A Topological Method for Finding Invariant Sets of Continuous Systems

Laurent Fribourg[2], Eric Goubault[1], Sameh Mohamed[21], Marian Mrozek[3] and Sylvie Putot[1]

[1] LIX, Ecole Polytechnique, 91128 Palaiseau Cedex, France
Email: {goubault,putot}@lix.polytechnique.fr
[2] LSV, ENS Cachan & CNRS, France
Email: {fribourg,mohamed}@lsv.ens-cachan.fr
[3] Division of Computational Mathematics, Jagiellonian University, Kraków, Poland
Email: marian.mrozek@uj.edu.pl

Abstract. A usual way to find positive invariant sets of ordinary differential equations is to restrict the search to predefined finitely generated shapes, such as linear templates, or ellipsoids as in classical quadratic Lyapunov function based approaches. One then looks for generators or parameters for which the corresponding shape has the property that the flow of the ODE goes inwards on its border. But for non-linear systems, where the structure of invariant sets may be very complicated, such simple predefined shapes are generally not well suited. The present work proposes a more general approach based on a topological property, namely Ważewski's property. Even for complicated non-linear dynamics, it is possible to successfully restrict the search for isolating blocks of simple shapes, that are bound to contain non-empty invariant sets. This approach generalizes the Lyapunov-like approaches, by allowing for inwards and outwards flow on the boundary of these shapes, with extra topological conditions. We developed and implemented an algorithm based on Ważewski's property, SOS optimization and some extra combinatorial and algebraic properties, that shows very nice results on a number of classical polynomial dynamical systems.

1 Introduction

This paper describes a new method for proving the existence of a *positive invariant* set (generally called invariant set in computer science - we will stick to the former terminology, classical in the dynamical systems community) of a dynamical system, inside some region of the state space. Positive invariant sets are central to control theory and to validation of systems, such as programs (when considering discrete dynamical systems), physical systems (considering continuous dynamical systems), or hybrid systems. In this paper, we are focusing on continuous dynamical systems, but part of the method described here makes sense in a discrete setting - in particular, Conley's index theory [20] can be developed for discrete systems, only the differential conditions we are giving in this paper have to be replaced by different conditions, which will be developed elsewhere.

Let us consider an autonomous polynomial differential equation, for the rest of this article :

$$\frac{dx}{dt} = f(x) \tag{1}$$

where x is a vector $(x_1, \ldots, x_n)$ of $\mathbb{R}^n$ and f is a vector of n polynomials in $x_1, \ldots, x_n$, e.g. for all $i = 1, \ldots, n$, $f_i \in \mathbb{R}[x_1, \ldots, x_n]$, the multivariate polynomial ring on n variables. As a polynomial function is locally Lipschitz, we know by the Cauchy-Lipschitz theorem that the vector field f generates a flow $\varphi : U \to \mathbb{R}^n$, where U is an open subset of $\mathbb{R} \times \mathbb{R}^n$ in such a way that $t \mapsto \varphi(x, t)$ is a solution of the differential equation.

A *positive invariant set* is a subset of the state-space such that if the initial state of the system belongs to this set, then the state of the system remains inside the set for all future time instances. An *invariant set* is a subset of the state-space which is positively invariant under the flow, and positively invariant under the opposite flow (i.e. it is also negatively invariant). The classical approach to find positive invariant sets of dynamical systems is through the determination of a Lyapunov function, generally polynomial, which decreases along trajectories of the dynamical system, positive in a neighborhood of the equilibrium point (the only point on which its value is zero). This approach is particularly well-suited to linear dynamical systems, where quadratic Lyapunov functions prove to be the right class of functions, but in the case of non-linear systems [30,15] the shape of the invariant set itself may be very complicated, in fact, far too complicated to be easily found in general by polynomial Lyapunov functions. Some authors, including the authors of the present article, have shown how to find, in some cases, rational functions [30], and even functions in the differential field extension of rational functions by some logarithms and exponentials, which could be candidate Lyapunov functions [11]. This is of course highly costly in computational power.

Recently, some authors have proposed to use piecewise linear [24] or piecewise quadratic [2] Lyapunov functions inspired both by abstract interpretation of programs [25] and recent results in hybrid systems theory, most notably in switched systems theory [26,3]. If these methods, based on "templates", are computationally tractable, they are limited by the fact that they can only find very specific shaped positive invariant sets ; they must be in particular convex, which is not always the case, even for simple classical systems.

We propose in this paper a method that also builds on templates. But instead of considering more complicated Lyapunov functions (or shapes of candidate invariant sets) as in [11], we relax the classical condition that, for a template (given by a piecewise linear, quadratic, and even more general polynomial Lyapunov function) to be positively invariant, the flow of the differential system must go inwards, condition that can be expressed as a negativity condition on a certain Lie derivative (a notion defined in Section 3). We relax this classical condition by asking only some parts of the boundary of the template to have inwards flows, relying on some simple techniques of Conley's index theory [8], and in particular

Ważewski's property, to show the existence of a positive invariant set within this template. The positive invariant set itself may be very complicated, but we do not need to precisely describe it ; the template serves as an outer approximation of this positive invariant set.

Let us for instance consider the case of Example 1, which will be our running example throughout this paper.

Example 1. (Ex 2.8 of [20]) : $\dot{x} = y, \dot{y} = y + (x^2 - 1)\left(x + \frac{1}{2}\right)$. This system has several invariant sets in $B = [-2, 2] \times [-2, 2] \subset \mathbb{R}^2$: there are in particular 3 fixed points $(-1, 0)$, $\left(-\frac{1}{2}, 0\right)$ and $(1, 0)$ for this system within this box. On this system, it would be difficult to find a linear template on which we can prove the inward flows property (in fact, there is a "natural" degree 4 polynomial Lyapunov function, see [20]), whereas we will see that boxes such as B can be easily shown to contain a positive invariant set, using our approach.

Claims and contents of this article. The main idea of our article is that even though invariant sets for nonlinear dynamics may be very complicated to represent, and thus to find (e.g. by an explicit Lyapunov function), there exist *topological* criteria to deduce that there exists a non-empty positive invariant set inside some region of the state space. Note that not all positive invariant sets contain a point onto which the dynamical system converges. There might be a limit cycle, or a more complicated recurrent sets, for instance. For more complicated asymptotic behaviours, within invariant sets, we rely on notions from the Conley index theory [20], which we quickly state in Section 2. The key useful notions here, are that of an *isolating block* and the Ważewski's theorem, which gives a sufficient condition to the existence of a non-empty (positive) invariant set within an isolating block.

Several approaches have been developed over the years to algorithmically compute such isolating sets and index pairs, but most of them have been derived for discrete time dynamical systems [29,21]. Most approaches for continuous-time systems reduce the problem to the discrete-time setting by constructing rigorous outer approximations of a map for the flow, which of course involves approximating the solution of the ordinary differential equation which describes the system.

In this work, we generalize the template based approach of [24], designed originally for linear systems, to derive some algebraic conditions for a template to be an isolating block. This is done in Section 3, in which we give a sufficient condition for a polynomial template to be an isolating block, expressed as conditions on the Lie derivatives of the polynomial functions involved, on the faces of the template, that generalize the conditions given in [28]. The main difficulty is in fact of a topological nature : most of Section 3 is concerned with proving that the so-called *exit set* on the template, under the flow given by Equation 1, i.e. the set of states leaving the template, on its faces, is closed. This is necessary for the template to be an isolating block. Note that our templatized isolating blocks are particular C^∞ isolating-block-with-corners of [14], that are as powerful as (generalized) Lyapunov functions for finding invariant sets (Theorem 2.4 of [14]);

but these isolating blocks are "robust": they are still isolating blocks for nearby flows (Theorem 3.5 of [14]), which make them more robust numerically. They are also close to the polyfacial sets of [23] attributed there to the original paper of Ważewski [31].

We remark then, that the conditions we gave for a template to be an isolating block can be solved, in particular, by Sum of Squares programming [17] using Stengle's nichtnegativstellensatz and even, most often, just Putinar's positivstellensatz [22], which is computationally tractable using a SdP (Semi Definite Programming) relaxation. This makes a second major difference with [28] where an interval-based method is used instead. Finally, an isolating block may only contain an empty invariant set, unless the conditions for Ważewski's principle are satisfied. For the purpose of this paper, we use a simpler condition, of a purely combinatorial nature, in Section 4. We end up by discussing the algorithm on simple examples from the literature, in Section 5. The first simple experiments obtained with our *Matlab* implementation are still quite costly, but we propose in the conclusion a number of possible algorithmical improvements.

2 Some basics of dynamical systems theory

The following definitions come from Conley index theory, and the qualitative description of nonlinear dynamics [20]. Let us call $\varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ the *flow* function, such that $\varphi(., x_1, \ldots, x_n)$ is the unique solution to the differential equation system (1) starting, at time 0, at state $(x_1, \ldots, x_n) \in \mathbb{R}^n$, meaning that $\varphi(0, x_1, \ldots, x_n) = (x_1, \ldots, x_n)$ and $\frac{d\varphi}{dt}(t, x_1, \ldots, x_n) = f \circ \varphi(t, x_1, \ldots, x_n)$. (Positive) invariant sets are invariant under the flow φ, for all (resp. positive) times, i.e. they are sets S such that (resp. $\varphi(\mathbb{R}^+, S) \subset S$) $\varphi(\mathbb{R}, S) \subset S$.

A subtle point is that the method is designed to find invariant sets S within a compact set N, but not quite positive invariant sets. But in fact, it is well-known that when we have one, we will have the other [5,23].

For non-linear dynamics, the shape of invariant sets can be very complicated. A central notion to our method is that of *isolating block*, that isolates invariant sets, meaning that invariant sets therein, if ever they exist, are necessarily in the *interior* of isolating blocks.

Definition 1. *(Isolating block). A compact set B is an* isolating block *if*

(a) $B^- = \{x \in B \mid \varphi([0, T), x) \not\subseteq B, \forall T > 0\}$ *is closed*
(b) $\forall T > 0, \{x \in B \mid \varphi([-T, T], x) \subseteq B\} \subseteq intB$

Condition (a) imposes that the *exit set* B^-, i.e. the set of states of B which leave B under flow φ, is closed in the topology of $\mathbb{R}^n$. When condition (b) is satisfied, B is called an isolating neighborhood. The combination of (a) and (b) guarantees that no trajectory is inner tangential to the boundary ∂B of B. A fundamental difficulty in computational topological dynamics is that isolating neighborhoods are generally much easier to construct than isolating blocks.

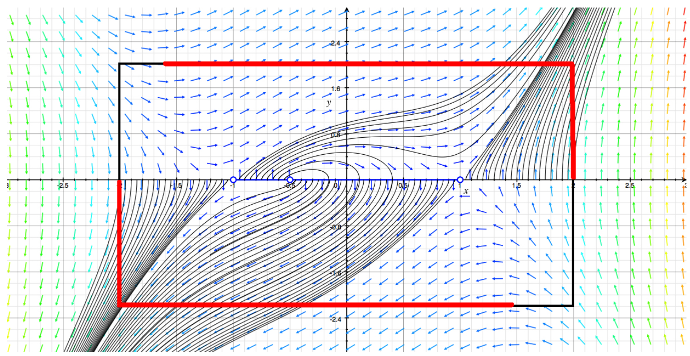

Fig. 1. The three fixed points and the exit set (in red) of the system of Example 1.

Example 2. For Example 1, we will see that for $B = [-2, 2] \times [-2, 2]$ the box of Figure 2, B^- is the set of four red segments, one on each face, on the same Figure, and we will prove (Example 5.) that B^- is closed, so that B is an isolating block for this system. Note that this is a robust notion : all $B = [-a, a] \times [-b, b]$ with $a > 1$ and $b > 1$, in particular, are isolating blocks.

Still, isolating blocks may not contain interesting (meaning non-empty) invariant sets. There is a simple topological condition on isolating blocks that implies the existence of a non empty invariant set therein.

Theorem 1. *(Ważewski Property [20]). If B is an isolating block and B^- is not a deformation retract of B then there exists a not-empty invariant set S in the interior of B.*

We will not define formally a deformation retract, for the sake of simplicity. Let us just say that a deformation retract of a topological space B is a subspace which is an "elastic" deformation of it, so that it retains its essential topological features. For the method we are developing here, we will content ourselves with the much weaker statement that among the topological features that are retained in a deformation retract, is the number of connected components.

Example 3. We saw in Example 2 that we had a square B with closed exit set B^- made of two connected components. Clearly B^- is not a deformation retract of B, as B^- is made of two connected components, and B of only one.

3 Isolating blocks : algebraic conditions

We are now giving a simple criterion for a compact set B, given as a general polynomial template, to be an isolating block for the dynamics given by Equation 1. Note that although the method can be defined for general polynomial templates, which is what we describe here, isolating blocks are robust properties that permit the use of very simple templates in general, which will be the case in the experiments presented here. The set $B \subseteq \mathbb{R}^n$ is defined, for some vector $c = (c_1, \ldots, c_m) \in \mathbb{R}^m$, by the m polynomial inequalities :

$$(P) \begin{cases} p_1(x_1, \ldots, x_n) \leq c_1 \\ \ldots \\ p_m(x_1, \ldots, x_n) \leq c_m \end{cases}$$

We call P_i^c the face of template B given by $\{(x_1, \ldots, x_n) | p_i(x_1, \ldots, x_n) = c_i\} \cap B$ which might be proper (non-empty) or not. In what follows, we suppose that each face of B is proper.

We call *minimal polynomial templates*, the templates B whose border ∂B is equal (and not just included as would be generally the case) to $\bigcup_{i=1}^s \{x | p_i(x) = c_i, p_j(x) \leq c_j \ \forall j \neq i\}$. For $x \in \partial B$, we note $I(x)$ the non-empty and maximal set of indices in $1, \ldots, m$ such that for all $i \in I(x)$, $p_i(x) = c_i$.

Let us now define Lie derivatives, that we will use hereafter.

Definition 2. *(Lie derivative and higher-order Lie derivatives). The Lie derivative of $h \in \mathbb{R}[x]$ along the vector field $f = (f_1, \ldots, f_n)$ is defined by*

$$\mathcal{L}_f(h) = \sum_{i=1}^n \frac{\partial h}{\partial x_i} f_i = \langle f, \nabla h \rangle$$

Higher-order derivatives are defined by $\mathcal{L}_f^{k+1}(h) = \mathcal{L}_f(\mathcal{L}_f^{(k)}(h))$ with $\mathcal{L}_f^0(h) = h$.

For polynomial dynamical systems, only a finite number of Lie derivatives are necessary to generate all higher-order Lie derivatives. Indeed, let $h \in \mathbb{R}[x_1, \ldots, x_n]$, we recursively construct an ascending chain of ideals of $\mathbb{R}[x_1, \ldots, x_n]$ by appending successive Lie derivatives of h to the list of generators:

$$< h > \subseteq < h, \mathcal{L}_f^1(h) > \subseteq \cdots \subseteq < h, \mathcal{L}_f^1(h), \ldots, \mathcal{L}_f^{(N)}(h) >$$

Since the ring $\mathbb{R}[x]$ is Noetherian [16], this increasing chain of ideals has necessarily a finite length: the maximal ideal is called the differential radical ideal of h and will be noted $\sqrt[\mathcal{L}]{< h >}$. Its order is the smallest N such that:

$$\mathcal{L}_f^{(N)}(h) \in < h, \mathcal{L}_f^{(1)}(h), \ldots, \mathcal{L}_f^{(N-1)}(h) > \tag{2}$$

Not surprisingly, this is a notion that has already been used to characterize algebraic positive invariant sets of dynamical systems [10].

This N is computationally tractable. If we note N_i the order of the differential radical ideal $\sqrt[\mathcal{L}]{< p_i >}$, then for face i we should compute the successive Lie derivatives until N_i. This can be done by testing if the *Gröbner basis* spanned by the derivatives changes. Indeed, two ideals are equal if they have the same reduced Gröbner basis (usually a Gröbner basis software produces reduced bases) [1]. If we denote by $\mathcal{G}(\{g_1, \cdots, g_n\})$ the Gröbner basis of $\{g_1, \cdots, g_n\}$, the first n s.t. $\mathcal{G}(\{\mathcal{L}_f^{(0)}(p_i), \cdots, \mathcal{L}_f^{(n)}(p_i)\}) = \mathcal{G}(\{\mathcal{L}_f^{(1)}(p_i), \cdots, \mathcal{L}_f^{(n+1)}(p_i)\})$ is equal to N_i. It can be observed that upper bounds for N_i could be used instead of computing Gröbner bases in some cases, see [27].

Example 4. If we take the first face P_1^c of the template $\{p_1 = -x, p_2 = x, p_3 = -y, p_4 = y\}$ with $\{c_1 = 2, c_2 = 2, c_3 = 2, c_4 = 2\}$ i.e $P_1^c = \{-x = 2, x \leq 2, -y \leq 2, y \leq 2\}$ then $\mathcal{L}_f^{(1)}(p_1) = -y$, $\mathcal{G}(\{\mathcal{L}_f^{(0)}(p_1), \mathcal{L}_f^{(1)}(p_1)\}) = \{-x, -y\}$; $\mathcal{L}_f^{(2)}(p_1) = -y - (x^2 - 1)\left(x + \frac{1}{2}\right)$, $\mathcal{G}(\{\mathcal{L}_f^{(0)}(p_1), \mathcal{L}_f^{(1)}(p_1), \mathcal{L}_f^{(2)}(p_1)\}) = \mathbf{1}$. We can already deduce from this that $N_1 = 3$.

Now, in order to find isolating blocks, we need to find and prove some topological properties on the exit sets, see Definition 1. For polynomial templates, we rely on Lemma 1:

Lemma 1. *Let x_0 be a point on the border ∂B of a minimal polynomial template B. Then x_0 is in the exit set B^- of B if and only if, for some $i_0 \in I(x)$,*

$$\exists k_0 > 0 \text{ such } \mathcal{L}_f^{(k_0)}(p_{i_0}) > 0 \text{ and } \forall 0 < k < k_0 \; \mathcal{L}_f^{(k)}(p_{i_0}) = 0$$

For a template B to be an isolating block, we know from Definition 1 that we need to check first that the exit set B^- is closed :

Lemma 2. *Let B be a compact minimal polynomial template defined by the set of inequalities (P) and let N_i be the order of the differential radical ideal $\sqrt[\mathcal{L}]{< p_i >}$ (i.e. the index defined by Equation 2) for the dynamical system of Equation 1.*

If for each face P_i^c of template B, for all $k \in \{1, \cdots, N_i - 2\}$,

$$(H_k^i) \;:\; \begin{cases} \{x \in P_i^c \mid \mathcal{L}_f^{(1)}(p_i)(x) = 0, \; \cdots, \; \mathcal{L}_f^{(k)}(p_i)(x) = 0, \\ \qquad\qquad \mathcal{L}_f^{(k+1)}(p_i)(x) < 0\} = \emptyset \end{cases}$$

then B^- is closed and is equal to $\bigcup_{i=1}^m \{x \in P_i^c \mid \mathcal{L}_f^{(1)}(p_i)(x) \geq 0\}$.

Proof. We begin to show that if for each face P_i^c of template B, for all $k \in \{1, \cdots, N_i - 2\}$, we have $\{x \in P_i^c \mid \mathcal{L}_f^{(1)}(p_i)(x) = 0, \; \cdots, \; \mathcal{L}_f^{(N_i-1)}(p_i)(x) = 0\} = \emptyset$ as well as (H_k^i) as above, then B^- is closed and is equal to $\bigcup_{i=1}^m \{x \in P_i^c \mid \mathcal{L}_f^{(1)}(p_i)(x) \geq 0\}$. If we set, for $k = 0, \ldots, N_i - 1$:

$$\begin{aligned} P_k &= \{x \mid \mathcal{L}_f^{(1)}(p_i)(x) = 0, \; \cdots, \; \mathcal{L}_f^{(k)}(p_i)(x) = 0, \mathcal{L}_f^{(k+1)}(p_i)(x) \geq 0\} \\ Q_k &= \{x \mid \mathcal{L}_f^{(1)}(p_i)(x) = 0, \; \cdots, \; \mathcal{L}_f^{(k)}(p_i)(x) = 0, \mathcal{L}_f^{(k+1)}(p_i)(x) > 0\} \\ R_k &= \{x \mid \mathcal{L}_f^{(1)}(p_i)(x) = 0, \; \cdots, \; \mathcal{L}_f^{(k)}(p_i)(x) = 0, \mathcal{L}_f^{(k+1)}(p_i)(x) < 0\} \end{aligned}$$

We then have, for $k = 1, \ldots, N_i - 2$, $P_k = Q_k \cup P_{k+1} \cup R_{k+1}$. Given that N_i is the index of the differential ideal $\sqrt[\mathcal{L}]{< p_i >}$, we know that $Q_{N_i-1} = \emptyset$, $R_{N_i-1} = \emptyset$, since $\mathcal{L}_f^{(1)}(p_i)(x) = 0, \; \cdots, \; \mathcal{L}_f^{(N_i-1)}(p_i)(x) = 0$ implies $\mathcal{L}_f^{(N_i)}(p_i)(x) = 0$. Given the hypotheses, we know also that for all $k = 1, \ldots, N_i - 2$, $R_k = \emptyset$, and $P_{N_i-1} = \emptyset$. This means that $P_k = \bigcup_{i=k}^{N_i-2} Q_i$.

By Lemma 1, x is in $B_- \cap P_i^c$ if and only if it is in $\cup_{i=0}^{\infty} Q_i \cup = \cup_{i=0}^{N_i-2} Q_i$. This last set is, by the equation above, equal to P_0, which is the inverse image by a continuous function (the higher Lie derivative) of the closed set $[0, max\; \mathcal{L}_f^{(1)}(p_i)(B)]$ (since B is compact in $\mathbb{R}^n$).

We can then notice that the exit set B_- is the union of $B_- \cap P_i^c$, for all $i = 1, \ldots, m$, each one of which is closed, hence is closed.

Note now that $p_i = c \wedge \mathcal{L}_f^1(p_i) \ \ldots \ \mathcal{L}_f^{N_i-1}(p_i) = 0$ is equivalent to saying that all solutions to the ODE are constant on face p_i. This implies that the face P_i^c is not an exit set, and the exit set relative to P_i^c is trivially closed (since empty). This means that, to check that a template is an isolating block, we only need to check that for all $k \in \{1, \cdots, N_i - 2\}$, :

$$p_i = c_i \wedge (p_j \leq c_j)_{j \neq i} \wedge \mathcal{L}_f^{(1)}(p_i)(x) = 0, \ \cdots, \ \mathcal{L}_f^{(k)}(p_i)(x) = 0, \ \Rightarrow \mathcal{L}_f^{(k+1)}(p_i)(x) \geq 0$$

Note that the criterion used in [28] strictly implies all (H_k^i).

Finally, for B a polynomial template to be an isolating block, we need to show (see Definition 1) that there is no inner tangential flow within it. It is easy to see that not having any inner tangential flow is equivalent to asking that for each of its faces P_i^c, for all $k \in \{0, \cdots, N_i - 1\}$, no $x \in P_i^c$ can satisfy the following set of equalities and inequalities :

$$\mathcal{L}_f^{(1)}(p_i)(x) = 0, \ \cdots, \ \mathcal{L}_f^{(2k-1)}(p_i)(x) = 0, \ \mathcal{L}_f^{(2k)}(p_i)(x) < 0 \tag{3}$$

This is clearly satisfied when the condition of Lemma 2 is satisfied.

The algorithm we are going to develop now thus relies on checking the condition of Lemma 2. This can be checked using Sum of Squares optimization [17] and Stengle's nichtnegativstellensatz, for increasing k from 1 to $N_i - 2$, for each face i of the template. This is done as follows. We determine polynomials α_j ($j = 0, \ldots, k$), SoS polynomials $\beta_{S,\mu}$ ($S \subseteq \{1, \ldots, i-1, i+1, \ldots, m\}, \mu \in \{0, 1\}$) and an integer l, such that

$$\sum_{j=0}^{k} \alpha_j \mathcal{L}_f^{(j)} \ + \sum_{S \subset \{1,\ldots,i-1,i+1,\ldots,m\}} \beta_{S,\mu} G_{S,\mu} + \left(\mathcal{L}_f^{(k+1)}\right)^{2l} = 0 \tag{4}$$

where $G_{S,\mu} = (-\mathcal{L}_f^{(k+1)})^{\mu} \Pi_{s \in S}(c_s - p_s)$ for any $S \subseteq \{1, \ldots, i-1, i+1, \ldots, m\}$ and $\mu \in \{0, 1\}$ and the convention that $\mathcal{L}_f^0(p_i) = c_i - p_i$. Practically speaking, this is done by bounding the degrees of the polynomials α_j and $\beta_{S,\mu}$ we are looking for, and taking low values for l (in all our examples, we took $l = 1$). Hence we get the following Proposition, at the heart of our algorithm :

Proposition 1. *For each face P_i^c, if for all $k = 1, \ldots, N_i - 2$, there exist polynomials α_j ($j \in \{0, \ldots, k\}$) and sum-of-squares polynomials $\beta_{S,\mu}$ ($S \subseteq \{1, \ldots, i-1, i+1, \ldots, m\}$ and $\mu \in \{0, 1\}$) such that Equation 4 holds, then the template P_i^c is an isolating block.*

To provide for faster results, in most cases, we begin, for a given k (and i), instead of solving (H_k^i) by Equation 4, by solving the simpler property $p_i = c_i \wedge (p_j \leq c_j)_{j \neq i} \wedge \mathcal{L}_f^{(1)}(p_i)(x) = 0, \ \cdots, \ \mathcal{L}_f^{(k)}(p_i)(x) = 0, \ \Rightarrow \mathcal{L}_f^{(k+1)}(p_i)(x) > 0$. If so, we can stop testing (H_k^i) for higher values of k since they are then trivially satisfied. We can test whether this equation above is true using Putinar's *positivstellensatz* [22] which is much less computationally demanding than Stengle's nichtnegativstellensatz, and which also stops the algorithm potentially before reaching $k = N_i - 2$. A sufficient condition for this to be true is to find polynomials α_l ($l = 1, \ldots, k$), γ_i and sum-of-squares polynomials β_j $j = 1, \ldots, m$, $j \neq i$ such that $\mathcal{L}_f^{(k+1)}(p_i) = \sum_{j=1}^{k} \alpha_j \mathcal{L}_f^{(j)}(p_i) + \beta_0 + \sum_{j=1, j\neq i}^{m} \beta_j (c_j - p_j) + \gamma_i (p_i - c_i)$. For each fixed integer $D > 0$, which we choose as a bound on the degree of polynomials α_l, γ_i and β_j, this can be tested by semidefinite programming (see [18] and the improvement of [19] for a discussion on the maximal degree for these problems).

Example 5. We take again the face P_1^c, and try to prove (H_1^1) for example. A sufficient condition is to find polynomials α, γ (for equality conditions) and sum-of-squares polynomials $\beta_0, \beta_1, \beta_2, \beta_3$ (for inequality conditions) such that

$$\mathcal{L}_f^{(2)}(p_1) = \alpha \mathcal{L}_f^{(1)}(p_1) + \beta_0 + \beta_1(c_2 - p_2) + \beta_2(c_3 - p_3) + \beta_3(c_4 - p_4) + \gamma(p_1 - c_1)$$

which is trivially satisfied with $\alpha = 1$, $\beta_0 = \frac{9}{2}$, $\beta_1 = \beta_2 = \beta_3 = 0$ and $\gamma = \left(\left(\frac{1}{2} + x\right)(2 - x) - 3\right)$. Using *SOSTools* with the SdP solver *SeDuMi* under *Matlab* gives more complicated solutions.

4 A simple combinatorial condition for proving the existence of (non-empty) invariant sets

Even though we found a compact minimal polynomial template B with closed exit set, i.e. an isolating block, it can be the case that the inner invariant set is empty. We use Ważewski's property, Theorem 1, to ensure that it is not empty.

It is difficult, in general, to test whether B^- is a deformation retract of B or not. We will use sufficient conditions to guarantee that B^- is not a deformation retract of B, in the simpler case where B is contractible (i.e. there is a deformation retract of B onto any of its points).

Using Lemma 2, we know that exit sets on each of the faces P_i^c is given as the set of points x on P_i^c such that $\mathcal{L}_f^{(1)}(p_i)(x) \geq 0$. Define G as the graph whose nodes are the P_i^c for which $P_i^c \cap B^-$ is non-empty and whose edges are given by pairs P_i^c, P_j^c of faces of B, such that $\mathcal{L}_f^{(1)}(p_i)(x) \geq 0 \ \wedge \ \mathcal{L}_f^{(1)}(p_j)(x) \geq 0$ is satisfiable on $P_i^c \cap P_j^c$ (when non-empty).

If G is not connected, then the exit set B^- is trivially not connected either, because G has a number of components less or equal than that of B^- (this can be stricly less if some $P_i^c \cap B^-$ is not connected). But B is connected because it is in particular contractible. Thus B^- cannot be a deformation retract of B. This is what we used in Example 3 to prove that there is a positive invariant set within B. Note that we can do the same for the complement of the exit set (i.e. the entrance set), combined, and that, by Alexander duality [13], these two connectedness tests are rather fine tests : the connected components of the entrance set give information on the first cohomology group of the exit set in dimension $n = 3$.

This leads to Proposition 2, that uses positivstellensatz once again as an algorithmic method to determine connectivity of (an abstraction of) graph G.

Proposition 2. *Let $G^\sharp$ be the graph whose nodes are given by the faces P_i^c of the template considered such that there exists an x with $p_i(x) = c_i$ and $\mathcal{L}_f^{(1)}(p_i)(x) = \beta_0 + \sum_{k=1,k\neq i}^m \beta_k(c_k - p_k) + \gamma_i(c_i - p_i)$ (where β_0, β_k are SoS polynomials and γ_i is any polynomial), and whose edges are given by pair of faces (P_i^c, P_j^c) such that there exists an x with $p_i(x) = c_i$, $p_j(x) = c_j$, and $-\mathcal{L}_f^{(1)}(p_i)(x) \times \mathcal{L}_f^{(1)}(p_j)(x) = \beta_0' + \sum_{k=1,k\neq i,k\neq j}^m \beta_k'(c_k - p_k) + \gamma_i(c_i - p_i) + \gamma_j(c_j - p_j)$ (where β_0', β_k' are SoS polynomials and γ_i is any polynomial). Then if $G^\sharp$ is disconnected and the template is an isolating block then its invariant subset is non-empty.*

Algorithmically, on top of the classical SdP relaxation for solving positivstellensatz, we use a simple depth-first traversal of the graph to compute the set of connected components of $G^\sharp$.

Example 6. We consider again Example 1. Each of the four faces of B is a node in $G^\sharp$. The faces are respectively given by $\{(-2, y) | -2 \leq y \leq 2\}$ (face P_1^c), $\{(2, y) | -2 \leq y \leq 2\}$ (face P_2^c), $\{(x, -2) | -2 \leq x \leq 2\}$ (face P_3^c) and $\{(x, 2) | -2 \leq x \leq 2\}$ (face P_4^c). We thus have non-empty intersections $P_1^c \cap P_3^c = \{(-2, -2)\}$, $P_1^c \cap P_4^c = \{(-2, 2)\}$, $P_2^c \cap P_3^c = \{(2, -2)\}$ and $P_3^c \cap P_4^c = \{(2, 2)\}$. Therefore we have an edge from P_1^c to P_3^c if and only if $\mathcal{L}_f^{(1)}(p_1)(-2, -2) = 2$ and $\mathcal{L}_f^{(1)}(p_3)(-2, -2) = \frac{13}{2}$ are both positive - which is true ; we have an edge from P_1^c to P_4^c if and only if $\mathcal{L}_f^{(1)}(p_1)(-2, 2) = -2$ and $\mathcal{L}_f^{(1)}(p_4)(-2, 2) = \frac{5}{2}$ are both positive - which is false ; and similarly, we obtain that there is no edge between P_2^c and P_3^c, and there is an edge between P_2^c and P_4^c.

We conclude that B^- has (at least) two connected components, and that there is a non empty invariant set within the square B.

5 Experiments

The algorithm was implemented in Matlab, with the *Symbolic Math Toolbox* to compute the Lie derivatives, and *MuPaD* for *Gröbner basis* manipulations. Sum of square problems are solved with the semi-definite programs solver *SeDumi*. Timings of the execution of our algorithm on classical examples are given for a MacBook Air (Processor) 1,3 GHz Intel Core i5 , (Memory) 4 GB 1600 MHz DDR3 and expressed in seconds as follows : t Gröbner is the time needed to find the order of the differential radical for a given face, t SoS optim is the time taken to solve the SoS optimization problems for each face. We also indicate in Figure 2 the order of the differential radical in column "N_i" and the maximal degree of polynomials in the corresponding Gröbner base.

Example A. It is our running example, Example 1 from Section 1. Just as a matter of comparison, if we had applied directly Stengle's nichtnegativstellensatz, the time it would have taken to prove closedness of the exit set, for each face, would have been around 130 seconds, in sharp contrast with the 4 seconds, using Putinar's positivstellensatz.

Example B. It is defined by $\left(\dot{x} = y, \dot{y} = -y - x + \frac{1}{3}x^3\right)$ (Example 4 of [7], with a quadratic Lyapunov function), with as template, the box defined by $c = (2.4, 2.4, 2.4, 2.4)^t$. We are able to show, using the method of Proposition 2, that this box contains a non-trivial invariant.

Example C. It is defined by $\left(\dot{x} = -\frac{1}{10}x + y - x^3, \dot{y} = -x - \frac{1}{10}y, \dot{z} = 5z\right)$ (Example 4.1, page 21, of [28]). There are three fixed points $p_0 = (-1, 0, 0)$, $p_1 = (1, 0, 0)$, and $p_2 = (0, 1, -1)$, and rather complicated dynamics between neighborhoods of these points. The only face we are considering is the sphere of radius $\frac{1}{5}$ centered at p_2, which is defined by the template $x^2 + (y-1)^2 + (z+1)^2 = \frac{1}{25}$. The exit set can be shown to have two connected components, but Proposition 2 fails to see that, because we have only one face. This can be solved by considering the two hemispheres, one with $z \geq -1$, the other with $z \leq -1$. Our method then finds one connected component in one of the hemispheres and the other in the opposite one : $G^\sharp$ is disconnected, hence contains a non-empty invariant. Note that the direct SoS approach seems to find only degree $\geq$3 polynomials [28].

Ex.	Face	t Gröbner	d^o G-base	N_i	t SoS optim
Example A	Face 1 $(-x)$	0.39	3	3	4.7
	Face 2 (x)	0.43	3	3	4.67
	Face 3 $(-y)$	0.45	3	3	5.01
	Face 4 (y)	0.45	3	3	5.07
Example B	Face 1 (x)	0.38	2	3	4.8
	Face 2 $(-x)$	0.43	2	3	4.7
	Face 3 (y)	0.36	2	3	4.9
	Face 4 $(-y)$	0.39	2	3	5.0
Example C	Face 1	0.51	3	4	14.6
Example D	Face 1	1.83	7	4	158.42

Fig. 2. Some benchmarks

Example D. It is the same as Example C (i.e. Example 4.1 of [28]), but using as template a 4-norm ellipsoid centered at the point $(0, \frac{1}{2}, \frac{1}{2})$ whose principal axes are pointing in the coordinate directions x, y, and z and have lengths $\frac{3}{2}$, 1, and 1, respectively. This is described by the only face $\left(\frac{2}{3}x\right)^4 + \left(y - \frac{1}{2}\right)^4 + \left(z + \frac{1}{2}\right)^4 = 1$. The exit set is connected but is not simply connected : this can be seen by noticing that the entrance set has two components. But as for Example C, Proposition 2 cannot help distinguish them right away as we have encoded the template by just one face.

6 Conclusion and future work

This paper is a first step towards more involved criteria for finding (positive) invariant sets. First, this can be generalized to switched systems, a particular class of hybrid systems which have regained recent interest in the control community. Also, the nature of the invariant sets isolated by our method can be more precisely characterized, using further the Conley index theory, making for instance the difference between a stable point or a limit cycle. We can also generalize the combinatorial criterion of Section 4 : we used the first step of a general nerve lemma [32], there might be an interest in going one step further.

Another direction of improvement concerns the choice of templates. For instance, classical linear templates are quite hard to use for Van der Pol's equation, but the results of [6] seem to indicate that refining them should be possible.

There are also numerous algorithmic improvements over the costly Gröbner base and SoS computations. Instead of SoS methods, we could think of using simpler but still precise inner [12] and outer [9] approximations of the image of a polynomial function on a box. Some quantifier elimination methods might also been useful, using Cylindrical Algebraic Decomposition [4].

Finally, turning the SoS problems that we used for finding solutions to some polynomial inequalities into real optimization problems will provide a way to find the vector c defining the faces of a template isolating block, instead of merely checking the property for a given c. For finite or regular control problems, a finite set of parameters defining a stabilizing control may also be found this way.

Acknowledgments. The authors were partially supported by Digiteo Project SIMS 2013-0544D, by ANR projects CAFEIN, ANR-12-INSE-0007 and MALTHY, ANR-13-INSE-0003, by iCODE and by the academic chair "Complex Systems Engineering" of École polytechnique-ENSTA-Télécom-Thalès-Dassault Aviation-DCNS-DGA-FX-Fondation ParisTech-FDO ENSTA.

References

1. William W. Adams and Philippe Loustaunau. *An introduction to Grobner bases.* Graduate studies in mathematics. American mathematical society, 1994.
2. A. Adjé and P.-L. Garoche. Automatic synthesis of piecewise linear quadratic invariants for programs. In *Proceedings of VMCAI*, pages 99–116, 2015.
3. A. Bacciotti and L. Mazzi. Stability of dynamical polysystems via families of Lyapunov functions. *Jour. Nonlin. Analysis*, 2007.
4. Saugata Basu, Richard Pollack, and Marie-Françoise Roy. *Algorithms in Real Algebraic Geometry.* Springer-Verlag, 2006.
5. N.P. Bhatia and O. Hájek. *Local Semi-dynamical Systems.* Number 90 in Lecture notes in mathematics. Springer, 1969.
6. E. Boczko, W. D. Kalies, and K. Mischaikow. Polygonal approximation of flows. *Topology and its applications*, 154, 2007.
7. Olivier Bouissou, Alexandre Chapoutot, Adel Djaballah, and Michel Kieffer. Computation of Parametric Barrier Functions for Dynamical Systems using Interval Analysis. In *IEEE CDC 2014*, 2014.
8. C. Conley. Isolated invariant sets and the Morse index. *Amer. Math. Soc., CBMS Regional Conf. Series Math.*, 38, 1978.
9. Thao Dang and Romain Testylier. Reachability analysis for polynomial dynamical systems using the Bernstein expansion. *Reliable Computing*, 17:128–152, 2012.

10. Khalil Ghorbal and André Platzer. Characterizing algebraic invariants by differential radical invariants. In *TACAS*, pages 279–294, 2014.
11. E. Goubault, J.-H. Jourdan, S. Putot, and S. Sankaranarayanan. Finding non-polynomial positive invariants and Lyapunov functions for polynomial systems through Darboux polynomials. In *American Control Conference*, 2014.
12. Eric Goubault, Olivier Mullier, Sylvie Putot, and Michel Kieffer. Inner approximated reachability analysis. In *HSCC*, pages 163–172, 2014.
13. Michiel Hazewinkel. Alexander duality. *Encyclopaedia of Mathematics*, 2002.
14. F. Wesley Wilson Jr. and J. A. Yorke. Lyapunov functions and isolating blocks. *Journal of Differential Equations*, 1973.
15. H.K. Khalil. *Nonlinear Systems*. Prentice Hall, 2002.
16. S. Lang. *Algebra*. Graduate Texts in Mathematics. Springer New York, 2002.
17. Jean-Bernard Lasserre. *Moments, positive polynomials and their applications*, volume 1. World Scientific, 2009.
18. Henri Lombardi. Une borne sur les degrés pour le theorème des zéros rééls effectifs. In *Real Algebraic Geometry*, volume 1524 of *Lecture Notes in Mathematics*, pages 323–345. Springer, 1992.
19. Henri Lombardi, Daniel Perrucci, and Marie-Franoise Roy. An elementary recursive bound for effective positivstellensatz and Hilbert 17-th problem, 2014.
20. K. Mischaikow and M. Mrozek. Conley index theory. *Handbook of Dynamical Systems II: Towards Applications, (B. Fiedler, ed.) North-Holland*, 2002.
21. Marian Mrozek. Index pairs algorithms. *Foundations of Computational Mathematics*, 6(4):457–493, 2006.
22. Mihai Putinar. Positive polynomials on compact semi-algebraic sets. *Indiana University Mathematics Journal*, 42(3):969–984, 1993.
23. Roman Srzednicki. Ważewski Method and Conley Index, 2004.
24. Sriram Sankaranarayanan, Thao Dang, and Franjo Ivancic. A policy iteration technique for time elapse over template polyhedra. In *HSCC*, pages 654–657, 2008.
25. Sriram Sankaranarayanan, Henny B. Sipma, and Zohar Manna. Scalable analysis of linear systems using mathematical programming. In *VMCAI*, volume 3385 of *LNCS*. Springer, 2005.
26. Robert Shorten, Fabian Wirth, Oliver Mason, Kai Wulff, and Christopher King. Stability criteria for switched and hybrid systems. *SIAM Rev.*, 49(4):545–592.
27. Guillermo Moreno Socías. Length of polynomial ascending chains and primitive recursiveness. 71:181–205, 1992.
28. T. Stephens and T. Wanner. Rigorous validation of isolating blocks for flows and their Conley indices. Technical report, IMA Preprint Series #2424, May 2014.
29. Andrzej Szymczak. A combinatorial procedure for finding isolating neighbourhoods and index pairs. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 127:1075–1088, 1997.
30. M. Vidyasagar. *Nonlinear Systems Analysis*. Prentice-Hall Networks Series. Prentice-Hall, 1978.
31. T. Ważewski. Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires. *Ann. Soc. Polon. Math.*, 10(25):279–313, 1947.
32. Volkmar Welker, Gunter M. Ziegler, and Rade T. Živaljević. Homotopy colimits – comparison lemmas for combinatorial applications. *J. Reine Angew. Math.*, 509:117–149, 1999.