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A TIME-HOMOGENEOUS DIFFUSION MODEL WITH TAX

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Abstract

We study the two-sided exit problem of a time-homogeneous diffusion process with tax payments of loss-carry-forward type and obtain explicit formulae for the Laplace transforms associated with the two-sided exit problem. The expected present value of tax payments until default, the two-sided exit probabilities, and, hence, the nondefault probability with the default threshold equal to the lower bound are solved as immediate corollaries. A sufficient and necessary condition for the tax identity in ruin theory is discovered.

Keywords: Diffusion; hitting time; Laplace transform; Markov property; tax; two-sided exit problem

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1. Introduction

We are interested in the two-sided exit problem of a time-homogeneous diffusion process with tax payments. Suppose that the value of a firm before taxation is modeled by a time-homogeneous diffusion process $X = \{X_t,\ t \geq 0\}$, defined on a filtered probability space $\{\Omega, \{\mathcal{F}_t,\ t \geq 0\}, \mathbb{P}\}$, with dynamics

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \qquad t \geq 0, \tag{1.1}$$

where $X_0 = x_0$ is the initial value, $\{W_t,\ t \geq 0\}$ is a standard Brownian motion, and $\mu(\cdot)$ and $\sigma(\cdot) > 0$ are two measurable functions on I, a relevant interval for the firm value. As usual, assume that $\mu(\cdot)$ and $\sigma(\cdot)$ satisfy the conditions of the existence and uniqueness theorem for a stochastic differential equation; namely, there exists a constant $K > 0$ such that, for all $x_1, x_2 \in I$,

$$|\mu(x_1) - \mu(x_2)| + |\sigma(x_1) - \sigma(x_2)| \leq K|x_1 - x_2|, \qquad \mu^2(x_1) + \sigma^2(x_1) \leq K^2(1 + x_1^2). \tag{1.2}$$

Then the unique solution of (1.1) possesses the strong Markov property. See Gīhman and Skorohod (1972, pp. 40, 107).

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Recently, ruin problems with tax have become an appealing research topic. Albrecher and Hipp (2007) first introduced tax payments at a constant rate at profitable times to the compound Poisson risk model and established a charming tax identity for the nonruin probability. Later, Albrecher *et al.* (2009) found a simple proof using downward excursions and extended the study to a value-dependent tax rate. Further extensions to the Lévy framework were done by Albrecher *et al.* (2008), Kyprianou and Zhou (2009), and Renaud (2009), among others. See also Hao and Tang (2009) for the study in the Lévy framework but under periodic taxation. To date, there has been little study beyond the Lévy framework, with difficulty mainly in the two-sided exit problem.

Following this new trend of ruin theory, we introduce a value-dependent tax rate to the time-homogeneous diffusion model (1.1). More precisely, whenever the process X coincides with its running maximum M^X, defined by $M_t^X = \sup_{0\le\tau\le t} X_\tau$, $t \ge 0$, the firm pays tax at rate $\gamma(M_t^X)$, where $\gamma(\cdot)\colon [x_0, \infty) \to [0, 1)$ is a measurable function. This is the so-called loss-carry-forward taxation. It is easy to understand that the value process after taxation satisfies

$$\mathrm{d}U_t = \mathrm{d}X_t - \gamma(M_t^X)\,\mathrm{d}M_t^X, \qquad t \ge 0, \tag{1.3}$$

with $U_0 = X_0 = x_0$.

We study the two-sided exit problem of the value process U. Throughout the paper, let

$$a < x_0 < b. \tag{1.4}$$

The lower bound a represents the default threshold, so the firm defaults whenever its value is below a. In particular, the threshold a is set to 0 in ruin theory. For a real number x, introduce the first hitting times of X and U respectively as

$$T^X(x) = \inf\{t \ge 0\colon X_t = x\} \quad \text{and} \quad T^U(x) = \inf\{t \ge 0\colon U_t = x\},$$

where $\inf \varnothing = \infty$ by convention. In particular, $T^U(a)$ stands for the time of default with tax. Our main goal is to solve the Laplace transforms associated with the two-sided exit problem:

$$\mathbb{E}_{x_0}[\mathrm{e}^{-\lambda T^U(b)}; T^U(b) < T^U(a)] \quad \text{and} \quad \mathbb{E}_{x_0}[\mathrm{e}^{-\lambda T^U(a)}; T^U(a) < T^U(b)].$$

Here and throughout the paper, for ease of notation, we write $\mathbb{E}_{x_0}[\cdot] = \mathbb{E}[\cdot \mid X_0 = U_0 = x_0]$ for the conditional expectation, $\mathbb{P}_{x_0}\{\cdot\}$ for the corresponding probability, and $\mathbb{E}_{x_0}[\cdot; C] = \mathbb{E}_{x_0}[\cdot 1_C]$ with 1_C denoting the indicator function of a set $C \subset \Omega$. Our idea of the proof of the main result stems from the work of Lehoczky (1977). As corollaries, we study the expected present value of tax payments until default and the two-sided exit probabilities. In particular, we examine the tax identity in the current situation.

The rest of this paper consists of two sections. In Section 2 we present our main result and its corollaries, and in Section 3 we prove these results.

2. Main results and related discussions

2.1. Preliminaries on time-homogeneous diffusion processes

The two-sided exit problem for the diffusion process X has been well studied in the literature. The exit probabilities from the interval $[a, b]$ can be expressed in terms of the function

$$G(y) = \exp\left\{-\int^y \frac{2\mu(x)}{\sigma^2(x)}\,\mathrm{d}x\right\}, \qquad y \in I.$$

Hereafter, the interval I is specified to $I = [a, b]$ and the lower bound of the integral above can be specified to any point in the interval $[a, b]$. More precisely, under (1.2), it is well known that

$$\mathbb{P}_{x_0}\{T^X(b) < T^X(a)\} = \frac{\int_a^{x_0} G(y)\,\mathrm{d}y}{\int_a^b G(y)\,\mathrm{d}y} \quad \text{and} \quad \mathbb{P}_{x_0}\{T^X(a) < T^X(b)\} = \frac{\int_{x_0}^b G(y)\,\mathrm{d}y}{\int_a^b G(y)\,\mathrm{d}y}; \tag{2.1}$$

see, e.g. Gīhman and Skorohod (1972, p. 110) or Klebaner (2005, Section 6.4). The nondefault probability of X follows immediately by letting $b \uparrow \infty$ in the first relation in (2.1):

$$\mathbb{P}_{x_0}\{T^X(a) = \infty\} = \frac{\int_a^{x_0} G(y)\,\mathrm{d}y}{\int_a^\infty G(y)\,\mathrm{d}y}. \tag{2.2}$$

Note that if $\int^\infty G(y)\,\mathrm{d}y = \infty$ then $\mathbb{P}_{x_0}\{T^X(a) = \infty\} = 0$ for all $x_0 \geq a$.

The Laplace transforms of $T^X(a)$ and $T^X(b)$ associated with the two-sided exit problem for a diffusion process X were first solved by Darling and Siegert (1953). Suppose that $g_{-,\lambda}(\cdot)$ and $g_{+,\lambda}(\cdot)$ are two independent, positive, and convex solutions of the equation

$$\tfrac{1}{2}\sigma^2(x)g''(x) + \mu(x)g'(x) = \lambda g(x), \qquad \lambda \geq 0, \tag{2.3}$$

with $g_{-,\lambda}(\cdot)$ decreasing and $g_{+,\lambda}(\cdot)$ increasing. For many particular diffusions of interest, the differential equation (2.3) yields explicit expressions for $g_{-,\lambda}(\cdot)$ and $g_{+,\lambda}(\cdot)$; see Borodin and Salminen (2002). Define

$$f_\lambda(y, z) = g_{-,\lambda}(y)g_{+,\lambda}(z) - g_{-,\lambda}(z)g_{+,\lambda}(y) \quad \text{and} \quad w_\lambda(y, z) = \frac{\partial}{\partial z} f_\lambda(y, z). \tag{2.4}$$

Note that the function $f_\lambda(y, z)$ is strictly decreasing in y and strictly increasing in z. Hence, $f_\lambda(y, z) = 0$ if and only if $y = z$. By the continuous dependence theorem,

$$f_0(y, z) = \lim_{\lambda \downarrow 0} f_\lambda(y, z) = \int_y^z G(x)\,\mathrm{d}x \quad \text{and} \quad w_0(y, z) = \lim_{\lambda \downarrow 0} w_\lambda(y, z) = G(z). \tag{2.5}$$

Lemma 2.1. (Theorem 3.2 of Darling and Siegert (1953).) *For $a < x_0 < b$ and $\lambda \geq 0$, we have*

$$\mathbb{E}_{x_0}[\mathrm{e}^{-\lambda T^X(b)}; T^X(b) < T^X(a)] = \frac{f_\lambda(a, x_0)}{f_\lambda(a, b)} \tag{2.6}$$

and

$$\mathbb{E}_{x_0}[\mathrm{e}^{-\lambda T^X(a)}; T^X(a) < T^X(b)] = \frac{f_\lambda(x_0, b)}{f_\lambda(a, b)}. \tag{2.7}$$

When $\lambda = 0$, by (2.5), the two relations in Lemma 2.1 are reduced to those in (2.1).

2.2. The main result

Recall the initial value x_0, the lower boundary a, and the upper boundary b as specified by (1.4). Following Kyprianou and Zhou (2009), we define

$$\bar{\gamma}(x) = x - \int_{x_0}^x \gamma(z)\,\mathrm{d}z = x_0 + \int_{x_0}^x (1 - \gamma(z))\,\mathrm{d}z, \qquad x \geq x_0,$$

which is strictly increasing and continuous in x with $\bar{\gamma}(x_0) = x_0$. Thus, its inverse function $\bar{\gamma}^{-1}(\cdot)$ is well defined on $[x_0, \bar{\gamma}(\infty))$. Note that both $x - \bar{\gamma}(x)$ and $\bar{\gamma}^{-1}(x) - x$ are nondecreasing and continuous functions. Trivially, $\bar{\gamma}(\infty) = \infty$ if we assume that

$$\int_{x_0}^{\infty} (1 - \gamma(z))\,\mathrm{d}z = \infty. \tag{2.8}$$

As before, denote by $M_t^U = \sup_{0\le\tau\le t} U_\tau$, $t \ge 0$, the running maximum of U. In terms of the function $\bar{\gamma}(\cdot)$, we can rewrite the process U in (1.3) as

$$U_t = X_t - M_t^X + \bar{\gamma}(M_t^X), \qquad t \ge 0. \tag{2.9}$$

As shown in Lemma 2.1 of Kyprianou and Zhou (2009), we have

$$M_t^U = M_t^X - \int_0^t \gamma(M_\tau^X)\,\mathrm{d}M_\tau^X = \bar{\gamma}(M_t^X), \qquad t \ge 0, \tag{2.10}$$

and, hence, $T^U(x) = T^X(\bar{\gamma}^{-1}(x))$ for $x \ge x_0$.

Our main result is the following.

Theorem 2.1. *For $a < x_0 < b$ and $\lambda > 0$, we have*

$$\mathbb{E}_{x_0}[\mathrm{e}^{-\lambda T^U(b)}; T^U(b) < T^U(a)] = \exp\bigg\{-\int_{x_0}^{\bar{\gamma}^{-1}(b)} \frac{w_\lambda(x - \bar{\gamma}(x) + a, x)}{f_\lambda(x - \bar{\gamma}(x) + a, x)}\,\mathrm{d}x\bigg\} \tag{2.11}$$

and

$$\begin{aligned}&\mathbb{E}_{x_0}[\mathrm{e}^{-\lambda T^U(a)}; T^U(a) < T^U(b)]\\ &\quad = \int_{x_0}^{\bar{\gamma}^{-1}(b)} \exp\bigg\{-\int_{x_0}^{y} \frac{w_\lambda(x - \bar{\gamma}(x) + a, x)}{f_\lambda(x - \bar{\gamma}(x) + a, x)}\,\mathrm{d}x\bigg\} \frac{w_\lambda(y, y)}{f_\lambda(y - \bar{\gamma}(y) + a, y)}\,\mathrm{d}y.\end{aligned} \tag{2.12}$$

The complete proof of Theorem 2.1 is deferred to Section 3.

One can check that Theorem 2.1 agrees with Lemma 2.1 in the case of no taxation, namely, $\bar{\gamma}(x) \equiv x$. Actually, it is clear that (2.11) is reduced to (2.6) when $\bar{\gamma}(x) \equiv x$. To check that (2.12) is reduced to (2.7) when $\bar{\gamma}(x) \equiv x$, we use the identity

$$f_\lambda(a, x_0) w_\lambda(b, b) = w_\lambda(x_0, b) f_\lambda(a, b) - f_\lambda(x_0, b) w_\lambda(a, b)$$

for all $a < x_0 < b$ and $\lambda > 0$, which can be verified by (2.4). In addition, by (2.9) we have

$$T^U(a) = \inf\{t \ge 0 \colon U_t \le a\} = \inf\{t \ge 0 \colon M_t^X - X_t \ge \bar{\gamma}(M_t^X) - a\}. \tag{2.13}$$

Therefore, under (2.8), our relation (2.12) with $b = \infty$ agrees with Relation (21) of Lehoczky (1977) with the function $u(\cdot) = \bar{\gamma}(\cdot) - a$ and $\alpha = 0$.

In the example below we show that, restricted to a Brownian motion and $a = 0$, our relation (2.11) coincides with Relation (1.5) of Kyprianou and Zhou (2009).

Example 2.1. Let $X_t = \mu t + \sigma W_t$ be a Brownian motion with positive drift μ and write $\mu_\lambda = \sqrt{\mu^2 + 2\lambda}$ for $\lambda > 0$. We have

$$g_{-,\lambda}(x) = \exp\bigg\{\frac{-\mu - \mu_\lambda}{\sigma^2} x\bigg\} \quad \text{and} \quad g_{+,\lambda}(x) = \exp\bigg\{\frac{-\mu + \mu_\lambda}{\sigma^2} x\bigg\}.$$

Then it follows that

$$\frac{w_\lambda(y, z)}{f_\lambda(y, z)} = \frac{((-\mu + \mu_\lambda)/\sigma^2)\exp\{\mu_\lambda(z - y)/\sigma^2\} + ((\mu + \mu_\lambda)/\sigma^2)\exp\{-\mu_\lambda(z - y)/\sigma^2\}}{\exp\{\mu_\lambda(z - y)/\sigma^2\} - \exp\{-\mu_\lambda(z - y)/\sigma^2\}}.$$

On the other hand, by inverting a corresponding Laplace transform, the scale function of X as a spectrally negative Lévy process is

$$W^{(\lambda)}(x) = \frac{\sigma^2}{\mu_\lambda}\left(\exp\left\{\frac{-\mu+\mu_\lambda}{\sigma^2}x\right\} - \exp\left\{\frac{-\mu-\mu_\lambda}{\sigma^2}x\right\}\right);$$

see Chapter 8 of Kyprianou (2006) for the definition of $W^{(\lambda)}$. It follows that

$$\frac{w_\lambda(y,z)}{f_\lambda(y,z)} = \frac{W^{(\lambda)\prime}(y-z)}{W^{(\lambda)}(y-z)}.$$

Then, by a change of variables, one can easily check that our relation (2.11) with $a = 0$ agrees with Relation (1.5) of Kyprianou and Zhou (2009).

As an application of Theorem 2.1, we derive a formula for the expected present value of tax payments until default. Parallel works in the Lévy framework include Theorem 3.2 of Albrecher *et al.* (2008), Theorem 1.2 of Kyprianou and Zhou (2009), and Theorem 3.1 of Renaud (2009).

Corollary 2.1. *Under (2.8), we have*

$$\mathbb{E}_{x_0}\left[\int_0^{T^U(a)} \mathrm{e}^{-\lambda t}\gamma(M_t^X)\,\mathrm{d}M_t^X\right] = \int_{x_0}^{\infty} \gamma(y)\exp\left\{-\int_{x_0}^{y} \frac{w_\lambda(x-\bar{\gamma}(x)+a,x)}{f_\lambda(x-\bar{\gamma}(x)+a,x)}\,\mathrm{d}x\right\}\mathrm{d}y.$$

The proof of Corollary 2.1 is deferred to Section 3.

2.3. Two-sided exit probabilities and the tax identity

Letting $\lambda \downarrow 0$ in Theorem 2.1 and using the convergence in (2.5), we obtain the following result.

Corollary 2.2. *It holds that*

$$\mathbb{P}_{x_0}\{T^U(b) < T^U(a)\} = \exp\left\{-\int_{x_0}^{\bar{\gamma}^{-1}(b)} \frac{G(x)}{\int_{x-\bar{\gamma}(x)+a}^{x} G(y)\,\mathrm{d}y}\,\mathrm{d}x\right\} \tag{2.14}$$

and that $\mathbb{P}_{x_0}\{T^U(a) < T^U(b)\} = 1 - \mathbb{P}_{x_0}\{T^U(b) < T^U(a)\}$.

A separate proof of Corollary 2.2 can be given by going along the same lines of the proof of Theorem 2.1 with $\lambda = 0$. Clearly, in the case of no taxation, namely, $\bar{\gamma}(x) \equiv x$, relation (2.14) agrees with the first relation in (2.1). Moreover, we point out that relation (2.14) is a special case of Relation (20) of Lehoczky (1977) with $u(\cdot) = \bar{\gamma}(\cdot) - a$. This is due to the observation that, by (2.10),

$$\mathbb{P}_{x_0}\{T^U(b) < T^U(a)\} = \mathbb{P}_{x_0}\{M^U_{T^U(a)} \geq b\} = \mathbb{P}_{x_0}\{M^X_{T^U(a)} \geq \bar{\gamma}^{-1}(b)\}$$

and relation (2.13).

Letting $b \uparrow \infty$ in (2.14) yields the nondefault probability of U as follows.

Corollary 2.3. *Under (2.8), it holds that*

$$\mathbb{P}_{x_0}\{T^U(a) = \infty\} = \exp\left\{-\int_{x_0}^{\infty} \frac{G(x)}{\int_{x-\bar{\gamma}(x)+a}^{x} G(y)\,\mathrm{d}y}\,\mathrm{d}x\right\}. \tag{2.15}$$

Tax payments increase default risk, of course, which can be observed by comparing (2.15) with (2.2). Thus, relation (2.15) provides us with a quantitative understanding of the impact of tax payments on default risk. In particular, the following example shows that the standard Black–Scholes model without tax has a positive probability to survive forever while any constant tax rate, no matter how small it is, will drive the firm to default eventually.

Example 2.2. Consider the geometric Brownian motion

$$\mathrm{d}X_t = \mu X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}W_t, \qquad t \geq 0,$$

where $X_0 = x_0 > 0$ is the initial value, and μ and σ are positive constants satisfying $\rho = 2\mu/\sigma^2 > 1$. In addition, we assume that the default threshold is $a > 0$. Then, by relation (2.2) with $G(y) = (a/y)^\rho$ for $y \geq a$, the nondefault probability without tax is

$$\mathbb{P}_{x_0}\{T^X(a) = \infty\} = 1 - \left(\frac{a}{x_0}\right)^{\rho-1} > 0.$$

However, in the presence of a constant tax rate $0 < \gamma < 1$, by (2.15) we have

$$\mathbb{P}_{x_0}\{T^U(a) = \infty\} = \exp\left\{-\int_{x_0}^{\infty} \frac{x^{-\rho}}{\int_{\gamma x - \gamma x_0 + a}^{x} y^{-\rho}\,\mathrm{d}y}\,\mathrm{d}x\right\} = 0.$$

In order to compare our Corollary 2.3 with Theorem 1.1 of Kyprianou and Zhou (2009), we can use the change of variables $x = \bar{\gamma}^{-1}(s)$ to rewrite relation (2.14). In particular, if $\gamma(\cdot) \equiv \gamma \in [0, 1)$ is constant then $\bar{\gamma}(x) = x - \gamma x + \gamma x_0$ and relation (2.14) is reduced to

$$\mathbb{P}_{x_0}\{T^U(b) < T^U(a)\} = \exp\left\{-\int_{x_0}^{b} \frac{G((s-\gamma x_0)/(1-\gamma))}{\int_{(\gamma s - \gamma x_0)/(1-\gamma)+a}^{(s-\gamma x_0)/(1-\gamma)} G(y)\,\mathrm{d}y}\,\mathrm{d}s\right\}^{1/(1-\gamma)}. \tag{2.16}$$

As mentioned in Section 1, for the case of a constant tax rate γ, the tax identity

$$\mathbb{P}_{x_0}\{T^U(0) = \infty\} = (\mathbb{P}_{x_0}\{T^X(0) = \infty\})^{1/(1-\gamma)} \tag{2.17}$$

has been established by researchers in various situations within the Lévy framework. However, relation (2.16) indicates that such an identity does not hold in general within the diffusion framework.

Slightly more generally, we now consider under what condition the identity

$$\mathbb{P}_{x_0}\{T^U(b) < T^U(a)\} = (\mathbb{P}_{x_0}\{T^X(b) < T^X(a)\})^{1/(1-\gamma)} \tag{2.18}$$

holds. Interestingly, the answer is that $\mu(\cdot)/\sigma^2(\cdot)$ has to be constant.

Corollary 2.4. *Consider constant tax rates.*

1. *For arbitrarily fixed x_0 and a with $a < x_0$, relation (2.18) holds for all $b > x_0$ and $0 \leq \gamma < 1$ if and only if $\mu(x)/\sigma^2(x)$ is constant for $x \geq a$.*
2. *For arbitrarily fixed a and b with $a < b$, relation (2.18) holds for all $a < x_0 < b$ and $0 \leq \gamma < 1$ if and only if $\mu(x)/\sigma^2(x)$ is constant for $a \leq x \leq b$.*

The proof of Corollary 2.4 is deferred to Section 3. By letting $b \uparrow \infty$ and $a = 0$ in part 2 of Corollary 2.4 and going along the same lines of its proof, we obtain the following result.

Corollary 2.5. *Consider constant tax rates, and assume that (2.8) holds and that* $\int^{\infty} G(y)\,\mathrm{d}y < \infty$. *Then relation (2.17) holds for all* $0 < x_0 < \infty$ *and* $0 \le \gamma < 1$ *if and only if* $\mu(x)/\sigma^2(x)$ *is constant for* $x \ge 0$.

The condition $\int^{\infty} G(y)\,\mathrm{d}y < \infty$ in Corollary 2.5 is necessary; otherwise, the probability $\mathbb{P}\{T^X(0) = \infty\}$ is equal to 0 and relation (2.17) becomes trivial. Note that the square-root process with dynamics

$$\mathrm{d}X_t = \mu X_t\,\mathrm{d}t + \sigma\sqrt{X_t}\,\mathrm{d}W_t, \qquad t \ge 0,$$

which is widely used in finance, satisfies the condition that $\mu(\cdot)/\sigma^2(\cdot)$ is constant in Corollaries 2.4 and 2.5.

Corollaries 2.4 and 2.5 confirm that some intrinsic properties of a Brownian motion can often be inherited by a time-homogeneous diffusion process with constant $\mu(\cdot)/\sigma^2(\cdot)$. A similar implication can be found in Lehoczky (1977). Relation (5) therein gives the distribution of the running maximum of a time-homogeneous diffusion process at the first time it falls a specified amount below its current maximum. Lehoczky (1977) observed that if $\mu(\cdot)/\sigma^2(\cdot)$ is constant then this result agrees with that for a Brownian motion.

3. Proofs

Clearly, in order for U to hit b before a, for every $s \in [x_0, b)$, after $T^U(s)$ the process U must enter (s, ∞) before it hits a. By relations (2.9) and (2.10), this fact can be restated in terms of X as follows. After $T^X(\bar{\gamma}^{-1}(s))$, the process X must enter $(\bar{\gamma}^{-1}(s), \infty)$ before it hits $\bar{\gamma}^{-1}(s) - s + a$. Thus, the event $(T^U(b) < T^U(a))$ necessitates a two-sided exit problem of X for every $s \in [x_0, b)$. Based on this intuition, we establish lower and upper discrete approximations for the event $(T^U(b) < T^U(a))$ in the following.

Lemma 3.1. *Let* $x_0 = s_0 < s_1 < \cdots < s_n = b$ *form a partition of the interval* $[x_0, b]$, $n \in \mathbb{N}$. *Then, almost surely,*

$$\bigcap_{i=1}^{n} A_i \subset (T^U(b) < T^U(a)) \subset \bigcap_{i=1}^{n} B_i, \tag{3.1}$$

where each A_i *denotes the event that, after* $T^X(\bar{\gamma}^{-1}(s_{i-1}))$, *the process* X *hits* $\bar{\gamma}^{-1}(s_i)$ *before* $\bar{\gamma}^{-1}(s_i) - s_i + a$ *while each* B_i *denotes the event that, after* $T^X(\bar{\gamma}^{-1}(s_{i-1}))$, *the process* X *hits* $\bar{\gamma}^{-1}(s_i)$ *before* $\bar{\gamma}^{-1}(s_{i-1}) - s_{i-1} + a$.

Proof. To prove the first inclusion in (3.1), assume that the path of X is continuous such that $\bigcap_{i=1}^{n} A_i$ holds. Arbitrarily choose $t \in [0, T^X(\bar{\gamma}^{-1}(b))]$ and suppose that t falls into the interval $[T^X(\bar{\gamma}^{-1}(s_{i-1})), T^X(\bar{\gamma}^{-1}(s_i))]$ for some $i = 1, \ldots, n$. Then $M_t^X \le \bar{\gamma}^{-1}(s_i)$ and, by relation (2.9), the monotonicity of $s - \bar{\gamma}(s)$, and the description of A_i, we have

$$U_t = X_t - (M_t^X - \bar{\gamma}(M_t^X)) \ge X_t - (\bar{\gamma}^{-1}(s_i) - \bar{\gamma}(\bar{\gamma}^{-1}(s_i))) > a.$$

To summarize, $U_t > a$ for all $t \in [0, T^X(\bar{\gamma}^{-1}(b))]$. Hence, $T^U(a) > T^X(\bar{\gamma}^{-1}(b)) = T^U(b)$.

To prove the second inclusion in (3.1), assume by contradiction that there exists some $i = 1, \ldots, n$ such that, after $T^X(\bar{\gamma}^{-1}(s_{i-1}))$, the path of X hits $\bar{\gamma}^{-1}(s_{i-1}) - s_{i-1} + a$ before $\bar{\gamma}^{-1}(s_i)$. Then at the moment of hitting $\bar{\gamma}^{-1}(s_{i-1}) - s_{i-1} + a$, by relation (2.9), the monotonicity of $s - \bar{\gamma}(s)$, and $M_t^X \geq \bar{\gamma}^{-1}(s_{i-1})$, we have

$$U_t = X_t - (M_t^X - \bar{\gamma}(M_t^X)) \leq (\bar{\gamma}^{-1}(s_{i-1}) - s_{i-1} + a) - (\bar{\gamma}^{-1}(s_{i-1}) - \bar{\gamma}(\bar{\gamma}^{-1}(s_{i-1}))) = a,$$

which contradicts $T^U(b) < T^U(a)$.

Proof of relation (2.11). Let $\{s_{n,i},\ i = 0, \dots, m_n\}$, $n \in \mathbb{N}$, constitute a sequence of increasing partitions of the interval $[x_0, b]$ with $x_0 = s_{n,0} < s_{n,1} < \cdots < s_{n,m_n} = b$ and the maximum length of subintervals $\Delta_n = \max_{1 \leq i \leq m_n}(s_{n,i} - s_{n,i-1}) \downarrow 0$ as $n \to \infty$. By Lemma 3.1 we have

$$\begin{aligned}
\mathbb{E}_{x_0}[\mathrm{e}^{-\lambda T^U(b)}; T^U(b) < T^U(a)] &= \mathbb{E}_{x_0}\Bigg[\prod_{i=1}^{m_n} \mathrm{e}^{-\lambda(T^U(s_{n,i}) - T^U(s_{n,i-1}))}; T^U(b) < T^U(a)\Bigg] \\
&\leq \mathbb{E}_{x_0}\Bigg[\prod_{i=1}^{m_n} \mathrm{e}^{-\lambda(T^X(\bar{\gamma}^{-1}(s_{n,i})) - T^X(\bar{\gamma}^{-1}(s_{n,i-1})))}; \bigcap_{i=1}^{m_n} B_{n,i}\Bigg] \\
&= E_n,
\end{aligned}$$

where each $B_{n,i}$, the same as in Lemma 3.1, denotes the event that, after $T^X(\bar{\gamma}^{-1}(s_{n,i-1}))$, the process X hits $\bar{\gamma}^{-1}(s_{n,i})$ before $\bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a$. Furthermore, by the strong Markov property of X,

$$\begin{aligned}
E_n = \prod_{i=1}^{m_n} \mathbb{E}_{x_0}[&\mathrm{e}^{-\lambda T^X(\bar{\gamma}^{-1}(s_{n,i}))}; \\
&T^X(\bar{\gamma}^{-1}(s_{n,i})) < T^X(\bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a) \mid \mathcal{F}_{T^X(\bar{\gamma}^{-1}(s_{n,i-1}))}].
\end{aligned}$$

For ease of notation, introduce

$$h(c_1, c_2 \mid c_0) = 1 - \mathbb{E}_{x_0}[\mathrm{e}^{-\lambda T^X(c_2)}; T^X(c_2) < T^X(c_1) \mid \mathcal{F}_{T^X(c_0)}], \qquad c_1 < c_0 < c_2,$$

so that

$$E_n = \exp\Bigg\{\sum_{i=1}^{m_n} \log(1 - h(\bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a, \bar{\gamma}^{-1}(s_{n,i}) \mid \bar{\gamma}^{-1}(s_{n,i-1})))\Bigg\}.$$

It can be shown that all the $h(\cdot, \cdot \mid \cdot)$ terms in E_n are uniformly small. In fact, by relation (2.6) and the monotonicity of $\bar{\gamma}^{-1}(s) - s$,

$$\begin{aligned}
&h(\bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a, \bar{\gamma}^{-1}(s_{n,i}) \mid \bar{\gamma}^{-1}(s_{n,i-1})) \\
&\quad = \frac{f_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a, \bar{\gamma}^{-1}(s_{n,i})) - f_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a, \bar{\gamma}^{-1}(s_{n,i-1}))}{f_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a, \bar{\gamma}^{-1}(s_{n,i}))} \\
&\quad \leq K \max_{1 \leq i \leq m_n}(\bar{\gamma}^{-1}(s_{n,i}) - \bar{\gamma}^{-1}(s_{n,i-1})),
\end{aligned} \tag{3.2}$$

where the constant K is defined as

$$K = \frac{\sup_{(y,z) \in D} w_\lambda(y, z)}{\inf_{(y,z) \in D} f_\lambda(y, z)} < \infty$$

with $D = \{(y, z) \colon a \leq y \leq \bar{\gamma}^{-1}(b) - b + a,\ x_0 \leq z \leq \bar{\gamma}^{-1}(b),\ z - y \geq x_0 - a\}$. Note that, over the closed set D, the function $f_\lambda(y, z)$ is strictly positive (hence, away from 0), and that the function $w_\lambda(y, z)$ is always continuous and strictly positive. Therefore, by the elementary relation $\log(1-h) \sim -h$ as $h \downarrow 0$, it holds for arbitrarily fixed $0 < \varepsilon < 1$ and all large n that

$$E_n \le \exp\bigg\{-(1-\varepsilon)\sum_{i=1}^{m_n} h(\bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a, \bar{\gamma}^{-1}(s_{n,i}) \mid \bar{\gamma}^{-1}(s_{n,i-1}))\bigg\}.$$

Since, for all large n and $i = 1, \dots, m_n$, the numerator of (3.2) is bounded below by

$$(1-\varepsilon)w_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a, \bar{\gamma}^{-1}(s_{n,i}))(\bar{\gamma}^{-1}(s_{n,i}) - \bar{\gamma}^{-1}(s_{n,i-1})),$$

it follows that

$$\begin{aligned}\limsup_{n\to\infty} E_n &\le \lim_{n\to\infty} \exp\bigg\{-(1-\varepsilon)^2 \sum_{i=1}^{m_n} \frac{w_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a, \bar{\gamma}^{-1}(s_{n,i}))}{f_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a, \bar{\gamma}^{-1}(s_{n,i}))} \\ &\qquad\qquad \times (\bar{\gamma}^{-1}(s_{n,i}) - \bar{\gamma}^{-1}(s_{n,i-1}))\bigg\} \\ &= \exp\bigg\{-(1-\varepsilon)^2 \int_{x_0}^{\bar{\gamma}^{-1}(b)} \frac{w_\lambda(x - \bar{\gamma}(x) + a, x)}{f_\lambda(x - \bar{\gamma}(x) + a, x)}\,\mathrm{d}x\bigg\}. \end{aligned} \tag{3.3}$$

The last equality in (3.3) is justified by changing each $s_{n,i}$ in the second step to $s_{n,i-1}$. By the arbitrariness of ε we have

$$\mathbb{E}_{x_0}[\mathrm{e}^{-\lambda T^U(b)}; T^U(b) < T^U(a)] \le \limsup_{n\to\infty} E_n \le \exp\bigg\{-\int_{x_0}^{\bar{\gamma}^{-1}(b)} \frac{w_\lambda(x - \bar{\gamma}(x) + a, x)}{f_\lambda(x - \bar{\gamma}(x) + a, x)}\,\mathrm{d}x\bigg\}.$$

The inequality for the lower bound can be established analogously by using the other part of Lemma 3.1.

Proof of relation (2.12). We employ the same partition of the interval $[x_0, b]$ as in the proof of relation (2.11). By considering the range of the running maximum of U before hitting a, we have

$$\begin{aligned}&\mathbb{E}_{x_0}[\mathrm{e}^{-\lambda T^U(a)}; T^U(a) < T^U(b)] \\ &\quad= \sum_{i=1}^{m_n} \mathbb{E}_{x_0}[\mathrm{e}^{-\lambda T^U(a)}; M^U_{T^U(a)} \in [s_{n,i-1}, s_{n,i})] \\ &\quad= \sum_{i=1}^{m_n} \mathbb{E}_{x_0}[\mathrm{e}^{-\lambda(T^U(s_{n,i-1}) + T^U(a) - T^U(s_{n,i-1}))}; T^U(s_{n,i-1}) < T^U(a) < T^U(s_{n,i})] \\ &\quad= \sum_{i=1}^{m_n} \mathbb{E}_{x_0}[\mathrm{e}^{-\lambda T^U(s_{n,i-1})}\mathbb{E}_{x_0}[\mathrm{e}^{-\lambda T^U(a)}; T^U(a) < T^U(s_{n,i}) \mid \mathcal{F}_{T^U(s_{n,i-1})}]; \\ &\qquad\qquad T^U(s_{n,i-1}) < T^U(a)],\end{aligned}$$

where the last step is due to the fact that $\mathcal{F}_{T^U(s_{n,i-1})} = \mathcal{F}_{T^X(\bar{\gamma}^{-1}(s_{n,i-1}))}$ and the strong Markov property of X. Clearly, after $T^U(s_{n,i-1})$, if the process U hits a before $s_{n,i}$ then the process X hits $\bar{\gamma}^{-1}(s_{n,i}) - s_{n,i} + a$ before $\bar{\gamma}^{-1}(s_{n,i})$ because, otherwise, for $t \in [T^U_{s_{n,i-1}}, T^U_{s_{n,i}}]$, by the monotonicity of $s - \bar{\gamma}(s)$,

$$U_t = X_t - (M^X_t - \bar{\gamma}(M^X_t)) \ge X_t - (\bar{\gamma}^{-1}(s_{n,i}) - \bar{\gamma}(\bar{\gamma}^{-1}(s_{n,i}))) > a.$$

Hence, conditional on $\mathcal{F}_{T^X(\bar{\gamma}^{-1}(s_{n,i-1}))}$,

$$T^X(\bar{\gamma}^{-1}(s_{n,i}) - s_{n,i} + a) < T^U(a).$$

Therefore, the inner expectation above is dealt with as

$$\begin{aligned}
&\mathbb{E}_{x_0}[\mathrm{e}^{-\lambda T^U(a)}; T^U(a) < T^U(s_{n,i}) \mid \mathcal{F}_{T^U(s_{n,i-1})}] \\
&\quad \le \mathbb{E}_{x_0}[\mathrm{e}^{-\lambda T^X(\bar{\gamma}^{-1}(s_{n,i}) - s_{n,i} + a)}; \\
&\qquad\qquad T^X(\bar{\gamma}^{-1}(s_{n,i}) - s_{n,i} + a) < T^X(\bar{\gamma}^{-1}(s_{n,i})) \mid \mathcal{F}_{T^X(\bar{\gamma}^{-1}(s_{n,i-1}))}] \\
&\quad = \frac{f_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}), \bar{\gamma}^{-1}(s_{n,i}))}{f_\lambda(\bar{\gamma}^{-1}(s_{n,i}) - s_{n,i} + a, \bar{\gamma}^{-1}(s_{n,i}))},
\end{aligned}$$

where the last step is due to relation (2.7). Substituting this into the above and applying relation (2.11), we obtain

$$\begin{aligned}
&\mathbb{E}_{x_0}[\mathrm{e}^{-\lambda T^U(a)}; T^U(a) < T^U(b)] \\
&\quad \le \sum_{i=1}^{m_n} \mathbb{E}_{x_0}[\mathrm{e}^{-\lambda T^U(s_{n,i-1})}; T^U(s_{n,i-1}) < T^U(a)] \frac{f_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}), \bar{\gamma}^{-1}(s_{n,i}))}{f_\lambda(\bar{\gamma}^{-1}(s_{n,i}) - s_{n,i} + a, \bar{\gamma}^{-1}(s_{n,i}))} \\
&\quad \le \sum_{i=1}^{m_n} \exp\biggl\{-\int_{x_0}^{\bar{\gamma}^{-1}(s_{n,i-1})} \frac{w_\lambda(x - \bar{\gamma}(x) + a, x)}{f_\lambda(x - \bar{\gamma}(x) + a, x)}\,\mathrm{d}x\biggr\} \\
&\qquad \times \frac{f_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}), \bar{\gamma}^{-1}(s_{n,i}))}{f_\lambda(\bar{\gamma}^{-1}(s_{n,i}) - s_{n,i} + a, \bar{\gamma}^{-1}(s_{n,i}))}. \qquad (3.4)
\end{aligned}$$

Since $f_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}), \bar{\gamma}^{-1}(s_{n,i-1})) = 0$ and the function $w_\lambda(y, z)$ is continuous and strictly positive, for arbitrarily fixed $0 < \varepsilon < 1$, it holds that, for all large n and $i = 1, \ldots, m_n$,

$$\begin{aligned}
&f_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}), \bar{\gamma}^{-1}(s_{n,i})) \\
&\quad = f_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}), \bar{\gamma}^{-1}(s_{n,i})) - f_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}), \bar{\gamma}^{-1}(s_{n,i-1})) \\
&\quad \le (1+\varepsilon) w_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}), \bar{\gamma}^{-1}(s_{n,i-1}))(\bar{\gamma}^{-1}(s_{n,i}) - \bar{\gamma}^{-1}(s_{n,i-1})). \qquad (3.5)
\end{aligned}$$

Substituting (3.5) into (3.4), changing each $s_{n,i}$ in (3.4) to $s_{n,i-1}$ based on the same reasoning as in deriving (3.3), and letting $n \to \infty$, we obtain

$$\begin{aligned}
&\mathbb{E}_{x_0}[\mathrm{e}^{-\lambda T^U(a)}; T^U(a) < T^U(b)] \\
&\quad \le (1+\varepsilon) \int_{x_0}^{\bar{\gamma}^{-1}(b)} \exp\biggl\{-\int_{x_0}^{y} \frac{w_\lambda(x - \bar{\gamma}(x) + a, x)}{f_\lambda(x - \bar{\gamma}(x) + a, x)}\,\mathrm{d}x\biggr\} \frac{w_\lambda(y, y)}{f_\lambda(y - \bar{\gamma}(y) + a, y)}\,\mathrm{d}y.
\end{aligned}$$

By the arbitrariness of ε, the desired upper bound for (2.12) follows.

The corresponding lower bound for (2.12) can be established analogously using the fact that, after $T^U(s_{n,i-1})$, if the process X hits $\bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a$ before $\bar{\gamma}^{-1}(s_{n,i})$ then the process U hits a before $s_{n,i}$.

Proof of Corollary 2.1. Using the change of variable $y = M_t^X$, we obtain

$$\begin{aligned}\mathbb{E}_{x_0}\bigg[\int_0^{T^U(a)} \mathrm{e}^{-\lambda t}\gamma(M_t^X)\,\mathrm{d}M_t^X\bigg] &= \mathbb{E}_{x_0}\bigg[\int_{x_0}^{\infty} \gamma(y)\mathrm{e}^{-\lambda T^X(y)} 1_{\{y\le M^X_{T^U(a)}\}}\,\mathrm{d}y\bigg] \\ &= \mathbb{E}_{x_0}\bigg[\int_{x_0}^{\infty} \gamma(y)\mathrm{e}^{-\lambda T^X(y)} 1_{\{T^X(y)<T^U(a)\}}\,\mathrm{d}y\bigg] \\ &= \mathbb{E}_{x_0}\bigg[\int_{x_0}^{\infty} \gamma(y)\mathrm{e}^{-\lambda T^U(\bar\gamma(y))} 1_{\{T^U(\bar\gamma(y))<T^U(a)\}}\,\mathrm{d}y\bigg] \\ &= \int_{x_0}^{\infty} \gamma(y)\mathbb{E}_{x_0}[\mathrm{e}^{-\lambda T^U(\bar\gamma(y))};\, T^U(\bar\gamma(y)) < T^U(a)]\,\mathrm{d}y.\end{aligned}$$

Applying relation (2.11) to the right-hand side above yields the desired result.

Proof of Corollary 2.4. Clearly, $\mu(x)/\sigma^2(x)$ is constant for $x \ge a$ if and only if

$$G(x) = c_1\mathrm{e}^{c_2 x}, \qquad x \ge a,$$

for some constants $c_1 > 0$ and c_2. The sufficiency of both parts can be checked directly. We now prove the necessity separately for both parts.

Part 1. For arbitrarily fixed x_0 and a with $a < x_0$, we assume that relation (2.18) holds for all $b > x_0$ and $0 \le \gamma < 1$. By (2.16), (2.18), and (2.1),

$$\int_{x_0}^{b} \frac{G((s-\gamma x_0)/(1-\gamma))}{\int_{(\gamma s-\gamma x_0)/(1-\gamma)+a}^{(s-\gamma x_0)/(1-\gamma)} G(y)\,\mathrm{d}y}\,\mathrm{d}s = \int_{x_0}^{b} \frac{G(s)}{\int_a^s G(y)\,\mathrm{d}y}\,\mathrm{d}s, \qquad b > x_0,\ 0 \le \gamma < 1.$$

It follows that

$$\frac{G((\gamma s-\gamma x_0)/(1-\gamma)+s)}{\int_{(\gamma s-\gamma x_0)/(1-\gamma)+a}^{(\gamma s-\gamma x_0)/(1-\gamma)+s} G(y)\,\mathrm{d}y} = \frac{G(s)}{\int_a^s G(y)\,\mathrm{d}y}, \qquad s > x_0,\ 0 \le \gamma < 1.$$

Using the change of variable $x = (\gamma s - \gamma x_0)/(1-\gamma)$ on the left-hand side above, upon some simple rearrangements we obtain

$$\frac{\int_a^s G(y)\,\mathrm{d}y}{G(s)} G(x+s) = \int_{x+a}^{x+s} G(y)\,\mathrm{d}y, \qquad s > x_0,\ x \ge 0.$$

By the continuity of $G(\cdot)$, it follows that

$$\frac{\int_a^s G(y)\,\mathrm{d}y}{G(s)} G(x+s) = \int_{x+a}^{x+s} G(y)\,\mathrm{d}y, \qquad s \ge x_0,\ x \ge 0. \tag{3.6}$$

Taking the derivative with respect to s, upon some simple rearrangements we obtain

$$\frac{G'(x+s)}{G(x+s)} = \frac{G'(s)}{G(s)}, \qquad s > x_0,\ x \ge 0.$$

This means that $G'(\cdot)/G(\cdot)$ is constant over the interval (x_0, ∞). Hence, by the positivity and continuity of $G(\cdot)$, it must hold that, for some constants $c_1 > 0$ and c_2,

$$G(x) = c_1\mathrm{e}^{c_2 x}, \qquad x \ge x_0. \tag{3.7}$$

Substituting (3.7) into (3.6) with $s = x_0$ yields

$$\mathrm{e}^{c_2 x} \int_a^{x_0} G(y)\,\mathrm{d}y = \int_{x+a}^{x+x_0} G(y)\,\mathrm{d}y, \qquad x \geq 0.$$

Taking the derivative with respect to x and using (3.7) and a change of variables, we have

$$G(x) = \mathrm{e}^{-c_2 a}\bigg(c_1 \mathrm{e}^{c_2 x_0} - c_2 \int_a^{x_0} G(y)\,\mathrm{d}y\bigg)\mathrm{e}^{c_2 x}, \qquad x \geq a.$$

Comparing this with (3.7), we must have $\mathrm{e}^{-c_2 a}(c_1 \mathrm{e}^{c_2 x_0} - c_2 \int_a^{x_0} G(y)\,\mathrm{d}y) = c_1$ since $G(\cdot)$ is continuous at x_0. One can also easily check this by substitution. Hence, $G(x) = c_1 \mathrm{e}^{c_2 x}$ is valid over $[a, \infty)$.

Part 2. For arbitrarily fixed a and b with $a < b$, we assume that relation (2.18) holds for all $x_0 \in (a, b)$ and $\gamma \in [0, 1)$. Similarly as in the proof of part 1, by (2.14), (2.18), and (2.1), we see that

$$\int_{x_0}^{(b-\gamma x_0)/(1-\gamma)} \frac{G(x)}{\int_{\gamma x - \gamma x_0 + a}^{x} G(y)\,\mathrm{d}y}\,\mathrm{d}x$$
$$= \frac{1}{1-\gamma} \int_{x_0}^{b} \frac{G(x)}{\int_a^x G(y)\,\mathrm{d}y}\,\mathrm{d}x, \qquad x_0 \in (a, b),\ \gamma \in [0, 1).$$

Taking the derivative with respect to x_0 and cancelling γ, we obtain, over the range $x_0 \in (a, b)$ and $\gamma \in (0, 1)$,

$$\frac{1}{1-\gamma} \frac{G(x_0)}{\int_a^{x_0} G(y)\,\mathrm{d}y} - \frac{1}{1-\gamma} \frac{G((b-\gamma x_0)/(1-\gamma))}{\int_{(\gamma b - \gamma x_0)/(1-\gamma)+a}^{(b-\gamma x_0)/(1-\gamma)} G(y)\,\mathrm{d}y}$$
$$= \int_{x_0}^{(b-\gamma x_0)/(1-\gamma)} \frac{G(x) G(\gamma x - \gamma x_0 + a)}{(\int_{\gamma x - \gamma x_0 + a}^{x} G(y)\,\mathrm{d}y)^2}\,\mathrm{d}x.$$

Letting $\gamma \to 0$ yields

$$\frac{G(x_0)}{\int_a^{x_0} G(y)\,\mathrm{d}y} - \frac{G(b)}{\int_a^{b} G(y)\,\mathrm{d}y} = \int_{x_0}^{b} \frac{G(x) G(a)}{(\int_a^x G(y)\,\mathrm{d}y)^2}\,\mathrm{d}x, \qquad x_0 \in (a, b).$$

Upon some rearrangements we obtain

$$\frac{G(b) - G(a)}{\int_a^b G(y)\,\mathrm{d}y} \int_a^{x_0} G(y)\,\mathrm{d}y = \int_a^{x_0} G'(y)\,\mathrm{d}y, \qquad x_0 \in (a, b),$$

which implies that

$$\frac{G(b) - G(a)}{\int_a^b G(y)\,\mathrm{d}y} G(x) = G'(x), \qquad x \in (a, b).$$

Therefore, it must hold that

$$G(x) = c_1 \mathrm{e}^{c_2 x}, \qquad x \in [a, b],$$

for some constants $c_1 > 0$ and c_2 by the positivity and continuity of $G(\cdot)$.

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