } i,j) \quad (17)$$

where

$$\rho_{ijkl_m} = \eta_{ijkl_m}/C_{ijkl_m} \quad \rho_{ij_p} = \eta_{ij_p}/\beta_{ij_p} \quad (\text{no sum on } i,j,k,l) \quad (18)$$

In the above, the η_{ijkl_m} and η_{ij_p} are dashpot coefficients and the C_{ijkl_m} and β_{ij_p} are spring constants. The ρ's are generally referred to as relaxation times. The reader is urged to note that the whenever a ρ or η possesses four subscripts, it is associated with the Wiechert model of a particular member of C_{ijkl} whereas those with two subscripts are associated with the Wiechert model of a particular member of β_{ij}. Hence the four-subscripted ρ's and η's are distinct from the two-subscripted variety. The use of ρ and η to represent relaxation times and dashpot coefficients for the Wiechert models of both C_{ijkl} and β_{ij} is admittedly potentially confusing, but should cause the reader no undue burdon with the foregoing note of caution.

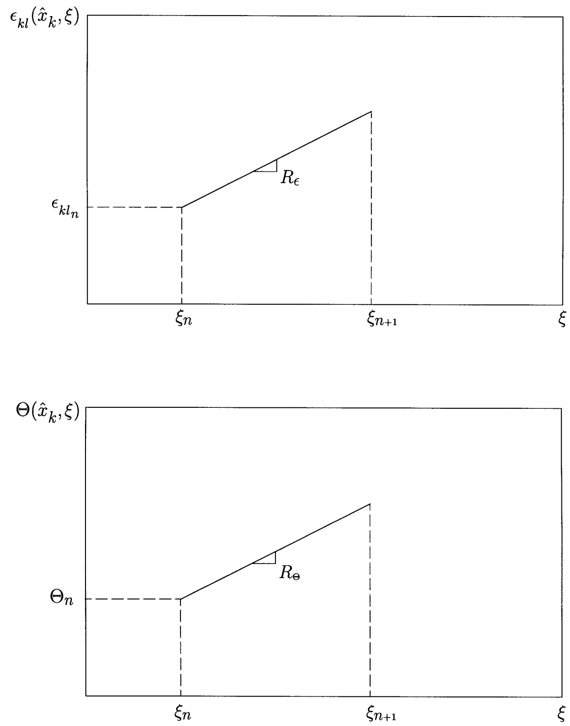

Figure 2. Approximations of $\varepsilon_{kl}(\hat{x}_k, \xi)$ and $\Theta(\hat{x}_k, \xi)$ over $\xi_n \rightarrow \xi_{n+1}$

In addition, let us suppose that $\varepsilon_{kl}(x_k, \xi)$ and $\Theta(x_k, \xi)$ can be approximated over the interval $\xi_n \leqslant \xi \leqslant \xi_{n+1}$ by the following:

$$\varepsilon_{kl}(x_k, \xi) = \varepsilon_{kl_n} + R_\varepsilon(\xi - \xi_n) H(\xi - \xi_n) \tag{19}$$

$$\Theta(x_k, \xi) = \Theta_n + R_\Theta(\xi - \xi_n) H(\xi - \xi_n) \tag{20}$$

where ε_{kl_n} and Θ_n represent the values at the beginning of the time interval, R_ε and R_Θ are constants representing the time rate of change over the interval, and $H(\xi - \xi_n)$ is the Heaviside step function. These approximations are depicted graphically in Figure 2.

With these two approximations, (13) may be integrated in closed form to produce:

$$\Delta\sigma_{ij} = C'_{ijkl}\Delta\varepsilon_{kl} - \beta'_{ij}\Delta\Theta + \Delta\sigma^{\mathrm{R}}_{ij} \tag{21}$$

where

$$C'_{ijkl} \equiv C_{ijkl_\infty} + \frac{1}{\Delta\xi}\sum_{m=1}^{M_{ijkl}} \eta_{ijkl_m}\left(1 - \mathrm{e}^{-\Delta\xi/\rho_{ijkl_m}}\right) \quad \text{(no sum on } i, j, k, l) \tag{22}$$

$$\beta'_{ij} \equiv \beta_{ij_\infty} + \frac{1}{\Delta\xi}\sum_{p=1}^{P_{ij}} \eta_{ij_p}\left(1 - \mathrm{e}^{-\Delta\xi/\rho_{ij_p}}\right) \quad \text{(no sum on } i, j) \tag{23}$$

$$\Delta\varepsilon_{kl} \equiv R_\varepsilon\Delta\xi, \quad \Delta\Theta \equiv R_\Theta\Delta\xi \tag{24}$$

Note that C'_{ijkl} and β'_{ij} are independent of time if $\Delta\xi$ remains constant. In that case, all time dependence in the material behaviour resides in $\Delta\sigma^{\mathrm{R}}_{ij}$. A fortuitous consequence of this is that in those analyses for which it is reasonable to use a constant time step, the stiffness matrix will remain constant and will not have to be regenerated on each time step (this will become apparent in the following section). There will of course be analyses (such as in example three presented later) for which the costs involved in using a variable time step are justified.

Let us now redirect our attention to $\Delta\sigma_{ij}^{\mathrm{R}}$ for the purpose of converting (14) into a more convenient form. The steps involved in affecting this conversion are set forth as follows. First, the Wiechert models of (16) and (17), along with analogous expressions for $C_{ijkl}(x_k, \xi_n - \xi')$ and $\beta_{ij}(x_k, \xi_n - \xi')$, are used in equations (11) and (12) so that ΔC_{ijkl} and $\Delta\beta_{ij}$ can be rewritten as:

$$\Delta C_{ijkl} = -\sum_{m=1}^{M_{ijkl}} C_{ijkl_m} \mathrm{e}^{-(\xi_n - \xi')/\rho_{ijkl_m}} \left(1 - \mathrm{e}^{-\Delta\xi/\rho_{ijkl_m}}\right) \quad (\text{no sum on } i, j, k, l) \tag{25}$$

$$\Delta\beta_{ij} = -\sum_{p=1}^{P_{ij}} \beta_{ij_p} \mathrm{e}^{-(\xi_n - \xi')/\rho_{ij_p}} \left(1 - \mathrm{e}^{-\Delta\xi/\rho_{ij_p}}\right) \quad (\text{no sum on } i, j) \tag{26}$$

Equations (25) and (26) are then substituted into (14) so that $\Delta\sigma_{ij}^{\mathrm{R}}$ may be expressed as:

$$\Delta\sigma_{ij}^{\mathrm{R}} = -\sum_{k=1}^{3}\sum_{l=1}^{3} A_{ijkl} + \sum_{p=1}^{P_{ij}} \left(1 - \mathrm{e}^{-\Delta\xi/\rho_{ij_p}}\right) B_{ij_p}(\xi_n) \quad (\text{no sum on } i, j) \tag{27}$$

where

$$A_{ijkl} = \sum_{m=1}^{M_{ijkl}} \left(1 - \mathrm{e}^{-\Delta\xi/\rho_{ijkl_m}}\right) S_{ijkl_m}(\xi_n) \quad (\text{no sum on } i, j, k, l) \tag{28}$$

$$S_{ijkl_m}(x_k, \xi_n) \equiv \int_0^{\xi_n} C_{ijkl_m} \left(\mathrm{e}^{-(\xi_n - \xi')/\rho_{ijkl_m}}\right) \frac{\partial\varepsilon_{kl}(x_k, \xi')}{\partial\xi'} \, \mathrm{d}\xi' \tag{29}$$

$$B_{ij_p}(x_k, \xi_n) \equiv \int_0^{\xi_n} \beta_{ij_p} \left(\mathrm{e}^{-(\xi_n - \xi')/\rho_{ij_p}}\right) \frac{\partial\Theta(x_k, \xi')}{\partial\xi'} \, \mathrm{d}\xi' \tag{30}$$

The final step in the conversion of (14) is to develop reasonable approximations to (29) and (30). If we assume that the partial derivatives appearing in (29) and (30) can be approximated as

$$\frac{\partial\varepsilon_{kl}(x_k, \xi')}{\partial\xi'} \approx R_\varepsilon \equiv \frac{\Delta\varepsilon_{kl}}{\Delta\xi} \quad \left(\xi_n - \Delta\xi \leqslant \xi' \leqslant \xi_n\right) \tag{31}$$

$$\frac{\partial\Theta(x_k, \xi')}{\partial\xi'} \approx R_\Theta \equiv \frac{\Delta\Theta}{\Delta\xi} \quad \left(\xi_n - \Delta\xi \leqslant \xi' \leqslant \xi_n\right) \tag{32}$$

where $\Delta\varepsilon_{kl}/\Delta\xi$ and $\Delta\Theta/\Delta\xi$ are determined from the previous time step, S_{ijkl_m} and B_{ij_p} may then be determined recursively as follows:

$$S_{ijkl_m}(x_k, \xi_n) = \mathrm{e}^{-\Delta\xi/\rho_{ijkl_m}} S_{ijkl_m}(x_k, \xi_n - \Delta\xi) + \eta_{ijkl_m} R_\varepsilon \left(1 - \mathrm{e}^{-\Delta\xi/\rho_{ijkl_m}}\right) \quad (\text{no sum on } i, j, k, l) \tag{33}$$

$$B_{ij_p}(x_k, \xi_n) = \mathrm{e}^{-\Delta\xi/\rho_{ij_p}} \; B_{ij_p}(x_k, \xi_n - \Delta\xi) + \eta_{ij_p} R_\Theta \left(1 - \mathrm{e}^{-\Delta\xi/\rho_{ij_p}}\right) \quad (\text{no sum on } i, j) \tag{34}$$

The fundamental step in deriving this recursive relationship was to divide the domain of integration $(0 \leqslant \xi' \leqslant \xi_n)$ in (29) and (30) into two parts: $(0 \leqslant \xi' \leqslant \xi_{n-1})$ and $(\xi_{n-1} \leqslant \xi' \leqslant \xi_n)$, similar to the step taken on going from (9) to (10). We have now completed the conversion of (14) into the form we desire. It is noted that the values of S and B from the previous time step must be kept in storage much as if they were internal variables.

3.1. Summary of incrementalization

In summary, we have now succeeded in converting the constitutive equations:

$$\sigma_{ij}(x_k, \xi) = \int_0^{\xi} C_{ijkl}\left(x_k, \xi - \xi'\right) \frac{\partial \varepsilon_{kl}(x_k, \xi')}{\partial \xi'} \mathrm{d}\xi' - \int_0^{\xi} \beta_{ij}\left(x_k, \xi - \xi'\right) \frac{\partial \Theta(x_k, \xi')}{\partial \xi'} \mathrm{d}\xi'$$

into an incremental form given by

$$\Delta\sigma_{ij} = C'_{ijkl}\Delta\varepsilon_{kl} - \beta'_{ij}\Delta\Theta + \Delta\sigma^{\mathrm{R}}_{ij}$$

where C'_{ijkl}, β'_{ij}, $\Delta\varepsilon_{kl}$ and $\Delta\Theta$ are given by

$$C'_{ijkl} \equiv C_{ijkl_\infty} + \frac{1}{\Delta\xi}\sum_{m=1}^{M_{ijkl}} \eta_{ijkl_m}\left(1 - \mathrm{e}^{-\Delta\xi/\rho_{ijkl_m}}\right) \quad (\text{no sum on } i, j, k, l)$$

$$\beta_{ij}{}' \equiv \beta_{ij_\infty} + \frac{1}{\Delta\xi}\sum_{p=1}^{P_{ij}} \eta_{ij_p}\left(1 - \mathrm{e}^{-\Delta\xi/\rho_{ij_p}}\right) \quad (\text{no sum on } i, j)$$

$$\Delta\varepsilon_{kl} \equiv R_\varepsilon \Delta\xi, \quad \Delta\Theta \equiv R_\Theta \Delta\xi$$

and $\Delta\sigma^{\mathrm{R}}_{ij}$ is given by:

$$\Delta\sigma^{\mathrm{R}}_{ij} = -\sum_{k=1}^{3}\sum_{l=1}^{3} A_{ijkl} + \sum_{p=1}^{P_{ij}}\left(1 - \mathrm{e}^{-\Delta\xi/\rho_{ij_p}}\right) B_{ij_p}(\xi_n) \quad (\text{no sum on } i, j)$$

where

$$A_{ijkl} = \sum_{m=1}^{M_{ijkl}}\left(1 - \mathrm{e}^{-\Delta\xi/\rho_{ijkl_m}}\right) S_{ijkl_m}(\xi_n) \quad (\text{no sum on } i, j, k, l)$$

$$S_{ijkl_m}(x_k, \xi_n) = \mathrm{e}^{-\Delta\xi/\rho_{ijkl_m}} S_{ijkl_m}(x_k, \xi_n - \Delta\xi) + \eta_{ijkl_m} R_\varepsilon\left(1 - \mathrm{e}^{-\Delta\xi/\rho_{ijkl_m}}\right) \quad (\text{no sum on } i, j, k, l)$$

$$B_{ij_p}(x_k, \xi_n) = \mathrm{e}^{-\Delta\xi/\rho_{ij_p}} B_{ij_p}(x_k, \xi_n - \Delta\xi) + \eta_{ij_p} R_\Theta\left(1 - \mathrm{e}^{-\Delta\xi/\rho_{ij_p}}\right) \quad (\text{no sum on } i, j)$$

This incremental form of the constitutive equations is well suited to implementation in a finite element program.

It must be recognized that the incremental reformulation of the constitutive equations just presented includes some approximations that can lead to error in the solution of the IBVP. Approximations were introduced in equations (19), (20), (31) and (32). The nature of each of these approximations is the same: that the variation in a quantity across some $\Delta\xi$ shall be assumed to be linear. This assumption introduces error if the relevant change is actually non-linear (of course no error is introduced if the relevant change is linear). Fortunately, for cases in which the relevant change is non-linear, the user can drive the error to as small a value as is considered acceptable merely by using small time steps. An additional note on convergence and time-step size is provided later. Another potential source of error is the assumption that each term of C_{ijkl} and β_{ij} can be fit with a Wiechert model. Fortunately, it has been the experience of the authors that these quantities can indeed be fit with a Wiechert model so accurately as to make the fit indistinguishable from the experimental data.

4. FINITE ELEMENT FORMULATION

We showed in the previous section how the constitutive equations (4) can be recast in an incrementalized form (as given in (21)). This incrementalized constitutive formula now becomes the basis for the following finite element formulation. Applying the method of weighted residuals, the governing differential equation which is given in (2) can be converted to the symmetric variational form given by

$$\int_{\Omega} \sigma_{ji}\mu_{ij}\,\mathrm{d}V = \int_{\Omega} \rho f_i v_i\,\mathrm{d}V + \int_{\partial\Omega_2} T_i v_i\,\mathrm{d}S \tag{35}$$

where v_i is an arbitrary admissible test function (in this case test displacement) and μ_{ij} is defined as follows:

$$\mu_{ij} \equiv \tfrac{1}{2}\left(v_{i,j} + v_{j,i}\right)$$

Equation (35), evaluated at time ξ_{n+1} (remember that we assume that the solution is known at time ξ_n) is given by

$$\int_{\Omega} \sigma_{ji}^{n+1}\mu_{ij}^{n+1}\,\mathrm{d}V = \int_{\Omega} \rho f_i^{n+1} v_i^{n+1}\,\mathrm{d}V + \int_{\partial\Omega_2} T_i^{n+1} v_i^{n+1}\,\mathrm{d}S \tag{36}$$

where the superscript 'n+1' denotes 'at reduced time ξ_{n+1}'. Since the stress–strain relations (21) are incrementalized, it is necessary to incrementalize (36). Let us define the following:

$$\begin{aligned}
\Delta\sigma_{ji} &\equiv \sigma_{ji}^{n+1} - \sigma_{ji}^{n} \;\Rightarrow\; \sigma_{ji}^{n+1} = \sigma_{ji}^{n} + \Delta\sigma_{ji}\\
\Delta\mu_{ij} &\equiv \mu_{ij}^{n+1} - \mu_{ij}^{n} \;\Rightarrow\; \mu_{ij}^{n+1} = \mu_{ij}^{n} + \Delta\mu_{ij}\\
\Delta u_{i} &\equiv u_{i}^{n+1} - u_{i}^{n} \;\Rightarrow\; u_{i}^{n+1} = u_{i}^{n} + \Delta u_{i}\\
\Delta v_{i} &\equiv v_{i}^{n+1} - v_{i}^{n} \;\Rightarrow\; v_{i}^{n+1} = v_{i}^{n} + \Delta v_{i}
\end{aligned} \tag{37}$$

Now recognizing that v_i^n and μ_{ij}^n are zero (a consequence of u_i^n being known), substitution of (37) into (36) yields

$$\int_{\Omega} \left(\sigma_{ji}^{n} + \Delta\sigma_{ji}\right)\Delta\mu_{ij}\,\mathrm{d}V = \int_{\Omega} \rho f_i^{n+1}\Delta v_i\,\mathrm{d}V + \int_{\partial\Omega_2} T_i^{n+1}\Delta v_i\,\mathrm{d}S \tag{38}$$

Or, upon rearranging terms:

$$\int_{\Omega} \Delta\sigma_{ji}\Delta\mu_{ij}\,\mathrm{d}V = \int_{\Omega} \rho f_i^{n+1}\Delta v_i\,\mathrm{d}V + \int_{\partial\Omega_2} T_i^{n+1}\Delta v_i\,\mathrm{d}S - \int_{\Omega} \sigma_{ji}^{n}\Delta\mu_{ij}\,\mathrm{d}V \tag{39}$$

We now reintroduce thermomechanical constitution (21) onto the formulation by way of substitution into (39); doing so yields

$$\begin{aligned}
&\int_{\Omega} \left[C'_{ijkl}\Delta\varepsilon_{kl} - \beta'_{ij}\Delta\Theta + \Delta\sigma_{ij}^{\mathrm{R}}\right]\Delta\mu_{ij}\,\mathrm{d}V\\
&\quad = \int_{\Omega} \rho f_i^{n+1}\Delta v_i\,\mathrm{d}V + \int_{\partial\Omega_2} T_i^{n+1}\Delta v_i\,\mathrm{d}S - \int_{\Omega} \sigma_{ji}^{n}\Delta\mu_{ij}\,\mathrm{d}V
\end{aligned} \tag{40}$$

Rearranging gives

$$\begin{aligned}
\int_{\Omega} C'_{ijkl}\Delta\varepsilon_{kl}\Delta\mu_{ij}\,\mathrm{d}V &= \int_{\Omega} \rho f_i^{n+1}\Delta v_i\,\mathrm{d}V + \int_{\partial\Omega_2} T_i^{n+1}\Delta v_i\,\mathrm{d}S\\
&\quad - \int_{\Omega} \sigma_{ji}^{n}\Delta\mu_{ij}\,\mathrm{d}V - \int_{\Omega} \Delta\sigma_{ij}^{\mathrm{R}}\Delta\mu_{ij}\,\mathrm{d}V + \int_{\Omega} \beta'_{ij}\Delta\Theta\Delta\mu_{ij}\,\mathrm{d}V
\end{aligned} \tag{41}$$

which can be equivalently expressed in matrix notation as:

$$\int_{\Omega}([D][\Delta v])^{\mathrm{T}}[C'][D][\Delta u]\,\mathrm{d}V=\int_{\Omega}[\Delta v]^{\mathrm{T}}\rho[f^{n+1}]\,\mathrm{d}V+\int_{\partial\Omega_2}[\Delta v]^{\mathrm{T}}[T^{n+1}]\,\mathrm{d}S$$
$$-\int_{\Omega}([D][\Delta v])^{\mathrm{T}}[\sigma^{n}]\,\mathrm{d}V-\int_{\Omega}([D][\Delta v])^{\mathrm{T}}[\Delta\sigma^{\mathrm{R}}]\,\mathrm{d}V+\int_{\Omega}([D][\Delta v])^{\mathrm{T}}[\beta']\,\mathrm{d}V \tag{42}$$

where $[D]$ is the typical strain–displacement relations operator. We now turn to a discussion of finding an approximate solution to the IBVP. The nature of the approximation is that we shall assume that the integrals appearing in (42) can be calculated as the sum of contributions furnished by each element in the mesh, and that the test and trial functions (which are infinite dimensional) can be approximated within an element by the following finite dimensional series:

$$\Delta u_{ih}^{\mathrm{e}}(x,y,z,\xi)=\sum_{I=1}^{N_{\mathrm{e}}}\Delta u_i^I\psi_I^{\mathrm{e}}(x,y,z,\xi) \tag{43}$$

$$\Delta v_{jh}^{\mathrm{e}}(x,y,z,\xi)=\sum_{J=1}^{N_{\mathrm{e}}}\Delta v_j^J\psi_J^{\mathrm{e}}(x,y,z,\xi) \tag{44}$$

where the range on i and j is three, Δu_i^I and Δv_j^J are the changes in displacement vectors at nodes I and J, respectively, and N_{e} is the number of shape functions (ψ's) used in the approximation for the element (N_{e} is also equal to the number of nodes in the element). The discretization of (42), written in terms of the finite element interpolants Δu_h^{e} and Δv_h^{e} is now expressed as:

$$\int_{\Omega_{\mathrm{e}}}([D][\Delta v_h^{\mathrm{e}}])^{\mathrm{T}}[C'^{e}][D][\Delta u_h^{\mathrm{e}}]\,\mathrm{d}V=\int_{\Omega_{\mathrm{e}}}[\Delta v_h^{\mathrm{e}}]^{\mathrm{T}}\rho[f^{n+1}]\,\mathrm{d}V+\int_{\partial\Omega_{2h}^{\mathrm{e}}}[\Delta v_h^{\mathrm{e}}]^{\mathrm{T}}[T^{n+1}]\,\mathrm{d}S$$
$$-\int_{\Omega_{\mathrm{e}}}([D][\Delta v_h^{\mathrm{e}}])^{\mathrm{T}}[\sigma^{n}]\,\mathrm{d}V-\int_{\Omega_{\mathrm{e}}}([D][\Delta v_h^{\mathrm{e}}])^{\mathrm{T}}[\Delta\sigma^{\mathrm{R}}]\,\mathrm{d}V+\int_{\Omega_{\mathrm{e}}}([D][\Delta v_h^{\mathrm{e}}])^{\mathrm{T}}[\beta'^{\mathrm{e}}]\,\mathrm{d}V \tag{45}$$

The matrices $[\Delta u_h^{\mathrm{e}}]$ and $[\Delta v_h^{\mathrm{e}}]$ which are introduced in (45) are given by

$$[\Delta u_h^{\mathrm{e}}]=[\psi^{\mathrm{e}}][\Delta u^{\mathrm{e}}],\quad [\Delta v_h^{\mathrm{e}}]=[\psi^{\mathrm{e}}][\Delta v^{\mathrm{e}}] \tag{46}$$

where $[\psi^{\mathrm{e}}]$ is the typical matrix of shape functions and $[\Delta u^{\mathrm{e}}]$, and $[\Delta v^{\mathrm{e}}]$ are vectors of the change in nodal displacement during $\Delta\xi$. To get (45) into the form we desire, we introduce the following:

$$[D][\Delta u_h^{\mathrm{e}}]=[D][\psi^{\mathrm{e}}][\Delta u^{\mathrm{e}}]=[B^{\mathrm{e}}][\Delta u^{\mathrm{e}}] \tag{47}$$

$$[D][\Delta v_h^{\mathrm{e}}]=[D][\psi^{\mathrm{e}}][\Delta v^{\mathrm{e}}]=[B^{\mathrm{e}}][\Delta v^{\mathrm{e}}] \tag{48}$$

Using (46)–(48), we may rewrite (45) as:

$$\int_{\Omega_{\mathrm{e}}}([B^{\mathrm{e}}][\Delta v^{\mathrm{e}}])^{\mathrm{T}}[C'][B^{\mathrm{e}}][\Delta u^{\mathrm{e}}]\,\mathrm{d}V=\int_{\Omega_{\mathrm{e}}}([\psi^{\mathrm{e}}][\Delta v^{\mathrm{e}}])^{\mathrm{T}}\rho[f^{n+1}]\,\mathrm{d}V$$
$$+\int_{\partial\Omega_{2h}^{\mathrm{e}}}([\psi^{\mathrm{e}}][\Delta v^{\mathrm{e}}])^{\mathrm{T}}[T^{n+1}]\,\mathrm{d}S-\int_{\Omega_{\mathrm{e}}}([B^{\mathrm{e}}][\Delta v^{\mathrm{e}}])^{\mathrm{T}}[\sigma^{n}]\,\mathrm{d}V$$
$$-\int_{\Omega_{\mathrm{e}}}([B^{\mathrm{e}}][\Delta v^{\mathrm{e}}])^{\mathrm{T}}[\Delta\sigma^{\mathrm{R}}]\,\mathrm{d}V+\int_{\Omega_{\mathrm{e}}}([B^{\mathrm{e}}][\Delta v^{\mathrm{e}}])^{\mathrm{T}}[\beta'^{e}]\,\mathrm{d}V \tag{49}$$

Acknowledging that $[\Delta v^{\mathrm{e}}]^{\mathrm{T}}$ is arbitrary, (49) simplifies to

$$[k^{\mathrm{e}}][\Delta u^{\mathrm{e}}]=[f_1^{\mathrm{e}}]+[f_2^{\mathrm{e}}]+[f_3^{\mathrm{e}}]+[f_4^{\mathrm{e}}]+[f_5^{\mathrm{e}}] \tag{50}$$

where

$$
\begin{aligned}
[k^{\mathrm{e}}] &= \int_{\Omega_{\mathrm{e}}} [B^{\mathrm{e}}]^{\mathrm{T}}\, [C'^{e}]\, [B^{\mathrm{e}}]\; \mathrm{d}V \\
[f_1^{\mathrm{e}}] &= \int_{\Omega_{\mathrm{e}}} [\psi^{\mathrm{e}}]^{\mathrm{T}}\, \rho [f^{n+1}]\, \mathrm{d}V \\
[f_2^{\mathrm{e}}] &= \int_{\partial\Omega_{2h}^{\mathrm{e}}} [\psi^{\mathrm{e}}]^{\mathrm{T}}\, [T^{n+1}]\, \mathrm{d}S \\
[f_3^{\mathrm{e}}] &= \int_{\Omega_{\mathrm{e}}} [B^{\mathrm{e}}]^{\mathrm{T}} [\sigma^{n}]\, \mathrm{d}V \\
[f_4^{\mathrm{e}}] &= \int_{\Omega_{\mathrm{e}}} [B^{\mathrm{e}}]^{\mathrm{T}} [\Delta\sigma^{\mathrm{R}}]\, \mathrm{d}V \\
[f_5^{\mathrm{e}}] &= \int_{\Omega_{\mathrm{e}}} [B^{\mathrm{e}}]^{\mathrm{T}} [\beta'^{e}]\, \mathrm{d}V
\end{aligned}
\tag{51}
$$

In the above, $[k^{\mathrm{e}}]$ is referred to as the element stiffness matrix, $[f_1^{\mathrm{e}}]$, $[f_2^{\mathrm{e}}]$, $[f_3^{\mathrm{e}}]$, $[f_4^{\mathrm{e}}]$ and $[f_5^{\mathrm{e}}]$, are contributions to the element load vector due to body forces, surface tractions, stresses at the start of the time step, change of stresses during the time step, and thermal effects, respectively.

Summation of the contributions from all elements yields,

$$[K]\,\{\Delta u\} = \{F\} \tag{52}$$

where $[K]$ is the global stiffness matrix, $\{F\}$ is the global load vector, and $\{\Delta u\}$ is the change in the displacement vector during the time step. The global stiffness matrix and load vector are arrived at through appropriate assembly of element contributions. Equation (52) is a system of linear algebraic equations which can be solved by Gauss elimination.

5. CODE DESCRIPTION AND VERIFICATION

The authors have incorporated the numerical methods outlined in the two previous sections into the FE code ORTHO3D. This program, which was written by the authors, now provides considerable capability for the solution of thermoviscoelastic IBVPs. Developed with the analysis of polymeric composites in mind, the code represents a versatile tool for the analysis of orthotropic media. The preprocessor is written so as to enable the user to work with equal ease in either cylindrical or cartesian co-ordinates (cylindrical co-ordinates are convenient when dealing with filament wound composites). All results will be output in cylindrical co-ordinates if the user has selected this option. The program is capable of predicting the response to complex loading/thermal histories. Phenomena such as creep, relaxation, and creep-and-recovery can all be predicted using this program. The code also includes automated mesh generators which enable convenient grid generation for problems involving internal boundaries such as matrix cracks or delaminations. The element that has been employed in ORTHO3D is an eight-node isoparametric brick. A more detailed description of the code may be found in Zocher.[1]

For verification purposes, we shall now present the solution to three illustrative example problems (several more example problems may be found in Reference 1). Examples one and two involve isotropic bodies which are constructed from a hypothetical material system: material system A. The uniaxial relaxation modulus of material system A is given as:

$$E(t) = E_{\infty} + E_1\; \mathrm{e}^{-t/\rho_1} \tag{53}$$

Table I. Wiechert constants for material system A

MATL.	ij	k	C_{ij_k}	η_{ij_k}
A	11, 22, 33	∞	134, 615	—
		1	538, 462	538, 462
	44, 55, 66	∞	38, 462	—
		1	153, 846	153, 846
	12, 13, 23	∞	57, 692	—
		1	230, 769	230, 769

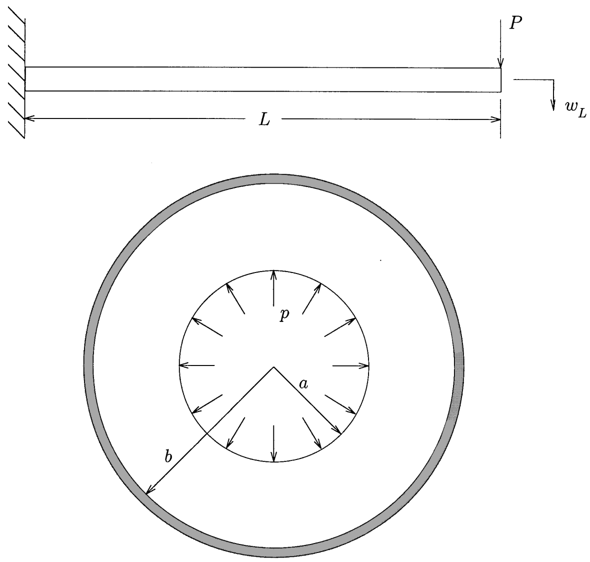

Figure 3. Beam and encased cylinder of Examples 1 and 2

The values of E_∞ and E_1 in the above are 0·1, and 0·4 MPa, respectively; the value of ρ_1 is 1·0. It is noted that the relaxation modulus of material system A is in the form of a standard linear solid (a one-element Wiechert model). Poisson's ratio of material system A is taken to be 0·3 (a constant). A full description of the corresponding Wiechert model for the C_{ijkl} (C_{ij} in Voigt notation) of (16) is provided in Table I. Example three involves a cylindrically anisotropic material system, properties of which will be given later.

5.1. *Example one* (*beam with tip load*)

Consider the cantilever beam shown in Figure 3. The beam has a length, L, of 20 and a cross-sectional area, A, of $1\,\mathrm{m}^2$ (an aspect ratio of 20 : 1). The beam is subjected to the tip load

$$P = P_0[H(t) - H(t - t_1)]$$

where $P_0 = 1\,\mathrm{N}$ and $t_1 = 10\,\mathrm{s}$. We seek the tip displacement w_L. This is similar to a creep-and-recovery test, but with spatially varying stress and strain.

It is easy to derive an analytical solution to the IBVP by applying the standard viscoelastic correspondence principle to the elastic solution for tip deflection from strength of materials. Doing so yields:

$$w_L = \frac{P_0 L^3}{3I}[D(t) - D(t - t_1)H(t - t_1)] \tag{54}$$

where I is the area moment of inertia of the beam (assumed to have a value of 1/12) and D is the creep compliance. The creep compliance is easily determined from the relaxation modulus (53) and is given by

$$D(t) = D_0 + D_1(1 - \mathrm{e}^{-t/\lambda_1}) \tag{55}$$

where

$$D_0 \equiv \frac{1}{E_0}, \quad E_0 \equiv E_\infty + E_1, \quad D_1 \equiv \left(\frac{1}{E_\infty} - \frac{1}{E_0}\right), \quad \lambda_1 \equiv \frac{E_0 \rho_1}{E_\infty}$$

The strength of materials solution is not exact but is considered a good approximation for a beam with an aspect ratio of 20 : 1. It follows that the analytical solution represented by (54) is not exact but is expected to be a good approximation.

Finite element results, produced by ORTHO3D, are compared to the analytical solution in Figure 4. The finite element prediction and the analytical solution are close (maximum difference 1·89 per cent). A Δt of 0·1 s was used in the finite element calculations. The mesh used in the analysis of example one is shown in Figure 5.

5.2. *Example two* (*encased cylinder*)

Consider a long thick-walled viscoelastic cylinder encased in a shell of infinite stiffness and subjected to internal pressure p (Figure 3). This geometry is representative of a solid propellant rocket motor. The viscoelastic cylinder represents the fuel and the stiff shell is representative of the rocket motor casing. Let the internal pressure, p, be given by $p = p_0 H(t)$ (similar to a creep test but involving spatial inhomogeneity).

Employing the elasticity solution along with the viscoelastic correspondence principle, it is easy to derive the following analytical solution for the radial displacement u_r:

$$u_r(r,t) = \frac{p_0 a^2 b(1+v)(1-2v)}{a^2 + (1-2v)b^2}\left(\frac{b}{r} - \frac{r}{b}\right) D(t) \tag{56}$$

For purposes of numerical calculations, let $a = 2\,\mathrm{m}$, $b = 4\,\mathrm{m}$, and $p_0 = 100\,\mathrm{Pa}$. Analytical and finite element results are presented in Figure 6 for the radial displacement of the mid-thickness datum. The finite element model used in the analysis employed a mesh consisting of 72 elements (Figure 5). Symmetry conditions were exploited so that the entire cylinder did not have to be modelled. A Δt of 0·1 s was used in the finite element calculations. It is seen that the analytical solution and the finite element prediction are in close agreement.

5.3. *Example three* (*orthotropic cylinder*)

In this example we investigate the response of a long thick-walled cylindrically anisotropic cylinder subjected to the internal pressurization $p = PH(t)$; a problem which has been previously investigated by Schapery.[47] In Schapery's analysis, the response of the cylinder was predicted by three different methods: (1) correspondence principle, (2) quasielastic method, and (3) collocation. In the present, we duplicate the first two analyses of Schapery (correspondence principle and

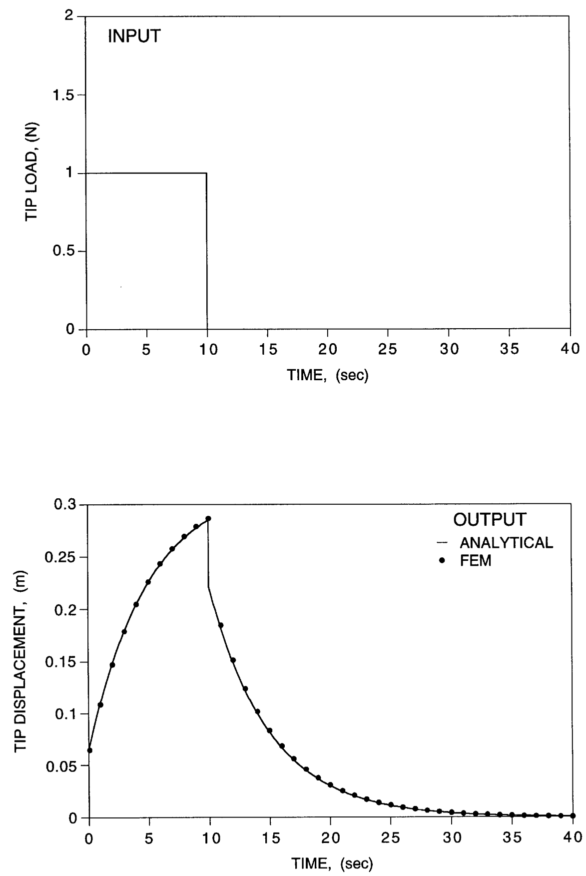

Figure 4. Loading history and response of Example 1

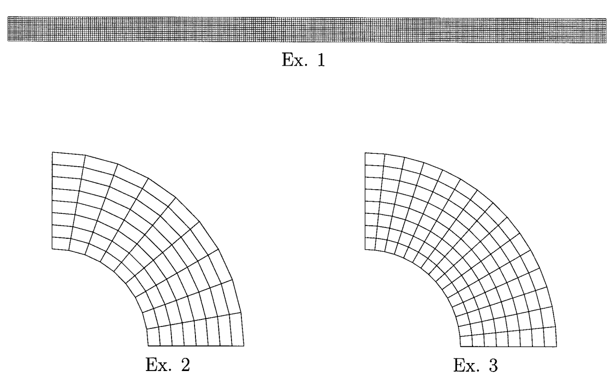

Figure 5. Finite element meshes used in Examples 1–3

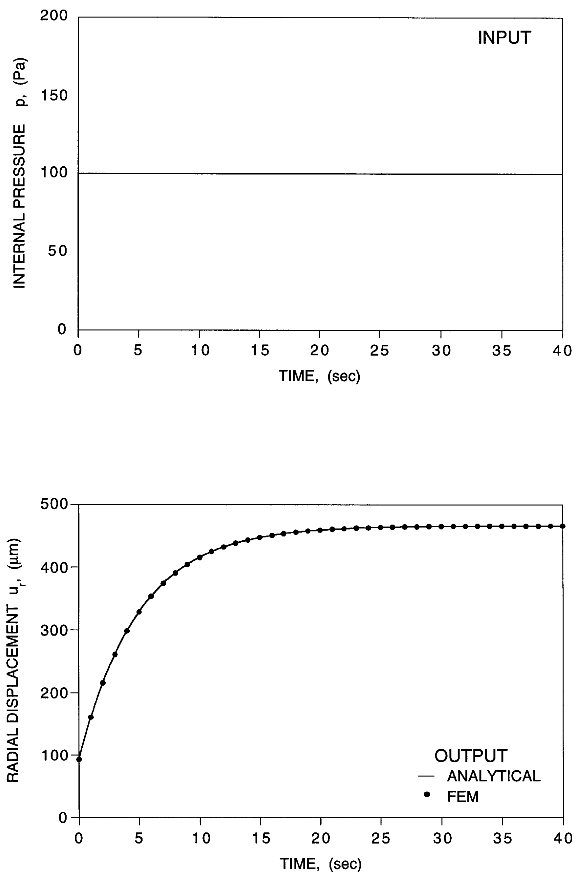

Figure 6. Loading history and response of Example 2

quasielastic method) and compare the results to finite element prediction. Following Schapery, we concern ourselves only with the prediction of the hoop stress, $\sigma_{\theta\theta}$ at the inner wall of the cylinder; at $r = a$ (the cylinder has inner radius a and outer radius b).

In his analysis, Schapery[47] assumed the relaxation moduli in the radial and circumferential (or hoop) directions to be given by the following:

$$E_r = E_{\mathrm{e}}\left[1 + 100\left(\frac{t}{t_0}\right)^{-0\cdot 5}\right]$$

$$E_\theta = E_{\mathrm{e}}\left[1 + 100\left(\frac{t}{t_0}\right)^{-0\cdot 1}\right] \tag{57}$$

Then using an elastic solution previously presented by Leknitskii,[48] Schapery showed the viscoelastic solution (as derived by correspondence principle) to be given by

$$\sigma_{\theta\theta}(a,t) = \tilde{\sigma}_{\theta\theta}(a,s)\big|_{s=1/2t} \tag{58}$$

Table II. Wiechert models of E_r and E_θ

	E_r		E_θ	
m	E_m	η_m	E_m	η_m
∞	1000	—	1200	—
1	0·96405 E07	0·19281 E04	0·57445 E05	0·11489 E02
2	0·29396 E07	0·58792 E04	0·44981 E05	0·89963 E02
3	0·93082 E06	0·18616 E05	0·35735 E05	0·71469 E03
4	0·29434 E06	0·58867 E05	0·28387 E05	0·56775 E04
5	0·93077 E05	0·18615 E06	0·22548 E05	0·45096 E05
6	0·29434 E05	0·58867 E06	0·17910 E05	0·35821 E06
7	0·93086 E04	0·18617 E07	0·14227 E05	0·28454 E07
8	0·29427 E04	0·58854 E07	0·11301 E05	0·22602 E08
9	0·93112 E03	0·18622 E08	0·89764 E04	0·17953 E09
10	0·29454 E03	0·58908 E08	0·71307 E04	0·14261 E10
11	0·91943 E02	0·18389 E09	0·56634 E04	0·11327 E11
12	0·30384 E02	0·60767 E09	0·44989 E04	0·89978 E11
13	0·85414 E01	0·17083 E10	0·35747 E04	0·71494 E12
14	0·48513 E01	0·97026 E10	0·28376 E04	0·56751 E13
15			0·22561 E04	0·45122 E14
16			0·17896 E04	0·35791 E15
17			0·14232 E04	0·28465 E16
18			0·11299 E04	0·22599 E17
19			0·89863 E03	0·17973 E18
20			0·71182 E03	0·14236 E19
21			0·56744 E03	0·11349 E20
22			0·44914 E03	0·89828 E20
23			0·35763 E03	0·71527 E21
24			0·28336 E03	0·56672 E22
25			0·22673 E03	0·45345 E23
26			0·17773 E03	0·35546 E24
27			0·14384 E03	0·28767 E25
28			0·11040 E03	0·22080 E26
29			0·95272 E02	0·19054 E27
30			0·59356 E02	0·11871 E28
31			0·83425 E02	0·16685 E29

where

$$\tilde{\sigma}_{\theta\theta}(a,s) = \frac{\tilde{k}}{1-\left(\frac{a}{b}\right)^{2\tilde{k}}}\left[1+\left(\frac{a}{b}\right)^{2\tilde{k}}\right]P$$

$$\tilde{k} = \sqrt{\frac{1+107(t_0)^{0\cdot 1}s^{0\cdot 1}}{1+177(t_0)^{0\cdot 5}s^{0\cdot 5}}}$$

The quasielastic solution to the IBVP (Schapery[47]) is given by

$$\sigma_{\theta\theta}(a,t) = \frac{k}{1-\left(\frac{a}{b}\right)^{2k}}\left[1+\left(\frac{a}{b}\right)^{2k}\right]P \tag{59}$$

where

$$k = \sqrt{\frac{1+100\left(\frac{t_0}{t}\right)^{0\cdot 1}}{1+100\left(\frac{t_0}{t}\right)^{0\cdot 5}}}$$

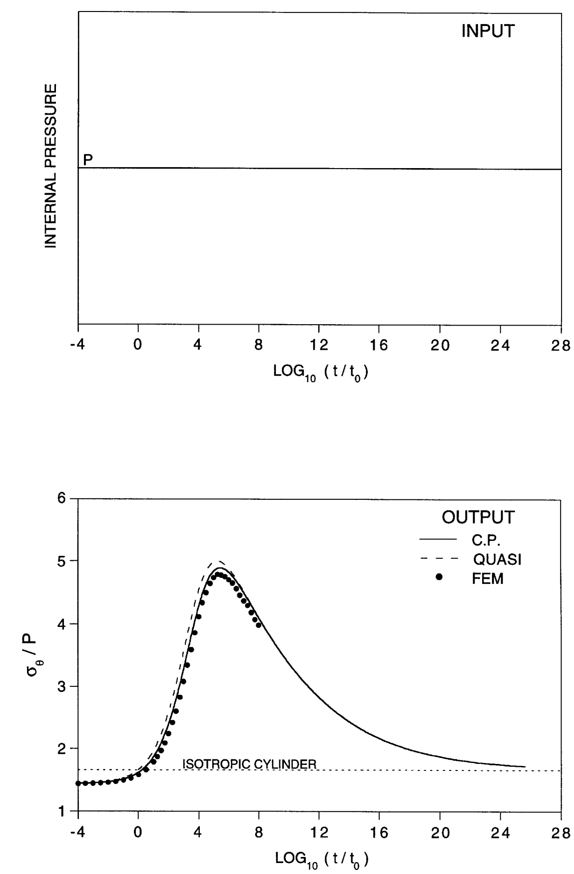

Figure 7. Loading history and response of Example 3

In order to solve this problem using ORTHO3D, we must first express E_r and E_θ in terms of Wiechert models. This is easily accomplished using collocation. The resultant Wiechert models for E_r and E_θ are given in Table II. The corresponding Wiechert models for the C_{ijkl} (C_{ij} in Voigt notation) of (16) are given as follows: $C_{11} = E_\mathrm{e} = 1000$, $C_{22} = E_\theta$, $C_{33} = E_r$, $C_{44} = C_{55} = C_{66} = 2000$, and $C_{12} = C_{13} = C_{23} = 0$. The value of 2000 for C_{44}, etc., is really an arbitrary number in this problem since its value has no influence on the results.

Finite element results are compared to the solution by correspondence principle (58) and by the quasielastic method (59) in Figure 7. Of the three methods compared in Figure 7, the finite element prediction is considered by the authors to be the more accurate. This claim is supported to some degree by Schapery's collocation results.[47] Figure 7 also shows that anisotropy can have a dramatic influence on the response of a viscoelastic body by including the response of an isotropic body for comparison. The finite element mesh used in the analysis consisted of 120 elements and is shown in Figure 5. The value of $\sigma_{\theta\theta}$ at the inner radius was determined through extrapolation of the value of $\sigma_{\theta\theta}$ at neighbouring integration points. Time steps of variable length were used in the analysis with the magnitude of the time step slowly increasing as time progressed. It is noted that this variability in time step size was not accomplished adaptively, but merely assigned in the input file.

5.4. A note on convergence

A formal convergence study pertaining to the sensitivity of time step size has not at this time been conducted. Consequently, such an endeavor represents an obvious arena for further study. While the authors are unable to give the reader formal guidance concerning temporal convergence issues, we can at least provide him with a qualitative sense of time step sensitivity by way of illustration with respect to examples one and two presented above. We will not address example three since this example involved a variable time step size and as such does not provide as clear an illustration. The result presented for example one was produced from calculations involving a Δt of 0·1 and as stated previously resulted in a maximum difference between finite element prediction and the analytical solution of 1·89 per cent. A Δt of 1·0 produces a maximum difference of 22·24 per cent and a Δt of 0·01 produces a maximum difference of 0·94 per cent. The result presented for example two was produced from calculations involving a Δt of 0·1, and had a maximum error of 1·96 per cent. A Δt of 1·0 produces a maximum error of 19·94 per cent and a Δt of 0·01 produces a maximum error of 0·21 per cent for this example.

6. CONCLUSIONS

A three dimensional FE formulation has been developed and incorporated into ORTHO3D. This development provides the analyst with a versatile tool with which he can easily predict the response of an orthotropic body (isotropic and transversely isotropic bodies are considered subsets) to a wide range of loading/temperature histories. The primary motivation behind the development was to enable accurate analysis of laminated polymeric composite structures subjected to a high-temperature environment. Such analysis is expected to be critical in developing predictions of component life.

The code developed here is easy to use and provides the capability of solving many interesting IBVPs involving viscoelastic media. It is, however, limited. Some of the most obvious limitations are as follows: (1) it has no dynamic capability, (2) it is linear, and (3) it is restricted to thermorheologically simple non-ageing materials. Having noted these restrictions, it should be recognized that the code is far more robust than many of its predecessors that have appeared in the literature. For example, some predict creep behaviour well but not relaxation, while the case is just the opposite for others. In addition, many programs (even some of the big commercial codes) restrict the user to very simple viscoelastic behaviour, such as standard linear solid or Maxwell. Moreover, very few viscoelastic codes have been developed to handle full orthotropic behaviour. In summary, ORTHO3D provides the user with the ability to handle a wider range of loading/temperature histories imposed on a wider range of materials than many of its predecessors.

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REFERENCES

1. M. A. Zocher, 'A thermoviscoelastic finite element formulation for the analysis of composites', *Ph.D. Dissertation*, Texas A&M University, College Station, TX, 1995.
2. E. H. Lee and T. G. Rogers, 'Solution of viscoelastic stress analysis problems using measured creep or relaxation functions', *J. Appl. Mech.*, **30,** 127–133 (1963).
3. I. L. Hopkins and R. W. Hamming, 'On creep and relaxation', *J. Appl. Phys.*, **28,** 906–909 (1957).
4. A. R. Zak, 'Structural analysis of realistic solid-propellant materials', *J. Spacecr. Rockets*, **5,** 270–275 (1968).

5. I. P. King, 'On the finite element analysis of two-dimensional problems with time dependent properties', *Ph.D. Dissertation*, University of California, Berkeley, CA, 1965.
6. R. L. Taylor and T. Y. Chang, 'An approximate method for thermoviscoelastic stress analysis', *Nucl. Eng. Des.*, **4,** 21–28 (1966).
7. T. Y. Chang, 'Approximate solutions in linear viscoelasticity', *Ph.D. Dissertation*, University of California, Berkeley, CA, 1966.
8. R. L. Taylor, K. S. Pister and G. L. Goudreau, 'Thermomechanical analysis of viscoelastic solids', *Int. j. numer. methods eng.*, **2,** 45–59 (1970).
9. O. C. Zienkiewicz and M. Watson, 'Some creep effects in stress analysis with particular reference to concrete pressure vessels', *Nucl. Eng. Des.*, **4,** 406–412 (1966).
10. Y. R. Rashid and W. Rockenhauser, 'Pressure vessel analysis by finite element techniques', *Proc of Conf. on Prestressed Concrete Pressure Vessels*, London, 1967, pp. 375–383.
11. J. L. White, 'Finite elements in linear viscoelasticity', *Proc. 2nd Conf. on Matrix Methods in Struct Mech*, AFFDL-TR-68-150 Wright Patterson AFB, Dayton, OH, 1968, pp. 489–516.
12. O. C. Zienkiewicz, M. Watson and I. P. King, 'A numerical method of visco-elastic stress analysis', *Int. J. Mech. Sci.*, **10,** 807–827 (1968).
13. G. A. Greenbaum and M. F. Rubinstein, 'Creep analysis of axisymmetric bodies using finite elements', *Nucl. Eng. Des.*, **7,** 379–397 (1968).
14. J. P. H. Webber, 'Stress analysis in viscoelastic bodies using finite elements and a correspondence rule with elasticity', *J. Strain Anal.*, **4,** 236–243 (1969).
15. F. de S. Lynch, 'A finite element method of viscoelastic stress analysis with application to rolling contact problems', *Int. j. numer. methods eng.*, **1,** 379–394 (1969).
16. R. C. Batra, M. Levinson and E. Betz, 'Rubber covered rolls—the thermoviscoelastic problem. A finite element solution', *Int. j. numer. methods eng.*, **10,** 767–785 (1976).
17. R. C. Batra, 'Cold sheet rolling, the thermoviscoelastic problem, a numerical solution', *Int. j. numer. methods eng.*, **11,** 671–682 (1977).
18. N. Purushothaman, I. D. Moore and B. S. Heaton, 'Finite element analysis of viscoelastic solids responding to periodic disturbances', *Int. j. numer. methods eng.*, **26,** 1471–1483 (1988).
19. N. A. Cyr and R. D. Teter, 'Finite element elastic-plastic-creep analysis of two-dimensional continuum with temperature dependent material properties', *Comput. Struct.*, **3,** 849–863 (1973).
20. H. O. Kim and R. L. Kuhlemeyer, 'A finite element formulation for creep analysis', *Int. j. numer. methods eng.*, **11,** 1865–1877 (1977).
21. P. Krishnaswamy, M. E. Tuttle, A. F. Emery and J. Ahmad, 'Finite element modelling of crack tip behavior in viscoelastic materials. Part I: linear behavior', *Int. j. numer. methods eng.*, **30,** 371–387 (1990).
22. P. Krishnaswamy, M. E. Tuttle, A. F. Emery and J. Ahmad, 'Finite element modeling of the time-dependent behavior of non-linear ductile polymers', *Polym. Eng. Sci.*, **32,** 1086–1096 (1992).
23. Z. P. Bažant, 'Matrix differential equation and higher-order numerical methods for problems of non-linear creep', *Int. j. numer. methods eng.*, **4,** 11–15 (1972).
24. W. C. Carpenter, 'Viscoelastic stress analysis', *Int. j. numer. methods eng.*, **4,** 357–366 (1972).
25. Y. R. Nagaraja and R. S. Alwar, 'Viscoelastic analysis of an adhesive-bonded plane lap joint', *Comput. Struct.*, **11,** 621–627 (1980).
26. S. Yadagiri, C. P. Reddy and T. S. Reddy 'Viscoelastic analysis of adhesively bonded joints', *Comput. Struct.*, **27,** 445–454 (1987).
27. S. Roy and J. N. Reddy, 'Finite-element models of viscoelasticity and diffusion in adhesively bonded joints', *Int. j. numer. methods eng.*, **26,** 2531–2546 (1988).
28. S. Roy and J. N. Reddy, 'A finite element analysis of adhesively bonded composite joints with moisture diffusion and delayed failure', *Comput. Struct.*, **29,** 1011–1031 (1988).
29. B. Moran and W. G. Knauss, 'Crack-tip stress and deformation fields in strain-softening nonlinearly viscoelastic materials', *J. Appl. Mech*., **59,** 95–101 (1992).
30. M. K. Warby, J. R. Walton and J. R. Whiteman, 'A finite element model of crack growth in a finite body in the context of mode I linear viscoelastic fracture', *Comput. Methods Appl. Mech. Eng.*, **97,** 375–397 (1992).
31. L. C. Brinson and W. G. Knauss, 'Finite element analysis of multiphase viscoelastic solids', *J. Appl. Mech.*, **59,** 730–737 (1992).
32. J. T. Oden, 'Finite element approximations in non-linear thermoviscoelasticity', *Lectures on Finite Element Methods in Continuum Mechanics* (*NATO Advanced Study Institute on Finite Element Methods in Continuum Mechanics*), Lisbon, 1971, pp, 77–119.
33. J. T. Oden and W. H. Armstrong, 'Analysis of non-linear dynamic coupled thermoviscoelastivity problems by the finite element method', *Comput. Struct.*, **1,** 603–621 (1971).
34. M. Henriksen, 'Nonlinear viscoelastic stress analysis—a finite element approach', *Comput. Struct.*, **18,** 133–139 (1984).
35. H. R. Srinatha and R. W. Lewis, 'A finite element method for thermoviscoelastic analysis of plane problems', *Comput. Methods Appl. Mech. Eng.*, **25,** 21–23 (1981).
36. H. R. Srinatha and R. W. Lewis, 'A finite element formulation of uncoupled thermoviscoelastic response of plane problems for all admissible values of Poisson's ratio', *Int. j. numer. methods eng.*, **18,** 765–774 (1982).

37. K. Y. Lin and I. H. Hwang, 'Thermo-viscoelastic analysis of composite materials', *J. Compos. Mater.*, **23,** 554–569 (1989).
38. K. Y. Lin and I. H. Hwang, 'Thermo-viscoelastic response of graphite/epoxy composites', *J. Eng. Mater. Tech.*, **110,** 113–116 (1988).
39. R. M. Jones, *Mechanics of Composite Materials*, McGraw-Hill, New York, 1975.
40. K. Y. Lin and S. Yi, 'Analysis of interlaminar stresses in viscoelastic composites', *Int. J. Solids Struct.*, **27,** 929–945 (1991).
41. H. H. Hilton and S. Yi, 'Anisotropic viscoelastic finite element analysis of mechanically and hygrothermally loaded composites', *Compos. Eng.*, **3,** 123–135 (1993).
42. S. Yi, 'Thermoviscoelastic analysis of delamination onset and free edge response in laminated composites', *AIAA J.*, **31,** 2320–2328 (1993).
43. T. C. Kennedy and M. Wang, 'Three-dimensional, non-linear viscoelastic analysis of laminated composites', *J. Compos. Mater.*, **28,** 902–925 (1994).
44. Y. C. Lou and R. A. Schapery, 'Viscoelastic characterization of a non-linear fiber reinforced plastic', *J. Compos. Mater.*, **5,** 208–234 (1971).
45. M. A. Zocher, D. H. Allen and S. E. Groves, 'Stress analysis of a matrix-cracked viscoelastic laminate', *Int. J. Solids Struct.*, to appear.
46. G. Ghazlan, S. Caperaa and C. Petit, 'An incremental formulation for the linear analysis of thin viscoelastic structures using generalized variables', *Int. j. numer. methods eng.*, **38,** 3315–3333 (1995).
47. R. A. Schapery, 'Stress analysis of viscoelastic composite materials', *J. Compos. Mater.*, **1,** 228–267 (1967).
48. S. G. Leknitskii, *Theory of Elasticity of an Anisotropic Body*, Mir, Moscow, 1981, pp. 39–43.