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A TINY PECULIAR FRÉCHET SPACE

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In this paper we construct a countable sequentially regular Fréchet space L which fails to be regular. We also show that L has some other peculiar properties.

1

There is an extensive literature on sequential and Fréchet spaces (cf. [ENG], [SIW]). These are also known as "topological spaces in which sequences suffice" ([FRA]). Closely related (via modification functors (cf. [FKO_1])) to sequential spaces are convergence spaces (cf. [NOV_2], [NOV_3], [FKO_2], [KOU]), i.e., closure spaces in which the closure operator is derived from a sequential convergence structure. The concept dates back to M. Fréchet, who introduced the notion of an $\mathscr{L}$-space in [FRE]. He assumes a set L equipped with a sequential convergence $\mathfrak{L}$ such that each convergent sequence has a unique $\mathfrak{L}$-limit, each constant sequence $\langle x \rangle$ converges to $x \in L$, and each subsequence of a convergent sequence converges to the same $\mathfrak{L}$-limit (axioms $(\mathscr{L}_0)$, $(\mathscr{L}_1)$, $(\mathscr{L}_2)$ in the notation of [NOV_2], or H, S, F in the Katowice notation (cf. [KAT])). Starting with an $\mathscr{L}$-space $(L, \mathfrak{L})$ we can define sequentially open sets. These sets form a topology for L and the resulting topological space is a sequential space. It need not be Hausdorff but it has unique sequential limits (cf. [FKO_2]). This is a very efficient way how to construct sequential and Fréchet spaces with prescribed properties (cf. [NOV_1], [NOV_2], [FKO_2]). However, in case we construct a Fréchet space it suffices to show that the closure operator λ for L derived from the sequential convergence $\mathfrak{L}$ in the usual way (i.e., for $A \subset L$ let λA be the set of all $\mathfrak{L}$-limit points of sequences of points of A) satisfies the fourth Kuratowski closure axiom $\lambda^2 = \lambda$, the other three axioms being satisfied automatically (cf. [NOV_2]). Throughout the paper we assume that all spaces have unique sequential limits.

Recall (cf. [NOV_2], [FRI_3]) that a sequential space L is E-sequentially regular, where E is a subspace of the real line R, iff the convergence of sequences in L is projectively defined by the set $C_E(L)$ of all continuous functions on L into E, i.e., $x_n \to x$ in L iff for each $\varphi \in C_E(L)$ we have $\varphi(x_n) \to \varphi(x)$. If $E = R$, then we speak of sequential regularity. For spaces determined by sequences, the sequential regularity is a sequential analogue of complete regularity. To show the difference between the complete regularity and the sequential regularity the notion of $\aleph_\alpha$-complete regularity can be used. A topological space is said to be $\aleph_\alpha$-completely regular if a point x and a subset A can be separated by a continuous function whenever $x \notin \operatorname{cl} A$ and $\operatorname{card} A \leqq \aleph_\alpha$ ([NOV$_3$]). Clearly, if L is $\aleph_\alpha$-completely regular, then it is also sequentially regular.

It has been known for a long time (cf. [NOV$_1$]) that there is a regular sequential space all continuous functions on which are constant. On the other hand, it is also known that a sequentially regular Fréchet space need not be regular. Known examples of such a space (in [NOV$_2$] under CH, in [FKO$_2$] without CH) are uncountable. Answering a question raised in [FKO$_2$] we construct a countable sequentially regular Fréchet space which fails to be regular.

2

It is known that the irrational numbers can be identified (e.g., using continued fractions) with the Baire space ${}^\omega\omega$ of all mappings of ω into ω. Let $\mathcal{N} = \{N_f;\ f \in {}^\omega\omega\}$ be an almost disjoint family of infinite subsets N_f of ω (note that each N_f can be realized via a one-to-one sequnece $\langle r^f_n \rangle$ of rational numbers converging to f). Arrange each N_f into a one-to-one sequence $\langle i^f_n \rangle$ of elements of ω.

Consider the set $L = (\omega + 1) \times (\omega + 1) \setminus \{(\omega, n);\ n \in \omega\}$. Define a sequential convergence $\mathfrak{L}$ for L:

(i) For each $x \in L$, the constant sequence $\langle x \rangle$ converges to x;

(ii) For each $m \in \omega$, each subsequence of the sequence $\langle (m, n) \rangle$ converges to (m, ω);

(iii) For each $f \in {}^\omega\omega$, each subsequence of the sequence $\langle (i^f_n, f(i^f_n)) \rangle$ converges to (ω, ω).

Denote by λ the closure operator for L derived from $\mathfrak{L}$. We shall prove that L equipped with λ is a countable sequentially regular Fréchet space which fails to be regular.

Proposition 1. (a) *λ satisfies the fourth Kuratowski axiom for a closure operator.*

(b) *The following subsets of L are clopen:*

$H(m, n)$ is a singleton $\{(m, n)\}$, where $m, n \in \omega$;

$H(m) = \{(m, n);\ n \in \omega + 1\}$, where $m \in \omega$;

$H(f) = \{(m, n);\ m \in N_f,\ n \in \omega + 1,\ n \neq f(m)\}$, where $f \in {}^\omega\omega$;

$H(B) = \{(m, n);\ m \in B,\ n \in \omega + 1\}$, where B is an infinite subset of ω such that the set $B \cap N_f$ is finite for each $f \in {}^\omega\omega$.

(c) *L is $\{0, 1\}$-sequentially regular.*

(d) *L is not regular.*

Proof. (a) and (b) follow immediately from the definition of $\mathfrak{L}$.

(c) It suffices to prove (cf. [KOU]) that points of L are separated by clopen sets and if $A \subset L$ is an infinite subset and $x \in L \setminus \lambda A$, then there is a clopen subset $H \subset L$ such that $x \in L \setminus H$ and the set $H \cap A$ is infinite. For, if a sequence $\langle x_n \rangle$ does not converge in L to a point x, then there are a neighbourhood U of x and a subsequence $\langle x'_n \rangle$ of $\langle x_n \rangle$ such that $x'_n \in L \setminus U$ for each $n \in \omega$. Hence, then there is a subsequence $\langle x''_n \rangle$ of $\langle x'_n \rangle$ and a clopen set H which separates x and $\langle x''_n \rangle$. Let φ be a function on L which equals 0 on H and 1 on $L \setminus H$. Clearly, $\varphi \in C_{\{0,1\}}(L)$ and $\langle \varphi(x_n) \rangle$ does not converge to $\varphi(x)$.

It is easy to see that points of L are separated by clopen sets. Now, let $A \subset L$ be an infinite subset and let $x \in L \setminus \lambda A$.

1. If $x = (m, n) \in \omega \times \omega$, then we put $H = L \setminus H(m, n)$.
2. If $x = (m, \omega)$, then we put $H = L \setminus H(m)$.
3. Let $x = (\omega, \omega)$. Denote $A_1 = A \cap (\omega \times \omega)$,

$A_2 = A \cap (\omega \times (\omega + 1))$. Since $x \in L \setminus A$, we have $A = A_1 \cup A_2$. If $(m, \omega) \in \lambda A_1$ for some $m \in \omega$, then we put $H = H(m)$. Now, suppose that $\lambda A_1 \cap (\omega \times (\omega + 1)) = \emptyset$. If A_2 is finite, then we put $H = A_1$. If A_2 is infinite, then there are two possibilities. First, there is an irrational $f \in {}^{\omega}\omega$ such that the set $\{m \in \omega;\ (m, \omega) \in A_2\} \cap N_f$ is infinite. In this case we put $H = H(f)$. Second, the set $\{m \in \omega;\ (m, \omega) \in A_2\} \cap N_f$ is finite for each $f \in {}^{\omega}\omega$. Then we put $H = H(\{m \in \omega;\ (m, \omega) \in A_2\})$. Since in all cases H is clopen and $H \cap A$ is infinite, the proof of (c) is complete.

(d) It suffices to show that in L there are a closed set A and a point $x \in I \setminus A$ such that x and A cannot be separated by disjoint open sets. The set $A = \{(m, \omega) \in L;\ m \in \omega\}$ is closed and $x = (\omega, \omega) \in L \setminus A$. If O_A is an open set containing A, then for some $f \in {}^{\omega}\omega$ we have $\{(m, n) \in L;\ m \in \omega,\ n > f(m)\} \subset O_A$. But then for $f + 1 = g \in {}^{\omega}\omega$ the sequence $\langle (i^g_n, g(i^g_n)) \rangle$ converges to x. Thus A and x cannot be separated by disjoint open sets. This completes the proof.

3

It has already been mentioned in section 1 that every $\aleph_\alpha$-completely regular space is sequentially regular. It was shown in [FRI$_1$] that the space Λ_∞ constructed by B. F. Jones in [JON] (a quotient of a sequence of Niemytzki planes) is a sequentially regular Fréchet space which fails to be $\aleph_0$-completely regular (a solution of Problem 7 in [NOV$_3$]). The space is regular and uncountable. Clearly, for countable spaces the notions of complete regularity and $\aleph_0$-complete regularity coincide. It follows immediately that the space L constructed in section 2 cannot be $\aleph_0$-completely regular (hence also solves Problem 7 in [NOV$_3$]). Further, in [FRI$_2$] it was shown that Λ_∞ has some peculiar properties (Propositions 4, 5, and 6 in [FRI$_2$]). In this section we show that our space L, albeit countable, has the same properties.

Proposition 2. *Let* $Z = \{x \in L;\ x = (m, \omega),\ m \in \omega + 1\}$. *Then* Z *is a closed discrete subset of* L *which is not* C^**-embedded in* L.

Proof. It follows from the definition of $\mathfrak{L}$ that Z is a closed discrete subset of L. Put $A = \{x \in L;\ x = (m, \omega),\ m \in \omega\}$, and define a function φ on Z by $\varphi[A] = 0$ and $\varphi((\omega, \omega)) = 1$. Then φ is a bounded continuous function on Z but it follows from the proof of (d) in Proposition 1 that φ cannot be continuously extended onto L.

Proposition 3. *For each closed discrete infinite subset* I *of* L, *there are infinite subset* I_1 *and* I_2 *of* I *and a function* $\varphi \in C_{\{0,1\}}(L)$ *such that* $\varphi[I_1] = 0$ *and* $\varphi[I_2] = 1$.

Proof. Let I be a closed discrete infinite subset of L. It suffices to prove that there are two infinite disjoint clopen subsets of L, each of which contains infinitely many points of I.

There are two possibilities. 1. For infinitely many n there is a natural number k_n such that $(n, k_n) \in I$. Arrange these points into a one-to-one sequence $\langle a_n \rangle$. Since (ω, ω) cannot be a limit point of the set $\bigcup_{n\in\omega} \{a_n\}$, sets $\bigcup_{n\in\omega} \{a_{2n}\}$ and $\bigcup_{n\in\omega} \{a_{2n-1}\}$ are disjoint clopen subsets of L.

2. The set $(\omega \times \omega) \cap I$ is finite. Denote $N_I = \{n \in \omega;\ (n, \omega) \in I\}$. If the sets $N_I \cap N_f$ are finite for each $f \in {}^{\omega}\omega$, then for each two disjoint infinite subsets B_1 and B_2 of N_I the sets $H(B_1)$ and $H(B_2)$ (see Proposition 1) are disjoint clopen subsets of L and both contain infinitely many points of I. If the set $N_I \cap N_f$ is infinite for some $f \in {}^{\omega}\omega$, then there are disjoint infinite subsets B_3 and B_4 of N_I such that the sets $H(B_3) \cap H(f)$ and $H(B_4) \cap H(f)$ are disjoint infinite clopen subsets of L and both contain infinitely many points of I. This completes the proof.

The almost disjoint family $\mathcal{N}$ used in the construction of the space L in section 2 has the prescribed cardinality of continuum but otherwise it is not specified. Some properties of L, however, might depend on a suitable choice of $\mathcal{N}$.

Denote by N_1 and N_2 the set of all odd and all even natural numbers. Let $\mathcal{N}_i = \{N_f^{(i)};\ f \in {}^{\omega}\omega\}$ be an almost disjoint family of infinite subsets $N_f^{(i)}$ of N_i, $i \in \{1, 2\}$. If we put $N_f = N_f^{(1)} \cup N_f^{(2)}$, then $\mathcal{N} = \{N_f;\ f \in {}^{\omega}\omega\}$ is an almost disjoint family of infinite subsets of ω. Consider the space L constructed via the just specified family $\mathcal{N}$.

Proposition 4. *In* L *there are two disjoint closed discrete infinite subsets* I_1 *nad* I_2 *which cannot be separated by continuous functions on* L.

Proof. Put $I_1 = \{(2n - 1, \omega);\ n \in \omega\}$ and $I_2 = \{(2n, \omega);\ n \in \omega\}$. Clearly, the sets I_1 and I_2 are disjoint closed discrete infinite subsets of L. Suppose that, on the contrary, there is a continuous function φ on L such that $\varphi[I_1] = 0$ and $\varphi[I_2] = 1$. Let ε be a positive real number. Since for each k, the sequence $\langle (k, n) \rangle$ converges in L to (k, ω), there is a natural number $n(\varepsilon, k)$ such that $|\varphi((k, n)) - \varphi((k, \omega))| < \varepsilon$ for all $n > n(\varepsilon, k)$. Define a function $f \in {}^{\omega}\omega$ by $f(k) = n(\varepsilon, k) + 1$. The sequence $\langle (i_n^f, f(i_n^f)) \rangle$ converges in L to (ω, ω). If $\varepsilon < \frac{1}{2}$, then it follows from the construction of $\mathcal{N}$ that φ attains on the points of the sequence values close to 0 and at the same time values close to 1. This is a contradiction.

We conclude with a quotient-type construction using our space L as a building block. Consider two disjoint copies of L, denote them L_1 and L_2. Further, if $x \in L$, denote by x_α the corresponding point in L_α, $\alpha \in \{1, 2\}$. Let M be the quotient space obtained from the disjoint topological sum of L_1 and L_2 by sticking together the corresponding points $(m, \omega)_1$ and $(m, \omega)_2$ for all $m \in \omega$.

Proposition 5. (a) *M is a countable Fréchet space.*

(b) *M is Hausdorff.*

(c) *Points $(\omega, \omega)_1$ and $(\omega, \omega)_2$ cannot be separated by continuous functions on M.*

Proof. (a) and (b) follow immediately from the construction of M.

(c) Let $\varphi \in C_R(M)$ and let ε be a positive real number. Denote by (k, ω) the point of M obtained by sticking together $(k, \omega)_1$ and $(k, \omega)_2$. Now, for each $k \in \omega$ and for each $\alpha \in \{1, 2\}$ the sequence $\langle (k, n)_\alpha \rangle$ converges to (k, ω). Hence, for each $k \in \omega$ there is a natural number $n(\varepsilon, k)$ such that $|\varphi((k, n)_\alpha) - \varphi((k, \omega))| < \varepsilon$ whenever $n > n(\varepsilon, k)$, $\alpha \in \{1, 2\}$. Define a function $f \in {}^\omega\omega$ by $f(k) = n(\varepsilon, k) + 1$. Then for each $\alpha \in \{1, 2\}$ the sequence $\langle (i_n^f, f(i_n^f))_\alpha \rangle$ converges to $(\omega, \omega)_\alpha$. Since $|\varphi((i_n^f, f(i_n^f))_1) - \varphi((i_n^f, f(i_n^f))_2)| < 2\varepsilon$, it follows that $\varphi((\omega, \omega)_1) = \varphi((\omega, \omega)_2)$.

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