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α-TIME FRACTIONAL BROWNIAN MOTION: PDE CONNECTIONS AND LOCAL TIMES *

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Abstract. For $0 < \alpha \le 2$ and $0 < H < 1$, an α-time fractional Brownian motion is an iterated process $Z = \{Z(t) = W(Y(t)), t \ge 0\}$ obtained by taking a fractional Brownian motion $\{W(t), t \in \mathbb{R}\}$ with Hurst index $0 < H < 1$ and replacing the time parameter with a strictly α-stable Lévy process $\{Y(t), t \ge 0\}$ in $\mathbb{R}$ independent of $\{W(t), t \in \mathbb{R}\}$. It is shown that such processes have natural connections to partial differential equations and, when Y is a stable subordinator, can arise as scaling limit of randomly indexed random walks. The existence, joint continuity and sharp Hölder conditions in the set variable of the local times of a d-dimensional α-time fractional Brownian motion $X = \{X(t), t \in \mathbb{R}_+\}$ defined by $X(t) = \big(X_1(t), \ldots, X_d(t)\big)$, where $t \ge 0$ and $X_1, \ldots, X_d$ are independent copies of Z, are investigated. Our methods rely on the strong local nondeterminism of fractional Brownian motion.

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1. Introduction

In recent years, iterated Brownian motion and related iterated processes have received much research interest. Such iterated processes are connected naturally with partial differential equations and have interesting probabilistic and statistical features such as self-similarity, non-Markovian dependence structure, non-Gaussian distributions; see [2,10–12,17,18,28,29,33,44] and references therein for further information. Inspired by these results, we consider a new class of iterated processes called α-time fractional Brownian motion (fBm) for $0 < \alpha \le 2$ and $0 < H < 1$. These are obtained by taking a fractional Brownian motion of index H and replacing the time parameter with a strictly α-stable Lévy process Y. More precisely, let $W = \{W(t), t \in \mathbb{R}\}$ be a fractional Brownian motion in $\mathbb{R}$ with index H, which is a centered, real-valued Gaussian process with covariance function

$$\mathbb{E}\left(W(t)W(s)\right) = \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big)$$

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and $W(0) = 0$ a.s. Here and in the sequel, $|\cdot|$ denotes the Euclidean norm. Let $Y = \{Y(t), t \geq 0\}$ be a real-valued strictly α-stable Lévy process, $0 < \alpha \leq 2$, starting from 0; see Section 3 for its definition and [9,36] for further information. We assume that W and Y are independent. Then a real-valued α-time fractional Brownian motion $Z = \{Z(t), t \geq 0\}$ is defined by

$$Z(t) \equiv W(Y(t)), \quad t \geq 0. \tag{1.1}$$

For $\alpha = 2$ and $H = 1/2$, this is the iterated Brownian motion of Burdzy [10]. When $0 < \alpha < 2$ and $H = 1/2$, Z is called an α-time Brownian motion by Nane [30]. Aurzada and Lifshits [3] and Linde and Shi [27] studied the small deviation problem for real-valued α-time Brownian motion. Nane [32] studied laws of the iterated logarithm for a version of Z. Moreover, when Y is symmetric, for $\alpha = 1$, 2 and $H = 1/2$ these processes have connections with partial differential operators as described in [2,31].

More generally, it is easy to verify that the process Z has stationary increments and is a self-similar process of index H/α. The latter means that, for every constant $c > 0$, the processes $\{Z(t) : t \geq 0\}$ and $\{c^{-H/\alpha} Z(ct) : t \geq 0\}$ have the same finite-dimensional distributions. Gaussian and stable self-similar processes have been studied extensively in recent years; see Samorodnitsky and Taqqu [35], Embrechts and Maejima [20] for further information. The α-time fractional Brownian motions form an important class of non-Markovian and non-stable self-similar processes, except in the special case when $H = 1/2$ and Y is a stable subordinator [In this case, Z is a symmetric stable Lévy process]. As will be shown in this paper, they have natural connections to partial differential equations and can arise as scaling limit of randomly indexed random walks with dependent jumps. Hence they can serve as useful stochastic models in many scientific areas including physics, insurance risk theory and communication networks. Moreover, because they are non-Markovian and have non-stable distributions, new methods are often needed in order to study their properties.

When $\alpha < 2$, the sample function of the α-time fractional Brownian motion Z is not continuous and its irregularity is closely related to those of W and Y. One of our motivations of this paper is to characterize the irregularity of Z in terms of the parameters H and α. We do this by studying the existence and regularity of the local times of α-time fractional Brownian motion $X = \{X(t), t \geq 0\}$ with values in $\mathbb{R}^d$ defined by

$$X(t) = \big(X_1(t), \dots, X_d(t)\big), \qquad (t \geq 0), \tag{1.2}$$

where $X_j = W_j(Y_j(t))$ $(j = 1, \dots, d)$. We assume that $W_1, \dots, W_d$ are independent copies of W, $Y_1, \dots, Y_d$ are independent copies of Y, and $\{W_j\}$ and $\{Y_j\}$ are independent. We will call $X = \{X(t), t \geq 0\}$ a d-dimensional α-time fractional Brownian motion. It is clear that X is also self-similar of index H/α and has stationary increments.

The rest of this paper is organized as follows. In Section 2, we study the PDE connections of α-time fractional Brownian motions, and prove that they can be obtained as scaling limit of randomly indexed random walks with dependent jumps. These results provide some analytic and physical interpretations for α-time fractional Brownian motions.

In Sections 3 and 4 we investigate the existence, joint continuity and sharp Hölder conditions in the set variable of the local times of a d dimensional α-time fractional Brownian motion X. In the special case of $d = 1$, W is Brownian motion and Y is a symmetric α-stable Lévy process with $\alpha > 1$, the existence and joint continuity of the local time of Z have been proved by Nane [30]. The methods used in this paper differ from those of Nane [30]. The latter uses the existence of local times of Brownian motion in $\mathbb{R}$ as well as the existence of local time of symmetric stable Lévy processes, which does not exist whenever $\alpha \leq 1$ or $d > 1$. Our Theorem 3.1 implies that, for the α-time Brownian motion X in $\mathbb{R}^d$, local times exist in the case $d = 1$ for $\alpha > 1/2$; in the case $d = 2$ for $\alpha > 1$ and in the case $d = 3$ for $\alpha > 3/2$. Moreover, Theorem 4.1 shows that these local times have jointly continuous versions.

The methods of Sections 3 and 4 rely on the Fourier analytic argument of Berman [7,8] and a chaining argument in Ehm [19]. In order to derive crucial moment estimates for the local times, we make use of the strong local nondeterminism (SLND) of fractional Brownian motion proved by Pitt [34] as well as several nontrivial modifications of the arguments in Xiao [43,44].

Finally, we provide some technical lemmas as an Appendix in Section 5.

Throughout the paper, we will use K to denote an unspecified positive finite constant which may not necessarily be the same in each occurrence.

2. PDE connections and scaling limits of randomly indexed random walks

In this section we show that α-time fractional Brownian motions have natural connections to partial differential equations. They may also arise as scaling limit of randomly indexed random walks with dependent jumps. These results provide some analytic and physical interpretations of α-time fractional Brownian motions. These results show that α-time fBm can serve as useful stochastic model in various scientific areas.

2.1. **PDE connections**

The domain of the infinitesimal generator $\mathcal{A}$ of a semigroup $T(t)$ defined on a Banach space $\mathcal{H}$ is the set of all $\varphi \in \mathcal{H}$ such that the limit

$$\lim_{t\to 0} \frac{T(t)\varphi(x) - \varphi(x)}{t}$$

exists in the strong norm of $\mathcal{H}$.

Let $\Delta = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ be the Laplacian operator, and let $\delta(x)$ be the Dirac-delta function. The density of Brownian motion W in $\mathbb{R}^d$ is $f(t,x) = \frac{1}{(2\pi t)^{d/2}} \mathrm{e}^{-|x|^2/2t}$. Let

$$T(t)\varphi(x) = \int_{\mathbb{R}^d} f(t, x-y)\varphi(y)\mathrm{d}y$$

be the semigroup of Brownian motion on $L^2(\mathbb{R}^d)$. Then the generator of $T(t)$ is Δ with the domain $Dom(\Delta) = \{\varphi \in L^2(\mathbb{R}^d) : \nabla\varphi \in L^2(\mathbb{R}^d)\}$, where $\nabla\varphi$ is the weak derivative of φ. See Section 31 in Sato [36] for more details and semigroups on other Banach spaces. Let φ be a function in the domain of the Laplacian. Then the function $u(t,x) = T(t)\varphi(x)$ is a solution of the heat equation

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= \frac{1}{2}\Delta u(t,x), \quad t>0,\ x\in\mathbb{R}^d, \\ u(0,x) &= \varphi(x), \quad x \in \mathbb{R}^d. \end{aligned} \tag{2.1}$$

In the case $H = \frac{1}{2}$ and $Y(t)$ is stable subordinator of index $\beta/2$, $0 < \beta \le 2$ with $\mathbb{E}(\mathrm{e}^{-sY(t)}) = \mathrm{e}^{-ts^{\beta/2}}$, $W(Y(t))$ is a symmetric stable process of index β in $\mathbb{R}^d$. The density of $W(Y(t))$ is given by

$$q(t,x) = \int_0^\infty f(s,x)p_t(s)\,\mathrm{d}s = \int_0^\infty \frac{\mathrm{e}^{-|x|^2/2s}}{(2\pi s)^{d/2}} p_t(s)\mathrm{d}s,$$

where $p_t(s)$ is the density of $Y(t)$. Then the function

$$u(t,x) = \mathbb{E}_x[\psi(W(Y(t)))] = \int_0^\infty [T(s)\psi(x)]p_t(s)\mathrm{d}s$$

is a solution of

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= -2^{-\beta/2}(-\Delta)^{\beta/2} u(t,x), \qquad t>0, \quad x \in \mathbb{R}^d \\ u(0,x) &= \psi(x), \qquad x \in \mathbb{R}^d, \end{aligned} \tag{2.2}$$

where $-(-\Delta)^{\beta/2}$ is the fractional Laplacian with Fourier transform

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle k,x\rangle}[-(-\Delta)^{\beta/2}\psi(x)]\mathrm{d}x = -|k|^\beta \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle k,x\rangle}\psi(x)\mathrm{d}x,$$

for functions ψ in the domain of the fractional Laplacian, see [36], Theorem 31.5 and Example 32.6.

For the case of $H = \frac{1}{2}$ and $\alpha = 2$, Allouba and Zheng [2] and DeBlassie [17] showed that, for any function φ in the domain of the Laplacian, the function $u(t,x) = \mathbb{E}_x[\varphi(W(Y(t)))]$ solves the Cauchy problem

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= \frac{\Delta\varphi(x)}{\sqrt{8\pi t}} + \frac{1}{8}\Delta^2 u(t,x), \qquad t>0, \quad x\in\mathbb{R}^d \\ u(0,x) &= \varphi(x), \qquad x\in\mathbb{R}^d. \end{aligned}$$

In this case $u(t,x) = \mathbb{E}_x[\varphi(W(Y(t)))]$ also solves the fractional Cauchy problem

$$\begin{aligned} \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}u(t,x) &= 2^{-3/2}\Delta u(t,x); \quad x\in\mathbb{R}^d, \ t>0 \\ u(0,x) &= \varphi(x), \ x\in\mathbb{R}^d. \end{aligned}$$

Here $\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}u(t,\cdot)$ is the Caputo fractional derivative with respect to t of order $\frac{1}{2}$, defined by (for fixed $x\in\mathbb{R}^d$)

$$\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}u(t,x) = \frac{1}{\sqrt{\pi}}\int_0^t \left[\frac{\partial u(s,x)}{\partial s}\right]\frac{\mathrm{d}s}{(t-s)^{\frac{1}{2}}} = \frac{1}{\Gamma(1-\frac{1}{2})}\int_0^t \left[\frac{\partial u(s,x)}{\partial s}\right]\frac{\mathrm{d}s}{(t-s)^{\frac{1}{2}}}, \tag{2.3}$$

see [4].

For the case $H = \frac{1}{2}$, $\alpha = 1$ and Y is a symmetric Cauchy process, Nane [31] showed that $u(t,x) = \mathbb{E}_x[\varphi(W(Y(t)))]$ solves

$$\begin{aligned} \frac{\partial^2}{\partial t^2}u(t,x) &= -\frac{\Delta\varphi(x)}{\pi t} - \frac{1}{4}\Delta^2 u(t,x), \qquad t>0, \quad x\in\mathbb{R}^d, \\ u(0,x) &= \varphi(x), \qquad x\in\mathbb{R}^d, \end{aligned} \tag{2.4}$$

where φ is a bounded measurable function in the domain of the Laplacian, with $\frac{\partial^2\varphi}{\partial x_i\partial x_j}$ bounded and Hölder continuous for all $1\le i,\ j\le d$.

Nane [31] has also established pde connection for the case $H = \frac{1}{2}$, and $\alpha = \frac{k}{m}$ for relatively prime integers k, m, see Theorem 2.5 in [31].

For the case $0<H<1$, $d=1$ and $\alpha=2$, D'ovidio and Orsingher [18] established the fact that the density of $Z(t) = W(Y(t))$

$$q(t,x) = 2\int_0^\infty \frac{\mathrm{e}^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}}\frac{\mathrm{e}^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}}\mathrm{d}s$$

is a solution of the first order PDE

$$t\frac{\partial q(t,x)}{\partial t} = -\frac{H}{2}\frac{\partial}{\partial x}(xq(t,x)), \quad t>0,\ x\in\mathbb{R}. \tag{2.5}$$

Now we consider an $\mathbb{R}^d$-valued α-time fractional Brownian motion $Z(t) = W(Y(t))$ with $\alpha = 1$. The following theorem answers a question in [18].

Theorem 2.1. *Let $W = \{W(t), t \in \mathbb{R}\}$ be an $\mathbb{R}^d$-valued fractional Brownian motion with index $H \in (0,1)$ and let $Y = \{Y(t), t \geq 0\}$ be a symmetric Cauchy process. Let $f^H(t,x)$ be the density function of $W(t)$, $p_t(s)$ be the density function $Y(t)$ and let $\delta(x)$ denote the Dirac-delta function. Then the density function*

$$q(t,x) = 2\int_0^\infty f^H(s,x)p_t(s)\mathrm{d}s = 2\int_0^\infty \frac{\mathrm{e}^{-\frac{|x|^2}{2s^{2H}}}}{(2\pi s^{2H})^{d/2}} \frac{t}{\pi(t^2+s^2)}\mathrm{d}s$$

of $W(Y(t))$ solves the PDE

$$\begin{aligned}\frac{\partial^2 q(t,x)}{\partial t^2} = &-\frac{2HI_{(0,1/2]}(H)}{\pi t}\Delta\delta(x) - H(2H-1)\Delta G_{(2H-2),t}q(t,x)\\ &- H^2\Delta^2 G_{(4H-2),t}q(t,x), \qquad x \in \mathbb{R}^d, t > 0,\end{aligned} \tag{2.6}$$

where

$$G_{\gamma,t}q(t,x) = 2\int_0^\infty s^\gamma p_t(s) f^H(s,x)\mathrm{d}s, \quad \gamma \neq 0,$$

and $G_{0,t}$ is the identity operator.

An operator similar to the operator $G_{\gamma,t}$ was introduced in [22]. We refer to their Proposition 3.6 and Remark 3.7 for some nice properties of that operator. For the case $H \neq \frac{1}{2}$, it might be a challenging problem to find the right class of functions ϕ and establish the Cauchy problem that is solved by $u(t,x) = \mathbb{E}_x(\phi(W(Y(t))))$. This is due to the fact that $v(t,x) = \mathbb{E}_x(\phi(W(t)))$ is not a semigroup on a Banach space. The general theory of semigroups and their generators will not apply in this case

Proof. Recall that the density function of symmetric Cauchy process $Y(t)$ is

$$p_t(s) = \frac{t}{\pi(t^2+s^2)}, \quad t \geq 0, \ s \in \mathbb{R}.$$

Since

$$\frac{\partial^2}{\partial t^2}p_t(s) = \frac{-2t(3s^2-t^2)}{(t^2+s^2)^3}$$

and for $t > 0$

$$2\int_0^\infty f^H(s,x)\left|\frac{\partial^2}{\partial t^2}p_t(s)\right|\mathrm{d}s = 2\int_0^\infty \frac{\mathrm{e}^{-\frac{|x|^2}{2s^{2H}}}}{(2\pi s^{2H})^{d/2}}\left|\frac{-2t(3s^2-t^2)}{(t^2+s^2)^3}\right|\mathrm{d}s < \infty,$$

we apply the Dominated Convergence Theorem to verify the following interchange of the second derivative in t:

$$\frac{\partial^2}{\partial t^2}q(t,x) = 2\int_0^\infty f^H(s,x)\frac{\partial^2}{\partial t^2}p_t(s)\mathrm{d}s. \tag{2.7}$$

By using integration by parts to (2.7) and the facts

$$\begin{aligned}&\left(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2}\right)p_t(s) = 0;\\ &\frac{\partial}{\partial s}f^H(s,x) = Hs^{2H-1}\Delta f^H(s,x),\end{aligned} \tag{2.8}$$

we derive

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}q(t,x) &= -2\int_0^\infty f^H(s,x)\frac{\partial^2}{\partial s^2}p_t(s)\mathrm{d}s \\
&= -2f^H(s,x)\frac{\partial}{\partial s}p_t(s)\Big|_0^\infty + 2\int_0^\infty \frac{\partial}{\partial s}f^H(s,x)\frac{\partial}{\partial s}p_t(s)\mathrm{d}s \\
&= 2p_t(s)\frac{\partial}{\partial s}f^H(s,x)\Big|_0^\infty - 2\int_0^\infty p_t(s)\frac{\partial^2}{\partial s^2}f^H(s,x)\mathrm{d}s \\
&= 2p_t(s)\frac{\partial}{\partial s}f^H(s,x)\Big|_0^\infty + 2\int_0^\infty p_t(s)\frac{\partial}{\partial s}\left(Hs^{2H-1}\Delta f^H(s,x)\right)\mathrm{d}s \\
&= -\frac{2HI_{(0,1/2]}(H)}{\pi t}\Delta\delta(x) \\
&\quad -2\int_0^\infty p_t(s)\left(H(2H-1)s^{2H-2}\Delta f^H(s,x)+H^2s^{4H-2}\Delta^2 f^H(s,x)\right)\mathrm{d}s \\
&= -\frac{2HI_{(0,1/2]}(H)}{\pi t}\Delta\delta(x) \\
&\quad -\Delta 2\int_0^\infty p_t(s)H(2H-1)s^{2H-2}f^H(s,x)ds + \Delta^2 2\int_0^\infty p_t(s)H^2s^{4H-2}f^H(s,x)\mathrm{d}s,
\end{aligned}$$

where the last line follows by the dominated convergence theorem. In the above we have used that

$$\begin{aligned}
\lim_{s\to 0} f^H(s,x)\frac{\partial}{\partial s}p_t(s) &= 0, \\
\lim_{s\to\infty} f^H(s,x)\frac{\partial}{\partial s}p_t(s) &= 0, \\
\lim_{s\to\infty} p_t(s)\frac{\partial}{\partial s}f^H(s,x) &= 0
\end{aligned} \tag{2.9}$$

and that

$$\begin{aligned}
\lim_{s\to 0} p_t(s)\frac{\partial}{\partial s}f^H(s,x) &= \lim_{s\to 0}\frac{H}{\pi t}s^{2H-1}\Delta f^H(s,x) \\
&= \begin{cases} 0 & \text{if } H > \frac{1}{2}, \\ \frac{H}{\pi t}\Delta\delta(x) & \text{if } 0 < H \le \frac{1}{2}. \end{cases}
\end{aligned}$$

This finishes the proof of (2.6). □

Remark 2.1. After we submitted our paper, we learned that Beghin *et al.* [6] has established that the density $W(Y(t))$ in the case $d = 1$, $0 < H < 1, \alpha = 1$

$$q(t,x) = 2\int_0^\infty \frac{\mathrm{e}^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}}\frac{t}{\pi(t^2+s^2)}\mathrm{d}s$$

solves

$$\frac{\partial^2}{\partial t^2}q(t,x) = -\frac{1}{t^2}\left[H(H-1)\frac{\partial}{\partial x}x - H^2\frac{\partial^2}{\partial x^2}x^2\right]q(t,x) - \frac{2HI_{(0,1/2]}(H)}{\pi t}\frac{\partial^2\delta(x)}{\partial x^2}. \tag{2.10}$$

It is interesting to compare equations (2.6) and (2.10).

Let $0 < \beta = \frac{k}{m} < 2$ for k, m relatively prime integers, and let $Y(t)$ be a stable subordinator of index $\beta/2$. In this case the density $p_t(s)$ is a solution of

$$\frac{\partial^{2m}}{\partial t^{2m}} p_t(s) = \frac{\partial^k}{\partial s^k} p_t(s), \quad s, t > 0, \tag{2.11}$$

see Lemma 3.1 in [16]. We have the next theorem which gives an extension of the PDE in (2.2).

Theorem 2.2. *Let $W = \{W(t), t \in \mathbb{R}\}$ be an $\mathbb{R}^d$-valued fractional Brownian motion with index $H \in (0,1)$ and let $Y(t)$ be a stable subordinator of index $\alpha = \frac{\beta}{2}$, where $\beta = \frac{1}{m}$ for $m = 2, 3, \dots$ Then the density $q(t,x) = \int_0^\infty f^H(s,x) p_t(s) \mathrm{d}s$ of $W(Y(t))$ is a solution of*

$$\frac{\partial^{2m}}{\partial t^{2m}} q(t,x) = -H \Delta V_{(2H-1),t}\, q(t,x), \tag{2.12}$$

where $V_{\gamma,t}\, q(t,x) = \int_0^\infty s^\gamma p_t(s) f^H(s,x) p_t(s) \mathrm{d}s$ for $\gamma \neq 0$ and $V_{0,t}$ is the identity operator.

Proof. The proof follows by integration by parts as in the proof of Theorem 2.1, and by using (2.11) with $k = 1$.

$$\begin{aligned}
\frac{\partial^{2m}}{\partial t^{2m}} q(t,x) &= \int_0^\infty f^H(s,x) \frac{\partial^{2m}}{\partial t^{2m}} p_t(s) \mathrm{d}s \\
&= \int_0^\infty f^H(s,x) \frac{\partial}{\partial s} p_t(s) \mathrm{d}s \\
&= f^H(s,x) p_t(s) \Big|_0^\infty - \int_0^\infty \frac{\partial}{\partial s} f^H(s,x) p_t(s) \mathrm{d}s \\
&= f^H(s,x) p_t(s) \Big|_0^\infty - \int_0^\infty p_t(s) \Big(H s^{2H-1} \Delta f^H(s,x) \Big) \mathrm{d}s \\
&= -H \int_0^\infty p_t(s) s^{2H-1} \Delta f^H(s,x)\, \mathrm{d}s, \\
&= -H \Delta \int_0^\infty p_t(s) s^{2H-1} f^H(s,x)\, \mathrm{d}s,
\end{aligned}$$

the last line follows by dominated convergence theorem. See equations (2.7)–(2.10) in [16] to show that the boundary terms are all zero. □

Letting $H = \frac{1}{2}$ in Theorem 2.2, the density $q(t,x)$ of symmetric stable process $W(Y(t))$ of index $\alpha = \frac{1}{m}$ is a solution of

$$\frac{\partial^{2m}}{\partial t^{2m}} q(t,x) = -\frac{1}{2} \Delta\, q(t,x). \tag{2.13}$$

Equations (2.2) and (2.13) should be compared. This result is a special case of the following result stated in Nane [31], Lemma 3.2 that is due to DeBlassie [16] originally: Let $H = \frac{1}{2}$ and $0 < \alpha = \frac{k}{m} < 2$, where k, m are relatively prime. Let Y be a stable subordinator of index $\alpha/2$. In this case $W(Y(t))$ is a symmetric stable process of index $\alpha = \frac{k}{m}$. Then the density $q(t,x)$ of $W(Y(t))$ is a solution of

$$\frac{\partial^{2m}}{\partial t^{2m}} q(t,x) = \frac{1}{2^k} (-\Delta)^k\, q(t,x), \quad s, t > 0. \tag{2.14}$$

We can work out a similar connection for the case $0 < H < 1$ and $\alpha = \frac{k}{m}$ (k and m are relatively prime integers) by using integration by parts, (2.8), and (2.14), which extends the PDE connection in Nane ([31], Thm. 2.5), for the case $H = \frac{1}{2}, \alpha = \frac{k}{m}$.

Alternatively, for the case $0 < H < 1$, $\alpha = \frac{k}{m}$, k, m relatively prime integers, we have $W(Y(t)) = W(B(U(t)))$, where B is a Brownian motion running twice the speed of standard Brownian motion and U is a stable subordinator of index $\alpha/2 = \frac{k}{2m}$. In this case using the methods of Theorem 2.1, equation (2.5) for the density of $W(B(t))$ and equation (2.11) for the density of $U(t)$ we can obtain the PDE solved by the density of $W(Y(t))$.

For many other PDE connections of different types of subordinate processes, we refer to [2,4,5,18,31,33].

2.2. Scaling limits of randomly indexed random walks

Now we prove that the α-time fBm $Z = \{W(Y(t)), t \geq 0\}$, where W is H-fractional Brownian motion with values in $\mathbb{R}$ and $\{Y(t), t \geq 0\}$ is a stable subordinator of index $\alpha \in (0, 1)$, can be approximated weakly in the Skorohod space $D([0, \infty), \mathbb{R})$ by normalized partial sums of randomly indexed random walks with dependent jumps.

Let $\{\xi_n, n \in \mathbb{Z}\}$ be a sequence of i.i.d. random variables with $\mathbb{E}[\xi] = 0$, $\mathbb{E}[\xi^2] = 1$, and let $\{a_n, n \in \mathbb{Z}_+\}$ be a sequence of real numbers such that

$$\sum_{n=0}^{\infty} a_n^2 < \infty.$$

We consider the linear stationary process $\{X_n, n \in \mathbb{N}\}$ defined by

$$X_n = \sum_{j=0}^{\infty} a_j \xi_{n-j}, \quad n \in \mathbb{N}. \tag{2.15}$$

Davydov [15] was the first to study the weak convergence of normalized partial sums of $\{X_n, n \in \mathbb{N}\}$ to fractional Brownian motion. The following result is taken from Whitt [42], Theorem 4.6.1.

Lemma 2.1. *Let $\{X_n, n \in \mathbb{N}\}$ be the linear stationary process defined by (2.15), and let $S_n = X_1 + \ldots + X_n$. If*

$$\mathrm{Var}(S_n) = n^{2H} L(n), \quad n \in \mathbb{N} \tag{2.16}$$

for some $H \in (0, 1)$, where $L(\cdot)$ is a slowly varying function, and

$$\mathbb{E}\left[|S_n|^{2a}\right] \leq K \cdot \left(\mathbb{E}[S_n^2]\right)^a \tag{2.17}$$

for some constants $a > 1/H$ and $K > 0$, then

$$\left\{\frac{1}{n^H\sqrt{L(n)}} S_{\lfloor nt \rfloor}, t \geq 0\right\} \Rightarrow \{W(t), t \geq 0\} \tag{2.18}$$

in the J_1-topology on $D([0, \infty), \mathbb{R})$, where W is a fractional Brownian motion with Hurst index H.

Example 2.2. As in ([42], pp. 123–124) we take $a_j = cj^{-\gamma}$ for some constants $c \in \mathbb{R}\backslash\{0\}$ and $\gamma \in (\frac{1}{2}, 1)$, then it can be verified that

$$\mathrm{Var}(S_n) \sim c_1 n^{3-2\gamma} \quad \text{as } n \to \infty, \tag{2.19}$$

where

$$c_1 = \frac{2c^2\,\Gamma(1-\gamma)\Gamma(2\gamma-1)}{\Gamma(\gamma)(3-2\gamma)^2}.$$

By applying Lemma 2.1, we have that

$$\left\{\frac{1}{\sqrt{c_1}\, n^H} S_{\lfloor nt \rfloor}, t \geq 0\right\} \Rightarrow \{W(t), t \geq 0\}, \tag{2.20}$$

in the J_1-topology on $D([0, \infty), \mathbb{R})$, where $H = \frac{3-2\gamma}{2}$.

Theorem 2.3. *Let $\{X_n, n \geq 1\}$ be the linear stationary process defined by (2.15) and satisfies (2.16) and (2.17) in Lemma 2.1. Let $\{J_n, n \geq 1\}$ be a sequence of i.i.d. random variables also independent of $\{\xi_n, n \in \mathbb{Z}_+\}$, which belongs to the domain of attraction of some stable law Y with index $\alpha \in (0, 1)$ and $Y > 0$ a.s. Denote by $\{b_n, n \geq 1\}$ a sequence of positive numbers such that $b_n T_n \Rightarrow Y$, where*

$$T_n = J_1 + \ldots + J_n, \quad \forall n \geq 1.$$

Then as $c \to \infty$

$$\left\{ \frac{1}{\big(b(c)\big)^{-H}\sqrt{L\big(b(c)^{-1}\big)}} S_{\lfloor T(ct) \rfloor},\, t \geq 0 \right\} \Rightarrow \big\{ W\big(Y(t)\big),\, t \geq 0 \big\} \tag{2.21}$$

in the J_1-topology on $D\left([0, \infty), \mathbb{R}\right)$, where $b(c) = b_{\lfloor c \rfloor}$ and $T(s) = T_{\lfloor s \rfloor}$.

Proof. It follows from Theorem 4.5.3 in Whitt [42] that

$$\{b(c)T(ct),\, t \geq 0\} \Rightarrow \{Y(t),\, t \geq 0\} \tag{2.22}$$

in the J_1-topology on $D\left([0, \infty), \mathbb{R}_+\right)$, where $\{Y(t),\, t \geq 0\}$ is a stable subordinator with index α.

Notice that $\{X_n,\, n \geq 1\}$ and $\{J_n\}$ are independent, we derive from Lemma 2.1 and (2.22) that as $c \to \infty$

$$\left\{ \left(\frac{1}{c^H \sqrt{L(c)}} S_{\lfloor ct \rfloor},\, b(c)T(ct) \right),\, t \geq 0 \right\} \Rightarrow \{(W(t),\, Y(t)),\, t \geq 0\} \tag{2.23}$$

in the J_1-topology on $D\left([0, \infty), \mathbb{R}\right) \times D\left([0, \infty), \mathbb{R}_+\right)$.

Since, for the limiting processes in (2.23), the sample function $W(t)$ is continuous and $Y(t)$ is strictly increasing, the conclusion of Theorem 2.3 follows from Theorem 13.2.2 in Whitt [42]. □

3. Existence of local times

Let $X = \{X(t), t \geq 0\}$ be an α-time fractional Brownian motion in $\mathbb{R}^d$ defined by (1.2). In this section, we study the existence of local times

$$L = \{L(x, B) : \; x \in \mathbb{R}^d, B \in \mathcal{B}(\mathbb{R}_+)\}$$

of X, where $\mathcal{B}(\mathbb{R}_+)$ is the Borel σ-algebra of $\mathbb{R}_+$. In Section 4, we will establish joint continuity and sharp Hölder conditions in the set variable for the local times.

We recall briefly the definition of local times. For an extensive survey, see Geman and Horowitz [21]. Let $X : \mathbb{R} \to \mathbb{R}^d$ be any Borel function and let $B \subset \mathbb{R}$ be a Borel set. The occupation measure of $X(t)$ on B is defined by

$$\mu_B(A) = \lambda_1\{t \in B : \quad X(t) \in A\} \tag{3.1}$$

for all Borel sets $A \subset \mathbb{R}^d$, where λ_1 is the Lebesgue measure on $\mathbb{R}$. If μ_B is absolutely continuous with respect to the Lebesgue measure λ_d on $\mathbb{R}^d$, we say that X has a local time on B and define its local time $L(x, B)$ to be the Radon-Nikodym derivative of μ_B. If $B = [0, t]$, we will simply write $L(x, B)$ as $L(x, t)$. If $I = [0, T]$ and $L(x, t)$ is continuous as a function of $(x, t) \in \mathbb{R}^d \times I$, then we say that X has a jointly continuous local time on I. In this latter case, the set function $L(x, \cdot)$ can be extended to be a finite Borel measure on the level set

$$X_I^{-1}(x) = \{t \in I : X(t) = x\}.$$

See Adler [1], Theorem 8.6.1. This fact has been used by many authors to study fractal properties of level sets, inverse image and multiple times of stochastic processes. Related to our paper, we mention that Xiao [44] and

Hu [23] have studied the Hausdorff dimension, and exact Hausdorff and packing measure of the level sets of iterated Brownian motion, respectively.

An α-stable Lévy process $Y = \{Y(t), t \geq 0\}$ with values in $\mathbb{R}$ is a stochastically continuous process with stationary independent increments, $Y(0) = 0$, and characteristic exponent ψ given by

$$\psi(z) = -\sigma|z|^\alpha \left(1 - i\beta \,\mathrm{sgn}(z) \tan \frac{\pi\alpha}{2}\right), \qquad \alpha \neq 1;$$
$$\psi(z) = -\sigma|z| \left(1 + i\frac{2}{\pi}\beta \,\mathrm{sgn}(z) \ln(|z|)\right), \qquad \alpha = 1, \tag{3.2}$$

where $0 < \alpha \leq 2$, $\sigma > 0$ and $-1 \leq \beta \leq 1$ are fixed constants (we have tacitly assumed that there is no drift term). See Bertoin [9] and Sato [36] for a systematic accounts on Lévy processes and stable laws, respectively.

Throughout this paper, we assume that $Y = \{Y(t), t \geq 0\}$ is strictly stable. That is, we assume $\beta = 0$ in (3.2) when $\alpha = 1$ so the asymmetric Cauchy process is excluded. Strictly stable Lévy processes of index α are $(1/\alpha)$-self-similar. Recall that, for $t > 0$, $p_t(x)$ is the density function of the random variable $Y(t)$. It is a bounded continuous function with the following scaling property:

$$p_t(x) = p_{rt}(r^{1/\alpha}x)\, r^{1/\alpha} \qquad \text{for every } r > 0. \tag{3.3}$$

As discovered in Taylor [41], it is natural to distinguish between two types of strictly stable processes: those of *Type A*, and those of *Type B*. A strictly stable process, Y, is of *Type A*, if

$$p_t(x) > 0, \qquad \forall t > 0,\, x \in \mathbb{R};$$

all other stable processes are of *Type B*. Taylor [41] has shown that if $\alpha \in (0, 1)$ and Y is of *Type B*, then either Y or $-Y$ is a subordinator, while all other strictly stable processes of index $\alpha \neq 1$ are of *Type A*. Hence, without loss of generality, we will assume Y is either a strictly stable process of type A, or a subordinator. It will be shown that the properties of local times of α-time fractional Brownian motion X in $\mathbb{R}^d$ depends on the type of Y.

The following existence theorem for square integrable local time of X is easily proved by using the Fourier analysis (see, *e.g.*, Berman [7], Geman and Horowitz [21] or Kahane [24]).

Theorem 3.1. *Let $X = \{X(t), t \geq 0\}$ be an α-time fractional Brownian motion in $\mathbb{R}^d$. Then for any $T > 0$, X has a local time $L(x, T) \in L^2(\mathbb{P} \times \lambda_d)$ almost surely if and only if $d < \alpha/H$.*

Remark 3.1. We conjecture that if $d \geq \alpha/H$ then X does not have a local time. It is known that an $\mathbb{R}^d$-valued fractional Brownian motion W with index H does not have a local time when $d \geq 1/H$ (this follows from Thm. 1.1 of Talagrand [38] and Thm. 3 of Talagrand [39]) and an $\mathbb{R}^d$-valued α-stable Lévy process Y does not have local time when $d \geq \alpha$ (this follows from Thm. 1 in Bertoin ([9], p. 126). However, the proofs of these results rely on special properties of W and Y, and new method is needed in order to prove an analogous result for α-time fractional Brownian motion X.

Proof of Theorem 3.1. Let $\mu_{[0,T]}$ be the occupation measure of X on $[0, T]$ defined by (3.1). Then its Fourier transform can be written as

$$\widehat{\mu}_{[0,T]}(u) = \int_0^T \exp(i\langle u, X(t)\rangle)\, \mathrm{d}t,$$

where $\langle \cdot, \cdot\rangle$ is the ordinary scalar product in $\mathbb{R}^d$. It follows from Fubini's theorem that

$$\mathbb{E}\int_{\mathbb{R}^d} |\widehat{\mu}_{[0,T]}(u)|^2 \mathrm{d}u = \int_0^T \int_0^T \int_{\mathbb{R}^d} \mathbb{E}\exp(i\langle u, X(t) - X(s)\rangle)\, \mathrm{d}u\mathrm{d}s\mathrm{d}t. \tag{3.4}$$

To evaluate the characteristic function in (3.4), we assume $0 < s < t$ (the other case is similar) and note that the density of $(Y(t), Y(s))$ is given by

$$p_{s,t}(x, y) = p_s(x)p_{t-s}(y - x).$$

Since $X_1, \ldots, X_d$ are independent copies of $Z = \{W(Y(t)), t \geq 0\}$, we have

$$\begin{aligned}
\mathbb{E}\exp(i\langle u, X(t) - X(s)\rangle) &= \prod_{k=1}^d \mathbb{E}\exp\left(iu_k(W(Y(t)) - W(Y(s)))\right) \\
&= \prod_{k=1}^d \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}\exp\big(iu_k(W(y) - W(x))\big)p_{s,t}(x, y)\,\mathrm{d}x\mathrm{d}y \\
&= \prod_{k=1}^d \int_{\mathbb{R}}\int_{\mathbb{R}} \exp\Big(-\frac{u_k^2}{2}|y - x|^{2H}\Big)\, p_{s,t}(x, y)\,\mathrm{d}x\mathrm{d}y \\
&= \prod_{k=1}^d \int_{\mathbb{R}} \exp\left(-\frac{u_k^2}{2}|z|^{2H}\right) p_{t-s}(z)\,\mathrm{d}z. \qquad (3.5)
\end{aligned}$$

To evaluate the integrals with respect to u, we make a change of variables to get

$$\int_{\mathbb{R}} \exp\Big(-\frac{u_k^2}{2}\,|z|^{2H}\Big)\,\mathrm{d}u_k = |z|^{-H}\int_{\mathbb{R}} \exp\Big(-\frac{u_k^2}{2}\Big)\,\mathrm{d}u_k. \qquad (3.6)$$

It follows from (3.4), (3.5) and (3.6) that

$$\begin{aligned}
\mathbb{E}\int_{\mathbb{R}^d} |\widehat{\mu}_{[0,T]}(u)|^2\,\mathrm{d}u &= \int_0^T\int_0^T \prod_{k=1}^d \left(\int_{\mathbb{R}} \exp\left(-\frac{u_k^2}{2}\right)\,\mathrm{d}u_k \int_{\mathbb{R}} |z|^{-H}p_{|t-s|}(z)\,\mathrm{d}z\right)\mathrm{d}s\mathrm{d}t \\
&= (2\pi)^{d/2}\left(\int_{\mathbb{R}} |z|^{-H}p_1(z)\mathrm{d}z\right)^d \int_0^T\int_0^T \frac{1}{|t - s|^{dH/\alpha}}\,\mathrm{d}s\mathrm{d}t. \qquad (3.7)
\end{aligned}$$

In the above, we have used the fact that $p_{|t-s|}(z) = |t - s|^{-1/\alpha}p_1(|t - s|^{-1/\alpha}z)$ and another change of variables.

The last integral in (3.7) is finite if and only if $dH/\alpha < 1$. Hence $\widehat{\mu}(\cdot) \in L^2(\mathbb{P} \times \lambda_d)$ if and only if $dH/\alpha < 1$. Therefore, Theorem 3.1 follows from Plancherel's theorem (see also Thm. 21.9 in Geman and Horowitz [21]). □

The local time $L(t, x)$ can be formally expressed as the inverse Fourier transform of $\widehat{\mu}_{[0,T]}(u)$, namely (*cf.* [24], p. 267)

$$\begin{aligned}
L(t, x) &= \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \exp(-i\langle u, x\rangle)\,\widehat{\mu}_{[0,t]}(u)\,\mathrm{d}u \\
&= \left(\frac{1}{2\pi}\right)^d \int_0^t\int_{\mathbb{R}^d} \exp\left(-i\langle u, x\rangle\right)\,\exp\left(i\langle u, X(s)\rangle\right)\,\mathrm{d}u\,\mathrm{d}s. \qquad (3.8)
\end{aligned}$$

This can be justified by defining the first integral in (3.8) as

$$\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^d} \exp(-i\langle u, x\rangle)\,\exp(-\varepsilon|u|^2)\,\widehat{\mu}_{[0,t]}(u)\,\mathrm{d}u,$$

which is convergent in $L^n(\mathbb{P})$ for all $n \geq 1$ ([24], p. 271). It follows from (3.8) and the above justification that for any $x, w \in \mathbb{R}^d$, $B \in \mathcal{B}(\mathbb{R}_+)$ and all integers $n \geq 1$, we have

$$\mathbb{E}[L(x,B)]^n = (2\pi)^{-nd} \int_{B^n} \int_{\mathbb{R}^{nd}} \exp\left(-i \sum_{j=1}^{n} \langle u_j, x\rangle\right) \mathbb{E} \exp\left(i \sum_{j=1}^{n} \langle u_j, X(t_j)\rangle\right) \mathrm{d}\bar{u}\, \mathrm{d}\bar{t} \tag{3.9}$$

and for all even integers $n \geq 2$,

$$\mathbb{E}[L(x+w,B) - L(x,B)]^n = (2\pi)^{-nd} \int_{B^n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^{n} \left(\exp(-i\langle u_j, x+w\rangle) - \exp(-i\langle u_j, x\rangle)\right) \times \mathbb{E} \exp\left(i \sum_{j=1}^{n} \langle u_j, X(t_j)\rangle\right) \mathrm{d}\bar{u}\, \mathrm{d}\bar{t}, \tag{3.10}$$

where $\bar{u} = (u_1, \ldots, u_n)$, $\bar{t} = (t_1, \ldots, t_n)$ and each $u_j \in \mathbb{R}^d$, $t_j \in B$ $(j = 1, \ldots, n)$. In the coordinate notation we then write $u_j = (u_j^1, \ldots, u_j^d)$. See also (25.5) and (25.7) of Geman and Horowitz [21] for a similar justification of (3.9) and (3.10).

4. Joint continuity and Hölder conditions

In this section, we establish the joint continuity and sharp Hölder conditions in the set variable for the local times of d-dimensional α-time fractional Brownian motion X. Then we apply these results to study the irregularities of the sample paths of $X(t)$.

We use methods which are similar to those in Ehm [19] and Xiao [43,44]. The following Lemmas 4.1, 4.2 and 4.3 give the crucial estimates for the moments of the local time of α-time fractional Brownian motion. Note that the estimates in the case Y is of type A (*i.e.*, (4.2) and (4.3)) are different from the case when Y is a stable subordinator (see Lem. 4.3).

We need the fact that fractional Brownian motion W satisfies the property of strong local nondeterminism (SLND), which was proved by Pitt [34]. More precisely, for any $y_1, \ldots, y_n \in \mathbb{R}$,

$$\operatorname{Var}\big(W(y_n)|W(y_1), \ldots, W(y_{n-1})\big) \geq K \min_{0 \leq j \leq n-1} |y_n - y_j|^{2H}, \tag{4.1}$$

where $y_0 = 0$ and $K > 0$ is an absolute constant.

Lemma 4.1. *Let $X = \{X(t), t \geq 0\}$ be a d-dimensional α-time fractional Brownian motion with $d < \alpha/H$ for which $Y(t)$ is of type A. For any $h > 0$, $B = [0, h]$, $x \in \mathbb{R}^d$, any integer $n \geq 1$, we have*

$$\mathbb{E}\big[L(x,B)\big]^n \leq K^n\, h^{(1-dH/\alpha)n}\, (n!)^{dH(1+1/\alpha)}, \tag{4.2}$$

where $K > 0$ is a finite constant depending on d, H and α only.

Proof. Thanks to the strong local nondeterminism (SLND) of fractional Brownian motion (*cf.* Eq. (4.1)), Lemma A.2 and the scaling property of $p_t(x)$ (*cf.* Eq. (3.3)), the proof of Lemma 4.1 follows along a similar line of the proof of equation (2.11) in Xiao [44] with obvious modifications. We omit the details. □

Lemma 4.2. *Under the conditions of Lemma 4.1, we have that for all even integers $n \geq 2$ and $0 < \gamma < \frac{1}{2}\min\{\alpha/(Hd) - 1, 1 - H\}$, $x, w \in \mathbb{R}^d$*

$$\mathbb{E}[L(x+w,B) - L(x,B)]^n \leq K^n |w|^{n\gamma} h^{n(1-(d+\gamma)H/\alpha)} (n!)^{d+\gamma+\frac{H(d+2\gamma)}{\alpha}}, \tag{4.3}$$

where $K > 0$ is a finite constant depending on d, H, γ and α only.

Proof. Even though the arguments for proving Lemma 4.2 are similar to that of equation (2.12) in Xiao [44], several essential modifications are needed.

By (3.10) and the elementary inequality

$$|\mathrm{e}^{\mathrm{i}u} - 1| \le 2^{1-\gamma}|u|^\gamma \quad \text{for all} \quad u \in \mathbb{R} \text{ and } \quad 0 < \gamma < 1,$$

we see that for any even integer $n \ge 2$ and any $0 < \gamma < 1$,

$$\mathbb{E}[L(x+w, B) - L(x, B)]^n \le |w|^{n\gamma} \int_{B^n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n |u_j|^\gamma \mathbb{E} \exp\left(i \sum_{j=1}^n \langle u_j, X(t_j)\rangle\right) \mathrm{d}\bar{u}\mathrm{d}\bar{t}. \tag{4.4}$$

By making the change of variables $t_j = hs_j$, $j = 1, \dots, n$ and $u_j = h^{-H/\alpha} v_j$, $j = 1, \dots, n$ and changing the letters s, v back to t, u, the self-similarity of X implies that the right-hand side of (4.4) equals

$$|w|^{n\gamma}\, h^{n(1-(d+\gamma)H/\alpha)} \int_{[0,1]^n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n |u_j|^\gamma \mathbb{E} \exp\left(i \sum_{j=1}^n \langle u_j, X(t_j)\rangle\right) \mathrm{d}\bar{u}\mathrm{d}\bar{t}. \tag{4.5}$$

We fix any distinct $t_1, \dots, t_n \in [0, 1]$ satisfying

$$0 = t_0 < t_1 < t_2 < \dots < t_n, \tag{4.6}$$

and consider the inside integral in (4.5). Since for any $0 < \gamma < 1$, $|a + b|^\gamma \le |a|^\gamma + |b|^\gamma$, we have

$$\prod_{j=1}^n |u_j|^\gamma \le \sum{}' \prod_{j=1}^n |u_j^{k_j}|^\gamma, \tag{4.7}$$

where the summation $\sum'$ is taken over all $(k_1, \dots, k_n) \in \{1, \dots, d\}^n$.

Let us fix a sequence $(k_1, \dots, k_n) \in \{1, \dots, d\}^n$, and consider the integral

$$\begin{aligned} J &= \int_{\mathbb{R}^{nd}} \prod_{j=1}^n |u_j^{k_j}|^\gamma \, \mathbb{E} \exp\left(i \sum_{j=1}^n \langle u_j, X(t_j)\rangle\right) \mathrm{d}\bar{u} \\ &= \int_{\mathbb{R}^{nd}} \prod_{j=1}^n |u_j^{k_j}|^\gamma \mathbb{E} \exp\left(i \sum_{\ell=1}^d \sum_{j=1}^n u_j^\ell W_\ell(Y_\ell(t_j))\right) \mathrm{d}\bar{u}, \end{aligned} \tag{4.8}$$

since $X(t_j) = (W_1(Y_1(t_j)), \dots, W_d(Y_d(t_j)))$. Now, we condition on $Y_\ell(t_j) = y_{\ell j}$, $\ell = 1, \dots, d, j = 1, \dots, n$. By independence of the processes Y_ℓ we have that the density of

$$(Y_\ell(t_j) = y_{\ell j} : \ \ell = 1, \dots, d, j = 1, \dots, n)$$

is given by

$$\widetilde{\mathbf{p}}_{t_1,\dots,t_n}(y_{11}, \dots, y_{1n}, y_{21}, \dots, y_{2n}, \dots, y_{d1}, \dots, y_{dn}) = \prod_{\ell=1}^d \prod_{j=1}^n p_{t_j - t_{j-1}}(y_{\ell j} - y_{\ell(j-1)}).$$

Let $\bar t = (t_1,\dots,t_n)$ and $\bar y = (y_{11},\dots,y_{1n},y_{21},\dots,y_{2n},\dots,y_{d1},\dots,y_{dn})$. By conditioning we have

$$J=\int_{\mathbb{R}^{nd}}\int_{\mathbb{R}^{nd}}\prod_{j=1}^n |u_j^{k_j}|^\gamma\,\mathbb{E}\exp\left(\mathrm{i}\sum_{\ell=1}^d\sum_{j=1}^n u_j^\ell W_\ell(y_{\ell j})\right)\,\widetilde{\mathbf{p}}_{\bar t}(\bar y)\,\mathrm{d}\bar u\,\mathrm{d}\bar y.$$

For any fixed $\bar y\in\mathbb{R}^{nd}$, let

$$\begin{aligned}I&=\int_{\mathbb{R}^{nd}}\prod_{j=1}^n |u_j^{k_j}|^\gamma\,\mathbb{E}\exp\left(\mathrm{i}\sum_{\ell=1}^d\sum_{j=1}^n u_j^\ell W_\ell(y_{\ell j})\right)\,\mathrm{d}\bar u\\&=\int_{\mathbb{R}^{nd}}\prod_{j=1}^n |u_j^{k_j}|^\gamma\exp\left(-\frac12\mathrm{Var}\left(\sum_{\ell=1}^d\sum_{j=1}^n u_j^\ell W_\ell(y_{\ell j})\right)\right)\,\mathrm{d}\bar u.\end{aligned}$$

Then by a generalized Hölder's inequality and Lemma A.4, we have

$$\begin{aligned}I&\le\prod_{j=1}^n\left[\int_{\mathbb{R}^{nd}}|u_j^{k_j}|^{n\gamma}\exp\left(-\frac12\mathrm{Var}\left(\sum_{\ell=1}^d\sum_{j=1}^n u_j^\ell W_\ell(y_{\ell j})\right)\right)\,\mathrm{d}\bar u\right]^{1/n}\\&\le\frac{(2\pi)^{\frac{nd-1}{2}}}{(\det\mathrm{Cov}(W_\ell(y_{\ell j}),1\le\ell\le d,\ 1\le j\le n))^{1/2}}\int_{\mathbb{R}}|v|^{n\gamma}\,\mathrm{e}^{-v^2/2}\,dv\prod_{j=1}^n\frac{1}{\sigma_{k_j,j}^\gamma}\\&\le\frac{K^n(n!)^\gamma}{(\det\mathrm{Cov}(W_\ell(y_{\ell j}),1\le\ell\le d,\ 1\le j\le n))^{1/2}}\prod_{j=1}^n\frac{1}{\sigma_{k_j,j}^\gamma}\\&=\frac{K^n(n!)^\gamma}{\prod_{\ell=1}^d(\det\mathrm{Cov}(W_\ell(y_{\ell j}),\ 1\le j\le n))^{1/2}}\prod_{j=1}^n\frac{1}{\sigma_{k_j,j}^\gamma},\end{aligned}$$

where

$$\begin{aligned}\sigma_{k_j,j}^2&=\mathrm{Var}(W_{k_j}(y_{k_j,j})\big|\ W_\ell(y_{\ell,i}):\quad \ell\ne k\ \text{ or }\ \ell=k_j,\ i\ne j)\\&=\mathrm{Var}(W_{k_j}(y_{k_j,j})\big|\ W_{k_j}(y_{k_j,i}):\quad i=0\ \text{ or }\ i\ne j).\end{aligned}\tag{4.9}$$

For any $\ell\in\{1,\dots,d\}$ and any $y_{\ell,1},\dots,y_{\ell,n}$, there exists a permutation π_ℓ of $\{1,\dots,n\}$ such that

$$y_{\ell,\pi_\ell(1)}\le y_{\ell,\pi_\ell(2)}\le\dots\le y_{\ell,\pi_\ell(n)}.$$

Hence, if we write $k_j=\ell$, then by SLND of fractional Brownian motion (4.1),

$$\begin{aligned}\sigma_{\ell,j}^2&=\mathrm{Var}(W_\ell(y_{\ell,j})|\ W_\ell(y_{\ell,i}):\quad i=0\ \text{ or }\ i\ne j)\\&\ge K\ \min\{|y_{\ell,\pi_\ell(j)}-y_{\ell,\pi_\ell(j-1)}|^{2H},\ |y_{\ell,\pi_\ell(j+1)}-y_{\ell,\pi_\ell(j)}|^{2H}\}.\end{aligned}\tag{4.10}$$

Hence

$$\prod_{j=1}^n\frac{1}{\sigma_{k_j,j}^\gamma}\le K^n\prod_{\ell=1}^d\prod_{j=1}^n\frac{1}{\min\{|y_{\ell,\pi_\ell(j)}-y_{\ell,\pi_\ell(j-1)}|,\ |y_{\ell,\pi_\ell(j+1)}-y_{\ell,\pi_\ell(j)}|\}^{H\eta_{\ell,j}\gamma}},$$

where $\eta_{\ell,j}=1$ if $k_j=\ell$ and $\eta_{\ell,j}=0$ otherwise. Note that

$$\sum_{\ell=1}^d\sum_{j=1}^n\eta_{\ell,j}=n.\tag{4.11}$$

Since

$$\begin{aligned}
&\prod_{\ell=1}^{d}\prod_{j=1}^{n}\frac{1}{\min\{|y_{\ell,\pi_\ell(j)}-y_{\ell,\pi_\ell(j-1)}|,\,|y_{\ell,\pi_\ell(j+1)}-y_{\ell,\pi_\ell(j)}|\}^{H\eta_{\ell,j}\gamma}}\\
&\le \prod_{\ell=1}^{d}\prod_{j=1}^{n}\left(\frac{1}{|y_{\ell,\pi_\ell(j)}-y_{\ell,\pi_\ell(j-1)}|^{H\eta_{\ell,j}\gamma}}+\frac{1}{|y_{\ell,\pi_\ell(j+1)}-y_{\ell,\pi_\ell(j)}|^{H\eta_{\ell,j}\gamma}}\right)\\
&=\sum{}^{''}\prod_{\ell=1}^{d}\prod_{j=1}^{n}\left(\frac{1}{|y_{\ell,\pi_\ell(j)}-y_{\ell,\pi_\ell(j-1)}|^{H\delta_{\ell,j}\gamma}}\right),
\end{aligned}$$

where the summation $\sum''$ is taken over 2^{nd} terms and $\delta_{\ell,j}\in\{0,1,2\}$ and, thanks to (4.11),

$$\sum_{\ell=1}^{d}\sum_{j=1}^{n}\delta_{\ell,j}\le 2\sum_{\ell=1}^{d}\sum_{j=1}^{n}\eta_{\ell,j}=2n, \tag{4.12}$$

we obtain

$$\prod_{j=1}^{n}\frac{1}{\sigma_{k_j,j}^{\gamma}}\le K^n\sum{}^{''}\prod_{\ell=1}^{d}\prod_{j=1}^{n}\left(\frac{1}{|y_{\ell,\pi_\ell(j)}-y_{\ell,\pi_\ell(j-1)}|^{H\delta_{\ell,j}\gamma}}\right). \tag{4.13}$$

Now we go back to estimating J. Let

$$\Delta_{\pi_\ell}=\big\{(y_{\ell,1},\dots,y_{\ell,n}):\ y_{\ell,\pi_\ell(1)}\le y_{\ell,\pi_\ell(2)}\le\dots\le y_{\ell,\pi_\ell(n)}\big\}.$$

Then

$$J\le K^n(n!)^\gamma\sum_{\{\pi_\ell\}}\sum{}^{''}\prod_{\ell=1}^{d}\left[\int_{\mathbb{R}^n\cap\Delta_{\pi_\ell}}\prod_{j=1}^{n}\left(\frac{p_{t_j-t_{j-1}}(y_{\ell,j}-y_{\ell,j-1})}{(y_{\ell,\pi_\ell(j)}-y_{\ell,\pi_\ell(j-1)})^{H(1+\delta_{\ell,j}\gamma)}}\right)\mathrm{d}\bar{y}\right]. \tag{4.14}$$

Fix $\ell\in\{1,\dots,d\}$ and a term in $\sum''$, we proceed to estimate the integral

$$\int_{\mathbb{R}^n\cap\Delta_{\pi_\ell}}\prod_{j=1}^{n}\frac{p_{t_j-t_{j-1}}(y_{\ell,j}-y_{\ell,j-1})}{(y_{\ell,\pi_\ell(j)}-y_{\ell,\pi_\ell(j-1)})^{H(1+\delta_{\ell,j}\gamma)}}\,\mathrm{d}\bar{y}_\ell, \tag{4.15}$$

where $\mathrm{d}\bar{y}_\ell=\mathrm{d}y_{\ell,1}\dots\mathrm{d}y_{\ell,n}$. Note that we can write (4.15) as

$$\int_{\mathbb{R}^n\cap\Delta_{\pi_\ell}}\prod_{j=1}^{n}\frac{p_{t_{\pi_\ell(j)}-t_{\pi_\ell(j)-1}}(y_{\ell,\pi_\ell(j)}-y_{\ell,\pi_\ell(j)-1})}{(y_{\ell,\pi_\ell(j)}-y_{\ell,\pi_\ell(j-1)})^{H(1+\delta_{\ell,j}\gamma)}}\mathrm{d}\bar{y}_\ell. \tag{4.16}$$

It will be helpful to notice the difference of $y_{\ell,\pi_\ell(j)-1}$ in the numerator and $y_{\ell,\pi_\ell(j-1)}$ in the denominator.

Since $p_t(x)=t^{-1/\alpha}p_1(x/t^{1/\alpha})$, for all $t>0$, $x\in\mathbb{R}$. Now (4.16) can be written as

$$\int_{\mathbb{R}^n\cap\Delta_{\pi_\ell}}\prod_{j=1}^{n}\left(\frac{1}{(t_{\pi_\ell(j)}-t_{\pi_\ell(j)-1})^{1/\alpha}}\frac{p_1\left(\frac{y_{\ell,\pi_\ell(j)}-y_{\ell,\pi_\ell(j)-1}}{(t_{\pi_\ell(j)}-t_{\pi_\ell(j)-1})^{1/\alpha}}\right)}{(y_{\ell,\pi_\ell(j)}-y_{\ell,\pi_\ell(j-1)})^{H(1+\delta_{\ell,j}\gamma)}}\right)\mathrm{d}\bar{y}_\ell. \tag{4.17}$$

We now integrate in the order $\mathrm{d}y_{\ell,\pi_\ell(n)},\mathrm{d}y_{\ell,\pi_\ell(n-1)},\dots,\mathrm{d}y_{\ell,\pi_\ell(1)}$.

A change of variables

$$y_{\ell,\pi_\ell(n)}-y_{\ell,\pi_\ell(n)-1}=(t_{\pi_\ell(n)}-t_{\pi_\ell(n)-1})^{1/\alpha}\,z_n$$

gives

$$\int_{y_{\ell,\pi_\ell(n-1)}}^{\infty} \frac{1}{(t_{\pi_\ell(n)} - t_{\pi_\ell(n)-1})^{1/\alpha}} \frac{p_1\left(\frac{y_{\ell,\pi_\ell(n)}-y_{\ell,\pi_\ell(n)-1}}{(t_{\pi_\ell(n)}-t_{\pi_\ell(n)-1})^{1/\alpha}}\right)}{(y_{\ell,\pi_\ell(n)} - y_{\ell,\pi_\ell(n-1)})^{H(1+\delta_{\ell,j}\gamma)}} \,\mathrm{d}y_{\ell,\pi_\ell(n)}$$
$$= \frac{1}{\left(t_{\pi_\ell(n)} - t_{\pi_\ell(n)-1}\right)^{\frac{H(1+\delta_{\ell,j}\gamma)}{\alpha}}} \int_{s_n}^{\infty} \frac{p_1(z_n)}{(z_n - s_n)^{H(1+\delta_{\ell,j}\gamma)}} \,\mathrm{d}z_n, \tag{4.18}$$

where

$$s_n = \frac{y_{\ell,\pi_\ell(n-1)} - y_{\ell,\pi_\ell(n)-1}}{(t_{\pi_\ell(n)} - t_{\pi_\ell(n)-1})^{1/\alpha}}.$$

Since we have assumed $0 < \gamma < \frac{1}{2}(1-H)$, so $H(1+\delta_{\ell,j}\gamma) < 1$ for all ℓ, j. Thus

$$\int_{s_n}^{\infty} \frac{p_1(z_n)}{(z_n - s_n)^{H(1+\delta_{\ell,j}\gamma)}} \mathrm{d}z_n \le K,$$

where K is a constant independent of s_n. This can be verified directly by splitting the interval $[s_n, \infty)$ into $[s_n, s_n+1]$ and $[s_n+1, \infty)$.

Continuing this procedure we derive

$$\int_{\mathbb{R}^n \cap \Delta_{\pi_\ell}} \prod_{j=1}^{n} \frac{p_{t_{\pi_\ell(j)}-t_{\pi_\ell(j)-1}}(y_{\ell,\pi_\ell(j)} - y_{\ell,\pi_\ell(j)-1})}{(y_{\ell,\pi_\ell(j)} - y_{\ell,\pi_\ell(j-1)})^{H(1+\delta_{\ell,j}\gamma)}} \,\mathrm{d}\bar{y}_\ell \le K^n \prod_{j=1}^{n} \frac{1}{\left(t_{\pi_\ell(j)} - t_{\pi_\ell(j)-1}\right)^{\frac{H(1+\delta_{\ell,j}\gamma)}{\alpha}}}. \tag{4.19}$$

Combining this inequality with equation (4.14) gives

$$\begin{aligned} J &\le K^n\,(n!)^\gamma \sum_{\pi_1,\dots,\pi_d} \prod_{\ell=1}^{d}\prod_{j=1}^{n} \frac{1}{\left(t_{\pi_\ell(j)} - t_{\pi_\ell(j)-1}\right)^{\frac{H(1+\delta_{\ell,j}\gamma)}{\alpha}}} \\ &\le K^n\,(n!)^\gamma \sum_{\pi_1,\dots,\pi_d} \prod_{j=1}^{n} \frac{1}{(t_j - t_{j-1})^{\frac{H}{\alpha}(d+\gamma\sum_{\ell=1}^{d}\delta_{\ell,\pi_\ell^{-1}(j)})}} \\ &\le K^n\,(n!)^{\gamma+d} \prod_{j=1}^{n} \frac{1}{(t_j - t_{j-1})^{\frac{H}{\alpha}(d+\gamma\epsilon_j)}}, \end{aligned} \tag{4.20}$$

where $0 \le \epsilon_j \le 2d$ and $\sum_{j=1}^{n} \epsilon_j \le 2n$, thanks to (4.12).

Hence we have shown

$$\begin{aligned} \mathbb{E}\,[L(x+w,B) - L(x,B)]^n &\le K^n\,|w|^{n\gamma} h^{n(1-(d+\gamma)H/\alpha)} (n!)^{d+\gamma+1} \\ &\quad \times \int_{0\le t_1\le\ldots\le t_n\le 1} \frac{\mathrm{d}\bar{t}}{\prod_{j=1}^{n}(t_j - t_{j-1})^{\frac{H}{\alpha}(d+\gamma\epsilon_j)}} \\ &\le K^n\,|w|^{n\gamma} h^{n(1-(d+\gamma)H/\alpha)} (n!)^{d+\gamma+1} \\ &\quad \times \frac{\prod_{j=1}^{n}\Gamma(1-\frac{H}{\alpha}(d+\gamma\epsilon_j))}{\Gamma\left(1+n-\sum_{j=1}^{n}\frac{H}{\alpha}(d+\gamma\epsilon_j)\right)}. \end{aligned} \tag{4.21}$$

In the above we use Lemma A.3 with the fact that $\frac{H}{\alpha}(d+\gamma\epsilon_j)<1$ because γ satisfies $0<\gamma<\frac{1}{2}(\alpha/(Hd)-1)$. It is now clear that (4.3) follows from (4.21) and Stirling's formula. This completes the proof. □

We have similar bounds for the moments of the local time in the case when Y is a stable subordinator. It should be noted that the power of $n!$ in (4.22) is different from that in Lemma 4.1. This will lead to different forms of laws of the iterated logarithm for the local times in the two cases.

Lemma 4.3. *Let $X=\{X(t),\, t\geq 0\}$ be a d-dimensional α-time fractional Brownian motion with $d<\alpha/H$ for which $Y(t)$ is a stable subordiantor with index $\alpha<1$. For any $h>0$, $B=[0,h]$, $x,w\in\mathbb{R}^d$, any even integer $n\geq 2$ and any $0<\gamma<\frac{1}{2}\min\{\alpha/(Hd)-1,\, 1-H\}$, we have*

$$\mathbb{E}[L(x,B)]^n \quad\leq\quad K^n h^{(1-dH/\alpha)n}(n!)^{dH/\alpha}, \tag{4.22}$$

$$\mathbb{E}[L(x+w,B)-L(x,B)]^n \quad\leq\quad K^n|w|^{n\gamma}h^{n(1-(d+\gamma)H/\alpha)}(n!)^{\gamma+\frac{H(d+2\gamma)}{\alpha}}, \tag{4.23}$$

where $K>0$ is a finite constant depending on d, H, γ and α only.

Proof. The proof of (4.22) is similar to the proof of Lemma 4.1. However, since the sample function $Y(t)$ is increasing, for $t_1,\ldots,t_n$ that satisfy (4.6), the corresponding $y_j=Y(t_j)$ $(j=1,\ldots,n)$ satisfy $y_1<\ldots<y_n$, which leads to some clear modifications to the proof. Equation (4.23) follows similarly from the proof of Lemma 4.2. □

Now we are ready to prove the joint continuity result of local times.

Theorem 4.1. *If $d<\alpha/H$, then almost surely $X=\{X(t),\, t\geq 0\}$ has a jointly continuous local time $L(x,\, t)$ $(x\in\mathbb{R}^d, t\geq 0)$.*

Proof. The proof follows from Lemmas 4.1–4.3 and Kolmogorov's continuity theorem. □

Remark 4.1. For $d=1$, $H=1/2$ and $\alpha>1$ this result was proved by Nane [30]. Theorem 4.1 implies that for $d=1$, $H=1/2$ and $\alpha>1/2$, almost surely $X(t)$ $(t\geq 0)$ has a jointly continuous local time $L(x,\, t)$ $(x\in\mathbb{R}^d,\, t\geq 0)$. Hence Theorem 4.1 is an improvement of the results in [30] and an extension of results in [44] obtained for multidimensional iterated Brownian motion.

The following tail probability estimates are used in deriving the sharp Hölder conditions in the set variable of the local times of α-time fractional Brownian motion.

Lemma 4.4. *Suppose Y is not a subordinator. For any $\lambda>0$, there exists a finite constant $A>0$, depending on λ, d, H and α only, such that for all $\tau\geq 0$, $h>0$, $B=[\tau,\tau+h]$, $x,w\in\mathbb{R}^d$, all $0<\gamma<\frac{1}{2}\min\{\alpha/(Hd)-1,\, 1-H\}$, and all $u>0$*

$$\mathbb{P}\Big\{L(x+X(\tau),B)\geq Ah^{1-dH/\alpha}u^{dH(1+1/\alpha)}\Big\} \quad\leq\quad \exp(-\lambda u), \tag{4.24}$$

$$\begin{aligned}\mathbb{P}\Big\{|L(x+w+X(\tau),B)-L(x+X(\tau),B)| \quad&\geq\quad A|w|^\gamma h^{1-(d+\gamma)H/\alpha}u^{C(H,\alpha)}\Big\}\\ &\leq\quad \exp(-\lambda u),\end{aligned} \tag{4.25}$$

where $C(H,\alpha)=d+\gamma+\frac{H(d+2\gamma)}{\alpha}$.

Proof. Since $X=\{X(t),t\geq 0\}$ has stationary increments, *i.e.*, for any $\tau\geq 0$, the processes $\{X(t+\tau)-X(\tau),t\geq 0\}$ and X have the same finite dimensional distributions. Hence Lemmas 4.1 and 4.2 can be reformulated as follows: For any $\tau \geq 0$, $h > 0$, $B = [\tau, \tau + h]$, $x, w \in \mathbb{R}^d$, any even integer $n \geq 2$ and any $0 < \gamma < \frac{1}{2}\min\{\alpha/(Hd) - 1, 1 - H\}$, we have

$$\mathbb{E}[L(x + X(\tau), B)]^n \leq K^n h^{(1-dH/\alpha)n}(n!)^{dH(1+1/\alpha)}, \tag{4.26}$$

$$\mathbb{E}[L(x + w + X(\tau), B) - L(x + X(\tau), B)]^n \leq K^n |w|^{n\gamma} h^{n(1-(d+\gamma)H/\alpha)}(n!)^{d+\gamma+\frac{H(d+2\gamma)}{\alpha}}, \tag{4.27}$$

where $K > 0$ is a finite constant depending on d, H, γ and α only.

Now, Lemma 4.4 is a direct consequence of (4.26), (4.27) and the Chebyshev's inequality. □

Lemma 4.5. *Suppose Y is a stable subordinator of index $\alpha < 1$. For any $\lambda > 0$, there exists a finite constant $A > 0$, depending on λ, d, H and α only, such that for all $\tau \geq 0$, $h > 0$, $B = [\tau, \tau + h]$, $x, w \in \mathbb{R}^d$, all $0 < \gamma < \frac{1}{2}\min\{\alpha/(Hd) - 1, 1 - H\}$, and $u > 0$*

$$\mathbb{P}\Big\{L(x + X(\tau), B) \geq Ah^{1-dH/\alpha}u^{dH/\alpha}\Big\} \leq \exp(-\lambda u), \tag{4.28}$$

$$\begin{aligned}\mathbb{P}\Big\{|L(x + w + X(\tau), B) - L(x + X(\tau), B)| &\geq A|w|^{\gamma} h^{1-(d+\gamma)H/\alpha}u^{D(H,\alpha)}\Big\} \\ &\leq \exp(-\lambda u),\end{aligned} \tag{4.29}$$

where $D(H, \alpha) = \gamma + \frac{H(d+2\gamma)}{\alpha}$.

The proof of Lemma 4.5 follows the same idea as that in the proof of Lemma 4.4, with an application of Lemma 4.3. We omit it here.

The next lemma shows that process the real-valued process $Z(t) = W(Y(t))$ has heavy tails as in the case of Y. This might make this process more desirable, since it has heavy tails without independence of increments and with the stationarity of the increments. We need the following lemma to prove Lemma 4.7 for a two sided estimate of $\mathbb{P}\Big\{\sup_{0\leq t\leq 1}|Z(t)| > u\Big\}$, which will be useful in proving Theorem 4.2.

Lemma 4.6. *Let $d = 1$, $0 < H < 1$ and $0 < \alpha \leq 2$, and let $0 \leq a \leq b$ then*

$$\lim_{u\to\infty} \frac{\mathbb{P}\Big\{|Z(b) - Z(a)| > u\Big\}}{u^{-\alpha/H}} = C(b - a)$$

for some finite constant $C > 0$.

Proof. By using the stationarity of the increments and the self-similarity of W and Y we get

$$\begin{aligned}\mathbb{P}\{|Z(b) - Z(a)| > u\} &= \mathbb{P}\{|W(Y(b - a))| > u\} \\ &= \mathbb{P}\Big\{(b - a)^{H/\alpha}|Y(1)|^H|W(1)| > u\Big\} \\ &= \int_{-\infty}^{\infty} \mathbb{P}\Big\{(b - a)^{H/\alpha}|Y(1)|^H|s| > u\Big\} f^H(s)\mathrm{d}s \\ &= \int_{-\infty}^{\infty} \mathbb{P}\Big\{|Y(1)| > u^{1/H}(b - a)^{-1/\alpha}|s|^{-1/H}\Big\} f^H(s)\mathrm{d}s,\end{aligned} \tag{4.30}$$

here $f^H(s) = \frac{\mathrm{e}^{-s^2/2}}{\sqrt{2\pi}}$ is the density of $W(1)$.

The following is a well-known result

$$\lim_{u\to\infty}\frac{\mathbb{P}\Big\{|Y(1)|>u\Big\}}{u^{-\alpha}}=k$$

for some $k>0$; see, for example, Bertoin [9]. Hence, for fixed $a\le b$, $s\in\mathbb{R}$, and as $x\to\infty$

$$\begin{aligned}P\Big\{|Y(1)|>x^{1/H}(b-a)^{-1/\alpha}|s|^{-1/H}\Big\}&\sim k(x^{1/H}(b-a)^{-1/\alpha}|s|^{-1/H})^{-\alpha}\\&=kx^{-\alpha/H}(b-a)|s|^{\alpha/H}.\end{aligned}\tag{4.31}$$

Now we apply the Dominated Convergence Theorem in equation (4.30) to get

$$\begin{aligned}\lim_{x\to\infty}\frac{\mathbb{P}\Big\{|Z(b)-Z(a)|>x\Big\}}{x^{-\alpha/H}}&=k(b-a)\int_{-\infty}^{\infty}|s|^{\alpha/H}f^H(s)\mathrm{d}s\\&=k(b-a)2^{\alpha/2H}\pi^{-1/2}\Gamma((\alpha+H)/2H).\end{aligned}\tag{4.32}$$

Hence the constant in the theorem is $C=k2^{\alpha/2H}\pi^{-1/2}\Gamma((\alpha+H)/2H)$. □

Lemma 4.7. *Let $d=1$, $0<H<1$ and $0<\alpha\le 2$. There exists a finite constant $K>0$ such that for $u\ge 1$,*

$$K^{-1}u^{-\alpha/H}\le\mathbb{P}\Big\{\sup_{0\le t\le 1}|Z(t)|>u\Big\}\le Ku^{-\alpha/H}.\tag{4.33}$$

Proof. Let $S(t)\equiv\sup_{0\le s\le t}|Y(s)|$. Then, by using the scaling property of W and conditioning, we have

$$\begin{aligned}\mathbb{P}\Big\{\sup_{0\le t\le 1}|Z(t)|>u\Big\}&\le\mathbb{P}\Big\{\sup_{|x|\le S(1)}|W(x)|>u\Big\}\\&=\mathbb{E}\bigg(\mathbb{P}\Big\{\sup_{|x|\le 1}|W(x)|>\frac{u}{S(1)^H}\Big|Y\Big\}\bigg).\end{aligned}\tag{4.34}$$

It is well known that, for any $\varepsilon>0$, there exists a finite constant K such that for all $u>0$

$$\mathbb{P}\left\{\sup_{|x|\le 1}|W(x)|>u\right\}\le K\,\exp\left(-\frac{u^2}{2+\varepsilon}\right).\tag{4.35}$$

See, for example, Lifshits [26], Section 14. Consequently

$$\mathbb{P}\left\{\sup_{0\le t\le 1}|Z(t)|>u\right\}\le K\,\mathbb{E}\exp\left(-\frac{u^2}{(2+\varepsilon)S(1)^{2H}}\right).\tag{4.36}$$

Since, for all $x>0$, the function $g(x)=\exp\big(-\frac{u^2}{(2+\varepsilon)x^{2H}}\big)$ has positive derivative

$$g'(x)=\frac{2H\,u^2}{2+\varepsilon}\,\exp\left(-\frac{u^2}{(2+\varepsilon)x^{2H}}\right)\frac{1}{x^{2H+1}},$$

we derive

$$\mathbb{P}\left\{\sup_{0\le t\le 1}|Z(t)|>u\right\}\le Ku^2\int_0^\infty \exp\left(-\frac{u^2}{(2+\varepsilon)x^{2H}}\right)\frac{1}{x^{2H+1}}\mathbb{P}\Big\{S(1)>x\Big\}\,\mathrm{d}x$$

$$=K\int_0^\infty \exp\left(-\frac{1}{(2+\varepsilon)y^{2H}}\right)\frac{1}{y^{2H+1}}\mathbb{P}\Big\{S(1)>u^{1/H}y\Big\}\,\mathrm{d}y, \tag{4.37}$$

where the last inequality follows from the change of variable $x=u^{1/H}y$. Now by using the well-known estimate

$$\mathbb{P}\big\{S(1)>y\big\}\le K\left(1\wedge y^{-\alpha}\right),\qquad \forall\, y>0,$$

we obtain that for all $u>1$,

$$\int_0^\infty \exp\left(-\frac{1}{(2+\varepsilon)y^{2H}}\right)\frac{1}{y^{2H+1}}\mathbb{P}\left\{S(1)>u^{1/H}y\right\}\,\mathrm{d}y\le K\,u^{-\alpha/H},$$

where $K>0$ is a finite constant. This and (4.37) together give the upper bound in (4.33).

The lower bound in (4.33) follows from Lemma 4.6 and the fact that

$$\mathbb{P}\big\{|Z(1)|>u\big\}=\sup_{0\le t\le 1}\mathbb{P}\big\{|Z(t)|>u\big\}\le\mathbb{P}\bigg\{\sup_{0\le t\le 1}|Z(t)|>u\bigg\}. \tag{4.38}$$

The first equality in Equation (4.38) follows from the fact that the function

$$t\to\mathbb{P}\big\{|Z(t)|>u\big\}=\mathbb{P}\Big\{|Z(1)|>t^{-H/\alpha}u\Big\}$$

is an increasing function for $t\in(0,1]$. □

The following theorems are for laws of the iterated logarithm for the maximum local time $L^*([\tau,\tau+h])=\sup_{x\in\mathbb{R}^d}L(x,[\tau,\tau+h])$ and uniform Hölder conditions of local times of α-time fractional Brownian motions.

Theorem 4.2. *Let $d<\alpha/H$ and suppose Y is not a subordinator.*

(1) *There exists a finite constant $K>0$ such that for any $\tau\ge 0$ with probability* 1

$$\limsup_{h\to 0}\sup_{x\in\mathbb{R}^d}\frac{L(x,\tau+h)-L(x,\tau)}{h^{1-dH/\alpha}(\log\log h^{-1})^{dH(1+1/\alpha)}}\le K. \tag{4.39}$$

(2) *For any $T>0$, there exists a positive constant K such that almost surely*

$$\limsup_{h\to 0}\sup_{0\le t\le T}\sup_{x\in\mathbb{R}^d}\frac{L(x,t+h)-L(x,t)}{h^{1-dH/\alpha}(\log 1/h)^{dH(1+1/\alpha)}}\le K. \tag{4.40}$$

Proof. Equation (4.39) follows from Lemma 4.7 and a chaining argument as that in the proof of Theorem 2 in [44]. The proof of equation (4.40), using Lemma 4.4, is very similar to that of Xiao [44], Theorem 3 and Ehm [19], Theorem 2.1. We omit the details. □

Theorem 4.3. *Let $d<\alpha/H$ and suppose Y is a stable subordinator of index $\alpha<1$.*

(1) *There exists a finite constant $K>0$ such that for any $\tau\ge 0$ with probability* 1

$$\limsup_{h\to 0}\sup_{x\in\mathbb{R}^d}\frac{L(x,\tau+h)-L(x,\tau)}{h^{1-dH/\alpha}(\log\log h^{-1})^{dH/\alpha}}\le K. \tag{4.41}$$

(2) *For any $T > 0$, there exists a finite constant $K > 0$ such that almost surely*

$$\limsup_{h\to 0} \sup_{0\le t\le T} \sup_{x\in\mathbb{R}^d} \frac{L(x,t+h)-L(x,t)}{h^{1-dH/\alpha}(\log h^{-1})^{dH/\alpha}} \le K. \tag{4.42}$$

The Hölder conditions for the local time of a stochastic process $X(t)$ are closely related to the irregularity of the sample paths of $X(t)$ (*cf.* Berman [7]). In the following, we will apply Theorems 4.2 and 4.3 to derive results about the degree of oscillation of the sample paths of $X(t)$.

Theorem 4.4. *Suppose Y is not a stable subordinator. Let $X = \{X(t), t \in \mathbb{R}_+\}$ be an α-time fractional Brownian motion in $\mathbb{R}^d$ with $H < \alpha$. For any $\tau \in \mathbb{R}_+$, there exists a finite constant $K > 0$ such that*

$$\liminf_{r\to 0} \sup_{s\in B(\tau,r)} \frac{|X(s)-X(\tau)|}{r^{H/\alpha}/(\log\log 1/r)^{H(1+1/\alpha)}} \ge K \quad a.s. \tag{4.43}$$

For any interval $T \subset \mathbb{R}_+$

$$\liminf_{r\to 0} \inf_{t\in T} \sup_{s\in B(t,r)} \frac{|X(s)-X(\tau)|}{r^{H/\alpha}/(\log 1/r)^{H(1+1/\alpha)}} \ge K \quad a.s. \tag{4.44}$$

In particular, $X(t)$ is almost surely nowhere differentiable in $\mathbb{R}_+$.

Proof. Clearly, it is sufficient to consider the case of $d = 1$, where the condition of Theorem 4.2 (i.e. $1 < \alpha/H$) is fulfilled. For any interval $Q \subset \mathbb{R}_+$,

$$\lambda_1(Q) = \int_{\overline{X(Q)}} L(x,Q)\mathrm{d}x \le L^*(Q)\left(\sup_{s,t\in Q} |X(s)-X(t)|\right). \tag{4.45}$$

Let $Q = B(\tau, r)$. Then (4.43) follows immediately from (4.39) and (4.45). Similarly (4.44) follows from (4.40) and (4.45). □

Remark 4.2. Theorem 4.4 extends partially the results obtained by Nane [30].

Theorem 4.5. *Suppose Y is a stable subordinator of index $\alpha < 1$. Let $X = \{X(t), t \in \mathbb{R}_+\}$ be α-time fractional Brownian motion in $\mathbb{R}^d$ with $H < \alpha$. For any $\tau \in \mathbb{R}_+$, there exists a finite constant $K > 0$ such that*

$$\liminf_{r\to 0} \sup_{s\in B(\tau,r)} \frac{|X(s)-X(\tau)|}{r^{H/\alpha}/(\log\log 1/r)^{H/\alpha}} \ge K \quad a.s. \tag{4.46}$$

For any interval $T \subset \mathbb{R}_+$

$$\liminf_{r\to 0} \inf_{t\in T} \sup_{s\in B(t,r)} \frac{|X(s)-X(\tau)|}{r^{H/\alpha}/(\log 1/r)^{H/\alpha}} \ge K \quad a.s. \tag{4.47}$$

In particular, $X(t)$ is almost surely nowhere differentiable in $\mathbb{R}_+$.

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A. Appendix

As an appendix, we provide the following lemmas, which are used in the proofs of our main results in Section 4. Lemma A.1 is from Xiao [44], which is used to prove Lemma A.2.

Lemma A.1. *Let $0<\gamma<1$ be a constant. Then for any integer $n\geq 1$ and any $x_1,\ldots,x_n\in\mathbb{R}$, we have*

$$\int_0^1 \frac{1}{\min\{|x-x_j|^\gamma, j=1,\ldots,n\}}\,\mathrm{d}x \leq K\,n^\gamma,$$

where $K>0$ is a finite constant depending only on γ.

Lemma A.2. *Let $0<\gamma<1$ be a constant. Then for any integer $n\geq 1$ and any $x_1,\ldots,x_n\in\mathbb{R}$, we have*

$$\int_{\mathbb{R}} \frac{p_1(x)}{\min\{|x-x_j|^\gamma, j=1,\ldots,n\}}\,\mathrm{d}x \leq K\,n^\gamma, \tag{A.1}$$

where $K>0$ is a finite constant depending only on γ and α.

Proof. We recall the following asymptotic bounds from [37] for the stable density function $p_1(x)$ as $x\to\infty$ (the asymptotics for the case $x\to-\infty$ are obtained by changing x to $-x$). For $0<\alpha<1$:

$$p_1(x)\leq K\,x^{-(1+\alpha)}, \quad \text{as} \quad x\to\infty.$$

For $\alpha=1$ and $\beta=0$ (this is the symmetric Cauchy case):

$$p_1(x)\leq K\,x^{-2} \;\; \textit{as} \;\; x\to\infty.$$

For $\alpha>1$ and $-1<\beta<1$:

$$p_1(x)\leq Kx^{-(1+\alpha)}, \quad \text{as} \quad x\to\infty.$$

For $\alpha>1$ and $\beta=-1,1$:

$$p_1(x)\leq K\,\max\Big\{x^{-(1+\alpha)},\, x^{-1+\alpha/2(\alpha-1)}\exp\big(-c(\alpha)x^{\alpha/(\alpha-1)}\big)\Big\}, \quad \text{as} \quad x\to\infty.$$

Now we observe that the left-hand side of (A.1) can be written as

$$\begin{aligned}\sum_{l\in\mathbb{Z}}\int_l^{l+1} \frac{p_1(x)}{\min\{|x-x_j|^\gamma, j=1,\ldots,n\}}\,\mathrm{d}x &\leq \max_{|x|\leq M} p_1(x)\int_{-M}^{M}\frac{1}{\min\{|x-x_j|^\gamma, j=1,\ldots,n\}}\,\mathrm{d}x\\ &+\sum_{|l|>M}\max_{l\leq x\leq l+1} p_1(x)\int_0^1\frac{1}{\min\{|x+l-x_j|^\gamma, j=1,\ldots,n\}}\,\mathrm{d}x. \end{aligned}\tag{A.2}$$

It can be verified that equation (A.1) follows from (A.2), the asymptotics of $p_1(x)$ and Lemma A.1. □

Lemma A.3 is taken from Ehm [19] and Lemma A.4 is due to Cuzick and DuPreez [14] (the current form is from Khoshnevisan and Xiao [25]).

Lemma A.3. *For any integer $n\geq 1$, and $\beta_j\in(0,1)$ for $1\leq j\leq n$, for all $h>0$, we have*

$$\int_{0\leq x_1\leq x_2\leq\ldots\leq x_n\leq h}\prod_{j=1}^n\frac{1}{(x_j-x_{j-1})^{\beta_j}}\mathrm{d}x_1\ldots\mathrm{d}x_n = h^{n-\sum_{j=1}^n\beta_j}\frac{\prod_{j=1}^n\Gamma(1-\beta_j)}{\Gamma(1+n-\sum_{j=1}^n\beta_j)}.$$

Lemma A.4. *Let $\xi_1, \dots, \xi_n$ be mean zero Gaussian variables which are linearly independent, then for any nonnegative function $g : \mathbb{R} \to \mathbb{R}_+$,*

$$\int_{\mathbb{R}^n} g(v_1) \exp\left[-\frac{1}{2}\mathrm{Var}\left(\sum_{j=1}^{n} v_j \xi_j\right)\right] \mathrm{d}v_1 \dots \mathrm{d}v_n = \frac{(2\pi)^{(n-1)/2}}{(\mathrm{detCov}(\xi_1, \dots, \xi_n))^{1/2}} \int_{-\infty}^{\infty} g\left(\frac{v}{\sigma_1}\right) \mathrm{e}^{-v^2/2} \, \mathrm{d}v, \tag{A.3}$$

where $\mathrm{detCov}(\xi_1, \dots, \xi_n)$ *denotes the determinant of the covariance matrix of the Gaussian random vector* $(\xi_1, \dots, \xi_n)$*, and where* $\sigma_1^2 = \mathrm{Var}(\xi_1 | \xi_2, \dots, \xi_n)$ *is the conditional variance of* ξ_1 *given* $\xi_2, \dots, \xi_n$*.*

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