A Time-dependent Stopping Problem with Application to Live Organ Transplants

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We consider a time-dependent stopping problem and its application to the decision-making process associated with transplanting a live organ. "Offers" (e.g., kidneys for transplant) become available from time to time. The values of the offers constitute a sequence of independent identically distributed positive random variables. When an offer arrives, a decision is made whether to accept it. If it is accepted, the process terminates. Otherwise, the offer is lost and the process continues until the next arrival, or until a moment when the process terminates by itself. Self-termination depends on an underlying lifetime distribution (which in the application corresponds to that of the candidate for a transplant). When the underlying process has an increasing failure rate, and the arrivals form a renewal process, we show that the control-limit type policy that maximizes the expected reward is a nonincreasing function of time. For non-homogeneous Poisson arrivals, we derive a first-order differential equation for the control-limit function. This equation is explicitly solved for the case of discrete-valued offers, homogeneous Poisson arrivals, and Gamma distributed lifetime. We use the solution to analyze a detailed numerical example based on actual kidney transplant data.

THIS WORK was motivated by a decision-making problem associated with transplanting a live organ—in this case, a kidney. The decision whether to transplant the organ depends on the degree of histocompatability between the "donor" and the recipient. One relevant criterion for compatability is the match level in the so-called HL-A antigeh system. Definitions of the various match-levels—$\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$ or $\underline{E}$—are presented by Barnes and Miettinen [1972], who derive formulas for calculating the probability of any match-level between a given recipient and a random graft. With each match-level, we associate a value, such as the probability of graft survival for at least 1 year, or the expected lifetime of the graft (see, for example, data presented in Brunner [1975] and in Dausset et al. [1974]). Another decision-making consideration is the time that the potential recipient has been under medical care. For example, in the kidney case, medical care would be home or hospital dialysis. The problem

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is to provide the physician with a quantitative tool specifying, for each combination of HL-A match and the recipient's time on dialysis, whether to perform a transplant or to reject (and lose) the graft and wait for a possibly better combination in the future. In this paper, we provide such a tool and give some quantitative results.

The problem belongs to the family of optimal stopping problems. Books by Chow et al. [1971], De Groot [1970], and others gave expositions of the theory of optimal stopping, together with extensive bibliographies. Time-dependent aspects of the problem with emphasis on obtaining explicit solutions have been studied by Elfving [1967], who considered a decreasing discount function and Poisson-type arrival of offers, and also by Mucci [1978], who extended the solution of the exponential discount case to a wide class of arrival processes. Albright [1974] generalized Elfving's result to an n-person assignment problem.

In Section 1 we formulate a general setting of an optimal-stopping problem whose underlying process has a failure-rate that depends on time. To illustrate some of the ideas developed in later sections, we first study situations in which offers arive at fixed instants, and show that the optimal policy is of a time-dependent control-limit type.

In Section 2 we assume that the arrival of offers is a renewal process. We show that if the lifetime distribution of the underlying process has an increasing failure rate, then the optimal control limit is a continuous nonincreasing function of time. We further show that the increasing failure rate property is necessary to ensure such monotonicity.

In Section 3 we study nonhomogeneous Poisson arrivals. We rederive Elfving's differential equation for the optimal control-limit function, using a different approach, which leads to a more tractable equation particularly suited to our model. With the aid of this equation, we develop properties of some special cases. In the case of homogeneous Poisson arrival, we derive in Section 4 an explicit solution of the equation, considering discrete-valued offers and Gamma distributed lifetime with shape parameter $a = 2$. The method we use is applicable to problems with offers admitting finitely many values and any increasing failure rate lifetime distribution.

In Section 5, we apply this explicit solution to determine the optimal policy for the kidney transplant problem, and present detailed calculations based on actual data.

1. THE MODEL

Consider a stopping problem with actions taken at (random or fixed) times $0 = t_0 < t_1 < t_2 < \cdots$. At instant t_j an offer X_j is available. We assume that $\{X_j\}_0^\infty$ is a sequence of independent-identically distributed, positive, bounded random variables having a distribution function $F(x) = P(X \leq x)$. An action at time t_j is a decision whether to accept or reject the offer. If the offer is accepted, the process is stopped and a reward $\beta(t_j)X_j$ is gained, where $\beta(t) \geq 0$ is a continuous nonincreasing discount function with $\beta(0) = 1$. If the offer is not accepted, it is lost and the process continues until the next offer, or until it terminates by itself ("dies")—whichever occurs first. The probability that the process terminates by itself before the new offer arrives at time t_{j+1} is given by the variable $1 - \alpha_{j+1} = P(T \leq t_{j+1} \mid T > t_j)$ defined by T, the lifetime of the underlying process. If the process terminates by itself, no reward is gained.

The objective is to characterize and find a stopping rule that maximizes the expected discounted reward from any time t onward.

The Case of Fixed Arrival Instants

Suppose that the sequence $t_0, t_1, t_2, \cdots, t_j, \cdots$ is a set of fixed numbers, and that the process is allowed to continue for at most n; if it has not been stopped or self-terminated by time t_n, the last offer, X_n, must be accepted. Suppose that at time t_j the process is still alive and an offer $X_j = x$ is available. Let $V_n^j(x)$ denote the maximal expected discounted reward attained from that situation, and write $\beta(t_j) = \beta_j$. We have

$$V_n^n(x) = \beta_n x$$

$$V_n^j(x) = \max[\beta_j x, \alpha_{j+1} \int_{y=0}^{\infty} V_n^{j+1}(y)\, dF(y)], \quad 0 \leq j < n.$$

Define $\lambda_n^n = 0$,

$$\lambda_n^j = \alpha_{j+1}[\lambda_n^{j+1} F(\lambda_n^{j+1}/\beta_{j+1}) + \int_{\lambda_n^{j+1}/\beta_{j+1}}^{\infty} \beta_{j+1} x\, dF(x)], \quad 0 \leq j \leq n. \quad (1)$$

λ_n^j is the maximum expected discounted reward if an offer at time t_j is rejected. Other authors (e.g., Ross [1970, p. 152]) have already commented that the probability of self-termination functions as (1 minus) a discount factor.

It is clear that the optimal strategy is a control-limit type policy with a set of controls $\{\lambda_n^j\}_{j=0}^{n}$ and that an offer X_j at time t_j is accepted if and only if $\beta_j X_j > \lambda_n^j$. It is also easy to show that, for each $j \leq n$, the set $\{\lambda_n^j\}_{n=0}^{\infty}$ is a nondecreasing bounded sequence of n and hence has a limit $l_j = \lim_{n\to\infty} \lambda_n^j$.

Letting $\gamma_j = l_j/\beta_j$, we see from Equation 1 that

$$l_j = \alpha_{j+1}\beta_{j+1}[\gamma_{j+1}F(\gamma_{j+1}) + \int_{\gamma_{j+1}}^{\infty} x\, dF(x)]. \tag{2}$$

Consequently, for an infinite-horizon problem, where $\overline{\lim}_{j\to\infty} \alpha_j < 1$, an offer at time t_j is accepted if and only if $X_j > \gamma_j$.

The sequence $\{l_j\}_0^\infty$ is determined by the sequences $\{\alpha_j\}_1^\infty$ and $\{\beta_j\}_0^\infty$, but is not readily calculated because l_j is forwardly defined in terms of l_{j+1}. David and Yechiali [1983] present an example for calculating the sequence $\{l_j\}$ when X is uniformly distributed on (0, 1). When the sequence $\{\alpha_j\}_1^\infty$ is nonincreasing and $\beta_j = 1$ for all j, one can show that the sequence of controls $\{l_j\}_0^\infty$ is also nonincreasing.

Now, if we let $\beta_j = 1$ and $\alpha_j = \alpha$ for all j, we can see intuitively that there is a unique control limit $l_j = l$ for all j. Indeed, in such a case, Equation 1 takes the form

$$\lambda_n^j = \alpha[\lambda_n^{j+1}F(\lambda_n^{j+1}) + \int_{\lambda_n^{j+1}}^{\infty} x\, dF(x)] \equiv f(\lambda_n^{j+1}).$$

Since $\lambda_n^n = 0$, one recursively obtains $\lambda_n^j = f^{(n-j)}(\lambda_n^n) = f^{(n-j)}(0)$, where $f^{(0)}(\lambda) = \lambda$ and $f^{(k)}(\lambda) = f(f^{(k-1)}(\lambda))$. Similarly, $\lambda_{n-j}^0 = f^{(n-j)}(\lambda_{n-j}^{n-j}) = f^{(n-j)}(0)$.

Thus, $\lambda_n^j = \lambda_{n-j}^0$ for all n and $j \le n$. Taking limits as $n \to \infty$, we have, for every j, $l_j = \lim_{n\to\infty}\lambda_n^j = \lim_{n\to\infty}\lambda_{n-j}^0 = l_0 \equiv l$. This limit can be calculated via Equation 2. For example, if X is uniformly distributed on (0, 1), Equation 2 reduces to $l = \alpha(1 + l^2)/2$ with a solution

$$l = \alpha/(1 + \sqrt{1 - \alpha^2}). \tag{3}$$

We turn now to the random-arrival case and derive analogous results.

2. DETERIORATING LIFETIME AND RENEWAL-TYPE ARRIVAL OF OFFERS

Let $G(t) = P(T \le t)$ be the probability distribution function of T, and let the interarrival times of offers constitute a renewal process with underlying distribution function $H(s) = P(t_{j+1} - t_j \le s)$ for $j \ge 0$. Suppose that the process has not been stopped or self-terminated by time t when an offer $X = x$ arrives. Let $V(t, x)$ be the optimal expected discounted reward from that instant on. Then

$$V(t, x) = \max\{\beta(t)x, \lambda(t)\} \tag{4}$$

where

$$\lambda(t) = \int_{s=0}^{\infty} \bar{G}(s \mid t)\left[\int_{y=0}^{\infty} V(t+s, y)\, dF(y)\right] dH(s). \qquad (5)$$

Here $\bar{G}(s \mid t) \equiv P(T > t + s \mid T > t)$ is the probability of survival beyond $t + s$, given that the process survives beyond t. Note that $\lambda(t)$ serves as a "control limit" at time t, and is equal to the future expected discounted reward if the offer is arbitrarily rejected at time t and an optimal strategy (if one exists) is applied thereafter.

Let

$$V(t) = \int_{y=0}^{\infty} V(t, y)\, dF(y). \qquad (6)$$

If an optimal strategy exists, $V(t)$ may be interpreted as its a priori expected discounted gain from time t on.

We now characterize the structure of the optimal policy for a special family of distributions $G(\cdot)$.

DEFINITION. *A lifetime distribution function G is called IFR (Increasing Failure Rate) if and only if $\bar{G}(s \mid t)$ is nonincreasing as a function of t for any $s \geq 0$.*

When G possesses a density $g(t)$, it is convenient to deal with the failure rate $r(t) = g(t)/[1 - G(t)]$. In these situations, this definition is equivalent to $r(t)$ being nondecreasing on $(0, \infty)$ (see Barlow and Proschan [1975]). To exclude trivialities, assume that $\int_0^\infty [1 - G(s)]\, dH(s) < 1$.

We are now in a position to derive the main result in this section.

THEOREM 1. *If G is IFR, then there exists an optimal policy characterized by a continuous nonincreasing real function $\lambda(t)$ on $[0, \infty)$, such that an offer x at time t is accepted if and only if $\beta(t)x \geq \lambda(t)$.*

Proof. Let BC denote the space consisting of all bounded continuous real-valued functions $u(t, x)$, define on $[0, \infty) \times [0, M]$, where M is the bound of the random variable X. Using the metric

$$\| u - v \| = \sup_{\substack{t \in [0,\infty), \\ x \in [0,M]}} | u(t, x) - v(t, x) |,$$

it is possible to use classical methods to show that BC is a complete metric space. Define an operator $T{:}BC \rightarrow BC$ by

$$Tu(t, x) = \max\left\{\beta(t)x, \int_0^{\infty} \bar{G}(s \mid t)\left[\int_0^{\infty} u(t+s, y)\, dF(y)\right] dH(s)\right\}. \qquad (7)$$

Now let $u, v \in BC$ be arbitrary, and fix $t \geq 0$ and $0 \leq x \leq M$. Then,

$$\begin{aligned}
&| Tu(t, x) - Tv(t, x) | \\
&= \left| \max\left\{\beta(t)x, \int_0^\infty \bar{G}(s \mid t)\left[\int_0^\infty u(t+s, y)\, dF(y)\right] dH(s)\right\} \right. \\
&\qquad \left. - \max\left\{\beta(t)x, \int_0^\infty \bar{G}(s \mid t)\left[\int_0^\infty v(t+s, y)\, dF(y)\right] dH(s)\right\} \right| \\
&\leq \left| \int_0^\infty \bar{G}(s \mid t)\left[\int_0^\infty u(t+s, y)\, dF(y)\right] dH(s) \right. \\
&\qquad \left. - \int_0^\infty \bar{G}(s \mid t)\left[\int_0^\infty v(t+s, y)\, dF(y)\right] dH(s) \right| \\
&= \left| \int_0^\infty \bar{G}(s \mid t)\left[\int_0^\infty (u(t+s, y) - v(t+s, y))\, dF(y)\right] dH(s) \right| \\
&\leq \sup_{\substack{t \in [0,\infty), \\ x \in [0,M]}} | u(t, x) - v(t, x) | \cdot \int_0^\infty \bar{G}(s \mid t)\, dH(s). \qquad (8)
\end{aligned}$$

Since G is IFR, $\int_0^\infty \bar{G}(s \mid t)\, dH(s) \leq \int_0^\infty \bar{G}(s \mid 0)\, dH(s) = \int_0^\infty [1 - G(s)]\, dH(s)$. By our assumption $\int_0^\infty [1 - G(s)]\, dH(s) \leq \alpha$ for some $\alpha < 1$. If we replace $\int_0^\infty \bar{G}(s \mid t)\, dH(s)$ by α, the right-hand side of (8) becomes independent of t and x. Hence, $\| Tu - Tv \| \leq \alpha \| u - v \|$, $\alpha < 1$, and T is a contraction. We deduce that there is a $v^* \in BC$ satisfying $Tv^* = v^*$. Furthermore, $T^n u_0 \to v^*$ for any $u_0 \in BC$. We claim that v^* is nonincreasing in t and x. It is sufficient to start with u_0 possessing that property. Since $\beta(t)x$ is nonincreasing in t for any x, and since G is IFR, iterating (7) shows that, for any integer $n \geq 0$, $T^n u_0$ is nonincreasing in t for any x. Hence, the limit v^* is nonincreasing as well. Define

$$\lambda^*(t) = \int_{s=0}^\infty \bar{G}(s \mid t)\left[\int_{y=0}^\infty v^*(t + s, y)\, dF(y)\right] dH(s).$$

This nonincreasing property of v^* and the IFR behavior of G imply that $\lambda^*(t)$ is nonincreasing in t. Since $Tv^* = v^*$, λ^* and v^* satisfy (4) and (5). Thus v^* is the optimal reward and the result follows. (See also Mine and Osaki [1970] on the role of the principle of contraction mappings in dynamic programming models.)

To strengthen the characterization, we prove the following result:

LEMMA 1. *In Theorem* 1 *the IFR assumption is necessary.*

Proof. For any non-IFR $G(\cdot)$, we can construct an example to show that the control limit is not a monotone decreasing function. To simplify the presentation, we assume here that G is differentiable. Thus, if G is not IFR, we deduce from the definition that for some $t < s$, there exists an interval $(0, \epsilon)$ such that, for all $x \in (0, \epsilon)$, $\overline{G}(x \mid s) > \overline{G}(x \mid t)$. Fix an integer $m = \min\{k \mid (s - t)/k < \epsilon, k \geq 1\}$, and let $h = (s - t)/m$. Let $G_s = \overline{G}(h \mid s)$ and $G_t = \overline{G}(h \mid t)$, i.e., $G_s > G_t$. Now, choose a real number $a > 1$ such that $1 < 1/G_s < a < 1/G_t$. Define $p^* = (1 - G_s)/(G_s(a - 1)) < 1$, and let p_0 satisfy $p^* < p_0 < 1$.

We consider a degenerate renewal process with constant interarrival time h, and assume that any offer X_n may take on two values: $x_1 > 0$ with probability $p_1 = 1 - p_0$, and $x_0 = ax_1$ with probability p_0. Assume further that $\beta(t) = 1$. Naturally, $V(\tau, x) \leq x_0$ for any τ and x. Applying (5) in this case, we have

$$\begin{aligned}\lambda(t) &= G_t[V(t + h, x_0)p_0 + V(t + h, x_1)p_1] \leq G_t[x_0p_0 + x_0p_1] \\ &= G_tx_0 < x_0/a = x_1.\end{aligned}$$

Using (4) and $\beta(t) = 1$ gives

$$\begin{aligned}\lambda(s) &= G_s[V(s + h, x_0)p_0 + V(s + h, x_1)p_1] \geq G_s[x_0p_0 + x_1p_1] \\ &= G_sx_1(ap_0 + (1 - p_0)) > x_1G_s(1 + p^*(a - 1)) = x_1.\end{aligned}$$

That is, for $t < s$, $\lambda(t) < x_1 < \lambda(s)$, and hence $\lambda(\cdot)$ is not a monotone decreasing function. This result completes the proof.

Note that in the above example an offer x_1 is accepted at time t while it is rejected at time $s > t$. This example also shows that a milder assumption on the deterioration of G (e.g., IFRA) is not sufficient to ensure the monotonicity of $\lambda(t)$.

3. NONHOMOGENEOUS POISSON ARRIVAL PROCESS

Suppose that the arrival process is a point process on $t \in [0, \infty)$ with positive and continuous intensity function $\mu(t)$. That is, $P\{$no arrivals in $(t, t + s)\} = \exp[-\int_t^{t+s} \mu(\tau)\, d(\tau)]$ for all $s \geq 0$. Assume further that $G(t)$ has a density $g(t)$, and let $r(t) = g(t)/[1 - G(t)]$ be the failure rate function. Also, let $V(t, x)$, $\lambda(t)$ and $V(t)$ be defined as in Equations 4, 5, and 6. We claim

LEMMA 2. *$\lambda(t)$ is continuous.*

Proof. For $\epsilon > 0$, we have

$$\begin{aligned}\lambda(t) &= e^{-\int_t^{t+\epsilon}\mu(\tau)d\tau}\overline{G}(\epsilon \mid t)\lambda(t + \epsilon) \\ &\quad + \int_{s=0}^{\epsilon} \overline{G}(s \mid t)V(t + s)e^{-\int_t^{t+s}\mu(\tau)d\tau}\mu(t + s)\, ds. \qquad (9)\end{aligned}$$

Equation 9 uses the properties of the Poisson process. The first term follows by conditioning on the event of no arrivals during $(t, t+\epsilon)$, while the second term follows by conditioning on the event that first arrival after t occurs at time $t+s$, for any $0 < s \leq \epsilon$. Taking limits of (9) as $\epsilon \downarrow 0$, and observing that $\bar{G}(\epsilon \mid t) \to 1$ as $\epsilon \downarrow 0$, right-continuity of $\lambda(t)$ follows. Left-continuity is shown similarly.

THEOREM 2. *$\lambda(t)$ satisfies the differential equation*

$$\lambda'(t) = r(t)\lambda(t) - \beta(t)\mu(t) \int_{\lambda(t)/\beta(t)}^{\infty} \bar{F}(x)\, dx \tag{10}$$

where $\bar{F}(x) = 1 - F(x)$.

Proof. For $\epsilon > 0$, we use (9) to write

$$\begin{aligned}\lambda(t+\epsilon) - \lambda(t) &= \lambda(t+\epsilon)[1 - e^{-\int_t^{t+\epsilon}\mu(\tau)d\tau}\bar{G}(\epsilon \mid t)] \\ &\quad - \int_{s=0}^{\epsilon} \bar{G}(s \mid t) V(t+s) e^{-\int_t^{t+s}\mu(\tau)d\tau}\mu(t+s)\, ds.\end{aligned} \tag{11}$$

Applying L'Hôpital's rule, we find

$$\begin{aligned}&\lim_{\epsilon\to 0}(1/\epsilon)\left\{1 - \exp\left[-\int_t^{t+\epsilon}\mu(\tau)\, d\tau\right]\bar{G}(\epsilon \mid t)\right\} \\ &\quad = \lim_{\epsilon\to 0}\left\{\exp\left[-\int_t^{t+\epsilon}\mu(\tau)\, d\tau\right]\mu(t+\epsilon)\bar{G}(\epsilon \mid t)\right. \\ &\quad \left. + g(t+\epsilon)/(1-G(t)) \exp\left[-\int_t^{t+\epsilon}\mu(\tau)\, d\tau\right]\right\} = \mu(t) + r(t).\end{aligned} \tag{12}$$

By the mean value theorem, we can write

$$\begin{aligned}&\lim_{\epsilon\to 0}(1/\epsilon)\left\{\int_{s=0}^{\epsilon} \bar{G}(s \mid t) V(t+s) \exp\left[-\int_t^{t+s}\mu(\tau)\, d\tau\right]\mu(t+s)\, ds\right\} \\ &= \lim_{s^*\to 0}\left\{\bar{G}(s^* \mid t) V(t+s^*) \exp\left[-\int_t^{t+s^*}\mu(\tau)\, d\tau\right]\mu(t+s^*)\right\} = V(t)\mu(t).\end{aligned} \tag{13}$$

Thus, dividing Equation 11 by ϵ, taking limits as $\epsilon \downarrow 0$, and using Equations 12 and 13, together with the continuity of $\lambda(t)$, we obtain

$$\begin{aligned}\lambda'(t) &= \lambda(t)[\mu(t) + r(t)] - V(t)\mu(t) \\ &= r(t)\lambda(t) - \mu(t)[V(t) - \lambda(t)].\end{aligned} \tag{14}$$

But $V(t) = \int_{x=0}^{\infty} V(t, x)\, dF(x) = \lambda(t)P[\beta(t)X \leq \lambda(t)] + \int_{\lambda(t)/\beta(t)}^{\infty} \beta(t)x\, dF(x)$, hence

$$V(t) - \lambda(t) = \beta(t)\left[-(\lambda(t)/\beta(t))\bar{F}(\lambda(t)/\beta(t)) + \int_{\lambda(t)/\beta(t)}^{\infty} x\, dF(x)\right]$$
$$= \beta(t) \int_{\lambda(t)/\beta(t)}^{\infty} \bar{F}(x)\, dx. \tag{15}$$

Combining (14) and (15) yields (10), which completes the proof.

Now let $\gamma(t) = \lambda(t)/\beta(t)$ and $\bar{A}(t) = \beta(t)\bar{G}(t)$, where $\bar{G}(t) = 1 - G(t)$. Then, $[\gamma(t)\bar{A}(t)]' = \lambda'(t)\bar{G}(t) - \lambda(t)g(t)$. Using (10), we derive

$$[\gamma(t)\bar{A}(t)]' = -\mu(t)\bar{A}(t) \int_{\gamma(t)}^{\infty} \bar{F}(x)\, dx. \tag{16}$$

Equation 16 could be compared with Equation 3.1 of Elfving, with the interpretation that $\bar{A}(t) = \beta(t)\bar{G}(t)$ is a "compound" discount function. Note that, in Theorem 2, the role of the lifetime distribution is explicitly exhibited by means of $r(t)$.

Special Cases and Examples

(i) *Constant failure rate, exponential discount and homogeneous Poisson arrival*

Suppose that the lifetime is exponentially distributed with parameter r (i.e., $r(t) \equiv r$), and assume that the arrival of offers constitutes a homogeneous Poisson process with intensity $\mu(t) \equiv \mu$. Assume further that the discount function is $\beta(t) = e^{-\beta t}$.

For $\gamma(t) = \lambda(t)/\beta(t)$, Equation 10 becomes

$$\gamma'(t) = (\beta + r)\gamma(t) - \mu \int_{\gamma(t)}^{\infty} \bar{F}(x)\, dx. \tag{17}$$

The unique bounded solution of (17) is $\gamma(t) \equiv \gamma_0$, where γ_0 satisfies

$$(\beta + r)\gamma_0 = \mu \int_{\gamma_0}^{\infty} \bar{F}(x)\, dx. \tag{18}$$

In other words, the exponential property of the discount function and the lifetime distribution determines a control limit for X that is independent of time (compare with Ross, pp. 156–158).

Since $\gamma'(t) = \beta\gamma(t) + e^{\beta t}\lambda'(t) = 0$, Equation 14 takes the form

$$-\beta\gamma_0 e^{-\beta t} = r\lambda(t) - \mu[V(t) - \lambda(t)].$$

Since $\lambda(t) = \gamma_0 e^{-\beta t}$, we finally obtain

$$V(t) = (\mu + r + \beta)/\mu \cdot \gamma_0 e^{-\beta t}. \tag{19}$$

Consequently, the ratio between $V(t)$ and $\lambda(t)$ is fixed, that is, $V(t)/\lambda(t) = 1 + (r + \beta)/\mu$.

As a specific example, we calculate γ_0 when X is uniformly distributed on [0, 1]. Substituting in (18) and arranging terms, we obtain

$$\alpha\gamma_0^2/2 - \gamma_0 + \alpha/2 = 0 \tag{20}$$

where $\alpha = \mu/(\beta + r + \mu)$. The solution of (20) is

$$\gamma_0 = \alpha/(1 + \sqrt{1 - \alpha^2}). \tag{21}$$

Note that result (21) is identical to result (3). In the present case, $\beta(t) = 1$ is equivalent to $\beta = 0$ so that $\alpha = \mu/(r + \mu)$. Indeed, in terms of Section 1, the probability of arrival of an offer before self-termination of the process is

$$\begin{aligned}\alpha &= P(T > t_{j+1} \mid T > t_j)\\ &= \int_0^\infty P(T > t_j + s \mid T > t_j)\mu e^{-\mu s}\, ds = \int_0^\infty e^{-rs}\mu e^{-\mu s}\, ds = \mu/(r + \mu).\end{aligned}$$

(ii) *Increasing failure rate*

When the lifetime distribution G is IFR and the arrival process is nonhomogeneous Poisson with nonincreasing intensity $\mu(t)$, one can apply analytical arguments on Equation 10 to prove directly that the control limit function $\lambda(t)$ is nonincreasing (see David and Yechiali). Note that, when $\mu(t) = \mu$, this result is an immediate consequence of Theorem 1. Furthermore, if $r(t)$ is strictly increasing, then $\lambda(t)$ is decreasing (David and Yechiali).

(iii) *A bound on* $\lambda(t)$

The above results can be used to obtain a simple bound, which is decreasing in t, for the critical curve in the IFR case.

LEMMA 3. *If* $r(t)$ *is increasing and* $\mu(t)$ *is nonincreasing, then*

$$0 < \lambda(t) \le [\mu(t)\beta(t)/r(t)]EX.$$

Proof. From the discussion in the previous subsection, $\lambda'(t) \le 0$. Then, using (10) we have

$$r(t)\lambda(t) \le \mu(t)\beta(t) \int_{\lambda(t)/\beta(t)}^\infty \bar{F}(x)\, dx \le \mu(t)\beta(t)EX.$$

4. AN EXPLICIT SOLUTION FOR $\lambda(t)$—AN EXAMPLE

In this section we develop an explicit solution for $\lambda(t)$ when the lifetime distribution is Gamma with shape parameter $a = 2$ (i.e., $G(\cdot)$ is IFR). We let $\beta(t) = 1$ and $\mu(t) = \mu$. The density is given by

$$g(t) = \theta^a t^{a-1} e^{-\theta t}/\Gamma(a) = \theta^2 t e^{-\theta t} \quad \text{for} \quad \theta > 0,$$

and the failure rate $r(t) = \theta^2 t/(1 + \theta t)$ is increasing in t. X is taken to be a discrete random variable. Then $\mu\overline{F}(\lambda)$ is a nonincreasing step-function of λ. Since $\lambda(t)$ is monotone decreasing in t (see Section 3(ii)), $\mu\overline{F}(\lambda(t))$ is a nondecreasing step-function of t. Consider a time interval where $\mu F(\lambda)$ is constant, e.g.,

$$\mu\overline{F}(\lambda(t)) \equiv c.$$

For such an interval, Equation 10 reduces to

$$\lambda'(t) = (r(t) + c)\lambda(t) + B \tag{22}$$

for some constant B. The general solution for this first-order linear differential equation can be obtained with the aid of an integrating factor $\exp[-\int_0^t (r(x) + c)\, dx]$, and is given by

$$\lambda(t)\exp\left[-\int_0^t (r(x) + c)\, dx\right] = B\int_0^t \exp\left[-\int_0^\tau (r(x) + c)\, dx\right] d\tau + K$$

for some constant K.

Using the relation $e^{-\int_0^t r(x)dx} = \overline{G}(t)$, we see that the integrating factor is $\overline{G}(t)e^{-ct}$. Thus,

$$\lambda(t)\overline{G}(t)e^{-ct} = B\int_0^t \overline{G}(x)e^{-cx}\, dx + K. \tag{23}$$

Substituting $\overline{G}(t) = e^{-\theta t}(1 + \theta t)$ and integrating the right-hand side of (23) yield

$$\lambda(t) = B/-(\theta + c) - B\theta/((\theta + c)^2(1 + \theta t)) + De^{(\theta+c)t}/(1 + \theta t) \tag{24}$$

for some constant D. Define

$$A = -(c + \theta). \tag{25}$$

Without loss of generality, let the scale parameter θ equal 1. Equation 24 is now rewritten as

$$\lambda(t) = B/A - B/[A^2(1 + t)] + De^{-At}/(1 + t). \tag{26}$$

The parameters A, B, and D are determined separately and uniquely for each interval where $\mu\overline{F}(\lambda(t))$ is constant. A is determined by (25).

Combining (22) and (10), we get

$$c\lambda(t) + B = -\mu \int_{\lambda(t)}^{\infty} \bar{F}(x)\, dx. \tag{27}$$

c and B are now readily found, since both sides of (27) are linear functions of $\lambda(t)$. Finally, D is determined by the continuity of $\lambda(t)$ at the endpoints of the interval. The next section presents the method of successive determination of the time endpoints of the above intervals, together with calculation of the corresponding constants A, B, and D.

5. A NUMERICAL EXAMPLE

We present an example using actual data related to the kidney transplant problem. We consider a case where the expected lifetime on dialysis is 5 years (see the 55–64-year age group in Brunner, pp. 16 and 18). Assume that the distribution of the lifetime T is Gamma with shape parameter $a = 2$ and scale parameter $\theta = 1/2.5$. That is, $ET = a/\theta = 5$.

TABLE I

THE PROBABILITY DISTRIBUTION OF THE OFFER x

Match	X	$P(x)$
$\underline{E}$	0.44	0.1758
$\underline{D}$	0.47	0.4073
$\underline{C}$	0.49	0.3134
$\underline{B}$	0.62	0.0941
$\underline{A}$	0.70	0.0094

To simplify calculations, we assume $\theta = 1$, bearing in mind that our time unit is 2.5 years. We also assume that the Poisson arrival intensity is $\mu = 16$ (i.e., an average of 16 admissible donors for a specific recipient during the time period of 2.5 years). Each kidney arrival results in an $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$ or $\underline{E}$ match, as described in the introduction. To evaluate $P(\underline{A})$, $P(\underline{B})$ etc., we use Barnes and Miettinen's formulas (1)–(5), and the HL-A gene frequencies of Allen et al. that they cite. Assuming the recipient's antigens to be #1 and #2 in the first series, and #7 and #12 in the second series, we get $P(\underline{A}) = 0.0094$, $P(\underline{B}) = 0.0941$, $P(\underline{C}) = 0.3134$, $P(\underline{D}) = 0.4073$, $P(\underline{E}) = 0.1758$.

The values of X are given in terms of the graft survival probability within 1 year, and are taken from Dausset et al. Table I combines the above data. With the aid of Table I, determining the cumulative density function $F(x)$ of X, we calculate and tabulate the results in Table II. The values of the parameters c and B for the various regions of $\lambda(t)$ are readily deduced from Table II and Equation 27. A is calculated via (25).

TABLE II

FORMULAS FOR $-\mu \int_{\lambda(t)}^{\infty} \bar{F}(x)\, dx$

No.	Region	$-\mu \int_{\lambda(t)}^{0.70} \bar{F}(x)\, dx$
I	$0 \le \lambda(t) \le 0.44$	$16\ \lambda(t) - 7.7961$
II	$0.44 \le \lambda(t) \le 0.47$	$13.1872\ \lambda(t) - 6.5585$
III	$0.47 \le \lambda(t) \le 0.49$	$6.6704\ \lambda(t) - 3.4956$
IV	$0.49 \le \lambda(t) \le 0.62$	$1.6544\ \lambda(t) - 1.0378$
V	$0.62 \le \lambda(t) \le 0.70$	$0.1504\ \lambda(t) - 0.1053$

We must still calculate the values of D, and determine the time intervals corresponding to the various regions of $\lambda(t)$. Then, by means of (26), the control limit function is completely determined. First, we note that the failure rate $t/(1 + t)$ approaches 1 as $t \to \infty$. Then, setting $\beta = 0$ and using limiting arguments similar to those that led to Equations 17–18, we get

$$\lim_{t\to\infty}\lambda(t) \equiv L = \mu \int_{L}^{0.70} \bar{F}(x)\, dx.$$

With the aid of Table II, we find $L = 0.462$. Since $0.44 \le L \le 0.47$, region I is not attained by $\lambda(t)$. Furthermore, $D = 0$ in region II, for otherwise (by (26)) $\lambda(t)$ is not bounded. Having A, B, D for this region, we substitute in (26) and solve for t^*, the leftmost t in that region. Specifically, as $\lambda(t)$ is decreasing, Table III gives $\lambda(t^*) = 0.47$. Hence,

$$t^* = (-B/A^2)/(0.47 - B/A) - 1 = 0.03258/(0.47 - 0.46228) - 1 = 3.2202.$$

To calculate D of region III, we use continuity of $\lambda(t)$ at t^*. Hence,

$$0.47 = B_{\mathrm{III}}/A_{\mathrm{III}} - B_{\mathrm{III}}/(A_{\mathrm{III}}^2(1 + t^*)) + D_{\mathrm{III}}e^{-A_{\mathrm{III}}t^*}/(1 + t^*).$$

By substituting values of A_{III} and B_{III} from Table III, we determine D_{III} and proceed as above to find t^{**}, the leftmost t in region III. Computation gives $t^{**} = 0.73349$. Continuing similarly, we find that, in $[0, t^{**}]$, $\lambda(t)$ stays in region IV, so region V is not attained by $\lambda(t)$.

TABLE III

VALUES OF A AND B FOR EQUATION 26

No.	Region	c	A	B
I	$0 \le \lambda(t) \le 0.44$	16.0	−17.0	−7.7961
II	$0.44 \le \lambda(t) \le 0.47$	13.1872	−14.1872	−6.5585
III	$0.47 \le \lambda(t) \le 0.49$	6.6704	−7.6704	−3.4956
IV	$0.49 \le \lambda(t) \le 0.62$	1.6544	−2.6544	−1.0378
V	$0.62 \le \lambda(t) \le 0.70$	0.1504	−1.1504	−0.1053

Finally, we return to 1-year time unit and obtain $2.5 \cdot t^* = 8.05$ and $2.5 \cdot t^{**} = 1.83$.

We summarize our results. The optimal policy uses the following decisions:

(i) From 0 to 1.83 years of "dialysis age"—wait for at least a $\underline{B}$-match.
(ii) From 1.83 to 8.05 years—wait for at least a $\underline{C}$-match.
(iii) Beyond 8.05 years—wait for at least a $\underline{D}$-match.

Note that an $\underline{E}$ match is never accepted, while on the other hand, a transplant is never conditioned on receiving an $\underline{A}$-match, that is, matches $\underline{A}$ and $\underline{B}$ are always accepted.

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