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LETTER TO THE EDITOR

A transfer-matrix approach to random resistor networks

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Abstract. We introduce a transfer-matrix formulation to compute the conductance of random resistor networks. We apply this method to strips of width up to 40, and use finite size scaling arguments to obtain an estimate for the conductivity critical exponent in two dimensions, $t = 1.28 \pm 0.03$.

1. Introduction

The calculation of the conductivity for a random mixture of insulating and conducting materials is a central problem in the theory of disordered systems, and a number of different approaches to the problem have been proposed (see Kirkpatrick (1979) for a review). In particular, since the work of Stinchcombe and Watson (1976), the singular behaviour of the conductivity near a percolation threshold has repeatedly been studied using renormalisation-group ideas.

The results obtained so far are not so convincing as for the purely geometric connectivity properties. In two dimensions, for instance, very accurate values for the geometric critical exponents have been calculated, in agreement with presumably exact conjectures (Stauffer 1981). On the other hand, one finds in the literature values for the conductivity critical exponent t ranging from 1.0 to 1.43. A discussion of the situation has recently been given by Gefen *et al* (1981).

We propose here a new method of computing the conductance of a resistor network. It is very similar to the transfer-matrix method of statistical mechanics, but involves nonlinear matrix recursion relations. This approach has several features that should make it of general interest in the study of disordered networks:

It gives the conductance of a given network *exactly*, in contrast with commonly used relaxation methods.

It requires much less memory storage than systematic node elimination (Fogelholm 1980), so it is useful for large arrays.

It does not rely on specific properties of the system considered (such as missing resistors), but these may be used to speed up the computations.

As an example, we compute the conductance per unit length of networks consisting of very long strips with resistors placed at random on a square lattice. The results obtained for varying strip widths (up to $N = 40$) are analysed in terms of a finite-size scaling hypothesis. They give an estimate for the two-dimensional conductivity critical exponent $t = 1.28 \pm 0.03$. This value is compared with various recent predictions and other numerical results.

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2. Transfer-matrix formulation

Let us describe first the matrix method that we have developed to calculate the conductivity of strips of random resistors. We have considered strips of width N for which all the sites of the first horizontal line and all the sites of the $(N+1)$th horizontal line are respectively connected by links of zero resistance (see figure 1(a)). The quantity that we could calculate is the conductivity Σ_N between the 1st and the $(N+1)$th horizontal line per unit horizontal length.

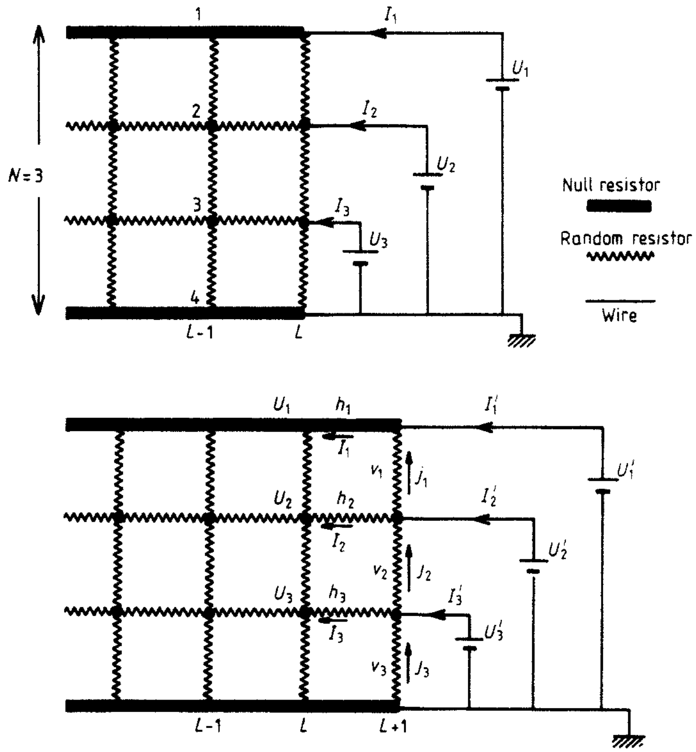

Figure 1. Recursive construction of a strip by adding the horizontal resistors h_i and the vertical resistors v_i.

As usual with transfer matrices, we need to introduce a matrix A_L which characterises the effect of the semi-infinite strip between $-\infty$ and column L. We fix the voltages, U_i, at sites i of column L by connecting these sites to external voltage sources by wires (figure 1). The $(N+1)$th site is always at voltage $U_{N+1}=0$. There will be a current I_i in each wire (see figure 1(a)). The matrix A_L gives by definition the currents I_i as functions of the U_i

$$\begin{pmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{pmatrix} = A_L \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_N \end{pmatrix}. \tag{1}$$

We now have to see how the matrix A_L is transformed into a new matrix A_{L+1} when one adds the horizontal resistors h_i and the vertical resistors v_i (see figure 1(*b*)). First, assume that we add only the resistors h_i. We now fix the voltages U_i' at sites i in column $L+1$ by external sources. Then we have

$$U_i = U_i' - h_i I_i. \tag{2}$$

Therefore the currents I_i and the voltages U_i' are related by a matrix B_{L+1}:

$$(I) = B_{L+1}(U'). \tag{3}$$

The equation (2) can be written in a matrix form

$$(U) = (U') - H(I), \tag{4}$$

where the matrix H is diagonal

$$H_{ij} = h_i \delta_{ij}, \tag{5}$$

and the resistor $h_1 = 0$ because all the sites of the first line are connected.

Then one sees that equations (1) and (4) give

$$(U) = (1 + HA_L)^{-1}(U'). \tag{6}$$

Therefore the matrix B_{L+1} is given by:

$$B_{L+1} = A_L(1 + HA_L)^{-1}. \tag{7}$$

Let us now add the vertical resistors v_i. If we impose voltages U_i' at the sites i of column $L+1$, there is a current j_i in the vertical resistor v_i

$$j_i = [U_{i+1}' - U_i']/v_i. \tag{8}$$

Therefore the current I_i' in the wire connected to the site i of column $L+1$ (to impose the voltage U_i') is given by

$$\begin{aligned} I_i' &= I_i + j_{i-1} - j_i \\ &= I_i + [1/v_i + 1/v_{i-1}]U_i' - [1/v_i]U_{i+1}' - [1/v_{i-1}]U_{i-1}'. \end{aligned} \tag{9}$$

The formula remains valid for $i = 1$ when one takes $1/v_0 = 0$. The equation (9) has again a matrix form

$$(I') = (I) + V(U'), \tag{10}$$

where V is a tri-diagonal matrix which represents the effect of the vertical bonds at column $L+1$:

$$V_{ij} = [1/v_i + 1/v_{i-1}]\delta_{ij} - [1/v_i]\delta_{j,i+1} - [1/v_{i-1}]\delta_{j,i-1}. \tag{11}$$

It is then clear that the matrix A_{L+1} is given by a recursion relation:

$$A_{L+1} = V + A_L(1 + HA_L)^{-1}. \tag{12}$$

The structure of the matrices H and V is such that the matrix A_L remains always symmetric. So A_L contains $N(N+1)/2$ numbers of interest. It is not surprising that we need only $N(N+1)/2$ numbers to characterise the semi-infinite strip: the semi-infinite strip is equivalent to $N(N+1)/2$ resistors, one for each pair of sites at column L. One can easily show that the conductivity Σ_N per unit length of a strip of width

N is given by:

$$\Sigma_N = \lim_{L\to\infty} A_L(1, 1)/L = \lim_{L\to\infty} (A_L^{-1}(1, 1))^{-1}/L. \tag{13}$$

Because the matrices H and V are random and change at every new column L, we did not find any analytic method which allows us to obtain the limit of $A_L(1, 1)/L$ for large L. So we simply performed the iterations of (12) numerically for long enough strips in order to approach the limit (13). We have generated randomly, at each column L, a new matrix H and a new matrix V and we have calculated the matrix A_{L+1} using (12). For long lengths, this method would require a lot of matrix inversions. To avoid this difficulty, it is preferable to add the resistors one by one: in this way, the inversion of the matrix becomes simple enough not to require a particular program to invert matrices.

In the present paper, we have chosen to study the case where the resistors are either cut with probability $1-p$ or present with probability p:

$$\begin{aligned} v_i \text{ or } h_i &= 1 \qquad \text{with probability } p \\ &= \infty \qquad \text{with probability } 1-p. \end{aligned} \tag{14}$$

The fact that some resistors are infinite can be used to speed up the calculations. However, the method described above can be used for any probability distribution of resistors.

3. Results for strips and finite-size scaling analysis

The transfer-matrix formulation presented above is particularly well suited to the case of random strips. It is the finite length L of the strip which can be studied in a given computer time that limits the accuracy of our calculations. The conductance per unit length converges towards its limit value Σ_N for an infinite strip as $L^{-1/2}$. We have found numerically that for a width $N=20$, at the percolation threshold, a length $L=40\,000$ gives a relative accuracy of 4 percent on Σ_N. The computing time in that case was about 1 minute or a CDC−7600.

We have carried out calculations of the conductance for strips of varying widths, at different concentrations p of active resistors. The results for concentrations in the vicinity of the percolation threshold ($p=p_c=\frac{1}{2}$) are displayed in figure 2. The data correspond to strips of length $L>5\times10^4$. Longer runs ($L>10^5$) were performed for $p=\frac{1}{2}$, up to 3.6×10^5 for $N=20$ and 2.4×10^5 for $N=30$, except for the largest widths: $L=6\times10^4$ for $N=32$, and 3×10^4 for $N=35$, 37 and 40.

One can see on the figure the change of behaviour of the conductance as a function of strip width, as one crosses the percolation threshold. Below p_c, Σ_N decreases exponentially fast with increasing N, while above p_c it decreases more slowly. One expects for large N a behaviour of the form

$$\Sigma_N \sim (A/N)(p-p_c)^t \qquad (p>p_c), \tag{15}$$

but it is apparent from figure 2 that for $p=0.52$, values of N much larger than 20 are necessary to observe the asymptotic regime where equation (15) is valid.

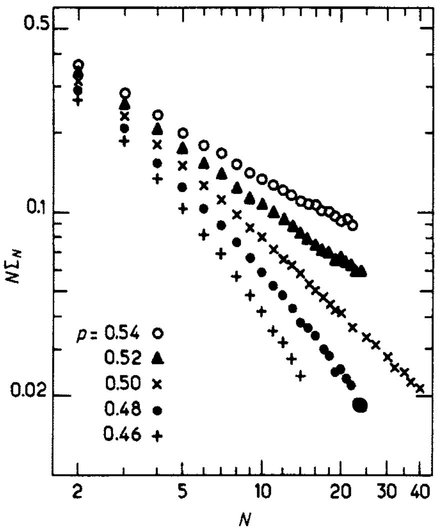

Figure 2. Log–log plot of the conductance per unit of a random resistor strip versus strip width N, in the vicinity of the percolation threshold ($p_c = \frac{1}{2}$ for the square lattice used here).

Precisely at p_c, finite-size scaling arguments (Lobb and Frank 1979 a, b, Mitescu *et al* 1982) suggest that

$$N\Sigma_N \sim CN^{-t/\nu} \qquad (p = p_c), \tag{16}$$

for large N, where ν is the correlation length exponent ($\nu = \frac{4}{3}$ is the accepted value for two-dimensional percolation). Equation (16) implies a linear variation of $\log(N\Sigma_N)$ versus $\log N$, which is indeed well verified on figure 2, for $7 < N < 40$. A least-squares fit of the data in this range yields an estimate for the slope:

$$t/\nu = 0.95 \pm 0.01.$$

The uncertainty comes from the fluctuations between the slopes measured in different sub-ranges of N. We have detected no systematic variation of the slope with N within the precision of the data.

4. Anisotropic media

A class of related problems that may easily be studied by the present approach concerns conduction in anisotropic media. We have considered systems where the probabilities of active resistors are different in the two lattice directions (Blanc *et al* 1980, Nakanishi *et al* 1981). For these systems, the percolation line is known exactly and corresponds to

$$p_V + p_H = 1 \tag{17}$$

if p_V and p_H denote respectively the concentrations of vertical resistors (transverse to the strip) and of horizontal ones (along the strip).

A finite-size scaling form similar to equation (16) is expected to hold for Σ_N on this line, with the same universal value of (t/ν). The data we have obtained for the two cases $p_V=\frac{1}{4}$ and $p_V=\frac{3}{4}$ show that the convergence towards the asymptotic form (equation (16)) is not as rapid as for the isotropic case. This may be seen by plotting for instance $-\log(N\Sigma_N)/\log N$ versus $1/\log N$ (figure 3). The intercepts of the three curves ($p_V=\frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$) with the vertical axis are expected to coincide at the common value of t/ν.

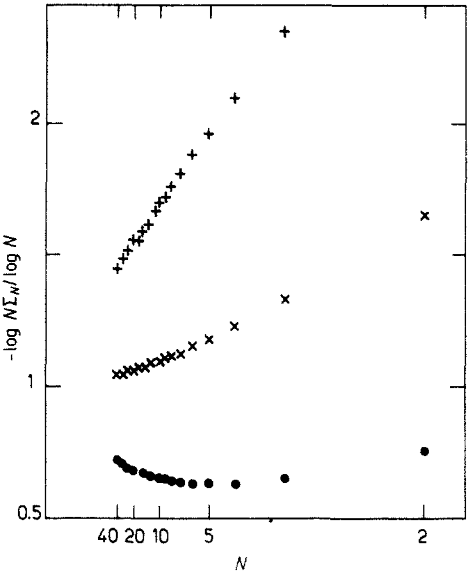

Figure 3. Plot of $-\log(N\Sigma_N)/\log N$ versus $(\log N)^{-1}$, for anisotropic systems on the critical line $p_V+p_H=1$: +, $p_V=\frac{1}{4}$; ×, $p_V=\frac{1}{2}$; ●, $p_V=\frac{3}{4}$.

As is clear from figure 3, the extrapolation of the results is not straightforward for the anisotropic systems: in one case, the curve is non-monotonic, in the other case, the slope is very large. A least-squares fit by a second-degree polynomial in $(1/\log N)$, for various ranges of N, gives as extrapolated values:

$$t/\nu \simeq 1.05 \pm 0.05, \qquad p_V=\tfrac{1}{4}$$
$$\simeq 0.96 \pm 0.02, \qquad p_V=\tfrac{1}{2}$$
$$\simeq 0.92 \pm 0.06, \qquad p_V=\tfrac{3}{4}.$$

This presumably indicates that in the presence of anisotropy the asymptotic regime is not yet reached for the strip widths considered here. Other ways to analyse the data give similar results. The strong dependence of the apparent conductivity exponent on anisotropy has already been noted in the study of finite-size effects (Blanc *et al* 1980) and in renormalisation-group calculations (Lobb *et al* 1981).

5. Discussion of the results

Finite-size scaling has been used in previous calculations of the conductivity (Lobb and Frank 1979b, Sarychev and Vinogradoff 1981, Mitescu *et al* 1982), but on finite networks such as squares. These calculations differ in an important respect from the present one: the conductance per unit length of a random strip is a number, whereas the conductance of finite systems is a random variable with a probability distribution.

In principle, one should renormalise this distribution (Stinchcombe and Watson 1976), though in practice the finite-size scaling analysis is often applied directly to some average. Our approach avoids that problem entirely. In addition, the study of connectivity properties by the transfer-matrix method (Derrida and de Seze 1982) has indicated that strips of moderate width give good results. Squares of much larger linear dimensions are needed to obtain comparable accuracy.

We have analysed carefully the results for the isotropic system at the percolation threshold, to obtain an accurate estimate of (t/ν). A particular motivation comes from the existence in the literature of two simple predictions for this ratio:

$$\text{(i)} \qquad t/\nu = 1, \tag{18}$$

which seems well supported by Monte Carlo simulations (Sarychev and Vinogradoff 1981) and finite-size scaling on squares of size 2 to 14 (Lobb and Frank 1979 b).

$$\text{(ii)} \qquad t/\nu = (\beta+\gamma)/2\nu = 91/96 = 0.9479\ldots, \tag{19}$$

which is a consequence of a recent conjecture for the density of states on the incipient infinite cluster at the percolation threshold (Alexander and Orbach 1982).

The results of various fits that we tried to obtain t/ν fluctuate too much to exclude one of these two predictions. However, the direct measure of the slope of $\log(N\Sigma_N)$ versus $\log N$ (figure 2) yielded $t/\nu = 0.95 \pm 0.01$, and the more elaborate fits in the variable $1/\log N$ (figure 3) gave 0.96 ± 0.02. Therefore, we conclude that a reasonable estimate is

$$t/\nu = 0.96 \pm 0.02,$$

which favours Alexander and Orbach's conjecture.

To confirm this tendency, an increase by at least one order of magnitude in the size of the numerical calculations seems necessary. This might be achieved for instance by combining the transfer matrix approach with an algorithm that first deletes all non-conducting dangling branches (Kirkpatrick 1979).

Acknowledgments

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