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A TRANSFORMATION OF RIGHT-DEFINITE S-HERMITIAN SYSTEMS TO CANONICAL SYSTEMS

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Abstract. It is shown that right-definite S-hermitian boundary value problems (im Normalfall), which were defined and thoroughly studied by Schäfke and Schneider in [5, 6], can be reduced to canonical systems with selfadjoint boundary conditions in such a way that the transformed boundary conditions become a special case of those considered by Dijksma, Langer and de Snoo in [1-3].

1. In this note, the first order system of differential equations

$$F_{11}y' + F_{12}y = \lambda(G_{11}y' + G_{12}y) \tag{1}$$

on the compact interval $I = [a,b]$ is considered. As in [5, 6] we suppose that the $n \times n$-matrix functions F_{1j}, G_{1j} as well as S_{1j} below, $j = 1, 2$, are continuous on I and that $F_{11}(x) - \lambda G_{11}(x)$ is invertible for all $x \in I$ and $\lambda \in \mathbb{R}$. The problem (1) is called S-hermitian (im Normalfall) with respect to the differential operator

$$S_1 y := S_{11}y' + S_{12}y$$

if there exists a continuously differentiable $n \times n$-matrix function H on I such that for all real λ the relationship

$$\begin{pmatrix} F_{11} - \lambda G_{11} & F_{12} - \lambda G_{12} \\ S_{11} & S_{12} \end{pmatrix}^* \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} F_{11} - \lambda G_{11} & F_{12} - \lambda G_{12} \\ S_{11} & S_{12} \end{pmatrix} = \begin{pmatrix} 0 & H \\ H & H' \end{pmatrix} \tag{2}$$

holds on I and $H(x)$ is invertible and skew-hermitian for all $x \in I$. If Y denotes the $n \times n$-matrix function which is the solution of the initial value problem

$$F_{11}Y' + F_{12}Y = 0, \quad Y(a) = I_n$$

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on I, then the matrix function H satisfies the equation

$$H(x) = Y(x)^{*-1} H(a) Y(x)^{-1}$$

and the relationship (2) can be written as

$$\begin{pmatrix} F_{11} - \lambda G_{11} & F_{12} - \lambda G_{12} \\ S_{11} & S_{12} \end{pmatrix} = \begin{pmatrix} I_n & -\lambda W_1 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ L & I_n \end{pmatrix} \begin{pmatrix} F_{11} & F_{12} \\ 0 & \overset{\circ}{S}_{12} \end{pmatrix} \tag{3}$$

where

$$\overset{\circ}{S}_{12} = -F_{11}^{*\,-1} H \tag{4}$$

and W_1, L are continuous $n \times n$-matrix functions such that $W_1(x)$, $L(x)$ are hermitian and $W_1(x)L(x)$ is nilpotent for all $x \in [a, b]$, (cf. [6], p.,72). The problem (1), (2) is called right-definite if $W_1 \geq 0$ on I.

It is the aim of this note to show that a right-definite S-hermitian system can be reduced to a canonical system, that is to the situation where $F_{12} = G_{11} = 0$, $F_{11} = J$ (J is an $n \times n$-matrix, independent of x, with the properties $J^* = J^{-1} = -J$) and $G_{12} = \Delta$ (a hermitian nonnegative matrix function on I). This reduction is done in two steps: The first one is to reduce the system (1) to a situation where $G_{11} = 0$; after a suitable symmetrization, a result of [4] can be applied to this new system which transforms it into a canonical system.

In [5, 6] the system (1),(2) is considered together with S-hermitian boundary conditions which depend linearly on the eigenvalue parameter λ. We show in section 3 that, by the transformation of the system (1), (2) to a canonical system, these S-hermitian boundary conditions become a special case of those conditions considered in [1] for canonical systems.

Similar results hold in the singular case when $I = [a, b)$ and the right endpoint b is in the limit point case for (1). Then the problem (1), (2) with a boundary condition at the regular point b can be included in the situation studied in [3], even in the case of arbitrary defect numbers. This will be considered elsewhere.

2. Proposition 1. *Suppose that the S-hermitian system* (1) *is right definite. If the function y satisfies* (1), *then it satisfies also the equation*

$$F_{11} y' + F_{12} y = \lambda W_1 \overset{\circ}{S}_{12} y \tag{5}$$

with $\overset{\circ}{S}_{12}$ and W_1 from (4), (3), *and*

$$\begin{pmatrix} F_{11} & F_{12} - \lambda W_1 \overset{\circ}{S}_{12} \\ 0 & \overset{\circ}{S}_{12} \end{pmatrix}^* \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} F_{11} & F_{12} - \lambda W_1 \overset{\circ}{S}_{12} \\ 0 & \overset{\circ}{S}_{12} \end{pmatrix} = \begin{pmatrix} 0 & H \\ H & H' \end{pmatrix}. \tag{6}$$

That is, the problem (5) is again S-hermitian with $S_{11} = 0$, $S_{12} = \overset{\circ}{S}_{12}$ and with the same function H as the original problem.

Proof: Consider $x_0 \in I$. As $W_1(x_0) \geq 0$, there exists an orthogonal decomposition of $\mathbf{C}^n$:

$$\mathbf{C}^n = \mathcal{L}_1 \oplus \mathcal{L}_2,$$

such that with respect to this decomposition we have

$$W_1(x_0) = \begin{pmatrix} \widehat{W}_1(x_0) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{with } \widehat{W}_1(x_0) > 0 \text{ if } \mathcal{L}_1 \neq \{0\}.$$

With a corresponding matrix representation of $L(x_0)$ it follows

$$W_1(x_0)L(x_0) = \begin{pmatrix} \widehat{W}_1(x_0)L_{11}(x_0) & \widehat{W}_1(x_0)L_{12}(x_0) \\ 0 & 0 \end{pmatrix}.$$

As this product is nilpotent we have $\sigma\big(\widehat{W}_1(x_0)L_{11}(x_0)\big) = \{0\}$. For arbitrary square matrices A, B the eigenvalues of AB and BA coincide. It follows that $\sigma\big(\widehat{W}_1(x_0)^{1/2}L_{11}(x_0)\widehat{W}_1(x_0)^{1/2}\big) = \{0\}$, hence $L_{11}(x_0) = 0$,

$$W_1(x_0)L(x_0) = \begin{pmatrix} 0 & \widehat{W}_1(x_0)L_{12}(x_0) \\ 0 & 0 \end{pmatrix}$$

and $W_1(x_0)L(x_0)W_1(x_0) = 0$. This relationship is independent of the choice of x_0, therefore

$$W_1(x)L(x)W_1(x) = 0 \quad \text{for all } x \in I. \tag{7}$$

The relationship (3) is equivalent to

$$G_{11} = W_1 L F_{11}, \quad G_{12} = W_1 L F_{12} + W_1 \overset{\circ}{S}_{12}, \tag{8}$$

$$S_{11} = L F_{11}, \qquad S_{12} = L F_{12} + \overset{\circ}{S}_{12}, \tag{9}$$

and it follows that

$$G_{11} = W_1 S_{11}, \quad G_{12} = W_1 S_{12}. \tag{10}$$

Now the system (1) can be written as

$$(I_n - \lambda G_{11} F_{11}^{-1}) F_{11} y' = \lambda G_{12} y - F_{12} y,$$

and it follows from (8) and (7) that

$$\begin{aligned} F_{11} y' &= (I_n - \lambda G_{11} F_{11}^{-1})^{-1} (\lambda G_{12} y - F_{12} y) = (I_n - \lambda W_1 L)^{-1} (\lambda G_{12} y - F_{12} y) \\ &= (I_n + \lambda W_1 L)(\lambda G_{12} y - F_{12} y). \end{aligned}$$

Further, (7) and (8) imply

$$W_1 L G_{12} = W_1 L (W_1 L F_{12} + W_1 \overset{\circ}{S}_{12}) = 0,$$

and we find

$$F_{11} y' = \lambda G_{12} y - \lambda W_1 L F_{12} y - F_{12} y = \lambda W_1 \overset{\circ}{S}_{12} y - F_{12} y,$$

that is (5). It remains to show that also (6) holds, which is equivalent to the relations

$$H = -F_{11}^* \overset{\circ}{S}_{12}, \tag{11}$$

$$H' = \overset{\circ}{S}{}_{12}^* F_{12} - F_{12}^* \overset{\circ}{S}_{12}. \tag{12}$$

The relationship (11) coincides with (4), and (2) yields

$$-F_{12}^* S_{12} + S_{12}^* F_{12} = H'.$$

With the second relationship in (9) and $L = L^*$ this implies (12). The proposition is proved.

Remark. Recall that the (complex) symplectic group $\Gamma(2n)$ is the set of all $2n \times 2n$-matrices $\mathfrak{D}$ such that

$$\mathfrak{D}^* \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \mathfrak{D} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

or, equivalently,

$$\mathfrak{D} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \mathfrak{D}^* = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

The relationships we have shown in the proof of Proposition 1 imply that

$$\mathfrak{A}(x) := \begin{pmatrix} I_n + \lambda W_1(x) L(x) & 0 \\ -L(x) & I_n - \lambda L(x) W_1(x) \end{pmatrix} \in \Gamma(2n)$$

(this is an immediate consequence of (7)), and that the relationship

$$\mathfrak{A} \begin{pmatrix} F_{11} - \lambda G_{11} & F_{12} - \lambda G_{12} \\ S_{11} & S_{12} \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} - \lambda W_1 \overset{\circ}{S}_{12} \\ 0 & \overset{\circ}{S}_{12} \end{pmatrix} \tag{13}$$

holds (this is a consequence of (7), (8), (9) and (10)). Thus the equivalence of (2) and (6) also follows if in (2) we replace $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ by $\mathfrak{A}^* \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \mathfrak{A}$ and observe (13).

Now we symmetrize the equation (5) by multiplying it from the left with $\overset{\circ}{S}{}_{12}^*$:

$$\overset{\circ}{S}{}_{12}^* F_{11} y' + \overset{\circ}{S}{}_{12}^* F_{12} y = \lambda \overset{\circ}{S}{}_{12}^* W_1 \overset{\circ}{S}_{12} y. \tag{14}$$

In the relation (6), this corresponds to a transformation with the matrix

$$\mathfrak{D}(x) = \begin{pmatrix} \overset{\circ}{S}{}_{12}^*(x) & 0 \\ 0 & \overset{\circ}{S}{}_{12}^{-1}(x) \end{pmatrix} \in \Gamma(2n),$$

that is, the matrix

$$\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

on the left hand side of (6) is replaced by

$$\mathfrak{D}^* \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \mathfrak{D},$$

and we obtain

$$\begin{pmatrix} \mathring{S}_{12}^* F_{11} & \mathring{S}_{12}^* F_{12} - \lambda \mathring{S}_{12}^* W_1 \mathring{S}_{12} \\ 0 & I_n \end{pmatrix}^* \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} \mathring{S}_{12}^* F_{11} & \mathring{S}_{12}^* F_{12} - \lambda \mathring{S}_{12}^* W_1 \mathring{S}_{12} \\ 0 & I_n \end{pmatrix}$$
$$= \begin{pmatrix} 0 & H \\ H & H' \end{pmatrix}. \tag{15}$$

We introduce the matrix functions

$$Q := -iH, \quad A := \mathring{S}_{12}^* W_1 \mathring{S}_{12}, \quad B := \mathrm{Re}(\mathring{S}_{12}^* F_{12}). \tag{16}$$

Then Q has a continuous derivative, A and B are continuous on I. The values of Q and B are hermitian matrices, the values of Q are invertible and A is nonnegative as the problem (1) was supposed to be right definite: $W_1 \geq 0$. With these functions, the equation (14) becomes

$$\tfrac{i}{2}\{(Qy)' + Qy'\} = \lambda Ay + By. \tag{17}$$

Now, as in [4] we represent $Q(x)$ as

$$iQ(x) = q(x)^* Jq(x) \tag{18}$$

where J is an $n \times n$-matrix with the properties $J^* = -J = J^{-1}$. From the construction of the $n \times n$-matrix function q in [4] it follows that this function is again continuously differentiable. With the matrix solution Z of the initial value problem

$$JZ' = q^{*-1}\{B + \tfrac{1}{2}(q^* Jq' - q'^* Jq)\}q^{-1}Z, \quad Z(a) = I_n \tag{19}$$

and

$$S(x) := q(x)^{-1}Z(x), \quad \tilde{y}(x) = S(x)^{-1}y(x) \quad (x \in I) \tag{20}$$

the equation (17) becomes

$$J\tilde{y}' = \lambda \Delta \tilde{y} \tag{21}$$

where J is given by (18), and

$$\Delta(x) := Z(x)^* q(x)^{*-1} A(x) q(x)^{-1} Z(x) \geq 0 \quad (x \in I).$$

Thus we have proved:

Proposition 2. *Suppose that the S-hermitian system* (1) *is right-definite. If the function y is a solution of* (1), *then the function $\tilde{y}$, defined by* (20), *is a solution of the canonical differential equation* (21). *The canonical differential equation* (21) *is again S-hermitian, and the relationship* (2) *becomes now*

$$\begin{pmatrix} J & -\lambda\Delta \\ 0 & I_n \end{pmatrix}^* \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} J & -\lambda\Delta \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}. \tag{22}$$

Below we shall use the following relation:

$$S(x)^* H(x) S(x) = J \quad (x \in I). \tag{23}$$

Indeed, from (16), (18), (20), we obtain

$$\begin{aligned} S(x)^* H(x) S(x) &= iS(x)^* Q(x) S(x) = S(x)^* q(x)^* J q(x) S(x) \\ &= Z(x)^* J Z(x) = J, \end{aligned} \tag{24}$$

where the last equality is a well-known property of the solution Z of the initial value problem (19). The relation (24) is equivalent to

$$S(x) J S(x)^* = -H(x)^{-1} \quad (x \in I). \tag{25}$$

3. Now we consider boundary conditions of the form

$$(F_{21} - \lambda G_{21}) y(a) + (F_{22} - \lambda G_{22}) y(b) = 0, \tag{26}$$

with $n \times n$-matrices F_{2j}, G_{2j}, $j = 1, 2$. Recall that the boundary condition (26) for the system (1),(2) is called S-hermitian if there exist $n \times n$-matrices S_{21}, S_{22} such that the relationship

$$\begin{aligned} &\begin{pmatrix} F_{21} - \lambda G_{21} & F_{22} - \lambda G_{22} \\ S_{21} & S_{22} \end{pmatrix}^* \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} F_{21} - \lambda G_{21} & F_{22} - \lambda G_{22} \\ S_{21} & S_{22} \end{pmatrix} \\ &= \begin{pmatrix} H(a) & 0 \\ 0 & -H(b) \end{pmatrix} \end{aligned} \tag{27}$$

holds for all real λ. Then there exists a hermitian $n \times n$-matrix W_2 such that

$$G_{21} = W_2 S_{21}, \quad G_{22} = W_2 S_{22}, \tag{28}$$

(see [6], p. 72). If we introduce the function $\tilde{y} = S^{-1} y$ according to (20), the boundary condition for the canonical system (21) becomes

$$(\widetilde{F}_{21} - \lambda \widetilde{G}_{21}) \tilde{y}(a) + (\widetilde{F}_{22} - \lambda \widetilde{G}_{22}) \tilde{y}(b) = 0, \tag{29}$$

with

$$\widetilde{F}_{21} = F_{21} S(a), \quad \widetilde{F}_{22} = F_{22} S(b), \quad \widetilde{G}_{21} = G_{21} S(a), \quad \widetilde{G}_{22} = G_{22} S(b). \tag{30}$$

Multiplying the relationship (27) by

$$\begin{pmatrix} S(a)^* & 0 \\ 0 & S(b)^* \end{pmatrix}$$

from the left and by

$$\begin{pmatrix} S(a) & 0 \\ 0 & S(b) \end{pmatrix}$$

from the right and observing (30) it follows

$$\begin{pmatrix} \widetilde{F}_{21} - \lambda \widetilde{G}_{21} & \widetilde{F}_{22} - \lambda \widetilde{G}_{22} \\ \widetilde{S}_{21} & \widetilde{S}_{22} \end{pmatrix}^* \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} \widetilde{F}_{21} - \lambda \widetilde{G}_{21} & \widetilde{F}_{22} - \lambda \widetilde{G}_{22} \\ \widetilde{S}_{21} & \widetilde{S}_{22} \end{pmatrix} = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} \tag{31}$$

with

$$\widetilde{S}_{21} = S_{21} S(a), \quad \widetilde{S}_{22} = S_{22} S(b). \tag{32}$$

Therefore, the canonical system (21) with the boundary condition (29) is S-hermitian with the companion matrix $\widetilde{H}(x) \equiv J$ $(x \in I)$ (observe (22)). The relationships (28), (30), and (32) imply that

$$\widetilde{G}_{21} = W_2 \widetilde{S}_{21}, \quad \widetilde{G}_{22} = W_2 \widetilde{S}_{22}, \tag{33}$$

that is, the hermitian matrix W_2 does not change under the transformation of (1) into a canonical system. Hence, if the boundary condition (26) is right-definite (that is $W_2 \geq 0$), also the boundary condition (29) for the canonical system is right-definite.

The next proposition implies that the S-hermitian boundary condition (29) is a special case of the boundary condition considered in [1].

Proposition 3. *If the boundary condition* (29) *is S-hermitian for the canonical system* (21) *(that is* (31) *is satisfied), then, with* $\widetilde{A}(\lambda) := \widetilde{F}_{21} - \lambda \widetilde{G}_{21}$, $\widetilde{B}(\lambda) := \widetilde{F}_{22} - \lambda \widetilde{G}_{22}$, *it holds:*

(i) $\operatorname{rank}(\widetilde{A}(\lambda), \widetilde{B}(\lambda)) = n \quad (\lambda \in \mathbf{R})$;

(ii) $\widetilde{A}(\lambda) J \widetilde{A}(\lambda)^* - \widetilde{B}(\lambda) J \widetilde{B}(\lambda)^* = 0 \quad (\lambda \in \mathbf{R})$;

(iii) $K_{\widetilde{A},B}(\lambda, l) := \dfrac{\widetilde{A}(\bar{l}) J \widetilde{A}(\overline{\lambda})^* - \widetilde{B}(\bar{l}) J \widetilde{B}(\overline{\lambda})^*}{\lambda - \bar{l}} = -\widetilde{F}_{21} J \widetilde{G}_{21}^* + \widetilde{F}_{22} J \widetilde{G}_{22}$

and this expression coincides with W_2 *in* (33):

$$W_2 = -\widetilde{F}_{21} J \widetilde{G}_{21}^* + \widetilde{F}_{22} J \widetilde{G}_{22}^*. \tag{34}$$

Proof: With $\widetilde{\mathfrak{A}}(\lambda) := \begin{pmatrix} \widetilde{F}_{21} - \lambda \widetilde{G}_{21} & \widetilde{F}_{22} - \lambda \widetilde{G}_{22} \\ \widetilde{S}_{21} & \widetilde{S}_{22} \end{pmatrix}$ the equation (31) is equivalent to

$$\widetilde{\mathfrak{A}}(\lambda) \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} \widetilde{\mathfrak{A}}(\lambda)^* = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \quad (\lambda \in \mathbf{R}),$$

that is

$$\widetilde{A}(\lambda)J\widetilde{A}(\lambda)^* - \widetilde{B}(\lambda)J\widetilde{B}(\lambda)^* = 0, \tag{35}$$

$$\widetilde{A}(\lambda)J\widetilde{S}_{21}^* - \widetilde{B}(\lambda)J\widetilde{S}_{22}^* = -I_n, \tag{36}$$

$$\widetilde{S}_{21}J\widetilde{S}_{21}^* - \widetilde{S}_{22}J\widetilde{S}_{22}^* = 0.$$

The relation (36) implies (i), and (ii) is equivalent to (35). The first equality in (iii) is easy to check. Further, (31) implies

$$\widetilde{G}_{21}^*\widetilde{S}_{21} - \widetilde{S}_{21}^*\widetilde{G}_{21} = \widetilde{G}_{22}^*\widetilde{S}_{21} - \widetilde{S}_{22}^*\widetilde{G}_{21} = \widetilde{G}_{22}^*\widetilde{S}_{22} - \widetilde{S}_{22}^*\widetilde{G}_{22} = 0,$$

and (36) yields

$$-\widetilde{F}_{21}J\widetilde{S}_{21}^* + \widetilde{F}_{22}J\widetilde{S}_{22}^* = I_n\,.$$

Therefore, we find for $j = 1, 2$

$$(-\widetilde{F}_{21}J\widetilde{G}_{21}^* + \widetilde{F}_{22}J\widetilde{G}_{22}^*)\widetilde{S}_{2j} = (-\widetilde{F}_{21}J\widetilde{S}_{21}^* + \widetilde{F}_{22}J\widetilde{S}_{22}^*)\widetilde{G}_{2j} = \widetilde{G}_{2j},$$

that is, the matrix defined in the first relationship of (iii) has the properties of W_2 in (33). As this matrix is uniquely determined, the relationship (34) follows.

Remark 1. The statement (iii) of Proposition 3 implies that the boundary condition (26) or (29) is right definite if and only if the kernel $K_{\widetilde{A},\widetilde{B}}(\lambda, l)$ is nonegative definite, that is, the extending space in the construction of [1] is equipped with a positive definite inner product.

Remark 2. The relations (25) and (30) imply that the matrix W_2 in (34) can be expressed in terms of the matrices in the boundary condition (26):

$$W_2 = F_{21}H(a)^{-1}G_{21}^* - F_{22}H(b)^{-1}G_{22}^*.$$

Remark 3. If $\widetilde{A}(\lambda)$, $\widetilde{B}(\lambda)$ are λ-linear $n \times n$-matrix functions as in Proposition 3 with the properties (i) and (ii), there do not necessarily exist matrices $\widetilde{S}_{21}$, $\widetilde{S}_{22}$ such that (31) holds. According to ([5], p.245), for this it is necessary and sufficient that we have additionally

$$\operatorname{rank}\begin{pmatrix} \widetilde{F}_{21} & \widetilde{F}_{22} \\ \widetilde{G}_{21} & \widetilde{G}_{22} \end{pmatrix} = n + \operatorname{rank}(\widetilde{G}_{21}, \widetilde{G}_{22}).$$

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