

## A TRANSMISSION PROBLEM FOR THERMOELASTIC PLATES

BY

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**Abstract.** In this paper we study a transmission problem for thermoelastic plates. We prove that the problem is well-posed in the sense that there exists only one solution which is as regular as the initial data. Moreover, we prove that the local thermal effect is strong enough to produce uniform rate of decay of the solution. More precisely, there exist positive constants  $C$  and  $\gamma$  such that the total energy  $E(t)$  satisfies

$$E(t) \leq CE(0)e^{-\gamma t}.$$

**1. Introduction.** From the point of view of applications, the suppression of vibration of elastic structures is one of the important topics in material science. For example, engineers at the Ford Motor Company designed a constrained-layer damping patch which was attached to an elastic plate. They compared the natural frequencies and mode shape of the plate with and without the patch to ascertain the effect of the patch. Due to the presence of the patches, the material properties of the structure, such as the elasticity moduli, damping coefficient, and Poisson ratio, are changed (see [14]). In particular, jump discontinuity at the location of the edges of the patches is usually introduced to these properties. In this direction we will consider the model which defines the oscillation of a plate which is composed of a thermoelastic part and an elastic part. This means that the thermal constant is discontinuous on the plate, positive over the thermoelastic part, and vanishing on the elastic part.

We will consider that the plate, in equilibrium, occupies a region  $\Omega$  which is a bounded open set in  $\mathbb{R}^n$  with boundary  $\partial\Omega = \Gamma_1 \cup \Gamma_2$  where  $\Gamma_1, \Gamma_2$  are two smooth surfaces such that  $\bar{\Gamma}_1 \cap \bar{\Gamma}_2 = \emptyset$ . We assume that the plate's particles in  $\Omega_1$  are sensitive to change of temperature and in its complementary part,  $\Omega_2 = \Omega \setminus \Omega_1$ , they are not. Let us denote

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by  $\Gamma_0$  the common smooth surface between  $\Omega_1$  and  $\Omega_2$ ; a region  $\Omega$  of this type is given by Fig. 1.

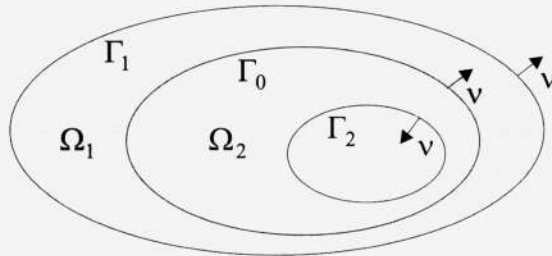


FIG. 1. The set  $\Omega$

Denoting by  $u(x, t)$  and  $v(x, t)$  the vertical displacements of the plate and by  $\theta(x, t)$  the difference of temperature, the corresponding model can be written as follows:

$$\rho_1 u_{tt} - \gamma_1 \Delta u_{tt} + \beta_1 \Delta^2 u + \mu \Delta \theta = 0 \quad \text{in } \Omega_1 \times \mathbb{R}^+, \tag{1.1}$$

$$\rho_0 \theta_t - \beta_0 \Delta \theta + \gamma_0 \theta - \mu \Delta u_t = 0 \quad \text{in } \Omega_1 \times \mathbb{R}^+, \tag{1.2}$$

$$\rho_2 v_{tt} - \gamma_2 \Delta v_{tt} + \beta_2 \Delta^2 v = 0 \quad \text{in } \Omega_2 \times \mathbb{R}^+. \tag{1.3}$$

We assume that the plate is clamped on the surfaces  $\Gamma_1, \Gamma_2$ , i.e.,

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_1 \times \mathbb{R}^+, \quad v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma_2 \times \mathbb{R}^+. \tag{1.4}$$

The transmission condition on the interface  $\Gamma_0$  is given by

$$u = v, \quad \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu}, \quad \beta_1 \Delta u + \mu \theta = \beta_2 \Delta v \quad \text{on } \Gamma_0 \times \mathbb{R}^+, \tag{1.5}$$

$$-\gamma_1 \frac{\partial u_{tt}}{\partial \nu} + \beta_1 \frac{\partial \Delta u}{\partial \nu} + \mu \frac{\partial \theta}{\partial \nu} = -\gamma_2 \frac{\partial v_{tt}}{\partial \nu} + \beta_2 \frac{\partial \Delta v}{\partial \nu} \quad \text{on } \Gamma_0 \times \mathbb{R}^+. \tag{1.6}$$

We consider the following condition for the temperature:

$$\theta = 0 \quad \text{on } \Gamma_0 \times \mathbb{R}^+, \quad \frac{\partial \theta}{\partial \nu} + \lambda \theta = 0 \quad \text{on } \Gamma_1 \times \mathbb{R}^+, \tag{1.7}$$

and the initial data

$$\begin{aligned} u(0) = u_0, \quad u_t(0) = u_1, \quad \theta(0) = \theta_0 \quad &\text{in } \Omega_1, \\ v(0) = v_0, \quad v_t(0) = v_1 \quad &\text{in } \Omega_2. \end{aligned} \tag{1.8}$$

Here, the coefficients  $\rho_i, \gamma_i, \beta_i$ , and  $\lambda$  are positive,  $\mu$  is different to zero, and  $u_0, u_1, \theta_0, v_0, v_1$  are prescribed functions. To fix ideas we consider  $\mu$  positive.

Controllability for transmission problems were studied by several authors; for example, the transmission problem for the wave equation was studied by Lions [6]. He applied the Hilbert Uniqueness Method (HUM) to show the exact controllability. Later, Lagnese [5], also applying HUM, extended this result; he showed the exact controllability for a class of hyperbolic systems which include the transmission problem for homogeneous anisotropic materials. The exact controllability for the plate equation was proved by Liu and Williams [9] and Aassila [1].

Concerning asymptotic stability, second order transmission problems were studied by Rivera and Oquendo [11], Liu and Williams [8], and Rivera and Ma To Fu [10], while for beams we have the works of Rivera and Oquendo [12, 13]. Thermoelastic plates were studied by Lagnese, Avalos, and Lasiecka. In [4], Lagnese obtained the exponential decay of solutions with the aid of a further mechanical dissipation on the boundary and in [2], Avalos and Lasiecka obtained the same result removing the boundary dissipation. It seems to us that there is no result concerning the asymptotic stability of solution for plates made of different types of materials. So to fill this gap we study this topic here.

The main result of this paper is to show that the dissipation given by the thermal part of the plate is strong enough to produce uniform stability of the solution, no matter how thin it is. To attain this goal we will assume that the material type in  $\Omega_1$  is more stiff than that in  $\Omega_2$ , that is

$$\rho_1 \geq \rho_2, \quad \gamma_1 \geq \gamma_2 \quad \text{and} \quad \beta_1 \leq \beta_2.$$

Additionally, some geometric assumptions on  $\Omega$  will be taken into account, as for example

$$\begin{aligned} (x - x_0) \cdot \nu(x) &\geq \delta_0 && \text{on } \Gamma_0, \\ (x - x_0) \cdot \nu(x) &\leq 0 && \text{on } \Gamma_2, \end{aligned}$$

for some  $x_0 \in \mathbb{R}^n$  and  $\delta_0 > 0$  small. In these conditions we will show that the total energy associated to the model decays exponentially as time goes to infinity. The idea we use to achieve our result is based on the energy method; to do so, we need that the solution enjoys the regularity property. Therefore, in the next section of the article, we will show that the solution of the above system has the m-regularity result. One of the main difficulties we have in showing the exponential decay is due to the boundary conditions. We avoid them using some localized multipliers and some technical ideas involving the compact embedding of the spaces  $H^{m-1} \subset H^m$ .

The remaining part of this article is organized as follows. In the next section we will show that the problem is well-posed in the sense of existence, uniqueness, and regularity of the solution. To do this we will use the semigroup approach. Finally in Sec. 3 we will prove that the solution of the system decays exponentially to zero.

**2. Existence of solutions.** To find a solution for the problem (1.1)–(1.8), we shall use the semigroup approach. Let us start analyzing the associated stationary problem. First we shall introduce some notation. Let us consider the following Hilbert spaces

$$\begin{aligned} H_T^1 &:= \{(\phi_1, \phi_2) \in H^1(\Omega_1) \times H^1(\Omega_2) : \phi_i = 0 \text{ on } \Gamma_i, \phi_1 = \phi_2 \text{ on } \Gamma_0\}, \\ H_T^2 &:= \left\{ (\phi_1, \phi_2) \in [H^2(\Omega_1) \times H^2(\Omega_2)] \cap H_T^1 : \frac{\partial \phi_i}{\partial \nu} = 0 \text{ on } \Gamma_i, \frac{\partial \phi_1}{\partial \nu} = \frac{\partial \phi_2}{\partial \nu} \text{ on } \Gamma_0 \right\}, \end{aligned}$$

$$H_D^1 := \{\phi \in H^1(\Omega_1) : \phi = 0 \text{ on } \Gamma_0\}, \quad H^0 := L^2(\Omega_1),$$

with the following inner products:

$$\begin{aligned}
 (\{w_1, w_2\}, \{\phi_1, \phi_2\})_{H_T^1} &:= \int_{\Omega_1} (\rho_1 w_1 \phi_1 + \gamma_1 \nabla w_1 \cdot \nabla \phi_1) dx \\
 &\quad + \int_{\Omega_2} (\rho_2 w_2 \phi_2 + \gamma_2 \nabla w_2 \cdot \nabla \phi_2) dx, \\
 (\{w_1, w_2\}, \{\phi_1, \phi_2\})_{H_T^2} &:= \int_{\Omega_1} \beta_1 \Delta w_1 \Delta \phi_1 dx + \int_{\Omega_2} \beta_2 \Delta w_2 \Delta \phi_2 dx, \\
 (w, \phi)_{H_D^1} &:= \int_{\Omega_1} \beta_0 \nabla w \cdot \nabla \phi + \gamma_0 w \phi dx + \int_{\Gamma_1} \beta_0 \lambda w \phi d\Gamma, \\
 (w, \phi)_{H^0} &:= \int_{\Omega_1} \rho_0 w \phi dx.
 \end{aligned}$$

Let us denote the dual space of  $H_T^s$  by  $H_T^{-s}$  for  $s = 1, 2$  and the dual space of  $H_D^1$  by  $H_D^{-1}$ . The following Lemma shows that the norm given by the inner product in  $H_T^2$  is equivalent to the usual norm of  $H^2(\Omega_1) \times H^2(\Omega_2)$ .

LEMMA 2.1. Let us take  $(f, g)$  in  $L^2(\Omega_1) \times L^2(\Omega_2)$ . Then there exists a unique couple

$$(u, v) \in H^2(\Omega_1) \times H^2(\Omega_2)$$

solution of

$$\Delta u = f \text{ on } \Omega_1, \quad \Delta v = g \text{ on } \Omega_2,$$

satisfying the boundary conditions

$$u = 0 \text{ on } \Gamma_1, \quad v = 0 \text{ on } \Gamma_2, \quad u = v \text{ and } \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} \text{ on } \Gamma_0.$$

Moreover, there exists  $C > 0$  such that

$$\|u\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)} \leq C(\|f\|_{L^2(\Omega_1)} + \|g\|_{L^2(\Omega_2)}).$$

*Proof.* See [3]. □

The corresponding stationary transmission problem for the plate equation is given by the equations

$$\Delta^2 u = f \text{ in } \Omega_1, \quad \Delta v^2 = g \text{ in } \Omega_2, \tag{2.1}$$

satisfying the boundary condition

$$u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_1, \quad v = \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma_2, \tag{2.2}$$

$$u = v, \quad \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu}, \quad \beta_1 \Delta u = \beta_2 \Delta v, \quad \beta_1 \frac{\partial \Delta u}{\partial \nu} = \beta_2 \frac{\partial \Delta v}{\partial \nu} \text{ on } \Gamma_0. \tag{2.3}$$

To find the variational formulation associated to this problem we multiply the first equation of (2.1) by  $\beta_1 \phi_1$  and the second by  $\beta_2 \phi_2$  with  $(\phi_1, \phi_2) \in H_T^2$ ; next we integrate by parts to obtain

$$\int_{\Omega_1} \beta_1 \Delta w_1 \Delta \phi_1 dx + \int_{\Omega_2} \beta_2 \Delta w_2 \Delta \phi_2 dx = \int_{\Omega_1} \beta_1 f \phi_1 dx + \int_{\Omega_2} \beta_2 g \phi_2 dx.$$

The existence and uniqueness of weak and strong solution for this problem is given by

LEMMA 2.2. The following items hold:

- (1) If  $(f, g) \in H_T^{-2}$ , then there exists a unique solution  $(u, v) \in H_T^2$  of (2.1)–(2.3). Moreover, there exists  $C > 0$  such that

$$\|(u, v)\|_{H_T^2} \leq C\|(f, g)\|_{H_T^{-2}}.$$

- (2) Let  $m \in \mathbb{Z}_0^+$ . If  $(f, g) \in H^m(\Omega_1) \times H^m(\Omega_2)$ , then there exists a unique solution  $(u, v) \in H^{m+4}(\Omega_1) \times H^{m+4}(\Omega_2)$  of (2.1)–(2.3). Moreover, there exists  $C > 0$  such that

$$\|u\|_{H^{m+4}(\Omega_1)} + \|v\|_{H^{m+4}(\Omega_2)} \leq C(\|f\|_{H^m(\Omega_1)} + \|g\|_{H^m(\Omega_2)}).$$

- (3) If  $(f, g) \in H_T^{-1}$ , then there exists a unique solution  $(u, v) \in H^3(\Omega_1) \times H^3(\Omega_2)$  of (2.1)–(2.3). Moreover, there exists  $C > 0$  such that

$$\|(u, v)\|_{H^3(\Omega_1) \times H^3(\Omega_2)} \leq C\|(f, g)\|_{H_T^{-1}}.$$

*Proof.* Item 1 is a consequence of Lax-Milgram Theorem; item 2, for the case  $m = 0$ , can be found in [9] and item 3 is a consequence of the Interpolation Theory. Item 2 is a well known result of elliptic regularity (see [15] for the transmission problem for general elliptic equations). In Appendix A of this paper we give a simple proof of this item when  $\beta_1/\beta_2$  is small or large.  $\square$

Now, we shall write system (1.1)–(1.7) in the abstract form of semigroups following the ideas of Lagnese [4]. Let us consider the operators

$$A_0 : H_T^2 \rightarrow H_T^{-2}, \quad A_1 : H_T^1 \times H^0 \rightarrow H_T^{-1} \times H^0 \quad \text{and} \quad B_0 : H_T^2 \times H_D^1 \rightarrow H_T^{-2} \times H_D^{-1}$$

given by

$$\begin{aligned} \langle A_0\{w_1, w_2\}, \{\phi_1, \phi_2\} \rangle &:= (\{w_1, w_2\}, \{\phi_1, \phi_2\})_{H_T^2}, \\ \langle A_1\{w_1, w_2, w_3\}, \{\phi_1, \phi_2, \phi_3\} \rangle &:= (\{w_1, w_2\}, \{\phi_1, \phi_2\})_{H_T^1} + (w_3, \phi_3)_{H^0}, \\ \langle B_0\{w_1, w_2, w_3\}, \{\phi_1, \phi_2, \phi_3\} \rangle &:= \int_{\Omega_1} \mu(w_3\Delta\phi_1 - \Delta w_1\phi_3) dx + (w_3, \phi_3)_{H_D^1}, \end{aligned}$$

and let us denote by

$$B_1\{\phi_1, \phi_2\} := \{A_0\{\phi_1, \phi_2\}, 0\}, \quad B_2\{\phi_1, \phi_2, \phi_3\} := A_0\{\phi_1, \phi_2\}.$$

Multiplying equations (1.1), (1.3), and (1.2) by  $\phi_1$ ,  $\phi_2$  and  $\phi_3$  respectively with  $\{\phi_1, \phi_2, \phi_3\} \in H_T^2 \times H_D^1$  and performing an integration by parts yields

$$\langle A_1 \frac{d}{dt} \{u_t, v_t, \theta\}, \{\phi_1, \phi_2, \phi_3\} \rangle = -\langle A_0\{u, v\}, \{\phi_1, \phi_2\} \rangle - \langle B_0\{u_t, v_t, \theta\}, \{\phi_1, \phi_2, \phi_3\} \rangle.$$

The above identity can be written as

$$A_1 \frac{d}{dt} \{u_t, v_t, \theta\} = -B_1\{u, v\} - B_0\{u_t, v_t, \theta\} \quad \text{in} \quad H_T^{-2} \times H_D^{-1}. \tag{2.4}$$

Taking into account that

$$A_0 \frac{d}{dt} \{u, v\} = A_0\{u_t, v_t\},$$

and denoting by  $w = (u, v, u_t, v_t, \theta)$ , Eq. (2.4) can be written in the following matrix form

$$\begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix} \frac{dw}{dt} + \begin{pmatrix} 0 & -B_2 \\ B_1 & B_0 \end{pmatrix} w = 0 \quad \text{in} \quad H_T^{-2} \times H_T^{-2} \times H_D^{-1}.$$

Denoting by

$$\mathbb{A} = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} 0 & -B_2 \\ B_1 & B_0 \end{pmatrix}$$

let us introduce the linear operator  $\Lambda := -\mathbb{A}^{-1}\mathbb{B}$  in  $\mathcal{H} := H_T^2 \times H_T^1 \times H^0$  with domain

$$D(\Lambda) = \{w \in H_T^2 \times H_T^2 \times H_D^1 : \{A_0\{w_1, w_2\}, 0\} + B_0\{w_3, w_4, w_5\} \in H_T^{-1} \times H^0\}.$$

If we denote by

$$D_0 := \left\{ w \in [H_T^2 \cap H^3(\Omega_1) \times H^3(\Omega_2)] \times H_T^2 \times [H^2(\Omega_1) \cap H_D^1] : \right. \\ \left. \beta_1 \Delta w_1 = \beta_2 \Delta w_2 \text{ on } \Gamma_0 \text{ and } \frac{\partial w_5}{\partial \nu} + \lambda w_5 = 0 \text{ on } \Gamma_1 \right\},$$

it is clear that  $D_0 \subset D(\Lambda)$  and  $D_0$  is dense in  $\mathcal{H}$ . The well-posedness of system (1.1)–(1.8) is given by the following Theorem.

**THEOREM 2.3.** The operator  $\Lambda$  is the generator of a semigroup of class  $C_0$  on  $\mathcal{H}$ .

*Proof.* We show that  $\mathbb{A}^{-1}\mathbb{B}$  is maximal accretive; our conclusion will follow by the well-known Lumer-Phillips’s Theorem.

**$\mathbb{A}^{-1}\mathbb{B}$  is accretive:** Let  $w \in D(\Lambda)$ , then

$$\begin{aligned} (\mathbb{A}^{-1}\mathbb{B}w, w)_{\mathcal{H}} &= (\mathbb{A}^{-1}[-A_0\{w_3, w_4\}, \{A_0\{w_1, w_2\}, 0\} + B_0\{w_3, w_4, w_5\}], w)_{\mathcal{H}} \\ &= (\{w_3, w_4\}, A_1^{-1}[\{A_0\{w_1, w_2\}, 0\} + B_0\{w_3, w_4, w_5\}], w)_{\mathcal{H}} \\ &= -(\{w_3, w_4\}, \{w_1, w_2\})_{H_T^2} + \langle A_0\{w_1, w_2\}, \{w_3, w_4\} \rangle_{H_T^{-2} \times H_T^2} \\ &\quad + \langle B_0\{w_3, w_4, w_5\}, \{w_3, w_4, w_5\} \rangle_{(H_T^{-2} \times H_D^{-1}) \times (H_T^2 \times H_D^1)} \\ &= (w_5, w_5)_{H_D^1} \geq 0. \end{aligned}$$

**$\mathbb{A}^{-1}\mathbb{B}$  is maximal:** We need to show that the operator  $I + \mathbb{A}^{-1}\mathbb{B}$  is surjective of  $D(\Lambda)$  on  $\mathcal{H}$ . Since  $\mathbb{A}$  is an isomorphism of  $\mathcal{H}$  onto  $\mathcal{H}'$ , it is sufficient to show that

$$\mathbb{A} + \mathbb{B} : D(\Lambda) \rightarrow \mathcal{H}' \quad \text{is surjective.}$$

Given  $f \in \mathcal{H}'$ , the equation  $(\mathbb{A} + \mathbb{B})w = f$  is equivalent to find  $w \in D(\Lambda)$  such that

$$\begin{aligned} A_0\{w_1, w_2\} - A_0\{w_3, w_4\} &= (f_1, f_2) \in H_T^{-2}, \\ A_1\{w_3, w_4, w_5\} + \{A_0\{w_1, w_2\}, 0\} + B_0\{w_3, w_4, w_5\} &= (f_3, f_4, f_5) \in H_T^{-1} \times H^0. \end{aligned} \tag{2.5}$$

Substituting the first equation of (2.5) into the second equation, we get

$$A_1\{w_3, w_4, w_5\} + \{A_0\{w_3, w_4\}, 0\} + B_0\{w_3, w_4, w_5\} = (f_3 - f_1, f_4 - f_2, f_5) \in H_T^{-2} \times H^0. \tag{2.6}$$

Note that the operator  $G : H_T^2 \times H_D^1 \rightarrow H_T^{-2} \times H_D^{-1}$ , given by

$$G\{w_3, w_4, w_5\} := A_1\{w_3, w_4, w_5\} + \{A_0\{w_3, w_4\}, 0\} + B_0\{w_3, w_4, w_5\}$$

is continuous and satisfies

$$\begin{aligned} \langle G\{w_3, w_4, w_5\}, \{w_3, w_4, w_5\} \rangle_{(H_T^{-2} \times H_D^{-1}) \times (H_T^2 \times H_D^1)} &= \|\{w_3, w_4, w_5\}\|_{H_T^{-1} \times H^0}^2 \\ &\quad + \|\{w_3, w_4\}\|_{H_T^2}^2 + \|w_5\|_{H_D^1}^2 \\ &\geq \|\{w_3, w_4, w_5\}\|_{H_T^2 \times H_D^1}^2. \end{aligned}$$

Therefore, the Lax-Milgram Theorem implies that there exists

$$\{w_3, w_4, w_5\} \in H_T^2 \times H_D^1 \tag{2.7}$$

such that relation (2.6) is satisfied. Since  $A_0$  is an isomorphism of  $H_T^2$  on  $H_T^{-2}$ , there exists

$$\{w_1, w_2\} \in H_T^2 \tag{2.8}$$

such that a first equation of (2.5) is satisfied. The second equation of (2.5) implies that

$$\{A_0\{w_1, w_2\}, 0\} + B_0\{w_3, w_4, w_5\} \in H_T^{-1} \times H^0;$$

therefore  $w = (w_1, \dots, w_5)$  given by (2.7)–(2.8) belong in  $D(\Lambda)$  and satisfy  $(A + B)w = f$ . □

**3. Exponential decay.** In this section we will show that the solution of the transmission problem (1.1)–(1.8) decays exponentially to zero as time goes to infinity. In Appendix B we prove that the functions in  $D(\Lambda^m)$  are regular, so if we take regular initial data in  $D(\Lambda^m)$ , then the solution is regular, too. Therefore we can apply multiplier techniques to this system. Through this section we denote by  $C$  a positive constant which will assume different values in different places. We shall assume that there exists  $x_0 \in \mathbb{R}^n$  such that the function  $m(x) : x - x_0$  satisfies

$$\begin{aligned} m \cdot \nu &\geq \delta_0 && \text{on } \Gamma_0, \\ m \cdot \nu &\leq 0 && \text{on } \Gamma_2, \end{aligned} \tag{3.1}$$

for  $\delta_0 > 0$  small. Let us consider the following equation:

$$\rho w_{tt} - \gamma \Delta w_{tt} + \beta \Delta^2 w = 0 \quad \text{in } \Omega_2 \times \mathbb{R}^+, \tag{3.2}$$

with boundary condition

$$w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } (\Gamma_2 \cup \Gamma_0) \times \mathbb{R}^+. \tag{3.3}$$

The energy associated to this system is given by

$$\hat{E}(t, w) := \frac{1}{2} \int_{\Omega_2} \rho |w_t|^2 + \gamma |\nabla w_t|^2 + \beta |\Delta w|^2 \, dx.$$

It is known that if  $w$  is a weak solution of system (3.2)–(3.3), that is to say,  $w \in L^\infty(0, \infty; H^2(\Omega_2)) \cap W^{1,\infty}(0, \infty; H^1(\Omega_2))$ , then  $\frac{d}{dt} \hat{E}(t, w) = 0$  for any  $t > 0$ ; therefore the function

$$t \mapsto \hat{E}(t, w) \quad \text{is constant.} \tag{3.4}$$

Some properties for the solution of this system, such as regularity of the trace and observability inequality, are given by the following Lemma. A similar result also can be found in [2, Lemma 2.3].

**LEMMA 3.1.** Let us suppose that  $w \in L^\infty(0, \infty; H^2(\Omega_2)) \cap W^{1,\infty}(0, \infty; H^1(\Omega_2))$  is a solution of (3.2)–(3.3); then the following properties hold:

(1) There exists  $C > 0$  such that

$$\int_0^T \int_{\Gamma_2 \cup \Gamma_0} |\Delta w|^2 \, d\Gamma \, dt \leq C \int_0^T \hat{E}(t, w) \, dt,$$

for  $T$  large enough.

(2) There exists  $C > 0$  such that

$$\int_0^T \hat{E}(t, w) \, dt \leq C \int_0^T \int_{\Gamma_0} m \cdot \nu |\Delta w|^2 \, d\Gamma \, dt,$$

for  $T$  large enough.

*Proof.* We will use multiplier techniques to show that the above inequalities hold for strong solutions, that is, for  $w \in L^\infty(0, T; H^4(\Omega_2)) \cap W^{1,\infty}(0, T; H^3(\Omega_2))$ , so our conclusion will follow using standard density arguments. Let us consider the vector field  $h = (h_1, \dots, h_n) \in [C^2(\Omega_2)]^n$  such that  $h(x) = \nu, \forall x \in \Gamma_2 \cup \Gamma_0$ . Multiplying Eq. (3.2) by  $h \cdot \nabla w$  and integrating by parts, we get

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega_2} \rho w_t h \cdot \nabla w + \gamma \nabla w_t \cdot \nabla (h \cdot \nabla w) \, dx \\ &= \frac{\beta}{2} \int_{\Gamma_2 \cup \Gamma_0} |\Delta w|^2 \, d\Gamma - \frac{1}{2} \int_{\Omega_2} \operatorname{div} h (\rho |w_t|^2 + \gamma |\nabla w_t|^2 - \beta |\Delta w|^2) \, dx \\ & \quad + \int_{\Omega_2} \left( \gamma \frac{\partial h_k}{\partial x_j} \frac{\partial w_t}{\partial x_j} \frac{\partial w_t}{\partial x_k} - 2\beta \Delta w \frac{\partial h_k}{\partial x_j} \frac{\partial w}{\partial x_k} \right) \, dx - \sum_{k=1}^n \int_{\Omega_2} \beta \Delta w \Delta h_k \frac{\partial w}{\partial x_k} \, dx. \end{aligned}$$

Integrating over  $[0, T]$  using Young’s inequality and (3.4), we obtain that

$$\begin{aligned} \frac{\beta}{2} \int_0^T \int_{\Gamma_2 \cup \Gamma_0} |\Delta w|^2 \, d\Gamma \, dt &\leq C \left\{ \int_0^T \hat{E}(t, w) \, dt + \hat{E}(T, w) + \hat{E}(0, w) \right\} \\ &\leq C \left\{ \int_0^T \hat{E}(t, w) \, dt + \frac{2}{T} \int_0^T \hat{E}(t, w) \, dt \right\}. \end{aligned}$$

From this inequality the first part of this Lemma follows. To show the second part, we multiply Eq. (3.2) by  $Kw := m(x) \cdot \nabla w + (n - 3)w/2$  and integrate by parts to obtain

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} (\rho w_t Kw + \gamma \nabla w_t \cdot \nabla Kw) \, dx &= -\frac{1}{2} \int_{\Omega} 3\rho |w_t|^2 + \gamma |\nabla w_t|^2 + \beta |\Delta w|^2 \, dx \\ & \quad + \frac{\beta}{2} \int_{\Gamma_0} m \cdot \nu |\Delta w|^2 \, d\Gamma + \frac{\beta}{2} \int_{\Gamma_2} m \cdot \nu |\Delta w|^2 \, d\Gamma. \end{aligned}$$

Integrating over  $[0, T]$  using Young’s inequality and hypothesis (3.1), we get

$$\begin{aligned} \int_0^T \hat{E}(t, w) \, dt &\leq \frac{\beta}{2} \int_0^T \int_{\Gamma_0} m \cdot \nu |\Delta w|^2 \, d\Gamma \, dt + C \{ \hat{E}(T, w) + \hat{E}(0, w) \} \\ &\leq \frac{\beta}{2} \int_0^T \int_{\Gamma_0} m \cdot \nu |\Delta w|^2 \, d\Gamma \, dt + \frac{2C}{T} \int_0^T \hat{E}(t, w) \, dt. \end{aligned}$$

Regrouping, we arrive at the following inequality:

$$\left( 1 - \frac{2C}{T} \right) \int_0^T \hat{E}(t, w) \, dt \leq \frac{\beta}{2} \int_0^T \int_{\Gamma_0} m \cdot \nu |\Delta w|^2 \, d\Gamma \, dt.$$



Now, taking  $T$  large, our conclusion follows. □

Let us consider the functionals  $E$  (first order energy) and  $F$  given by

$$\begin{aligned}
 E(t, u, v, \theta) &:= \frac{1}{2} \int_{\Omega_1} \rho_1 |u_t|^2 + \gamma_1 |\nabla u_t|^2 + \beta_1 |\Delta u|^2 + \rho_0 |\theta|^2 dx \\
 &\quad + \frac{1}{2} \int_{\Omega_2} \rho_2 |v_t|^2 + \gamma_2 |\nabla v_t|^2 + \beta_2 |\Delta v|^2 dx, \\
 F(t, u) &:= \int_{\Omega_1} |\nabla u|^2 + |u_t|^2 dx,
 \end{aligned}$$

and denote by  $\mathcal{E}$  and  $\mathcal{F}$  as follows:

$$\mathcal{E}(t, u, v, \theta) := \sum_{i=0}^3 E(t, u^{(i)}, v^{(i)}, \theta^{(i)}), \quad \mathcal{F}(t, u) := \sum_{i=0}^3 F(t, u^{(i)}), \tag{3.5}$$

where  $w^{(i)} = \frac{d^i w}{dt^i}$ ,  $w^{(0)} = w$ . The next Lemma will play an important role in the sequel.

LEMMA 3.2. Given  $\eta > 0$ , there exists a positive constant  $C_\eta > 0$  such that

$$\int_0^T \mathcal{F}(t, u) dt \leq \eta \int_0^T \mathcal{E}(t, u, v, \theta) dt + C_\eta \sum_{i=0}^3 \int_0^T \int_{\Omega_1} \beta_0 |\nabla \theta^{(i)}|^2 dx dt, \tag{3.6}$$

for any solution  $(u, v, \theta)$  of (1.1)–(1.8) such that  $(u_0, v_0, u_1, v_1, \theta_0) \in D(\Lambda^5)$ .

*Proof.* To establish our result we assume that the hypothesis of this Lemma is not true and derive a contradiction. Let us suppose that the inequality (3.6) does not hold, that is to say, there exist  $\eta_0 > 0$  and  $(u_0^n, v_0^n, u_1^n, v_1^n, \theta_0^n)_{n \in \mathbb{N}}$  in  $D(\Lambda^5)$  such that the solutions  $(u^n, v^n, \theta^n)$  of (1.1)–(1.8) satisfy

$$\eta_0 \int_0^T \mathcal{E}(t, u^n, v^n, \theta^n) dt + n \sum_{i=0}^3 \int_0^T \int_{\Omega_1} \beta_0 |\nabla \theta^{n(i)}|^2 dx dt < \int_0^T \mathcal{F}(t, u^n) dt, \tag{3.7}$$

for any  $n \in \mathbb{N}$ . Without loss of generality, since the system (1.1)–(1.7) is linear, we can suppose that

$$\int_0^T \mathcal{F}(t, u^n) dt = 1. \tag{3.8}$$

Hence the inequality (3.7) can be written as

$$\eta_0 \int_0^T \mathcal{E}(t, u^n, v^n, \theta^n) dt + n \sum_{i=0}^3 \int_0^T \int_{\Omega_1} \beta_0 |\nabla \theta^{n(i)}|^2 dx dt < 1, \tag{3.9}$$

for any  $n \in \mathbb{N}$ . This inequality implies that  $(u^{n(i)}, v^{n(i)}, \theta^{n(i)})$  is bounded in  $L^2(0, T; H_T^2 \times H^0)$  for  $i = 0, \dots, 3$ ; therefore there exists a subsequence, which we still denote in the same way, such that

$$(u^{n(i)}, v^{n(i)}, \theta^{n(i)}) \rightharpoonup (u^{(i)}, v^{(i)}, \theta^{(i)}) \quad \text{in } L^2(0, T; H_T^2 \times H^0).$$

Since  $(u^{n(i)}, v^{n(i)}, \theta^{n(i)})$  is a solution of (1.1)–(1.7), then  $(u^{(i)}, v^{(i)}, \theta^{(i)})$  is a solution too, so it satisfies the equations

$$\rho_1 u_{tt} - \gamma_1 \Delta u_{tt} + \beta_1 \Delta^2 u + \mu \Delta \theta = 0 \quad \text{in } \Omega_1 \times \mathbb{R}^+, \tag{3.10}$$

$$\rho_0 \theta_t - \beta_0 \Delta \theta + \gamma_0 \theta - \mu \Delta u_t = 0 \quad \text{in } \Omega_1 \times \mathbb{R}^+, \tag{3.11}$$

$$\rho_2 v_{tt} - \gamma_2 \Delta v_{tt} + \beta_2 \Delta^2 v = 0 \quad \text{in } \Omega_2 \times \mathbb{R}^+, \tag{3.12}$$

and the boundary conditions

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_1 \times \mathbb{R}^+, \quad v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma_2 \times \mathbb{R}^+, \tag{3.13}$$

$$u = v, \quad \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu}, \quad \beta_1 \Delta u = \beta_2 \Delta v \quad \text{on } \Gamma_0 \times \mathbb{R}^+, \tag{3.14}$$

$$-\gamma_1 \frac{\partial u_{tt}}{\partial \nu} + \beta_1 \frac{\partial \Delta u}{\partial \nu} + \mu \frac{\partial \theta}{\partial \nu} = -\gamma_2 \frac{\partial v_{tt}}{\partial \nu} + \beta_2 \frac{\partial \Delta v}{\partial \nu} \quad \text{on } \Gamma_0 \times \mathbb{R}^+, \tag{3.15}$$

$$\theta = 0 \quad \text{on } \Gamma_0 \times \mathbb{R}^+, \quad \frac{\partial \theta}{\partial \nu} + \lambda \theta = 0 \quad \text{on } \Gamma_1 \times \mathbb{R}^+. \tag{3.16}$$

Using (3.8) and Lions-Aubin’s Compactness Theorem (see [7]), we can prove that

$$\int_0^T \mathcal{F}(t, u) dt = 1. \tag{3.17}$$

The inequality (3.9) implies that

$$\nabla \theta = 0 \quad \text{in } \Omega_1 \times \mathbb{R}^+.$$

From Poincaré’s inequality we have that  $\theta = 0$  in  $\Omega_1 \times \mathbb{R}^+$ . Equation (3.11) implies that  $\Delta u_t = 0$  in  $\Omega_1 \times \mathbb{R}^+$ . Differentiating Eq. (3.10) with respect to  $t$  we obtain that  $u_{ttt} = 0$  in  $\Omega_1 \times \mathbb{R}^+$ . Differentiating Eq. (3.12)–(3.14) in  $t$  we have that  $v_{ttt}$  is a weak solution of (3.2)–(3.3), satisfying

$$\Delta v_{ttt} = \frac{\beta_1}{\beta_2} \Delta u_{ttt} = 0 \quad \text{on } \Gamma_0.$$

Lemma 3.1 implies that  $v_{ttt} = 0$  in  $\Omega_2 \times \mathbb{R}^+$ . Differentiating Eqs. (3.10)–(3.16) in  $t$ , we find that  $(u_t, v_t)$  satisfies the hypothesis of Lemma 2.2 with  $(f, g) = (0, 0)$ ; therefore  $(u_t, v_t) = (0, 0)$ . In these conditions, from equations (3.10)–(3.16) we conclude that  $(u, v) = (0, 0)$ . This is contradictory to (3.17), hence our conclusion follows.  $\square$

The exponential decay for regular solutions is given by the following theorem.

**THEOREM 3.3.** Let us take  $(u_0, v_0, u_1, v_1, \theta_0) \in D(\Lambda^3)$ . If

$$\rho_1 \geq \rho_2, \quad \gamma_1 \geq \gamma_2 \quad \text{and} \quad \beta_1 \leq \beta_2, \tag{3.18}$$

then there exist positive constants  $C$  and  $\kappa$  such that

$$\mathcal{E}(t, u, v, \theta) \leq C \mathcal{E}(0, u, v, \theta) e^{-\kappa t},$$

for any solution  $(u, v, \theta)$  of (1.1)–(1.8).

We shall prove this result for initial data  $(u_0, v_0, u_1, v_1, \theta_0) \in D(\Lambda^5)$ ; the general case follows using a standard density argument. Let us denote by  $B_\delta(\Omega_2) := \{x \in \mathbb{R}^n :$

$\inf_{y \in \Omega_2} |y - x| < \delta$  with  $\delta > 0$  small and let us consider the non-negative functions  $\varphi_i, \psi_i, i = 1, \dots, 4$ , of class  $C^\infty(\mathbb{R}^n)$  given by

$$\varphi_i(x) = \begin{cases} 0 & \text{if } x \in B_{i\delta/2}(\Omega_2) \\ 1 & \text{if } x \in \mathbb{R}^n \setminus B_{i\delta}(\Omega_2) \end{cases}, \quad \psi_i(x) = \begin{cases} 1 & \text{if } x \in B_{i\delta}(\Omega_2) \\ 0 & \text{if } x \in \mathbb{R}^n \setminus B_{2i\delta}(\Omega_2). \end{cases}$$

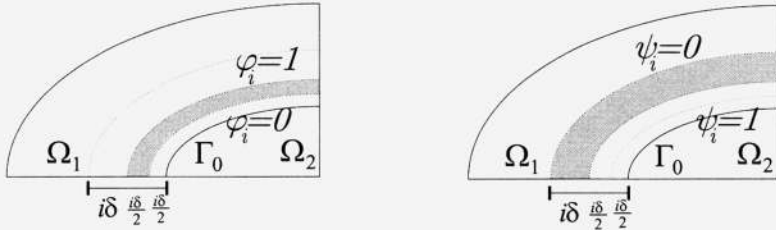


FIG. 2. The functions  $\varphi_i, \psi_i$

Decomposing the set  $\Omega_1$  by  $\Omega_1 = U_i \cup V_i$  with

$$U_i = \Omega_1 \setminus B_{i\delta}(\Omega_2), \quad V_i = \Omega_1 \cap B_{i\delta}(\Omega_2) \quad \text{for } i = 1, \dots, 4,$$

we get that  $\varphi_i = 1$  in  $U_i$  and  $\psi_i = 1$  in  $V_i$ . The proof of this Theorem will be a consequence of the following lemmas. Let us denote by  $R_1$  the functional

$$R_1(t, u, \theta) := \int_{\Omega_1} \rho_0 \varphi_1 \theta u_t \, dx.$$

LEMMA 3.4. Given  $\eta > 0$  there exists a positive constant  $C_\eta$  such that

$$\begin{aligned} \frac{d}{dt} R_1(t, u, \theta) &\leq \eta \left\{ \int_{\Gamma_1} |\Delta u|^2 \, d\Gamma + \int_{\Omega_1} |\Delta u|^2 + |\nabla u_t|^2 \, dx \right\} \\ &\quad + C_\eta \int_{\Omega_1} |u_t|^2 + |\nabla \theta|^2 \, dx - \mu \int_{U_1} |\nabla u_t|^2 \, dx. \end{aligned}$$

*Proof.* Multiplying Eq. (1.2) by  $\varphi_1 u_t$  and integrating by parts, we get

$$\begin{aligned} \frac{d}{dt} R_1(t, u, \theta) &= \rho_0 \int_{\Omega_1} \varphi_1 \theta u_{tt} \, dx - \int_{\Omega_1} \beta_0 \nabla \theta \cdot \nabla (\varphi_1 u_t) + \gamma_0 \theta (\varphi_1 u_t) \, dx \\ &\quad - \mu \int_{\Omega_1} \varphi_1 |\nabla u_t|^2 \, dx - \mu \int_{\Omega_1} u_t \nabla \varphi_1 \cdot \nabla u_t \, dx. \end{aligned} \tag{3.19}$$

Let  $w$  be the solution of the stationary problem

$$\begin{aligned} (\rho_1 I - \gamma_1 \Delta) w &= \rho_0 \varphi_1 \theta \quad \text{in } \Omega_1, \\ w &= 0 \quad \text{on } \Gamma_1, \quad \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \Gamma_0. \end{aligned}$$

From the Trace Theorem and the elliptic regularity it follows that there exists  $C > 0$  such that

$$\|w\|_{H^2(\Omega_1)}^2 + \int_{\Gamma_1} \left| \frac{\partial w}{\partial \nu} \right|^2 \, d\Gamma \leq C \int_{\Omega_1} |\theta|^2 \, dx. \tag{3.20}$$

Multiplying Eq. (1.1) by  $w$  and performing an integration by parts, we get

$$\rho_0 \int_{\Omega_1} \varphi_1 \theta u_{tt} dx = \beta_1 \int_{\Gamma_1} \Delta u \frac{\partial w}{\partial \nu} d\Gamma - \beta_1 \int_{\Omega_1} \Delta u \Delta w dx + \mu \int_{\Omega_1} \nabla \theta \cdot \nabla w dx;$$

substitution of this identity into (3.19) yields

$$\begin{aligned} \frac{d}{dt} R_1(t, u, \theta) &= \beta_1 \int_{\Gamma_1} \Delta u \frac{\partial w}{\partial \nu} d\Gamma - \beta_1 \int_{\Omega_1} \Delta u \Delta w dx + \mu \int_{\Omega_1} \nabla \theta \cdot \nabla w dx \\ &\quad - \int_{\Omega_1} \beta_0 \nabla \theta \cdot \nabla (\varphi_1 u_t) + \gamma_0 \theta (\varphi_1 u_t) dx - \mu \int_{\Omega_1} \varphi_1 |\nabla u_t|^2 dx \\ &\quad - \mu \int_{\Omega_1} u_t \nabla \varphi_1 \cdot \nabla u_t dx. \end{aligned}$$

Applying Young and Poincaré’s inequalities and (3.20), our conclusion follows. □

Let us consider a vector field  $h = (h_1, \dots, h_n) \in [C^2(\Omega)]^n$  satisfying

$$h(x) := \begin{cases} -\nu(x) & \text{if } x \in \Gamma_1 \\ 0 & \text{if } x \in B_\delta(\Omega_2) \end{cases},$$

and let us introduce the following functionals:

$$\begin{aligned} J_1(t, u) &:= \int_{\Omega_1} \rho_1 u_t h \cdot \nabla u + \gamma_1 \nabla u_t \cdot \nabla (h \cdot \nabla u) dx, \\ R_2(t, u, \theta) &:= R_1(t, u, \theta) + \frac{2\eta}{\beta_1} J_1(t, u). \end{aligned}$$

LEMMA 3.5. Given  $\eta > 0$  there exists a positive constant  $C_\eta > 0$  such that

$$\begin{aligned} \frac{d}{dt} R_2(t, u, \theta) &\leq \eta C \left\{ \int_{\Omega_1} |\Delta u|^2 + |\nabla u_t|^2 dx \right\} + \int_{\Omega_2} |\Delta v|^2 dx \\ &\quad + C_\eta \int_{\Omega_1} |u_t|^2 + |\nabla \theta|^2 dx - \mu \int_{U_1} |\nabla u_t|^2 dx. \end{aligned}$$

*Proof.* Multiplying Eq. (1.1) by  $h \cdot \nabla u$  and integrating by parts, we get

$$\begin{aligned} \frac{d}{dt} J_1(t, u) &= -\frac{\rho_1}{2} \int_{\Omega_1} \operatorname{div} h |u_t|^2 dx + \gamma_1 \int_{\Omega_1} \nabla u_t \cdot \nabla h_i \frac{\partial u_t}{\partial x_i} dx \\ &\quad - \frac{\gamma_1}{2} \int_{\Omega_1} \operatorname{div} h |\nabla u_t|^2 dx - \frac{\beta_1}{2} \int_{\Gamma_1} |\Delta u|^2 d\Gamma + \frac{\beta_1}{2} \int_{\Omega_1} \operatorname{div} h |\Delta u|^2 dx \\ &\quad - \beta_1 \int_{\Omega_1} \Delta u \left( 2\nabla h_i \cdot \nabla \frac{\partial u}{\partial x_i} + \Delta h_i \frac{\partial u}{\partial x_i} \right) dx + \mu \int_{\Omega_1} \nabla \theta \cdot \nabla (h \cdot \nabla u) dx. \end{aligned}$$

Using Young and Poincaré’s inequality once more, we find that

$$\begin{aligned} \frac{d}{dt} J_1(t, u) &\leq -\frac{\beta_1}{2} \int_{\Gamma_1} |\Delta u|^2 d\Gamma \\ &\quad + C \left\{ \int_{\Omega_1} (|\Delta u|^2 + |\nabla u_t|^2 + |\nabla \theta|^2) dx + \int_{\Omega_2} |\Delta v|^2 dx \right\}. \end{aligned}$$

Combining this inequality with Lemma 3.4, we conclude that

$$\begin{aligned} \frac{d}{dt}R_2(t, u, \theta) &\leq \eta C \left\{ \int_{\Omega_1} |\Delta u|^2 + |\nabla u_t|^2 dx \right\} + \int_{\Omega_2} |\Delta v|^2 dx \\ &\quad + C_\eta \int_{\Omega_1} |u_t|^2 + |\nabla \theta|^2 dx - \mu \int_{U_1} |\nabla u_t|^2 dx. \end{aligned}$$

This completes the proof. □

Let us introduce the following functionals:

$$\begin{aligned} J_2(t, u) &:= \int_{\Omega} \rho_1 u_t \varphi_2 u + \gamma_1 \nabla u_t \cdot \nabla (\varphi_2 u) dx, \\ R_3(t, u, \theta) &:= R_2(t, u, \theta) + k_0 J_2(t, u), \end{aligned}$$

where the constant  $k_0$  is given by (3.22).

LEMMA 3.6. Given  $\eta > 0$ , there exists a positive constant  $C_\eta > 0$  such that

$$\begin{aligned} \frac{d}{dt}R_3(t, u, \theta) &\leq -\frac{k_0 \beta_1}{2} \int_{U_2} |\Delta u|^2 + |\nabla u_t|^2 dx + C_\eta \int_{\Omega_1} |\nabla u|^2 + |u_t|^2 + |\nabla \theta|^2 dx \\ &\quad + \eta C \left\{ \int_{\Omega_1} |\Delta u|^2 + |\nabla u_t|^2 dx \right\} + \int_{\Omega_2} |\Delta v|^2 dx. \end{aligned}$$

*Proof.* Multiplying Eq. (1.1) by  $\varphi_2 u$  and integrating by parts, we get

$$\begin{aligned} \frac{d}{dt}J_2(t, u) &= \rho_1 \int_{\Omega_1} \varphi_2 |u_t|^2 dx + \gamma_1 \int_{\Omega_1} \nabla \varphi_2 \cdot \nabla u_t u_t dx + \gamma_1 \int_{\Omega_1} \varphi_2 |\nabla u_t|^2 dx \\ &\quad - \beta_1 \int_{\Omega_1} \varphi_2 |\Delta u|^2 dx - \beta_1 \int_{\Omega_1} \Delta u (2 \nabla \varphi_2 \cdot \nabla u + \Delta \varphi_2 u) dx \\ &\quad - \mu \int_{\Omega_1} \theta \Delta (\varphi_2 u) dx. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{d}{dt}J_2(t, u) &\leq -\frac{\beta_1}{2} \int_{U_2} |\Delta u|^2 dx + C \int_{U_1} |\nabla u_t|^2 dx \\ &\quad + \eta \int_{\Omega_1} |\Delta u|^2 dx + C_\eta \int_{\Omega_1} |\nabla u|^2 + |\nabla \theta|^2 dx. \end{aligned} \tag{3.21}$$

Let  $k_0$  be a positive constant satisfying

$$\mu = k_0 \left( \frac{\beta_1}{2} + C \right). \tag{3.22}$$

From Lemma 3.5 and inequality (3.21), we find that

$$\begin{aligned} \frac{d}{dt}R_3(t, u, \theta) &\leq -\frac{k_0 \beta_1}{2} \int_{U_2} |\Delta u|^2 + |\nabla u_t|^2 dx + C_\eta \int_{\Omega_1} |\nabla u|^2 + |u_t|^2 + |\nabla \theta|^2 dx \\ &\quad + \eta C \left\{ \int_{\Omega_1} |\Delta u|^2 + |\nabla u_t|^2 dx \right\} + \int_{\Omega_2} |\Delta v|^2 dx, \end{aligned}$$

from where our conclusion follows. □

Let us denote by  $Ku := m \cdot \nabla u + (n-3)u/2$  and let us consider the following functionals:

$$\begin{aligned}
 J_3(t, u, v) &:= \int_{\Omega_1} \rho_1 u_t \psi_4 K u + \gamma_1 \nabla u_t \cdot \nabla (\psi_4 K u) \, dx \\
 &\quad + \int_{\Omega_2} \rho_2 v_t \psi_4 K v + \gamma_2 \nabla v_t \cdot \nabla (\psi_4 K v) \, dx, \\
 R_4(t, u, v, \theta) &:= R_3(t, u, \theta) + k_1 J_3(t, u, v),
 \end{aligned}$$

where the constant  $k_1$  is given by (3.25).

LEMMA 3.7. There exist positive constants  $k_2$  and  $C$  such that

$$\frac{d}{dt} R_4(t, u, v, \theta) \leq -\frac{k_2}{2} E(t, u, v, \theta) + C \int_{\Omega_1} \{ |\nabla u|^2 + |u_t|^2 + |\nabla \theta|^2 \} \, dx.$$

*Proof.* Multiplying Eq. (1.1) by  $\psi_4(x)Ku$ , Eq. (1.3) by  $\psi_4(x)Kv$ , and integrating by parts, we get

$$\begin{aligned}
 &\frac{d}{dt} J_3(t, u, v) \\
 &= -\frac{(\rho_1 - \rho_2)}{2} \int_{\Gamma_0} m \cdot \nu |u_t|^2 \, d\Gamma - \frac{(\gamma_1 - \gamma_2)}{2} \int_{\Gamma_0} m \cdot \nu |\nabla u_t|^2 \, d\Gamma \\
 &\quad - \beta_1 \int_{\Gamma_0} \Delta u m \cdot \frac{\partial \nabla(u-v)}{\partial \nu} \, d\Gamma + \frac{\beta_1}{2} \int_{\Gamma_0} m \cdot \nu |\Delta u|^2 \, d\Gamma - \frac{\beta_2}{2} \int_{\Gamma_0} m \cdot \nu |\Delta v|^2 \, d\Gamma \\
 &\quad + \mu \int_{\Omega_1} \nabla \theta \cdot \nabla (\psi_4 K u) \, dx - \frac{3\rho_1}{2} \int_{V_4} |u_t|^2 \, dx - \frac{\gamma_1}{2} \int_{V_4} |\nabla u_t|^2 \, dx - \frac{\beta_1}{2} \int_{V_4} |\Delta u|^2 \, dx \\
 &\quad - \frac{\rho_1}{2} \int_{U_4} \xi |u_t|^2 \, dx + \gamma_1 \int_{U_4} \nabla u_t \cdot \nabla (\psi_4 m_i) \frac{\partial u_t}{\partial x_i} \, dx + \frac{(n-3)\gamma_1}{2} \int_{U_4} \nabla u_t \cdot \nabla \psi_4 u_t \, dx \\
 &\quad - \frac{\gamma_1}{2} \int_{U_4} \xi |\nabla u_t|^2 \, dx + \frac{\beta_1}{2} \int_{U_4} \xi |\Delta u|^2 \, dx - \beta_1 \int_{U_4} \frac{\partial u}{\partial x_i} \Delta u \Delta (\psi_4 m_i) \, dx \\
 &\quad - 2\beta_1 \int_{U_4} \Delta u \nabla \frac{\partial u}{\partial x_i} \cdot \nabla (\psi_4 m_i) \, dx - \frac{(n-3)\beta_1}{2} \int_{U_4} \Delta u (\Delta \psi_4 u + 2\nabla u \cdot \nabla \psi_4) \, dx \\
 &\quad + \frac{1}{2} \int_{\Gamma_2} m \cdot \nu (\rho_2 |v_t|^2 + \gamma_2 |\nabla v_t|^2 + \beta_2 |\Delta v|^2) \, d\Gamma - \frac{3\rho_2}{2} \int_{\Omega_2} |v_t|^2 \, dx \\
 &\quad - \frac{\gamma_2}{2} \int_{\Omega_2} |\nabla v_t|^2 \, dx - \frac{\beta_2}{2} \int_{\Omega_2} |\Delta v|^2 \, dx,
 \end{aligned} \tag{3.23}$$

where  $\xi := \operatorname{div}(\psi_4 m) - (n-3)\psi_4$ . The transmission boundary conditions imply that

$$\begin{aligned}
 -\beta_1 \int_{\Gamma_0} \Delta u m \cdot \frac{\partial \nabla(u-v)}{\partial \nu} \, d\Gamma &= -\beta_1 \int_{\Gamma_0} \Delta u m_i \nu_j \frac{\partial^2(u-v)}{\partial x_i \partial x_j} \, d\Gamma \\
 &= -\beta_1 \int_{\Gamma_0} \Delta u m_i \nu_i \nu_j^2 \Delta(u-v) \, d\Gamma \\
 &= -\beta_1 \int_{\Gamma_0} \Delta u m \cdot \nu (\Delta u - \Delta v) \, d\Gamma \\
 &\leq -\frac{\beta_1}{2} \int_{\Gamma_0} m \cdot \nu (|\Delta u|^2 - |\Delta v|^2) \, d\Gamma.
 \end{aligned}$$

Substitution of the above inequality into the above identity (3.23), and using Young’s inequality, relations (3.18), and Lemma 2.1, we arrive at

$$\begin{aligned} \frac{d}{dt} J_3(t, u, v) \leq & \eta \left\{ \int_{\Omega_1} |\Delta u|^2 dx + \int_{\Omega_2} |\Delta v|^2 dx \right\} + C_\eta \int_{\Omega_1} |\nabla \theta|^2 dx \\ & + C \int_{U_2} |u_t|^2 + |\nabla u_t|^2 + |\Delta u|^2 dx + C \int_{\Omega_1} |\nabla u|^2 dx \\ & - \frac{3\rho_1}{2} \int_{V_4} |u_t|^2 dx - \frac{\gamma_1}{2} \int_{V_4} |\nabla u_t|^2 dx - \frac{\beta_1}{2} \int_{V_4} |\Delta u|^2 dx \\ & - \frac{3\rho_2}{2} \int_{\Omega_2} |v_t|^2 dx - \frac{\gamma_2}{2} \int_{\Omega_2} |\nabla v_t|^2 dx - \frac{\beta_2}{2} \int_{\Omega_2} |\Delta v|^2 dx. \end{aligned} \tag{3.24}$$

Let  $k_1$  be a positive constant such that

$$\frac{k_0 \beta_1}{2} = k_1(C + 1). \tag{3.25}$$

From Lemma 3.6 and inequality (3.24), it follows that the functional  $R_4$  satisfies

$$\begin{aligned} \frac{d}{dt} R_4(t, u, v, \theta) \leq & \eta C \left\{ \int_{\Omega_1} (|\Delta u|^2 + |\nabla u_t|^2) dx + \int_{\Omega_2} |\Delta v|^2 dx \right\} \\ & + C_\eta \int_{\Omega_1} \{ |\nabla u|^2 + |u_t|^2 + |\nabla \theta|^2 \} dx \\ & - k_1 \int_{U_2} (|u_t|^2 + |\nabla u_t|^2 + |\Delta u|^2) dx \\ & - \frac{3k_1 \rho_1}{2} \int_{V_4} |u_t|^2 dx - \frac{k_1 \gamma_1}{2} \int_{V_4} |\nabla u_t|^2 dx - \frac{k_1 \beta_1}{2} \int_{V_4} |\Delta u|^2 dx \\ & - \frac{3k_1 \rho_2}{2} \int_{\Omega_2} |v_t|^2 dx - \frac{k_1 \gamma_2}{2} \int_{\Omega_2} |\nabla v_t|^2 dx - \frac{k_1 \beta_2}{2} \int_{\Omega_2} |\Delta v|^2 dx. \end{aligned}$$

Since  $V_4 \supset V_2$ , we have that  $\Omega_1 = U_2 \cap V_4$ . Therefore, there exists a small positive constant  $k_2$  such that

$$\begin{aligned} \frac{d}{dt} R_4(t, u, v, \theta) \leq & \eta C \left\{ \int_{\Omega_1} (|\Delta u|^2 + |\nabla u_t|^2) dx + \int_{\Omega_2} |\Delta v|^2 dx \right\} \\ & + C_\eta \int_{\Omega_1} \{ |\nabla u|^2 + |u_t|^2 + |\nabla \theta|^2 \} dx - k_2 E(t, u, v, \theta). \end{aligned}$$

Taking  $\eta$  small enough, our conclusion follows. □

**Proof of Theorem 3.3:** Multiplying Eq. (1.1) by  $u_t$ , Eq. (1.2) by  $\theta$ , Eq. (1.3) by  $v_t$ , and performing an integration by parts, we obtain that

$$\frac{d}{dt} E(t, u, v, \theta) = - \int_{\Omega_1} \beta_0 |\nabla \theta|^2 + \gamma_0 |\theta|^2 dx - \int_{\Gamma_1} \beta_0 \lambda |\theta|^2 d\Gamma. \tag{3.26}$$

Let us define the Lyapunov’s functional  $L$ , given by

$$L(t, u, v, \theta) := NE(t, u, v, \theta) + R_4(t, u, v, \theta),$$

where  $N$  denotes a large positive constant to be fixed later. Taking  $N$  large and combining Lemma 3.7 and identity (3.26), we get

$$\frac{d}{dt}L(t, u, v, \theta) \leq -\frac{k_2}{2}E(t, u, v, \theta) - \frac{N}{2} \int_{\Omega_1} \beta_0 |\nabla \theta|^2 dx + CF(t, u).$$

Since the system (1.1)–(1.8) is linear and  $(u_0, v_0, u_1, v_1, \theta_0) \in D(\Lambda^5)$ , then  $(u^{(i)}, v^{(i)}, \theta^{(i)})$  are strong solutions for  $i = 1, 2, 3$ ; therefore, using the same procedure, we have

$$\frac{d}{dt}L(t, u^{(i)}, v^{(i)}, \theta^{(i)}) \leq -\frac{k_2}{2}E(t, u^{(i)}, v^{(i)}, \theta^{(i)}) - \frac{N}{2} \int_{\Omega_1} \beta_0 |\nabla \theta^{(i)}|^2 dx + CF(t, u^{(i)}).$$

Denoting by  $\mathcal{L}(t, u, v, \theta) = \sum_{i=1}^3 L(t, u^{(i)}, v^{(i)}, \theta^{(i)})$ , we conclude that

$$\frac{d}{dt}\mathcal{L}(t, u, v, \theta) \leq -\frac{k_2}{2}\mathcal{E}(t, u, v, \theta) - \frac{N}{2} \sum_{i=0}^3 \int_{\Omega_1} \beta_0 |\nabla \theta^{(i)}|^2 dx + CF(t, u),$$

where  $\mathcal{E}$  and  $\mathcal{F}$  are given by (3.5). Integrating over  $[0, T]$ , we get

$$\begin{aligned} \mathcal{L}(T, u, v, \theta) &\leq \mathcal{L}(0, u, v, \theta) - \frac{k_2}{2} \int_0^T \mathcal{E}(t, u, v, \theta) dt \\ &\quad - \frac{N}{2} \sum_{i=0}^3 \int_0^T \int_{\Omega_1} \beta_0 |\nabla \theta^{(i)}|^2 dx dt + C \int_0^T \mathcal{F}(t, u) dt. \end{aligned}$$

Using Lemma 3.2 with  $\eta$  small enough and taking  $N$  large enough, we obtain that

$$\begin{aligned} \mathcal{L}(T, u, v, \theta) &\leq \mathcal{L}(0, u, v, \theta) - \frac{k_2}{4} \int_0^T \mathcal{E}(t, u, v, \theta) dt \\ &\leq \mathcal{L}(0, u, v, \theta) - \frac{k_2 T}{4} \mathcal{E}(T, u, v, \theta). \end{aligned} \tag{3.27}$$

Using Young’s inequality, we can prove that

$$\frac{N}{2} \mathcal{E}(t, u, v, \theta) \leq \mathcal{L}(t, u, v, \theta) \leq 2N \mathcal{E}(t, u, v, \theta), \tag{3.28}$$

for  $N$  large enough. Combining (3.27) and (3.28), we conclude that

$$\mathcal{L}(T, u, v, \theta) \leq \mathcal{L}(0, u, v, \theta) - \frac{k_2 T}{8N} \mathcal{L}(T, u, v, \theta).$$

Denoting by  $\alpha := (1 + k_2 T / (8N))^{-1}$ , this inequality can be written as

$$\mathcal{L}(T, u, v, \theta) \leq \alpha \mathcal{L}(0, u, v, \theta).$$

Since  $\alpha \in ]0, 1[$ , the semigroup property of the system (1.1)–(1.7) implies that there exist positive constants  $C$  and  $\kappa$  such that

$$\mathcal{L}(t, u, v, \theta) \leq C \mathcal{L}(0, u, v, \theta) e^{-\kappa t}.$$

From (3.28) we conclude that

$$\mathcal{E}(t, u, v, \theta) \leq C \mathcal{E}(0, u, v, \theta) e^{-\kappa t},$$

from where our conclusions follow. □



COROLLARY 3.8. If  $(u_0, v_0, u_1, v_1, \theta_0) \in H_T^2 \times H_T^1 \times H^0$ , then there exist positive constants  $C, \kappa$  such that

$$E(t, u, v, \theta) \leq CE(0, u, v, \theta)e^{-\kappa t}$$

for any weak solution  $(u, v, \theta)$  of (1.1)–(1.8).

*Proof.* Let us consider  $\Theta_0$  the solution of the elliptic problem

$$\begin{aligned} -\beta_0 \Delta \Theta_0 + \gamma_0 \Theta_0 &= \rho_0 \theta_0 - \mu \Delta u_0 \quad \text{in } \Omega_1, \\ \Theta_0 &= 0 \quad \text{on } \Gamma_0, \quad \frac{\partial \Theta_0}{\partial \nu} + \lambda \Theta_0 = 0 \quad \text{on } \Gamma_1, \end{aligned} \tag{3.29}$$

and let  $(U_0, V_0)$  be the solution of the variational transmission problem

$$\int_{\Omega_1} \beta_1 \Delta U_0 \Delta \varphi \, dx + \int_{\Omega_2} \beta_2 \Delta V_0 \Delta \psi \, dx = \langle f, \varphi \rangle + \langle g, \psi \rangle, \tag{3.30}$$

for every  $(\varphi, \psi) \in H_T^2$ , where  $(f, g) \in H_T^{-1}$  is given by

$$\begin{aligned} \langle f, \varphi \rangle &= \int_{\Omega_1} \rho_1 u_1 \varphi + \gamma_1 \nabla u_1 \cdot \nabla \varphi + \mu \nabla \Theta_0 \cdot \nabla \varphi \, dx, \\ \langle g, \psi \rangle &= \int_{\Omega_2} \rho_1 v_1 \psi + \gamma_1 \nabla v_1 \cdot \nabla \psi \, dx. \end{aligned}$$

Let us define the following functions:

$$\begin{aligned} U(x, t) &:= \int_0^t u(x, s) \, ds + U_0(x), \\ V(x, t) &:= \int_0^t v(x, s) \, ds + V_0(x), \\ \Theta(x, t) &:= \int_0^t \theta(x, s) \, ds + \Theta_0(x). \end{aligned}$$

Since system (1.1)–(1.7) is linear, we have that  $(U, V, \Theta)$  also is solution of this system; moreover, it verifies

$$(U_t, V_t, \Theta_t) = (u, v, \theta) \quad \text{and} \quad (U(0), V(0), \Theta(0)) = (U_0, V_0, \Theta_0).$$

From Lemma 2.2 applied to (3.29)–(3.30), it follows that there exists a constant  $C > 0$  such that

$$\sum_{i=0}^1 E(0, U^{(i)}, V^{(i)}, \Theta^{(i)}) \leq CE(0, u, v, \theta).$$

Using the same above reasoning, we can make a solution  $(U, V, \Theta)$  such that

$$(U^{(3)}, V^{(3)}, \Theta^{(3)}) = (u, v, \theta) \quad \text{and} \quad \sum_{i=0}^3 E(0, U^{(i)}, V^{(i)}, \Theta^{(i)}) \leq CE(0, u, v, \theta). \tag{3.31}$$

Applying Theorem 3.3 to  $(U, V, \Theta)$ , we get that

$$\sum_{i=0}^3 E(t, U^{(i)}, V^{(i)}, \Theta^{(i)}) \leq C \sum_{i=0}^3 E(0, U^{(i)}, V^{(i)}, \Theta^{(i)}) e^{-\kappa t}.$$

In view of (3.31), we conclude that

$$E(t, u, v, \theta) \leq CE(0, u, v, \theta)e^{-kt}.$$

This completes the proof. □

**Appendix A.** In this section we give a simple proof of the elliptic regularity for the stationary transmission problem.

LEMMA 3.9. Let  $\lambda$  be given by (A.6). If  $\beta_1/\beta_2$  satisfies

$$\frac{\beta_1}{\beta_2} < \frac{1}{\lambda} \quad \text{or} \quad \frac{\beta_1}{\beta_2} > \lambda, \tag{A.1}$$

then item 2 of Lemma 2.2 holds.

*Proof.* Let us take  $(f, g) \in H^m(\Omega_1) \times H^m(\Omega_2)$  and  $(w, z) \in \mathbb{I}^m := H^{m+7/2}(\Gamma_0) \times H^{m+5/2}(\Gamma_0)$ . Consider  $u \in H^{m+4}(\Omega_1)$  a unique solution of

$$\Delta^2 u = f \quad \text{in } \Omega_1, \tag{A.2}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_1, \quad u = w \quad \text{and} \quad \frac{\partial u}{\partial \nu} = z \quad \text{on } \Gamma_0. \tag{A.3}$$

Then  $(\Delta u|_{\Gamma_0}, \frac{\partial \Delta u}{\partial \nu}|_{\Gamma_0}) \in H^{m+3/2}(\Gamma_0) \times H^{m+1/2}(\Gamma_0)$ . Now, let us consider  $v \in H^{m+4}(\Omega_2)$  the unique solution of

$$\Delta^2 v = g \quad \text{in } \Omega_2, \tag{A.4}$$

$$v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma_2, \quad \Delta v = \frac{\beta_1}{\beta_2} \Delta u \quad \text{and} \quad \frac{\partial \Delta v}{\partial \nu} = \frac{\beta_1}{\beta_2} \frac{\partial \Delta u}{\partial \nu} \quad \text{on } \Gamma_0. \tag{A.5}$$

Then  $(v|_{\Gamma_0}, \frac{\partial v}{\partial \nu}|_{\Gamma_0}) \in \mathbb{I}^m$ . In these conditions we can define the operator  $T_{fg} : \mathbb{I}^m \rightarrow \mathbb{I}^m$  given by

$$(w, z) \mapsto T_{fg}(w, z) := \left( v|_{\Gamma_0}, \frac{\partial v}{\partial \nu}|_{\Gamma_0} \right).$$

Our conclusion will follow, showing that there exists a fixed point to  $T_{fg}$ . Let us take  $(w_1, z_1)$  and  $(w_2, z_2)$  in  $\mathbb{I}^m$  and consider  $(u_i, v_i) \in H^{m+4}(\Omega_1) \times H^{m+4}(\Omega_2)$ ,  $i = 1, 2$  solutions of

$$\Delta^2 u_i = f \quad \text{in } \Omega_1,$$

$$u_i = \frac{\partial u_i}{\partial \nu} = 0 \quad \text{on } \Gamma_1, \quad u_i = w_i \quad \text{and} \quad \frac{\partial u_i}{\partial \nu} = z_i \quad \text{on } \Gamma_0,$$

$$\Delta^2 v_i = g \quad \text{in } \Omega_2,$$

$$v_i = \frac{\partial v_i}{\partial \nu} = 0 \quad \text{on } \Gamma_2, \quad \Delta v_i = \frac{\beta_1}{\beta_2} \Delta u_i \quad \text{and} \quad \frac{\partial \Delta v_i}{\partial \nu} = \frac{\beta_1}{\beta_2} \frac{\partial \Delta u_i}{\partial \nu} \quad \text{on } \Gamma_0,$$

and denote by

$$(w, z) := (w_1, z_1) - (w_2, z_2),$$

$$u := u_1 - u_2,$$

$$v := v_1 - v_2.$$

It is not difficult to show that  $(u, v)$  satisfies (A.2)–(A.5) with  $(f, g) = (0, 0)$ . Therefore

$$T_{fg}(w_1, z_1) - T_{fg}(w_2, z_2) = T_{00}(w, z).$$

Moreover, from the Trace Theorem and elliptic regularity it follows that there exist  $\lambda_1, \dots, \lambda_4$  such that

$$\begin{aligned} \|T_{00}(w, z)\|_{\mathbb{I}^m} &\leq \lambda_1 \frac{\beta_1}{\beta_2} \|v\|_{H^{m+4}(\Omega_2)} \\ &\leq \lambda_1 \lambda_2 \frac{\beta_1}{\beta_2} \left\| \left( \Delta u|_{\Gamma_0}, \frac{\partial \Delta u}{\partial \nu} \Big|_{\Gamma_0} \right) \right\|_{H^{m+3/2}(\Gamma_0) \times H^{m+1/2}(\Gamma_0)} \\ &\leq \lambda_1 \lambda_2 \lambda_3 \frac{\beta_1}{\beta_2} \|u\|_{H^{m+4}(\Omega_1)} \\ &\leq \lambda_1 \lambda_2 \lambda_3 \lambda_4 \frac{\beta_1}{\beta_2} \|(w, z)\|_{\mathbb{I}^m}. \end{aligned}$$

This implies that

$$\|T_{fg}(w_1, z_1) - T_{fg}(w_2, z_2)\|_{H^m} \leq \lambda \frac{\beta_1}{\beta_2} \|(w_1, z_1) - (w_2, z_2)\|_{\mathbb{I}^m}, \tag{A.6}$$

where  $\lambda = \lambda_1 \lambda_2 \lambda_3 \lambda_4$ . If  $\lambda \frac{\beta_1}{\beta_2} < 1$ , we have that  $T_{fg}$  has a unique fixed point in  $\mathbb{I}^m$ . Using this reasoning and swapping  $u$  and  $v$  in (A.2)–(A.5), we obtain the same result when  $\lambda \frac{\beta_2}{\beta_1} < 1$ . Therefore  $T_{fg}$  has a unique fixed point when  $\beta_1/\beta_2$  satisfies (A.1).  $\square$

**Appendix B.** In this section we prove that the functions belonging to  $D(\Lambda^m)$  are regular.

LEMMA 3.10. We have that  $D(\Lambda) = D_0$  and for any  $m \in \mathbb{N}$  with  $m \geq 2$  we get

$$D(\Lambda^m) \subset H^{m+2}(\Omega_1) \times H^{m+2}(\Omega_2) \times H^{m+1}(\Omega_1) \times H^{m+1}(\Omega_2) \times H^{m+1}(\Omega_1).$$

*Proof.* Let us take  $w \in D(\Lambda)$ , then  $w \in H^2_{\Gamma} \times H^2_{\Gamma} \times H^1_D$  and

$$\{A_0\{w_1, w_2\}, 0\} + B_0\{w_3, w_4, w_5\} = \{f_1, f_2, f_3\} \in H^{-1}_{\Gamma} \times H^0. \tag{B.1}$$

Multiplying this equation by  $\{0, 0, \phi\} \in H^1_{\Gamma} \times H^1_D$ , we obtain that

$$(w_5, \phi)_{H^1_D} = (f_3, \phi)_{H^0} + \int_{\Omega_1} \mu \Delta w_3 \phi \, dx.$$

Since  $f_3 + \mu \Delta w_3 \in L^2(\Omega_1)$ , elliptic regularity implies that

$$w_5 \in H^2(\Omega_1) \quad \text{and} \quad \frac{\partial w_5}{\partial \nu} + \lambda w_5 = 0 \text{ on } \Gamma_1. \tag{B.2}$$

On the other hand, multiplying Eq. (B.1) by  $\{\phi_1, \phi_2, 0\} \in H^2_{\Gamma} \times H^0$ , we get

$$\begin{aligned} (\{w_1, w_2\}, \{\phi_1, \phi_2\})_{H^2_{\Gamma}} &= \langle \{f_1, f_2\}, \{\phi_1, \phi_2\} \rangle_{H^{-1}_{\Gamma} \times H^1_{\Gamma}} \\ &\quad + \underbrace{\int_{\Omega_1} \mu \nabla w_5 \cdot \nabla \phi_1 \, dx}_{\langle F, \{\phi_1, \phi_2\} \rangle} \quad \forall \{\phi_1, \phi_2\} \in H^2_{\Gamma}. \end{aligned}$$

Since  $\{f_1, f_2\} + F \in H^{-1}_{\Gamma}$ , Lemma 2.2 implies that

$$\{w_1, w_2\} \in H^3(\Omega_1) \times H^3(\Omega_2) \quad \text{and} \quad \beta_1 \Delta w_1 = \beta_2 \Delta w_2 \text{ on } \Gamma_0. \tag{B.3}$$

From (B.2)–(B.3) we have that  $w \in D_0$ ; therefore, the first part of this Lemma is proved. We shall prove the second part using induction. Let  $m \in \mathbb{N}$  with  $m \geq 2$  and suppose that

$$D(\Lambda^m) \subset H^{m+2}(\Omega_1) \times H^{m+2}(\Omega_2) \times H^{m+1}(\Omega_1) \times H^{m+1}(\Omega_2) \times H^{m+1}(\Omega_1),$$

and let us take  $w \in D(\Lambda^{m+1})$  such that  $\Lambda w = f \in D(\Lambda^m)$ . Applying the operator  $-A$ , we get

$$(-A_0\{w_3, w_4\}, \{A_0\{w_1, w_2\}, 0\} + B_0\{w_3, w_4, w_5\}) = -(A_0\{f_1, f_2\}, A_1\{f_3, f_4, f_5\}).$$

Hence we find that

$$\{w_3, w_4\} = \{f_1, f_2\} \in H^{m+2}(\Omega_1) \times H^{m+2}(\Omega_2), \tag{B.4}$$

and

$$\{A_0\{w_1, w_2\}, 0\} + B_0\{w_3, w_4, w_5\} = -A_1\{f_3, f_4, f_5\}. \tag{B.5}$$

Multiplying Eq. (B.5) by  $\{0, 0, \phi\} \in H_1^1 \times H_0^1$ , we obtain that

$$(w_5, \phi)_{H_0^1} = -(f_5, \phi)_{H^0} + \int_{\Omega_1} \mu \Delta w_3 \phi \, dx.$$

Since  $-f_5 + \mu \Delta w_3 \in H^m(\Omega_1)$ , elliptic regularity implies that

$$w_5 \in H^{m+2}(\Omega_1). \tag{B.6}$$

Multiplying Eq. (B.5) by  $\{\phi_1, \phi_2, 0\} \in H_1^2 \times H^0$ , we get

$$(\{w_1, w_2\}, \{\phi_1, \phi_2\})_{H_1^2} = -(\{f_3, f_4\}, \{\phi_1, \phi_2\})_{H_1^1} - \int_{\Omega_1} \mu w_5 \Delta \phi_1 \, dx. \tag{B.7}$$

Since  $\{f_3, f_4, w_5\} \in H^{m+1}(\Omega_1) \times H^{m+1}(\Omega_2) \times H^{m+1}(\Omega_1)$ , we have that

$$h := \gamma_1 \frac{\partial f_3}{\partial \nu} - \gamma_2 \frac{\partial f_4}{\partial \nu} - \mu \frac{\partial w_5}{\partial \nu} \in H^{m-\frac{1}{2}}(\Gamma_0).$$

Let  $z$  be a function such that

$$z \in H^{m+3}(\Omega_1) \quad z = \frac{\partial z}{\partial \nu} = \Delta z = 0 \quad \text{on} \quad \Gamma_1 \cap \Gamma_0, \quad \frac{\partial \Delta z}{\partial \nu} = \begin{cases} 0 & \text{on} \quad \Gamma_1 \\ \frac{1}{\beta_1} h & \text{on} \quad \Gamma_0 \end{cases}. \tag{B.8}$$

and let us denote by  $\{\tilde{w}_1, \tilde{w}_2\}$  the solution of the stationary problem

$$\begin{aligned} \Delta^2 \tilde{w}_1 = F & := \frac{1}{\beta_1} [-\rho_1 f_3 + \gamma_1 \Delta f_3 - \mu \Delta w_5] - \Delta^2 z \quad \text{in} \quad \Omega_1, \\ \Delta^2 \tilde{w}_2 = G & := \frac{1}{\beta_2} [-\rho_2 f_4 + \gamma_2 \Delta f_4] \quad \text{in} \quad \Omega_2, \end{aligned}$$

with boundary conditions

$$\begin{aligned} \tilde{w}_1 = \tilde{w}_2, \quad \frac{\partial \tilde{w}_1}{\partial \nu} = \frac{\partial \tilde{w}_2}{\partial \nu}, \quad \beta_1 \Delta \tilde{w}_1 = \beta_2 \Delta \tilde{w}_2, \quad \beta_1 \frac{\partial \Delta \tilde{w}_1}{\partial \nu} = \beta_2 \frac{\partial \Delta \tilde{w}_2}{\partial \nu} & \text{on} \quad \Gamma_0, \\ \tilde{w}_1 = \frac{\partial \tilde{w}_1}{\partial \nu} = 0 & \text{on} \quad \Gamma_1, \quad \tilde{w}_2 = \frac{\partial \tilde{w}_2}{\partial \nu} = 0 & \text{on} \quad \Gamma_2, \end{aligned}$$

Since  $\{F, G\} \in H^{m-1}(\Omega_1) \times H^{m-1}(\Omega_2)$ , Lemma 2.2 implies that

$$\{\tilde{w}_1, \tilde{w}_2\} \in H^{m+3}(\Omega_1) \times H^{m+3}(\Omega_2). \tag{B.9}$$

The solution of variational equation (B.7) is unique; therefore  $w_1 = z + \tilde{w}_1$ ,  $w_2 = \tilde{w}_2$ . From (B.8)–(B.9) we have

$$\{w_1, w_2\} \in H^{m+3}(\Omega_1) \times H^{m+3}(\Omega_2). \quad (\text{B.10})$$

Now, from (B.4), (B.6), and (B.10), we conclude that

$$w \in H^{m+3}(\Omega_1) \times H^{m+3}(\Omega_2) \times H^{m+2}(\Omega_1) \times H^{m+2}(\Omega_2) \times H^{m+2}(\Omega_1),$$

which completes the proof.  $\square$

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