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## A Topos Perspective on the Kochen-Specker Theorem: II. Conceptual Aspects, and Classical Analogues

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### Abstract

In a previous paper, we have proposed assigning as the value of a physical quantity in quantum theory, a certain kind of set (a sieve) of quantities that are functions of the given quantity. The motivation was in part physical—such a valuation illuminates the Kochen-Specker theorem; and in part mathematical—the valuation arises naturally in the topos theory of presheaves.

This paper discusses the conceptual aspects of this proposal. We also undertake two other tasks. First, we explain how the proposed

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valuations could arise much more generally than just in quantum physics; in particular, they arise as naturally in classical physics. Second, we give another motivation for such valuations (that applies equally to classical and quantum physics). This arises from applying to propositions about the values of physical quantities some general axioms governing partial truth for any kind of proposition.

# 1 Introduction

In a previous paper [1]—referred to hereafter as (I)—we proposed assigning as the value of a physical quantity in quantum theory, a certain kind of set (a sieve) of quantities that are functions of the given quantity. Our motivation was in part physical—such a valuation illuminates the Kochen-Specker theorem; and in part mathematical—the valuation arises naturally in the topos theory of presheaves. These aspects were closely linked. We interpreted a valuation as assigning truth-values to propositions ‘ $A \in \Delta$ ’ asserting that the value of the quantity  $A$  lies in the Borel subset  $\Delta$  of the spectrum of the operator  $\hat{A}$  that represents  $A$ . The fact that one quantity can be a function, or coarse-graining, of another implies that there is a natural presheaf associated with these propositions. And the theory of presheaves gives a natural generalization of the *FUNC* property (viz. that the value of a function of a given quantity is the function of the value of the quantity), which plays a central role in the Kochen-Specker theorem.

In this paper, we first show how sieve-valued valuations obeying our generalization of *FUNC* arise much more generally than just as the values of quantities in quantum physics; and accordingly, how the principal results of (I) can be generalized. In fact, we claim that they are one of the most natural notions of valuation for any presheaf of propositions, no matter what their topic. From a physical perspective, a mathematical structure of this type is indicated whenever the idea of ‘contextual’ statements about the system is (i) physically appropriate; and (ii) is so in such a way that the set of all possible such contexts can be regarded as the objects in a category, which then forms the base category over which the presheaves are defined. As we shall see, in making this claim we assume about valuations on propositions only the basic idea that they must be some sort of structure-preserving function from the set of propositions (with operations such as negation, conjunction *etc.* defined on it) to the set of truth values, which is to be some sort of logical algebra.

That is the task of Section 3—where we show that sieve-valued valuations arise naturally in classical physics; and Section 4—where we show how such valuations can arise in even more general contexts. But first, to facilitate reading the paper, there is a short review of the elements of the theory of presheaves (more concise than in (I), but with some extra heuristic material), and of how these ideas were applied in (I) to quantum physics.

The paper ends with a presentation of another motivation for such valuations (Section 5). We will argue that intuitive ideas about what might be meant by the notion of ‘partial truth’ (applying to any type of proposition) make sieve-valued valuations very natural. Among these principles, the main one will be that a proposition is nearer to ‘total truth’, the larger the subset of its consequences that are themselves totally true. This argument, and the principles it refers to, is conceptual, not mathematical: indeed, it will not need the mathematical notions of Section 2, except the idea of a category—that *is* compulsory, in order to make sense of the notion of a sieve. But the argument and its principles can be made precise most naturally by using the ideas of presheaf theory; in particular, the idea of ‘consequence’ (entailment) can be made precise in terms of the generalized notion of coarse-graining introduced in Section 4. Again, we shall see that the proposals of (I) arise from applying these general ideas to propositions about the values of quantum physical quantities.

We remark incidentally that there are still other motivations for sieve-valued valuations obeying a generalized *FUNC*. We discuss philosophical ones in [2], and physical ones in [3], in each case adding further material. For example, semantics for intuitionistic logic of the Kripke-Beth type assigns to each formula as its interpretation, a sieve on a poset; points of which are, intuitively, possible states of knowledge, so that paths represent possible courses of enquiry. In [2], we describe how this kind of construction suggests, as an analogy, our own valuations. In [3], the motivation concerns assigning to a quantity a Borel subset (rather than an element) of its spectrum; it also is foreshadowed in (I).

## 2 Review of Part I

In the first two subsections, we review elements of the theory of presheaves. In the third, we summarize how this theory was applied in (I) to the Kochen-Specker theorem, and to the idea of generalised valuations on the physical quantities in a quantum theory.

## 2.1 Categories, Presheaves and Subobjects

A *presheaf*  $\mathbf{X}$  on a small<sup>4</sup> category  $\mathcal{C}$  is a function that:

1. assigns to each  $\mathcal{C}$ -object  $A$ , a set  $\mathbf{X}(A)$ ;
2. assigns to each  $\mathcal{C}$ -morphism  $f : B \rightarrow A$ , a set-function,  $\mathbf{X}(f) : \mathbf{X}(A) \rightarrow \mathbf{X}(B)$ ; and
3. makes these assignments in a ‘meshing’ way, *i.e.*,  $\mathbf{X}(\text{id}_A) = \text{id}_{\mathbf{X}(A)}$ ; and, if  $g : C \rightarrow B$ , and  $f : B \rightarrow A$  then

$$\mathbf{X}(f \circ g) = \mathbf{X}(g) \circ \mathbf{X}(f) \tag{2.1}$$

where  $f \circ g : C \rightarrow A$  denotes the composition of  $f$  and  $g$ .

So intuitively, a presheaf is a collections of sets that vary in a meshing way between ‘stages’ or ‘contexts’  $A, B, \dots$  that are objects in the category  $\mathcal{C}$ . In terms of contravariant and covariant functors, a presheaf on  $\mathcal{C}$  is a contravariant functor from  $\mathcal{C}$  to the category  $\text{Set}$  of normal sets. Equivalently, it is a covariant functor from the ‘opposite’ category<sup>5</sup>  $\mathcal{C}^{\text{op}}$  to  $\text{Set}$ .

To make the collection of presheaves on  $\mathcal{C}$  into the objects of a category, we recall that a morphism between two presheaves  $\mathbf{X}$  and  $\mathbf{Y}$  is defined to be a *natural transformation*  $N : \mathbf{X} \rightarrow \mathbf{Y}$ , by which is meant a family of maps (called the *components* of  $N$ )  $N_A : \mathbf{X}(A) \rightarrow \mathbf{Y}(A)$ ,  $A$  an object in  $\mathcal{C}$ , such that if  $f : B \rightarrow A$  is a morphism in  $\mathcal{C}$ , then the composite map  $\mathbf{X}(A) \xrightarrow{N_A} \mathbf{Y}(A) \xrightarrow{\mathbf{Y}(f)} \mathbf{Y}(B)$  is equal to  $\mathbf{X}(A) \xrightarrow{\mathbf{X}(f)} \mathbf{X}(B) \xrightarrow{N_B} \mathbf{Y}(B)$ . In other words, we have the commutative diagram

$$\begin{array}{ccc} \mathbf{X}(A) & \xrightarrow{\mathbf{X}(f)} & \mathbf{X}(B) \\ \downarrow N_A & & \downarrow N_B \\ \mathbf{Y}(A) & \xrightarrow{\mathbf{Y}(f)} & \mathbf{Y}(B) \end{array} \tag{2.2}$$

The category of presheaves on  $\mathcal{C}$  equipped with these morphisms is denoted  $\text{Set}^{\mathcal{C}^{\text{op}}}$ .

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<sup>4</sup>A category is said to be *small* if the collection of objects, and the collection of all morphisms between a pair of objects, is a set.

<sup>5</sup>The ‘opposite’ of a category  $\mathcal{C}$  is a category, denoted  $\mathcal{C}^{\text{op}}$ , whose objects are the same as those of  $\mathcal{C}$ , and whose morphisms are defined to be the same as those of  $\mathcal{C}$  but with each arrow reversed in direction.

Since  $\mathcal{C}$  is small, it follows that  $\text{Set}^{\mathcal{C}^{\text{op}}}$  is a topos. But we will not need the full definition of a topos here[4, 5]: it suffices that it is a category that behaves much like the category  $\text{Set}$ , in particular as regards ‘subobjects’—the analogue of subsets. To this we now turn.

## 2.2 Subobjects, Sieves and Sections

**1. Subobjects:** The key idea about subobjects in a topos, which will be used throughout this paper, is this. Just as an object in  $\text{Set}$ , *i.e.*, a set  $X$ , has subsets  $K \subseteq X$  that are in one-to-one correspondence with set-functions  $\chi^K : X \rightarrow \{0, 1\}$ , from  $X$  to the special set  $\{0, 1\}$ , where  $\chi^K(x) = 1$  if  $x \in K$ , and  $\chi^K(x) = 0$  otherwise; so in a topos, the subobjects  $K$  of an object  $X$  are in one-to-one correspondence with morphisms  $\chi^K : X \rightarrow \Omega$ , where the special object  $\Omega$  in the topos—called the ‘subobject classifier’—forms an object of possible truth-values, just as  $\{0, 1\}$  does in the category of sets.

We turn to the exact definitions. An object  $\mathbf{K}$  is said to be a *subobject* of  $\mathbf{X}$  in the category of presheaves if there is a morphism in the category (*i.e.*, a natural transformation)  $i : \mathbf{K} \rightarrow \mathbf{X}$  with the property that, for each stage  $A$ , the component map  $i_A : \mathbf{K}(A) \rightarrow \mathbf{X}(A)$  is a subset embedding, *i.e.*,  $\mathbf{K}(A) \subseteq \mathbf{X}(A)$ . Thus, if  $f : B \rightarrow A$  is any morphism in  $\mathcal{C}$ , we get:

$$\begin{array}{ccc} \mathbf{K}(A) & \xrightarrow{\mathbf{K}(f)} & \mathbf{K}(B) \\ \downarrow & & \downarrow \\ \mathbf{X}(A) & \xrightarrow{\mathbf{X}(f)} & \mathbf{X}(B) \end{array} \quad (2.3)$$

where the vertical arrows are subset inclusions. In particular, it follows that  $\mathbf{K}(f)$  is the restriction of  $\mathbf{X}(f)$  to  $\mathbf{K}(A)$ .as

It is clear in what way the definitions above generalise the ideas of set and subset. Namely, a presheaf over the category  $\mathcal{C}$  consisting of a *single* object  $O$  corresponds to a set  $X := \mathbf{X}(O)$ ; and a subobject of this presheaf corresponds to a subset of  $X$ .

**2. Sieves and the Subobject Classifier  $\Omega$ :** To give the generalization for presheaves of an ordinary subsets’ characteristic function  $\chi^K : X \rightarrow \{0, 1\}$ , we first need the idea of a sieve. A *sieve* on an object  $A$  in  $\mathcal{C}$  is defined to be a collection  $S$  of morphisms in  $\mathcal{C}$  with codomain  $A$ , and with

the property that if  $f : B \rightarrow A$  belongs to  $S$ , and if  $g : C \rightarrow B$  is any morphism, then  $f \circ g : C \rightarrow A$  also belongs to  $S$ .

With the idea of a sieve, one can immediately define the *subobject classifier*. It is the presheaf  $\Omega : \mathcal{C} \rightarrow \text{Set}$  defined by:

1. if  $A$  is an object in  $\mathcal{C}$ , then  $\Omega(A)$  is the set of all sieves on  $A$ ;
2. if  $f : B \rightarrow A$ , then  $\Omega(f) : \Omega(A) \rightarrow \Omega(B)$  is defined as

$$\Omega(f)(S) := \{h : C \rightarrow B \mid f \circ h \in S\} \quad (2.4)$$

for all  $S \in \Omega(A)$ ; the sieve  $\Omega(f)(S)$  is often written as  $f^*(S)$ , and is known as the *pull-back* to  $B$  of the sieve  $S$  on  $A$  by the morphism  $f : B \rightarrow A$ .

There are two main aspects to the idea that  $\Omega$  supplies an object of generalized truth-values. Both arise from the basic idea mentioned in Section 1: that a valuation on propositions (of any sort, not necessarily about the values of physical quantities) must be some sort of structure-preserving function from the set of propositions (with some such operations as negation, conjunction etc. defined on it) to the set of truth values, which is to be some sort of logical algebra.

The first aspect is the fact that for any  $A$  in  $\mathcal{C}$ , the set  $\Omega(A)$  of sieves on  $A$  is a *Heyting algebra*. This is a logical algebra that is distributive, but more general than a Boolean algebra: the main difference being in the behaviour of negation. In this paper, we shall not need the exact definition: we need only to remark that the Heyting algebra structure of  $\Omega(A)$  is very natural; and then to make an ensuing conceptual point.

Specifically,  $\Omega(A)$  is a Heyting algebra where the partial ordering is defined on  $S_1, S_2$  in  $\Omega(A)$  by  $S_1 \leq S_2$  if  $S_1 \subseteq S_2$ ; so that the unit element  $1_{\Omega(A)}$  in  $\Omega(A)$  is the *principal sieve*  $\downarrow A := \{f : B \rightarrow A\}$  of all arrows whose codomain is  $A$ , and the null element  $0_{\Omega(A)}$  is the empty sieve  $\emptyset$ . The connectives for conjunction and disjunction are defined as<sup>6</sup>

$$S_1 \wedge S_2 := S_1 \cap S_2 \quad (2.5)$$

$$S_1 \vee S_2 := S_1 \cup S_2. \quad (2.6)$$

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<sup>6</sup>The other key connective is the pseudo-complement of  $S_1$  relative to  $S_2$ . This is defined as  $S_1 \Rightarrow S_2 := \{f : B \rightarrow A \mid \text{for all } g : C \rightarrow B \text{ if } f \circ g \in S_1 \text{ then } f \circ g \in S_2\}$ . The negation of an element  $S$  is defined as  $\neg S := S \Rightarrow 0$ ; so that  $\neg S := \{f : B \rightarrow A \mid \text{for all } g : C \rightarrow B, f \circ g \notin S\}$ .

The conceptual point is significant. It is that if for some reason a set of propositions is associated with each  $A$  in a category  $\mathcal{C}$  (perhaps, but not necessarily, as a presheaf)—so that one is concerned to define *contextual* valuations, *i.e.*, valuations associated with each ‘context’ or ‘stage of truth’  $A$ —then the set  $\Omega(A)$ , being a Heyting algebra, forms a ‘algebraically well-behaved’ target space for such a valuation associated with  $A$ .

The second aspect will be prominent in this Section and beyond. It is the idea of generalizing to presheaves the way that the subsets,  $K$ , of an ordinary set  $X$  are in one-to-one correspondence with characteristic functions  $\chi^K : X \rightarrow \{0, 1\}$ , from  $X$  to the two classical truth-values  $\{0, 1\}$ . More precisely: the presheaf  $\Omega$  plays a role for the topos  $\text{Set}^{\mathcal{C}^{\text{op}}}$  analogous to the set  $\{0, 1\}$ . That is to say, subobjects of any object  $\mathbf{X}$  in this topos (*i.e.*, subobjects of any presheaf on  $\mathcal{C}$ ) are in one-to-one correspondence with morphisms  $\chi : \mathbf{X} \rightarrow \Omega$ .

First, let  $\mathbf{K}$  be a subobject of  $\mathbf{X}$ . Then there is an associated *characteristic morphism*  $\chi^{\mathbf{K}} : \mathbf{X} \rightarrow \Omega$ , whose component  $\chi_A^{\mathbf{K}} : \mathbf{X}(A) \rightarrow \Omega(A)$  at each  $A$  in  $\mathcal{C}$  is defined as

$$\chi_A^{\mathbf{K}}(x) := \{f : B \rightarrow A \mid \mathbf{X}(f)(x) \in \mathbf{K}(B)\} \quad (2.7)$$

for all  $x \in \mathbf{X}(A)$ .

That the right hand side of Eq. (2.7) actually *is* a sieve on  $A$  follows from the defining properties of a subobject. Thus, in each ‘branch’ of the category  $\mathcal{C}$  going ‘down’ from the stage  $A$ ,  $\chi_A^{\mathbf{K}}(x)$  picks out the first member  $B$  in that branch for which  $\mathbf{X}(f)(x)$  lies in the subset  $\mathbf{K}(B)$ , and the commutative diagram Eq. (2.3) then guarantees that  $\mathbf{X}(f \circ h)(x)$  will lie in  $\mathbf{K}(C)$  for all  $h : C \rightarrow B$ . Hence each  $A$  in  $\mathcal{C}$  serves as a possible ‘context’ or ‘stage of truth’ for an assignment to each  $x \in \mathbf{X}(A)$  of a generalised truth-value which is a sieve, belonging to the Heyting algebra  $\Omega(A)$ , rather than an element of the Boolean algebra  $\{0, 1\}$  of normal set theory.

Conversely, each morphism  $\chi : \mathbf{X} \rightarrow \Omega$  (*i.e.*, each natural transformation between the presheaves  $\mathbf{X}$  and  $\Omega$ ) defines a subobject  $\mathbf{K}^\chi$  of  $\mathbf{X}$  by defining for each stage of truth  $A$

$$\mathbf{K}^\chi(A) := \chi_A^{-1}\{1_{\Omega(A)}\} = \{x \in \mathbf{X}(A) \mid \chi_A(x) = \downarrow A\} \quad (2.8)$$

and by defining for each  $f : B \rightarrow A$ , the map  $\mathbf{K}^\chi(f) : \mathbf{K}^\chi(A) \rightarrow \mathbf{K}^\chi(B)$  to be the restriction of  $\mathbf{X}(f)$  to  $\mathbf{K}^\chi(A)$ :

$$\mathbf{K}^\chi(f) := \mathbf{X}(f)|_{\mathbf{K}^\chi(A)}. \quad (2.9)$$



Note that the fact that principal sieves pull back to principal sieves ensures that Eq. (2.9) implies that, for any  $x \in \mathbf{K}^\chi(A)$ ,

$$\chi_B(\mathbf{X}(f)(x)) = \mathbf{\Omega}(f)(\chi_A(x)) = \mathbf{\Omega}(f)(\downarrow A) = \downarrow B \quad (2.10)$$

so that  $\mathbf{X}(f)(x) \in \mathbf{K}^\chi(B)$ , *i.e.*,  $\mathbf{K}^\chi$  is indeed a subobject of  $\mathbf{X}$ .

Note how this correspondence between subobjects and characteristic morphisms simplifies in the special case mentioned above (Section 2.2.1), of presheaves on the category with a single object. In effect, one gets just two truth-values—the unit element  $1_{\mathbf{\Omega}(O)}$  and the null element  $0_{\mathbf{\Omega}(O)}$ , at the single stage of truth  $O$ ; and the component of the characteristic morphism at this single stage is just the characteristic function of a subset of  $X := \mathbf{X}(O)$ .

**3. Sections of a Presheaf:** In any category, a *terminal object* is defined to be an object  $1$  such that, for any object  $X$  in the category, there is a unique morphism  $X \rightarrow 1$ . A *global element* of an object  $X$  is defined to be any morphism  $1 \rightarrow X$ . The motivation for this definition is that, in the case of the category of sets, a terminal object is any singleton set  $\{*\}$ ; and then there is a one-to-one correspondence between the elements of a set  $X$  and functions from  $\{*\}$  to  $X$ .

For the category of presheaves on  $\mathcal{C}$ , a terminal object  $\mathbf{1} : \mathcal{C} \rightarrow \text{Set}$  can be defined by  $\mathbf{1}(A) := \{*\}$  at all stages  $A$  in  $\mathcal{C}$ ; if  $f : B \rightarrow A$  is a morphism in  $\mathcal{C}$  then  $\mathbf{1}(f) : \{*\} \rightarrow \{*\}$  is defined to be the map  $* \mapsto *$ . A global element of a presheaf  $\mathbf{X}$  is also called a *global section*. As a morphism  $\gamma : \mathbf{1} \rightarrow \mathbf{X}$  in the topos  $\text{Set}^{\mathcal{C}^{\text{op}}}$ , a global section corresponds to a choice of an element  $\gamma_A \in \mathbf{X}(A)$  for each stage  $A$  in  $\mathcal{C}$ , such that, if  $f : B \rightarrow A$ , the ‘matching condition’

$$\mathbf{X}(f)(\gamma_A) = \gamma_B \quad (2.11)$$

is satisfied.

As discussed in the next Subsection, the Kochen-Specker theorem is equivalent to the statement that certain presheaves that arise naturally in quantum theory have no global sections. But on the other hand, a presheaf may have ‘partial’, or ‘local’, elements even if there are no global ones. In general, a *partial element* of an object  $X$  in a category with a terminal object is defined to be a morphism  $U \rightarrow X$ , where  $U$  is a subobject of the terminal object  $1$ . In the category of sets, there are no nontrivial subobjects of  $1 := \{*\}$ , but this is not the case in a general topos. In particular, in the case

of presheaves on  $\mathcal{C}$ , a partial element of a presheaf  $\mathbf{X}$  is an assignment  $\gamma$  of an element  $\gamma_A$  to a certain *subset* of objects  $A$  in  $\mathcal{C}$ —what we shall call the *domain*  $\text{dom } \gamma$  of  $\gamma$ —with the properties that (i) the domain is closed downwards in the sense that if  $A \in \text{dom } \gamma$  and  $f : B \rightarrow A$ , then  $B \in \text{dom } \gamma$ ; and (ii) for objects in this domain, the matching condition Eq. (2.11) is satisfied.

### 2.3 Some Applications to Quantum Physics

In this final Subsection, we will illustrate how the notions reviewed in this Section can be applied to the topic of valuations in quantum theory. Again, we will be concise and pick out just some of the main ideas of (I), leaving some to be generalized in later Sections, and some wholly unmentioned.

**1. Categories of Quantities and the Kochen-Specker Theorem:** We first introduce the set  $\mathcal{O}_d$  of all bounded self-adjoint operators with purely discrete spectra,  $\hat{A}, \hat{B}, \dots$  on the Hilbert space  $\mathcal{H}$  of a quantum system. We turn  $\mathcal{O}_d$  into a category by defining the objects to be the elements of  $\mathcal{O}_d$ , and saying that there is a morphism from  $\hat{B}$  to  $\hat{A}$  if there exists a real-valued function  $f$  on  $\sigma(\hat{A}) \subset \mathbb{R}$ , the spectrum of  $\hat{A}$ , such that  $\hat{B} = f(\hat{A})$  (with the usual definition of a function of a self-adjoint operator, using the spectral representation). If  $\hat{B} = f(\hat{A})$ , for some  $f : \sigma(\hat{A}) \rightarrow \mathbb{R}$ , then the corresponding morphism in the category  $\mathcal{O}_d$  will be denoted  $f_{\mathcal{O}_d} : \hat{B} \rightarrow \hat{A}$ .

We next form a presheaf on the category  $\mathcal{O}_d$  from the spectra of its objects. The *spectral presheaf* on  $\mathcal{O}_d$  is the covariant functor  $\Sigma : \mathcal{O}_d^{\text{op}} \rightarrow \text{Set}$  defined as follows:

1. On objects:  $\Sigma(\hat{A}) := \sigma(\hat{A})$ —the spectrum of  $\hat{A}$ .
2. On morphisms: If  $f_{\mathcal{O}_d} : \hat{B} \rightarrow \hat{A}$ , so that  $\hat{B} = f(\hat{A})$ , then  $\Sigma(f_{\mathcal{O}_d}) : \sigma(\hat{A}) \rightarrow \sigma(\hat{B})$  is defined by  $\Sigma(f_{\mathcal{O}_d})(\lambda) := f(\lambda)$  for all  $\lambda \in \sigma(\hat{A})$ .

With these definitions, we can state one version of the Kochen-Specker theorem in terms of presheaves. For recall that one form of the theorem asserts that if  $\dim \mathcal{H} > 2$ , there does not exist an assignment  $V$  to each object of  $\mathcal{O}_d$  (*i.e.*, each bounded self-adjoint operator on  $\mathcal{H}$  with a discrete spectrum) of a member of its spectrum, such that the so-called ‘functional composition principle’ (for short: *FUNC*) holds, viz. that for any pair  $\hat{A}, \hat{B}$

such that  $\hat{B} = f(\hat{A})$ :

$$V(\hat{B}) = f(V(\hat{A})). \quad (2.12)$$

But this is precisely the ‘matching condition’, Eq. (2.11), in the definition of a global element, as applied to the spectral presheaf. Thus, in this form, the Kochen-Specker theorem is equivalent to the statement that, if  $\dim \mathcal{H} > 2$ , there are no global elements of the spectral presheaf  $\Sigma : \mathcal{O}_d^{\text{op}} \rightarrow \text{Set}$ . Note that we have restricted attention to operators whose spectra are purely discrete on the grounds that it is not physically meaningful to assign an *exact* value to a quantity that lies in the continuous part of the spectrum of the associated operator.

**2. From Partial Valuations to Generalised Valuations:** Our next observation is that the Kochen-Specker theorem permits the spectral presheaf to have partial elements, as defined in Section 2.2.3. In more usual, physical language: it permits partial valuations, *i.e.*, an assignment to each element  $\hat{A}$  in some subset,  $\text{dom } V$ , of  $\mathcal{O}_d$ , of a member  $V(\hat{A})$  of  $\sigma(\hat{A})$ , such that: (i)  $\text{dom } V$  is closed under taking functions of its members (‘closed under coarse-graining’); and (ii) for all  $\hat{A}, \hat{B} \in \text{dom } V$ , with  $\hat{B} = f(\hat{A})$ , *FUNC*, Eq. (2.12), holds. And there are many such partial valuations (whatever  $\dim \mathcal{H}$ ). For example, each choice of (i) an operator  $\hat{M}$  with a purely discrete spectrum, and (ii) one of its eigenvalues  $m \in \sigma(\hat{M})$ , defines a partial valuation: one just takes  $\text{dom } V$  to be the set of operators  $\hat{A}$  that are functions of  $\hat{M}$ ,  $\hat{A} = f(\hat{M})$ ; and one defines  $V(\hat{M}) := m$ , and  $V(\hat{A}) := f(V(\hat{M})) = f(m)$ .

The idea of a partial valuation brings us to our main claims from (I) (which remain central in this paper). There are in effect three, which we will state briefly here, but explain in order in this Paragraph and the next two.

1. Given such a partial valuation, there is a natural associated valuation that:
  - (a) is defined on *all* propositions ‘ $A \in \Delta$ ’, stating that the value of the quantity  $A$  (represented by the operator  $\hat{A}$ ) lies in  $\Delta$ , a Borel subset of  $\sigma(\hat{A})$ ;
  - (b) assigns to such a proposition as its value, a sieve on  $\hat{A}$  in the category  $\mathcal{O}_d$ .

These valuations have various properties, in particular an analogue for sieves of the property *FUNC*: an analogue that involves the idea of a pull-back.

2. We then use these properties to generalise the notion of a valuation. That is, we define a *generalised valuation* as a map that (i) assigns a sieve on  $\hat{A}$  to each proposition ‘ $A \in \Delta$ ’,  $\Delta \subseteq \sigma(\hat{A})$ , and (ii) has these properties.

This definition has the desirable property that it can readily be extended to the category  $\mathcal{O}$  of *all* bounded self-adjoint operators on the Hilbert space: specifically, a proposition of the type ‘ $A \in \Delta$ ’ is physically (and mathematically) meaningful irrespective of whether or not the spectrum of  $\hat{A}$  is purely discrete.

3. We show that a quantum state (a pure state or a mixture) defines such a generalised valuation in a natural way.

As to (1), the main idea is that—in the discrete case—even if  $\hat{A}$  is not in the domain,  $\text{dom } V$ , of a partial valuation  $V$ , for given  $\Delta$  there might be one or more functions  $f$  such that (i)  $f(\hat{A})$  *does* lie in  $\text{dom } V$ ; and, (ii)  $V(f(\hat{A})) \in f(\Delta)$ . This situation prompts three observations.

- First: if a function  $f$  satisfies conditions (i) and (ii), then so does  $g \circ f$  for any  $g$ ; so the set of morphisms in  $\mathcal{O}_d$  associated with such functions determines a *sieve* on  $\hat{A}$  in  $\mathcal{O}_d$ .
- Second: condition (ii) means that  $V$  in effect assigns *true* (in the usual classical sense!) to the proposition ‘ $f(A) \in f(\Delta)$ ’.
- Third: the proposition ‘ $f(A) \in f(\Delta)$ ’ is weaker than the original proposition ‘ $A \in \Delta$ ’, both intuitively (since functions are generally not injective) and mathematically, in the sense that its representing projector is larger in the lattice of projectors on  $\mathcal{H}$ :  $\hat{E}[A \in \Delta] \leq \hat{E}[f(A) \in f(\Delta)]$ .

Putting these observations together, we propose to assign to ‘ $A \in \Delta$ ’ as a *contextual, partial truth-value* at the stage  $\hat{A}$ , the sieve on  $\hat{A}$  determined by the functions obeying (i) and (ii). Formally, we define a *generalised valuation* associated with a partial valuation  $V$  by

$$\nu^V(A \in \Delta) := \{f_{\mathcal{O}_d} : \hat{B} \rightarrow \hat{A} \mid \hat{B} \in \text{dom } V, V(\hat{B}) \in f(\Delta)\}. \quad (2.13)$$

This generalizes the values assigned by  $V$  itself, in that  $V$ 's assignments correspond to those propositions ' $A \in \Delta$ ' to which  $\nu^V$  assigns the principal sieve,  $\downarrow \hat{A} := \{f_{\mathcal{O}_d} : \hat{B} \rightarrow \hat{A}\}$ , *i.e.*, the unit  $1_{\Omega(\hat{A})}$  of the Heyting algebra  $\Omega(\hat{A})$ . We call this the *totally true* truth-value,  $\text{true}_A$ . Thus  $\nu^V(A \in \Delta) = \text{true}_A$  if (i)  $\hat{A}$  lies in the domain of  $V$ ; and (ii) the value of  $\hat{A}$  assigned by  $V$  lies in the subset  $\Delta \subseteq \sigma(\hat{A})$ .

These definitions imply that the partial truth-value of ' $A \in \Delta$ ' at stage  $\hat{A}$  is determined by those weaker propositions ' $f(A) \in f(\Delta)$ ' that are each totally true at *their* stage  $f(\hat{A})$ . For this partial truth-value just *is* the sieve on  $\hat{A}$  of coarse-grainings  $f(\hat{A})$  of  $\hat{A}$ , at which ' $f(A) \in f(\Delta)$ ' is totally true. (We shall return to this idea in Section 5.)

Generalised valuations associated with partial valuations have various properties, of which we here mention just one, since it will be significant in all later Sections (other properties are listed in the next Paragraph). This property is the analogue for sieves of the property *FUNC*. Roughly speaking, it is that the value of a function of a quantity is the pull-back of the quantity's value. To be precise: If  $f_{\mathcal{O}_d} : \hat{B} \rightarrow \hat{A}$ , so that  $\hat{B} = f(\hat{A})$ , then

$$\nu^V(B \in f(\Delta)) = f_{\mathcal{O}_d}^*(\nu^V(A \in \Delta)). \quad (2.14)$$

This property has two welcome consequences. First, we can express the point of the previous paragraph in terms of pull-backs. For note that for any category  $\mathcal{C}$ , with objects  $A, B, \dots$ , if  $S$  is a sieve on  $A$ , and if  $f : B \rightarrow A$  belongs to  $S$ , then

$$f^*(S) := \{h : C \rightarrow B \mid f \circ h \in S\} = \{h : C \rightarrow B\} = \downarrow B. \quad (2.15)$$

Thus, for any category, the pull-back of a sieve on  $A$  by any morphism from  $B$  to  $A$  that belongs to the sieve, is the principal sieve on  $B$ . Hence the pull-back of the truth-value of ' $A \in \Delta$ ' by a morphism within it, is total truth at the context (stage of truth) that is the morphism's domain. Second, there is a special, and especially intuitive, case of the first point. Since, for any category, the pull-back of any principal sieve by any morphism is the principal sieve, we can say: if ' $A \in \Delta$ ' is totally true (at stage  $\hat{A}$ ), then every weaker proposition ' $f(A) \in f(\Delta)$ ' is totally true (at its stage  $f(\hat{A})$ ).

**3. Generalised Valuations and the Coarse-Graining Presheaf:** We turn now to claim (2), and use the various properties possessed by generalised

valuations associated with partial valuations to define a wider notion of a *generalised valuation* that is applicable to the category  $\mathcal{O}$  of all bounded, self-adjoint operators. Thus a generalised valuation is defined to be any map that (i) assigns a sieve on  $\hat{A}$  to each proposition ‘ $A \in \Delta$ ’; and (ii) has these properties. For the sake of completeness, we state these properties here; though in the rest of this paper we shall only make substantial use of the first—which is the sieve-analogue of *FUNC* (see (I) for the motivation for the other three):

(i) *Functional composition:*

For any Borel function  $f : \sigma(\hat{A}) \rightarrow \mathbb{R}$  we have

$$\nu(f(A) \in f(\Delta)) = f^*(\nu(A \in \Delta)). \quad (2.16)$$

(ii) *Null proposition condition:*

$$\nu(A \in \emptyset) = \emptyset = 0_{\Omega(A)} \quad (2.17)$$

(iii) *Monotonicity:*

If  $\Delta_1 \subseteq \Delta_2$  then  $\nu(A \in \Delta_1) \leq \nu(A \in \Delta_2)$ , i.e.  $\nu(A \in \Delta_1) \subseteq \nu(A \in \Delta_2)$ . (2.18)

(iv) *Exclusivity:*

If  $\Delta_1 \cap \Delta_2 = \emptyset$  and  $\nu(A \in \Delta_1) = \text{true}_A$ , then  $\nu(A \in \Delta_2) < \text{true}_A$ . (2.19)

For later use in this paper, we also note that our collection of sets of propositions ‘ $A \in \Delta$ ’ at each stage  $\hat{A}$  can be made more precise; indeed, it can be regarded as a presheaf, which we call the *coarse-graining presheaf*  $\mathbf{G}$  on  $\mathcal{O}$ . It is defined as follows:

(i) For each  $\hat{A}$  in  $\mathcal{O}$ , the set  $\mathbf{G}(\hat{A})$  is defined to be the spectral algebra of  $\hat{A}$ , i.e., the algebra of spectral projectors  $\hat{E}[A \in \Delta]$  for the various Borel sets  $\Delta \subseteq \sigma(\hat{A})$ ; thus,  $\mathbf{G}(\hat{A})$  can be viewed as the Boolean algebra of all propositions of the form ‘ $A \in \Delta$ ’.

(ii) For each morphism  $f_{\mathcal{O}} : \hat{B} \rightarrow \hat{A}$ : the map  $\mathbf{G}(f_{\mathcal{O}}) : \mathbf{G}(\hat{A}) \rightarrow \mathbf{G}(\hat{B})$  is defined by

$$\mathbf{G}(f_{\mathcal{O}})(\hat{E}[A \in \Delta]) := \hat{E}[f(A) \in f(\Delta)] \quad (2.20)$$

or, equivalently, on propositions:

$$\mathbf{G}(f_{\mathcal{O}})('A \in \Delta') := 'f(A) \in f(\Delta)'. \quad (2.21)$$

Note that the proposition ' $f(A) \in f(\Delta)$ ' is equivalent to the proposition ' $A \in f^{-1}(f(\Delta))$ ', so the action of  $\mathbf{G}(f_{\mathcal{O}})$  can also be viewed as the explicit coarse-graining operation

$$\mathbf{G}(f_{\mathcal{O}})('A \in \Delta') := 'A \in f^{-1}(f(\Delta))' \quad (2.22)$$

in which  $\mathbf{G}(\hat{B})$  is identified as the appropriate subset of  $\mathbf{G}(\hat{A})$ .

In (I) we remarked on the fact that, as it stands, the right hand side of Eq. (2.20) is not well-defined if the function  $f$  and the Borel subset  $\Delta \subseteq \sigma(\hat{A})$  are such that  $f(\Delta)$  is not a *Borel* subset of  $\mathbb{R}$ . The way around this is to note that if  $f(\Delta)$  is a Borel subset, then we have

$$\begin{aligned} \hat{E}[f(A) \in f(\Delta)] &= \inf_{K \subseteq \sigma(f(\hat{A}))} \{ \hat{E}[f(A) \in K] \mid \hat{E}[A \in \Delta] \leq \hat{E}[f(A) \in K] \} \\ &= \inf_{K \subseteq \sigma(f(\hat{A}))} \{ \hat{E}[f(A) \in K] \mid \hat{E}[A \in \Delta] \leq \hat{E}[A \in f^{-1}(K)] \} \\ &= \inf_{K \subseteq \sigma(f(\hat{A}))} \{ \hat{E}[f(A) \in K] \mid \Delta \subseteq f^{-1}(K) \} \end{aligned} \quad (2.25)$$

where the *infimum* is taken over all Borel subsets  $J$  of  $\sigma(f(\hat{A}))$ . If  $f(\Delta)$  is *not* a Borel subset of  $\mathbb{R}$ , then we use Eq. (2.23) as the *definition* of  $\hat{E}[f(A) \in f(\Delta)]$  for the category of operators  $\mathcal{O}$ .

As we shall see in Section 4 *et seq.*, the presheaf of propositions discussed above, and its generalizations, play a central role in the motivations for, and properties of, sieve-valued valuations such as the generalised valuations just defined.

**4. Generalised Valuations from Quantum States:** We turn to our claim (3) above: that quantum states naturally define generalised valuations in the above sense. This proceeds as follows.

The standard minimal interpretation of quantum theory holds that a quantity  $A$  possesses a value  $a$  if, and only if, the state  $\psi$  is an eigenvector of  $\hat{A}$  with eigenvalue  $a$ ; *i.e.*,  $\hat{A}\psi = a\psi$ : or more generally, that ' $A \in \Delta$ ' is true, if and only if,  $\hat{E}[A \in \Delta]\psi = \psi$ . In terms of probability, it holds that ' $A \in \Delta$ '

is true if and only if  $\text{Prob}(A \in \Delta; \psi) = 1$  where  $\text{Prob}(A \in \Delta; \psi)$  denotes the usual quantum mechanical (Born-rule) probability that the result of a measurement of  $A$  will lie in  $\Delta \subseteq \sigma(\hat{A}) \subset \mathbb{R}$ , given that the quantum state is  $\psi$ .

But in view of the discussion above, it is natural to reflect that even if  $\psi$  is not an eigenvector of  $\hat{A}$ , it is an eigenvector of coarse-grainings  $f(\hat{A})$  of  $\hat{A}$  (for example, the unit operator  $\hat{1}$  is always such a function<sup>7</sup>); and hence we are led to propose that we should assign to the proposition ‘ $A \in \Delta$ ’ the sieve of such coarse-grainings for which  $\psi$  is in the range of the corresponding spectral projector  $\hat{E}[f(A) \in f(\Delta)]$ . Thus we define the generalised valuation  $\nu^\psi$  associated with a vector  $\psi \in \mathcal{H}$  as

$$\begin{aligned} \nu^\psi(A \in \Delta) &:= \{f_{\mathcal{O}} : \hat{B} \rightarrow \hat{A} \mid \hat{E}[B \in f(\Delta)]\psi = \psi\} \\ &= \{f_{\mathcal{O}} : \hat{B} \rightarrow \hat{A} \mid \text{Prob}(B \in f(\Delta); \psi) = 1\} \end{aligned} \quad (2.26)$$

where  $\Delta$  is a Borel subset of the spectrum  $\sigma(\hat{A})$  of  $\hat{A}$ . One can check that  $\nu^\psi$  has all the properties Eqs. (2.16–2.19) required in the definition of a generalised valuation.

Furthermore, one can give an exactly analogous definition of the generalised valuation  $\nu^\rho$  associated with a density matrix  $\rho$ . One defines:

$$\begin{aligned} \nu^\rho(A \in \Delta) &:= \{f_{\mathcal{O}} : \hat{B} \rightarrow \hat{A} \mid \text{Prob}(B \in f(\Delta); \rho) = 1\} \\ &= \{f_{\mathcal{O}} : \hat{B} \rightarrow \hat{A} \mid \text{tr}(\rho \hat{E}[B \in f(\Delta)]) = 1\}. \end{aligned} \quad (2.27)$$

Again, all the properties required of a generalised valuation are satisfied.

### 3 Sieve-valued Valuations in Classical Physics

Before developing the conceptual aspects of the ideas of Section 2 in a very general setting (in the next Section), it is worth seeing how they apply to classical physics. In the first Subsection, we give a presheaf perspective on ordinary classical valuations. We first make a category out of the real-valued functions on phase space that represent classical physical quantities; we then introduce the analogue of the spectral presheaf, and contrast the

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<sup>7</sup>If desired, and as explained in (I), such ‘trivial’ functions can be removed by replacing  $\mathcal{O}$  with the category  $\mathcal{O}_*$ : defined to be  $\mathcal{O}$  minus all real multiples of the unit operator.



classical existence of global sections with the Kochen-Specker theorem; and we remark that quantisation can be represented as a functor. In the second Subsection, we motivate sieve-valuations in terms of classical macrostates. In the third, we generalize this motivation, and present the classical analogue of generalised valuations associated with a partial valuation.

### 3.1 A Presheaf Perspective on Orthodox Classical Valuations

One usually thinks of quantities and their values in classical physics as follows. If  $\mathcal{S}$  is the state space of some classical system, a physical quantity  $A$  is represented by a measurable real-valued function  $\bar{A} : \mathcal{S} \rightarrow \mathbb{R}$ ; and then the value  $V^s(A)$  of  $A$  in any state  $s \in \mathcal{S}$  is simply

$$V^s(A) = \bar{A}(s). \quad (3.1)$$

Thus all physical quantities possess a value in any state. Furthermore, if  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a measurable function, a new physical quantity  $f(A)$  can be defined by requiring the associated function  $\overline{f(A)}$  to be

$$\overline{f(A)}(s) := f(\bar{A}(s)) \quad (3.2)$$

for all  $s \in \mathcal{S}$ ; *i.e.*,  $\overline{f(A)} := f \circ \bar{A} : \mathcal{S} \rightarrow \mathbb{R}$ . Thus by definition, the values of the quantities  $f(A)$  and  $A$  satisfy a classical version of *FUNC*:

$$V^s(f(A)) = f(V^s(A)) \quad (3.3)$$

for all states  $s \in \mathcal{S}$ .

In terms of propositions of the form ‘ $A \in \Delta$ ’, where  $\Delta$  is a Borel subset of  $\mathbb{R}$ : to each microstate  $s \in \mathcal{S}$ , there corresponds a valuation defined by

$$V^s(A \in \Delta) = \begin{cases} 1 & \text{if } s \in \bar{A}^{-1}[\Delta] \\ 0 & \text{otherwise.} \end{cases} \quad (3.4)$$

Thus the proposition ‘ $A \in \Delta$ ’ is assigned the value ‘true’(1) by  $V^s$  if, and only if,  $\bar{A}(s) \in \Delta$ .

We turn now to rendering these ideas in terms of presheaves. The notions introduced in the next two Paragraphs will also be used in later subsections where we discuss sieve-valued valuations for classical physics.

**1. The Category of Measurable Functions on  $\mathcal{S}$ :** Let  $\mathcal{S}$  be a classical state space, and let  $\mathcal{M}$  denote the set of all real-valued measurable functions on  $\mathcal{S}$ ; thus each quantity  $A$  is represented by one<sup>8</sup> such function  $\bar{A} : \mathcal{S} \rightarrow \mathbb{R}$ . We now regard  $\mathcal{M}$  as a category where: (i) the objects are the real-valued measurable functions on  $\mathcal{S}$ ; and (ii) we say there is a morphism from  $\bar{B}$  to  $\bar{A}$  if there exists a measurable function  $f : \mathcal{S}(\bar{A}) \rightarrow \mathbb{R}$  such that  $\bar{B} = f \circ \bar{A}$  (*i.e.*,  $\bar{B}(s) = f(\bar{A}(s))$ , for all  $s \in \mathcal{S}$ ), where

$$\mathcal{S}(\bar{A}) := \bar{A}(\mathcal{S}) = \{r \in \mathbb{R} \mid \exists s \in \mathcal{S}, r = \bar{A}(s)\} \quad (3.5)$$

is the set of all possible values that the physical quantity  $A$  could take; it is the classical analogue of the spectrum  $\sigma(\hat{A})$  of a self-adjoint operator  $\hat{A}$  in the quantum theory. The morphism in  $\mathcal{M}$  corresponding to  $f : \mathcal{S}(\bar{A}) \rightarrow \mathbb{R}$  will be denoted  $f_{\mathcal{M}} : \bar{B} \rightarrow \bar{A}$ .

**2. The Value Presheaf:** The analogue of the spectral presheaf in quantum theory is now the following. We define the *value presheaf* on  $\mathcal{M}$  to be the covariant functor  $\Upsilon : \mathcal{M}^{\text{op}} \rightarrow \text{Set}$  such that:

1. On objects:  $\Upsilon(\bar{A}) := \mathcal{S}(\bar{A})$ —the set of all possible values of the quantity  $A$ .
2. On morphisms: If  $f_{\mathcal{M}} : \bar{B} \rightarrow \bar{A}$ , so that  $\bar{B} = f \circ \bar{A}$ , then  $\Upsilon(f_{\mathcal{M}}) : \mathcal{S}(\bar{A}) \rightarrow \mathcal{S}(\bar{B})$  is defined by  $\Upsilon(f_{\mathcal{M}})(\lambda) := f(\lambda)$  for all  $\lambda \in \mathcal{S}(\bar{A})$ .

We now observe that a global section of the value presheaf  $\Upsilon$  is a function  $\gamma$  that assigns to each object  $\bar{A}$  in the category  $\mathcal{M}$ , an element  $\gamma_{\bar{A}} \in \mathcal{S}(\bar{A})$  in such a way that if  $f_{\mathcal{M}} : \bar{B} \rightarrow \bar{A}$  (so that  $\bar{B} = f \circ \bar{A}$ ), then  $\Upsilon(f_{\mathcal{M}})(\gamma_{\bar{A}}) = \gamma_{\bar{B}}$ ; in other words

$$\gamma_{\bar{B}} = f(\gamma_{\bar{A}}). \quad (3.6)$$

Thus each global section corresponds to a classical valuation that satisfies classical *FUNC*, as in Eq. (3.3). Conversely, each such valuation determines a global section of the value presheaf. Clearly, the key difference from the situation in quantum theory (the Kochen-Specker theorem) is that the classical presheaf *does* have global sections: namely, each microstate  $s \in \mathcal{S}$ ,

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<sup>8</sup>Strictly speaking, functions that differ only on a set of Lebesgue measure zero should be identified; but we shall ignore this subtlety here.

determines a global section  $\gamma^s$  defined by

$$\gamma_{\bar{A}}^s := \bar{A}(s) \tag{3.7}$$

for all stages of truth  $\bar{A}$ .

**3. Quantisation as a Functor From  $\mathcal{M}$  to  $\mathcal{O}$ :** We remark incidentally that we can represent in terms of  $\mathcal{M}$  one of the main practical problems in quantum physics; viz. knowing how to ‘quantise’ a given classical system. More precisely, one wants to associate to each measurable function  $\bar{A} : \mathcal{S} \rightarrow \mathbb{R}$ , a self-adjoint operator  $\hat{A}$ ; or, perhaps, one seeks to do this for some special subset of classical variables. There is no universal way of performing such a quantisation; but it is generally agreed that if a physical quantity represented by  $\bar{A}$  is associated in some way with a particular operator  $\hat{A}$ , then, for any measurable function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , the function  $f(\bar{A})$  should be associated with the operator  $f(\hat{A})$ .

This preservation of functional relations can be represented neatly in the language of category theory by saying that a quantisation of the set of all classical quantities corresponds to a *covariant functor*  $\mathbf{Q} : \mathcal{M} \rightarrow \mathcal{O}$  that is defined (i) on an object  $\bar{A}$  in  $\mathcal{M}$  as  $\mathbf{Q}(\bar{A}) := \hat{A}$ ; and (ii) on a morphism  $f_{\mathcal{M}} : \bar{B} \rightarrow \bar{A}$ , by

$$\mathbf{Q}(f_{\mathcal{M}}) := f_{\mathcal{O}}. \tag{3.8}$$

This is because  $f_{\mathcal{M}} : \bar{B} \rightarrow \bar{A}$ , means that  $\bar{B} = f \circ \bar{A}$ ; and  $f_{\mathcal{O}} : \hat{B} \rightarrow \hat{A}$ , means that  $\hat{B} = f(\hat{A})$ .

### 3.2 Motivating Sieve-valued Valuations for Classical Physics

Since classical physics suffers no ‘Kochen-Specker prohibitions’ on global valuations of the orthodox kind, the motivation in Section 2.3.2 for sieve-valued valuations—as being naturally associated with *partial* valuations—seems not to apply to classical physics. But, in fact, the notion of a classical macrostate motivates the classical analogue of a partial valuation, and thereby leads to the associated sieve-valued valuations. This Subsection describes the role of the notion of a macrostate; and the next Subsection develops the idea so as to give the exact classical analogue of the partial valuations discussed in Section 2.3.2, and of the associated generalised valuations.

So suppose we are given, not a microstate  $s \in \mathcal{S}$ , but only a *macrostate*, represented by some Borel subset  $R \subseteq \mathcal{S}$ : what then can be said about the ‘value’ of a quantity  $A$ , or the truth-value of a proposition ‘ $A \in \Delta$ ’? Various responses are possible<sup>9</sup>: for example, the obvious choice is simply to say that the proposition ‘ $A \in \Delta$ ’ is true in the macrostate  $R$  if  $\bar{A}(R) \subseteq \Delta$ , and false otherwise. Thus ‘ $A \in \Delta$ ’ is defined to be true if, for *all* microstates  $s$  in  $R$ , the value  $\bar{A}(s)$  lies in the subset  $\Delta$ .

However, one may feel that this assignment of true and false is rather indiscriminating in so far as the proposition ‘ $A \in \Delta$ ’ is adjudged false irrespective of whether  $\bar{A}(s)$  fails to be in  $\Delta$  for all  $s \in R$ , or does so only for a ‘few’ points. For this reason, a more refined response is to say that one wants the proposition ‘ $A \in \Delta$ ’ to be ‘more true’, the greater the set of such points  $s$ : an idea that can be implemented by defining, for example, a generalised truth-value  $v^R(A \in \Delta)$  of the proposition ‘ $A \in \Delta$ ’ to be the set of such points:

$$v^R(A \in \Delta) := R \cap \bar{A}^{-1}[\Delta]. \quad (3.9)$$

Thus the set of possible truth-values of ‘ $A \in \Delta$ ’ is the Boolean algebra of Borel subsets of  $R \subseteq \mathcal{S}$ ; the actual truth-value being the subset of  $R$  in which the value of  $\bar{A}$  does belong to  $\Delta$ . (So ‘totally true’ corresponds to the first response’s ‘true’, and is represented by  $R$  itself; whereas ‘totally false’ is represented by the empty set.) These two responses are certainly workable; we discuss the second in another paper [2].

But a third response is much more similar to what we have discussed in the quantum case. Namely, we note that even if  $\bar{A}(R)$  is not a subset of  $\Delta$ , there will be<sup>10</sup> functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  with the property that the ‘coarse-grained’ function  $f(\bar{A}) := f \circ \bar{A} : \mathcal{S} \rightarrow \mathbb{R}$  satisfies the *weaker* condition  $f \circ \bar{A}(R) \subseteq f(\Delta)$ ; (so that, according to the first response above, the weaker

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<sup>9</sup>The most familiar response is that in order to assign values, we must do statistical physics; *i.e.*, we must have some probability measure  $\mu$  defined on  $\mathcal{S}$ , so that we say the *probability* that the value of  $A$  lies in  $\Delta$ , given that the macrostate is  $R$ , is:  $\text{Prob}(A \in \Delta; R) = \mu(R \cap \bar{A}^{-1}[\Delta]) / \mu(R)$ . But we are asking what can be said about values, supposing we are *not* doing statistical physics.

<sup>10</sup>The assertion ‘will be’ is on the assumption that constant functions on  $\mathcal{S}$  are admitted. Such trivial functions are the classical analogues of real multiples of the unit operator  $\hat{1}$  in quantum physics, and—if desired—they can be removed from the base category  $\mathcal{M}$ ; just as the quantum category  $\mathcal{O}$  can be replaced with the category  $\mathcal{O}_*$  in which multiples of  $\hat{1}$  are removed.

proposition ‘ $f(A) \in f(\Delta)$ ’ is true in the macrostate  $R$ ). And we then define the generalised truth-value  $\nu^R(A \in \Delta)$  of the original proposition ‘ $A \in \Delta$ ’ to be the set of all such coarse-grainings of  $\bar{A}$ . Formally, in terms of the category  $\mathcal{M}$ :

$$\nu^R(A \in \Delta) := \{f_{\mathcal{M}} : \bar{B} \rightarrow \bar{A} \mid \bar{B}(R) \subseteq f(\Delta)\}. \quad (3.10)$$

It is straightforward to check that the right hand side of Eq. (3.10) is a sieve on  $\bar{A}$ . Furthermore,  $\nu^R(A \in \Delta)$  has (classical analogues of) all the other properties listed in Section 2.3.3 as clauses of the general definition of a generalised valuation—as we shall discuss in the next Subsection.

### 3.3 The Classical Analogue of Generalised Valuations

We will now generalize the use in Section 3.2 of macrostates to motivate sieve-valued valuations, to obtain the classical analogue of the generalised valuations in Section 2.3.2 associated with *any* partial valuation. All the properties discussed in Sections 2.3.2 and 2.3.3 (and incorporated as clauses of the definition of a generalised valuation)—in particular, the sieve-analogue of *FUNC*—will carry over to this classical setting.

We begin by noting that a macrostate  $R \subseteq \mathcal{S}$  is naturally associated with a classical partial valuation  $V^R$  (*i.e.*, an assignment to some quantities of numbers as values, obeying a classical version of *FUNC*). First we define the domain of  $V^R$  to be the set of all measurable functions on  $\mathcal{S}$  that are constant on the subset  $R$ :

$$\text{dom } V^R := \{\bar{A} : \mathcal{S} \rightarrow \mathbb{R} \mid \forall s_1, s_2 \in R, \bar{A}(s_1) = \bar{A}(s_2)\}. \quad (3.11)$$

Then we define the value of a quantity  $A$  whose representative function  $\bar{A}$  lies in the domain of  $\text{dom } V^R$ , by

$$V^R(\bar{A}) := \bar{A}(s_0) \quad (3.12)$$

for any  $s_0 \in R$ ; since  $\bar{A}$  is constant on  $R$ , the result does not depend on the choice of  $s_0$  in  $R$ . So  $V^R(\bar{A}) \in \mathcal{S}(\bar{A})$ . It also follows that  $\text{dom } V^R$  is closed under coarse-graining, and that the values of  $V^R$  obey *FUNC*. That is, we have, just as in the definition of a partial valuation in Section 2.3.2: if  $\bar{A} \in \text{dom } V^R$  and  $\bar{B} = f(\bar{A})$  then (i)  $\bar{B} \in \text{dom } V^R$ ; and (ii)  $V^R(\bar{B}) = f(V^R(\bar{A}))$ .

This prompts us to define a classical partial valuation in general (*i.e.*, regardless of specifying a macrostate) as an assignment  $V$  to each element  $\bar{A}$  belonging to some subset  $\text{dom } V$  of  $\mathcal{M}$ , of a member of  $\mathcal{S}(\bar{A})$ , such that if  $\bar{B} = f(\bar{A})$  then (i)  $\bar{B} \in \text{dom } V$  and (ii)  $V(\bar{B}) = f(V(\bar{A}))$ . With this definition, claims (1) and (2) of Section 2.3.2 and 2.3.3 carry over completely to the classical case: we simply substitute  $\mathcal{M}$  for  $\mathcal{O}$  (and so  $\bar{A}$  for  $\hat{A}$  etc.) and the ‘classical spectrum’  $\mathcal{S}(\bar{A})$  for  $\sigma(\hat{A})$ . Thus we claim:

1. Given such a partial valuation  $V$ , there is a natural associated valuation that: (i) is defined on *all* propositions ‘ $A \in \Delta$ ’; and (ii) assigns to such a proposition as its value, a sieve on  $\bar{A}$  in the category  $\mathcal{M}$ . Namely:

$$\nu^V(A \in \Delta) := \{f_{\mathcal{M}} : \bar{B} \rightarrow \bar{A} \mid \bar{B} \in \text{dom } V, V(\bar{B}) \in f(\Delta)\}. \quad (3.13)$$

Furthermore, the properties of these valuations, in particular the analogue for sieves of the property *FUNC*, carry over completely from the quantum to the classical case.

2. Accordingly, we can use these properties to generalise the notion of a valuation, *i.e.*, to define a *generalised valuation* as a map that (i) assigns a sieve on  $\bar{A}$  to each proposition ‘ $A \in \Delta$ ’ and (ii) has these properties.

We can also present our collection of sets of propositions at each stage  $\bar{A}$  as a classical coarse-graining presheaf  $\mathbf{G}$  that (i) assigns to each  $\bar{A}$  the Boolean algebra of propositions of the form ‘ $A \in \Delta$ ’ (or, equivalently, the algebra of characteristic functions  $\chi_{\Delta}$ : the classical analogue of the spectral projectors in quantum theory) identified as the Boolean algebra of Borel subsets  $\Delta \subseteq \mathcal{S}(\bar{A})$ ; and (ii) acts on morphisms  $f_{\mathcal{M}} : B \rightarrow A$  such that  $\mathbf{G}(f_{\mathcal{M}})$  coarsens propositions, in exact analogy to Eq. (2.22).

But we shall not rehearse all the definitions, and verifications of properties, substantiating these claims. For firstly, they carry over directly from the discussion in (I) of the quantum case; and secondly, we shall see in the Sections to follow that many of these definitions and properties apply much more widely than in classical and quantum physics.

Something that is worth developing a little further however, is the observation that—as in the quantum case—the situation can arise in which  $f(\Delta)$

is not a Borel subset of  $\mathbb{R}$ , even though  $\Delta$  is. In this context, we note that the central reason why it is feasible to regard Eq. (2.23) as a *definition* of  $\hat{E}[f(A) \in f(\Delta)]$  if  $f(\Delta)$  is not Borel, is that the lattice of projection operators is *complete*, and hence the right hand side of Eq. (2.23) is well-defined. A natural analogue of this construction in the classical case would be to start with the Hilbert space  $L^2(S, d\mu)$ , where  $d\mu$  is the natural measure on the classical state space (a  $2n$ -dimensional symplectic manifold, where  $n$  is the number of degrees of freedom)  $S$  formed by taking the wedge-product  $n$ -times of the basic symplectic 2-form on  $S$ . Any proposition  $A \in \Delta$  can then be associated with a corresponding *projection operator* on this Hilbert space: namely, the projection onto the Borel subset  $A^{-1}(\Delta)$ . An analogous trick to that in Eq. (2.23) can then be applied by using the projection lattice on the separable Hilbert space  $L^2(S, d\mu)$ . However, we shall not go into the mathematical details here since, in the present paper, our invocation of the classical example is intended primarily to be of pedagogical value as an illustration of the general concepts that will be discussed in the next Section.

Finally, for the sake of completeness, we remark on the classical analogue of claim (3) of Section 2.3.4: the claim that an orthodox quantum state, a vector  $\psi \in \mathcal{H}$  or a density matrix  $\rho$ , induces a generalised valuation. For  $\psi \in \mathcal{H}$ , we defined (see Eq. (2.26))

$$\begin{aligned} \nu^\psi(A \in \Delta) &:= \{f_{\mathcal{O}} : \hat{B} \rightarrow \hat{A} \mid \hat{E}[B \in f(\Delta)]\psi = \psi\} \\ &= \{f_{\mathcal{O}} : \hat{B} \rightarrow \hat{A} \mid \text{Prob}(B \in f(\Delta); \psi) = 1\} \end{aligned} \quad (3.14)$$

where  $\Delta$  is a Borel subset of the spectrum  $\sigma(\hat{A})$  of  $\hat{A}$ . In the classical case, for  $s \in \mathcal{S}$ , and  $\Delta$  a Borel subset of the ‘classical spectrum’  $\mathcal{S}(\bar{A})$ , the analogue of Eq. (3.14) is clearly

$$\nu^s(A \in \Delta) := \{f_{\mathcal{M}} : \bar{B} \rightarrow \bar{A} \mid \chi_{[B \in f(\Delta)]}(s) = 1\} = \{f_{\mathcal{M}} : \bar{B} \rightarrow \bar{A} \mid f(\bar{A}(s)) \in f(\Delta)\} \quad (3.15)$$

where  $\chi_{[B \in f(\Delta)]}$  is the characteristic function for  $\bar{B}^{-1}(f(\Delta))$ . It is easy to see that the sieve  $\nu^s(A \in \Delta)$  is the principal sieve,  $1_{\Omega(A)} = \downarrow A$  (so that in the language of Section 2.3.2, ‘ $A \in \Delta$ ’ is totally true) if and only if  $\bar{A}(s) \in \Delta$ . One can check that  $\nu^s$  has all the properties required in the definition of a generalised valuation (items (i) to (iv) in Section 2.3.3).

Furthermore, there is a corresponding classical analogue of the definition of the generalised valuation  $\nu^\rho$  associated with a density matrix  $\rho$ :

$$\nu^\rho(A \in \Delta) := \{f_{\mathcal{O}} : \hat{B} \rightarrow \hat{A} \mid \text{Prob}(B \in f(\Delta); \rho) = 1\} \quad (3.16)$$

$$= \{f_{\mathcal{O}} : \hat{B} \rightarrow \hat{A} \mid \text{tr}(\rho \hat{E}[B \in f(\Delta)]) = 1\}.$$

Namely: the classical analogue is that, with  $\rho$  now representing a classical mixed state, *i.e.*, a probability measure on  $\mathcal{S}$ :

$$\nu^{\rho}(A \in \Delta) := \{f_{\mathcal{M}} : \bar{B} \rightarrow \bar{A} \mid \text{Prob}_{\rho}(B \in f(\Delta)) = 1\} \quad (3.17)$$

where  $\text{Prob}_{\rho}(B \in f(\Delta))$  is the classical statistical probability, according to  $\rho$ , that ‘ $B \in f(\Delta)$ ’, *i.e.*, the  $\rho$ -measure  $\rho(\bar{B}^{-1}(f(\Delta)))$  of  $\bar{B}^{-1}(f(\Delta))$ . It follows that the sieve  $\nu^{\rho}(A \in \Delta)$  is the principal sieve,  $1_{\Omega(A)} = \downarrow A$  if and only if  $\rho(\bar{A}^{-1}(\Delta)) = 1$ , *i.e.* if and only if ‘ $A \in \Delta$ ’ is certain according to  $\rho$ .

## 4 General Properties of Sieve-valued Valuations

In this Section and the next, we turn to showing how sieve-valued valuations arise much more generally than just in the examples of quantum and classical physics discussed earlier. Indeed, we claim that they are one of the most natural notions of valuation for any presheaf of propositions, no matter what their topic. In claiming this we will assume about valuations only the basic idea that they must be some sort of structure-preserving function from the sets of contextualised propositions (with some such operations as negation, conjunction etc. defined on it) to the corresponding sets of truth values, which are to be some sort of logical algebra.

In this Section, we will argue for this claim by displaying how some of the principal ideas and results of Section 4.2 and Section 5 of (I)—specifically, the sieve-version of *FUNC* already emphasised in Sections 2 and 3 above, and the notion of ‘coarse-graining’—can be greatly generalized so that, for the most part, they apply to *any* presheaf of propositions. Another argument for the claim will be presented in Section 5 of the present paper.

### 4.1 The Role of *FUNC*

In this Subsection, we introduce our most general version of *FUNC*; and motivate it and the idea of a sieve-valued valuation on an arbitrary presheaf of propositions  $\mathbf{G}$ , by showing that together they define natural transformations from  $\mathbf{G}$  to  $\Omega$ , and hence subobjects of  $\mathbf{G}$ .



Let  $\mathcal{C}$  be any small category, with objects  $A, B, \dots$ ; and let  $\mathbf{G}$  be any presheaf on  $\mathcal{C}$ , with the set  $\mathbf{G}(A)$  having elements  $d, e, \dots$ . We think of the pair  $[A, d]$  as specifying a proposition at the context, or stage of truth,  $A$ ; and so of  $\mathbf{G}$  as a presheaf of propositions. We call a function  $\nu$  that assigns to each choice of object  $A$  and each  $d \in \mathbf{G}(A)$ , a set of morphisms in  $\mathcal{C}$  to  $A$  (*i.e.*, morphisms with  $A$  as codomain), a *morphism-valued valuation* on  $\mathbf{G}$ . We write the values of this function as  $\nu(A, d)$ .

Note that for any set  $S$  of morphisms to  $A$  (not necessarily a sieve), and any  $f : B \rightarrow A$ , we can define a pull-back to  $B$  of  $S$  by Eq. (2.4):

$$f^*(S) := \{h : C \rightarrow B \mid f \circ h \in S\} \quad (4.1)$$

although we note that there is no compelling reason for this definition if the sets  $S$  are totally unrestricted. However, this caveat notwithstanding, we will say that a morphism-valued valuation satisfies *generalized functional composition*—for short, *FUNC*—if for all  $A, B$  and  $f : B \rightarrow A$  and all  $d \in \mathbf{G}(A)$ , it obeys

$$\nu(B, \mathbf{G}(f)(d)) = f^*(\nu(A, d)). \quad (4.2)$$

We call a morphism-valuation on  $\mathbf{G}$  a *sieve-valued valuation* on  $\mathbf{G}$  if its values are all sieves; in this case Eq. (4.1) is much better motivated since the pull-back of a sieve is itself a sieve. The discussion in Section 2 already supplies us with two motivations for using sieve-valuations in this very general setting. First, from a logical perspective: if we think of  $\mathbf{G}(A)$  as a set of propositions, we expect a value  $\nu(A, d)$  of such a proposition to be some sort of truth-value. And we saw in Section 2 how  $\Omega$  supplies a well-behaved set of contextual and generalized truth-values.

Second, and more generally: for any presheaf  $\mathbf{G}$ , a natural notion of a valuation on  $\mathbf{G}$  is a subobject of  $\mathbf{G}$ . For think, as in logic, of a valuation as specifying the ‘selected’ or ‘winning’ elements  $d$  in each  $\mathbf{G}(A)$ . One naturally imagines that these selected elements might form a subobject of  $\mathbf{G}$ . But we saw in Section 2 that subobjects are in one-one correspondence with morphisms, *i.e.*, natural transformations,  $N : \mathbf{G} \rightarrow \Omega$ . So one expects that at least some sieve-valued valuations will define such a natural transformation by  $N_A^\nu(d) := \nu(A, d)$ .

This motivation for sieve-valued valuations leads directly to *FUNC*. For it turns out that *FUNC* is exactly the condition a sieve-valued valuation must obey in order to thus define a natural transformation, *i.e.*, a subobject of  $\mathbf{G}$ . Specifically, we have (*cf.* Theorem 4.2 of (I)),

**Theorem 4.1** *A sieve-valued valuation  $\nu$  on  $\mathbf{G}$  obeys FUNC if and only if the functions at each stage of truth  $A$*

$$N_A^\nu(d) := \nu(A, d) \tag{4.3}$$

define a natural transformation  $N^\nu$  from  $\mathbf{G}$  to  $\mathbf{\Omega}$ .

**Proof**

Suppose  $f : B \rightarrow A$ , so that naturalness means that the composite map  $\mathbf{G}(A) \xrightarrow{N_A^\nu} \mathbf{\Omega}(A) \xrightarrow{\mathbf{\Omega}(f)} \mathbf{\Omega}(B)$  is equal to  $\mathbf{G}(A) \xrightarrow{\mathbf{G}(f)} \mathbf{G}(B) \xrightarrow{N_B^\nu} \mathbf{\Omega}(B)$ . But given that  $N_A^\nu(d) := \nu(A, d)$ , this is the condition that

$$\mathbf{\Omega}(f)(\nu(A, d)) = (N_B^\nu \circ \mathbf{G}(f))(d) = \nu(B, \mathbf{G}(f)(d)) \tag{4.4}$$

which is exactly *FUNC*. **Q.E.D.**

To sum up: we conclude that sieve-valued valuations obeying *FUNC* are a very natural notion of valuation on any presheaf of propositions

## 4.2 Coarse-Graining Presheaves

In this Subsection, we will generalize one of the main notions in Section 5 of (I): the idea of generalized coarse-graining. Our generalization of this notion involves the use of a new map, called the *comparison functor*; this will also be needed in Section 5 in our general discussion of the logic of partial truth.

There are two main ways in which we shall generalize the idea of coarse-graining:

1. In (I), the set of ‘propositions’  $\mathbf{G}(\hat{A})$  at each stage  $\hat{A}$  was a Boolean algebra (of Borel subsets of  $\sigma(\hat{A})$ , or equivalently of  $\hat{A}$ ’s spectral projectors; and similarly for the classical case, *cf.* Section 3). However, here we shall assume only that  $\mathbf{G}(A)$  is a poset with a 0 and a 1. Indeed, much of what follows could be generalized to the case where  $\mathbf{G}(A)$  is just a poset; but we will also require a 0 and a 1, to link to the null, exclusivity and monotonicity clauses of the definition of a generalised valuation, Eqs. (2.17–2.19) (see Eqs. (4.7–4.9) below).
2. In (I), generalized coarse-graining was defined so as to use, for the case where  $f_{\mathcal{O}} : \hat{B} \rightarrow \hat{A}$ , the identity map on  $\mathbf{G}(\hat{B})$  to embed the Boolean

algebra  $\mathbf{G}(\hat{B})$  into its superset (larger Boolean algebra)  $\mathbf{G}(\hat{A})$  (this was used in writing Eq. (2.22)). In the generalization in this Subsection to any presheaf of propositions  $\mathbf{G}$  on any small category  $\mathcal{C}$ , such that  $\mathbf{G}(A)$  is a poset with a 0 and 1, we will again need a map acting in the opposite direction to  $\mathbf{G}(f)$ . But it need not be the identity map, since the poset  $\mathbf{G}(B)$  need not be a subset of  $\mathbf{G}(A)$ . So we will simply assume that there is some such map (given by the *comparison* functor introduced in Paragraph 1 below).

We should also note another way in which the exposition to follow differs from that in Section 5 of (I). There, our discussion took as the base-category, not  $\mathcal{O}$ , but the poset  $\mathcal{W}$  of all Boolean subalgebras of the projection lattice  $\mathcal{P}(\mathcal{H})$  of the Hilbert space  $\mathcal{H}$ . In this category<sup>11</sup>, the objects are defined to be the subalgebras  $W \in \mathcal{W}$ ; and a morphism is defined to exist from  $W_2$  to  $W_1$  if  $W_2 \subseteq W_1$ : thus there is at most one morphism between any two objects. In some respects  $\mathcal{W}$  is a more natural category to work with than  $\mathcal{O}$ , since it ‘identifies’ quantities that are each a function of the other, and hence have the same spectral algebra.

For this reason, in (I) we sometimes worked with  $\mathcal{W}$ , rather than  $\mathcal{O}$ . In particular, the Kochen-Specker theorem gets as natural an expression in terms of  $\mathcal{W}$ , as it does in terms of  $\mathcal{O}$ . But, for the sake of brevity, in the review of (I) in the present paper we have used only the category  $\mathcal{O}$  (and its classical analogue  $\mathcal{M}$ ). And again in this Subsection, while generalizing Section 5 of (I), we will present our definitions and results in terms of the category  $\mathcal{C}$ , which we have hitherto thought of as generalizing  $\mathcal{O}$ . So for the rest of this Subsection, we assume as in Section 4.1 that  $\mathcal{C}$  is any small category, with objects  $A, B, \dots$ ; and that  $\mathbf{G}$  is a presheaf on  $\mathcal{C}$ , with the set  $\mathbf{G}(A)$  having elements  $d, e, \dots$ . We also assume that at each  $A$ ,  $\mathbf{G}(A)$  is a poset with a 0 and a 1.

**1. The Comparison Functor:** In Section 5 below, given a morphism  $f : B \rightarrow A$ , we shall need to be able to ‘push-forward’ a proposition  $d \in \mathbf{G}(B)$

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<sup>11</sup>In a similar way, one can make a category out of any poset; in particular, the corresponding category for classical physics will consist of all Boolean subalgebras of the algebra (itself Boolean!) of all Borel subsets of the classical state space  $\mathcal{S}$ , again ordered by subalgebra inclusion.

to  $\mathbf{G}(A)$ , for comparison of ‘logical strength’ (*i.e.*, comparison according to the partial order  $<$  in the poset  $\mathbf{G}(A)$ ) with propositions in  $\mathbf{G}(A)$ .

In (I), this presented no problem since, in using the base category  $\mathcal{W}$ , we have that  $d \in \mathbf{G}(B)(= W_2)$  is itself also a member of  $\mathbf{G}(A)(= W_1)$ , (if  $f : W_2 \rightarrow W_1$ , so that  $W_2 \subseteq W_1$ ). But with a general category  $\mathcal{C}$ , this fails since there is no *a priori* embedding of  $\mathbf{G}(B)$  in  $\mathbf{G}(A)$ .

Accordingly, we now assume that such a map is given. More precisely, we assume that whenever  $f : B \rightarrow A$  in  $\mathcal{C}$ , we are given a map from  $\mathbf{G}(B)$  to  $\mathbf{G}(A)$ , which need not be injective. For much of the argument to follow, we do *not* need to assume that these maps mesh under composition so as to give a (covariant) functor from  $\mathcal{C}$  to  $\text{Set}$ , but for simplicity, we will do so. Thus we assume that there is a covariant functor,  $\mathbf{C}$ , from  $\mathcal{C}$  to  $\text{Set}$ , called the *comparison functor* (‘ $\mathbf{C}$ ’ for ‘comparison’), with the same action on objects  $A$  in  $\mathcal{C}$  as has  $\mathbf{G}$ . To sum up:

- $\mathbf{C}(A) := \mathbf{G}(A)$  at all  $A$ ;
- if  $f : B \rightarrow A$ , there is a map  $\mathbf{C}(f) : \mathbf{C}(B) \rightarrow \mathbf{C}(A)$ .

**2. Coarse-Graining Presheaves:** We turn now to the main topic of this Subsection, which is to generalize the discussion in (I), Section 5, of generalized coarse-graining. In effect, that discussion proceeded by noting three properties of the original coarse-graining presheaf  $\mathbf{G} : \mathcal{O} \rightarrow \text{Set}$  (defined in Section 2.3.3 above); and then defining *a* coarse-graining presheaf to be any presheaf with these properties. These properties were called ‘coarse-graining’, ‘retraction’ and ‘monotonicity’; but we need not list them. (We say ‘in effect’, just because the definition was in terms of the category  $\mathcal{W}$ , not  $\mathcal{O}$ .)

Here, we will generalize to any small category  $\mathcal{C}$ . The idea is to take the comparison functor  $\mathbf{C}$  to be given *ab initio*, and then to define a presheaf  $\mathbf{G}$  to be a ‘coarse-graining’ with respect to  $\mathbf{C}$ , if it has these three properties—or rather, their generalizations, to allow for  $\mathbf{C}(f)$  not necessarily being a subset inclusion map.

So we assume we are given a covariant functor  $\mathbf{C}$  from  $\mathcal{C}$  to  $\text{Set}$ , with all the  $\mathbf{C}(A)$  being posets with a 0 and 1. Then we define a *coarse-graining with respect to  $\mathbf{C}$*  to be a presheaf  $\mathbf{G}$  on  $\mathcal{C}$  (*i.e.*, a contravariant functor from  $\mathcal{C}$  to  $\text{Set}$ ), with the following properties:

- 1)  $\mathbf{G}$  has the same action on objects as  $\mathbf{C}$ , *i.e.*,  $\mathbf{G}(A) := \mathbf{C}(A)$ ;
- 2) ‘coarse-graining’: if  $f : B \rightarrow A$ , then for all  $d \in \mathbf{G}(A)$ ,

$$d \leq \mathbf{C}(f)[\mathbf{G}(f)(d)]; \quad (4.5)$$

- 3) ‘monotonicity’: if  $f : B \rightarrow A$ , and  $d \leq e$  in  $\mathbf{G}(A)$ , then  $\mathbf{G}(f)(d) \leq \mathbf{G}(f)(e)$  in  $\mathbf{G}(B)$ .

In (I) we also added the condition

- 4) ‘generalized retraction’: if  $f : B \rightarrow A$ , then for all  $d \in \mathbf{G}(B)$ ,

$$\mathbf{G}(f)[\mathbf{C}(f)(d)] = d \quad (4.6)$$

but we note that if this extra condition Eq. (4.6) is imposed, then the map  $\mathbf{C}(f) : \mathbf{G}(B) \rightarrow \mathbf{G}(A)$  is necessarily injective (*i.e.*, it is one-to-one); and hence we have only a marginal generalisation of the situation in (I) in which  $\mathbf{G}(B)$  is an explicit subset of  $\mathbf{G}(A)$ . On the other hand, the motivation for imposing the generalised retraction condition in the first place was closely linked to the fact that  $\mathbf{G}(B)$  is a subset of  $\mathbf{G}(A)$  in the example of quantum theory; therefore it is legitimate to consider removing this condition, with a concomitant freeing up of possibilities for the comparison-functor maps  $\mathbf{C}(f) : \mathbf{G}(B) \rightarrow \mathbf{G}(A)$ .

### 3. Generalised Valuations on a general Coarse-graining Presheaf:

The general notion of a coarse-graining presheaf just introduced admits generalised valuations of the *FUNC*-obeying kind originally envisaged in (I) and in Section 4.1.

The first step is to define a *local valuation* of the poset  $\mathbf{G}(A)$  in the Heyting algebra  $\mathbf{\Omega}(A)$ . This is to be a map  $\phi : \mathbf{G}(A) \rightarrow \mathbf{\Omega}(A)$  such that the following conditions are satisfied:

$$\text{Null proposition condition : } \phi(0_{\mathbf{G}(A)}) = 0_{\mathbf{\Omega}(A)} \quad (4.7)$$

$$\text{Monotonicity : } \alpha \leq \beta \text{ implies } \phi(\alpha) \leq \phi(\beta) \quad (4.8)$$

$$\text{Exclusivity : } \text{ If } \alpha \wedge \beta = 0_{\mathbf{G}(A)} \text{ and } \phi(\alpha) = 1_{\mathbf{\Omega}(A)}, \text{ then } \phi(\beta) < 1_{\mathbf{\Omega}(A)} \quad (4.9)$$

which are the appropriate analogues of Eq. (2.17), Eq. (2.18) and Eq. (2.19) respectively.

Now we define a *generalised valuation* on  $\mathcal{C}$  associated with a coarse-graining presheaf  $\mathbf{G}$  ( $\mathbf{G}$  being with respect to some comparison functor  $\mathbf{C}$ ) to be a family of local valuations  $\phi_A : \mathbf{G}(A) \rightarrow \mathbf{\Omega}(A)$ , at each  $A$ , such that if  $f : B \rightarrow A$  then, for all  $d \in \mathbf{G}(A)$ ,

$$\phi_B(\mathbf{G}(f)(d)) = f^*(\phi_A(d)). \quad (4.10)$$

Bearing in mind that this equation is essentially *FUNC*, Eq. (4.2), and that local valuations obey the null proposition, monotonicity and exclusivity conditions in Eqs. (4.7–4.8), we see that this definition directly generalizes the generalised valuations on  $\mathcal{O}$  of Section 2.3.3. So the definition is non-empty. In particular: in (I), Section 5.3.4, we showed that a density matrix defines such a generalised valuation on the specific category  $\mathcal{W}$ , associated with any coarse-graining presheaf on  $\mathcal{W}$ . A similar result can be proved for the classical case, using the material at the end of Section 3 above, especially Eq. (3.17).

Finally, we remark that since these generalised valuations for an arbitrary coarse-graining presheaf  $\mathbf{G}$  (with respect to an arbitrary comparison functor) obey *FUNC*, the discussion of Section 4.1 applies. That is: each such generalised valuation,  $\Phi$  say, (a family of local valuations  $\phi_A$ ) defines a natural transformation  $N^\Phi$  from the coarse-graining presheaf  $\mathbf{G}$  to the subobject classifier  $\mathbf{\Omega}$ , by defining the components:

$$N_A^\Phi(d) := \phi_A(d) \quad (4.11)$$

As emphasised in Section 2.2, such natural transformations are in one-to-one correspondence with subobjects. Thus each such generalised valuation defines a subobject of  $\mathbf{G}$ .

## 5 The Logic of Partial Truth

We turn now to give our final motivation for the use of sieve-valued valuations. We start from a handful of general intuitive requirements about how the truth-values of propositions should reflect their logical relations, and argue that sieve-valued valuations are the natural way to satisfy these

requirements. More precisely: valuations taking sieves as their values are determined in a natural way, for any category  $\mathcal{C}$  of ‘contexts’, once we require the following:

- (i) each object in the category has an associated family of propositions, with different families corresponding to different objects families meshing suitably;
- (ii) the valuation is to represent partial truth (degrees of truth), subject to some weak conditions, the most important being that the partial truth-value of a proposition at a stage  $A$  in  $\mathcal{C}$  is to be determined by which of its consequences (weaker propositions) are totally true at their own stage.

The concrete valuations discussed in Sections 2.3 and 3 (and in (I)) arise from applying these requirements to propositions about the values of physical quantities.

We emphasise that although we think conditions in (ii) on partial truth are very reasonable, we make no claim that they are obligatory. In the philosophical literature, partial truth is modelled in various ways, and indeed often rejected altogether (for example, [6]). We discuss this more in [2]. Here, suffice it to say in defence of our own notion that at least it is tightly controlled by the notion of total truth, in the sense that the partial truth-value of any proposition is determined by which propositions are totally true.

Our argument will be very general: indeed, the only precise mathematical notion that is needed is that of a sieve in a category. Otherwise, the argument can be formulated intuitively: for example, in its use of the idea of one proposition being a consequence of (logically weaker than) another. Of course, by assuming mathematical notions in addition to that of a category, these intuitive ideas can be made precise. But it seems to us best to emphasise the generality of the intuitive argument by assuming these further notions only after giving the argument.

We will therefore proceed in two subsections. Subsection 1 will give the intuitive argument that assumes only the notion of a category, and leads to sieve-valued valuations. Subsection 2 will comment on the argument, and exhibit one natural way of making its intuitive ideas precise: in particular, making consequence (entailment) precise by having the families of

propositions at each stage be posets, and having embedding maps like the comparison functor introduced in Section 4.2.

## 5.1 The Intuitive Argument for Sieve-valued Valuations

Suppose we are given some category  $\mathcal{C}$ , and that to each object  $A \in \mathcal{C}$  is associated a set  $\mathcal{P}(A)$  whose elements  $d$  we will call ‘items’. We allow that for different objects  $A, B$  in  $\mathcal{C}$ , the sets  $\mathcal{P}(A)$ ,  $\mathcal{P}(B)$  can differ. For each  $A$  and  $d \in \mathcal{P}(A)$  we think of  $[A, d]$  as a proposition. We do not require that for fixed  $A$ , the family  $\{[A, d] \mid d \in \mathcal{P}(A)\}$  is a Boolean algebra; nor, for the moment, that it have any other structure—for example, that of a poset. But we do require the following assumptions.

(A) The morphisms in the category are associated with maps between propositions for different objects, as follows. If there is a morphism  $f : B \rightarrow A$  from  $B$  to  $A$ , then there is a function from the family of propositions  $\{[A, d] \mid d \in \mathcal{P}(A)\}$  to the corresponding family  $\{[B, e] \mid e \in \mathcal{P}(B)\}$  associated with  $B$ . We represent this map associated with  $f$  by  $f^\dagger$  acting on the items  $d$ . So given a morphism  $f : B \rightarrow A$ , then  $[B, f^\dagger(d)]$  is the ‘ $B$ -proposition’ that ‘corresponds by  $f$ ’ to  $[A, d]$ . Furthermore, recalling that every object  $A$  in a category has an identity morphism,  $\text{id}_A : A \rightarrow A$ , we require that the map on propositions associated with the identity morphism be the identity map on propositions. That is: we require that for any  $A$ ,  $(\text{id}_A)^\dagger = \text{id}_{\mathcal{P}(A)}$ . Two remarks about this assumption.

(a) We do not initially require that the associations  $A \mapsto \mathcal{P}(A)$  and  $f \mapsto f^\dagger$  together define a presheaf on  $\mathcal{C}$ . That is: we do not need to assume that, given morphisms  $f : B \rightarrow A$  and  $g : C \rightarrow B$ , and so a morphism  $f \circ g : C \rightarrow A$ , we have:  $g^\dagger(f^\dagger(d)) = (f \circ g)^\dagger(d)$ .

However, we note *en passant* that if this presheaf condition is *not* satisfied, then the ‘ $\dagger$ ’-operation is ‘path-dependent’ in the following sense: If a morphism  $k : C \rightarrow A$  can be factored in the form  $C \xrightarrow{g} B \xrightarrow{f} A$ , then the pull-back  $k^\dagger(d)$  of  $d \in \mathcal{P}(A)$  may not equal the composite pull-back  $g^\dagger(f^\dagger(d))$  obtained by factoring  $k$



through the intermediate object  $B$ . In most physical situations, such a behaviour would be considered distinctly pathological.

- (b) To use the notation  $\mathbf{G}(f)$  instead of  $f^\dagger$  would echo the notation in Section 4.2 (and its special cases, Definitions 5.3 and 5.4 of (I)). But we use  $f^\dagger$  to indicate that we do not require a presheaf. See the next subsection for how the argument to come can be carried over to any coarse-graining presheaf in the sense of Section 4.2.
- (B) For any morphism  $f : B \rightarrow A$ , and any proposition  $[A, d]$ , the corresponding  $B$ -proposition  $[B, f^\dagger(d)]$  is intuitively logically weaker than (a consequence of)  $[A, d]$ . Again, two remarks about this assumption.
- (a) To accommodate the identity morphism, and the requirement of (A) that  $(\text{id}_A)^\dagger = \text{id}_{\mathcal{P}(A)}$ , we note that ‘weaker’ here means ‘strictly weaker *or* the same as’, just as ‘ $\leq$ ’ means ‘is less than or equal to’. Similarly for ‘consequence’.
  - (b) Again: it is enough at this stage to use ‘logically weaker’ in an intuitive sense, so as to motivate the requirements in (C) below. Subsection 2 will make it precise, in terms of each object’s family of propositions being a poset and there being a comparison functor between them.
- (C) We propose to assign to each proposition  $[A, d]$  a truth-value  $\nu(A, d)$ . There is to be one truth-value, called ‘total truth’ (as against the other ‘partially true’ values), that is subject to the following intuitive requirements:
- (a) If  $[A, d]$  is totally true, so are all its weakenings (consequences)  $[B, f^\dagger(d)]$ ; (note that since assumption (A) required  $(\text{id}_A)^\dagger = \text{id}_{\mathcal{P}(A)}$ ,  $[A, d]$  is one of its own weakenings, and so the converse statement is automatic).
  - (b) If  $[A, d]$  is partially true (*i.e.*, has one of the other truth-values), it is in some intuitive sense ‘more true’, or ‘nearer being totally true’, the more of its weakenings  $[B, f^\dagger(d)]$  are totally true.
  - (c) The truth-value  $\nu(A, d)$ , is to be determined by which of the weakenings  $[B, f^\dagger(d)]$  of  $[A, d]$  is totally true: determined in some way that obeys (a) and (b) above.

Three remarks about assumption (C). First: part (c) is perhaps less intuitive than parts (a) and (b); but it can be motivated by the philosophical idea that the semantic value or ‘content’ of a sentence is determined by the set of those of its consequences that are true (in the usual classical two-valued sense)—we discuss this in [2]. Second: part (c) can also be defended as likely to mollify sceptics about partial truth. For it makes the notion of partial truth tightly controlled by the more acceptable notion of total truth: once the maps  $f^\dagger$  and the set of totally true propositions is given, the partial truth-values of all propositions are fixed—and fixed ‘individually’ in that the partial truth-value of  $[A, d]$  depends only on which of *its* weakenings are totally true. In any case, we now assume (c). Third: one might propose as intuitive a variant of (b), namely (b’): if  $[A, d]$  is partially true, it is more true (*i.e.*, nearer total truth), the more of its weakenings  $[B, f^\dagger(d)]$  are near to total truth. But we will make no use of this.

Given these assumptions, the intuitive argument proceeds in two steps. First, these assumptions, especially part (c) of (C), prompt a very natural suggestion for what  $\nu(A, d)$  should be. Namely:

(M):  $\nu(A, d)$  is to be the set of those morphisms  $f : B \rightarrow A$  with the property that the corresponding proposition,  $[B, f^\dagger(d)]$  is totally true. In symbols:

$$\nu(A, d) = \{ f : B \rightarrow A \mid [B, f^\dagger(d)] \text{ is totally true } \} \quad (5.1)$$

(We write ‘(M)’ for ‘morphisms’.) This suggestion makes  $\nu(A, d)$  determined by which weakenings of  $[A, d]$  are totally true, as required by (C) part (c): indeed, determined very simply.

Second, one naturally asks: what is it for a proposition, whether  $[B, f^\dagger(d)]$  or  $[A, d]$ , to be totally true? That is not yet settled. But again there is a very natural suggestion, obeying parts (a) and (b) of (C). Namely:

(T): For any proposition  $[A, d]$ , total truth is just  $\nu(A, d)$  being the set of *all* morphisms,  $f : B \rightarrow A$ , to  $A$ , *i.e.*, the principal sieve on  $A$ . In symbols:

$$[A, d] \text{ is totally true if } \nu(A, d) = \downarrow A \quad (5.2)$$

(We write ‘(T)’ for ‘total truth’.) This suggestion is natural, because when taken together with (M):

- a) it follows that part (a) of (C) holds;
- b) it follows that part (b) of (C) holds, in a very natural sense of the phrase ‘ $[A, d]$  is more true’, namely that  $\nu(A, d)$  is a larger subset of  $\downarrow A$ .

Finally, to complete the intuitive argument: it follows immediately from (M) and (T) taken together that  $\nu(A, d)$  is a sieve. For recall that, for any object  $A$  in a category  $\mathcal{C}$ , a set  $S$  of morphisms to  $A$  is a sieve if and only if the pull-back along any morphism in  $S$  is the principal sieve.

## 5.2 Assessing the Argument

We will make two comments on the argument in the last Subsection, and then describe how to make it precise using the ideas in Section 4.2.

First, we emphasise that the intuitive argument is not a genuine *deduction* of valuations being sieve-valued. It only claims that (M) and (T) (and therefore, sieve-valued valuations) are natural, given (A) to (C). One could perhaps get a genuine deduction of  $\nu(A, d)$  being a sieve, but only at the price of some strong premises. Indeed, the obvious stronger premises that one might consider do not quite imply (M) and (T); they just make them even more natural than they were in Subsection 1. Thus suppose we added as premises, both:

- (D) The truth-value  $\nu(A, d)$  is some set of morphisms to  $A$ ;
- (T) Total truth is to be just  $\nu(A, d)$  being the principal sieve on  $A$ . In symbols:  $[A, d]$  is totally true if and only if  $\nu(A, d) = \downarrow A$ .

Even these two do not *imply* (M); though they make it extremely natural to accept (M), and therefore to accept (as in the argument in Subsection 1)  $\nu(A, d)$  being a sieve.

More generally, we agree that essentially the same argument can be given different versions; and we make no claim to the version in Subsection 1 being the unique best balance between premises being plausible and the inference being rigorously deductively valid. (We admit that in philosophical argument, we tend to weigh the former more highly, as shown by our choice of version in Subsection 1.)

Second, we note that the fact that  $\nu(A, d)$  is a sieve implies the special case of our sieve-version of *FUNC*, viz. the case when  $f : B \rightarrow A \in \nu(A, d)$ . For by the construction above:

$$\nu(B, f^\dagger(d)) = \downarrow B; \quad (5.3)$$

while by the definition of a sieve, and that fact that  $f : B \rightarrow A \in \nu(A, d)$ ,

$$f^*(\nu(A, d)) = \downarrow B. \quad (5.4)$$

But nothing in the argument implies our sieve-version of *FUNC* in full generality, *i.e.*, the principle that even when  $f : B \rightarrow A \notin \nu(A, d)$

$$\nu(B, f^\dagger(d)) = f^*(\nu(A, d)). \quad (5.5)$$

On the other hand, as we saw in Section 4.1, *FUNC* can be motivated by the requirement that a valuation determines a subobject of  $\mathbf{G}$ .

Finally, we round off this Section by showing how to make the intuitive argument precise by using the notions of Section 4.2. We only need to make assumptions (A) to (C) precise: the argument then proceeds as in Section 5.1. So, first: we can make (A) precise by requiring that (i) each of the sets  $\mathcal{P}(A)$  be a poset with a 0 and a 1; and (ii) the map  $A \mapsto \mathcal{P}(A)$  define a presheaf—as explained above, this requirement is natural in view of the likely existence of factorisations of morphisms  $k : C \rightarrow A$ . From now on, we call this presheaf  $\mathbf{G}$ , as in Section 4.2 (and earlier). So  $\mathcal{P}(A) = \mathbf{G}(A)$  and  $f^\dagger = \mathbf{G}(f)$ . Note that the fact that  $\mathbf{G}$  is a presheaf now implies our requirement that  $(\text{id}_A)^\dagger = \text{id}_{\mathcal{P}(A)}$ , *i.e.*,  $\mathbf{G}(\text{id}_A) = \text{id}_{\mathbf{G}(A)}$ .

Second, to make (B) precise: whenever  $f : B \rightarrow A$ , we need to be able to ‘push-forward’ a proposition  $[B, \mathbf{G}(f)(d)]$ —or, in our other notation,  $\mathbf{G}(f)(d) \in \mathbf{G}(B)$ —to  $\mathbf{G}(A)$  for comparison of ‘logical strength’ (*i.e.*, comparison according to the partial order  $<$  in  $\mathbf{G}(A)$ ) with the proposition  $[A, d]$ , *i.e.*, with  $d \in \mathbf{G}(A)$ . And the pushed proposition is required to be weaker (*i.e.*, higher in the partial order) than  $[A, d]$ . So we require:

- There is a comparison functor in the (weak) sense of Section 4.2.1, *i.e.*, a covariant functor  $\mathbf{C}$  from  $\mathcal{C}$  to  $\text{Set}$  with the same action on objects  $A$  in  $\mathcal{C}$  as has  $\mathbf{G}$ :  $\mathbf{C}(A) := \mathbf{G}(A)$ .
- The functors  $\mathbf{C}$  and  $\mathbf{G}$  together obey generalized coarse-graining, Eq. (4.5), *i.e.*:

$$d \leq \mathbf{C}(f)[\mathbf{G}(f)(d)]. \quad (5.6)$$

The generalized retraction and monotonicity clauses in the definition in Section 4.2 of a generalized coarse-graining presheaf are not needed *a priori*, although we note that the monotonicity condition is particularly natural.

Finally, (C) can be rendered precise by requiring that the set of truth-values be a poset with a 1 (representing ‘totally true’); and that assignments of truth values,  $\nu$ , should obey the following conditions:

(a) If  $\nu(A, d) = 1$ , then for any  $f : B \rightarrow A$ ,  $\nu(B, \mathbf{G}(f)(d)) = 1$ .

(b) Suppose  $[A, d]$  and  $[A, e]$  are such that whenever  $f : B \rightarrow A$ , if  $\nu(B, \mathbf{G}(f)(d)) = 1$ , then also  $\nu(B, \mathbf{G}(f)(e)) = 1$ . Then:

$$\nu(A, d) < \nu(A, e). \quad (5.7)$$

(c)  $\nu(A, d)$  is determined by the set  $\{[B, \mathbf{G}(f)(d)] \mid f : B \rightarrow A, \text{ and } \nu(B, \mathbf{G}(f)(d)) = 1\}$ .

Given these assumptions, the argument for sieve-valued valuations, *i.e.*, for the set of truth-values at each  $A$  in  $\mathcal{C}$  being  $\Omega(A)$ , can proceed just as in Section 5.1, though with the new notation,  $\mathbf{G}$ ,  $\mathbf{C}$ , etc.

## 6 Conclusion

To conclude, let us summarize some of our main proposals (both in (I) and this paper), referring mainly to the physical cases (quantum and classical).

(1) We consider the set of physical quantities as a mathematical category, with morphisms given by coarse-graining, *i.e.*, taking functions of quantities. So in quantum theory, we consider the category  $\mathcal{O}$  of bounded self-adjoint operators on a Hilbert space  $\mathcal{H}$ , with a morphism from one such operator,  $\hat{B}$ , to another,  $\hat{A}$ , whenever  $\hat{B}$  is a function of  $\hat{A}$ . Correspondingly, for classical physics, we consider the category  $\mathcal{M}$  of real-valued measurable functions on a classical state space  $\mathcal{S}$ , with a morphism from one such function,  $\bar{B}$ , to another,  $\bar{A}$ , whenever  $\bar{B}$  is a function of  $\bar{A}$ .

(2) We assign to each proposition, ‘ $A \in \Delta$ ’, (that says the value of the quantity  $A$  lies in the Borel set  $\Delta$ ) as its value: a sieve on  $A$ —a sieve on  $A$  being a set of morphisms to  $A$ ,  $f : B \rightarrow A$  that is closed under further coarse-graining. (Here and in what follows, we use ‘ $A$ ’ to stand indifferently for a quantum or classical quantity, represented by  $\hat{A}$  or  $\bar{A}$  respectively.)

(3) The previous paper motivated this proposal, for quantum theory, by linking the Kochen-Specker theorem to the theory of presheaves. For our category  $\mathcal{O}_d$  of discrete-spectrum operators, the theorem states that if the dimension of  $\mathcal{H}$  is greater than 2, then there are no real-valued functions  $V$  on  $\mathcal{O}_d$  that have the *FUNC* property,

$$V(f(\hat{A})) = f(V(\hat{A})). \quad (6.1)$$

On the other hand, a presheaf is an assignment, to each object in a category, of a set, such that the sets assigned to objects that are related by a morphism, ‘mesh’ with each other by having a corresponding set-morphism, *i.e.*, a function, between them.

The Kochen-Specker theorem turns out to be a statement about the presheaf on  $\mathcal{O}$  that assigns to each operator, its spectrum: the meshing of this presheaf turns out to be very closely related to the meshing of values given by Eq. (6.1). Namely, the Kochen-Specker theorem says that this presheaf has no global elements; where a global element is the analogue, for presheaves, of the ordinary idea of an element of a set.

This situation suggests partial valuations on  $\mathcal{O}_d$ , *i.e.*, real-valued functions on a subset of  $\mathcal{O}_d$  that obey Eq. (6.1); and this led us to our proposed sieve-valued valuations on all of  $\mathcal{O}$ . These have a corresponding *FUNC* property (expressed in terms of pull-backs of sieves) and other natural properties (like Null proposition, and monotonicity); and yet they are defined on *all* quantities.

(4) In this paper, we have motivated these proposals in three other ways. First, we showed that they apply equally well to classical physics: in the absence of Kochen-Specker prohibitions, we considered how to define a valuation given only a macrostate (Section 3). Second, we showed how some of our main proposals carry over directly to the very general setting of any presheaf of propositions on any small category: *e.g.* the equivalence of *FUNC* to a sieve-valued valuation specifying a subobject (Section 4). Third, we showed that our sieve-valued valuations are a very natural way to satisfy some general intuitive requirements about partial truth, as applied to a presheaf of propositions defined on any (small) category (Section 5). Here we emphasised the point that for our valuations, the partial truth-value of a proposition is determined by which of its weakenings are totally true.

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