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SUPPLEMENT

A supplement to A tractable approximation of non-convex chance constrained optimization with non-Gaussian uncertainties

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1. Introduction

This online publication is a supplement to the article *A tractable approximation of non-convex chance constrained optimization with non-Gaussian uncertainties* by A. Geletu, M. Klöppel, A. Hoffmann and P. Li. This supplement provides proofs of major theorems that are left out from the main manuscript. To make the reading of this document easier, theorems are restated again here along with their corresponding proofs. Basic mathematical definitions are also provided with some additional references. All other theorems, propositions, assumptions, etc., referenced here are found in the original manuscript.

2. Proofs propositions and theorems

PROPOSITION 3.3 *Suppose Assumption 3.2 holds true.* Then

(1) $\Theta(\tau, u, \cdot)$ *is a strictly increasing function w.r.t.* $s \in \mathbb{R}$*;*
(2) $\Theta(\cdot, u, s)$ *is a non-decreasing function w.r.t.* τ *for* $0 < \tau < 1$*;*
(3) *and*

$$\lim_{\tau \searrow 0^+} \Theta(\tau, u, s) = \begin{cases} 1, \text{ if } s \geq 0, \\ 0, \text{ if } s < 0, \end{cases}$$

uniformly for $u \in \mathcal{U}$ *and uniformly for* $s \in (-\infty, -\varepsilon] \cup [0, +\infty)$ *and arbitrary* $\varepsilon > 0$.

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Proof Part (1) follows through an elementary analysis. Since u is fixed, the dependence of m_1 and m_2 on u can be dropped for clarity of presentation. Thus, to verify part (2), differentiate $\Theta(\tau, u, s)$ w.r.t. τ to obtain

$$\begin{aligned}\frac{\partial\Theta(\tau,u,s)}{\partial\tau} &= \frac{m_1}{\left[1+m_2\tau e^{-s/\tau}\right]} + \frac{1+m_1\tau}{\left[1+m_2\tau e^{-s/\tau}\right]^2}\left[-m_2 e^{-s/\tau} - m_2\left(\frac{s}{\tau}\right)e^{-s/\tau}\right] \\ &= \frac{1}{\left[1+m_2\tau e^{-s/\tau}\right]^2}\left[m_1\left(1+m_2\tau e^{-s/\tau}\right) - m_2(1+m_1\tau)\left(1+\frac{s}{\tau}\right)e^{-s/\tau}\right].\end{aligned}$$

Considering only the factor on the right hand side, it follows that

$$\begin{aligned}&\left[m_1\left(1+m_2\tau e^{-s/\tau}\right) - m_2(1+m_1\tau)\left(1+\frac{s}{\tau}\right)e^{-s/\tau}\right] \\ &\qquad = m_1 - m_2\left(1+\frac{s}{\tau}\right)e^{-s/\tau} - m_1 m_2 s e^{-s/\tau} \\ &\qquad = m_1 - m_2\left(1+\frac{s}{\tau}\right)e^{-s/\tau} - m_1 m_2 \tau\left(\frac{s}{\tau}\right)e^{-s/\tau}.\end{aligned}$$

The standard inequality $(1+t)e^{-t} \leq 1$ holds true for any $t \in \mathbb{R}$. Hence, replacing t by $\frac{s}{\tau}$ in the inequality, it follows that

$$\begin{aligned}&\left[m_1\left(1+m_2\tau e^{-s/\tau}\right) - m_2(1+m_1\tau)\left(1+\frac{s}{\tau}\right)e^{-s/\tau}\right] \\ &\qquad \geq m_1 - m_2 - m_1 m_2 \tau \\ &\qquad > m_1 - m_2(1+m_1) \quad (\text{since } 0 < \tau < 1) \\ &\qquad \geq m_1 - \frac{m_1}{1+m_1}(1+m_1) = 0.\end{aligned}$$

Consequently, $\frac{\partial\Theta(\tau,u,s)}{\partial\tau} > 0$. Therefore, for any value of s (positive or negative) and u, the function $\Theta(\tau, u, s)$ is strictly increasing with respect to τ.
The simple estimation

$$|\Theta(\tau,u,s) - 1| \leq \tau C_1, \text{ for } s \geq 0, \tag{1}$$

$$\Theta(\tau,u,s) \leq \frac{1+C_1\tau}{1+C_3\tau\exp\left(\frac{\varepsilon}{\tau}\right)}, \text{ for } s \leq -\varepsilon, \tag{2}$$

yields part (3).

■

The proof of Theorem 4.2. below needs definitions and properties of convergence of sequences of sets. Hence, given a sequence $\{A_k\}_{k=1}^{+\infty}$ of (closed) sets in $\mathbb{R}^m$, basically there are two types of limits (cf. Aubin & Cellina (1984), Aubin & Frankwoska (1990), Kisielewicz (1991) for details):

$$\limsup_{k\to\infty} A_k = \left\{ x \in \mathbb{R}^n \;\middle|\; \liminf_{k\to\infty}\; dist\,(x, A_k) = 0 \right\}, \tag{3}$$

and

$$\liminf_{k\to\infty} A_k = \left\{ x \in \mathbb{R}^n \;\middle|\; \lim_{k\to\infty}\; dist\,(x, A_k) = 0 \right\}, \tag{4}$$

where $dist(x, A) = \inf_{z\in A} \|x - z\|$ and $\|\cdot\|$ is the Euclidean norm. If $\limsup_{k\to\infty} A_k = \liminf_{k\to\infty} A_k$, then it is said that the limit of the sequence $\{A_k\}_{k=1}^{+\infty}$ exists and is denoted by $\lim_{k\to\infty} A_k = \limsup_{k\to\infty} A_k = \liminf_{k\to\infty} A_k$.

THEOREM 4.2 *The compactness of $\mathcal{U}$, the monotonicity and continuity of $\psi(\cdot, u)$ with respect to τ as well as properties P1-P3 imply that, for each decreasing zero sequence $\{\tau_k\}_{k\in\mathbb{N}}$ and for a regular chance constraint, the relation*

$$\lim_{k\to\infty} M(\tau_k) = \mathcal{K}$$

holds true.

Proof Take an arbitrary decreasing zero sequence $\{\tau_k\}_{k\in\mathbb{N}}$. Using property P3, due to the compactness of $\mathcal{U}$ and continuity of $\psi(\tau, u)$, $\{M(\tau_k)\}_{k\in\mathbb{N}}$ is a monotonic sequence of compact sets (i.e. $M(\tau_k) \subset M(\tau_{k+1})$). It follows that

$$\liminf_{k\to\infty} M(\tau_k) = \limsup_{k\to\infty} M(\tau_k). \tag{5}$$

Since $\limsup_{k\to\infty} M(\tau_k) \subset \mathcal{K}$ is obvious, it remains to show that $\limsup_{k\to\infty} M(\tau_k) \supset \mathcal{K}$. Suppose $u \in \mathcal{K}$, then there are two cases:

Case 1: $E[h(u, \xi)] < 1 - \alpha$.

Since $E[h(u, \xi)] = \inf_{\tau\in(0,1)} \psi(\tau, u)$ and $\psi(\cdot, u)$ is non-decreasing and continuous, there is some k_0 such that $\psi(\tau_k, u) < 1 - \alpha$, for all $k \geq k_0$. Hence, $u \in M(\tau_k)$ for all $k \geq k_0$ which yields $u \in \limsup\limits_{k\to\infty} M(\tau_k)$.

Case 2: $E[h(u, \xi)] = 1 - \alpha$.

Using the regularity Assumption 4.1, there is a sequence $\{u_k\}_{k\in\mathbb{N}}$ in $\mathcal{U}$ such that $\lim_{k\to\infty} u_k = u$ and $Pr\,\{g(u_k, \xi) \leq 0\} > \alpha$. This implies that $E[h(u_k, \xi)] < 1-\alpha$. Hence, by property P2, for each u_k there is a sufficiently small τ_k such that $E[h(u_k, \xi)] < \psi(\tau_k, u_k) < 1 - \alpha$. Hence, $u_k \in M(\tau_k)$ for all k. This implies that $u \in \limsup_{k\to\infty} M(\tau_k)$.

Observe that, by using property P3 in conjunction with P2, the parameters τ_k can be chosen so that $\{\tau_k\}_{k\in\mathbb{N}}$ is a decreasing zero sequence. Therefore, $\lim_{k\to\infty} M(\tau_k) = \mathcal{K}$. ■

COROLLARY 4.3 *Let $\{\tau_k\}_{k\in\mathbb{N}} \subset (0, 1)$ be any sequence that decreases to zero. Then Assumption 4.1 implies that*

$$\lim_{k\to\infty} H\,(M(\tau_k), \mathcal{K}) = 0.$$

Proof Since $\{M(\tau_k)\}_{k\in\mathbb{N}}$ is a sequence of compact sets, $M(\tau_k) \subset \mathcal{U}$, $\mathcal{U}$ is a compact set, $\lim_{k\to\infty} M(\tau_k) = \mathcal{K}$ and $\mathcal{K}$ is a compact set, it follows (see Remark 1.2 of Kisielewicz (1991), also Theorem 5.2.4 of Aubin (1999)) that

$$\lim_{k\to\infty} H\left(M(\tau_k), \mathcal{K}\right) = 0.$$

■

THEOREM 4.4 *(Approximability of optimal solutions)*

(1) *Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence such that u_k is a local optimal solution of the NLP$_k$, $k = 1, 2, \ldots$ Then there is a subsequence $\{u_{k_l}\}_{l\in\mathbb{N}}$ of $\{u_k\}_{k\in\mathbb{N}}$ such that $u_{k_l} \to u^*$, there is an open ball $B(u^*)$ around u^*, $u^* \in \mathcal{K} \cap B(u^*)$ and*

$$E\left[f(u^*, \xi)\right] = \min_{u\in\mathcal{K}\cap B(u^*)} E\left[f(u, \xi)\right]; \tag{6}$$

i.e., u^* is a local optimal solution of (CCOPT).

(2) *Conversely, if u_0 is a strict local minimizer of CCOPT, then there is a sequence of local minimizers u_k of NLP_k which converges to u_0.*

Proof

(1) Since $\mathcal{U}$ is a compact set and $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{U}$, there is a convergent subsequence $\{u_{k_l}\}_{l\in\mathbb{N}}$ such that $u_{k_l} \to u^*$. Since $u_{k_l} \in M(\tau_{k_l})$ and (according to Theorem 4.2) $\limsup_{k\to\infty} M(\tau_k) = \mathcal{K}$, it follows that $u^* \in \mathcal{K}$. For a sufficiently large l, there is a ball $B(u^*)$ such that $u_{k_l} \in \mathcal{K} \cap B(u^*)$ are minimizers of NLP$_{k_l}$.

Assume now that $u^* \in \mathcal{K} \cap B(u^*)$ is not a local optimum point of CCOPT. This implies that there is $\widetilde{u} \in \mathcal{K} \cap B(u^*)$ such that

$$E\left[f(u^*, \xi)\right] > E\left[f(\widetilde{u}, \xi)\right].$$

Using Corollary 4.3, there is a sequence $\{z_k\}_{k\in\mathbb{N}}$ such that $z_k \in M(\tau_k) \cap B(u^*)$ and $z_k \to \widetilde{u}$. Thus, for a subsequence $\{z_{k_l}\}$ it follows that

$$E\left[f(z_{k_l}, \xi)\right] \geq E\left[f(u_{k_l}, \xi)\right] \quad \text{(by the local optimality of } u_{k_l}\text{)}.$$

This implies that

$$E\left[f(\widetilde{u}, \xi)\right] = \lim_{k_l\to\infty} E\left[f(z_{k_l}, \xi)\right] \geq \lim_{k_l\to\infty} E\left[f(u_{k_l}, \xi)\right] = E\left[f(u^*, \xi)\right] > E\left[f(\widetilde{u}, \xi)\right].$$

But this is a contradiction. Therefore, u^* should be a local optimal solution of CCOPT.

(2) Suppose u_0 be a strict local optimal solution of CCOPT. Let $F(u) > F(u_0)$ for all $u \in \mathrm{cl}B(u_0)$ with $p(u) \geq \alpha$ for a bounded ball $B(u_0)$ around u_0. Let further that $F(u) \geq F(u_k)$ for all $u \in \mathrm{cl}B(u_0)$ with $1 - \psi(\tau_k, u) \geq \alpha$. By the compactness of the closed ball such a u_k exists. Without loss of generality, $\lim_{k\to\infty} u_k =: u^*$. Since $H(M(\tau_k) \cap \mathrm{cl}B(u_0), \mathcal{K} \cap \mathrm{cl}B(u_0)) \to 0$, the continuity of F implies that

$$F(u) \geq F(u^*) \text{ for all } u \in \mathrm{cl}B(u_0) \text{ with } p(u) \geq \alpha.$$

Hence $F(u^*) = F(u_0)$. If $u^* \neq u_0$, then this leads to a contradiction to the assumption that u_0 is a strict local minimizer. Consequently, u_k converge to u_0 and there is a k_0 such for all $k > k_0$, u_k belongs to the interior of $B(u_0)$. These u_k are all local solutions of NLP$_k$.

■

THEOREM 4.9 *Let W be an open bounded subset of $\mathcal{U}$ and $\tilde{\Omega}$ be an open set with $\Omega \subset \tilde{\Omega}$. Define $B(u,\varepsilon) := \{\xi \in \tilde{\Omega} \mid |g(u,\xi)| < \varepsilon\}$. Then for each $u \in W$ and each $\varepsilon > 0$ it follows that*

i)
$$\lim_{k\to\infty} \sup_{u\in W} \left| \int_{\tilde{\Omega}\setminus B(u,\varepsilon)} \left[\frac{\partial\Theta(\tau_k,u,s)}{\partial m_1} \nabla_u m_1(u) + \frac{\partial\Theta(\tau_k,u,s)}{\partial m_2} \nabla_u m_2(u) \Big|_{s=g(u,\xi)} + \frac{\partial\Theta(\tau_k,u,s)}{\partial s}\Big|_{s=g(u,\xi)} \nabla_u g(u,\xi) \right] \phi(\xi) d\xi \right| = 0;$$

ii)
$$\lim_{k\to\infty} \sup_{u\in W} \left| \int_{B(u,\varepsilon)} \left[\frac{\partial\Theta(\tau_k,u,s)}{\partial m_1} \nabla_u m_1(u) + \frac{\partial\Theta(\tau_k,u,s)}{\partial m_2} \nabla_u m_2(u) \right|_{s=g(u,\xi)} \phi(\xi) d\xi \right| = 0;$$

iii)
$$\lim_{k\to\infty} \nabla\psi(\tau_k,u) = \lim_{k\to\infty} \int_{B(u,\varepsilon)} \left[\frac{\partial\Theta(\tau_k,u,s)}{\partial s}|_{s=g(u,\xi)} \nabla_u g(u,\xi) \right] \phi(\xi) d\xi$$

uniformly on W.

Proof A detailed proof of this theorem takes several pages. Thus, this section provides only a sketch of the idea of the proof.

- Referring to Proposition 3.3 it should be noted that the function $\lim_{\tau\searrow 0^+} \Theta(\tau,u,s) = \begin{cases} 1, \text{ if } s \geq \varepsilon, \\ 0, \text{ if } s < -\varepsilon, \end{cases}$ for an $\varepsilon > 0$. Consequently, for $|s| > \varepsilon$ the limit of the derivative $\lim_{\tau\searrow 0^+} \nabla_u \Theta(\tau,u,s) = 0$. Thus, the main contribution to the derivative comes when $-\varepsilon < s < \varepsilon$. Thus, replacing $s = g(u,\xi)$ gives an idea on how the proof should proceed.
- Let $u_0 \in W$ and let

$$\Gamma_0(u_0) = \{\xi \in \widetilde{\Omega} \mid g(u_0,\xi) = 0\}.$$

Assume that $\Gamma_0(u_0) \neq \emptyset$ (for $\Gamma_0(u_0) = \emptyset$ the discussion below follows trivially). Hence, there is $\tilde{\xi} \in \widetilde{\Omega}$ such that $(u_0,\tilde{\xi}) \in \Gamma_0(u_0)$. Furthermore, define the set

$$\Gamma_g = \{(u,\xi,s) \mid s = 0, g(u,\xi) = 0, (u,\xi) \in W \times \widetilde{\Omega}\}.$$

Since $\Gamma_0(u_0) \neq \emptyset$, it follows that $\Gamma_g \neq \emptyset$.
- The set Γ_g is a compact set. For each $(\tilde{u},\tilde{\xi},\tilde{s}) \in \Gamma_g$ it follows that $g(\tilde{u},\tilde{\xi}) = 0$ and $\tilde{s} = 0$. According to Assumption 4.5, $\nabla_\xi g(\tilde{u},\tilde{\xi}) \neq 0$. This implies there is a component ξ_β of $\xi = (\xi_1,\xi_2,\ldots,\xi_p)$ such that $\frac{\partial}{\partial\xi_\beta} g(\tilde{u},\tilde{\xi}) \neq 0$. Without loss of generality, let $\xi_\beta = \xi_p$ and define $\eta = (\xi_1,\xi_2,\ldots,\xi_{p-1})$. Hence,

(a) according to the Implicit Function and Open Mapping Theorems, there are open neighborhoods $\widetilde{U}, \widetilde{V}$ and $(-\varepsilon, \varepsilon)$ of $\widetilde{u}, \tilde{\eta}$ and $\tilde{s} = 0$, respectively; an interval (a, b) with $a < \tilde{\xi}_\beta < b$ and a unique surjective mapping q

$$q : \tilde{U} \times \tilde{V} \times (-\varepsilon, \varepsilon) \to (a, b)$$

such that $q(\tilde{u}, \tilde{\eta}, \tilde{s}) = q(\tilde{u}, \tilde{\eta}, 0) = \tilde{\xi}_\beta$ and $g(u, (\eta, \underbrace{q(u, \xi, s)}_{=\xi_\beta})) = s$, for each $(u, \eta, s) \in \tilde{U} \times \tilde{V} \times (-s_0, s_0)$.

(b) The family of sets $\left\{\left(\tilde{U} \times \left(\tilde{V} \times (a, b)\right) \times (-\varepsilon, \varepsilon)\right) \mid (u, \xi, 0) \in \Gamma_g\right\}$ defines an open covering of the compact set Γ_g. Hence, there is a finite collection $\left\{\left(\tilde{U}_i \times \left(\tilde{V}_i \times (a_i, b_i)\right) \times (-\varepsilon_i, \varepsilon_i))\right) \mid i = 1, \ldots, z\right\}$ that covers Γ_g with a corresponding set of surjective mappings $\{q_i \mid i = 1, \ldots, z\}$. Define now that

$$\varepsilon_0 := \min\{\bar{\varepsilon}, \varepsilon_1, \ldots, \varepsilon_z\}.$$

Let $s = g(u, \xi)$. Then, for each $u \in \bigcup_{i=1}^{z} \tilde{U}_i$

$$B(u, \varepsilon_0) \subset \bigcup_{i=1}^{z} \left(\tilde{V}_i \times (a_i, b_i)\right)$$

with $B(u, \varepsilon_0) = \{\xi \mid -\varepsilon_0 < g(u, \xi) < \varepsilon_0\}$. Figure 1 indicates that the set $\Gamma_0(u_0)$ is contained in the open set $B(u, \varepsilon_0)$.

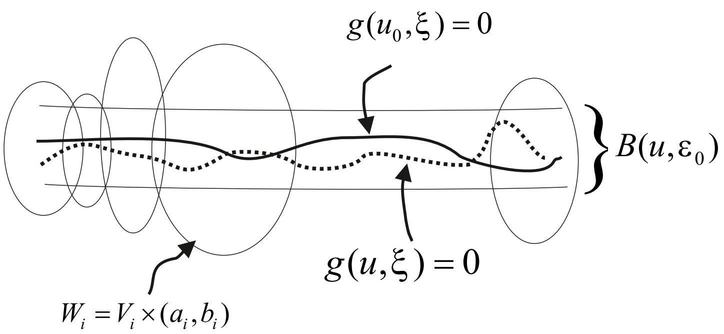

Figure 1. An ε-tube around $\Gamma_0(u_0)$

(c) Let $M := \cup_{i=1}^{z} \left(\tilde{U}_i \times \left(\tilde{V}_i \times (a_i, b_i)\right)\right)$ and the family $\varphi_i \in C^\infty(M, \mathbb{R}), i = 1, \ldots, z$, be a partition of unity of the system $\left\{\tilde{U}_i \times \left(\tilde{V}_i \times (a_i, b_i)\right) \mid i = 1, \ldots, z\right\}$; i.e., $\sum_{i=1}^{z} \varphi_i(u, \xi) = 1$ on M, $0 \leq \varphi_i(u, \xi) \leq 1, i = 1, \ldots, z$, and compact support $supp\ \varphi_i \subset \tilde{U}_i \times \left(\tilde{V}_i \times (a_i, b_i)\right), i = 1, \ldots, z$.

- Hence, it follows that

$$\int_{B(u,\varepsilon_0)} \nabla_u \Theta(\tau_k, u, s)|_{s=g(u,\xi)}\, \phi(\xi) d\xi$$
$$= \sum_{i=1}^{z} \int_{V_i \times (a_i,b_i), |g(u,\xi)|<\varepsilon_0} \nabla_u \Theta(\tau_k, u, s)]_{s=g(u,\xi)}\, \phi(\xi)\varphi_i(u,\xi) d\xi.$$

Furthermore, use the coordinate transformation

$$\xi_j = \begin{cases} \eta_j & \text{if } j \neq \beta \\ q\,(u,\eta,s) & \text{if } j = \beta \end{cases}, \quad j = 1, 2, ..., p,$$

to write

$$\sum_{i=1}^{z} \int_{V_i \times (a_i,b_i), |g(u,\xi)|<\varepsilon_0} \nabla_u \Theta(\tau_k, u, s)|_{s=g(u,\xi)}\, \phi(\xi)\varphi_i(u,\xi) d\xi$$
$$= \sum_{i=1}^{z} \int_{V_i \times (a_i,b_i), |g(u,\xi)|<\varepsilon_0} Q(\tau_k, u, \xi, g(u,\xi))\mu_i(u,\xi) d\xi,$$

where

$$Q(\tau_k, u, \xi, g(u,\xi)) := \left[\frac{\partial \Theta(\tau_k, u, s)}{\partial m_1} \nabla_u m_1(u) + \frac{\partial \Theta(\tau_k, u, s)}{\partial m_2} \nabla_u m_2(u) \right.$$
$$\left. + \frac{\partial \Theta(\tau_k, u, s)}{\partial s} \nabla_u g(u,\xi) \right]_{s=g(u,\xi)}$$

and $\mu_i(u,\xi) := \phi(\xi)\varphi_i(u,\xi)$ so that

$$\sum_{i=1}^{z} \int_{V_i \times (a_i,b_i), |g(u,\xi)|<\varepsilon_0} Q(\tau_k, u, \xi, g(u,\xi))\phi(\xi)\varphi_i(u,\xi) d\xi$$
$$= \sum_{i=1}^{z} \int_{V_i} \int_{-\varepsilon_i}^{\varepsilon_i} Q(\tau_k, u, (\eta, q_i(u,\eta,s)))\mu_i(u,\eta,q_i(u,\eta,s)) ds d\eta$$
$$- \sum_{i=1}^{z} \int_{V_i} \int_{\varepsilon_0}^{\varepsilon_i} Q(\tau_k, u, (\eta, q_i(u,\eta,s)))\mu_i(u,\eta,q_i(u,\eta,s)) ds d\eta$$
$$- \sum_{i=1}^{z} \int_{V_i} \int_{-\varepsilon_i}^{-\varepsilon_0} Q(\tau_k, u, (\eta, q_i(u,\eta,s)))\mu_i(u,\eta,q_i(u,\eta,s)) ds d\eta.$$

The rest of the proof uses a sequence of technical Lemmata for elementary properties (which hold uniformly with respect $u \in W$) of the expressions $\frac{\partial\Theta(\tau,u,s)}{\partial m_1}$, $\frac{\partial\Theta(\tau,u,s)}{\partial m_2}$, $\frac{\partial\Theta(\tau,u,s)}{\partial s}$, their integrals w.r.t. s and $\nabla_u g(u,\xi)$ corresponding to $|s| \geq \varepsilon_0$, $|s| < \varepsilon_0$ and $\tau_k \searrow 0^+$, respectively, as well as the properties of the coordinate transformation. Based on the

Mean-Value Theorem, the inner integration can be removed and thus the delta distributional term $\frac{\partial\Theta(\tau,u,s)}{\partial s}$ remains. Hereby the continuity of the density $\phi(\xi)$ is required. ■

THEOREM 4.13 *Let* $\{\tau_k\}_{k\in\mathbb{N}} \subset (0,1)$ *be any sequence with* $\tau_k \searrow 0^+$ *and let Assumptions 4.11 and 4.12 be satisfied. Then the following hold true*

(1) *If Assumption 4.11 (MFCQ) holds, then*
  (a) $\limsup_{k\to\infty} \Lambda_k \subset \Pi$ *and*
  (b) $\limsup_{k\to\infty} \Lambda_k^0 \subset \Pi_0$.
(2) *If Assumption 4.12 (LICQ) holds, then* $\Pi_1 \subset \limsup_{k\to\infty} \Lambda_k^0$.

Proof

(1) Take any sequence $\{\tau_k\}_{k\in\mathbb{N}} \subset (0,1)$ such that $\tau_k \searrow 0^+$. Let $(u_k, \eta_k) \in \Lambda_k$ such that $(u_k, \eta_k) \longrightarrow (u_0, \eta_0)$. For each $j \in \{1, \dots, q\}$ using the estimation

$$\|\nabla\psi_{k,j}(u_k) + \nabla p(u_0)\| \le \|\nabla\psi_{k,j}(u_k) + \nabla p_j(u_k)\| + \| - \nabla p_j(u_k) + \nabla p_j(u_0)\|$$

and applying Proposition 4.9 and the continuity of $\nabla p_j(\cdot)$ it follows that

$$\lim_{k\to\infty} \|\nabla\psi_{k,j}(u_k) + \nabla p(u_k)\| \le \lim_{k\to\infty} \sup_{u\in\mathcal{U}} \|\nabla\psi_{k,j}(u) + \nabla p_j(u)\| = 0$$

and $\lim_{k\to\infty} \| - \nabla p(u_k) + \nabla p(u_0)\| = 0$. Consequently,

$$\sum_{j=1}^{q} \eta_{k,j} \nabla\psi_{k,j}(u_k) \underset{k\to\infty}{\longrightarrow} -\sum_{j=1}^{q} \eta_{0,j} \nabla p_j\left(u_k\right),$$

where $\eta_{j,k}$ and $\eta_{j,0}$ are the j-th components of η_k and η_0, respectively. Furthermore, the continuous differentiability of $F(u)$ yields that

$$\nabla F(u_k) + \sum_{j=1}^{q} \eta_{k,j} \nabla\psi_{k,j}(u_k) = 0 \quad \underset{k\to\infty}{\longrightarrow} \quad \nabla F(u_0) - \sum_{j=1}^{q} \eta_{0,j} \nabla p_j\left(u_0\right) = 0.$$

The complementarity and non-negativity conditions in Π are also consequences of the continuity and non-negativity properties of $p_j(u)$ and $\psi_{k,j}(u)$. Hence, it follows that $(x_0, \eta_0) \in \Pi$ and

$$\limsup_{k\to\infty} \Lambda_k \subset \Pi.$$

The proof for $\limsup_{k\to\infty} \Lambda_k^0 \subset \Pi_0$ is the same as in part (1a).

(2) It remains to prove that $\Pi_1 \subset \limsup_{k\to\infty} \Lambda_k^0$. Suppose now that $(u_0, \eta_0) \in \Pi_1$. Since u_0 is a strict local optimal solution of CCOPT, Theorem 4.4(2) implies that there is sequence $\{u_k\}_{k\in\mathbb{N}}$ such that u_k is a local optimal solution of NLP_k and $\lim_{k\to\infty} u_k = u_0$.

For each $k \in \{1, 2, 3, \dots\}$, the local optimality of u_k to NLP_k yields that there is $\eta_k \in \mathbb{R}_+^q$ such that $(u_k, \eta_k) \in \Lambda_k^0$.

Claim: There is a subsequence $\{\eta_{k_l}\}$ of $\{\eta_k\}$ such that

$$\eta_{k_l} \longrightarrow \eta_0. \tag{7}$$

Now, from the local optimality of u_0 and u_k, it follows that

$$\left\| \nabla F(u_k) + \sum_{j=1}^{q} \eta_{k,j} \nabla \psi_{k,j}(u_k) - \nabla F(u_0) + \sum_{j=1}^{q} \eta_{0,j} \nabla p_j(u_0) \right\| \le \varepsilon, k = 1, 2, \ldots$$

for an arbitrary $\varepsilon > 0$. This implies,

$$\left\| \nabla F(u_k) + \sum_{j=1}^{q} (\eta_{k,j} - \eta_{0,j}) \nabla \psi_{k,j}(u_k) - \nabla F(u_0) \right.$$
$$\left. + \sum_{j=1}^{q} \eta_{0,j} \left(\nabla p_j(u_0) + \nabla \psi_j(\tau_k, u_k) \right) \right\| \le \varepsilon,$$
$$k = 1, 2, \ldots$$

This yields that

$$\left\| \sum_{j=1}^{q} (\eta_{k,j} - \eta_{0,j}) \nabla \psi_{k,j}(u_k) \right\| \le \varepsilon + \|\nabla F(u_k) - \nabla F(u_0)\|$$
$$+ \left\| \sum_{j=1}^{q} \eta_{0,j} \left(\nabla p_j(u_0) + \nabla \psi_{k,j}(u_k) \right) \right\|,$$
$$k = 1, 2, \ldots$$

Consequently,

$$\|\eta_k - \eta_0\| \left\| \sum_{j=1}^{q} \frac{(\eta_{k,j} - \eta_{0,j})}{\|\eta_k - \eta_0\| + 1} \nabla \psi_{k,j}(u_k) \right\| \le$$
$$\varepsilon + \|\nabla F(u_k) - \nabla F(u_0)\| + \left\| \sum_{j=1}^{q} \eta_{0,j} \left(\nabla p_j(u_0) + \nabla \psi_{k,j}(u_k) \right) \right\|$$
$$k = 1, 2, \ldots.$$

Let $\delta_{k.j} := \frac{(\eta_{k,j} - \eta_{0,j})}{\|\eta_k - \eta_0\| + 1}, j = 1, 2, \ldots, q$. Then the sequence $\{\delta_{k,j}\}$, for each $j \in \{1, \ldots, q\}$, is bounded. Hence, there is a subsequence $\{\delta_{k_l,j}\}$ which is convergent and $\delta_{k_l,j} \longrightarrow \delta_{0,j}$ for some $\delta_{0,j} \in \mathbb{R}$. Since, $\{\psi_{k,j}(u_k)\}$ is a convergent sequence, the sequence $\{\delta_{k_l,j} \psi_{k_l,j}(u_k)\}$ is convergent and $\delta_{k_l,j} \psi_{k_l,j}(u_k) \longrightarrow \delta_{0,j} \nabla p_j(u_0)$. As a result it follows that

$$\left\| \sum_{j=1}^{q} \frac{(\eta_{k_l,j} - \eta_{0,j})}{\|\eta_{k_l} - \eta_0\| + 1} \nabla \psi_{k_l,j}(u_k) \right\| \longrightarrow \left\| \sum_{j=1}^{q} \delta_{0,j} \nabla p_j(u_0) \right\|.$$

To show that $\left\| \sum_{j=1}^{q} \delta_{0,j} \nabla p_j(u_0) \right\| = 0$ only if $\delta_{0,j} = 0$ for each $j \in \{1, \ldots, q\}$. Assume that there is $j' \in \{1, \ldots, q\}$ such that $\delta_{0,j'} \neq 0$ and $\left\| \sum_{j=1}^{q} \delta_{0,j} \nabla p_j(u^*) \right\| = 0$. This implies $\sum_{j=1}^{q} \delta_{0,j} \nabla p_j(u^*) = 0$ violating the Assumption 4.12 (LICQ). Consequently,

(i) either $\delta_{0,j} = 0$ for each $j \in \{1, \ldots, q\}$;

(ii) or $\left\| \sum_{j=1}^{q} \delta_{0,j} \nabla p_j(u^*) \right\| \neq 0$.

The latter case yields

$$\|\eta_{k_l} - \eta_0\| \leq \frac{\varepsilon + \|\nabla F(u_{k_l}) - \nabla F(u_0)\| + \left\| \sum_{j=1}^{q} \eta_{0,j} \left(\nabla p_j(u_0) + \nabla \psi_{k_l,j}(u_{k_l}) \right) \right\|}{\left\| \sum_{j=1}^{q} \delta_{k,l} \nabla \psi_{k_l,j}(u_{k_l}) \right\|}.$$

Consequently, both cases (i) and (ii) imply that $\|\eta_{k_l} - \eta_0\| \longrightarrow 0$ which verifies the claim and concludes the proof.

■

3. Additional References

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