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Abstract

It is shown that, for any round-robin tournament, one can find a pairing of the teams and allocate home and away matches so that only one member of each pair plays at home in each round.

1. Introduction

A (single) round-robin tournament for n teams consists of the $\frac{1}{2}n(n-1)$ matches between distinct pairs of the teams, together with some type of schedule. We assume the most common form of schedule: the matches are placed in rounds, such that every team plays exactly once per round (except that, if the number of teams is odd, exactly one team has a bye in each round).

In many two-team sports, such as football, baseball, cricket, *etc.*, there is a "home team" and an "away team" in each match. This is often important—there may be a "home ground" advantage, or the home team may receive the greater share of the gate receipts. So in each match home and away teams must be designated.

Suppose two teams share the same home ground. Then those two teams cannot both be "home" in the same round. To handle the most demanding case we are interested in single round-robin tournaments with the following property:

> there is a way of pairing the teams such that only one member of each pair is at home in each round.

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Such tournaments also arise in a paper of Beecham and Hurley [1], as a stepping-stone in the construction of more complicated tournaments. (However, as they point out in a footnote to their paper, the existence of these tournaments is not necessary.) They give an example for 4 teams. In this note we prove that a suitable pairing and allocation of home matches exists for any single round-robin tournament on n teams, where n is even; by the simple method of adding one "dummy" team and granting a bye to the team scheduled to play the dummy, the odd cases are handled as well.

2. Graph-theoretic interpretation

We use standard graph-theoretic ideas (see, for example, [2]). A graph consists of a set of *points* together with a set of *edges*: each edge is an unordered pair of distinct points of the graph. This definition does not allow loops or multiple edges. A *complete* graph on n points has all $\frac{1}{2}n(n-1)$ possible edges. A *one-factor* in a graph is a set of edges which between them contain every point exactly once (such a thing can only exist in a graph with an even number of points).

Clearly one can represent a single round-robin tournament for n teams by a complete graph on n points: the points represent the teams, and edge $\{x, y\}$ represents the match "Team x plays Team y". Each round is represented by a one-factor, and the tournament schedule is a way of decomposing the complete graph into pairwise disjoint one-factors.

A cycle of length k is a graph with k distinct points, say $p_1, p_2, \ldots, p_k$, and edges joining them in sequence: $\{p_1, p_2\}, \{p_2, p_3\}, \ldots, \{p_{k-1}, p_k\}, \{p_k, p_1\}$. It is clear that, if one takes the union of two disjoint one-factors in the same complete graph, the result is a union of disjoint cycles. (Since a cycle is a graph, not just a set of edges, "disjoint cycles" implies that the point-sets are disjoint as well as the edge-sets.) Moreover each such cycle will be of *even* length. For suppose the one-factors were F_1 and F_2, and suppose their union contained a cycle with length k, whose points (in order) are $p_1, p_2, \ldots, p_k$. Suppose $\{p_1, p_2\}$ is in F_1. No two successive edges are in the same one-factor, so $\{p_2, p_3\}$ is in F_2, $\{p_3, p_4\}$ is in F_1, and so on in alternation. So $\{p_k, p_1\}$ is in F_2 if k is even and F_1 if k is odd. But $\{p_k, p_1\}$ and $\{p_1, p_2\}$ cannot both be in F_1. So k is even.

3. Main result

Given any single round-robin tournament for n teams (n even), select one round at random. Call it the *base round*. The pairs of teams which play together in this round are the pairs referred to in (P). No matter how home grounds are allocated in the base round one member of each pair will be at home.

Now select any other round. Let the one-factors associated with the base round and this other round be F_1 and F_2 respectively. Consider the union of F_1 and F_2. In each of the cycles comprising it, select one of the points at random; the teams corresponding to it and to every second point as you go around the cycle from it are the home teams in the round under discussion.

Exactly one home team has been selected in every edge of F_1, and exactly one in every edge of F_2. So there is exactly one home team in every match of the new round, and exactly one from each of the pairs. So (P) is satisfied.

4. An unsolved problem

If a double round-robin tournament is to be run, the usual requirement is that in the two matches where A meets B, A is the home team in one match and B in the other. If the tournament consists of two identical halves, the solution is easy: allocate pairing and home-and-away for the first half so that (P) is satisfied, and run the second half the same way but with "home" and "away" interchanged.

Not every double round-robin consists of two replicates of one single round-robin. Table 1 exhibits the three non-isomorphic double round-robin tournaments for six teams. Of these the first consists of two copies of a single round-robin, the second is the union of two different round-robins, and the third cannot be reduced in this way. An exhaustive search shows that there are no other cases.

Each tournament has been written so that, if the left-hand team in each match is the home team, then the pairing 1–2, 3–4, 5–6 satisfies (P). The results of Section 3 prove that such an allocation and pairing exist for the first case, but do not guarantee the second and third solutions.

Does there exist for any n a double round-robin tournament for n teams in which (P) cannot be satisfied by any allocation of home and away?

Table 1. Double round robins for six teams

12	13	14	15	16	21	31	41	51	61
34	52	62	42	32	43	25	26	24	23
56	46	35	63	54	65	64	53	36	45
12	13	14	51	61	21	31	41	15	16
34	52	62	24	23	43	26	25	32	42
56	46	35	36	45	65	54	63	64	53
12	21	31	13	14	41	51	15	16	61
34	35	24	62	52	26	23	42	32	25
56	64	65	45	36	53	46	63	54	43

References

[1] A. F. Beecham and A. C. Hurley, "A scheduling problem with a simple graphical solution", *J. Austral. Math. Soc. Ser. B* 21 (1980), 486–495.

[2] F. Harary, *Graph Theory* (Addison-Wesley, Reading, Mass., 1969).